# GENERALIZATIONS OF THE FUNDAMENTAL THEOREM OF PROJECTIVE GEOMETRY

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By

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## Abstract

The fundamental theorem of projective geometry states that every projective plane (satisfying sufficiently many axioms) is a projective plane over a commutative field, and every line-preserving bijection of such a projective plane arises through the projective action of the general linear group (possibly composed with a mapping induced by an automorphism of the underlying field). This theorem extends to higher-dimensional projective spaces, and has an affine version, which may be due to Darboux, which states that every line-preserving bijection of an affine space is composed of a linear transformation and a translation and a mapping induced by an automorphism of the underlying field, or, in modern terminology, is an affine map composed with a mapping induced by an automorphism of the underlying field. In the 1800s, properties of maps such as conformal and Möbius transformations were worked out, and, in particular, Liouville showed that sufficiently smooth conformal mappings of the real plane arose from the action of the conformal group. [1] In the 1950s and 1960s, more work was done on conformal mapping, and the regularity assumptions of Liouville's theorem were relaxed by analysts such as Gehring [10] and Reshetnyak [20], and theorems were proved about conformal mappings defined on open subsets of Euclidean space, rather than on the whole space, or the one-point compactification of the whole space. Earlier, in the 1930s, Carathéodory [3] had considered maps of spheres preserving circles in the spheres, and maps of subsets of the sphere preserving arcs as well. Curiously enough, maps of subsets of the real plane preserving line segments did not seem to have been considered until quite recently (except that mappings of the disc that preserve line segments can be interpreted as mappings of the Klein model of the hyperbolic plane that preserve geodesics, and it was apparently well-known that these come from the isometry group of the hyperbolic plane).

In the 1970s, Tits [23] extended the fundamental theorem of projective geometry to show that certain bijections of the spherical Tits building associated to a semisimple group $G$ over


[1] Liouville's contribution is discussed in [7].

a general field of rank at least two come from the action of $G$, possibly composed with a mapping induced by an automorphism of the underlying field; namely, the bijections that preserves the fibrations of the building associated to the fibrations of $G$ by the cosets of parabolic subgroups $P$ containing a fixed minimal parabolic subgroup $P_0$. A little later, Mostow [17] considered certain mappings of $G/P_0$ in the proof of his "rigidity theorem", and showed that these came from the action of the group $G$; this involved real Lie groups only. In the 1990s, others, including Gromov and Schoen [12], and Corlette [6] extended Mostow's ideas using geometric tools such as harmonic mappings; some of these extensions dealt with local fields. At around the same time, in the case of real Lie groups, Yamaguchi [24] proved that smooth, locally defined mappings of the spaces $G/P$ that preserved the fibrations considered by Tits also arose from the action of the group $G$.

This thesis unifies the geometric themes of the classical authors with the group-theoretic approach of Tits. Associated to the classical real Lie groups $G$, there are spaces of the form $G/P$ for some parabolic subgroup $P$ of geometric significance; for example, some $G/P$ are quasi-spheres. We consider bijections of these spaces that preserve their geometry, such as transformations of quasi-spheres that preserve quasi-circles, and show geometrically that these arise from the $G$-action. Thus Tits' results follow from purely geometric considerations. However, we do more: we prove local versions of these theorems (along the lines of the result of Carathéodory mentioned above), and thereby generalize the work of Yamaguchi. Moreover, we prove local theorems over arbitrary non-discrete fields, thereby extending theorems that seem to rely on real differential geometry into a much broader context: indeed, our work seems to hint that there might be some form of "topological geometry" that lies between algebraic geometry and differential geometry.

Some recent work was important for this thesis. In the paper [4], Čap, Cowling, De Mari, Eastwood, and McCallum show that maps of open sets in the plane that preserve collinearity come from projective transformations, using an argument that involves order; we also show that this implies a local version of Tits' "fundamental theorem of projective geometry" for the group $\mathrm{SL}(3, \mathbb{R})$. In this thesis, we give an alternative proof of the collinearity-preserving result that works for an arbitrary non-discrete topological field. We reproduce part of the paper [4] in Chapter 2 of this thesis.

The reader of this thesis will encounter a review of the "fundamental theorem of projective geometry" over the real and complex numbers $\mathbb{R}$ and $\mathbb{C}$, as well as the quaternions $\mathbb{H}$ in Part I. I discuss both global and local theorems; the former are well-known but the latter are new. In some sense, the key result is that mappings defined on open subsets of the real plane, that preserve line segments, whose range have sufficiently many points in general position, are projective mappings, and I give two proofs of this: the proof of Čap et al. [4] is given in Chapter 2 (Theorem 2.5), and my own is in Chapter 3 (Theorem 3.4). In Part II, I consider extensions to other classical groups. Here there are a number of key geometric results; the first of these, Theorem 4.8, which extends Carathéodory's theorem, is that mappings that send quasi-circles into quasi-circles come from the group $G$. In this part, we also consider flag manifolds appropriate to the various classical groups and show how Theorem 4.8 can be interpreted as a result about mappings of a flag manifold.

Part III of the thesis deals with non-discrete topological fields, and more general rings, including the adeles. We extend many of the results for classical Lie groups and geometries into this generality. The major result here, Lemma 6.8, is both topological and algebraic in nature: "local homomorphisms" of these extend to global homomorphisms. This enables us to extend the local version of the fundamental theorem of projective geometry to topological fields and division rings and even some rings that are not division rings. Finally, in Part IV, we consider some results where the hypotheses involve measurable sets rather than open sets.

# Contents

# Part I

# The General Linear Group in the Real, Complex, and Quaternionic Cases

# Chapter 1

# Introduction

## 1.1 Notation

We use $\mathbb{R}$ to denote the field of real numbers, $\mathbb{C}$ to denote the field of complex numbers, $\mathbb{H}$ to denote the ring of quaternions.

Throughout the thesis all modules over a division ring are right modules unless explicitly stated otherwise. Also, all projective spaces are right projective spaces unless explicitly stated otherwise. (The definitions of right projective spaces and left projective spaces are given below.) If $\mathbb{K}$ is a division ring then we view the affine space $\mathbb{K}^n$ as a right module over $\mathbb{K}$.

**Definition 1.1.** *Given a module $V$ over a division ring $\mathbb{K}$, we may consider the equivalence relation $R$ induced on $V \setminus \{0\}$ by right multiplication; that is, $vRw$ if and only if there exists $\lambda \in \mathbb{K} \setminus \{0\}$ such that $v = w\lambda$. The quotient of $V \setminus \{0\}$ by $R$ is a right projective space over $\mathbb{K}$. If $V = \mathbb{K}^{n+1}$, then the projective space is denoted by $\mathbb{K}P^n$. The equivalence class containing $(x_1, x_2, \ldots, x_{n+1})$ is denoted $(x_1 : x_2 : \ldots : x_{n+1})$ and these are said to be the homogeneous co-ordinates of that point.*

**Definition 1.2.** *Given a left module $V$ over a division ring $\mathbb{K}$, we may consider the equivalence relation $R$ induced on $V \setminus \{0\}$ by left multiplication; that is, $vRw$ if and only if there exists $\lambda \in \mathbb{K} \setminus \{0\}$ such that $v = \lambda w$. The quotient of $V \setminus \{0\}$ by $R$ is a left projective* *space over $\mathbb{K}$. If $V = \mathbb{K}^{n+1}$, then the corresponding left projective space is denoted by $\mathbb{K}P^n_L$. Homogeneous co-ordinates are defined similarly as in the case of right projective spaces.*

**Definition 1.3.** *Suppose that $\mathbb{K}$ is a division ring and consider $\mathbb{K}^n$ as a right module. A straight line in $\mathbb{K}^n$ is defined to be a set of the form $\{\boldsymbol{a}+\boldsymbol{b}\cdot k \mid k \in \mathbb{K}\}$, where $\boldsymbol{a} \in \mathbb{K}^n$, $\boldsymbol{b} \in \mathbb{K}^n$. A straight line in $\mathbb{K}P^n$ is defined to be the image, under the canonical projection $\mathbb{K}^{n+1}\setminus\{0\} \to \mathbb{K}P^n$, of a set of the form $W \setminus \{0\}$ where $W$ is a submodule of $\mathbb{K}^{n+1}$ of rank two.*

We shall sometimes speak of the slope of a line in $\mathbb{K}^n$. If $\mathbb{K}^n$ is viewed as a right module, then the slope of a line in $\mathbb{K}^n$ has to be viewed as an element of a left projective space. (Of course in the case where $\mathbb{K}$ is commutative it makes no difference whether we speak of a left projective space or a right projective space.) By the slope of a line in $\mathbb{K}^n$ we shall mean an element of $\mathbb{K}P^{n-1}_L$ defined as follows: the slope of the line $\{\mathbf{a} + \mathbf{b} \cdot k \mid k \in \mathbb{K}\}$ is the image of $\mathbf{b}$ under the canonical projection from $\mathbb{K}^n$ (viewed as a bimodule) to $\mathbb{K}P^{n-1}_L$. It is easy to see that this is well-defined.

If $\mathbb{K}$ is a division ring, then $\mathrm{GL}(n, \mathbb{K})$ is the group of invertible $n$-by-$n$ matrices with entries in $\mathbb{K}$, which may be thought of as acting on $\mathbb{K}^n$ on the left. In the case where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, this is a classical group. Other classical groups will be introduced and defined in Chapter 4, in Definition 4.1.

In this thesis we shall sometimes speak of the usual topology on $\mathbb{K}P^n$, or $\mathbb{K}P^n_L$, or $\mathrm{GL}(n, \mathbb{K})$, where $\mathbb{K}$ is a topological division ring. This is not the Zariski topology (which makes sense in the case where $\mathbb{K}$ is commutative). Rather, the space $\mathbb{K}^{n+1}$ is given the product topology and this induces a topology on $\mathbb{K}P^n$ and $\mathbb{K}P^n_L$. In the case of $\mathrm{GL}(n, \mathbb{K})$, the space $M_{nn}(\mathbb{K})$ is identified with $\mathbb{K}^{n^2}$ and given the product topology, and then $\mathrm{GL}(n, \mathbb{K})$ is given the subspace topology. When we speak of topologies on these spaces the topology in question should be understood to be one of these topologies unless explicitly indicated otherwise.

The notions of hyperbolic space, Euclidean space, and Möbius transformation are reviewed in Appendix A.

## 1.2 The fundamental theorem of projective geometry: projective spaces

Our aim in this section is to explain a version of the fundamental theorem of projective geometry which deals with mappings from projective spaces to themselves. In this thesis we shall be exploring generalizations of a number of geometrical theorems. We give three of those theorems below. An account of them can be found in [15]. [1]

**Theorem 1.4.** *Given a hyperbolic space $H^n$, where $n > 1$, a bijection $H^n \to H^n$ which maps geodesics into geodesics is an isometry.*

**Theorem 1.5.** *(Darboux.) Given a Euclidean space $E^n$, where $n > 1$, a bijection $E^n \to E^n$ which maps straight lines into straight lines is an affine transformation.* [2]

**Theorem 1.6.** *Given a sphere $S^n$, where $n > 1$, a bijection $S^n \to S^n$ which maps circles into circles is a Möbius transformation.* [3]

The definitions of hyperbolic space, Euclidean space, and Möbius transformations are reviewed in Appendix A.

In Part I we shall be looking at generalizations of Theorem 1.5. The two-dimensional case of this result was known to Darboux in the nineteenth century [15]. In fact, for Darboux, an affine transformation of $E^n$, where $n > 1$, was a bijection $E^n \to E^n$ which mapped straight lines into straight lines, so that for Darboux our statement of Theorem 1.5 would be a tautology. However, Darboux also knew that the class of affine transformations, so

[1] In [15] the phrase "preserves geodesics" is used rather than "maps geodesics into geodesics" in the statement of Theorem 1.4, and similarly with the other theorems. This could be interpreted as "maps geodesics into geodesics" or "maps geodesics onto geodesics". It is easy to show and will become clear in Section 3.1 that the two are equivalent assuming that the mapping is a bijection. We believe that our interpretation is probably what was intended in [15] and it fits in with our exposition later.

[2] In [15] it is stated that the two-dimensional case was known to Darboux in the nineteenth century. The higher-dimensional case follows by a very easy induction argument and must have been discovered shortly afterwards.

[3] In [15] it is stated that this result was known to Coxeter and was almost certainly known earlier. See [1], [5], [8], [9], [11], [14], [3]. In [15] the theorem refers to $\mathbb{R}^n_\infty$ rather than $S^n$; as discussed in Appendix A this makes no difference.

defined, was equal to the class of affine transformations as we define it today, at least in the two-dimensional case. [4]

It is possible to state Theorem 1.5 with a weaker hypothesis, namely, the following is true: if $n > 1$, and $\phi$ is a mapping $E^n \to E^n$, such that the range of $\phi$ contains $n+1$ points in general position, and $\phi$ maps straight lines into straight lines, then $\phi$ is an affine transformation. There is also another closely related theorem: denote by $\mathbb{R}P^n$ real $n$-dimensional projective space, where $n > 1$. If $\phi$ is a mapping $\mathbb{R}P^n \to \mathbb{R}P^n$, such that the range of $\phi$ contains $n+1$ points in general position, and $\phi$ maps straight lines into straight lines, then $\phi$ is a projective transformation. Below we shall discuss how to generalize this result first to arbitrary fields and then to arbitrary division rings.

The first generalization it is natural to inquire about is: what happens if we vary the underlying field? In that case, we have to allow for the fact that the field may admit nontrivial self-homomorphisms, as we shall explain in what follows. The field $\mathbb{R}$ does not, as we shall see in Appendix B. Suppose that $\mathbb{K}$ is a field and suppose that $n$ is an integer such that $n > 1$. If $\sigma$ is a unital field homomorphism $\mathbb{K} \to \mathbb{K}$, denote by $\sigma^*$ the mapping $\mathbb{K}P^n \to \mathbb{K}P^n$ which maps $(x_0 : x_1 : \ldots : x_n) \mapsto (\sigma(x_0) : \sigma(x_1) : \ldots : \sigma(x_n))$ (this can easily be seen to be well-defined). If $g \in \mathrm{GL}(n+1, \mathbb{K})$, then denote by $g^*$ the projective transformation $\mathbb{K}P^n \to \mathbb{K}P^n$ with matrix $g$. Suppose that $\phi$ is of the form $g^* \circ \sigma^*$ where $g \in \mathrm{GL}(n+1, \mathbb{K})$ and $\sigma$ is a unital homomorphism $\mathbb{K} \to \mathbb{K}$. Then $\phi$ is a mapping $\mathbb{K}P^n \to \mathbb{K}P^n$, such that the range of $\phi$ contains $n+1$ points in general position, and $\phi$ maps straight lines into straight lines. The mapping $\phi$ will be injective, and will be bijective if $\sigma$ is surjective.

The definition of $\mathbb{K}P^n$ can be generalized to the case where $\mathbb{K}$ is a non-commutative division ring, as discussed in Section 1.1. One proceeds as follows. Consider $\mathbb{K}^{n+1}$ as a right module over $\mathbb{K}$ of rank $n+1$. Denote by $R$ the equivalence relation on $\mathbb{K}^{n+1} \setminus \{\mathbf{0}\}$ induced by right multiplication: namely we have $vRw$ if and only if there exists a $\lambda \in \mathbb{K} \setminus \{0\}$ such

[4] The author communicated with Jeffers, but was unable to locate the original citation to Darboux.

that $v\lambda = w$. We define $\mathbb{K}P^n$ to be the quotient of $\mathbb{K}^{n+1} \setminus \{\mathbf{0}\}$ by $R$. If $W$ is a right submodule of $\mathbb{K}^{n+1}$ of rank two, the projection of $W \setminus \{\mathbf{0}\}$ to $\mathbb{K}P^n$ is said to be a straight line. Projective transformations are induced by $\mathbb{K}$-module automorphisms of $\mathbb{K}^{n+1}$ acting on the left. Understanding projective spaces over non-commutative division rings in this way, we can now give a result known as the fundamental theorem of projective geometry. It appears in Chapter 3 as Theorem 3.1 and the proof is given there.

**Theorem 3.1.** *Any mapping $\phi : \mathbb{K}P^n \to \mathbb{K}P^n$, whose range contains $n+1$ points in general position, which maps straight lines into straight lines, is of the form $g^* \circ \sigma^*$, where $g$ is some $\mathbb{K}$-module automorphism of $\mathbb{K}^{n+1}$, and $\sigma$ is some unital homomorphism $\mathbb{K} \to \mathbb{K}$.*

For the new results in Part I of this thesis (which will appear as Theorems 2.1 to 2.2 and 3.3 to 3.7) we shall be concerned only with the cases $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. In Part III we will generalize the new results of Part I to other topological division rings. It may be helpful briefly to discuss the set of unital homomorphisms $\mathbb{K} \to \mathbb{K}$ in each of the cases where $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. We do this in Appendix B.

As observed in Appendix B, the only unital homomorphism $\mathbb{R} \to \mathbb{R}$ is the identity, and every unital homomorphism $\mathbb{H} \to \mathbb{H}$ is an inner automorphism. As a consequence of this, in these two cases we can give a statement of the fundamental theorem of projective geometry which makes no mention of unital ring homomorphisms. This is clear in the case $\mathbb{K} = \mathbb{R}$. Regarding the case $\mathbb{K} = \mathbb{H}$, suppose that $\sigma$ is a unital homomorphism $\mathbb{H} \to \mathbb{H}$. Then there exists a unit quaternion $r$ such that $\sigma(q) = rqr^{-1}$ for all quaternions $q$. In that case $\sigma^*$ is the projective transformation induced by the matrix $D$ which is a diagonal matrix whose diagonal entries are all $r$. This is because

$$\sigma^*(x_1 : x_2 : \ldots : x_{n+1}) = (rx_1r^{-1} : rx_2r^{-1} : \ldots : rx_{n+1}r^{-1}) = (rx_1 : rx_2 : \ldots : rx_{n+1}).$$

Hence in the cases where $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$ the mapping $\phi$ is guaranteed to be simply a projective transformation.

This completes the discussion of the fundamental theorem of projective geometry, understood as a result about mappings from projective spaces to themselves. There is another way to understand it which seems to be due to Bertini. We discuss this in the next section.

## 1.3 The fundamental theorem of projective geometry: spaces of full flags

In this section we discuss an equivalent formulation of the fundamental theorem of projective geometry which deals with mappings from spaces of full flags to themselves. The result is due to Bertini and appears in [2]. We define the necessary concepts below and then give the statement of the theorem.

Suppose that $\mathbb{K}$ is a division ring and suppose that $n$ is an integer greater than two.

**Definition 1.7.** *A full flag in $\mathbb{K}^n$ is a sequence $(V_1, V_2, \dots V_n)$ where each $V_i$ is a submodule of $\mathbb{K}^n$ of rank $i$, and $V_i \subset V_j$ if $i < j$. A partial flag in $\mathbb{K}^n$ is a sequence $(V_{i_1}, V_{i_2}, \dots V_{i_j})$ where each $V_{i_k}$ is a submodule of $\mathbb{K}^n$ of rank $i_k$, and $i_{k_1} < i_{k_2}$ if $k_1 < k_2$, and $V_{i_{k_1}} \subset V_{i_{k_2}}$ if $k_1 < k_2$. The space of full flags, $F$, is the set of full flags in $\mathbb{K}^n$.*

We can identify $F$ with the left coset space $G/P$, where $G = \mathrm{GL}(n, \mathbb{K})$ and $P$ is the subgroup of $G$ consisting of the lower triangular matrices. In the case where $\mathbb{K}$ is commutative, $G$ is an algebraic group, and $P$ is a Zariski closed subgroup. In the case where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, $G$ is a Lie group, and $P$ is a closed subgroup (hence $G/P$ can be given a real-analytic manifold structure, so that in this case we speak of the "flag manifold" whereas in the more general case we speak of the "space of flags").

We will now show that we may identify $F$ with $G/P$. There is a natural transitive action of $G$ on $F$, namely the action whereby $g(V_1, V_2, \dots V_n) = (gV_1, gV_2, \dots gV_n)$ for all $g \in G$. If we write $\{\mathbf{e}_1, \mathbf{e}_2, \dots \mathbf{e}_n\}$ for the standard basis for $\mathbb{K}^n$, and suppose that $V_i$ is the module spanned by $\mathbf{e}_n, \mathbf{e}_{n-1}, \dots \mathbf{e}_{n-i+1}$ for $i = 1, 2, \dots n$, then the stabilizer of $(V_1, V_2, \dots V_n)$ is the group $P$ of lower triangular matrices, and as observed the action is transitive. In this way,

$F$ may be identified with $G/P$. The element of $F$ corresponding to $gP$ will be $(V_1, V_2, \dots V_n)$ where $V_i$ is the module spanned by the rightmost $i$ columns of the matrix $g$.

We shall now discuss spaces of partial flags.

**Definition 1.8.** *For $k = 1, 2, \dots n-1$, denote by $P_k$ the subgroup of $G$ consisting of those matrices $(m_{ij})$ such that $m_{ij} = 0$ if $j > i$ and $(i, j) \neq (k, k+1)$. This is a subgroup of $G$ containing $P$. The space $G/P_k$ may be identified with the space of partial flags $(V_1, V_2, \dots, V_{n-k-1}, V_{n-k+1}, \dots, V_n)$, where each $V_i$ is a submodule of $\mathbb{K}^n$ of rank $i$, and $V_i \subset V_j$ if $i < j$. These flags are like full flags except that they are missing a module of rank $n-k$. There is a natural projection from $G/P$ to $G/P_k$: the fibres for this projection are called type $k$ fibres.*

If $Q$ is a subgroup of $G$ generated by some finite set of the $P_k$s, then $G/Q$ may be identified with the space of partial flags $(V_{i_1}, V_{i_2}, \dots, V_{i_l})$, where each $V_{i_j}$ is a submodule of $\mathbb{K}^n$ of rank $i_j$, and $V_{i_j} \subset V_{i_k}$ if $j < k$, and $(i_1, i_2, \dots, i_l)$ is some increasing finite sequence of integers depending on $Q$, such that $1 \leq i_j \leq n$ for all $j$ such that $1 \leq j \leq l$. If we let $Q$ be the subgroup generated by the $P_k$ for $k = 1, 2, \dots n-2$, then $G/Q = \mathbb{K}P^{n-1}$.

**Definition 1.9.** *If $Q_1$ and $Q_2$ are two subgroups of $G$, each generated by some finite set of the $P_k$, such that $Q_1 \subset Q_2$, then there is a natural projection $G/Q_1 \to G/Q_2$. In particular if $F = G/P$ then there is a natural projection $F \to \mathbb{K}P^{n-1}$. We shall say that $n$ points in $F$ are in general position if their images under this projection are in general position.*

In Section 1.2 we discussed a version of the fundamental theorem of projective geometry which dealt with mappings from projective spaces to themselves. In the current section we are expounding a version of the fundamental theorem of projective geometry which deals with mappings from spaces of full flags to themselves. We will demonstrate that these two versions are equivalent in Chapter 3. We outline the main idea of the argument now. A crucial observation is the following.

**Definition 1.10.** *A mapping $\phi : G/P \to G/P$ is a good mapping if its range has $n$ points in general position and it maps type $k$ fibres into type $k$ fibres for $k = 1, 2, \ldots n-1$. A mapping $\psi : \mathbb{K}P^{n-1} \to \mathbb{K}P^{n-1}$ is a good mapping if its range has $n$ points in general position and it maps straight lines into straight lines.*

Given a good mapping $\phi : G/P \to G/P$, we may define a good mapping $\phi^* : \mathbb{K}P^{n-1} \to \mathbb{K}P^{n-1}$ as follows. Suppose that $p$ is the natural projection $G/P \to \mathbb{K}P^{n-1}$. It can be shown that if $p(x_1) = p(x_2)$, then $p(\phi(x_1)) = p(\phi(x_2))$. Then we can let $\phi^*$ be the mapping which maps $p(x)$ to $p(\phi(x))$, since this is well-defined by the foregoing. It can be shown that $\phi^*$ is a good mapping, and that given any good mapping $\psi : \mathbb{K}P^{n-1} \to \mathbb{K}P^{n-1}$ there exists a unique good mapping $\phi : G/P \to G/P$ such that $\phi^* = \psi$. We shall prove this in Chapter 3.

From this result, it will be possible to show that the fundamental theorem of projective geometry described in Section 1.2 can equivalently be formulated as follows. Define $F$ to be $G/P$. Given a unital homomorphism $\sigma : \mathbb{K} \to \mathbb{K}$, by abuse of notation let $\sigma$ also be the mapping $G \to G$ acting via the original $\sigma$ on the entries of the matrix (which clearly maps $P$ into $P$), and let $\sigma^*$ be the mapping thereby induced from $F$ to itself. Given $g \in G$, let $g^*$ be the mapping from $G/P \to G/P$ which maps $hP \mapsto ghP$.

**Theorem 3.2.** *Any good mapping $\phi : F \to F$ is of the form $g^* \circ \sigma^*$, for some $g \in G$ and some unital homomorphism $\sigma : \mathbb{K} \to \mathbb{K}$.*

This version of the result appears in Chapter 3 as Theorem 3.2 and is proved there.

## 1.4 Local versions of the fundamental theorem of projective geometry

All of the results discussed so far are already known. Next we discuss some new local versions of these results which are the product of team-based research, with contributions from Michael Eastwood, Andreas Čap, Filippo de Mari, and Michael Cowling. The results of the team-based part of the research is presented in [4]. In Part I, we state and prove these local results in the cases $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. We shall describe these results in this section.

Let us consider a result related to the simplest case of Theorem 3.1: a mapping from the real plane to the real plane, whose range has three points in general position, which maps straight lines into straight lines, is an affine transformation. Now, suppose we have a mapping from a nonempty open subset of the real plane into the real plane, whose range has three points in general position, which maps collinear points to collinear points. What can be said then?

It turns out that the mapping must be a projective transformation. This result remains true if we make the codomain the projective plane, rather than the affine plane. It also remains true if we make the domain an open connected subset of the projective plane, rather than of the affine plane. The result then generalizes to higher dimensions and Theorem 1.4 is an immediate corollary of this, using the Klein model for hyperbolic space. The result also generalizes to the complex numbers and the quaternions. (In the complex case, one obviously must allow for the fact that the field of complex numbers admits nontrivial unital self-homomorphisms.) This will be proved as Theorem 3.5 in Chapter 3.

**Theorem 3.5.** *Suppose that $n$ is an integer such that $n > 1$. Suppose that $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Suppose that $\phi$ is a mapping defined on an open subset $U$ of $\mathbb{K}P^n$, such that the range of $\phi$ has $n+1$ points in general position, and such that $\phi$ maps collinear points to collinear points. Then $\phi$ is the restriction to $U$ of a mapping of the form $\psi \circ \sigma^*$, where $\psi$ is a projective transformation and $\sigma$ is a unital homomorphism $\mathbb{K} \to \mathbb{K}$.*

There is also a result which deals with the flag manifold. In preparation for stating this result, we mention a different version of Theorem 3.5 which makes use of the notion of a connected set. This result weakens the hypothesis in one respect and strengthens it in another. Specifically, the hypothesis is weakened in that we require only that the mapping maps connected subsets of straight lines into straight lines. If that were the only change we made, then the result would no longer be true, because there could be two disjoint open sets and a different projective transformation acting on each of them. So we must also strengthen the hypotheses in another respect, namely we require that the domain of the mapping be connected as well as open.

**Theorem 1.11.** *Suppose that $n$ is an integer such that $n > 1$. Suppose that $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Suppose that $\phi$ is a mapping defined on an open connected subset $U$ of $\mathbb{K}^n$, such that the range of $\phi$ has $n+1$ points in general position, and such that $\phi$ maps connected subsets of straight lines contained in $U$ into straight lines. Then $\phi$ is the restriction to $U$ of a mapping of the form $\psi \circ \sigma^*$, where $\psi$ is a projective transformation and $\sigma$ is a unital homomorphism $\mathbb{K} \to \mathbb{K}$.*

We will discuss how to generalize the results of this section to other topological division rings in Part III; the matter of how to generalize these results dealing with connected sets to rings which are not locally connected is quite delicate.

There is another way in which the hypothesis can be strengthened, which is crucial in order to get the corresponding result for the flag manifold. We can topologize the set of straight lines in a projective space over a topological division ring $\mathbb{K}$ as follows. Let $G = \mathrm{GL}(n, \mathbb{K})$ and let $P'$ be the subgroup generated by $P$ and the $P_k$s for $i = 1, 2, \ldots n-3$ and $i = n-1$, where the $P_k$s are as in Definition 1.8. There is a natural isomorphism between the space of lines in $\mathbb{K}P^n$ and $G/P'$. Via this isomorphism, the usual topology on $G/P'$ induces a topology on the space of lines in $\mathbb{K}P^n$.

We can require that the mapping only map connected subsets of straight lines $l$ into straight lines if $l$ is a member of some fixed open set $V$ in this topology, with the property that given any point $p$ in the domain of the mapping there exists a line $l$ through $p$ such that $l$ is a member of $V$. Even then, the theorem that deals with connected sets still remains true in the cases $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$, as will be proved as Theorem 3.6, in Chapter 3. Again, the generalization of this version of the result to other topological division rings will be discussed in Part III.

**Theorem 3.6.** *Suppose that $n$ is an integer such that $n > 1$. Suppose that $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Suppose that $\phi$ is a mapping defined on an open connected subset $U$ of $\mathbb{K}P^n$, such* *that the range of $\phi$ has $n+1$ points in general position, and such that $\phi$ maps connected subsets of straight lines $l \in V$ into straight lines, where $V$ is a fixed open set in the topology described in Section 1.4, with the property that for every $p \in U$, there exists a line $l \in V$ such that $p \in l$. Then $\phi$ is the restriction to $U$ of a mapping of the form $\psi \circ \sigma^*$, where $\psi$ is a projective transformation and $\sigma$ is a unital homomorphism $\mathbb{K} \to \mathbb{K}$.*

Thanks to Theorem 3.6, we can also state a theorem dealing with the flag manifold. Suppose that $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Suppose that $G = \mathrm{GL}(n, \mathbb{K})$ where $n > 2$, and that $P$ is the subgroup of lower triangular matrices, and denote $G/P$ by $F$. Suppose that $\phi$ is a mapping defined on a nonempty open connected subset $U$ of $F$, whose range has $n$ points in general position, and which maps connected subsets of type $k$ fibres into type $k$ fibres, for $k = 1, 2, \ldots n-1$. It follows that $\phi$ is the restriction to $U$ of a mapping of the form $g^* \circ \sigma^*$, where $g \in G$ and $\sigma$ is a unital homomorphism $\mathbb{K} \to \mathbb{K}$. This will be proved as Theorem 3.7 in Chapter 3.

**Theorem 3.7.** *Suppose that $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Suppose that $n > 2$. Suppose that $\mathbb{K}$, $G$, $P$, $F$ are as in the statement of Theorem 3.2. Suppose that $U$ is a nonempty open connected subset of $F$ and that $\phi$ is a mapping $U \to F$, which maps connected subsets of type $k$ fibres contained in $U$ into type $k$ fibres, and whose range contains $n$ points in general position. Then $\phi$ is a restriction of a mapping of the form $g^* \circ \sigma^*$, where $\sigma$ is a unital ring homomorphism $\mathbb{K} \to \mathbb{K}$ and $g \in G$. If $\mathbb{K} = \mathbb{C}$ and $\phi$ is measurable, then $\sigma$ is either the identity or conjugation.*

We have now stated the local versions of the fundamental theorem of projective geometry which we seek to prove for $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ in Part I. We begin in the next chapter by presenting some results about the real projective plane.

# Chapter 2

## Some results about the real projective plane

In this chapter, we shall focus on the case of the real projective plane. The material appears in [4]. Theorems 2.1 and 2.2 were known in 2002, before the author began work on this research project. We wish to prove two results, given below. We say that three points in $\mathbb{R}P^2$ are in general position if they do not lie on a straight line.[1] The topology on $\mathbb{R}P^2$ in this chapter is that described in Section 1.1; that is, its customary topology as a real manifold.

**Theorem 2.1.** *Suppose that $\phi$ is a mapping defined on an open subset $U$ of $\mathbb{R}P^2$, such that the range of $\phi$ has three points in general position, and such that $\phi$ maps collinear points to collinear points. Then $\phi$ is the restriction to $U$ of a projective transformation.*

**Theorem 2.2.** *Suppose that $\phi$ is a mapping defined on an open connected subset $U$ of $\mathbb{R}P^2$, such that the range of $\phi$ has three points in general position, and such that $\phi$ maps connected subsets of straight lines $l \in V$ into straight lines, where $V$ is a fixed open set in the topology described in Section 1.4, and such that given any $p \in U$, there exists an $l \in V$ such that $p \in l$. Then $\phi$ is the restriction to $U$ of a projective transformation.*[2]

One useful observation in proving these two theorems is that if these results can be proved for a certain $U$, then they also follow for any $U'$ which contains this $U$. We shall now prove this.

[1] We generalize this definition to higher dimensions in Chapter 3.
[2] These two theorems are similar to Theorems 3.1 and 3.2 in [4].

**Lemma 2.3.** *Suppose there is an open set $U$ for which Theorem 2.1 holds for any mapping $\phi$ satisfying the hypotheses of Theorem 2.1. Suppose that $U'$ is another open set such that $U \subset U'$. Then Theorem 2.1 also holds for $U'$.*

*Proof.* Suppose that $U$ is an open set for which the hypothesis of the lemma holds, and $U'$ is an open set such that $U \subset U'$, and that $\phi$ is a mapping defined on $U'$ satisfying the hypotheses of Theorem 2.1. Then the restriction of $\phi$ to $U$ is equal to the restriction to $U$ of a projective transformation, say $\psi$. We will show that the restriction of $\phi$ to $U'$ is determined by its restriction to $U$. Any point in $U'$ can be obtained as the intersection of two distinct straight lines which meet $U$. This is because if we consider the family of straight lines passing through a given point $p \in U'$, which may be topologized in such a way that it is homeomorphic to a projective line in the obvious way, then the set of lines which meet $U$ is nonempty and open, and a nonempty open subset of the circle is guaranteed to contain at least two distinct points. The intersection of each straight line with $U$ will contain at least two distinct points. The intersections of these straight lines with $U$ will be mapped into distinct straight lines, because the restriction of $\phi$ to $U$ is a projective transformation. For each straight line, two distinct points lying on the straight line will be mapped to two distinct points, for the same reason. Hence the restriction of $\phi$ to $U$ determines the restriction of $\phi$ to $U'$, and it will agree with $\psi$ at each point in $U'$, because $\phi$ and $\psi$ agree on $U$, and both $\phi$ and $\psi$ map collinear points to collinear points. So if $\phi$ agrees with a projective transformation $\psi$ on $U$, it also agrees with the same projective transformation $\psi$ on $U'$. $\square$

**Lemma 2.4.** *Suppose there is a connected open set $U$ and a set of straight lines $V$ such that $V$ satisfies the hypotheses of Theorem 2.2 with respect to $U$ and such that Theorem 2.2 holds for any mapping $\phi$ satisfying the hypotheses of Theorem 2.2 with respect to $U$ and $V$. Suppose that $U'$ is another connected open set such that $U \subset U'$ and $V$ also satisfies the hypotheses of Theorem 2.2 with respect to $U'$. Then Theorem 2.2 also holds for $U'$ and $V$.*

*Proof.* Recall that each point in $\mathbb{R}P^2$ can be specified by homogeneous coordinates $(x : y : z)$, unique up to a scalar multiple. Let us define the line at infinity to be the set $\{(x : y : z) \in \mathbb{R}P^2 \mid z = 0\}$; this is indeed a straight line. Let us define the affine part of the real projective plane to be the complement of the line at infinity in $\mathbb{R}P^2$. Given a point $p$ in the affine part of the real projective plane, there exist unique real numbers $x, y$ such that $p = (x : y : 1)$, and so the correspondence $(x : y : 1) \mapsto (x, y)$ is a bijection from the affine part of $\mathbb{R}P^2$ to the affine plane $\mathbb{R}^2$; we shall identify the affine part of $\mathbb{R}P^2$ with $\mathbb{R}^2$ via this bijection, regarding $\mathbb{R}^2$ as a subset of $\mathbb{R}P^2$. Note that the usual topology for $\mathbb{R}^2$ is the same as the subspace topology it inherits from $\mathbb{R}P^2$ when it is regarded as a subset of $\mathbb{R}P^2$ in this way. A line which meets $\mathbb{R}^2$ has a well-defined slope, which may be regarded as an element of $\mathbb{R}P^1$, and two such lines with the same slope meet at the line at infinity. Now, $U$ must meet $\mathbb{R}^2$ because $U$ is nonempty and open and $\mathbb{R}^2$ is a dense subset of $\mathbb{R}P^2$. Since $V$ is nonempty and open, there exists a nonempty open subset $W$ of $\mathbb{R}P^1$ and an open set $U'' \subset U$ such that every line which meets $U''$ and has slope in $W$ is in $V$. We can assume without loss of generality that $U = U''$ and we will do so in what follows.

Suppose that $U$ and $U'$ are as in the hypotheses of the lemma and that $\phi$ is a mapping defined on $U'$ which satisfies the hypotheses of Theorem 2.2, with respect to $V$. Then $\phi$ agrees with some projective transformation $\psi$ on $U$. Let us consider the set $U^{(3)} \subset U'$ of points $p$ such that $\phi$ agrees with $\psi$ in a neighbourhood of $p$. $U^{(3)}$ is open by definition. Let us say that a point $p \in U'$ has property $P$ if it has points $q, r \in U^{(3)}$ arbitrarily close to it, distinct from each other and distinct from $p$, such that the line from $q$ to $p$ and the line from $r$ to $p$ both lie in $V$. We will now show that if a point $p \in U'$ has property $P$, then $p \in U^{(3)}$. For suppose that $p$ has property $P$. By a basic neighbourhood of a point $p \in \mathbb{R}P^2$ we shall mean a neighbourhood of $p$ which is the image of a product of three bounded open intervals contained in $\mathbb{R}$, not containing the zero vector, under the standard projection $\mathbb{R}^3 \setminus \{\mathbf{0}\} \to \mathbb{R}P^2$. Consider a basic neighbourhood $N$ of $p$ contained in $U'$, such that there exist points $q, r \in U^{(3)}$, distinct from each other and distinct from $p$, such that the line from $q$ to $p$ is contained in $V$ and there exists a connected subset of it containing $q$ and $p$ which is contained in $N$, and the same is true of the line from $r$ to $p$. Then there will be a neighbourhood $N_1$ of $p$, a neighbourhood $N_2 \subset U^{(3)}$ of $q$, and a neighbourhood $N_3 \subset U^{(3)}$ of $r$, with $N_i \subset N$ for $i = 1, 2, 3$, such that, given any $p' \in N_1$, there will exist $q' \in N_2, r' \in N_3$, such that the line from $q'$ to $p'$ is contained in $V$ and a connected subset of it containing $q'$ and $p'$ is contained in $N$, and the same is true of the line from $r'$ to $p'$. It will then follow that $\phi$ agrees with $\psi$ at $p'$. It follows that $p \in U^{(3)}$.

To summarize what we have observed about $U^{(3)}$ so far: (1) $U^{(3)}$ is open. (2) If a point $p \in U'$ has points $q, r \in U^{(3)}$ arbitrarily close to it, distinct from each other and distinct from $p$, such that the line from $q$ to $p$ and the line from $r$ to $p$ both lie in $V$, then $p \in U^{(3)}$.

I claim that, from these two properties, given that $U'$ and $V$ are open, it follows that if $l \in V$, then $l \cap U^{(3)}$ is closed in $l \cap U'$. To see this, consider a point $p$ in the closure of $l \cap U^{(3)}$ in $l \cap U'$. There are points in $l \cap U^{(3)}$ arbitrarily close to $p$. Given such a point $q$, I can find an open neighbourhood $N$ of $q$ contained in $U^{(3)}$, such that, given two points $r$ and $s$ contained in $N$, the line from $r$ to $p$ and the line from $s$ to $p$ are both in $V$. (The line from $q$ to $p$ is equal to $l$ and so is in $V$ by assumption.) Furthermore $N$ is certainly guaranteed to contain at least two distinct points. It follows by (2) that $p \in U^{(3)}$. Hence our claim is true.

We shall now show that it follows that $U^{(3)}$ is closed in $U'$. To see this, let $p$ be a point in the closure of $U^{(3)}$ in $U'$. We may assume without loss of generality that $p \in \mathbb{R}^2$ and that horizontal lines are in $V$. Pick a bounded open rectangle $R$ (that is, a product of two bounded open intervals), contained in $U' \cap \mathbb{R}^2$, containing $p$. If $q$ is in $R \cap U^{(3)}$ and $l$ is the horizontal line containing $q$, then $l \cap U^{(3)}$ is both open and closed in $l \cap U'$ and contains $q$, and therefore contains all of $l \cap R$. So $R \cap U^{(3)}$ contains line segments of the form $l \cap R$, where $l$ is a horizontal line, arbitrarily close to $p$. From this and the fact that $V$ is open it follows, by similar reasoning to that given in the previous two paragraphs, that $p \in U^{(3)}$. So

$U^{(3)}$ is both open and closed in $U'$. So it must be all of $U'$. This shows that $\phi$ agrees with $\psi$ on all of $U'$ which is what we set out to prove. [3] $\square$

It now follows from the two preceding lemmas that in order to establish Theorem 2.1 and Theorem 2.2 it is sufficient to show the following.

**Theorem 2.5.** *Suppose that $U$ is an open rectangle in $\mathbb{R}^2$, and suppose that $W$ is a nonempty open subset of $\mathbb{R}P^1$. Suppose that $\phi$ is a mapping $U \to \mathbb{R}P^2$, such that the range of $\phi$ has three points in general position, and such that $\phi$ maps connected subsets contained in $U$ of straight lines whose slope is a member of $W$ into straight lines. Then $\phi$ is the restriction to $U$ of a projective transformation.*

We may take $W$ to be any particular nonempty open interval in $\mathbb{R}P^1$ without loss of generality. For we may compose $\phi$ on the right with an appropriate projective transformation $\psi$, adjusting $W$ accordingly and shrinking it, if necessary, and replacing $U$ by $\psi^{-1}(U)$, and shrinking this, if necessary, to make it an open rectangle again. In this way $W$ may be transformed into any open interval in $\mathbb{R}P^1$ without affecting the hypotheses or the conclusion of Theorem 2.5. Proving the result for the case where $W$ equals the open interval $(-1, 1)$ is sufficient, for example.

Having simplified the problem in this way we now give a proof of Theorem 2.5.

As noted earlier, Theorems 2.1 and 2.2 were known in 2002, with contributions from Andreas Čap, Filippo de Mari, Michael Eastwood and Michael Cowling. In 2003 the author independently proved the result with an additional assumption about $\phi$: namely that $\phi$ is continuous, or alternatively that $\phi$ is order-preserving for lines with slope in $W$. We say that $\phi$ is order-preserving for a line $l$ if, whenever $a, b, c$ and $d$ lie on $l$, and $a$ and $c$ divide $b$ and $d$ (that is, $b$ and $d$ are two distinct points and $a$ and $c$ lie in different components of the

[3] In [4], the proofs of Lemmas 2.3 and 2.4 are not given explicitly. ("For then an analytic continuation argument shows that $\phi$ is projective on $U$", Čap *et al.*, Section 3, p. 46.) Here I have written out my own argument explicitly.

complement of $\{b, d\}$ in the line $bd$), then $\phi(a)$ and $\phi(c)$ divide $\phi(b)$ and $\phi(d)$. One interesting feature of the author's proof, which will be given shortly, making use of the exposition in [4], was that it was only necessary to assume that $\phi$ was collinearity-preserving for lines with any of six possible slopes in $W$, rather than arbitrary slopes in $W$. It was also necessary to assume that $\phi$ was order-preserving for lines with two of these six slopes. We shall present this argument first and then present elaborations which are due to the other authors of [4] which show how to eliminate the assumption that $\phi$ is continuous or order-preserving. We shall be making use of fragments of text from the proof of Theorem 3.2 in [4]. That theorem is slightly different to our theorem because the domain of the mapping is an arbitrary open connected set rather than an open rectangle and the authors of [4] are working in the affine plane rather than the projective plane.

*Proof of Theorem 2.5.* We begin by making the assumption that $\phi$ is continuous, or alternatively we may make the assumption that $\phi$ is order-preserving for lines with slope in $W$ in the sense described above. First of all it is easy to see that we may assume without loss of generality that $U$ contains an equilateral triangle with sides of slope 0 and $\pm\sqrt{3}$, and that $\phi$ fixes the vertices and centroid of this triangle, and furthermore that $\phi$ is collinearity-preserving and order-preserving for lines of slope 0, $\infty$, $\pm\sqrt{3}$ and $\pm 1/\sqrt{3}$. (We may achieve this by composing on the right and left with a projective transformation. When that is done $U$ will be the image of an open rectangle under a projective transformation. However, by making the triangle small and shrinking $U$, we can assume that $U$ is an open rectangle again.) The task is now to show that $\phi$ is the identity on all of $U$; that will suffice to prove Theorem 2.5 under the extra assumptions about $\phi$.

Suppose that the triangle is as in Figure 2.1. Since $\phi$ sends chords with slopes 0, $\infty$, $\pm\sqrt{3}$ and $\pm 1/\sqrt{3}$ into lines, $\phi$ preserves the midpoints of the sides. For example, the chord $CP$ is mapped into the line $CP$ and the chord $AB$ is mapped into the line $AB$, and $F = CP \cap AB$, so $\phi(F) = F$. Similarly, $\phi$ also preserves $AD \cap EF$ and $BE \cap DF$, that is, the midpoints of $EF$ and $DF$, so the horizontal chord through these points is sent into the line through

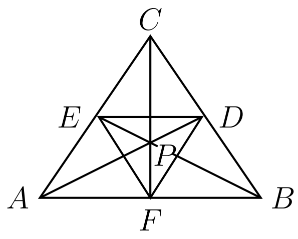


Figure 2.1: Subdivision

these points. It follows that the midpoints of $AE$ and $DB$ are also preserved. In this way, we deduce that not only does $\phi$ preserve the vertices of the four smaller equilateral triangles $AFE$, $FBD$, $DEF$ and $EDC$, but also $\phi$ preserves the midpoints of their sides, and their centroids. We can therefore continue to subdivide and find a dense subset of the triangle of points that are fixed by $\phi$. From either of the additional assumptions about $\phi$ it now easily follows that $\phi$ is the identity on all of the triangle. Then, using a continuation argument like that in the proof of Lemma 2.4, it follows that $\phi$ is the identity on all of $U$. So Theorem 2.5, with the additional assumption that $\phi$ is continuous or order-preserving for lines with slope in $W$, is now proved. Note that we actually only need to assume that $\phi$ is collinearity-preserving for lines with any one of six slopes, and order-preserving for lines with any one of two of these six slopes.

We now show how to eliminate the need for any extra assumption about $\phi$.

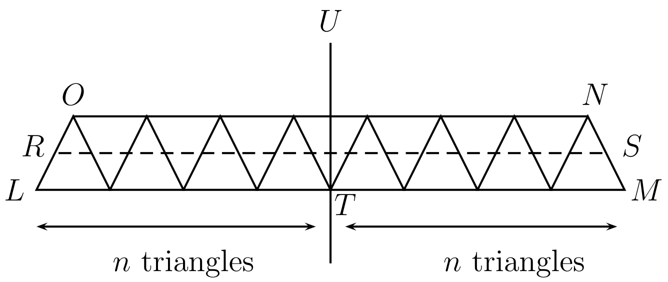


Figure 2.2: Horizontal chords

We first show, given a horizontal line $l$ that meets the interior of the triangle, $l \cap U$ is sent into a horizontal line, or more precisely, that a horizontal chord $RS$ (where $R$ and $S$

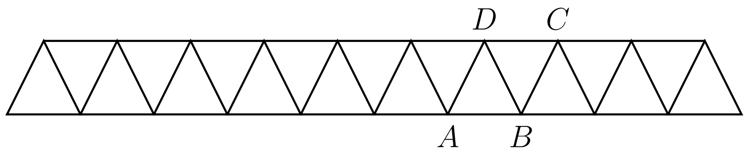


Figure 2.3: A rhombus

lie on the edges of the triangle) is sent into a horizontal line $R'S'$ (where $R'$ and $S'$ lie on the edges of the triangle, possibly produced). For this part of the argument we assume (as we may without loss of generality) that $\phi$ is collinearity-preserving for lines whose slope is contained in some open subset of $\mathbb{R}P^1$, which contains 0, $\pm\sqrt{3}$, $\pm 1/\sqrt{3}$, and $\infty$.

If the chord is one of those that arises in the subdivision, we are done, by construction. Otherwise, for some arbitrarily large positive integer $n$, we can find a subdivision as shown in Figure 2.2, so that the chord $RS$ passes through $2n$ congruent equilateral triangles with horizontal bases, and $2n-1$ inverted congruent equilateral triangles with horizontal bases. (Or it may be the case that $RS$ passes through $2n+1$ congruent equilateral triangles and $2n$ inverted congruent equilateral triangles; this will not affect the argument). If $RS$ is as shown, then the absolute values of the slopes of $LS$ and $RM$ are less than $1/(2n-1)$, so if $n$ is large enough, the images of the chords $LS$ and $RM$ lie in lines. Further, no matter what slopes are involved, if $R'$ and $S'$ lie on $AC$ and $CB$ respectively, possibly produced, where $A, B$, and $C$ are as in Figure 2.1, and $R' \neq A$ and $S' \neq B$, then $R'S'$ is parallel to $LM$ if and only if $LS' \cap R'M$ lies on the vertical line $TU$ through the midpoint of $LM$. (Here we recall that two lines are parallel if and only if they are equal or if the point where they meet is on the line at infinity; special attention needs to be paid to the case where $R'$ or $S'$ lie on the line at infinity). Since $TU$ is also preserved, it follows that the image of a horizontal chord [which meets the interior of the triangle] is horizontal.

We can repeat this argument to deduce that the images of chords with slopes $\pm\sqrt{3}$ that meet the interior of the triangle also have slopes of $\pm\sqrt{3}$.

We now show that $\phi$ is continuous on the triangle, which will suffice for the desired result. To do this, it suffices to show that each small rhombus like $ABCD$ in [Figure 2.3] is preserved; since these can be made arbitrarily small, $\phi$ is continuous on the triangle. After a linear transformation centred at $A$, doing this is equivalent to proving Lemma 2.6, and its proof will conclude the proof of the theorem. We need to get one detail out of the way first. The image of the triangle under $\phi$ is contained in $\mathbb{R}^2$. For suppose some point in the triangle were mapped to a point on the line at infinity; then the chords of slope $0, \pm\sqrt{3}$ passing through this point would be mapped into the line at infinity, as would the chords of slope $0, \pm\sqrt{3}$ passing through any point on these chords, and we would get a contradiction with our assumption that the range of $\phi$ contains three points in general position. The proof of Theorem 2.5 is now complete except for the proof of Lemma 2.6, stated below. $\square$

**Lemma 2.6.** *Let $\Sigma$ denote the square $ABCD$, where the coordinates of $A$, $B$, $C$ and $D$ are $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, and suppose that $\phi : \Sigma \to \mathbb{R}^2$ preserves the vertices of $\Sigma$, sends chords of $\Sigma$ with slope $0$, $\pm 1$ and $\infty$ into lines of the same slope, and sends chords of $\Sigma$ with slope in $(0, \infty)$ into lines. Then the image of $\phi$ is contained in the set $\Sigma$.* [4]

*Proof of Lemma 2.6.* Suppose that $0 < y < 1$ and take $s$ in $(0,1)$ so that $y = s^2$. Then $y$ is the ordinate of the point $Q$ constructed as the intersection of $AS$ and $UV$ in the top left-hand diagram below, in which the ordinate of $S$ is $s$, the line $TS$ is horizontal, the line $TU$ has slope $-1$, and the line $UV$ is vertical.

The line $AS$ has gradient in $(0,1)$, so maps into a line. Now $\phi(A) = A$ and $\phi(S)$ lies on $BC$ (possibly produced), and so has coordinates $(1,t)$ for some real $t$. Next, the line $\phi(T)\phi(S)$ is horizontal and $\phi(T)$ lies on $AD$ (possibly produced), so $\phi(T)$ has coordinates $(0,t)$, and the line $\phi(T)\phi(U)$ has gradient $-1$ and $\phi(U)$ lies on $AB$ (possibly produced), and hence $\phi(U)$ has coordinates $(t,0)$. Now $\phi(U)\phi(V)$ is vertical, and so $\phi(Q)$, which is the intersection of $\phi(U)\phi(V)$ and $\phi(A)\phi(S)$, has ordinate $t^2$, and lies above $AB$. Since $\phi$ sends horizontal chords into horizontal lines, the horizontal chord through $Q$ maps into the

[4] This is identical to Lemma 3.3 in [4].

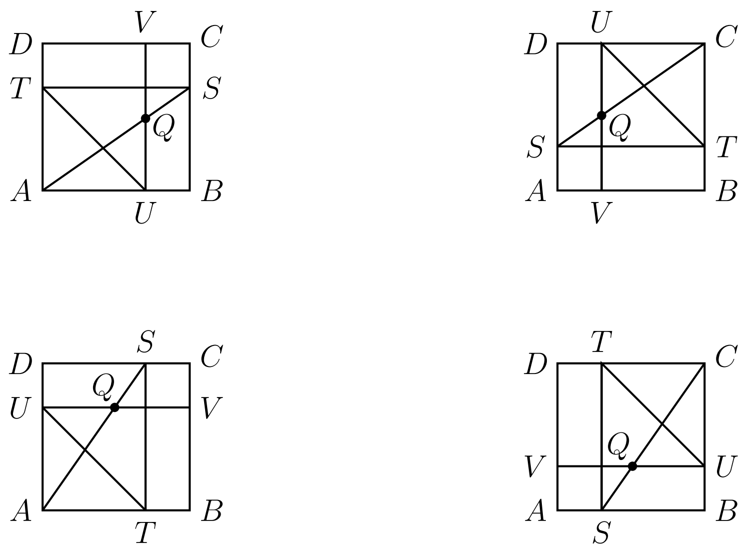


Figure 2.4: Constructions

horizontal line through the image of $\phi(Q)$, and the square maps into the half-plane above the line $AB$.

Analogous constructions, presented in the three other diagrams in Figure 2.4, show that the square maps into the half planes below $DC$, left of $AD$ and right of $BC$, together forcing the image to be in the intersection of these four half-planes, that is, $\Sigma$. $\square$

We have now proved Theorem 2.5, and from that we may infer Theorem 2.1 and Theorem 2.2. We shall make use of generalizations of them in the next chapter to obtain the local versions of the fundamental theorem of projective geometry in the cases $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, as described in Section 1.4.

# Chapter 3

# The local and global versions of the fundamental theorem of projective geometry

In this chapter we prove all the results discussed in Chapter 1. Theorems 3.1 and 3.2 are already known and Theorems 3.3 to 3.7 are new. The latter theorems extend the global versions of the fundamental theorem of projective geometry to local versions. A version of Theorem 3.7 appeared in [13] but it required that the mapping be $C^2$. Our results do not require regularity assumptions on the mappings.

## 3.1 The global versions of the theorems

We first discuss the global versions of the fundamental theorem of projective geometry. In the discussion below, up until the end of the proof of Theorem 3.2, let us suppose that $\mathbb{K}$ is an arbitrary division ring.

From here until the end of the proof of Theorem 3.1, we define $V$ to be a module over a division ring $\mathbb{K}$ whose rank $n+1$ is finite and greater than two. We define $R$ to be the equivalence relation on $V \setminus \{0\}$ induced by right multiplication. We define $S$ to be the quotient of $V \setminus \{0\}$ by $R$. We say that $S$ is a finite-dimensional projective space over $\mathbb{K}$ with dimension one less than the rank of $V$. If $W$ is a submodule of $V$ of rank two, then the quotient of $W \setminus \{0\}$ by $R$ will be called a line in $S$. Note that there is a unique line joining any two distinct points. Given a $\mathbb{K}$-module automorphism $\phi$ of $V$, we let $\phi^*$ denote the induced bijection of $S$. Such a bijection is called a projective transformation. Now, suppose a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n+1}\}$ of $V$ is given. Then for any $\mathbf{x} \in V$ there is a unique sequence $(x_1, x_2, \ldots x_{n+1})$ such that $\mathbf{x} = \mathbf{e}_1 x_1 + \mathbf{e}_2 x_2 + \ldots + \mathbf{e}_{n+1} x_{n+1}$. If $\mathbf{x} \in V \setminus \{0\}$ then the equivalence class of $\mathbf{x}$ under $R$ is an element of $S$, and then $(x_1 : x_2 : \ldots : x_{n+1})$ may be said to be the homogeneous coordinates of this point of $S$ with respect to the basis of $V$ in question. Homogeneous coordinates are unique up to multiplication on the right by a nonzero scalar. Then, given a unital ring homomorphism $\sigma : \mathbb{K} \to \mathbb{K}$, we denote by $\sigma^*$ the mapping $(x_1 : x_2 : \ldots : x_{n+1}) \mapsto (\sigma(x_1) : \sigma(x_2) : \ldots : \sigma(x_{n+1}))$, which is easily seen to be well-defined. Note that, for this notation to be well-defined, we must specify a basis of $V$.

We define a subspace of $S$ to be a nonempty subset $T$ of $S$ with the property that, whenever $x, y \in T$, and $x \neq y$, the line joining $x$ and $y$ is contained in $T$. If $S$ is the quotient of $V \setminus \{0\}$ by the usual equivalence relation $R$, then for some non-negative integer $k$ there exists a submodule $W$ of $V$ of rank $k+1$ such that $T$ is the quotient of $W \setminus \{0\}$ by $R$. We then call $k$ the dimension of $T$. If $n$ is the dimension of $S$, we say that $n+1$ points are in general position if they are not contained in any hyperplane of $S$, a hyperplane of $S$ being a subspace of $S$ of dimension $n-1$.

Then we have the following theorem (the fundamental theorem of projective geometry):

**Theorem 3.1.** *Suppose that $S$ is a finite-dimensional projective space over a division ring $\mathbb{K}$ with dimension $n$, where $n > 1$, arising from a module $V$, with some basis of $V$ given (so that the notation $\sigma^*$ makes sense). Then any mapping $\phi : S \to S$, whose range contains $n+1$ points in general position, which maps straight lines into straight lines, is of the form $g^* \circ \sigma^*$, where $g$ is some $\mathbb{K}$-module automorphism of $V$, and $\sigma$ is some unital homomorphism $\mathbb{K} \to \mathbb{K}$.*

*Remark.* When $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$, it is redundant to mention self-homomorphisms of $\mathbb{K}$. This is because $\mathbb{R}$ has no nontrivial self-homomorphisms, and the mappings induced by self-homomorphisms of $\mathbb{H}$ are all projective transformations anyway, as discussed in Section 1.2.

In [15], Jeffers proves a result related to a special case of Theorem 3.1 dealing with affine spaces over $\mathbb{R}$, which was known to Darboux in the nineteenth century. Generalizing the ideas of this proof to get Theorem 3.1 is relatively simple, and we follow [15].

*Proof of Theorem 3.1.* Suppose that $\phi$ is a mapping from $S$ to $S$, and that the range of $\phi$ contains $n+1$ points in general position, and that $\phi$ maps straight lines into straight lines. An easy induction on the dimension shows that, given a $k$-dimensional subspace, where $k > 0$, its image under $\phi$ is contained in at least one $k$-dimensional subspace. When $k = 1$ this is just part of the hypothesis on $\phi$. Suppose that we have proved that the image of any $k$-dimensional subspace under $\phi$ is contained in at least one $k$-dimensional subspace. Given a $(k+1)$-dimensional subspace $T$, there exists a $k$-dimensional subspace $T'$ and a point $p$ such that $T$ is the union of all lines $l$ joining $p$ and $q$ where $q \in T'$. $T'$ is mapped into some $k$-dimensional subspace and each of the aforementioned lines is mapped into a line, consequently $T$ is mapped into some $(k+1)$-dimensional subspace. This completes the induction. Further, we claim that, when $k > 0$, a $k$-dimensional subspace cannot be mapped into a $(k-1)$-dimensional subspace. We shall now prove this claim. If it were to happen that, for some $k$ such that $k > 0$, a $k$-dimensional subspace were mapped into a $(k-1)$-dimensional subspace, then essentially the same induction on the dimension as before would show that the range of $\phi$ would be contained in a subspace of strictly smaller dimension than $S$, but this would contradict the fact that the range of $\phi$ contains $n+1$ points in general position, and so our claim is true. From this it follows that, given a $k$-dimensional subspace $T$, there exists a unique $k$-dimensional subspace in which $\phi(T)$ is contained.

In what follows we shall show how to reduce Theorem 3.1 to a theorem about affine spaces. Suppose that $S$ is the quotient of $V \setminus \{0\}$ by $R$ and some basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n, \mathbf{e}_{n+1}\}$ for $V$ is given. We define the hyperplane at infinity to be the image of the submodule generated by $\{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n\}$, with the zero vector removed, under the canonical projection from $V \setminus \{0\}$ to $S$. It follows from the previous paragraph that $\phi$ will map the hyperplane at infinity onto a set such that the smallest subspace containing it is a hyperplane. By composing on the left with a projective transformation, we can assume that $\phi$ maps the hyperplane at infinity onto a set such that the smallest subspace containing it is the hyperplane at infinity. Then, if $S$ is $n$-dimensional, we can identify the complement of the hyperplane at infinity in $S$ with $\mathbb{K}^n$, (here the way in which the identification is done will depend on the choice of basis for $V$), and the restriction of $\phi$ to the complement in $S$ of the hyperplane at infinity will be a mapping $\mathbb{K}^n \to \mathbb{K}^n$ which maps lines into lines, and whose range contains $n+1$ points in general position, which by abuse of notation we shall also call $\phi$. Lines in $\mathbb{K}P^n$ are said to be parallel if they meet in the hyperplane at infinity. Lines in $\mathbb{K}^n$ are said to be parallel if the unique lines in $\mathbb{K}P^n$ in which they are contained are parallel. Our mapping $\phi : \mathbb{K}^n \to \mathbb{K}^n$ will map parallel lines into parallel lines, since the original $\phi$ mapped the hyperplane at infinity into the hyperplane at infinity.

We must show that on these assumptions $\phi$ is of the form $\psi \circ \sigma^*$, where $\psi$ is an affine transformation and $\sigma$ is a unital homomorphism $\mathbb{K} \to \mathbb{K}$, and where now by $\sigma^*$ we mean the mapping $(x_1, x_2, \ldots, x_n) \mapsto (\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_n))$. Here by an affine transformation we mean a $\mathbb{K}$-module automorphism of $\mathbb{K}^n$ composed on the left with a translation. We think of transformations as acting on the left. We begin by proving this in the case where $n = 2$.

By composing with an affine transformation on the left, we can assume without loss of generality that $\phi$ fixes $(0,0), (0,1)$, and $(1,0)$. A line parallel to the $x$-axis is mapped into a line parallel to the $x$-axis, and a line parallel to the $y$-axis is mapped into a line parallel to the $y$-axis. Hence $\phi$ also fixes $(1,1)$. Let $\alpha$ and $\beta$ be the maps such that $\phi(x,0) = (\alpha(x), 0)$ and $\phi(0,y) = (0, \beta(y))$. We have $\phi(x,y) = (\alpha(x), \beta(y))$. Since $\phi$ maps the line joining $(0,0)$ and $(1,1)$ into itself, we see that $\alpha = \beta$. It remains only to prove that $\alpha$ is a unital ring homomorphism $\mathbb{K} \to \mathbb{K}$.

Suppose $s$ and $t$ are two points in $\mathbb{K}^2$ not collinear with 0. Then clearly $\phi(s+t) = \phi(s) + \phi(t)$. This follows from the parallelogram law for vector addition and the fact that $\phi$ maps parallel lines into parallel lines. If 0, $s$ and $t$ are collinear, choose a point $u$ not lying on the line through 0, $s$ and $t$. Then $\phi(s+t+u) = \phi(s+t) + \phi(u)$, and also $\phi(s+t+u) = \phi(s) + \phi(t+u) = \phi(s) + \phi(t) + \phi(u)$. Thus $\phi(s+t) = \phi(s) + \phi(t)$. Thus $\alpha$ is a group homomorphism $\mathbb{K} \to \mathbb{K}$, where $\mathbb{K}$ is considered as a group under addition. We also know that $\alpha$ fixes 0 and 1, so it remains only to prove that $\alpha$ is multiplicative.

Consider the line $l = \{(x, y) \in \mathbb{K}^2 \mid y = ax\}$. The points $(0, 0)$ and $(1, a)$ lie on this line, and $\phi(1, a) = (1, \alpha(a))$. Thus $\phi$ maps the line $l$ into the line $l' = \{(x, y) \in \mathbb{K}^2 \mid y = \alpha(a)x\}$. So $\phi(b, ab) = (\alpha(b), \alpha(a)\alpha(b))$, since the $x$-coordinate of $\phi(b, ab)$ is $\alpha(b)$ and $\phi(b, ab)$ lies on $l'$, and it also equals $(\alpha(b), \alpha(ab))$, since in general $(x, y)$ is mapped to $(\alpha(x), \alpha(y))$. Thus $\alpha(ab) = \alpha(a)\alpha(b)$. This proves that $\alpha$ is a unital ring homomorphism $\mathbb{K} \to \mathbb{K}$.

That completes the proof in the case $n = 2$. We now prove the result for higher dimensions by induction. For example, in the case $n = 3$, suppose the affine hull of the image under $\phi$ of the $xy$-plane is some plane $\pi$ and the affine hull of the image under $\phi$ of the $yz$-plane is another plane $\pi'$. By composing on the left with an appropriate affine transformation and a left inverse of $\sigma^*$ where $\sigma$ is an appropriately chosen unital homomorphism $\mathbb{K} \to \mathbb{K}$, we can assume that all the points in the $xy$-plane are fixed. Then all the points on the $y$-axis will be fixed and so by composing on the left with an appropriate affine transformation we can assume that all the points in the $yz$-plane will be fixed. Then it is easy to show that every point is fixed, for every point outside these two planes can be obtained as the intersection of two lines joining points in these planes, provided that $\mathbb{K}$ has more than two elements. In the case where $\mathbb{K}$ is the two-element field, there is only one point outside the two planes and it is forced to be fixed by the requirement that, when $k > 0$, a $k$-dimensional subspace cannot be mapped into a $(k-1)$-dimensional subspace (because that forces $\phi$ to be injective in the case where $\mathbb{K}$ has only two elements). By generalizing this argument, we obtain the inductive step, and thereby obtain the general result. $\square$

That completes the proof of the version of the theorem dealing with projective spaces. We mentioned a version of the theorem dealing with affine spaces: namely, given a mapping

$\phi : \mathbb{K}^n \to \mathbb{K}^n$, where $\mathbb{K}$ is any division ring and $n$ is an integer greater than one, such that the range of $\phi$ contains $n+1$ points in general position and $\phi$ maps straight lines into straight lines, $\phi$ is of the form $\psi \circ \sigma^*$ where $\psi$ is an affine transformation and $\sigma$ is a unital ring homomorphism $\mathbb{K} \to \mathbb{K}$. To prove this version of the theorem, we need to show that given the hypotheses on $\phi$, $\phi$ will map parallel lines into parallel lines.[1] The proof of Theorem 2.5 shows that we can assume this if we compose on the left with an appropriate projective transformation. Then once we have the final conclusion (namely, that $\phi$ is of the desired form) we can show that this projective transformation must preserve the affine part of $\mathbb{K}P^n$, and then the desired result follows. The argument given in the proof of Theorem 2.5 uses the notion of a midpoint; this only makes sense on the assumption that $\mathbb{K}$ does not have characteristic two. However, the argument can easily be modified to cover the characteristic two case; we shall show how in Part III.

As discussed in Chapter 1, there is another version of the result, due to Bertini [2], dealing with spaces of full flags rather than projective spaces, which we shall now discuss. In what follows, suppose that $n$ is an integer and that $n > 2$. Suppose that $G = \mathrm{GL}(n, \mathbb{K})$ and suppose that $P$ is the subgroup consisting of all lower triangular matrices. Denote by $F$ the space $G/P$ of left cosets of $P$ in $G$. In what follows we make use of Definitions 1.8 to 1.10. Given $g \in G$, let $g^*$ be the mapping $F \to F$ which maps $hP \mapsto ghP$; this is clearly well-defined. Given a unital ring homomorphism $\sigma : \mathbb{K} \to \mathbb{K}$, by abuse of notation let $\sigma$ also denote the mapping $G \to G$ which acts via the original $\sigma$ on the entries of the matrix. This preserves $P$ and so there is an induced mapping $F \to F$ which we denote by $\sigma^*$.

**Theorem 3.2.** *Any good mapping $\phi : F \to F$ is of the form $g^* \circ \sigma^*$, for some unital ring homomorphism $\sigma : \mathbb{K} \to \mathbb{K}$ and $g \in G$.*

[1] In [15], Jeffers was able to do this very simply because he was assuming that $\phi$ was surjective, but we are not. Because we are not assuming that $\phi$ is surjective, $\sigma$ is not necessarily an automorphism, it may be a self-homomorphism which is not surjective. This does not matter in the cases $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$, but it does make a difference in the case $\mathbb{K} = \mathbb{C}$, as shown in Appendix B.

*Proof of Theorem 3.2.* In Section 1.3, we defined the notion of a good mapping $F \to F$ and a good mapping $\mathbb{K}P^{n-1} \to \mathbb{K}P^{n-1}$ and showed how to associate with each good mapping $\phi : F \to F$ a mapping $\phi^* : \mathbb{K}P^{n-1} \to \mathbb{K}P^{n-1}$. We promised to prove that this mapping is well-defined and a good mapping, and that given a good mapping $\psi : \mathbb{K}P^{n-1} \to \mathbb{K}P^{n-1}$, there exists a unique good mapping $\phi : F \to F$ such that $\phi^* = \psi$. We shall do this shortly, starting in the next paragraph. Furthermore, it is clear that given a mapping $\phi$ of the desired form in Bertini's theorem, Theorem 3.2, $\phi^*$ is a mapping of the desired form in Theorem 3.1, and that if $\psi$ is a mapping of the desired form in Theorem 3.1, there exists a mapping $\phi$ such that $\phi^* = \psi$, and $\phi$ is of the desired form in Theorem 3.2. It will follow that Bertini's theorem is equivalent to Theorem 3.1, which has been proved.

In what follows, let $A$ be the family of subsets of $\mathbb{K}P^{n-1}$ which are images of type $n-1$ fibres in $F$ under the canonical projection $F \to \mathbb{K}P^{n-1}$. These are precisely the straight lines. For it follows from our discussion in Section 1.3 that the type $n-1$ fibres can be thought of as sets of full flags of the form $\{(V_1, V_2, \dots V_n) \mid V_1 \text{ is a module of rank } 2\}$, where all $V_i$ are fixed except $V_1$. The image of such a set under the projection to $\mathbb{K}P^{n-1}$ is equal to the image of $V_2 \setminus \{0\}$ under the canonical projection $\mathbb{K}^n \setminus \{0\} \to \mathbb{K}P^{n-1}$, which is a straight line. Consider a good mapping $\phi : F \to F$. Now, $\phi$ maps type $k$ fibres into type $k$ fibres for $k = 1, 2, \dots n-2$, so it maps projections of left cosets of $Q$ to $G/P$ into projections of left cosets of $Q$ to $G/P$. For $Q$ is generated by the subgroups $P_k, k = 1, 2, \dots n-2$. Hence $\phi$ induces a mapping $\phi^* : G/Q \to G/Q$ whose range contains $n$ points in general position, and which maps members of $A$ into members of $A$ (because $\phi$ maps type $n-1$ fibres into type $n-1$ fibres). As observed, the members of $A$ are precisely the straight lines. Hence $\phi^*$ is indeed well-defined and good.

We must now show that, given a good mapping $\psi : \mathbb{K}P^{n-1} \to \mathbb{K}P^{n-1}$, there exists a unique good mapping $\phi : F \to F$ such that $\phi^* = \psi$. Existence follows from Theorem 3.1, but can also be proved independently, as we will now do. As observed, $G/Q$ may be identified with $\mathbb{K}P^{n-1}$ and $G/P$ may be identified with the set of full flags in $\mathbb{K}^n$. Furthermore, as observed in the proof of Theorem 3.1, given a $(k-1)$-dimensional subspace $T$ of $\mathbb{K}P^{n-1}$, where $1 \leq k \leq n$, there exists a unique $(k-1)$-dimensional subspace containing $\psi(T)$. Thus $\psi$ determines an action on the $(k-1)$-dimensional subspaces of $\mathbb{K}P^{n-1}$ for every $k$ from 1 to $n$, and hence determines an action $\phi$ on the full flags such that $\phi^* = \psi$. This proves existence.

Now we shall prove uniqueness without making use of Theorem 3.1. When $H$ and $K$ are two subgroups of a group $G$, by $HK$ we shall mean $\{hk \mid h \in H, k \in K\}$.

Suppose we are given $\psi$ and we want to determine the putative $\phi$ such that $\phi^* = \psi$. Let $Q_j$ be the subgroup of $G$ generated by $P$, and the $P_k$ where $k < j$. We shall prove by downward induction that the action of $\phi$ on $G/Q_j$ is determined. The base case will be $j = n-1$, and the final case will be $j = 1$, which will yield the desired result. We know the action of $\phi$ on left cosets of $Q_{n-1}$. From this we can determine the action of $\phi$ on left cosets of $Q_{n-2}$. Take a coset $gQ_{n-2}$. Recalling the discussion in Chapter 1 where we showed that a left coset of a subgroup $Q$ generated by some finite set of the $P_k$s could be thought of as corresponding to a partial flag, we see that the coset $gQ_{n-2}$ corresponds to a flag $(V_1, V_2)$ where $V_i$ is a submodule of $\mathbb{K}^n$ of rank $i$ for $i = 1, 2$. The projection of $gP_{n-1}Q_{n-2}$ to $G/Q_{n-1}$ will correspond to a line $l$ in $\mathbb{K}P^{n-1}$, which will be the projection of $V_2 \setminus \{0\}$ to $\mathbb{K}P^{n-1}$. (The coset $gQ_{n-2}$ can be thought of as an ordered pair consisting of a point $p$ and a line $l$ in $\mathbb{K}P^{n-1}$, where $p$ lies on $l$. The point $p$ corresponds to the coset $gQ_{n-1}$.) The line $l$ will be mapped by $\psi$ to some other line which we may denote by $l'$, which will be equal to, say, the projection of $hP_{n-1}Q_{n-2}$ to $G/Q_{n-1}$. Take the intersection of $hP_{n-1}Q_{n-2}$ with $\psi(gQ_{n-1})$. We have $hP_{n-1}Q_{n-2}$ corresponding to the line $l'$ and $gQ_{n-1}$ identified with the point $p$. Intuitively, what we are doing is forming the ordered pair $(p', l')$ where $p' = \psi(p)$. In this way we are determining a coset of $Q_{n-2}$. By this means we get the coset of $Q_{n-2}$ to which $\phi$ maps $gQ_{n-2}$. Thus, given the action of $\phi$ on left cosets of $Q_{n-1}$ the action of $\phi$ on left cosets of $Q_{n-2}$ is determined.

A similar argument shows that if the action of $\phi$ is determined on cosets of $Q_{j'}$ for $j' > j$, it is also determined on cosets of $Q_j$, and thus the induction goes through.

For example, suppose that $n > 3$ and take the case $j = n - 3$. We pick a $g$ and let $\psi'$ be the action induced by $\phi$ on $gQ_{n-1}/Q_{n-2}$, which may be identified with $\mathbb{K}P^{n-2}$. (Since we know the action of $\phi$ on cosets of $Q_{n-2}$, we know enough about the action of $\phi$ to determine this.) We show in a similar way to the way we did previously that knowledge of the action of $\phi$ on left cosets of $Q_{n-2}$ contained in $gQ_{n-1}$ gives us knowledge of the action of $\phi$ on left cosets of $Q_{n-3}$ contained in $gQ_{n-1}$. Thus, let $g'Q_{n-3}$ be contained in $gQ_{n-1}$. (Here $g'Q_{n-3}$ corresponds to an ordered pair consisting of a point and a line in $\mathbb{K}P^{n-2}$ in the same way as before.) The projection of $g'P_{n-2}Q_{n-3}$ to $gQ_{n-1}/Q_{n-2}$ will correspond to a line in $\mathbb{K}P^{n-2}$, and so will be mapped by $\psi'$ to another coset of $P_{n-2}Q_{n-3}$ corresponding to a different line in $\mathbb{K}P^{n-2}$, say $h'P_{n-2}Q_{n-3}$. Take the intersection of $h'P_{n-2}Q_{n-3}$ with $\psi(g'Q_{n-2})$ and we get the coset of $Q_{n-3}$ to which $\phi$ maps $g'Q_{n-3}$. Thus we have determined the action of $\phi$ on cosets of $Q_{n-3}$ contained in $gQ_{n-1}$. All this has the same geometric interpretation as before, only in a projective space of lower dimension.

Generalizing this argument for arbitrary $j$ gives us the inductive step. Hence by downward induction, given the action of $\phi$ on $G/Q_{n-1}$ we can obtain complete information about the action of $\phi$ on $F = G/P$. This proves uniqueness. This completes the proof that given a good mapping $\psi : \mathbb{K}P^{n-1} \to \mathbb{K}P^{n-1}$ there exists a unique good mapping $\phi : G/P \to G/P$ such that $\phi^* = \psi$. We have now shown that Bertini's theorem (Theorem 3.2) is equivalent to Theorem 3.1, a generalization of a theorem of Darboux, and this theorem has been proved. This completes the proof of Theorem 3.2.

$\square$

We have now proved the two global versions of the fundamental theorem of projective geometry, which are already known.

## 3.2 The local versions of the theorems

Our goal is to prove the local versions of the fundamental theorem of projective geometry stated in Section 1.4, in the real, complex, and quaternionic cases. In this section, $\mathbb{K}$ is $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. We begin by generalizing the results of Chapter 2 to the complex numbers and quaternions.

We start by generalizing the results of Chapter 2, dealing with the two-dimensional case, to the case where $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

**Theorem 3.3.** *Suppose that $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Suppose that $\phi$ is a mapping defined on an open subset $U$ of $\mathbb{K}P^2$, such that the range of $\phi$ has three points in general position, and such that $\phi$ maps collinear points to collinear points. Then $\phi$ is the restriction to $U$ of a mapping of the form $\psi \circ \sigma^*$, where $\psi$ is a projective transformation and $\sigma$ is a unital homomorphism $\mathbb{K} \to \mathbb{K}$.*

**Theorem 3.4.** *Suppose that $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Suppose that $\phi$ is a mapping defined on an open connected subset $U$ of $\mathbb{K}P^2$, such that the range of $\phi$ has three points in general position, and such that $\phi$ maps connected subsets of straight lines $l \in V$ into straight lines, where $V$ is a fixed open set in the topology described in Section 1.4, such that for all $p \in U$, there exists an $l \in V$ such that $p \in l$. Then $\phi$ is the restriction to $U$ of a mapping of the form $\psi \circ \sigma^*$, where $\psi$ is a projective transformation and $\sigma$ is a unital homomorphism $\mathbb{K} \to \mathbb{K}$.*

*Proof of Theorems 3.3 and 3.4.* In Chapter 2, we made use of the notion of a "bounded open interval" in $\mathbb{R}$. Let us define a "bounded open interval" in $\mathbb{C}$ or $\mathbb{H}$ to be a set of the form $U_1 + U_2 i$ or $U_1 + U_2\mathbf{i} + U_3\mathbf{j} + U_4\mathbf{k}$, respectively, where the $U$s are bounded open intervals in $\mathbb{R}$. Then we define an open rectangle in $\mathbb{K}^n$ to be an open subset of $\mathbb{K}^n$ which is a product of $n$ bounded open intervals. A basic open subset of $\mathbb{K}P^n$ is defined to be an image of an open rectangle in $\mathbb{K}^{n+1}$ not containing the zero vector under the canonical projection $\mathbb{K}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{K}P^n$, and a basic neighbourhood of a point $p \in \mathbb{K}P^n$ is defined to be a basic open subset of $\mathbb{K}P^n$ containing $p$. The proofs of Lemmas 2.3 and 2.4 now generalize to the complex and quaternionic cases without any change in the argument.

Hence, as in Chapter 2 when Theorems 2.1 and 2.2 were reduced to Theorem 2.5, it is sufficient to prove the theorems in the case where $U$ is an open rectangle in $\mathbb{K}^2$, where, as before, we identify $\mathbb{K}^2$ with the affine part of $\mathbb{K}P^2$, and we may also assume that there exists an open set $W \subset \mathbb{K}P^1$ such that the lines in $V$ which meet $U$ are precisely the lines which meet $U$ with slope in $W$. We may further assume that $W$ contains $[-2, \infty]$. If $\mathbb{K} = \mathbb{C}$, we further assume that $W$ contains $i$, and if $\mathbb{K} = \mathbb{H}$, we assume that $W$ contains $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$. The reason we can do this is as follows. We are free to replace $W$ with the image of $W$ under any projective transformation of $\mathbb{K}P^1$. We can achieve this by composing $\phi$ on the right with a projective transformation and shrinking $U$ if necessary to make it an open rectangle again. Our remarks regarding what we can assume about $W$ now follow. We can then replace $U$ with its image under any real dilation or translation by composing on the right with a real dilation or translation; this will not affect $W$. So we can also assume that $U$ is a Cartesian power $I^n$ where $I$ is a bounded open interval in $\mathbb{K}$.

Furthermore we can assume that $U$ contains the vertices of an isosceles right triangle with sides of slope 0 and $\infty$ and that the vertices of this triangle are all fixed by $\phi$. Indeed, we can assume that the vertices of this triangle are $(0, 0), (1, 0)$, and $(0, 1)$. (It will be convenient to work with an isosceles right triangle rather than an equilateral triangle.) We can also assume that $U$ contains $(1, -1)$ and that this is also fixed by $\phi$. This completes our discussion of the assumptions which we may make without loss of generality.

To complete the proofs of Theorems 3.3 and 3.4 it is sufficient to prove, from these assumptions, that $\phi$ is of the form $\sigma^*$ where $\sigma$ extends to a unital homomorphism $\mathbb{K} \to \mathbb{K}$. Let us embark on that task. Now, as before, using the same argument as in the proof of Theorem 2.5, it may be seen that lines of slope 0, -1, or $\infty$ which meet $U$ are mapped into lines of the same slope. One point requires some discussion; the argument we gave in the proof of Theorem 2.5 about lines of slope 0 is not valid when we are dealing with a line of slope 0 passing through $(0,1)$, because in that case the points $R$ and $S$ in Figure 2.2, discussed in the proof of Theorem 2.5, will be equal. However, in that case we may make use of the same argument regarding the triangle with vertices $(0,0),(0,1)$, and $(1,-1)$ and the desired conclusion will still follow.

From this we shall derive the conclusion that $\phi = \sigma^*$, where $\sigma$ is a map $I \to \mathbb{K}$ such that if $x, y, x+y \in I$, then $\sigma(x+y) = \sigma(x) + \sigma(y)$, or, as we might put it, where $\sigma$ is a locally additive map. To see this, let $\sigma$ be such that when $x \in I$, $\phi(0,x) = (0, \sigma(x))$. This is possible because $\phi$ maps the line from $(0,0)$ to $(0,1)$ into itself. (It might seem conceivable at first that the range of $\sigma$ could include $\infty$, but actually that is impossible, because if $\phi(0,x)$ lies on the line at infinity, then the lines of slope 0, $-1$, and $\infty$ passing through $(0,x)$ are all mapped into the line at infinity, and this contradicts the assumption that the range of $\phi$ contains three points in general position.) Since $\phi$ maps the line from $(0,0)$ to $(1,0)$ into itself as well and also maps lines of slope $-1$ which meet $U$ into lines of slope $-1$, it follows that $\phi(x,0) = (\sigma(x), 0)$, provided $x \in I$. It now follows, as we may see by considering the rectangle with corners $(0,0),(x,0),(0,y)$, and $(x,y)$, that $\phi(x,y) = (\sigma(x), \sigma(y))$; or, in other words, that $\phi = \sigma^*$. It remains to prove that $\sigma$ is locally additive. Now, supposing that $x, y, x+y \in I$, the line from $(0, x+y)$ to $(y, x)$ has slope $-1$ and meet $U$, and so is mapped into a line with slope $-1$. From this we may conclude that $\sigma(x+y) = \sigma(x) + \sigma(y)$. We have now shown that $\sigma$ is locally additive. We also have that $I$ contains $[-1,1]$ and $\sigma$ fixes 0 and 1. We can also assume that $I$ contains $i$ if $\mathbb{K} = \mathbb{C}$, and contains $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ if $\mathbb{K} = \mathbb{H}$.

We shall now show that $\sigma$ is locally multiplicative. Now, if $x \in I$, then $\phi(1,x) = (1, \sigma(x))$, and since $\sigma$ is additive on $I$ it follows that parallel lines that meet $U$ with slope in $W$ are mapped to parallel lines. So if $x \in I \cap W$, then lines of slope $x$ that meet $U$ are mapped to lines of slope $\sigma(x)$. From this it follows that if $x, y, xy \in I \cap W$, then $\sigma(xy) = \sigma(x)\sigma(y)$. In this sense $\sigma$ is locally multiplicative.

We have established that $\sigma$ is defined on an open subset $I$ of $\mathbb{K}$ containing 0 and 1, fixes 0 and 1, and is additive on $I$ and multiplicative on $I \cap W$. We also have that $I \cap W$ contains $[-1, 1]$, and contains $i$ if $\mathbb{K} = \mathbb{C}$, and contains $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ if $\mathbb{K} = \mathbb{H}$. Our task was to establish that $\sigma$ extends to a unital homomorphism on all of $\mathbb{K}$. This can now be seen as follows. Since $\sigma$ is defined and additive on $[-1, 1]$, it must uniquely extend to an additive homomorphism on all of $\mathbb{R}$. Hence, since $I \cap W$ contains $i$ if $\mathbb{K} = \mathbb{C}$, and $I \cap W$ contains $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ if $\mathbb{K} = \mathbb{H}$, and since $\{1, i\}$ generates $\mathbb{C}$ as an $\mathbb{R}$-module and $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ generates $\mathbb{H}$ as an $\mathbb{R}$-module, it follows that $\sigma$ extends to a unique additive homomorphism on all of $\mathbb{K}$. Finally, it easily follows from our observations at the start of this paragraph that this homomorphism is unital and multiplicative. We have now shown that $\phi$ is of the form $\sigma^*$ where $\sigma$ extends to a unital ring homomorphism $\mathbb{K} \to \mathbb{K}$. This completes the proof of Theorems 3.3 and 3.4.

□

The next step is to generalize to higher dimensions.

**Theorem 3.5.** *Suppose that $n$ is an integer such that $n > 1$. Suppose that $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Suppose that $\phi$ is a mapping defined on an open subset $U$ of $\mathbb{K}P^n$, such that the range of $\phi$ has $n + 1$ points in general position, and such that $\phi$ maps collinear points to collinear points. Then $\phi$ is the restriction to $U$ of a mapping of the form $\psi \circ \sigma^*$, where $\psi$ is a projective transformation and $\sigma$ is a unital homomorphism $\mathbb{K} \to \mathbb{K}$.*

**Theorem 3.6.** *Suppose that $n$ is an integer such that $n > 1$. Suppose that $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Suppose that $\phi$ is a mapping defined on an open connected subset $U$ of $\mathbb{K}P^n$, such that the range of $\phi$ has $n + 1$ points in general position, and such that $\phi$ maps connected subsets of straight lines $l \in V$ into straight lines, where $V$ is a fixed open set in the topology described in Section 1.4, with the property that for every $p \in U$, there exists a line $l \in V$ such that $p \in l$. Then $\phi$ is the restriction to $U$ of a mapping of the form $\psi \circ \sigma^*$, where $\psi$ is a projective transformation and $\sigma$ is a unital homomorphism $\mathbb{K} \to \mathbb{K}$.*

*Proof of Theorems 3.5 and 3.6.* We have just proved these theorems in the case $n = 2$. We will generalize to higher dimensions by induction. We can again assume without loss of generality that $U$ is an open rectangle in $\mathbb{K}^n$, this being identified with the affine part of $\mathbb{K}P^n$. (It is obvious how to generalize the notion of the affine part of a projective space to higher dimensions.) As before we may assume without loss of generality that all connected subsets of lines meeting $U$ with slopes in a certain open set $W$ are mapped into lines. Consider $n+1$ points in general position contained in $U$. The intersection of a hyperplane (a subspace of $\mathbb{K}P^n$ of dimension $n-1$) with $U$ will be a nonempty open connected subset of the hyperplane, because $U$ is an open rectangle. Now by induction we can assume that the map is of the required form on the hyperplanes generated by any $n$ of the points, and from this we get that the map is of the required form on all of $U$. For from the images of the $n+1$ points we can determine a projective transformation $\psi$ which agrees with the map at those points. By composing on the right with $\sigma^*$ for an appropriate ring homomorphism $\sigma$, we can obtain a mapping which agrees with the map on all the hyperplanes generated by any $n$ of the points. (The same choice of $\sigma$ will work for all the hyperplanes since any two of the hyperplanes have an intersection with dimension greater than zero, and furthermore this intersection meets $U$ because the original $n+1$ points are all in $U$.) And, since any point in $U$ can be obtained as the intersection of two lines with slope in the required set, each of which meets two of the hyperplanes in two distinct points, it follows that $\psi$ agrees with $\phi$ on all of $U$.

□

Finally we have a theorem dealing with the flag manifold. Recall from Section 3.1 that $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, $G = \mathrm{GL}(n, \mathbb{K})$, $P$ is the subgroup of lower triangular matrices, and $F = G/P$.

**Theorem 3.7.** *Suppose that $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Suppose that $n > 2$. Suppose that $\mathbb{K}$, $G$, $P$, $F$ are as in the statement of Theorem 3.2. Suppose that $U$ is a nonempty open connected subset of $F$ and that $\phi$ is a mapping $U \to F$, which maps connected subsets of type $k$ fibres contained in $U$ into type $k$ fibres, and whose range contains $n$ points in general position. Then $\phi$ is a* *restriction of a mapping of the form $g^* \circ \sigma^*$, where $\sigma$ is a unital ring homomorphism $\mathbb{K} \to \mathbb{K}$ and $g \in G$. If $\mathbb{K} = \mathbb{C}$ and $\phi$ is measurable, then $\sigma$ is either the identity or conjugation.*

*Proof.* We shall deduce Theorem 3.7 from Theorem 3.6. In Section 3.1 we defined $\phi^*$ given a mapping $\phi : F \to F$ which maps type $k$ fibres into type $k$ fibres for $k = 1, 2, \ldots, n-1$ and whose range has $n$ points in general position. We say that an open subset of the flag manifold $F$ is an open rectangle if it is a projection from the intersection with $G$ of an open rectangle in $M_{nn}(\mathbb{K})$ (which here may be identified with $\mathbb{K}^{n^2}$, so that we may use the same definition of "open rectangle" as before) to $G/P$. We call a mapping $\phi$ from an open rectangle $U \subset F$ into $F$, which maps connected subsets of type $k$ fibres for $k = 1, 2, \ldots, n-1$, a local good mapping, and for such a mapping $\phi$ we define $\phi^*$ in the same way as we defined it in the global case. This works because any two points in $U$ will be connected by a finite sequence of connected subsets of fibres. $\phi^*$ will be a mapping from an open connected subset $U^* \subset \mathbb{K}P^{n-1}$ into $\mathbb{K}P^{n-1}$, which maps connected subsets, contained in $U^*$, of lines, lying in a certain open set $V$ in the natural topology on the set of lines, with the property that for any $p \in U$ there exists an $l \in V$ such that $p \in l$, into lines. We will again call such a mapping a local good mapping. We claim that, given such a mapping $\psi$, there exists a unique local good mapping $\phi : U \to F$ such that $\phi^* = \psi$. We further claim that the range of $\phi$ will contain $n$ points in general position if and only if the range of $\phi^*$ does. This is a local version of a result which appeared earlier in the chapter, and may be proved using the same argument as in the global case, which appeared in the argument for Theorem 3.2. We will be making use of generalizations of this result about liftings of local good mappings in Part II. From this result, together with Theorem 3.6, Theorem 3.7 follows for open rectangles, (apart from the remark about what follows from measurability in the case $\mathbb{K} = \mathbb{C}$), and then follows for arbitrary connected open subsets of $F$ by a continuation argument. We have now proved Theorem 3.7 apart from the remark that in the case where $\mathbb{K} = \mathbb{C}$ measurability suffices for $\sigma$ to be either the identity or conjugation. This follows from the known fact that the only measurable unital field homomorphisms $\mathbb{C} \to \mathbb{C}$ are the identity and conjugation; see [16]. $\square$

This completes our discussion of the local versions of the fundamental theorem of projective geometry in the cases $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. In Part II we discuss how to generalize these results to other classical groups.

# Part II

# The Other Classical Groups

# Chapter 4

## Generalizations of Carathéodory's theorem

In Part I, we discussed some results about $\mathrm{GL}(n, \mathbb{K})$ where $n > 2$ and $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. In this chapter, we will discuss some similar results that deal with the groups $\mathrm{O}(p, q)$, $\mathrm{SU}(p, q)$, and $\mathrm{Sp}(p, q)$, where $p, q > 0$, and $p + q > 2$ and $p + q > 3$ if $\mathbb{K} = \mathbb{R}$, and discuss how they are related to a result that was known to Carathéodory.

### 4.1 Definitions

Let us define the aforementioned groups. In what follows, we let $Q_{p,q}$ denote the form such that $Q_{p,q}(x) = |x_1|^2 + \ldots + |x_p|^2 - |x_{p+1}|^2 - \ldots - |x_{p+q}|^2$, where $x = (x_1, ..., x_{p+q})$ is a vector in $\mathbb{K}^{p+q}$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

**Definition 4.1.** *Suppose $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ respectively. The groups $\mathrm{O}(p, q), \mathrm{SU}(p, q)$, and $\mathrm{Sp}(p, q)$ respectively are defined to be the groups of matrices of $\mathbb{K}$-module automorphisms of $\mathbb{K}^{p+q}$ which preserve the form $Q_{p,q}$. In the case of $\mathrm{SU}(p, q)$ we further require that the matrices have determinant 1.*

*Remark.* The requirement that the matrices have determinant 1 in the case of $\mathrm{SU}(p, q)$ is an inessential restriction since we are only interested in the action of these matrices on projective spaces.

We will now define a certain subgroup of each of the groups defined in the previous section, which will enable us to define various generalized flag manifolds. Denote by $G$ one of $\mathrm{O}(p, q)$ where $p, q > 0$ and $p + q > 3$, $\mathrm{SU}(p, q)$ where $p, q > 0$ and $p + q > 2$, and $\mathrm{Sp}(p, q)$

where $p, q > 0$ and $p + q > 2$. We will now define a certain subgroup $P$ of $G$. Suppose that $\mathbb{K}$ is $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ respectively.

**Definition 4.2.** *A submodule of $\mathbb{K}^{p+q}$ is null if $Q_{p,q}$ vanishes on it.*

**Definition 4.3.** *A flag is a sequence of nonzero submodules of $\mathbb{K}^{p+q}$ of strictly increasing rank. The set of ranks which are represented in the flag is called the signature of the flag.*

The group $G$ acts on the flags of a given signature in the obvious way.

**Definition 4.4.** *A full flag is a flag in which all ranks from 1 to $p + q$ are represented.*

**Definition 4.5.** *A submodule $M$ of $\mathbb{K}^{p+q}$ is good under the following circumstances. Suppose that $p \geq q$ and the rank of $M$ is $d$. If $d \leq q$ and $M$ is null, then $M$ is good. If $q < d < p+q$, and $M$ contains $d+1-q$ null submodules of rank $q$, any two of which intersect in the same submodule of rank $q-1$, then $M$ is good. If $p < q$, then we reverse the roles of $p$ and $q$ in the foregoing.*

**Definition 4.6.** *A flag is good if it consists entirely of good submodules.*

We may see that good full flags $f$ exist as follows. Assume without loss of generality that $p \geq q$. We define $x_{ji}$ for $1 \leq i \leq p+q, 1 \leq j \leq p+q$ as follows. For integers $j$ such that $1 \leq j \leq q$, $x_{jj} = 1$ and $x_{j(p+j)} = 1$, and if $i \neq j, p+j$ then $x_{ji} = 0$. For integers $j$ such that $q < j \leq p$, $x_{j(p+1)} = 1$ and $x_{jj} = 1$, and if $i \neq j, p+1$ then $x_{ji} = 0$. For integers $j$ such that $p < j \leq p+q$, $x_{j1} = 1$ and $x_{jj} = 1$, and if $i \neq j, 1$ then $x_{ji} = 0$. For $1 \leq j \leq p+q$ we define $\mathbf{v}_j = (x_{j1}, x_{j2}, \dots x_{j(p+q)})$. For $1 \leq j \leq q$ we define the module $M_j$ to be the module spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_j\}$, and if $q < j < p+q$, we define the module $M'_j$ to be the module spanned by $\{\mathbf{v}_2, \mathbf{v}_3, \dots \mathbf{v}_q, \mathbf{v}_j\}$. Suppose that for $1 \leq i \leq p+q$ the module of rank $i$ in $f$ is the module spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_i\}$. We see that, for $1 \leq j \leq q$ the module $M_j$ is null, and if $q < j < p+q$, the module $M'_j$ is null and its intersection with the module $M_q$ and with any module $M'_{j'}$ such that $q < j' \leq p, j \neq j'$, is the same null module of rank $q-1$. Hence on these assumptions $f$ is a good full flag. Denote by $f$ the unique good full flag which satisfies these assumptions and denote by $P$ the stabilizer of $f$ under the action of $G$. We have now completed our task of defining the subgroup $P$.

We shall now define the notion of a quasi-sphere, quasi-circle, complex quasi-circle, and quaternionic 3-quasi-sphere. These are key terms in the statements of Theorems 4.1 and 4.3. Let $s$ be the submodule of rank one of $\mathbb{K}^{p+q}$ which occurs in $f$. Denote by $Q$ the stabilizer of $s$. Then $G/Q$ may be identified with the quasi-sphere

$$S = \{\mathbf{x} = (x_1 : ... : x_{p+q-1} : x_{p+q}) \in \mathbb{K}P^{p+q-1} \mid Q_{p,q}(\mathbf{x}) = 0\}.$$

For example, if $\mathbb{K} = \mathbb{R}$, $p = 3$, $q = 1$, then $S$ is the ordinary real 2-sphere living in $\mathbb{R}P^3$.

**Definition 4.7.** *A quasi-circle, complex quasi-circle, or quaternionic 3-quasi-sphere is the intersection of a real plane, complex line, or quaternionic line with the real, complex, or quaternionic quasi-sphere.*

*Remark.* Denote by $R$ the stabilizer of a flag consisting just of the submodule of $\mathbb{K}^{p+q}$ which occurs in $f$ with rank one (or two if $\mathbb{K} = \mathbb{R}$). In the case $q = 1$, left cosets of $R$ in $G$ project down to circles, complex circles, or quaternionic 3-spheres in $G/Q$.

We defined a topology on the set of complex lines and quaternionic lines in Part I, on pages 10 and 11. We may also obtain a topology on the set of real planes, by identifying the set of real planes with $G/P'$ where $G = \mathrm{GL}(n, \mathbb{R})$ and $P'$ is a different subgroup containing the subgroup $P$ of lower triangular matrices. Thus we get a topology on the family of real planes, complex lines, or quaternionic lines. From this we get a topology on the set of quasi-circles, complex quasi-circles, or quaternionic 3-quasi-spheres.

## 4.2 Generalizations of the theorems of Chapter 3

Now we can state some generalizations of the theorems in Chapter 3, which will appear as Theorems 4.8–4.13 below. We shall state these theorems and then prove them, stating the global theorems first, then the local ones, as in Chapter 3. In these theorems, if $g$ is a matrix in some subgroup of $\mathrm{GL}(n, \mathbb{K})$ then we denote by $g^*$ the induced bijection of $\mathbb{K}P^{n-1}$. We also denote by $g^*$ the mapping $G/P \to G/P$ which maps $hP \mapsto ghP$. If $\sigma$ is a unital ring homomorphism $\mathbb{K} \to \mathbb{K}$, and the value of $n$ is understood from the context, then we denote by $\sigma^*$ the induced mapping $\mathbb{K}P^{n-1} \to \mathbb{K}P^{n-1}$.

For the statement of Theorems 4.8 and 4.12, denote by $S$ the quasi-sphere $\{\mathbf{x} = (x_1 : ... : x_{p+q-1} : x_{p+q}) \in \mathbb{K}P^{p+q-1} \mid Q_{p,q}(\mathbf{x}) = 0\}$.

**Theorem 4.8.** *Suppose that $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Suppose that $p$ and $q$ are integers such that $p, q > 0$ and $p + q > 2, p + q > 3$ if $\mathbb{K} = \mathbb{R}$. Suppose that $G = \mathrm{O}(p, q), \mathrm{SU}(p, q)$, or $\mathrm{Sp}(p, q)$, respectively. Any mapping $\phi : S \to S$ which maps quasi-circles, complex quasi-circles, or quaternionic 3-quasi-spheres, respectively, into quasi-circles, complex quasi-circles, or quaternionic 3-quasi-spheres, respectively, and whose range has $p + q$ points in general position, is of the form $g^*$, or $g^* \circ \sigma^*$ if $\mathbb{K} = \mathbb{C}$, where $g \in G$, and if $\mathbb{K} = \mathbb{C}$ then $\sigma$ is either the identity or conjugation.*

The case of this theorem with $\mathbb{K} = \mathbb{R}$, $q = 1$ appeared earlier as Theorem 1.6 and is proved in [15]. To be more exact it appeared earlier as Theorem 1.6 and is proved in [15] with the further assumption that $\phi$ is surjective, which is easy to eliminate, as will be seen in the proof of Theorem 4.8 below. A related result was known to Carathéodory and is proved in [3]. In [3] Carathéodory proves that an injective function defined on a domain in the complex plane that maps circles lying entirely in the domain onto circles, must be Möbius. As will become clear, this is closely related to the case $\mathbb{K} = \mathbb{R}$, $p = 2$, $q = 1$.

For the statement of Theorems 4.11 and 4.13, let $\mathbb{K}$, $p$, $q$, $f$, $s$, $S$, $P$, $Q$, $G$, $F$ be as defined in Section 4.1.

**Definition 4.9.** *A finite set of points in $F$ is in general position if its projection to $S = G/Q$ is in general position.*

Let $P_k$ be the stabilizer of the flag that results by deleting the submodule of dimension $p + q - k$ from $f$.

**Definition 4.10.** *The projections of left cosets of $P_k$ in $G$ to $F$ are called type $k$ fibres.*

**Theorem 4.11.** *Any mapping $\phi : F \to F$ which maps type $k$ fibres into type $k$ fibres for $k = 1, 2, \ldots, p+q-1$, and whose range contains $p+q$ points in general position, is of the form $g^*$, or $g^* \circ \sigma^*$ if $\mathbb{K} = \mathbb{C}$, where $g \in G$, and if $\mathbb{K} = \mathbb{C}$ then $\sigma$ is either the identity or conjugation.*

This result was known to Jacques Tits and follows from material in Chapter 5 of [23]. There it appears as a special case of a result about the building of an algebraic group. It is equivalent to Theorem 4.8 by an argument similar to the argument discussed in Chapter 3 for the equivalence between Theorem 3.1 and Theorem 3.2; however in the case where $q > 1$ the argument requires a bit more elaboration. We will discuss this later in the chapter, when discussing the proofs of Theorems 4.11 and 4.13.

For the statement of Theorem 4.12, let $\mathbb{K}$, $p$, $q$, and $S$ be as in the statement of Theorem 4.8.

**Theorem 4.12.** *Suppose that $T$ is a subset of the family of quasi-circles, complex quasi-circles, or quaternionic 3-quasi-spheres, which is open in the natural topology on this family, defined in Section 4.1, and that $U$ is a nonempty open connected subset of $S$, and that, for every $x \in U$, there exists a $c \in T$ such that $x \in c$. Suppose that $\phi$ is a mapping $U \to S$, whose range contains $p+q$ points in general position, which maps connected subsets, contained in $U$, of quasi-circles, complex quasi-circles or quaternionic 3-quasi-spheres, belonging to $T$, into quasi-circles, complex quasi-circles or quaternionic 3-quasi-spheres. Then $\phi$ is the restriction to $U$ of a mapping of the form $g^*$, or $g^* \circ \sigma^*$ if $\mathbb{K} = \mathbb{C}$, where $g \in G$, and if $\mathbb{K} = \mathbb{C}$ then $\sigma$ is either the identity or conjugation. The result remains true if we drop all occurrences of the words "connected" and remove the requirement that the quasi-circles, complex quasi-circles, or quaternionic 3-quasi-spheres belong to $T$.*

For the statement of Theorem 4.13, let $\mathbb{K}$, $p$, $q$, $f$, $s$, $S$, $P$, $Q$, $G$, and $F$ be as in the statement of Theorem 4.11. We will make use of Definitions 4.9 and 4.10.

**Theorem 4.13.** *Suppose that $\phi$ is a mapping from a nonempty open connected subset $U$ of $F$ into $F$, which maps connected subsets of type $k$ fibres contained in $U$ into type $k$ fibres, for $k = 1, 2, \ldots p + q - 1$, and whose range contains $p + q$ points in general position. Then $\phi$ is of the form $g^*$, or $g^* \circ \sigma^*$ if $\mathbb{K} = \mathbb{C}$, where $g \in G$, and if $\mathbb{K} = \mathbb{C}$ then $\sigma$ is either the identity or conjugation.*

Before proceeding to prove Theorems 4.8 to 4.13, let us start by discussing the equivalence of Theorems 4.8 and 4.11, and how Theorem 4.13 follows from Theorem 4.12, in the case where $q = 1$. Fix $\mathbb{K}, p$, and $q$ as in the statements of Theorems 4.8 and 4.11, with the further requirement that $q = 1$. Denote by $S$ the quasi-sphere as in the statement of Theorem 4.8, and define a good mapping $S \to S$ to be one which satisfies the hypotheses of Theorem 4.8. Let $F$ be as defined in Section 4.1, and define a good mapping $F \to F$ to be one which satisfies the hypotheses of Theorem 4.11. We have $F = G/P$ and $S = G/Q$ where $G, P$, and $Q$ are defined as in the statement of Theorem 4.11. Given a good mapping $\phi : F \to F$ we can define a mapping $\phi^* : S \to S$ much as we did before: namely, if $p$ is the canonical projection $F \to S$, then we define $\phi^*(p(x)) = p(\phi(x))$, which can be shown to be well-defined. We can then prove, as in the proof of Theorem 3.2, that if $\phi : F \to F$ is good then $\phi^*$ is good, and if $\psi : S \to S$ is good, then there exists a unique good mapping $\phi : F \to F$ such that $\phi^* = \psi$. These two results show that Theorems 4.8 and 4.11 are equivalent in the case where $q = 1$. Similarly, as in the proof of Theorem 3.7, we define an open rectangle to be the image of $R \cap G$ under the canonical projection $G \to F$ where $R$ is an open rectangle contained in $\mathbb{K}^{n^2}$. Then, given a mapping $\phi$ of the kind required in Theorem 4.13 defined on an open rectangle, we can define a mapping $\phi^*$ of the kind required in Theorem 4.12, in the same way. We know that $\phi^*$ is well-defined because any two points in the open rectangle will be connected by a finite sequence of connected subsets of fibres. Then the same argument as in the proof of Theorem 3.7, on p. 36, shows that Theorem 4.13 follows from Theorem 4.12 in the case where $q = 1$. We shall discuss the case where $q > 1$ later, after the proofs of Theorems 4.8 and 4.12.

We shall start by proving Theorem 4.8. Then we will show how to strengthen it to Theorem 4.12.

*Proof of Theorem 4.8.* Let us begin by discussing the case $\mathbb{K} = \mathbb{R}$, and $q = 1$. This is the case discussed in [15], on the further assumption that $\phi$ is surjective. Recall that since $\mathbb{K} = \mathbb{R}$ we are assuming that $p > 2$. Let us think of $S$ as embedded in $\mathbb{R}P^p$ and call the point $(0 : 0 : \ldots : 0 : 1 : 1)$ the north pole $n$. Clearly, we can assume that the north pole is fixed without loss of generality. This is because the Möbius transformation group acts transitively on the sphere, and so we may compose $\phi$ on the left with an appropriate Möbius transformation so that the north pole becomes fixed, without affecting the hypotheses on $\phi$ or the conclusion of the theorem.

**Definition 4.14.** *Real stereographic projection from the north pole is the bijection $\tau : S \setminus \{n\} \to \mathbb{R}^{p-1}$ such that $\tau(x_1 : x_2 : \ldots : x_p : 1) = (x_1, x_2, \ldots, x_{p-1})/(1 - x_p)$.*

So, given a mapping $\phi$ of the kind specified in Theorem 4.8 which fixes the north pole, we obtain a mapping $\phi^* = \tau \circ \phi \mid_{S \setminus \{n\}} \circ \tau^{-1} : \mathbb{R}^{p-1} \to \mathbb{R}^{p-1}$. Now, $\tau$ maps circles passing through the north pole (minus the north pole) onto lines, and circles that do not pass through the north pole onto circles. (This is a standard property of real stereographic projection. We discuss this in Appendix A; a proof can be found in [19].) So $\phi^*$ maps lines into lines and circles into circles. Hence $\phi^*$ is certainly affine, by the affine version of Theorem 3.1, since it maps lines into lines and its range has a sufficient number of points in general position. Further, since it maps circles into circles, it is actually a multiple of an isometry with respect to the Euclidean distance. In particular, it is a Möbius transformation, so $\phi$ is Möbius, and the result follows in this case.

Let us now talk about the case $\mathbb{K} = \mathbb{C}, q = 1$. We will start by giving an outline of the argument. We will reduce the case under discussion to a number of lemmas, which we will then prove.

We are dealing with a $(2p-1)$-sphere $S$ embedded in $\mathbb{C}P^p$ where $p > 1$. Let us call $(0 : 0 : \ldots : i : 1)$ the north pole $n$ and $(0 : 0 : \ldots : -i : 1)$ the south pole $s$. We start by showing that $\mathrm{SU}(p, 1)$ acts transitively on ordered pairs of distinct points in the $(2p-1)$-sphere. This means that we can assume without loss of generality that the north pole and the south pole are fixed, and we shall do so in what follows.

Next we define six maps, $\sigma_1, \sigma_2, \tau_1, \tau_2, \iota_1$, and $\iota_2$.

**Definition 4.15.** *We define:*
$\sigma_1 : S \setminus \{n\} \to \mathbb{R}^{2p-1}, \sigma_2 : S \setminus \{s\} \to \mathbb{R}^{2p-1}, \tau_1 : S \setminus \{n\} \to \mathbb{C}^{p-1}, \tau_2 : S \setminus \{s\} \to \mathbb{C}^{p-1}, \iota_1 : \mathbb{C}^{p-1} \to S, \iota_2 : \mathbb{C}^{p-1} \to S,$

*by*

$\sigma_1(x_1 + iy_1 : x_2 + iy_2 : \ldots : x_p + iy_p : 1) = (x_1, y_1, x_2, y_2, \ldots, x_p)/(1 - y_p),$
$\sigma_2(x_1 + iy_1 : x_2 + iy_2 : \ldots : x_p + iy_p : 1) = (x_1, y_1, x_2, y_2, \ldots, x_p)/(y_p + 1),$
$\tau_1(z_1 : z_2 : \ldots : z_p : 1) = (z_1, z_2, \ldots, z_{p-1})/(i - z_p),$
$\tau_2(z_1 : z_2 : \ldots : z_p : 1) = (z_1, z_2, \ldots, z_{p-1})/(z_p + i),$
$\iota_1(z_1, z_2, \ldots z_{p-1}) = (2z_1 : 2z_2 : \ldots : 2z_{p-1} : i(-1 + l) : 1 + l),$
$\iota_2(z_1, z_2, \ldots z_{p-1}) = (2z_1 : 2z_2 : \ldots : 2z_{p-1} : -i(-1 + l) : 1 + l),$
*where* $l = |z_1^2| + \ldots + |z_{p-1}|^2$.

*We call $\sigma_1$ real stereographic projection from the north pole, $\sigma_2$ real stereographic projection from the south pole, $\tau_1$ complex stereographic projection from the north pole, and $\tau_2$ complex stereographic projection from the south pole.*

*Remark.* The maps $\iota_1$ and $\iota_2$ are right inverses to $\tau_1$ and $\tau_2$ respectively.

Now, $\phi$ induces two maps $\mathbb{R}^{2p-1} \to \mathbb{R}^{2p-1}$, namely $\phi_1^* = \sigma_1 \circ \phi \circ \sigma_1^{-1}$ and $\phi_2^* = \sigma_2 \circ \phi \circ \sigma_2^{-1}$. Also, the family of sets of the form $C \cap (S \setminus \{n\})$ such that $C$ is a complex circle passing through the north pole consisting of more than one point is a partition $P_1$ of $S \setminus \{n\}$, because there is a unique complex circle passing through any two distinct points in $S$, namely the intersection of the unique complex line passing through the two points in question with $S$. From this we get a partition $P_1^* = \{\sigma_1(C) \mid C \in P_1\}$ of $\mathbb{R}^{2p-1}$ into straight lines. Now, $\phi$ maps any member of $P_1$ into a member of $P_1$. Therefore $\phi_1^*$ maps any member of $P_1^*$ into a member of $P_1^*$. Similarly, the family of sets of the form $\{C \cap (S \setminus \{s\})$, such that $C$ is a complex circle passing through the south pole consisting of more than one point, is a partition $P_2$ of $S \setminus \{s\}$, and from this we get a partition $P_2^* = \{\sigma_2(C) \mid C \in P_2\}$, which again is a partition of $\mathbb{R}^{2p-1}$ into straight lines. Again, $\phi_2^*$ maps any member of $P_2^*$ into a member of $P_2^*$.

Next we prove that each member of $P_1^*$ meets $\{(x_1, x_2 \ldots x_{2p-1}) \in \mathbb{R}^{2p-1} \mid x_{2p-1} = 0\}$ in exactly one point. Given a point $\mathbf{z} \in \mathbb{C}^{p-1}$, define $P_1(\mathbf{z})$ to be the member of $P_1$ passing through $\mathbf{z}$. The corresponding result will of course immediately follow for $P_2^*$. (We will not actually give the proof now; at this stage we are merely outlining the argument.) Let us identify the set $\{(x_1, x_2 \ldots x_{2p-1}) \in \mathbb{R}^{2p-1} \mid x_{2p-1} = 0\}$ with $\mathbb{C}^{p-1}$. Now, $\phi_1^*$ induces a permutation of $P_1^*$ and each member of $P_1^*$ meets $\mathbb{C}^{p-1}$ in a unique point, so this means that $\phi_1^*$ induces a map $\phi_1^{**} : \mathbb{C}^{p-1} \to \mathbb{C}^{p-1}$. We have $\phi_1^{**} = \tau_1 \circ \phi \circ \iota_1$, and given a point $\mathbf{z} \in \mathbb{C}^{p-1}$, $\phi_1^*(P_1(\mathbf{z})) = P_1(\phi_1^{**}(\mathbf{z}))$. Similarly, $\phi_2^*$ induces a map $\phi_2^{**} : \mathbb{C}^{p-1} \to \mathbb{C}^{p-1}$ with similar properties.

Next we prove that the family of sets $\tau_1(C)$, such that $C$ is a complex circle contained in $S$ not passing through the north pole consisting of more than one point, is equal to the set of complex circles in $\mathbb{C}^{p-1}$, a complex circle in $\mathbb{C}^{p-1}$ being defined to be a circle contained in a complex line. A similar result of course immediately follows for $\tau_2$. Since $\phi$ maps complex circles into complex circles, it follows that $\phi_1^{**}$ and $\phi_2^{**}$ also map complex circles into complex circles.

We will prove that $\phi$ fixes $\iota_1(\mathbb{C}^{p-1})$ setwise, and this means

$$\phi_2^{**} \mid_{\mathbb{C}^{p-1} \setminus \{\mathbf{0}\}} = \rho \circ \phi_1^{**} \mid_{\mathbb{C}^{p-1} \setminus \{\mathbf{0}\}} \circ \rho$$

where $\rho$ is inversion about the origin (as defined in Definition A.1 in Appendix A). We also have that $\phi_1^{**}$ and $\phi_2^{**}$ both fix the origin, and that their ranges contain $p$ points in general position.

So $\phi_1^{**}$ and $\phi_2^{**}$ both fix the origin, are related in the way we described in the previous paragraph, their ranges contain $p$ points in general position, and they map complex circles into complex circles. We will prove that this means that $\phi_1^{**} = rA, \phi_2^{**} = (1/r)A$, possibly both composed on the right with complex conjugation, where $r$ is a positive real number and $A$ is the linear transformation induced by a matrix in $\mathrm{U}(p-1)$.

Now, every point in $S$ other than the north pole and the south pole can be obtained as an intersection of a complex circle passing through the north pole and a complex circle passing through the south pole. Say that the point $p$ is the intersection of the complex circle $C_1$ passing through the north pole and the complex circle $C_2$ passing through the south pole. If we know $\phi_1^{**}$ and $\phi_2^{**}$, then we know $\phi(C_1)$ and $\phi(C_2)$ and $\phi(p) = \phi(C_1) \cap \phi(C_2)$. Thus, knowing $\phi_1^{**}$ and $\phi_2^{**}$ gives us complete information about $\phi$; there is at most one $\phi$ giving rise to a given pair $(\phi_1^{**}, \phi_2^{**})$. So it will just remain to prove that every pair $(\phi_1^{**}, \phi_2^{**})$ of the form described above comes from a $\phi$ of the desired form, and we will be done.

That concludes the outline of the argument. We have reduced Theorem 4.8 in the case $\mathbb{K} = \mathbb{C}, q = 1$ to a number of lemmas, stated below as Lemmas 4.16–4.22. In the statements and proofs of the lemmas, we will be making use of Definition 4.15.

**Lemma 4.16.** *The group* $\mathrm{SU}(p, 1)$ *acts transitively on ordered pairs of distinct points in the* $(2p-1)$*-sphere* $S$.

**Lemma 4.17.** *Suppose that* $P_1 = \{C \cap (S \setminus \{n\}) \mid C$ *is a complex circle passing through the north pole consisting of more than one point*$\}$. *Let* $P_1^* = \{\sigma_1(C) \mid C \in P_1\}$. *Each straight line in* $P_1^*$ *meets* $\{(x_1, x_2 \ldots x_{2p-1}) \in \mathbb{R}^{2p-1} \mid x_{2p-1} = 0\}$ *in exactly one point.*

**Lemma 4.18.** *Suppose that $\phi$ is as in the statement of Theorem 4.8. Suppose that $\phi_1^{**} = \tau_1 \circ \phi \circ \iota_1$, and that $P_1$ is as in the statement of Lemma 4.17. Denote by $P_1(\boldsymbol{z})$ the member of $P_1$ passing through $\boldsymbol{z}$. Given a point $\boldsymbol{z} \in \mathbb{C}^{p-1}$, $\phi_1^*(P_1(\boldsymbol{z})) = P_1(\phi_1^{**}(\boldsymbol{z}))$.*

**Lemma 4.19.** *The family of sets $\tau_1(C)$, such that $C$ is a complex circle, contained in $S$, not passing through the north pole, and consisting of more than one point, is equal to the set of complex circles in $\mathbb{C}^{p-1}$.*

**Lemma 4.20.** *Suppose that $\phi$ is as in the statement of Theorem 4.8. Define $\phi_1^{**} = \tau_1 \circ \phi \circ \iota_1, \phi_2^{**} = \tau_2 \circ \phi \circ \iota_2$. Suppose that $\rho : \mathbb{C}^{p-1} \to \mathbb{C}^{p-1}$ is inversion about the origin. We have $\phi_2^{**} |_{\mathbb{C}^{p-1} \setminus \{\boldsymbol{0}\}} = \rho \circ \phi_1^{**} |_{\mathbb{C}^{p-1} \setminus \{\boldsymbol{0}\}} \circ \rho$.*

**Lemma 4.21.** *Let $\phi_1^{**}$ and $\phi_2^{**}$ be two maps $\mathbb{C}^{p-1} \to \mathbb{C}^{p-1}$ whose ranges contain $p$ points in general position, which fix the origin and map complex circles into complex circles, such that $\phi_2^{**} |_{\mathbb{C}^{p-1} \setminus \{\boldsymbol{0}\}} = \rho \circ \phi_1^{**} |_{\mathbb{C}^{p-1} \setminus \{\boldsymbol{0}\}} \circ \rho$, where $\rho$ is inversion about the origin. Then $\phi_1^{**} = rA, \phi_2^{**} = (1/r)A$, possibly both composed on the right with complex conjugation, where $r$ is a positive real number and $A$ is the linear transformation induced by a matrix in* $\mathrm{U}(p-1)$.

**Lemma 4.22.** *Suppose $\phi_1^{**}$ and $\phi_2^{**}$ are as in the conclusion of the Lemma 4.21. Then there exists $\phi$ as in the conclusion of Theorem 4.8 such that $\phi_1^{**} = \tau_1 \circ \phi \circ \iota_1$, and $\phi_2^{**} = \tau_2 \circ \phi \circ \iota_2$.*

*Proof of Lemma 4.16.* Let $z = (z_1 : z_2 : \ldots : z_{p+1})$ and $w = (w_1 : w_2 : \ldots : w_{p+1})$ be two distinct points in the $(2p-1)$-sphere $S$. We must find a matrix $M \in \mathrm{SU}(p,1)$ such that $M \cdot (1, 0, \ldots, 0, 1)^T$ is a scalar multiple of $\mathbf{z} = (z_1, z_2, \ldots, z_{p+1})^T$ and $M \cdot (0, 1, \ldots, 0, 1)^T$ is a scalar multiple of $\mathbf{w} = (w_1, w_2, \ldots, w_{p+1})^T$. It is sufficient to let $M \cdot (1, 0, \ldots, 0, 1)^T = \lambda \mathbf{z}$ and $M \cdot (0, 1, \ldots, 0, 1)^T = \mu \mathbf{w}$ where $\lambda \mu Q_{p,1}(\mathbf{z}, \mathbf{w}) = -1$. It is easy to see that there exists a matrix $M \in \mathrm{SU}(p,1)$ with this property. This completes the proof of Lemma 4.16. □

*Proof of Lemma 4.17.* Proving this lemma is equivalent to showing that if $l$ is a complex line in the affine space $\mathbb{C}^p$ which meets the $(2p-1)$-sphere $S$ in the north pole and in at least one other point, then there are exactly two points on $l$ such that the real part of the last co-ordinate is zero (one of them of course being the north pole). Let us view $\mathbb{C}^p$ as a

$2p$-dimensional vector space over $\mathbb{R}$. We have that $l$ is a translate of a 2-dimensional subspace and the set of points $U$, such that the real part of the last co-ordinate is zero, is a translate of a $(2p-1)$-dimensional subspace. To prove the lemma, we must prove that $l \cap U$ meets $S$ in two distinct points. If we suppose that $l$ is a subset of $U$, then it is easy to see that $l$ only meets $S$ at the north pole, contrary to hypothesis. So we conclude that $l$ is not a subset of $U$. We then conclude by linear algebra that $V = l \cap U$ is a translate of a 1-dimensional subspace. Again, given that $l$ is a complex line, it is easy to see that if $V$ only met $S$ at the north pole, then $l$ would only meet $S$ at the north pole, contrary to hypothesis. It follows that $V$ meets $S$ in two distinct points. This completes the proof of Lemma 4.17. $\square$

*Proof of Lemma 4.18.* Suppose that $\phi$ is as in the statement of Theorem 4.8. Define $\phi_1^{**} = \tau_1 \circ \phi \circ \iota_1$. Suppose that $P_1$ is as in the statement of Lemma 4.17. In what follows, denote by $P_1(\mathbf{z})$ the member of $P_1$ passing through $\mathbf{z}$. We identify $\mathbb{C}^{p-1}$ with the set of points in $\mathbb{R}^{2p-1}$ whose last co-ordinate is zero. Suppose that $\mathbf{z} \in \mathbb{C}^{p-1}$. Define $b$ to be the member of $P_1(\mathbf{z})$, and $c$ to be $P_1(\phi_1^{**}(\mathbf{z}))$. We wish to show that $\phi_1^*(b) = c$. We have that $\phi_1^*(b)$ is equal to the image under $\sigma_1 \circ \phi$ of the complex circle passing through the north pole and $\sigma_1^{-1}(\mathbf{z})$; let us call this latter point $a$. Now, $\iota_1$ is an inverse to the restriction of $\sigma_1$ to the set of points in $S$ of all $(z_1 : z_2 : \ldots : z_p : 1)$ such that $\mathrm{Re}\, z_p = 0$. So $a = \iota_1(\mathbf{z})$. Thus $\phi_1^*(P_1(\mathbf{z}))$ is equal to the image under $\sigma_1$ of the complex circle passing through the north pole and $\phi(a)$. The complex circle passing through the north pole and $\phi(a)$ is the same as the complex circle passing through the north pole and

$$\iota_1(\tau_1(\phi(a)) = \iota_1(\phi_1^{**}(\tau_1(a))) = \iota_1(\phi_1^{**}(\mathbf{z})).$$

Hence $\phi_1^*(b) = c$. This completes the proof of Lemma 4.18. $\square$

*Proof of Lemma 4.19.* Define $\iota$ so that $\iota(z_1, z_2, \ldots, z_{p-1}) = (z_1, z_2, \ldots, z_{p-1}, 0)$. In this way $\mathbb{C}^{p-1}$ is embedded in $\mathbb{C}^p$. For any $a \in S \setminus \{n\}$, we have that $\iota(\tau_1(a))$ is the intersection with $\iota(\mathbb{C}^{p-1})$ of $\sigma_1(C)$ where $C$ is the unique complex circle passing through $n$ and $a$. So, if $C$ is a complex circle contained in $S$, not passing through $n$ and consisting of more than one point, then $\iota(\tau_1(C))$ is contained in a unique complex line contained in $\mathbb{C}^{p-1}$. Suppose that $\kappa$ is any complex-affine map from this line to $\mathbb{C}$. Then $\kappa(\iota(\tau_1(C))$ is an image of a circle under a linear fractional transformation and hence a circle. It follows that $\tau_1(C)$ is a complex circle. So the family of all sets $\tau_1(C)$, such that $C$ is a complex circle, contained in $S$, not passing through the north pole, and consisting of more than one point, certainly consists only of complex circles; it remains to prove that it contains all complex circles. For a moment let us relax the requirement that $\phi$ fix the south pole, and just assume that it fixes the north pole. Now $\phi_1^{**}$ is still well-defined and it no longer necessarily fixes the origin. Furthermore, the set of all $\tau_1(C)$ such that $C$ is a complex circle, contained in $S$, not passing through the north pole, and consisting of more than one point, contains all complex circles of the form $\phi_1^{**}(C)$, where $C$ is some fixed complex circle, say the unit circle about the origin in $\{(z_1, 0, 0, \ldots, 0) \mid z_1 \in \mathbb{C}\}$, and $\phi_1^{**}$ arises from a $\phi$ satisfying the hypotheses of Theorem 4.8 and fixing the north pole. It follows from Lemma 4.22, which will be proved without making any use of any of the other lemmas, that $\phi_1^{**}$ can be any unitary linear transformation or any dilation. Let us show that it can also be any translation, then the proof of the lemma will be complete. Consider the real 3-space in $\mathbb{C}^{p+1}$ generated by $(0, 0, \ldots, i, 1), (0, 0, \ldots, -i, 1)$, and $(1, 0, \ldots, 0)$. This projects down to a real projective plane in $\mathbb{C}P^p$, and its intersection with $S$ will be a circle. Consider the element $g$ of $\mathrm{SU}(p, 1)$ which acts on the three vectors in such a way as to induce a transformation of the circle which fixes the point $(0 : 0 : \ldots : i : 1)$, and, when conjugated by $\sigma_1$, is a translation by one unit in the direction of $(1, 0, \ldots, 0)$. Suppose further that $g$ acts trivially on the subspace which is orthogonal to these three vectors with respect to the Hermitian form $Q_{p,1}$. It may be calculated that this matrix $g$ can be chosen to fix $(0, 0, \ldots, i, 1)$, map $(0, 0, \ldots, -i, 1)$ to $(2, 0, \ldots, -i, 1) + r(0, 0, \ldots, i, 1)$ for some real number $r$, and map $(1, 0, \ldots, 0)$ to $(1, 0, \ldots, 0) + s(0, 0, \ldots, i, 1)$ for some real number $s$. Thus, in general, $g$ maps $(a, b, \ldots, -i, 1)$ to $(a + 2, b, \ldots, -i, 1) + \alpha(0, 0, \ldots, i, 1)$ for some complex number $\alpha$, and so maps the complex line through $(0, 0, \ldots, i, 1)$ and $(a, b, \ldots, -i, 1)$ to the complex line through $(0, 0, \ldots, i, 1)$ and $(a + 2, b, \ldots, -i, 1)$. It follows that $\phi_1^{**}$ is a translation by $(1, 0, \ldots, 0)$. A similar argument works to show that $\phi_1^{**}$ may be taken to be a translation by any other vector in $\mathbb{C}^{p-1}$. This completes the proof of the lemma. $\square$

*Proof of Lemma 4.20.* It is easy to check that $\tau_2 \mid_T = \rho \circ \tau_1 \mid_T$, where $T = \iota_1(\mathbb{C}^{p-1}) \setminus \{s\} = \iota_2(\mathbb{C}^{p-1}) \setminus \{n\}$. Thus in order to prove the lemma it is sufficient to show that any $\phi$ which satisfies the hypotheses of the lemma fixes $T$ setwise. Suppose that $a \in T$ and $\phi(a) \notin T$. We shall derive a contradiction from this and that will complete the proof of the lemma. If we let $b = \iota_1(r \cdot \tau_1(a))$, where $r$ is a positive real number which we can choose to be as small as we like, then we will still have $b \in T$, and $\phi(b) \notin T$ by the complex-linearity of $\phi_1^{**}$ and $\phi_2^{**}$. This follows from Lemma 4.19, because it follows from Lemma 4.19 that $\phi_1^{**}$ and $\phi_2^{**}$ must map complex circles into complex circles. This was discussed in the outline of the proof of the case $\mathbb{K} = \mathbb{C}, q = 1$, before the statements of Lemmas 4.16–4.22, when we stated that since $\phi$ maps complex circles into complex circles, it follows that $\phi_1^{**}$ and $\phi_2^{**}$ also map complex circles into complex circles. It follows from this that $\phi_1^{**}$ and $\phi_2^{**}$ map complex lines into complex lines. We also know that their ranges have $p$ points in general position, and that they fix the origin. Hence we can make use of Theorem 3.1 to conclude that $\phi_1^{**}$ and $\phi_2^{**}$ are complex-linear. Hence we can make $b$ as close to the south pole as we like by making $r$ sufficiently small. Now, since $\phi$ fixes the north pole and the south pole, the complex circle through the north pole and $\iota_1(\phi_1^{**}(b))$ has to intersect with the complex circle through the south pole and $\iota_2(\phi_2^{**}(b))$. Let us now think about the images of these two complex circles under $\sigma_1$. The first one becomes a straight line $l$. As $r$ gets arbitrarily small, $l$ gets arbitrarily close to being perpendicular to $\mathbb{C}^{p-1}$ (viewed as being embedded in $\mathbb{R}^{2p-1}$ in the obvious way). The second complex circle becomes a circle $c$ passing through the origin of $\mathbb{C}^{p-1}$ and $\rho(\phi_2^{**}(b)) = r \cdot \rho(\phi_2^{**}(a))$. The line joining these two points is a diameter of $c$, by symmetry. As $r$ gets arbitrarily small, the maximum distance of $c$ from $\mathbb{C}^{p-1}$ gets arbitrarily small, because the complex line between the south pole and $b$ gets arbitrary close to a line contained in the tangent $(p-1)$-space to $S$ at $s$. Thus, if $\rho(\phi_2^{**}(a)) \neq \phi_1^{**}(a)$, $l$ and $c$ will fail to meet each other if $r$ is sufficiently small, contrary to what was shown earlier.

Thus, the assumption that $\rho(\phi_2^{**}(a)) \neq \phi_1^{**}(a)$ becomes untenable. This completes the proof of Lemma 4.20. $\square$

*Proof of Lemma 4.21.* As discussed in the proof of Lemma 4.20, once we have Lemma 4.19 it is easy to see that $\phi_1^{**}$ maps complex lines into complex lines, and we are assuming that its range contains $p$ points in general position and that it fixes the origin. It follows from Theorem 3.1 that it is of the form $\psi \circ \sigma^*$, where $\psi$ is a complex-linear transformation and $\sigma$ is a unital field homomorphism $\mathbb{C} \to \mathbb{C}$. The homomorphism $\sigma$ is certainly $\mathbb{Q}$-linear, and we also know that it maps circles into circles, since $\phi_1^{**}$ maps complex circles into complex circles. From the fact that $\sigma$ maps circles into circles and is an additive homomorphism it follows that it is bounded on the unit disc, because any point in the unit disc lies on a circle of unit radius whose centre lies on the unit circle. Hence $\sigma$ is continuous so it is either the identity or conjugation. Suppose that there were two linearly independent vectors $u, v \in \mathbb{C}^{p-1}$, such that $|\phi_1^{**}(u)| = r_1 \cdot |u|, |\phi_1^{**}(v)| = r_2 \cdot |v|, r_1 > r_2$, where in general $|(z_1, z_2, \ldots z_{p-1})| = (z_1^2 + z_2^2 + \ldots + z_{p-1}^2)^{1/2}$. Then $\phi_1^{**}(u+v)$ would have a component in $u$ with larger modulus than that of the component of $v$, but $\phi_2^{**}(u+v)$ would have a component in $u$ with smaller modulus than that of the component of $v$. This contradicts Lemma 4.20. We can therefore conclude that $r_1 = r_2$ always and Lemma 4.21 now easily follows. $\square$

*Proof of Lemma 4.22.* Suppose $\phi_1^{**} = \psi \circ \sigma^*$ where the matrix of $\psi$ is $rM$, with $r > 0, M \in \mathrm{U}(p-1)$, and $\sigma$ is either the identity or conjugation. Let $\phi = g^* \circ \sigma^*$ where $g^*$ is the projective transformation induced by a block-diagonal matrix from $\mathrm{SU}(p,1)$ with $M$ as the first block and the second block being a matrix which multiplies $\begin{pmatrix} i \\ 1 \end{pmatrix}$ by $\sqrt{r}$ and $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ by $1/\sqrt{r}$. This is the $\phi$ which we seek. $\square$

We have now proved Theorem 4.8 in the case $\mathbb{K} = \mathbb{C}$, $q = 1$. Now let us discuss the case $\mathbb{K} = \mathbb{H}$, $q = 1$. The modifications to the argument just given for the case $\mathbb{K} = \mathbb{C}$, $q = 1$ required are quite minor. We are now dealing with a $(4p-1)$-sphere $S$ living in $\mathbb{H}P^p$, and with the group $\mathrm{Sp}(p, 1)$. We must make the following modifications to Definition 4.15.

We let $\sigma_1$ and $\sigma_2$ be real stereographic projection from the north pole and the south pole respectively. (The north pole is defined to be $(0 : 0 : \ldots : \mathbf{k} : 1)$ and the south pole is defined to be $(0 : 0 : \ldots : -\mathbf{k} : 1)$.) In both cases the codomain is a real $(4p-1)$-space. Then we let $\tau_1$ and $\tau_2$ be quaternionic stereographic projection from the north pole and the south pole respectively, the codomain being $\mathbb{H}^{p-1}$ in both cases. Define $S'$ to be

$$\{(w_1 : w_2 : \ldots : w_p : 1) \in S \mid w_p = r\mathbf{k} \text{ for some real number } r\}.$$

If $i = 1, c$ is the north pole, or $i = 2, c$ is the south pole, and in either case $a \neq c$, then we define $\tau_i(a)$ to be $\sigma_i(b)$ where $b$ is the unique point in $S'$ for some real number $r\}$ lying on the unique quaternionic line containing $c$ and $a$. Then we let $\iota_i$ be a right inverse to $\tau_i$ for $i = 1, 2$ whose range is $S'$, the set defined immediately above. The modification of the statements of all the lemmas and the entire argument given above to yield the result for $\mathbb{K} = \mathbb{H}, q = 1$ is now quite trivial. The only point that calls for comment is that we can eliminate reference to the automorphism group of $\mathbb{H}$ since any mapping induced by an automorphism of $\mathbb{H}$ will be equal to a projective transformation with a matrix from $\mathrm{Sp}(p, 1)$.

We have now established Theorem 4.8 in the case $q = 1$. Let us discuss the case where $q > 1$ now.

First let us consider the case $\mathbb{K} = \mathbb{R}$. We can assume without loss of generality that $p \geq q$, because the case $p < q$ can be converted to the case $p > q$ by a projective transformation. So the first new case is $p = 2, q = 2$. We are considering the quasi-sphere $S = \{(x_1 : x_2 : x_3 : x_4) \in \mathbb{R}P^3 \mid x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0\}$. Call $(1 : 0 : 0 : 1)$ the north pole, or $n$. Let $\mathbf{n}$ denote $(1, 0, 0, 1)$. Given a line $l$ passing through the north pole, there exists a unique vector $\mathbf{v} = (a, b, c, 0) \in \mathbb{R}^4$ such that the projection of $\mathrm{span}_{\mathbb{R}}\{\mathbf{n}, \mathbf{v}\}$ to $\mathbb{R}P^3$ is the line $l$. If $Q_{2,2}(\mathbf{u}, \mathbf{v}) = 0$ and $Q_{2,2}(\mathbf{v}, \mathbf{v}) = 0$, then $\mathbf{v}$ is a scalar multiple of $(0, 1, 1, 0)$ or $(0, 1, -1, 0)$, and the line $l$ is contained in $S$. If $Q_{2,2}(\mathbf{u}, \mathbf{v}) = 0$ and $Q_{2,2}(\mathbf{v}, \mathbf{v}) \neq 0$, then the line $l$ only meets $S$ at the north pole. Otherwise, $Q_{2,2}(\mathbf{u}, \mathbf{v}) \neq 0$ and the line meets $S$ in exactly two places. Thus we have two lines $m_1$ and $m_2$ passing through the north pole contained in $S$. We also have a stereographic projection $\sigma$ from the north pole which maps $S \setminus (m_1 \cup m_2)$ onto $\mathbb{R}^2$. Considering a mapping $\phi : S \to S$ satisfying the hypotheses of the theorem, we can assume without loss of generality that $\phi$ fixes the north pole and maps $m_i$ into $m_i$ for $i = 1, 2$. This point requires some discussion. A quasi-circle is mapped by $\phi$ into a quasi-circle, and the range of $\phi$ contains four points in general position. From this it can be deduced that the intersection of a real line with the quasi-sphere is mapped by $\phi$ into an intersection of the real line with the quasi-sphere. Furthermore the mapping must be injective. So a straight line contained in the quasi-sphere is mapped into a straight line contained in the quasi-sphere. Our remark now follows. We can then conjugate $\phi \mid_{S\setminus(m_1\cup m_2)}$ by $\sigma$. We thereby obtain a mapping $\mathbb{R}^2 \to \mathbb{R}^2$ which maps straight lines into straight lines and circles into circles and whose range has three points in general position. Thus it is an orthogonal affine transformation and that establishes the result in this case.

We have now established Theorem 4.8 for the case $\mathbb{K} = \mathbb{R}, p + q = 4$. To establish it for the case $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}, p + q = 3$, we merely need to observe that it follows from our earlier remark that we can assume without loss of generality that $p \geq q$. We will now generalize to higher dimensions by induction.

Suppose that $n > 3$ and that we have established the theorem for all cases where $p+q < n$ and we wish to establish the theorem for the case where $p+q = n$. Consider a quasi-sphere for values of $p$ and $q$ satisfying the constraints of Theorem 4.8 and such that $p + q = n$. Given a mapping $\phi$ satisfying the hypotheses of the theorem, we can assume without loss of generality that $\phi$ preserves the intersections with $S$ of the hyperplanes $\{(x_1 : x_2 : \ldots : x_{p+q}) \mid x_i = 0\}$ for all $i$ such that $1 \leq i \leq p + q$. This point requires some discussion. As observed, the intersection of a line with the quasi-sphere is mapped into the intersection of a line with the quasi-sphere. We must show that if the intersection of a line with the quasi-sphere contains only two points, then it is mapped into a line $l$ whose intersection with the quasi-sphere contains only two points. If not, then $l$ is contained in some $k$-dimensional subspace $T$ with $k > 1$ such that the intersection of $T$ with $S$ is the union of two $(k-1)$-dimensional subspaces. We can then get an instance of a $k$-dimensional subspace being mapped into a $(k-1)$-dimensional subspace, which is impossible. Our remark follows. Then, on each of the intersections with $S$ of the hyperplanes, $\phi$ agrees with a mapping of the desired form $g_i^*$, or $g_i^* \circ \sigma_i^*$ if $\mathbb{K} = \mathbb{C}$, and in the latter case all the $\sigma_i$ are equal, because the intersection of any two of the hyperplanes has dimension greater than zero. Thus there is a unique mapping $\psi$ of the desired form such that $\phi$ agrees with $\psi$ on all the hyperplanes; it remains to show that it agrees with $\psi$ everywhere. To see this, note that any point in the quasi-sphere can be obtained as the intersection of two quasi-circles, complex quasi-circles, or quaternionic 3-quasi-spheres which meet two of the hyperplanes in question. This shows that $\phi$ agrees with $\psi$ on all of the quasi-sphere. This completes the proof of Theorem 4.8.

□

We have now proved Theorem 4.8. Next we will prove Theorem 4.12, the local version of Theorem 4.8. After that we will proceed to Theorems 4.11 and 4.13.

*Proof of Theorem 4.12.* First consider the case $\mathbb{K} = \mathbb{R}, q = 1$. We can assume that the north pole is contained in the domain of $\phi$ and is fixed, by composing on the right and on the left with an appropriate rotation of the sphere. Then, if we conjugate $\phi$ by stereographic projection from the north pole and shrink its domain, we obtain a mapping $\phi^* : U \to \mathbb{R}^{p-1}$, where $U$ is a connected open subset of $\mathbb{R}^{p-1}$, and $\phi^*$ and $U$ satisfy the hypotheses of Theorem 3.3. Furthermore $\phi^*$ maps connected open subsets of circles into circles, provided the circles come from an open set $T$ in the appropriate topology on the family of circles, such that for all $p \in U$ there exists a $c \in T$ such that $p \in c$. By Theorem 3.3 it then follows that $\phi^*$ agrees with some orthogonal affine transformation on $U$. Theorem 4.12 follows in the case $\mathbb{K} = \mathbb{R}, q = 1$. A similar argument can be applied to the case $\mathbb{K} = \mathbb{R}, p = 2, q = 2$. To generalize to higher dimensions, suppose the result has been proved for $\mathbb{K} = \mathbb{R}, p + q < n$; let us try to prove the case $\mathbb{K} = \mathbb{R}, p + q = n$. We can assume that the north pole $(1 : 0 : \ldots : 0 : 1)$ of the quasi-sphere is contained in the domain of $\phi$ and is fixed. It then follows that the mapping is of the desired form on two hyperplanes passing through the north pole, and from this it follows that the mapping is of the desired form in a neighbourhood of the north pole. Theorem 4.12 then follows in the case $\mathbb{K} = \mathbb{R}, p + q = n$ by a continuation argument. Theorem 4.12 then follows in the case $\mathbb{K} = \mathbb{R}$ by induction.

Let us now consider the case $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}, q = 1$. We can find two distinct points in the domain of $\phi$ such that the complex quasi-circle or quaternionic 3-quasi-sphere joining them is in $T$ and is contained in the domain of $\phi$. We can assume without loss of generality that they are the north pole and the south pole and that they are fixed. We will also need to assume that complex quasi-circles or quaternionic 3-quasi-spheres joining the south pole to nearby points in $\iota_1(\mathbb{C}^{p-1})$ are contained in $T$; we can assume this without loss of generality. We can then repeat the reasoning given in the proof of Theorem 4.8 in the case $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}, q = 1$, making use of Theorem 3.6. Specifically, from $\phi$ we obtain two maps $\phi_1^{**}$ and $\phi_2^{**}$, defined on two open subsets of $\mathbb{C}^{p-1}$, which are conjugates of each other by inversion, as in the proof of Theorem 4.8, and these maps satisfy the hypotheses of Theorem 3.6. Then, using Theorem 3.6, the reasoning given in the proof of Theorem 4.8 goes through and shows that $\phi$ is of the desired form. Using Theorem 3.5, we can obtain the statement in Theorem 4.12 about what happens if the word "connected" is removed. We can then generalize to the case $q > 1$ using the same reasoning which we used to generalize to higher dimensions in the case $\mathbb{K} = \mathbb{R}$. This completes the proof of Theorem 4.12.

□

We have now completed the proof of Theorem 4.12. Next we need to discuss Theorems 4.11 and 4.13.

*Proofs of Theorem 4.11 and 4.13.* Earlier we discussed the equivalence of Theorems 4.8 and 4.11 in the case $q = 1$, proving it by an argument similar to that used in Section 3.1 to prove the equivalence of Theorems 3.1 and 3.2. The case $q > 1$ requires somewhat more discussion. We define "good mappings" as in Definition 1.10, only with the good mappings from the quasi-sphere to the quasi-sphere being required to send quasi-circles into quasi-circles, complex quasi-circles into complex quasi-circles, or quaternionic 3-quasi-spheres into quaternionic 3-quasi-spheres, depending on what $\mathbb{K}$ is. Now a quasi-circle, complex quasi-circle, or quaternionic 3-quasi-sphere can be obtained as an intersection of two good submodules of sufficiently high dimension. Hence given a good mapping $\phi : F \to F$ we can define a mapping $\phi^* : S \to S$ as before, and the usual required results can be obtained by the same argument as in Section 3.1. This proves Theorem 4.11. The argument can be localized so as to obtain Theorem 4.13 in the same way as in Section 3.2. $\square$

We will generalize to a larger class of classical groups in the next chapter.

# Chapter 5

# The remaining classical groups

We have obtained local versions of the fundamental theorem of projective geometry dealing with various classical groups. In this chapter we shall generalize our results to a larger class of classical groups. We will just state the local versions; the global versions will be immediate corollaries. The global versions were known to Jacques Tits and follow from material in Chapter 5 of [23], to do with the buildings of algebraic groups.

## 5.1 Definitions

**Definition 5.1.** *The group* $G_1 = \mathrm{SO}(n, \mathbb{C})$ *is the group of all orthogonal n-by-n matrices with entries in* $\mathbb{C}$ *with determinant one.*

Suppose that $n$ is an integer greater than three. Define $S_1 = \{(z_1 : z_2 : \ldots : z_n) \in \mathbb{C}P^{n-1} \mid z_1^2 + z_2^2 + \ldots + z_n^2 = 0\}$. We may view $S_1$ as $G_1/Q_1$ where $Q_1$ is the stabilizer of the point $(i : 0 : \ldots : 0 : 1)$ with respect to the natural action of $G$ on $\mathbb{C}P^n$. We may define $F_1$ to be $G_1/P_1$ where $G_1$ is as before and $P_1$ is the stabilizer of a full flag $f$ in $\mathbb{C}^n$, where the space of dimension one contained in $f$ is spanned by $(i, 0, \ldots, 0, 1)$. Call a sphere an intersection of a complex two-dimensional subspace of $\mathbb{C}P^{n-1}$ with $S_1$ (here we obviously mean two complex dimensions) consisting of more than one point. There is a natural topology on the family of complex two-dimensional subspaces coming from the topology on $\mathrm{GL}(n, \mathbb{C})/Q$ where $Q$ is generated by an appropriate family of the $P_k s$ defined in Definition 1.8. This induces a topology on the family of spheres.

Suppose that $n$ is an integer greater than two. Given a quaternion $w = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}$, define $\widetilde{w} = a + b\mathbf{i} - c\mathbf{j} + d\mathbf{k}$. The map $w \mapsto \widetilde{w}$ is an antiautomorphism of $\mathbb{H}$. Let $S_2 = (w_1 : w_2 : \ldots : w_n) \in \mathbb{H}P^{n-1} \mid \widetilde{w_1}w_1 + \widetilde{w_2}w_2 + \ldots + \widetilde{w_n}w_n = 0\}$.

**Definition 5.2.** *Define $G_2 = \mathrm{O}(n;\mathbb{H})$ to be the group of matrices in $\mathrm{GL}(n,\mathbb{H})$ which preserve the form $\widetilde{w_1}w_1 + \widetilde{w_2}w_2 + \ldots + \widetilde{w_n}w_n$.*

Pick a full flag $f$ in $\mathbb{H}^n$ where the module in $f$ of rank one is null; that is, the form $\widetilde{w_1}w_1 + \widetilde{w_2}w_2 + \ldots + \widetilde{w_n}w_n$ vanishes on it. Denote by $P_2$ the stabilizer of this flag $f$ with respect to the natural action of $G_2$ on the full flags in $\mathbb{H}^n$. Denote by $Q_2$ the stabilizer of the module in $f$ of rank one with respect to the natural action of $G_2$ on $\mathbb{H}P^{n-1}$. Then we may let $F_2 = G_2/P_2$ and $S_2 = G_2/Q_2$.

**Definition 5.3.** *Call an intersection of a quaternionic line with $S_2$ consisting of more than one point, a special subset of $S_2$.*

*Remark.* As before, there is a natural topology on the family of quaternionic lines and this induces a topology on the family of special subsets of $S_2$.

Suppose that $n$ is an integer greater than one. Suppose that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Define $S_3$ to be $\{(x_1 : x_2 : \ldots : x_{2n}) \in \mathbb{K}P^{2n-1} \mid x_1x_{n+1} + x_2x_{n+2} + \ldots + x_nx_{2n} = 0\}$.

**Definition 5.4.** *Let $G_3 = \mathrm{Sp}(2n,\mathbb{K})$ be the group of matrices which preserve the form $x_1x_{n+1} + x_2x_{n+2} + \ldots + x_nx_{2n}$.*

**Definition 5.5.** *In the case $\mathbb{K} = \mathbb{R}$, we say that a submodule $M$ of $\mathbb{K}^{2n}$ is good under the following circumstances. Suppose that the rank of $M$ is $d$. If $d \leq n$ and $M$ is null, then $M$ is good. If $n < d < 2n$, and $M$ contains $d + 1 - n$ null submodules of rank $n$, any two of which intersect in the same submodule of rank $n - 1$, then $M$ is good. In the case $\mathbb{K} = \mathbb{C}$, we say that a submodule $M$ is good if it contains a null submodule of rank one. A flag is said to be good if it consists entirely of good submodules.*

**Definition 5.6.** *Call a special subset of $S_3$ an intersection of a two-dimensional subspace of $\mathbb{K}P^{2n-1}$ with $S_3$ consisting of more than one point.*

*Remark.* As in the other cases there is a topology on the family of special subsets of $S_3$ coming from the topology on the family of lines.

Let $f$ be a good full flag and denote the stabilizer of $f$ under the action of $G_3$ by $P_3$. (The existence of good full flags in the case $\mathbb{K} = \mathbb{C}$ is obvious, and the existence of good full flags in the case $\mathbb{K} = \mathbb{R}$ follows by the same argument as in Chapter 4, immediately after Definitions 4.5–4.6, together with the fact that the quadratic form mentioned above is equivalent under an appropriate change of basis to the quadratic form $x_1^2 + x_2^2 + \ldots + x_n^2 - x_{n+1}^2 - \ldots - x_{2n}^2$.) Let $Q$ be the stabilizer of the space in $f$ of dimension one with respect to the natural action of $G$ on $\mathbb{K}P^{2n-1}$. Then we may let $F_3 = G_3/P_3$ and $S_3 = G_3/Q_3$.

## 5.2 Generalizations of the results of Chapter 4 to other classical groups

We have obtained local versions of the fundamental theorem of projective geometry which deal with the groups $\mathrm{GL}(n, \mathbb{K})$, $n > 2$, $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and $\mathrm{O}(p, q)$, $p, q > 0$, $p + q > 3$ and $\mathrm{SU}(p, q)$, $\mathrm{Sp}(p, q)$, $p, q > 0$, $p + q > 2$. We shall now obtain results dealing with the groups $\mathrm{SO}(n, \mathbb{C})$, $n > 3$, $\mathrm{O}(n; \mathbb{H})$, $n > 3$, and $\mathrm{Sp}(2n, \mathbb{K})$, $n > 1$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Specifically, our goal is to obtain the following six theorems.

**Theorem 5.7.** *Suppose that $n > 3$. Suppose that $\phi$ is a mapping defined on some open connected subset $U$ of $S_1$, whose range has n points in general position, which maps connected subsets of spheres contained in $U$ – coming from some fixed set $V$ which is open in the aforementioned topology on the family of spheres, such that for all $p \in U$ there exists an $s \in V$ such that $p \in s$ – into spheres. Then $\phi$ is the restriction to $U$ of a mapping of the form $g^* \circ \sigma^*$ where $g^*$ is a mapping induced by a matrix $g \in G_1$ and $\sigma^*$ is induced by a unital homomorphism $\sigma : \mathbb{C} \to \mathbb{C}$. The result remains true if we drop the word "connected" everywhere and remove the requirement that the spheres belong to $V$.*

Let $n$, $G_1$, $P_1$, and $F_1$ be as in Section 5.1. Recall that $n$ points in $F_1$ are in general position if their projections to $S_1 = G_1/Q_1$ are in general position, and $P_1$ is the stabilizer of some fixed full flag $f$. For $k = 1, 2, \ldots n-1$, we define $P_{1k}$ to be the stabilizer of the flag $f$ with the space of dimension $n-k$ deleted. The type $k$ fibres are projections of left cosets of $P_{1k}$ in $G_1$ to $F_1 = G_1/P_1$.

**Theorem 5.8.** *Suppose that $\phi$ is a mapping defined on some open connected subset $U$ of $F_1$, whose range has $n$ points in general position, which maps connected subsets of type $k$ fibres contained in $U$ into type $k$ fibres for $k = 1, 2, \ldots n-1$. Then $\phi$ is the restriction to $U$ of a mapping of the form $g^* \circ \sigma^*$ where $g^*$ is the mapping induced by a matrix $g \in G_1$ and $\sigma^*$ is induced by a unital homomorphism $\sigma : \mathbb{C} \to \mathbb{C}$.*

**Theorem 5.9.** *Suppose that $n > 2$. Suppose that $\phi$ is a mapping defined on some open connected subset $U$ of $S_2$, whose range has $n$ points in general position, which maps connected subsets of special sets contained in $U$, coming from some fixed set $V$ which is open in the topology on the family of special sets defined in Section 5.1, such that for all $p \in U$ there exists a special set $s \in V$ such that $p \in s$, into special sets. Then $\phi$ is the restriction to $U$ of a mapping of the form $g^*$ where $g^*$ is a mapping induced by a matrix $g \in G_2$. The result remains true if we drop the word "connected" everywhere and remove the requirement that the special sets belong to $V$.*

Let $n, f, G_2, P_2$, and $F_2$ be as in Section 5.1. Recall that $n$ points in $F_2$ are in general position if their projections to $S_2 = G_2/Q_2$ are in general position. Again $P_{2k}$ is defined to be the stabilizer of the flag $f$ with the module of rank $n-k$ deleted. The type $k$ fibres are projections of left cosets of $P_{2k}$ in $G_2$ to $F = G_2/P_2$.

**Theorem 5.10.** *Suppose that $\phi$ is a mapping defined on some open connected subset $U$ of $F_2$, whose range has $n$ points in general position, which maps connected subsets of type $k$ fibres contained in $U$ into type $k$ fibres for $k = 1, 2, \ldots n-1$. Then $\phi$ is the restriction to $U$ of a mapping of the form $g^*$ where $g^*$ is the mapping induced by a matrix $g \in G_2$.*

**Theorem 5.11.** *Suppose that $n > 1$, and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Suppose that $\phi$ is a mapping defined on some open connected subset $U$ of $S_3$, whose range has $n$ points in general position, which maps connected subsets of special sets contained in $U$ – coming from some fixed set $V$ which is open in the topology on the family of special sets defined in Section 5.1, such that for all $p \in U$ there exists a special set $s \in V$ such that $p \in s$ – into special sets. Then $\phi$ is the restriction to $U$ of a mapping of the form $g^* \circ \sigma^*$ where $g^*$ is a mapping induced by a matrix $g \in G_3$ and $\sigma^*$ is induced by a unital homomorphism $\mathbb{K} \to \mathbb{K}$. The result remains true if we drop the word "connected" everywhere and remove the requirement that the special sets belong to $V$.*

Let $n, f, G_3, P_3$ and $F_3$ be as in Section 5.1. We say that $n$ points in $F_3$ are in general position if their projections to $S_3 = G_3/Q_3$ are in general position. Again $P_{3k}$ is defined to be the stabilizer of the flag $f$ with the subspace of rank $n-k$ deleted. The type $k$ fibres are projections of left cosets of $P_{3k}$ in $G_3$ to $F = G_3/P_3$.

**Theorem 5.12.** *Suppose that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Suppose that $\phi$ is a mapping defined on some open connected subset $U$ of $F_3$, whose range has $n$ points in general position, which maps connected subsets of type $k$ fibres contained in $U$ into type $k$ fibres for $k = 1, 2, \ldots n-1$. Then $\phi$ is the restriction to $U$ of a mapping of the form $g^* \circ \sigma^*$ where $g^*$ is the mapping induced by a matrix $g \in G_3$ and $\sigma^*$ is induced by a unital homomorphism $\mathbb{K} \to \mathbb{K}$.*

As in Chapters 3 and 4, Theorems 5.8, 5.10, and 5.12 follow from Theorems 5.7, 5.9, and 5.11 respectively, so it suffices to prove the latter three.

*Proofs of Theorems 5.7 and 5.11.* Theorem 5.7 is proved by a stereographic projection argument similar to the arguments used to prove Theorem 4.8 in the case $\mathbb{K} = \mathbb{R}, q = 1$, or $\mathbb{K} = \mathbb{R}, p = 2, q = 2$. Given a point $n'$, hereafter referred to as the north pole, there is a stereographic projection from $S \setminus \{n'\}$ onto $\mathbb{C}P^{n-2} \setminus \mathbb{C}P^{n-3}$, where $\mathbb{C}P^{n-3}$ is a certain $(n-3)$-dimensional subspace of $\mathbb{C}P^{n-2}$; namely, if the line joining $n'$ and $p$ is the projection to $\mathbb{C}P^{n-1}$ of the space spanned by the vector whose co-ordinates are the homogeneous co-ordinates of $n'$, and another vector $v$ whose last co-ordinate is zero, then $p$ is mapped to the element of $\mathbb{C}P^{n-2}$ whose homogeneous co-ordinates are the first $n-1$ co-ordinates of $v$. Now, if $\phi$ is as in the statement of Theorem 5.7, one can assume without loss of generality that it fixes the north pole, and then conjugate it by stereographic projection and reason as in the proof of Theorem 4.8 in the case $\mathbb{K} = \mathbb{R}, q = 1$. We need to use a version of Theorem 3.1 which deals with the case where the mapping maps two-dimensional subspaces into two-dimensional subspaces, rather than lines into lines; this is easy to deduce from Theorem 3.1. Theorem 5.11 follows from Theorem 4.12 and Theorem 5.7 because the groups $\mathrm{Sp}(2n, \mathbb{K})$ are conjugate to the groups $\mathrm{O}(n, n)$ or $\mathrm{SO}(2n, \mathbb{C})$, depending on what $\mathbb{K}$ is. That completes the proofs of Theorems 5.7 and 5.11. $\square$

It remains to discuss Theorem 5.9.

*Proof of Theorem 5.9.* We prove this theorem by an argument similar to that used to prove Theorem 4.8 in the case $\mathbb{K} = \mathbb{H}, q = 1$. We first consider the global case. We fix two distinct points $n', s$ in $S$, the north pole and the south pole. The quaternionic lines passing through $n'$ may be parametrized by $\mathbb{H}P^{n-2}$. Specifically, we fix a copy of $\mathbb{H}P^{n-2}$ inside $\mathbb{H}P^{n-1}$ and let the element $p$ of $\mathbb{H}P^{n-2}$ correspond to the line through $n'$ and $p$. Thus we get a (surjective) quaternionic stereographic projection $\tau_1 : S \to \mathbb{H}P^{n-2} \setminus \mathbb{H}P^{n-3}$ from the north pole, where $\mathbb{H}P^{n-3}$ consists of those elements of $\mathbb{H}P^{n-2}$ that are orthogonal to $n'$ under the form $\widetilde{w_1}w_1 + \widetilde{w_2}w_2 + \ldots + \widetilde{w_n}w_n$. If we conjugate $\phi$ by $\tau_1$ then we get a bijection $\phi_1^{**} : \mathbb{H}P^{n-2} \setminus \mathbb{H}P^{n-3} \to \mathbb{H}P^{n-2} \setminus \mathbb{H}P^{n-3}$. If $C$ is a special set, then we call $\tau_1(C)$ a special subset of $\mathbb{H}P^{n-2} \setminus \mathbb{H}P^{n-3}$. It is not hard to show that the special subsets of $\mathbb{H}P^{n-2} \setminus \mathbb{H}P^{n-3}$ are precisely the circles contained in quaternionic lines. This can be seen as follows: there certainly exists at least one circle contained in a quaternionic line which is a special set, and $\{\phi_1^{**} \mid \phi \in G_2\}$ acts transitively both on the special subsets of $\mathbb{H}P^{n-2} \setminus \mathbb{H}P^{n-3}$ and on the circles contained in quaternionic lines. (The latter result may be seen by reasoning similar to that used to prove Lemma 4.19.) Thus $\phi_1^{**}$ has the property that it maps circles contained in quaternionic lines into circles contained in quaternionic lines. This allows us to conclude that it is a projective transformation. We may also construct a quaternionic stereographic projection from the south pole, using the same copy of $\mathbb{H}P^{n-2}$ inside $\mathbb{H}P^{n-1}$, which we denote by $\tau_2$. If we conjugate $\phi$ by $\tau_2$ then we obtain another bijection $\phi_2^{**} : \mathbb{H}P^{n-2} \setminus (\mathbb{H}P^{n-3})^* \to \mathbb{H}P^{n-2} \setminus (\mathbb{H}P^{n-3})^*$ which maps circles contained in quaternionic lines into circles contained in quaternionic lines, where here $(\mathbb{H}P^{n-3})^*$ is the set of elements of $\mathbb{H}P^{n-2}$ which are orthogonal to $s$ under the aforementioned form. Now as in the proof of Lemma 4.20 we consider the set of points $q \in S$ such that $\tau_1(q) = \tau_2(q)$ and prove that it is fixed setwise. This enables us to prove that $\phi_1^{**}$ agrees with $\phi_2^{**}$ on the intersection of their domains. Since $(\phi_1^{**}, \phi_2^{**})$ determines $\phi$, and our reasoning may be localized as in the proof of Theorem 4.12, the result follows. That completes the proof of Theorem 5.9. $\square$

The proofs of the six theorems are now complete.

This completes our discussion of how to generalize the local versions of the fundamental theorem of projective geometry to classical groups other than $\mathrm{GL}(n, \mathbb{K})$. In the next part we shall discuss how to generalize the results of Part I and Part II to rings other than $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

# Part III

# Other Rings

# Chapter 6

# Division rings

In Part I we presented some results that were already known regarding projective spaces and spaces of flags, the spaces of flags being quotient spaces of general linear groups. We also explored some localizations of those results in the case where the underlying ring is $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. In Part II we explored other results about certain subsets of projective spaces and generalized flag manifolds, the generalized flag manifolds being quotient spaces of other classical groups. The aim of Part III is to generalize the results of Parts I and II to a larger class of rings. In this chapter we explore generalizations of the results of Chapters 3, 4, and 5 to various topological division rings.

We begin by considering how the arguments for Theorems 3.3–3.7 may be generalized to topological division rings $\mathbb{K}$ other than $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. To do this it will be necessary to review the arguments of Chapter 2. We begin by presenting some definitions and then stating the generalizations of Theorems 2.1–2.2 and Lemmas 2.3–2.4.

**Definition 6.1.** *A base $B$ for the topology on a topological division ring $\mathbb{K}$ is a good base if $B$ is closed under taking the image under an affine transformation.*

Every topological division ring admits a good base, since $B$ may be taken to be the entire topology. If $\mathbb{F}$ is a local field with a valuation $v$, the family of open balls with respect to the metric arising from $v$ is a good base.

**Definition 6.2.** *Given a topological division ring $\mathbb{K}$ with a good base $B$ for the topology, a subset of $\mathbb{K}^n$ is a basic open set with respect to $B$ if it is a Cartesian product of members of $B$, and a subset of $\mathbb{K}P^{n-1}$ is a basic open set with respect to $B$ if it is a projection of a basic open subset of $\mathbb{K}^n$ (minus the zero vector if necessary).*

**Definition 6.3.** *An open subset $U$ of $\mathbb{K}^n$ or $\mathbb{K}P^{n-1}$ is quasi-connected with respect to $B$ if, given any two points $p_1, p_2 \in U$, there exists a finite sequence $(U_1, U_2, \ldots U_k)$ of basic open sets with respect to $B$, all contained in $U$, such that $p_1 \in U_1, p_2 \in U_k$, and $U_i$ and $U_{i+1}$ have nonempty intersection for integers $i$ such that $1 \leq i < k$.*

We will now state the generalizations of Theorems 2.1–2.2, and Lemmas 2.3–2.4, to arbitrary non-discrete topological division rings. We will not prove the generalizations of Theorems 2.1–2.2 straight away, but they will follow from later generalizations of Theorems 3.5–3.6. However it will be necessary to prove the generalizations of Lemmas 2.3–2.4 in order to obtain these later theorems.

**Theorem 6.4.** *Suppose that $\mathbb{K}$ is a non-discrete topological division ring. Suppose that $\phi$ is a mapping defined on an open subset $U$ of $\mathbb{K}P^2$, such that the range of $\phi$ has three points in general position, and such that $\phi$ maps collinear points to collinear points. Then $\phi$ is the restriction to $U$ of a projective transformation, possibly composed with a mapping induced by a self-homomorphism of $\mathbb{K}$.*

This is a generalization of Theorem 2.1. To see that the result is not always true in the discrete case, suppose that $\mathbb{K} = \mathbb{Q}$ and that $U$ contains five points, four of which are in general position and the fifth of which we shall denote by $p$. The action of $\phi$ on the four points in general position extends to a unique projective transformation, and $\mathbb{Q}$ has no nontrivial self-homomorphisms. Hence it is possible to choose a value of $\phi(p)$ which gives a counterexample.

In the statement of the next lemma, and Lemma 6.7, we fix a good base $B$ for the topology on $\mathbb{K}$ and use "quasi-connected" to mean "quasi-connected with respect to the base $B$".

**Theorem 6.5.** *Suppose that $\mathbb{K}$ is a non-discrete topological division ring. Suppose that $\phi$ is a mapping defined on an open quasi-connected subset $U$ of $\mathbb{K}P^2$, such that the range of $\phi$ has three points in general position, and such that $\phi$ maps quasi-connected open subsets of straight lines $l \in V$ into straight lines, where $V$ is a fixed open set in the topology described in Section 1.4, and such that given any $p \in U$, there exists an $l \in V$ such that $p \in l$. Then $\phi$ is the restriction to $U$ of a projective transformation, possibly composed with a mapping induced by a self-homomorphism of $\mathbb{K}$.*

This is a generalization of Theorem 2.2.

If the good base $B$ for the topology in Theorem 6.5 is the entire topology, then every open set is quasi-connected with respect to $B$ and Theorem 6.5 reduces to a slightly strengthened version of Lemma 6.4. However we get more information by considering other cases, such as the case, previously mentioned, where $\mathbb{K}$ is a local field with a valuation $v$ and $B$ is the set of open balls with respect to the metric induced by $v$. If $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and $B$ is taken to be the set of open balls with respect to the Euclidean metric on $\mathbb{K}$, then "quasi-connected with respect to the base $B$" is equivalent to "connected".

**Lemma 6.6.** *Suppose there is an open set $U$ for which Theorem 6.4 holds for any mapping $\phi$ satisfying the hypotheses of Theorem 6.4. Suppose that $U'$ is another open set such that $U \subset U'$. Then Theorem 6.4 also holds for $U'$.*

This is a generalization of Lemma 2.3.

**Lemma 6.7.** *Suppose that there is a quasi-connected open set $U$ and a set of straight lines $V$ such that $V$ satisfies the hypotheses of Theorem 6.5 with respect to $U$ and such that Theorem 6.5 holds for any mapping $\phi$ satisfying the hypotheses of Theorem 6.5 with respect to $U$ and $V$. Suppose that $U'$ is another quasi-connected open set such that $U \subset U'$ and $V$ also satisfies the hypotheses of Theorem 6.5 with respect to $U'$. Then Theorem 6.5 also holds for $U'$ and $V$.*

This is a generalization of Lemma 2.4.

Let us now proceed to generalize the arguments of Chapters 2 and 3.

*Proof of Lemma 6.6.* When we review the argument for Lemma 2.3, we see that use is made of the fact that an open subset of $\mathbb{K}$ contains more than one point; this is clearly a necessary assumption in order for the generalization of Theorem 2.3 to hold, and is in fact the only assumption on the ring $\mathbb{K}$ that is necessary for the argument. This condition is equivalent to non-discreteness. Thus the argument for Lemma 2.3 holds for any non-discrete topological division ring. Thus Lemma 6.6 is proved. $\square$

*Proof of Lemma 6.7.* If we replace "connected" with "quasi-connected" (with respect to some fixed good base $B$), then the argument for Lemma 2.4 goes through for any non-discrete topological division ring $\mathbb{K}$. Thus Lemma 6.7 is proved. $\square$

We will now proceed to obtain generalizations of Theorems 3.5–3.7; these generalizations, stated below as Theorems 6.9–6.11 respectively, will entail Theorems 6.4 and 6.5 (which were stated in order to facilitate the statements of Lemmas 6.6 and 6.7) without further argument. Recall that Theorems 3.3 and 3.4 are the two-dimensional cases of Theorems 3.5 and 3.6 respectively; we will proceed to the generalizations to higher dimensions later. Let us first investigate the extent to which the arguments for Theorems 3.3 and 3.4 can be generalized.

We spoke of the centroid of the triangle in the proof of Theorem 2.5, which is generalized in the proofs of Theorems 3.3 and 3.4; this is not appropriate if $\mathbb{K}$ has characteristic three, so in order to deal with the characteristic three case a slight modification to the argument is needed. Instead of the centroid we use $(1, -1)$. It is sufficient that $U$ contains $(0, 0), (0, 1), (1, 0)$, and $(1, -1)$, and that $\phi$ fixes all of these. The other parts of the arguments for Theorems 3.3 and 3.4 generalize without difficulty, except for the final parts where we need to prove that the locally additive and multiplicative map $\sigma$ extends to a unique unital homomorphism. In order to generalize the final parts of the arguments for Theorems 3.3 and 3.4 to arbitrary non-discrete topological division rings, we need to prove the following lemma, which is the key to generalizing the results of Chapters 2 and 3 to arbitrary non-discrete topological division rings, because it is the only place where generalizing the arguments involves any special difficulty.

**Lemma 6.8.** *Suppose that $\mathbb{K}$ is any non-discrete topological division ring. There exists an open set $U' \subset \mathbb{K}P^1$ containing 0 and $\infty$ with the following properties: (1) Given any nonempty open set $U$, there exists a projective transformation $\psi$ such that $U' \subset \psi(U)$. (2) Given a mapping $\sigma$ defined on $\mathbb{K} \cap U'$ such that $\sigma(0) = 0$ and $\sigma(1) = 1$ and (i) if $x, y, x+y \in \mathbb{K} \cap U'$ then $\sigma(x+y) = \sigma(x)+\sigma(y)$, (ii) if $x, y, xy \in \mathbb{K} \cap U'$ then $\sigma(xy) = \sigma(x)\sigma(y)$, $\sigma$ uniquely extends to a unital homomorphism $\mathbb{K} \to \mathbb{K}$. Also, in (ii) we can replace $\sigma(xy) = \sigma(x)\sigma(y)$ by $\sigma(xy) = \sigma(x)\tau(y)$ for some fixed function $\tau$ defined on $\mathbb{K} \cap U'$.*

*Proof of Lemma 6.8.* We let $U'$ be the union of two open neighbourhoods $V$ and $W$ such that $0 \in V$ and $\infty \in W$, and such that any $k \in \mathbb{K}$ is equal to $vw$ for some $v \in V$ and $w \in W$. To see that this is possible, pick an open neighbourhood $V$ of 0 and let $W$ be an open neighbourhood of $\infty$ containing the image of $V$ under multiplicative inversion. Given any $k \in \mathbb{K}$, let $v$ be a nonzero element of $V$ sufficiently close to zero that $v^{-1}k$ is contained in $W$. Then, letting $w = v^{-1}k$, we have $k = vw$ where $v \in V, w \in W$. This completes the proof that there exist two open neighbourhoods $V$ and $W$ satisfying the required conditions. Let $\sigma$ be a mapping satisfying the hypotheses of the lemma. Given any $w \in W \cap \mathbb{K}$, we may define $\sigma_w : Vw \to \mathbb{K}$ by $\sigma_w(vw) = \sigma(v)\sigma(w)$. If $w_1, w_2 \in W \cap \mathbb{K}$, then $\sigma_{w_1}$ and $\sigma_{w_2}$ agree on the intersection of their domains. We may glue all these maps together to obtain the unique unital homomorphism to which $\sigma$ extends. If we have the modified version of condition (ii), we can now observe that $\tau$ agrees with the unique extension of $\sigma$ on $\mathbb{K} \cap U'$. This completes the proof. ☐

*Remark.* The observation about how the condition (ii) may be changed will be useful in proving Lemma 8.4.

We now state the new versions of the results of Theorems 3.5–3.7, which appear below as Theorems 6.9–6.11 respectively. We will then give the proof of the generalizations of Theorems 3.3 and 3.4, (that is, the two-dimensional cases of Theorems 6.9 and 6.10). Then Theorems 6.9–6.11 will easily follow.

**Theorem 6.9.** *Suppose that $n$ is an integer such that $n > 1$. Suppose that $\mathbb{K}$ is a non-discrete topological division ring. Suppose that $\phi$ is a mapping defined on an open subset $U$ of $\mathbb{K}P^n$, such that the range of $\phi$ has $n+1$ points in general position, and such that $\phi$ maps collinear points to collinear points. Then $\phi$ is the restriction to $U$ of a mapping of the form $\psi \circ \sigma^*$, where $\psi$ is a projective transformation and $\sigma$ is a unital homomorphism $\mathbb{K} \to \mathbb{K}$.*

This is a generalization of Theorem 3.5.

**Theorem 6.10.** *Suppose that $n$ is an integer such that $n > 1$. Suppose that $\mathbb{K}$ is a non-discrete topological division ring and that $B$ is a good base for the topology on $\mathbb{K}$. Throughout, by "quasi-connected" we shall mean "quasi-connected with respect to $B$". Suppose that $\phi$ is a mapping defined on an open quasi-connected subset $U$ of $\mathbb{K}P^n$, such that the range of $\phi$ has $n+1$ points in general position, and such that $\phi$ maps quasi-connected subsets of straight lines $l \in V$ into straight lines, where $V$ is a fixed open set in the topology described in Section 1.4, with the property that for every $p \in U$, there exists a line $l \in V$ such that $p \in l$. Then $\phi$ is the restriction to $U$ of a mapping of the form $\psi \circ \sigma^*$, where $\psi$ is a projective transformation and $\sigma$ is a unital homomorphism $\mathbb{K} \to \mathbb{K}$.*

This is a generalization of Theorem 3.6.

**Theorem 6.11.** *Suppose that $\mathbb{K}$ is a non-discrete topological division ring and that $B$ is a good base for the topology on $\mathbb{K}$. Throughout, by "quasi-connected" we shall mean "quasi-connected with respect to $B$". Suppose that $n > 2$. Suppose that $\mathbb{K}$, $G$, $P$, $F$ are as in the statement of Theorem 3.2. Suppose that $U$ is a nonempty open quasi-connected subset of $F$ and that $\phi$ is a mapping $U \to F$, which maps connected subsets of type $k$ fibres contained* *in $U$ into type $k$ fibres, and whose range contains $n$ points in general position. Then $\phi$ is a restriction of a mapping of the form $g^* \circ \sigma^*$, where $\sigma$ is a unital ring homomorphism $\mathbb{K} \to \mathbb{K}$ and $g \in G$.*

This is a generalization of Theorem 3.7.

We will now proceed to generalize the arguments for Theorems 3.3 and 3.4 (the two-dimensional cases of Theorems 3.5 and 3.6, which are generalized above as Theorems 6.9 and 6.10) to arbitrary non-discrete topological division rings. Recall that in the argument for Theorems 3.3 and 3.4, $U$ was the domain of $\phi$, and, in the argument for Theorem 3.4, $W$ was the set of slopes for which $\phi$ preserved line segments. Letting $U$ and $W$ be as in the argument for Theorem 3.4, we may replace them by their images under any projective transformation. So letting $U'$ be as in Lemma 6.8, we can arrange it so that $U$ contains $(x, y)$ whenever $x, y \in U' \cap \mathbb{K}$, and also $W$ contains $U'$. Then the argument for Theorem 3.4 goes through for an arbitrary non-discrete topological division ring by Lemma 6.8. The argument for Theorem 3.3 goes through for an arbitrary non-discrete topological division ring, by an easy modification of the foregoing argument, in which one simply drops reference to $W$.

By reviewing the arguments given in Chapter 3 for Theorems 3.5–3.7, it may now be seen that Theorems 6.9–6.11 are true. In particular Theorems 6.9–6.11 apply to $p$-adic number fields and finite extensions thereof. As observed before, Theorems 6.4 and 6.5 now follow from Theorems 6.9 and 6.10.

We will now state generalizations of the results in Chapters 4 and 5. For the generalizations of the results of Chapter 4, since we are not necessarily dealing with formally real fields (fields in which any finite nonempty sum of nonzero squares is nonzero), in order to prove these results it is necessary to make use of arguments involving stereographic projections whose codomain is a projective space, as in Chapter 5.

**Definition 6.12.** *We say that $\mathbb{K}$ is a field of type 1 if $\mathbb{K}$ is a non-discrete topological field. In that case, we define $Q_{p,q}(\boldsymbol{x}) = x_1^2 + x_2^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2$ for $\boldsymbol{x} = (x_1, x_2, \ldots, x_{p+q}) \in \mathbb{K}^{p+q}$.*

**Definition 6.13.** *We say that $\mathbb{K}$ is a field of type 2, or a field of type 2 with respect to $\mathbb{L}$, if $\mathbb{K}$ is a non-discrete topological field which is a quadratic extension of a subfield $\mathbb{L}$. In that case, we define $|x|^2 = x \cdot \sigma(x)$ for $x \in \mathbb{K}$ where $\sigma$ is the unique nonidentity element of $\mathrm{Gal}(\mathbb{K} \mid \mathbb{L})$ and we define $Q_{p,q}(\boldsymbol{x}) = |x_1|^2 + |x_2|^2 + \ldots + |x_p|^2 - |x_{p+1}|^2 - \ldots - |x_{p+q}|^2$ for $\boldsymbol{x} = (x_1, x_2, \ldots, x_{p+q}) \in \mathbb{K}^{p+q}$.* [1]

**Definition 6.14.** *We say that $\mathbb{K}$ is a ring of type 3, or a ring of type 3 with respect to $\mathbb{L}$, if $\mathbb{K}$ is a non-discrete topological division ring which has a subfield $\mathbb{L}$ in which $x^2 + y^2 + z^2 + w^2$ cannot be zero unless $x = y = z = w = 0$, and $\mathbb{K} = \mathbb{L}(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ where $\boldsymbol{i}^2 = -1, \boldsymbol{j}^2 = -1, \boldsymbol{k}^2 = -1, \boldsymbol{ij} = \boldsymbol{k}, \boldsymbol{jk} = \boldsymbol{i}, \boldsymbol{ki} = \boldsymbol{j}, \boldsymbol{ji} = -\boldsymbol{k}, \boldsymbol{kj} = -\boldsymbol{i}, \boldsymbol{ik} = -\boldsymbol{j}$. In that case, we define $|x + y\boldsymbol{i} + z\boldsymbol{j} + w\boldsymbol{k}|^2 = x^2 + y^2 + z^2 + w^2$ for $x, y, z, w \in \mathbb{L}$ and we define $Q_{p,q}(\boldsymbol{x}) = |x_1|^2 + |x_2|^2 + \ldots + |x_p|^2 - |x_{p+1}|^2 - \ldots - |x_{p+q}|^2$ for $\boldsymbol{x} = (x_1, x_2, \ldots, x_{p+q}) \in \mathbb{K}^{p+q}$.*

*Remark.* We need to insert the condition about the sum of four nonzero squares not being zero in order to ensure that the resulting structure is indeed a division ring.

We generalize the definitions of quasi-circles, complex quasi-circles, or quaternionic 3-quasi-spheres, given in Definition 4.7, to quasi-circles, quasi-circles of type 2, or 3-quasi-spheres of type 3 in the obvious way. We define the natural topology on the family of quasi-circles, quasi-circles of type 2, or 3-quasi-spheres of type 3 in the obvious way, in analogy to the definition of the topology on the family of quasi-circles, complex quasi-circles, or quaternionic 3-quasi-spheres in Section 4.1. We will now state the generalizations of Theorems 4.12 and 4.13. Here we are just stating the local versions of the results. The global versions of the results are immediate corollaries and were known to Jacques Tits [23].

[1] In an early version of this thesis we only allowed quadratic extensions by a square root of $-1$; I am grateful to an earlier reader of the thesis for pointing out that arbitrary quadratic extensions can be allowed.

**Theorem 6.15.** *Suppose that $\mathbb{K}$ is a field of type 1, a field of type 2, or a ring of type 3, respectively and that $B$ is a good base for its topology. Suppose that $p+q > 2$, $p+q > 3$ if $\mathbb{K}$ is of type 1. Suppose that $G$ is the group of matrices of $\mathbb{K}$-module automorphisms of $\mathbb{K}^{p+q}$ which preserve the form $Q_{p,q}$. Throughout, by "quasi-connected" we shall mean "quasi-connected with respect to $B$". Suppose that $T$ is a subset of the family of quasi-circles, quasi-circles of type 2, or 3-quasi-spheres of type 3, respectively, which is open in the natural topology on this family, and that $U$ is a nonempty open quasi-connected subset of $S$, and that, for every $x \in U$, there exists a $c \in T$ such that $x \in c$. Suppose that $\phi$ is a mapping $U \to S$, whose range contains $p+q$ points in general position, which maps quasi-connected subsets, contained in $U$, of quasi-circles, quasi-circles of type 2 or 3-quasi-spheres of type 3, belonging to $T$, into quasi-circles, quasi-circles of type 2, or 3-quasi-spheres of type 3, respectively. Then $\phi$ is the restriction to $U$ of a mapping of the form $g^*$, or $g^* \circ \sigma^*$ if $\mathbb{K}$ is of type 2, where $g \in G$, and if $\mathbb{K}$ is of type 2 with respect to $\mathbb{L}$ then $\sigma \in \mathrm{Gal}(\mathbb{K} \mid \mathbb{L})$. The result remains true if we drop all occurrences of the words "quasi-connected" and remove the requirement that the quasi-circles, quasi-circles of type 2, or 3-quasi-spheres of type 3 belong to $T$.*

This is a generalization of Theorem 4.12.

For the statement of Theorem 6.16, let $\mathbb{K}, p, q, B$ be as in the statement of Theorem 6.15, use "quasi-connected" in the same way as in the statement of that theorem, and define $f, s, S, P, Q, G$, and $F$ as in the statement of Theorem 4.11, generalizing in the obvious way. We will make use of the obvious generalizations of Definitions 4.9 and 4.10, of the notions of "general position" and "type $k$ fibres".

**Theorem 6.16.** *Suppose that $\phi$ is a mapping from a nonempty open quasi-connected subset $U$ of $F$ into $F$, which maps quasi-connected subsets of type $k$ fibres contained in $U$ into type $k$ fibres, for $k = 1, 2, \ldots p+q-1$, and whose range contains $p+q$ points in general position. Then $\phi$ is of the form $g^*$, or $g^* \circ \sigma^*$ if $\mathbb{K}$ is of type 2, where $g \in G$, and if $\mathbb{K}$ is of type 2 with respect to $\mathbb{L}$ then $\sigma \in \mathrm{Gal}(\mathbb{K} \mid \mathbb{L})$.*

This is a generalization of Theorem 4.13.

*Proofs of Theorems 6.15 and 6.16.* We shall indicate the points where it is not obvious how to generalize the arguments of Chapter 4 to prove these theorems. Recall that Theorems 5.7 and 5.8 dealt with the case of $\mathrm{SO}(n, \mathbb{C})$, which is like the case of Theorems 4.12 and 4.13 dealing with the classical group $\mathrm{O}(p, q)$ except that the underlying field has a square root of $-1$. The arguments for Theorems 5.7 and 5.8, which dealt with $\mathrm{SO}(n, \mathbb{C})$, may be easily generalized to yield the cases of Theorems 6.15 and 6.16 where $\mathbb{K}$ is a field of type 1. For the other cases we may use the arguments for Theorems 4.12 and 4.13, of which the above theorems are generalizations. However if $\mathbb{K}$ is a field of type 2 with respect to a field $\mathbb{L}$ which is not formally real, or if $\mathbb{K}$ is a ring of type 3 with respect to a subfield $\mathbb{L}$ which is not formally real, then we must use a stereographic projection of the kind used in the arguments for Theorems 5.7 and 5.8, where the codomain is a projective space. In all of the generalized arguments we have just discussed, use is made of the generalizations of Theorems 6.9–6.10; thus these new results may be seen as corollaries of these previous results. $\square$

This completes the discussion of the generalizations of the local results of Chapter 4. (As we observed, the generalizations of the global results were known to Jacques Tits.) We shall now proceed to generalize the results of Chapter 5 in similar ways.

The results of Chapter 5 were about various classical groups defined over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. In generalizing these results, we replace $\mathbb{R}$ by "field of type 1", we replace $\mathbb{C}$ by "field of type 2", and we replace $\mathbb{H}$ by "ring of type 3". Theorems 5.7 and 5.8 have already been generalized in Theorems 6.15 and 6.16. Recall that Theorems 5.9 and 5.10 concerned the classical group $\mathrm{O}(n; \mathbb{H})$. In the generalizations of these theorems in Theorems 6.17 and 6.18, instead of the classical group $\mathrm{O}(n; \mathbb{H})$ we shall speak of $\mathrm{O}(n; \mathbb{K})$ where $\mathbb{K}$ is a ring of type 3. It is obvious how to generalize the definition of $\mathrm{O}(n; \mathbb{H})$, given in Definition 5.2, to $\mathrm{O}(n; \mathbb{K})$ where $\mathbb{K}$ is a ring of type 3. We used $\widetilde{w}$ to denote the image of $w$ under a certain antiautomorphism of $\mathbb{H}$; we shall use the same notation to denote the corresponding antiautomorphism of $\mathbb{K}$. We have already defined $\mathrm{Sp}(2n, \mathbb{K})$ for an arbitrary field $\mathbb{K}$, in Definition 5.4. The appropriate generalizations of the notations and terminology of Chapter 5, and the definitions of the topologies on the families of special sets, are all obvious. We shall refer to the topologies on the families of special sets so defined as the natural topologies on these families. We shall now state the generalizations of Theorems 5.9–5.12.

**Theorem 6.17.** *Suppose that $\mathbb{K}$ is a ring of type 3. Suppose that $n > 2$. Suppose that $\phi$ is a mapping defined on some open quasi-connected subset $U$ of $S_2$, whose range has $n$ points in general position, which maps quasi-connected subsets of special sets contained in $U$ – coming from some fixed set $V$ which is open in the natural topology on the family of special sets, such that for all $p \in U$ there exists a special set $s \in V$ such that $p \in s$ – into special sets. Then $\phi$ is the restriction to $U$ of a mapping of the form $g^*$ where $g^*$ is a mapping induced by a matrix $g \in G_2$. The result remains true if we drop the word "quasi-connected" everywhere in the statement of the theorem and remove the requirement that the special sets belong to $V$.*

This is a generalization of Theorem 5.9.

**Theorem 6.18.** *Suppose that $\mathbb{K}$ is a ring of type 3. Suppose that $\phi$ is a mapping defined on some open quasi-connected subset $U$ of $F_2$, whose range has $n$ points in general position, which maps quasi-connected subsets of type $k$ fibres contained in $U$ into type $k$ fibres for $k = 1, 2, \dots n-1$. Then $\phi$ is the restriction to $U$ of a mapping of the form $g^*$ where $g^*$ is the mapping induced by a matrix $g \in G_2$.*

This is a generalization of Theorem 5.10.

**Theorem 6.19.** *Suppose that $\mathbb{K}$ is a field of type 1. Suppose that $n > 1$. Suppose that $\phi$ is a mapping defined on some open quasi-connected subset $U$ of $S_3$, whose range has $n$ points in general position, which maps quasi-connected subsets of special sets contained in $U$ – coming from some fixed set $V$ which is open in the natural topology on the family of special sets, such* *that for all $p \in U$ there exists a special set $s \in V$ such that $p \in s$ – into special sets. Then $\phi$ is the restriction to $U$ of a mapping of the form $g^* \circ \sigma^*$ where $g^*$ is a mapping induced by a matrix $g \in G_3$ and $\sigma^*$ is induced by a unital homomorphism $\mathbb{K} \to \mathbb{K}$. The result remains true if we drop the word "quasi-connected" everywhere and remove the requirement that the special sets belong to $V$.*

This is a generalization of Theorem 5.11.

**Theorem 6.20.** *Suppose that $\mathbb{K}$ is a field of type 1. Suppose that $\phi$ is a mapping defined on some open quasi-connected subset $U$ of $F_3$, whose range has $n$ points in general position, which maps quasi-connected subsets of type $k$ fibres contained in $U$ into type $k$ fibres for $k = 1, 2, \dots n-1$. Then $\phi$ is the restriction to $U$ of a mapping of the form $g^* \circ \sigma^*$ where $g^*$ is the mapping induced by a matrix $g \in G_3$ and $\sigma^*$ is induced by a unital homomorphism $\mathbb{K} \to \mathbb{K}$.*

This is a generalization of Theorem 5.12.

Reviewing the arguments for Theorems 5.9–5.12, keeping in mind Theorems 6.9–6.10, we may see that Theorems 6.17–6.20 are true. This completes our discussion of how to generalize the results of Parts I and II to various division rings. In the next chapter we discuss rings which are not division rings.

# Chapter 7

## Rings which are not division rings

In this chapter we discuss how to generalize the results of Part I to some rings which are not division rings.

**Definition 7.1.** *The ring of adeles $A$ over $\mathbb{Q}$ is the restricted product $\mathbb{R} \times \prod' \mathbb{Q}_p$, where $\mathbb{Q}_p$ is the field of $p$-adic numbers and by "restricted product" we mean that we take all the elements of the full product in which all but finitely many of the entries are $p$-adic integers.*

In what follows, let $R$ be a topological ring in which the set of units is dense and everything which is not a left divisor of zero has a multiplicative inverse. For example, if we let $A$ be the ring of adeles over $\mathbb{Q}$ then we may take $R$ to be the localization $AS^{-1}$ where $S$ is the set of elements of $A$ which are not divisors of zero.

**Definition 7.2.** *An affine transformation of $R$ is one of the form $y = mx + c$ where $m$ is invertible.*

**Definition 7.3.** *A good base $B$ for the topology on $R$ is one which is closed under taking the image under an affine transformation.*

We need to clarify how to give appropriate generalizations of the notions of a projective space and a line in a projective space for the rings under discussion in this chapter. We define $RP^n$ to be the set of maximal free submodules of rank 1 of the free module $R^{n+1}$. A line in $RP^n$ is the projection to $RP^n$ of a maximal free submodule of rank 2 of $R^{n+1}$. It will not in general be true that every two distinct points determine a line or that every two distinct lines determine a point. However, under certain circumstances we will be able to say that two lines intersect in a point. We say that $n+1$ points in $RP^n$ are in general position if the $(n+1)$-by-$(n+1)$ matrix whose columns are the homogeneous co-ordinate vectors of the $n+1$ points is invertible.

Our goal is to prove the following theorems, which are generalizations of Theorems 3.5–3.7 to the rings under discussion in this chapter.

**Theorem 7.4.** *Suppose that $n$ is an integer such that $n > 1$. Suppose that $R$ is a non-discrete topological ring in which the set of units is dense and in which everything that is not a left divisor of zero has a multiplicative inverse. Suppose that $\phi$ is a mapping defined on an open subset $U$ of $RP^n$, such that the range of $\phi$ has $n+1$ points in general position, and such that $\phi$ maps collinear points to collinear points. Then $\phi$ is the restriction to $U$ of a mapping of the form $\psi \circ \sigma^*$, where $\psi$ is a projective transformation and $\sigma$ is a unital homomorphism $R \to R$.*

This is a generalization of Theorem 3.5.

**Theorem 7.5.** *Suppose that $n$ is an integer such that $n > 1$. Suppose that $R$ is a non-discrete topological ring in which the set of units is dense and in which everything that is not a left divisor of zero has a multiplicative inverse. Suppose that $B$ is a good base for the topology on $R$. Throughout, by "quasi-connected" we shall mean "quasi-connected with respect to $B$". Suppose that $\phi$ is a mapping defined on an open quasi-connected subset $U$ of $RP^n$, such that the range of $\phi$ has $n+1$ points in general position, and such that $\phi$ maps quasi-connected subsets of straight lines $l \in V$ into straight lines, where $V$ is a fixed open set in the topology described in Section 1.4, appropriately generalized, with the property that for every $p \in U$, there exists a line $l \in V$ such that $p \in l$. Then $\phi$ is the restriction to $U$ of a mapping of the form $\psi \circ \sigma^*$, where $\psi$ is a projective transformation and $\sigma$ is a unital homomorphism $R \to R$.*

This is a generalization of Theorem 3.6.

**Theorem 7.6.** *Suppose that $R$ is a non-discrete topological ring in which the set of units is dense and in which everything that is not a left divisor of zero has a multiplicative inverse. Suppose that $B$ is a good base for the topology on $R$. Throughout, by "quasi-connected" we shall mean "quasi-connected with respect to $B$". Suppose that $n > 2$. Suppose that $G = \mathrm{GL}(n, R)$, and that $P$ and $F$ are as in the statement of Theorem 3.2, appropriately generalized. Define type $k$ fibres as in Definition 1.8, appropriately generalized. Suppose that $U$ is a nonempty open quasi-connected subset of $F$ and that $\phi$ is a mapping $U \to F$, which maps quasi-connected subsets of type $k$ fibres contained in $U$ into type $k$ fibres, for $k = 1, 2, \ldots n-1$, and whose range contains $n$ points in general position. Then $\phi$ is a restriction of a mapping of the form $g^* \circ \sigma^*$, where $\sigma$ is a unital ring homomorphism $R \to R$ and $g \in G$.*

This is a generalization of Theorem 3.7.

We now show how to generalize the arguments of Chapter 3 to the rings under discussion in this chapter. Take Theorems 3.3 and 3.4, for example. Suppose we have a mapping from a nonempty open quasi-connected subset $U$ (with respect to some fixed good base $B$) of $RP^2$ into $RP^2$ which maps quasi-connected subsets of lines, contained in $U$, coming from a certain open set in the topology on the set of lines discussed in Section 1.4, appropriately generalized, into lines. The affine part of the projective plane is the set of points with homogeneous co-ordinates $(x : y : z)$ such that $z$ is not a left divisor of zero, and it is an affine space over $R$. Given two points in the affine part of the plane not collinear with the origin $(0 : 0 : 1)$ we can form their sum by joining the origin to the first point $p_1$ and taking the intersection with the line at infinity $q_1$ (which can easily be seen to exist, although intersections of lines do not always exist), and joining the origin to the second point $p_2$ and taking the intersection with the line at infinity $q_2$, and now joining $p_1$ to $q_2$ and $p_2$ to $q_1$ and taking the intersection, which can be shown to exist. The sum, so defined, of $(a : b : 1)$ and

$(c : d : 1)$ will be $(a + c : b + d : 1)$. However, if we try to form the sum of a point in the affine part of the plane and a point not in the affine part of the plane in this way, it can be seen that the intersection will not exist. This allows us to show that the affine part of the plane will be preserved by our mapping, if we assume that $U$ is a basic open quasi-connected set containing the origin, and one other point in the affine part of the plane, and two points such that the unique line passing through them is the line at infinity and that these are all fixed, and that the lines in question are such that quasi-connected subsets of them contained in $U$ are mapped into lines.

So we may assume without loss of generality that $U$ is such a basic open set, and that the part of it that meets $R^2$ is mapped by $\phi$ into $R^2$, and that the restriction of $\phi$ to $U \cap R^2$ is an additive homomorphism, and that $\phi$ maps quasi-connected subsets of lines whose slope lies in some open subset of $RP^1$ into lines. Now, by composing on the right by an appropriate affine transformation (which will not alter our assumptions) we may assume that $U$ contains $(0 : 0 : 1)$, $(1 : 0 : 1)$, $(0 : 1 : 1)$, and $(1 : -1 : 1)$. Furthermore we may modify the (open) set of slopes for which line segments are mapped into lines, acting on it via a projective transformation. Furthermore $\phi$ agrees with a transformation $\psi$ induced by some linear mapping on the four points mentioned. Let $\phi = \psi \circ \chi$. Reviewing the arguments for Lemma 6.8, it may be seen that the argument for that lemma – which states that local homomorphisms extend to global homomorphisms – also applies to the rings which we consider in this chapter, and so $\chi$ is induced by a unital homomorphism from $R$ to itself. Then, from the fact that the range of $\phi$ contains $n + 1$ points in general position, we have that $\psi$ is invertible, and this proves the generalizations of Theorems 3.3 and 3.4 to these other rings.

It may now be seen, arguing as in the proofs of Theorems 6.9–6.11, that Theorems 7.4–7.6 are true.

This concludes our discussion of how to generalize the results of Part I and II to a larger class of rings. In Part IV we shall discuss some results dealing with the Haar measure on a non-discrete locally compact Hausdorff topological division ring.

# Part IV

# Results dealing with Haar measure

# Chapter 8

## Results dealing with the Haar measure on a non-discrete locally compact Hausdorff topological division ring

In Parts I-III, we presented results dealing with generalizations of the fundamental theorem of projective geometry dealing with mappings defined on open sets. In Part IV, we present some results dealing with the general linear group over a non-discrete locally compact Hausdorff topological division ring, concerning mappings defined on sets of positive measure, the measure being induced by the Haar measure with respect to addition.

In this chapter, we let $\mathbb{K}$ denote any non-discrete locally compact Hausdorff topological division ring and we let $\mu$ denote a Haar measure on this ring with respect to addition. We can define a product measure on $\mathbb{K}^n$ in the obvious way. Let us say that the inner measure or outer measure of a subset of $\mathbb{K}P^n$ is equal to that of its intersection with the affine part of $\mathbb{K}P^n$. We shall prove the following:

**Theorem 8.1.** *Suppose that $\mathbb{K}$ is a nondiscrete locally compact Hausdorff topological division ring and that $n$ is an integer such that $n > 1$. Suppose that $\mu$ is a Haar measure on $\mathbb{K}$ with respect to addition. Suppose that $S$ is a subset of $\mathbb{K}P^n$ which has positive inner measure. Suppose that $\phi : S \to \mathbb{K}P^n$ is a mapping whose range contains $n+1$ points in general position, and such that it sends subsets of lines which are contained in $S$ into lines. Then $\phi$ is a restriction of a mapping of the form $g^* \circ \sigma^*$, where $g \in \mathrm{GL}(n, \mathbb{K})$ and $\sigma$ is a unital ring homomorphism $\mathbb{K} \to \mathbb{K}$.*

**Theorem 8.2.** *Suppose that $S$ is a subset of $\mathbb{K}P^n$ of full measure in $\mathbb{K}P^n$, and suppose that $W$ is a subset of $\mathbb{K}P^{n-1}$ with positive inner measure. Suppose that $\phi$ is a mapping $S \to \mathbb{K}P^n$, such that whenever $l$ is a straight line meeting the affine part of $\mathbb{K}P^n$ with slope coming from $W$, $\phi$ maps $l \cap S$ into a straight line, and such that the range of $\phi$ contains $n+1$ points in general position. Then $\phi$ is the restriction to $S$ of a mapping of the form $g^* \circ \sigma^*$, where $g \in \mathrm{GL}(n, \mathbb{K})$ and $\sigma$ is a unital ring homomorphism $\mathbb{K} \to \mathbb{K}$.*

We begin by proving a lemma.

**Lemma 8.3.** *Suppose that $U$ is a subset of $\mathbb{K}P^n$ of positive measure. Suppose that $W$ is a subset of $\mathbb{K}P^{n-1}$ of full measure. Denote by $U^*$ the set of all points $p$ such that there exist two straight lines with slope in $W$, containing $p$, whose intersections with $U$ have positive one-dimensional measure. Then we have $U^* = \mathbb{K}P^n$.*

*Proof of Lemma 8.3.* Suppose that $U$ is a subset of $\mathbb{K}P^n$ with positive measure, suppose that $W$ is a subset of $\mathbb{K}P^{n-1}$ of full measure and denote by $p$ any point in $\mathbb{K}^n$ (the affine part of $\mathbb{K}P^n$). Then, if we consider the set of points $q$ in $\mathbb{K}^n$ such that the line joining $p$ and $q$ has slope in $W$, it is clear that this will have full measure. Hence the intersection of the set with $U$ will have positive measure. Consider spherical co-ordinates with centre $p$. The image of $U$ under the transformation to spherical co-ordinates also has positive measure, and therefore it contains two subsets of lines with constant sets of angular co-ordinates each corresponding to an element of $V$, that have positive one-dimensional measure. Going back to the original co-ordinates, there exist two lines with slope in $V$ with intersection $p$ such that the intersections of these lines with $U$ have positive one-dimensional measure. This shows that $U^*$ contains $p$, and $p$ was arbitrary. Hence $U^*$ contains all of $\mathbb{K}^n$. Fix any slope in $W$; it is easy to see that there exist two distinct lines of that slope whose intersection with $U$ has positive one-dimensional measure. This shows that $U^*$ contains every point at infinity and so contains all of $\mathbb{K}P^n$. This proves Lemma 8.3.

$\square$

We can now return to the proof of Theorem 8.1.

*Proof of Theorem 8.1.* In order to prove Theorem 8.1, it suffices to prove it on the assumption that there exists an $\epsilon$ such that $0 < \epsilon < 1$, such that there exists a basic open set $T$ such that $S \cap T$ has measure $1 - \epsilon^n$ times that of $T$.

Let us begin by considering the case $n = 2$. When $S$ satisfies the above condition for some $T$ and some $\epsilon$, a useful observation may be made. We may put a measure on the family of horizontal lines which meet $T$, for there is a natural projection from this family to a vertical line, and there is a one-dimensional measure on this vertical line which induces a measure on the original family. Then there will exist a subfamily $F$ of the family of horizontal lines which meet $T$ such that the measure of $F$ is $\epsilon$ times that of the family of all horizontal lines which meet $T$, and, for all the horizontal lines $l$ which meet $T$ and are not in $F$, the one-dimensional measure of $l \cap S \cap T$ is at least $1 - \epsilon$ times the one-dimensional measure of $l \cap T$. Similar comments apply to vertical lines.

Now we shall prove the following lemma.

**Lemma 8.4.** *Assume the hypotheses of Theorem 8.1, where $n = 2$, and suppose that, for every $\epsilon$ such that $0 < \epsilon < 1$, there exists a basic open set $T$ such that the measure of $S \cap T$ is at least $1 - \epsilon^2$ times that of $T$. We can find a set contained in $\mathbb{K}$ of positive measure $A$, such that $A \times A$ is contained in $S \cap T$ and $\phi$ is a mapping of the desired form on $A \times A$.*

*Remark.* Note that, by Lemma 8.3, this is sufficient to prove Theorem 8.1 in the case $n = 2$.

*Proof of Lemma 8.4.* We may assume without loss of generality that $S \subset T$ where $T$ and $\epsilon$ satisfy the hypotheses of the lemma. By composing on the right by an appropriate affine transformation, and changing $S$ and $T$ accordingly, we may modify the set of slopes $V$ for which lines are mapped into lines, acting on it via a projective transformation from $\mathbb{K}P^1$

to $\mathbb{K}P^1$. So we can assume without loss of generality that intersections with $S \cap T$ of straight lines with slope equal to $0, -1$, or $\infty$ are mapped into straight lines. By composing on the right by an appropriate affine transformation, and changing $S$ and $T$ accordingly, and on the left by an appropriate projective transformation, assume that $S$ contains the points $(0,0), (1,0), (0,1), (1,-1)$, and that these are fixed. We may also assume that the intersections of the lines joining these points with $S$ have measure at least $1-\epsilon$ times the measure of their intersection with $T$. To show this, one observes that one may select two straight lines of slope 0 and $-2$ such that the one-dimensional measure of their intersections with $S$ is at least $1-\epsilon$ times the one-dimensional measure of their intersections with $T$. Then one can select two straight lines of slope $\infty$, with equal horizontal distance from the point of intersection of the original two straight lines, such that the one-dimensional measure of their intersections with $S$ is at least $1-\epsilon$ times the one-dimensional measure of their intersections with $T$. We may choose these lines from a set of vertical lines of relative measure at least $1-\epsilon$, and so, by choosing them appropriately, we can ensure that the other lines joining the points of intersection of these lines have high relative measure as well. Then by composing on the right with an affine transformation, these lines can become the lines joining our four points. (We compose on the left with an appropriate projective transformation to ensure that the points are fixed again.)

Let us call intersections of lines with $S$ that have one-dimensional measure equal to $1-\epsilon$ of the one-dimensional measure of their intersection with $T$, "line segments". Consider a neighbourhood $N$ of $(0,0)$ of the form $T \cap \{(x,y) \mid y \in N'\}$, where $N'$ is a neighbourhood of 0 of measure $\delta$. It may be seen that, if we choose $N'$ to be sufficiently small, a line segment of slope 0 which meets $N$ is mapped into a line of slope 0 with probability $1-\psi(\epsilon)$, where $\psi$ is a function of $\epsilon$ such that $\psi(0) = 0$ and $\psi$ is continuous at 0. Call such a function a good function of $\epsilon$. That is, there is a natural measure on the set of line segments of slope 0 which meet $N$ with total measure $2\delta$ (provided every horizontal line in $N$ meets $T$), and we say that the measure of the set of line segments of slope 0 which meet $N$ which are mapped into lines of slope 0 will be $2\delta(1-\psi(\epsilon))$. Clearly the line segment contained in the line from $(0,0)$ to $(1,0)$ will be mapped into a line of slope 0. Take a line segment not contained in this line, which meets $N$, and let the line containing it have intersection $P$ with the line from $(0,0)$ to $(0,1)$, and intersection $Q$ with the line from $(1,0)$ to $(0,1)$. We may require that $P$ and $Q$ are contained in $S$, since the probability of this happening will be 1 minus a good function of $\epsilon$. Let $R$ be the intersection of the line from $(0,0)$ to $Q$ with the line from $(1,0)$ to $P$. $R$ lies on the line from $(0,1)$ to $(1,-1)$, which we have assumed has intersection with $S$ of measure at least $1-\epsilon$ times that of its intersection with $T$. If the $y$-coordinates of $P$ and $Q$ are both $\alpha$, then $R$ will be $((1-\alpha)(2-\alpha)^{-1}, \alpha(2-\alpha)^{-1})$. Thus, in general, a line not passing through $(0,1)$ has slope 0 or is the line at infinity if and only if the $R$ obtained from it in this way lies on the line from $(0,1)$ to $(1,-1)$. By choosing $\alpha$ from a subset of a small neighbourhood of zero together with a small neighbourhood of 1, whose relative measure is 1 minus a good function of $\epsilon$, we may assume that $R$ is in $S$. Thus line segments of slope 0 close to the line from $(0,0)$ to $(1,0)$ or close to $(0,1)$ are mapped into lines of slope 0 or into the line at infinity, with probability $1-\psi(\epsilon)$ for some good function $\psi$. We can exclude the case of the line at infinity by making the neighbourhood of 0 and the neighbourhood of 1 sufficiently small. By a similar argument, the property of having slope $\infty$ is preserved, at least for lines close to the line from $(0,0)$ to $(0,1)$ and the line from $(1,0)$ to $(1,-1)$, with probability 1 minus a good function of $\epsilon$. This shows that there exists a set which is a product $A\times A$, where $A$ is a nonempty open set $M$ minus a set of measure $\chi(\epsilon)$ times that of $M$ for some good function $\chi$, on which $\phi$ preserves lines of slope 0 and $\infty$. We will have that, with probability 1 minus a good function of $\epsilon$, $\phi$ preserves lines of slope $-1$ whose intersections with the line from $(0,0)$ to $(0,1)$ and the line from $(1,0)$ to $(1,-1)$ are both contained in $A\times A$. Thus, by making $A$ smaller if necessary, we can ensure that this is always the case, and we can still force $A$ to have positive measure by taking $\epsilon$ small. Thus, we will have $\phi(x,y)=(\sigma(x),\sigma(y))$ for some function $\sigma$ defined on $U$, such that $\sigma(x+y)=\sigma(x)+\sigma(y)$ whenever $x,y,x+y\in A$. Also, given a line $l$ of slope $-1$, whenever we can find four points $(a,b),(a+c,b),(a,b+c),(a-c,b+c)\in A\times A$, with $a,b,c\in A$, such that the intersection of $l$ with the line from $(a, b)$ to $(a, b+c)$ and with the line from $(a+c, b)$ to $(a+c, b-c)$ are both contained in $A \times A$, $l$ will be mapped to a line of slope $-1$. This will happen with probability $1 - \chi'(\epsilon)$ for some continuous function $\chi'$, so by shrinking $A$ further we can arrange that all lines of slope $-1$ are mapped to lines of slope $-1$. We need to shrink $A$ to $A'$ where $A' \subset A$ and $A' + A'$ is contained in a certain set $B$ of relative measure $1 - \chi'(\epsilon)$. We do this by letting $A' = A \cap (A - B)$, where $A - B$ means $\{x - y \mid x \in A, y \in B\}$.

Then, by taking $\epsilon$ to be sufficiently small we can ensure that $A$ is still a set of positive measure. When this is done, we will have that $\sigma(x) + \sigma(y)$ depends only on $x + y$, even when $x + y \notin A$.

Hence $\phi(\alpha, \beta) = (\sigma(\alpha), \sigma(\beta))$, where $\sigma$ is a local injective additive homomorphism defined on a set $A$ which is an open set minus a set of small relative measure. We also have that $\sigma(xa) = \sigma(x)\tau(a)$, when $a \in W$ and $x, xa \in A$, for some fixed function $\tau$. But, because $\sigma(x) + \sigma(y)$ depends only on $x + y$ for $x, y \in A$, these statements remain true when we replace $A$ by $A + A$ and $W$ by $((W \cap \mathbb{K}) + (W \cap \mathbb{K})) \cup \{\infty\} = \mathbb{K}P^1$. Now, if $A$ is a set of positive measure, then $A + A$ has nonempty interior. This is true for the Haar measure on any locally compact abelian group. This is well-known, and follows fairly easily from Theorem (d) in [21], p. 4. That theorem states that if $G$ is a locally compact abelian group and we define $f * g(x) = \int_G f(x - y)g(y)dy$, and $f$ and $g$ are in $L^p(G)$ for some $p$ such that $1 < p < \infty$, then $f * g \in C_0(G)$. So, if $A$ has positive measure, then $\chi_A * \chi_A \in C_0(\mathbb{K})$, and, by Fubini's theorem, its integral over $\mathbb{K}$ is positive. Hence its support has nonempty interior, but its support is certainly contained in $A + A$. So, if we let $B = A + A$, we may extend $\sigma$ to a local injective additive homomorphism defined on $B$, which has nonempty interior, with the property that $\sigma(xb) = \sigma(x)\tau^*(b)$ whenever $x, xb \in B$, for some fixed function $\tau^*$ which is an extension of our original $\tau$. We may achieve this by defining $\sigma(x + y) = \sigma(x) + \sigma(y)$ when $x, y \in A$. As we have pointed out, this does not depend on the choice of $x$ and $y$. By composing on the right and on the left with an appropriate affine transformation, and making the appropriate change to $\sigma$, we may assume that $B$ contains $I$ where $I$ is an open set containing 0 and 1. It can now be shown by Lemma 6.8 that $\sigma$ will uniquely extend to a unital ring homomorphism of all of $\mathbb{K}$ into itself, and this is sufficient to prove the lemma. $\square$

It is easy to generalize Lemma 8.4 to higher dimensions by induction and so Theorem 8.1 is proved.

$\square$

*Proof of Theorem 8.2.* We will prove the case $n = 2$; generalizing to higher dimensions by induction is not difficult. Again we can assume without loss of generality that $W$ contains $0, 1$ and $\infty$. The arguments used in the proof of Lemma 8.4 now show that $S$ contains $U \times U$ where $U$ is a set of full measure in $\mathbb{K}$, where we can have, without loss of generality, that $U \times U$ contains $(0,0), (0,1), (1,0)$, and $(1,-1)$, and that these are fixed, and for $x, y \in U$ we have $\phi(x,y) = (\sigma(x), \sigma(y))$ where $\sigma$ is an additive homomorphism such that $\sigma(x) + \sigma(y)$ depends only on $x + y$, and, whenever $x \in U, a \in W, xa \in U$, we have $\sigma(xa) = \sigma(x)\tau(a)$ for some fixed function $\tau$. Since $U + U = \mathbb{K}$ we can easily extend $\sigma$ to an additive homomorphism defined on all of $\mathbb{K}$ with this property, which by abuse of notation we also call $\sigma$. Then we can replace $W$ by $(W \cap \mathbb{K}) + (W \cap \mathbb{K})$, which has nonempty interior. It can now be shown that $\sigma$ is a unital ring homomorphism, and this suffices to prove the theorem. $\square$

This completes our proofs of the resuts dealing with the Haar measure on a non-discrete locally compact Hausdorff topological division ring.

# Part V

# Conclusions

# Chapter 9

# Conclusions

The fundamental theorem of projective geometry and its generalizations to classical groups, which were known to Tits [23], are usually conceived of as aspects of a deep result in algebraic geometry. Our investigations indicate that for a large class of topological rings and fields, there exist local versions of these results. We have also obtained results dealing with sets of positive measure. One of the innovations of this thesis is that we have obtained geometric results which make use of a topological structure on the field without making use of a differentiable structure. This suggests that there may exist a field of "topological geometry", intermediate between algebraic geometry and differential geometry.

Besides the classical groups, there remain some exceptional groups to which our results might generalize further. We shall pursue this matter further in future research. This may require different techniques.

We have observed connections between the results of this thesis and ideas in mathematical logic which it may prove fruitful to investigate further. It turns out that the fundamental theorem of projective geometry can be seen as a corollary of the fact that the theory of projective geometry can be interpreted in the theory of division rings and vice versa, and that the local results of this thesis can be seen as due to a model-theoretic property of the various geometries, whereby the theory of the entire model is interpretable in the theory of an open subset of the model. We intend to present this material in future publications.

# Appendix A

## Review of hyperbolic space, Euclidean space, and Möbius transformations

Given any integer $n > 0$, there is the Euclidean space of dimension $n$, which we denote by $\mathbb{R}^n$ or $E^n$. This is the affine space of dimension $n$ over $\mathbb{R}$, $\{(x_1, x_2, \dots x_n) \mid x_i \in \mathbb{R}$ for all $i$ such that $1 \leq i \leq n\}$, endowed with the metric $d$ such that if $\mathbf{x} = (x_1, x_2, \dots x_n), \mathbf{y} = (y_1, y_2, \dots y_n)$, then $d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}$. The geodesics with respect to this metric are the straight lines, the translates of subspaces of $\mathbb{R}^n$ of dimension one, where $\mathbb{R}^n$ is being viewed as a vector space.

Given any integer $n > 0$ and any ordered field $\mathbb{K}$, we have the hyperbolic space of dimension $n$ over the ordered field $\mathbb{K}$, which we denote by $H^n(\mathbb{K})$. If mention of the field $\mathbb{K}$ is suppressed, then it is assumed that $\mathbb{K} = \mathbb{R}$. In that case the hyperbolic space is denoted by $H^n$. In what follows, we shall assume that $\mathbb{K} = \mathbb{R}$ but our discussion applies equally well to any ordered field. (The definition of the metric does not make sense for arbitrary ordered fields but we can define the notion of the distance between $a$ and $b$ being equal to that between $c$ and $d$, and that suffices.)

We endow $\mathbb{R}^{n+1}$ with the Lorentzian quadratic form $\circ$ such that if $\mathbf{x} = (x_1, x_2, \dots x_{n+1})$, $\mathbf{y} = (y_1, y_2, \dots y_{n+1})$, then $\mathbf{x} \circ \mathbf{y} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n - x_{n+1} y_{n+1}$. $H^n$ is taken to be the upper sheet of the hyperboloid $\{\mathbf{x} \mid \mathbf{x} \circ \mathbf{x} = -1\}$. That is, we define $H^n = \{\mathbf{x} \mid \mathbf{x} \circ \mathbf{x} = -1, x_{n+1} > 0\}$. Then given $\mathbf{x}, \mathbf{y} \in H^n$, we define $d(\mathbf{x}, \mathbf{y}) = \cosh^{-1}(-\mathbf{x} \circ \mathbf{y})$. This gives the metric on hyperbolic space. The geodesics with respect to this metric are the nonempty intersections of two-dimensional subspaces of $\mathbb{R}^{n+1}$ with $H^n$. The isometries of $H^n$ are induced by those linear transformations of $\mathbb{R}^{n+1}$ which preserve the Lorentzian quadratic form and map $H^n$ into $H^n$. The group of matrices of these transformations is sometimes denoted $\mathrm{PO}(n,1)$.

Using radial projection from the point $(0,0,\ldots,0,-1)$, we can radially project the upper half of the hyperboloid onto the disc $\{(x_1,x_2,\ldots x_{n+1}) \mid x_1^2+x_2^2+\ldots+x_n^2<1, x_{n+1}=1\}$. This gives the Poincaré disc model of hyperbolic geometry, in which the geodesics are circular arcs which intersect the boundary of the disc at right angles. Taking the image of this under an inversion in an appropriate sphere whose centre is a point on the boundary of the disc, and ignoring the last co-ordinate, one obtains the upper-half-space model for hyperbolic geometry, where the underlying space is $\{(x_1,x_2,\ldots x_n) \mid x_n>0\}$, and the geodesics are circular arcs which meet the boundary of the upper half space at right angles. (The notion of an inversion is defined below, in Definition A.1.) One virtue of these two models is that they are conformal; that is, angles are represented correctly. In the case of either model, the isometries are precisely those Möbius transformations which fix the underlying space setwise. We shall discuss Möbius transformations below.

Alternatively one can use radial projection from the origin, and radially project the upper half of the hyperboloid onto the disc $\{(x_1,x_2,\ldots x_{n+1}) \mid x_1^2+x_2^2+\ldots+x_n^2<1, x_{n+1}=0\}$. This gives the Klein model of hyperbolic geometry. Here the geodesics are straight lines and the isometries are the projective transformations which fix the disc setwise.

Let us now discuss Möbius transformations. If we take the one-point compactification of $\mathbb{R}^n$, adding a point at infinity which we denote by $\infty$, we obtain a space called $\mathbb{R}^n_\infty$ which is homeomorphic to the sphere $S^n$. One canonical homeomorphism between the two spaces is stereographic projection. Denote by $S^n$ the space $\{(x_1,x_2,\ldots x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2+x_2^2+\ldots+x_{n+1}^2=1\}$, endowed with the subspace topology. Let $n$ be the north pole

$(0,0,\ldots,0,1)$. Given any point $p$ in $S^n \setminus \{n\}$, we may define $\phi(p)$ to be the unique point in $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = 0\}$ which meets the line joining $n$ and $p$. We define $\phi(n) = \infty$. This defines a homeomorphism $\phi : S^n \to \mathbb{R}^n_\infty$ known as stereographic projection. The circles and $k$-spheres contained in $S^n$, where $1 < k < n$, are mapped onto circles and $k$-spheres respectively, or onto lines and $k$-planes respectively with $\infty$ added. We call the latter subsets of $\mathbb{R}^n_\infty$ circles and $k$-spheres respectively as well.

**Definition A.1.** *Given any $(n-1)$-sphere $S'$ with centre $c$ contained in $\mathbb{R}^n_\infty$ we have inversion $\rho$ in $S'$, which may be defined as follows. If $\infty \notin S'$, then $\rho$ swaps $c$ and $\infty$, and every other point $p$ is mapped to the unique point $q$ on the ray joining $c$ to $p$ such that the product of the Euclidean distance from $c$ to $p$ with the Euclidean distance from $c$ to $q$ is equal to the square of the radius of $S'$. If $\infty \in S'$, then $S' \cap \mathbb{R}^n$ is an $(n-1)$-plane $P$. In this case $\rho$ fixes $\infty$ and $\rho\,|_{\mathbb{R}^n}$ is simply reflection in $P$.*

Mappings which can be obtained as the composition of a finite sequence of such inversions are called Möbius transformations $\mathbb{R}^n_\infty \to \mathbb{R}^n_\infty$. The results of conjugating such transformations by $\phi^{-1}$ are called Möbius transformations $S^n \to S^n$. In either case they map circles onto circles and $k$-spheres onto $k$-spheres whenever $1 < k < n$. If we view $S^n$ as embedded in $\mathbb{R}^{n+1}$ as before, a Möbius transformation of $S^n$ is the restriction to $S^n$ of a projective transformation of $\mathbb{R}^{n+1}$ with a matrix in $\mathrm{O}(n+1,1)$. This group is defined in Chapter 4, in Definition 4.1.

All of the material in this appendix is discussed and proved in detail in [19].

# Appendix B

## Self-homomorphisms of $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$

In this appendix, we discuss the unital homomorphisms $\mathbb{K} \to \mathbb{K}$ when $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

The only unital homomorphism $\mathbb{R} \to \mathbb{R}$ is the identity. This is why in the case where the underlying field was $\mathbb{R}$ we were able to give a simpler statement of the fundamental theorem of projective geometry. This can be seen as follows. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a unital homomorphism. It is easy to see that $\phi$ fixes every element of $\mathbb{Q}$. Also, $\phi$ maps squares to squares, and therefore maps non-negative elements of $\mathbb{R}$ to non-negative elements, and is therefore order-preserving (since it is easy to see that a unital homomorphism must be injective). Since $\phi$ fixes every element of $\mathbb{Q}$ and is order-preserving, and $\mathbb{Q}$ is dense in $\mathbb{R}$, it follows that $\phi$ fixes every element of $\mathbb{R}$.

On the other hand, there are very many unital homomorphisms $\mathbb{C} \to \mathbb{C}$, and they are not necessarily surjective. This can be seen as follows. We shall refer to Zermelo-Fraenkel set theory without the axiom of choice as ZF. Assuming the axiom of choice, there exists a transcendence basis $B$ for $\mathbb{C}$ over $\mathbb{Q}$ (this can be proved using Zorn's lemma). Now, any mapping $\phi$ defined on $B$ whose range is a set of algebraically independent elements extends to a unital homomorphism $\mathbb{C} \to \mathbb{C}$, unique up to composition on the left by an element of the absolute Galois group of $\mathbb{Q}(\phi(B))$ (where we are thinking of $\phi$ as acting on the left). The proof of this again uses the axiom of choice. Here $\phi$ can certainly be any injective mapping from $B$ into $B$, for example. Thus we see that the number of unital homomorphisms from $\mathbb{C} \to \mathbb{C}$ is $2^{\mathfrak{c}}$ where $\mathfrak{c}$ is the cardinality of the continuum. For if we assume the axiom of choice, then the continuum can be well-ordered and it can be shown using easy arguments in the arithmetic of alephs that given an infinite set $B$ of algebraically independent complex numbers, the algebraic closure of $\mathbb{Q}(B)$ has the same cardinality as $B$. Hence if $B$ is a transcendence basis for $\mathbb{C}$ over $\mathbb{Q}$, then $B$ has cardinality $\mathfrak{c}$. Using transfinite induction on a well-ordering of the continuum of minimal length, one can prove that there exists a bijection between the set of injective mappings $B \to B$ and the set of functions $B \to B$. It then follows by easy and well-known results in cardinal arithmetic that the set of injective mappings $B \to B$ has cardinality $2^{\mathfrak{c}}$. So now we know (using the axiom of choice again) that there is an injective mapping from the set of injective mappings $B \to B$, which has cardinality $2^{\mathfrak{c}}$, into the set of unital homomorphisms $\mathbb{C} \to \mathbb{C}$. This in turn is a subset of the set of all functions $\mathbb{C} \to \mathbb{C}$, which has cardinality $2^{\mathfrak{c}}$. It now follows by the Schröder–Bernstein theorem (which is a theorem of ZF) that the set of homomorphisms $\mathbb{C} \to \mathbb{C}$ has cardinality $2^{\mathfrak{c}}$. This argument has relied heavily on the axiom of choice.

One useful observation is that of all these unital homomorphisms, only two are measurable: namely, the identity and conjugation. This is proved in [16]. We will make use of this fact to observe what happens if we add measurability to the hypotheses of our theorems in the case where the underlying field is $\mathbb{C}$.

As an aside, we may note an interesting metamathematical fact. The axiom of depending choice is the assertion that, if $R$ is a binary relation on a set $A$, with the property that for all $x \in A$ there exists a $y \in A$ such that $xRy$, then for all $a \in A$ there exists a sequence $\{x_n \mid n \in \mathbb{N}\}$, such that $x_0 = a$, and, for all $n \in \mathbb{N}$, $x_n R x_{n+1}$. This axiom is slightly stronger than the axiom of countable choice and considerably weaker than the full axiom of choice. (To prove that the axiom of depending choice is strictly stronger than the axiom of countable choice and the axiom of choice is strictly stronger than the axiom of depending choice, in the presence of the axioms of ZF, one of course needs to use the hypothesis that ZF is consistent.) The axiom of depending choice plays an important role in measure theory. For example, it is necessary for the proof of the basic properties of Lebesgue measure. In set theory we have the notion of an inaccessible cardinal. We define $V_0 = \emptyset, V_{\alpha+1} = P(V_\alpha)$ for all ordinals $\alpha$, and if $\alpha$ is a limit ordinal, we define $V_\alpha$ to be the union of all $V_\beta$ such that $\beta < \alpha$. An inaccessible cardinal is defined to be an uncountable limit ordinal $\kappa$ such that if $\alpha < \kappa$ and $\phi$ is a mapping $V_\alpha \to \kappa$ then the range of $\phi$ is not cofinal in $\kappa$. Now, it can be proved in ZF with the axiom of depending choice that the only two measurable unital homomorphisms $\mathbb{C} \to \mathbb{C}$ are the identity and conjugation. And there is a famous metamathematical result of Solovay: namely, Solovay showed in [22] how to prove, in Bounded Arithmetic, that if it is consistent with ZF that an inaccessible cardinal exists, then it is consistent with ZF plus the axiom of depending choice that every set of reals is Lebesgue measurable (and hence every function $\mathbb{C} \to \mathbb{C}$ is Lebesgue measurable). We may conclude that, if it is consistent with ZF that an inaccessible cardinal exists, it is consistent with ZF plus the axiom of depending choice that there are only two unital homomorphisms $\mathbb{C} \to \mathbb{C}$, namely the identity and conjugation. Hence (with the aforementioned consistency hypothesis) the use of some form of choice stronger than depending choice in the argument given two paragraphs previously was necessary.

Now let us consider the case of $\mathbb{H}$. It turns out that every unital homomorphism $\mathbb{H} \to \mathbb{H}$ is surjective; thus the set of such homomorphisms forms a group, the automorphism group of $\mathbb{H}$. It turns out to be isomorphic to SO(3). Given a quaternion $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, we call $a$ its real part and $b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ its pure part, and call a quaternion pure if and only if it is equal to its pure part. Now let $G = \mathrm{SO}(3)$ and define an action $\phi : G \times \mathbb{H} \to \mathbb{H}$ as follows. We define $\phi(g, a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) = a + b'\mathbf{i} + c'\mathbf{j} + d'\mathbf{k}$, where $g \cdot \begin{pmatrix} b \\ c \\ d \end{pmatrix} = \begin{pmatrix} b' \\ c' \\ d' \end{pmatrix}$. This is the action of the automorphism group of $\mathbb{H}$ on $\mathbb{H}$. Thus, the claim is that every automorphism $\rho$ is of the form $\phi(g, \cdot)$ for some $g \in G$.

This is proved in [18], or at any rate it is proved there on the assumption that $\rho$ is surjective; we shall show how to dispense with that assumption, which is quite easy. To begin with, let $\rho$ be a unital homomorphism $\mathbb{H} \to \mathbb{H}$. Now, $\mathbb{R}$ is the centre of $\mathbb{H}$, so $\rho(\mathbb{R})$ commutes with every element of $\rho(\mathbb{H})$. In particular it commutes with $\rho(q)$ whenever $q$ is a square root of $-1$. Since $\rho$ is injective and fixes every element of $\mathbb{Q}$, $\rho(\mathbf{i})$ and $\rho(\mathbf{j})$ are two square roots of $-1$ which are linearly independent over $\mathbb{Q}$. Each element of $\rho(\mathbb{R})$ commutes with both of these two square roots of $-1$. It can be shown (and will become clear below) that the set of quaternions which commute with $\rho(\mathbf{i})$, for example, is equal to $\mathbb{R} + \mathbb{R}\rho(\mathbf{i})$. A similar remark applies to $\rho(\mathbf{j})$. So it follows that $\rho(\mathbb{R})$ is contained in the intersection of $\mathbb{R}+\mathbb{R}\rho(\mathbf{i})$ and $\mathbb{R}+\mathbb{R}\rho(\mathbf{j})$, which is equal to $\mathbb{R}$, so that $\rho\mid_{\mathbb{R}}$ is a unital homomorphism $\mathbb{R} \to \mathbb{R}$, and must therefore be the identity. Let us denote by $\mathrm{re}(q)$ the real part of a quaternion $q$ and denote by $\mathrm{pu}(q)$ its pure part. We have $q^2 = (\mathrm{re}(q))^2 + 2(\mathrm{re}(q))(\mathrm{pu}(q)) + (\mathrm{pu}(q))^2$. We have $(b\mathbf{i}+c\mathbf{j}+d\mathbf{k})^2 = -(b^2+c^2+d^2)$, so the square of a pure quaternion is a non-positive real number, and a quaternion is a square root of $-1$ if and only if it is a unit pure quaternion. Suppose $q$ and $r$ are pure quaternions. If we let $q = b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}$ and $r = b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}$, then the real part of $qr$ and $rq$ is $-(b_1b_2 + c_1c_2 + d_1d_2)$, whereas the pure parts of $qr$ and $rq$ are negatives of each other. Thus $q$ and $r$ anti-commute; that is $qr = -rq$, if and only if $\begin{pmatrix} b_1 \\ c_1 \\ d_1 \end{pmatrix}$ and $\begin{pmatrix} b_2 \\ c_2 \\ d_2 \end{pmatrix}$ are orthogonal. It now becomes clear how to prove our earlier remark that the set of quaternions which commute with every element of $\mathbb{R} + \mathbb{R}q$, where $q$ is a square root of $-1$, is equal to $\mathbb{R} + \mathbb{R}q$. Namely, it is certainly clear that the set in question contains $\mathbb{R} + \mathbb{R}q$, and if $r$ commutes with $q$, then we can now see that the component of $r$ orthogonal to $\mathbb{R} + \mathbb{R}q$ must be zero, hence the set in question is also contained in $\mathbb{R} + \mathbb{R}q$. Also, if $q$ and $r$ are pure quaternions and anti-commute, then $qr = b_3\mathbf{i} + c_3\mathbf{j} + d_3\mathbf{k}$ where $\begin{pmatrix} b_3 \\ c_3 \\ d_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ c_1 \\ d_1 \end{pmatrix} \times \begin{pmatrix} b_2 \\ c_2 \\ d_2 \end{pmatrix}$. This means that $\rho(\mathbf{i}), \rho(\mathbf{j})$, and $\rho(\mathbf{k})$ must be unit pure quaternions and if we let them be $b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}, b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}, b_3\mathbf{i} + c_3\mathbf{j} + d_3\mathbf{k}$ respectively, then $\begin{pmatrix} b_1 \\ c_1 \\ d_1 \end{pmatrix}$ and $\begin{pmatrix} b_2 \\ c_2 \\ d_2 \end{pmatrix}$ must be orthogonal and $\begin{pmatrix} b_1 \\ c_1 \\ d_1 \end{pmatrix} \times \begin{pmatrix} b_2 \\ c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} b_3 \\ c_3 \\ d_3 \end{pmatrix}$. This shows that $\rho$ is of the form claimed.

Another useful observation is that every automorphism of $\mathbb{H}$ is inner, that is, given an automorphism $\rho$ of $\mathbb{H}$, there exists a unit quaternion $q$ such that $\rho(r) = q^{-1}rq$ for all quaternions $r$. This is also proved in [18]. To prove this, one starts by proving that every rotation of $\mathbb{R}^3$ is the composite of two plane reflections. To start with, if $a_1$ and $b_1$ are two unit vectors in $\mathbb{R}^3$, then there exists a plane reflection mapping $a_1$ to $b_1$. This is clear if $a_1 = b_1$, and if $a_1 \neq b_1$ then one uses the reflection in the plane orthogonal to $a_1 - b_1$. Then if $a_2$ and $b_2$ are unit vectors orthogonal to $b_1$, there exists a plane reflection mapping $a_2$ to $b_2$ and fixing $b_1$. So if $a_1, a_2, a_3$ are unit vectors which are mutually orthogonal and $a_1 \times a_2 = a_3$, and similarly for $b_1, b_2$ and $b_3$, then one can find a rotation which is a composite of two plane reflections mapping $a_i$ to $b_i$ for $i = 1, 2, 3$. This shows that every rotation is the composite of two plane reflections.

Now, if $q$ is a unit pure quaternion and one considers the action of the map $r \mapsto -q^{-1}rq$ on the space of pure quaternions, it is not hard to check that it is a reflection in the plane perpendicular to $q$. Thus, since every rotation is the composite of two plane reflections, this shows that given any rotation of the space of pure quaternions, there exists a unit quaternion $q$ such that the rotation is induced by the map $r \mapsto q^{-1}rq$. Combining this with the earlier characterization of the automorphisms of $\mathbb{H}$, this shows that every automorphism of $\mathbb{H}$ is inner.

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