Functional calculus and coadjoint orbits.

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Functional Calculus and Coadjoint Orbits

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Abstract

Let G be a compact Lie group and let π be an irreducible representation of G of highest weight λ. We study the operator-valued Fourier transform of the product of the j-function and the pull-back of π by the exponential mapping. We show that the set of extremal points of the convex hull of the support of this distribution is the coadjoint orbit through $\lambda + \delta$. The singular support is furthermore the union of the coadjoint orbits through $\lambda + w\delta$, as w runs through the Weyl group.

Our methods involve the Weyl functional calculus for noncommuting operators, the Nelson algebra of operants and the geometry of the moment set for a Lie group representation. In particular, we re-obtain the Kirillov-Duflo correspondence for compact Lie groups, independently of character formulae. We also develop a "noncommutative" version of the Kirillov character formula, valid for noncentral trigonometric polynomials. This generalises work of Cazzaniga, 1992.

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Contents

Chapter 1

Introduction

Let G be a semisimple compact connected Lie group and let $\mathfrak{g}$ be its Lie algebra. We fix a Cartan subalgebra $\mathfrak{t}$ and we write $\mathfrak{g}^*$ and $\mathfrak{t}^*$ for the vector space duals of $\mathfrak{g}$ and $\mathfrak{t}$ respectively. We denote by δ half-the-sum of the positive roots.

The inverse Fourier transform of Liouville measure on a coadjoint orbit of G in $\mathfrak{g}^*$ is an invariant eigendistribution with respect the universal enveloping algebra of $\mathfrak{g}$ acting as differential operators and thus defines an analytic function on $G \cdot \mathfrak{t}_r$, where $\mathfrak{t}_r$ denotes the set of regular elements of $\mathfrak{t}$, which is given explicitly by a well-known formula of Harish-Chandra [20, Theorem 2].

If $\lambda \in \mathfrak{t}^*$ is such that $\lambda - \delta$ is a dominant integral weight then the Kirillov character formula [27] asserts that the inverse Fourier transform of Liouville measure on the coadjoint orbit through λ equals the product of the pull-back to the Lie algebra by the exponential mapping of the character of the unitary irreducible representation of G of highest weight $\lambda - \delta$ and the so-called "j-function", which is the analytic square-root of the Jacobian of the exponential map (chosen so that $j(0) = 1$). This relates the formula of Harish-Chandra [20, Theorem 2] above to the Weyl character formula.

Extending the work of Cazzaniga [12] for $SU(2)$, we develop in the sequel a "non-commutative" generalisation of this fact: we show that the distributional Fourier transform of the product of the j-function and the pull-back of an arbitrary matrix coefficient of a unitary irreducible representation of G of highest weight λ is a finite linear combination of derivatives in the root, toral and radial directions applied to a measure supported on the convex hull of the coadjoint orbit through $\lambda+\delta$ and invariant under the coadjoint action of G; furthermore, the singular support of this distribution is a finite union of coadjoint orbits, namely of those through the orbits of $\lambda + w \cdot \delta$ as w runs through the Weyl group, and every point in this set is a singularity of this distribution for some choice of matrix coefficient. This result can be seen as a new demonstration of the orbit correspondence for compact Lie groups, insofar as the "outermost" member of this union is the orbit through $\lambda + \delta$, independently of the Kirillov formula. We also show that each point in the convex hull of the coadjoint orbit through $\lambda + \delta$ there is a matrix coefficient for which the above distribution is supported at that point. An earlier version of these results was published in [15].

Our proofs make fundamental use of an explicit formula of Edward Nelson [35] for the Weyl functional calculus of a d-tuple of self-adjoint operators on a finite dimensional Hilbert space as well as his construction of a certain commutative Banach algebra of "operants" the spectrum of which describes the support of the Weyl calculus distribution. It will be shown that the support of the Weyl calculus of the infinitesimal generators of a unitary representation of a compact Lie group is contained in the image of the moment map of the representation [4, 44]. This is the starting point of our investigations. Our results then follow from structural and analytic properties of Nelson's formula and as consequences of spectral properties of the Weyl calculus and numerical range techniques. The two streams will be developed independently in the exposition.

In Chapter 2 we collect the necessary background from the theory of numerical ranges and spectra in Banach algebras from various sources in the literature [8, 2, 10]. We introduce the concept of an *operating algebra* [2] which will be used later to motivate and describe Nelson's algebra of operants (Section 3.2).

In Chapter 3 is a unified exposition of the theory of the Weyl functional calculus for noncommutative self-adjoint operators following the original sources [3, 35, 41], but expressed in the generality of hermitian elements of Banach algebras.

Chapter 4 contains our version of Nelson's proof of an explicit formula for the Weyl calculus for hermitian matrices. While Nelson's statement was concerned only with the calculus of a basis of the Lie algebra $\mathfrak{u}(n)$, the same expression is valid for arbitrary d-tuples of hermitian matrices. We also include a generalisation to arbitrary d-tuples of complex matrices due to B. Jefferies [23].

Chapter 6 and Chapter 7 bring together the techniques of the previous chapters to derive the new results discussed above.

As techniques from numerical ranges and the Weyl functional calculus [3] are not commonly used in the orbit method, we include for the sake of completeness proofs of some known results in these areas, when indicated formulating results and their proofs in greater generality than the versions found in the literature.

Chapter 2

The Numerical Range

In this chapter we review some elementary notions from the theory of numerical ranges and some of its consequences [2], collecting those results which will be used in the sequel. We assume some familiarity with the rudiments of C^*-algebra theory, which can be found in [14].

In Section 2.1 we derive a criterion (Theorem 2.1.5) for an element of a Banach algebra to be hermitian in terms of the reality of its numerical range. This will be used in Chapter 6 in the case of the infinitesimal generators of a unitary representation of a compact Lie group. The main result of Section 2.2 is the coincidence between numerical range and the convex hull of the spectrum, in the setting of C^*-algebras (Corollary 2.2.8).

In Section 2.3 an introduction Albrecht's theory [2] of operating algebras is given, generalising ideas of Nelson [35] to wider families of functional calculi and facilitating a strengthening of some of the results of Section 2.2. This theory will be used in the next chapter to describe the algebra of operants.

2.1 Numerical Range and Hermitian Elements

Let $\mathfrak{R}$ be a complex unital Banach algebra with unit $1_{\mathfrak{R}}$ and let $\mathfrak{R}^*$ be the dual space of continuous linear functionals on $\mathfrak{R}$.

By the Hahn-Banach theorem the following set is not void,

$$D(\mathfrak{R}, 1) := \{f \in \mathfrak{R}^* \,:\, f(1_{\mathfrak{R}}) = 1 = \|f\|\}.$$

This enables us to make the next

Definition 2.1.1 *Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$. The* numerical range *of x with respect to $\mathfrak{R}$ is the set $V_{\mathfrak{R}}(x) := \{(f(x_1), \ldots, f(x_d)) \,|\, f \in D(\mathfrak{R}, 1)\}$.*

We note that if $\mathfrak{R}$ is a C^*-algebra then $D(\mathfrak{R},\, 1)$ is exactly the set of states of $\mathfrak{R}$.

Let $\mathcal{H}$ be a Hilbert space and denote by $L(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. It is well-known that the defining representation is the only irreducible representation of $L(\mathcal{H})$; hence, by the GNS construction, the pure states of $L(\mathcal{H})$ are exactly the set of functionals $\phi\colon L(\mathcal{H}) \to \mathbb{C}$ of the form $\phi(x) := \langle xu,\, u\rangle$ $(x \in L(\mathcal{H}))$ for some fixed unit vector $u \in \mathcal{H}$.

Let $A \subseteq \mathbb{C}^n$. We write $\operatorname{co} A$ for the convex hull of A. By the preceeding discussion and the Krein-Milman theorem, we immediately have the following

Theorem 2.1.2 *Let $\mathfrak{R} = L(\mathcal{H})$ and let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$. Then*

$$V_{L(\mathcal{H})}(x) := \operatorname{co}\{(\langle x_1 u,\, u\rangle, \ldots, \langle x_d u,\, u\rangle) \,:\, u \in \mathcal{H},\, \|u\| = 1\}. \tag{2.1}$$

We see below that some of the obvious properties of $V_{L(\mathcal{H})}(x)$ have a more general validity.

Lemma 2.1.3 *Let $T\colon \mathbb{C}^d \to \mathbb{C}^d$ be a linear mapping. Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$. Then $V_{\mathfrak{R}}(Tx) = TV_{\mathfrak{R}}(x)$.*

Proof: For each $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$, define a mapping $\Phi(x)\colon D(\mathfrak{R}, 1) \to \mathbb{C}^d$ given by $\Phi(x)f := (f(x_1), \ldots, f(x_d))$. Then $V_{\mathfrak{R}}(x) = \Phi(x)D(\mathfrak{R}, 1)$. On representing T as a matrix with complex entries, it is easily verified that

$\Phi(Tx) = T\Phi(x)$. Then

$$\begin{aligned} V_{\mathfrak{R}}(Tx) &= \Phi(Tx)D(\mathfrak{R},1) \\ &= T\Phi(x)D(\mathfrak{R},1) \\ &= TV_{\mathfrak{R}}(x) \end{aligned}$$

as required. $\diamond$

Theorem 2.1.4 ([8], Theorem 1.2.3) *Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$. The numerical range $V_{\mathfrak{R}}(x)$ is a compact, convex subset of $\mathbb{C}^d$.*

Proof: See [8], Theorem 3, Chapter 1, Section 2. $\diamond$

An element $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$ satisfying the following equivalence will be called **hermitian**.

Theorem 2.1.5 *Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$. Then $V_{\mathfrak{R}}(x) \subseteq \mathbb{R}^d$ if and only if $\|e^{i\xi\cdot x}\| = 1$ for all $\xi \in \mathbb{R}^d$.*

A statement and proof of Theorem 2.1.5 in the case $d = 1$ can be found in [8], Lemma 2, Chapter 2, Section 5. We observe that if $V_{\mathfrak{R}}(x) \subseteq \mathbb{R}^d$ then $V_{\mathfrak{R}}(x_j) \subseteq \mathbb{R}$ for each $1 \le j \le d$, from which it follows that $V_{\mathfrak{R}}(\lambda_1 x_1 + \cdots + \lambda_d x_d) \subseteq \mathbb{R}$ for all $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$. Hence, it suffices to prove Theorem 2.1.5 in the case $d = 1$ case. We do this after a sequence of lemmas, which are proved in [8]. We include proofs of these for completeness.

Fix $x \in \mathfrak{R}$ and let $\mu := \max\{\operatorname{Re} f(x) : f \in D(\mathfrak{R},1)\}$. We note that $\mu \ge \|x\|$.

Lemma 2.1.6 *For all $\xi > 0$ and $y \in \mathfrak{R}$,*

$$\|(1-\xi x)y\| \ge (1-\xi\mu)\|y\|.$$

Proof: By the properties of the norm $\|\cdot\|$, it suffices to consider $y \in \mathfrak{R}$ with $\|y\| = 1$. Let $f \in \mathfrak{R}^*$ such that $f(y) = 1 = \|f\|$. Then,

$$\operatorname{Re} f((1-\xi x)y) \le \|f\|\|(1-\xi x)y\| = \|(1-\xi x)y\|.$$

Setting $g(y') := f(y'y)$ for all $y' \in \mathfrak{R}$, we have $g \in D(\mathfrak{R}, 1)$ and hence,

$$\begin{aligned} \mathrm{Re}\, f((1-\xi x)y) &= \mathrm{Re}\,(f(y)) - \xi \mathrm{Re}\, f(xy) \\ &= 1 - \xi \mathrm{Re}\, g(x) \\ &\geq 1 - \xi\mu \end{aligned}$$

from which the lemma follows. ◇

Lemma 2.1.7 ([8], Lemma 1.2.5) $\mu = \lim_{\xi \to 0^+} \frac{1}{\xi}\{\|1+\xi x\| - 1\}$.

Proof: For $f \in D(\mathfrak{R}, 1)$,

$$f(x) = \frac{1}{\xi}\{f(1+\xi x) - 1\}.$$

Hence,

$$\begin{aligned} \mathrm{Re}\, f(x) &= \frac{1}{\xi}\{\mathrm{Re}\, f(1+\xi x) - 1\} \\ &\leq \frac{1}{\xi}\{\|f(1+\xi x)\| - 1\} \\ &\leq \frac{1}{\xi}\{\|f\|\|1+\xi x\| - 1\} \\ &= \frac{1}{\xi}\{\|1+\xi x\| - 1\} \end{aligned}$$

from which it follows that

$$\mathrm{Re}\, f(x) \leq \inf_{\xi>0} \frac{1}{\xi}\{\|1+\xi x\| - 1\}.$$

Therefore,

$$\mu \leq \inf_{\xi>0} \frac{1}{\xi}\{\|1+\xi x\| - 1\} \tag{2.2}$$

as the left-hand-side does not depend on ξ.

By Lemma 2.1.6, for ξ sufficiently small,

$$\begin{aligned} \|1+\xi x\| &\leq (1-\xi\mu)^{-1}\|1-\xi^2 x^2\| \\ &\leq (1-\xi\mu)^{-1}(1+\xi^2\|x\|^2) \end{aligned}$$

and after some manipulation,

$$\frac{1}{\xi}\{\|1+\xi x\|-1\} \leq \frac{\mu+\xi\|x\|^2}{1-\xi\mu}. \tag{2.3}$$

By (2.3), on taking limits,

$$\lim_{\xi\to 0^+}\frac{1}{\xi}\{\|1+\xi x\|-1\} \leq \mu. \tag{2.4}$$

The lemma now follows by (2.2) and (2.4). $\diamond$

Lemma 2.1.8 ([8], Lemma 1.3.4) *For every $x \in \mathfrak{R}$,*

$$\max\{\mathrm{Re}\, f(x) \,:\, f \in D(\mathfrak{R}, 1)\} = \sup\{\frac{1}{\xi}\log\|\exp(\xi x)\| \,:\, \xi > 0\}.$$

Proof: Fix $x \in \mathfrak{R}$ and set $\lambda(\xi) := \|\exp(\xi x)\| - \|1+\xi x\|$. When $\xi \in [0,1]$,

$$\begin{aligned}
|\lambda(\xi)| &= \big|\|\exp(\xi x)\| - \|1+\xi x\|\big| \\
&\leq \|\exp(\xi x) - (1+\xi x)\| \\
&= \|\sum_{n=2}^{\infty}\frac{\xi^n x^n}{n!}\| \\
&\leq \sum_{n=2}^{\infty}\frac{\xi^n\|x\|^n}{n!} \\
&\leq e^{\|x\|}\xi^2
\end{aligned}$$

For $t > 0$ we have the inequality $\log t \geq \frac{t-1}{t}$ which implies

$$\sup\{\frac{1}{\xi}\log\|\exp(\xi x)\| \,:\, \xi > 0\} \geq \frac{\frac{1}{\xi}\{\|1+\xi x\|-1\} + \frac{1}{\xi}\lambda(\xi)}{\|1+\xi x\| + \lambda(\xi)},$$

the right-hand-side of which converges to μ as $\xi \to 0^+$.

An induction based on Lemma 2.1.6 shows that

$$\|(1-\xi\mu)^n y\| \geq (1-\xi x)^n\|y\|$$

for all $y \in \mathfrak{R}$ and $n \in \mathbb{N}$, whenever $1 - \xi\mu \geq 0$. On replacing ξ with ξ/n and taking limits,

$$\|\exp(-\xi x)y\| \geq \exp(-\xi\mu)\|y\|.$$

Setting $y := \exp(\xi x)$, we have

$$\|\exp(\xi x)\| \leq \exp(\xi \mu)$$

from which it follows that

$$\sup\{\frac{1}{\xi}\log\|\exp(\xi x)\| \,:\, \xi > 0\} \leq \mu.$$

$\diamond$

We are now ready to prove Theorem 2.1.5.

Proof of Theorem 2.1.5: We have $V_{\mathfrak{R}}(x) \subseteq \mathbb{R}$ if and only if

$$\begin{aligned}\max\{\operatorname{Re} f(ix) \,:\, f \in D(\mathfrak{R},1)\} &= \max\{\operatorname{Re} f(-ix) \,:\, f \in D(\mathfrak{R},1)\} \\ &= 0.\end{aligned}$$

Hence, by Lemma 2.1.8,

$$\log\|\exp(i\xi x)\| = 0 \qquad \text{for all } \xi \in \mathbb{R}$$

and the theorem follows. $\diamond$

2.2 Spectra and Numerical Ranges

Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$. In this section, following [8], we investigate the relationship between $V_{\mathfrak{R}}(x)$ and the spectrum $\sigma_{\mathfrak{R}}(x)$ of x in $\mathfrak{R}$, as defined below. In particular, we show (Corollary 2.2.8) that when $\mathfrak{R}$ is a unital C^*-algebra the convex hull of $\sigma_{\mathfrak{R}}(x)$ coincides with $V_{\mathfrak{R}}(x)$.

Definition 2.2.1 *Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$. The two sets*

$$\sigma^L_{\mathfrak{R}}(x) := \{\lambda \in \mathbb{C}^d \,:\, \sum_i \mathfrak{R}\cdot(\lambda_i 1_{\mathfrak{R}} - x_i) \neq \mathfrak{R}\}$$

and

$$\sigma^R_{\mathfrak{R}}(x) := \{\lambda \in \mathbb{C}^d \;:\; \sum_i (\lambda_i 1_{\mathfrak{R}} - x_i) \cdot \mathfrak{R} \neq \mathfrak{R}\}$$

are called respectively the left spectrum *and the* right spectrum *of x with respect to $\mathfrak{R}$. The set $\sigma_{\mathfrak{R}}(x) := \sigma^L_{\mathfrak{R}}(x) \cap \sigma^R_{\mathfrak{R}}(x)$ is called the* spectrum *of x with respect to $\mathfrak{R}$.*

The following lemma is easily verified.

Lemma 2.2.2 *Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a bijective linear mapping and let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$. Then*

$$\sigma_{\mathfrak{R}}(Tx) = T\sigma_{\mathfrak{R}}(x).$$

The following result can be found in [8], Theorem 12, Chapter 1, Section 2. We include the proof for completeness.

Theorem 2.2.3 ([8], Theorem 1.2.12) *Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$. Then $\operatorname{co} \sigma_{\mathfrak{R}}(x) \subseteq V_{\mathfrak{R}}(x)$.*

Proof: Let $\lambda \in \sigma^L_{\mathfrak{R}}(x)$ and $J = \sum_i \mathfrak{R} \cdot (\lambda_i 1_{\mathfrak{R}} - x_i)$. Then $J \subsetneq \mathfrak{R}$. We show that also $\bar{J} \subsetneq \mathfrak{R}$. Let $\eta \in \bar{J}$ and suppose that $\|1_{\mathfrak{R}} - \eta\| < 1$. Then $\{s_n := \sum_{k=1}^n (1-\eta)^k\}$ is a Cauchy sequence in $\mathfrak{R}$. Let $s := \lim s_n \in \mathfrak{R}$. Now,

$$s - s\eta = (1_{\mathfrak{R}} - \eta)s = s - 1_{\mathfrak{R}}$$

and hence $s\eta = 1_{\mathfrak{R}}$. Therefore, $1_{\mathfrak{R}} \in J$ which is false; hence $\|1_{\mathfrak{R}} - \eta\| \geq 1$ for all $\eta \in J$. In particular, $1_{\mathfrak{R}}$ is not in $\bar{J}$. Hence, by the Hahn-Banach theorem, there exists $f \in D(\mathfrak{R}, 1)$ such that $f(\bar{J}) = 0$, and we have $f(\lambda_i 1_{\mathfrak{R}} - x_i) = 0$ from which it follows that $f(x_i) = \lambda_i$ for all i. Therefore $\lambda \in V_{\mathfrak{R}}(x)$. The argument for $\lambda \in \sigma^R_{\mathfrak{R}}(x)$ is analogous. The result now follows from Theorem 2.1.4. $\diamond$

Following [8], we will denote the set of all algebra norms p equivalent to the given norm by N. We write $V^{(p)}_{\mathfrak{R}}(x)$ for the numerical range of x with respect to p.

Lemma 2.2.4 ([8], Lemma 1.2.7) *Let S be a bounded multiplicative semigroup in $\mathfrak{R}$ containing the unit $1_{\mathfrak{R}}$. Then there exists $p \in N$ such that $p(s) \leq 1$ for all $s \in S$.*

Proof: For arbitrary $x \in \mathfrak{R}$ let $q(x) := \sup\{\|sx\| \,:\, s \in S\}$ and define

$$p(x) := \sup\{q(xy) \,:\, y \in \mathfrak{R}, q(y) \leq 1\}.$$

It is straightforward to verify that $p \in N$ and satisfies the required condition. $\diamond$

We recall the **spectral radius** $\rho(x)$ of an element $x \in \mathfrak{R}$, given by the following well-known equivalence [13],

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \max\{|\lambda| \,:\, \lambda \in \sigma_{\mathfrak{R}}(x)\}.$$

The following two results appear in [8], but we include proofs for the sake of completeness.

Lemma 2.2.5 ([8], Lemma 1.2.8) *Let $x_1, \ldots, x_n \in \mathfrak{R}$ be pairwise commuting elements. For each $\epsilon > 0$ there exists $p \in N$ such that*

$$p(x_k) < \rho(x_k) + \epsilon$$

for $k = 1, \ldots, n$.

Proof: Choose $\epsilon > 0$. Define $y_k = \frac{x_k}{\rho(x_k)+\epsilon}$ for $k = 1, \ldots, n$. We have

$$\begin{aligned}
\rho(y_k) &= \lim_{n\to\infty} \|y_k^n\|^{\frac{1}{n}} \\
&= \frac{1}{\rho(x_k)+\epsilon} \lim_{n\to\infty} \|x_k^n\|^{\frac{1}{n}} \\
&= \frac{\rho(x_k)}{\rho(x_k)+\epsilon} \\
&< 1
\end{aligned}$$

for each k. Hence, in particular, there exists $M_1, \ldots, M_n > 0$ such that $\|y_k^l\| < M_k$ for $k = 1, \ldots, n$ and all $l > 0$, and it follows that the multiplicative semigroup S generated by the y_k is bounded. Therefore, by Lemma 2.2.4 there is a $p \in N$ for which $p(y_k) \leq 1$ for all k. The required result follows. $\diamond$

For the next result, we will need the following theorem about convex sets.

Theorem 2.2.6 ([26], Theorem 1.2.10) *Let $X, Y \subseteq \mathbb{C}^n$ be disjoint, non-empty, closed convex sets at least one of which is compact. Then there exists $a, b \in \mathbb{R}$ and a continuous linear functional ρ on $\mathbb{C}^n$ such that*

$$Re\,\rho(w) \leq a < b \leq Re\,\rho(z)$$

for all $w \in X$ and $z \in Y$.

Theorem 2.2.7 ([8], Theorem 1.2.13) *Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$ be a d-tuple of commuting elements. Then*

$$\operatorname{co} \sigma_{\mathfrak{R}}(x) = \bigcap_{p \in N} V_{\mathfrak{R}}^{(p)}(x).$$

Proof: By Theorem 2.2.3,

$$\operatorname{co} \sigma_{\mathfrak{R}}(x) \subseteq \bigcap_{p \in N} V_{\mathfrak{R}}^{(p)}(x).$$

Suppose $\alpha \in \mathbb{C}^d \backslash \operatorname{co} \sigma_{\mathfrak{R}}(x)$. By Theorem 2.2.6 we can find $r \in \mathbb{R}$ and a non-zero linear functional ϕ on $\mathbb{C}^d$ such that

$$Re\,\phi(\zeta) < r < Re\,\phi(\alpha)$$

for all $\zeta \in \operatorname{co} \sigma_{\mathfrak{R}}(x)$. Let T be an invertible $d \times d$ matrix with first row $(t_1, \ldots, t_d)$ determined by the requirement $\phi(\zeta) = t_1\zeta_1 + \cdots + t_d\zeta_d$ for all $\zeta = (\zeta_1, \ldots, \zeta_d) \in \mathbb{C}^d$. If $\zeta \in \sigma_{\mathfrak{R}}(T^{-1}x)$ then

$$Re\,\zeta_1 < r < Re\,\beta_1$$

where $\beta = (\beta_1, \ldots, \beta_d) := T^{-1}\alpha$. Hence, by the compactness of $\sigma_{\mathfrak{R}}(x)$, there is an open polydisc $D \subseteq \mathbb{C}^d$ with $\sigma_{\mathfrak{R}}(T^{-1}x) \subseteq D$ but $T^{-1}\alpha \in \mathbb{C}^d \setminus D$. More precisely, we can find $c_1, \ldots, c_d \in \mathbb{C}$ and $R_1, \ldots, R_d > 0$ such that

$$|\lambda_i - c_i| < r_i$$

for all $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_{\mathfrak{R}}(T^{-1}x)$ and $i = 1, \ldots, d$. Let $y = (y_1, \ldots, y_d) := T^{-1}(x)$. We have,

$$\begin{aligned}\rho(y_i - c_i) &= \max\{|\lambda| \,:\, \lambda \in \sigma_{\mathfrak{R}}(y_i - c_i)\} \\ &= \max\{|\lambda - c_i| \,:\, \lambda \in \sigma_{\mathfrak{R}}(y_i)\} \\ &< r_i.\end{aligned}$$

Employing Lemma 2.2.5 we have $p \in N$ satisfying $p(y_i - c_i) < R_i$ from which it follows that $V_{\mathfrak{R}}^{(p)}(T^{-1}x) \subseteq D$. Hence $T^{-1}\alpha \in \mathbb{C}^d \setminus V_{\mathfrak{R}}^{(p)}(T^{-1}x)$ and by Lemma 2.1.3 $\alpha \in \mathbb{C}^d \setminus V_{\mathfrak{R}}^{(p)}(x)$. This completes the proof. $\diamond$

As the norm on C^*-algebras is unique, we have the following

Corollary 2.2.8 *Suppose that $\mathfrak{R}$ is a unital C^*-algebra and let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$ be a d-tuple of pairwise commuting elements. Then*

$$V_{\mathfrak{R}}(x) = \operatorname{co} \sigma_{\mathfrak{R}}(x).$$

Let $\pi_i : \mathbb{C}^d \to \mathbb{C}$ be the coordinate projection $\pi_i(z) = z_i$ $(i = 1 \ldots, d)$.

Definition 2.2.9 *Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$ and let $y \in \mathfrak{R}$. The* local resolvent *$\rho_{\mathfrak{R}}(y; x)$ of y with respect to x and $\mathfrak{R}$ is the set of points $z \in \mathbb{C}^d$ for which there exists an open neighbourhood U of z and functions $u_1, \ldots, u_d$ holomorphic on U such that*

$$y = \sum_{i=1}^{d} (\pi_i 1_{\mathfrak{R}} - x_i) u_i$$

on U. The local spectrum *$\sigma_{\mathfrak{R}}(y; x)$ of y with respect to x and $\mathfrak{R}$ is the set $\mathbb{C}^d \setminus \rho_{\mathfrak{R}}(y; x)$.*

For each $z \in \mathbb{C}^d$, let $\mathcal{O}_z(\mathfrak{R})$ denote the algebra of $\mathfrak{R}$-valued functions holomorphic in an open neighbourhood of z [13]. The following alternative characterization of left and right spectra is well-known.

Theorem 2.2.10 *Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$. Then*

$$\sigma_{\mathfrak{R}}^{L}(x) := \{z \in \mathbb{C}^d \,:\, \sum_i \mathcal{O}_z(\mathfrak{R}) \cdot (\pi_i 1_{\mathfrak{R}} - x_i) \neq \mathcal{O}_z(\mathfrak{R})\}$$

and

$$\sigma^R_{\mathfrak{R}}(x) := \{z \in \mathbb{C}^d \,:\, \sum_i (\pi_i 1_{\mathfrak{R}} - x_i) \cdot \mathcal{O}_z(\mathfrak{R}) \neq \mathcal{O}_z(\mathfrak{R})\}.$$

Proof: See Lemme 1 in Chapter 1, Section 4 of [10]. $\diamond$

Lemma 2.2.11 *Suppose that $\mathfrak{R}$ is a commutative unital complex Banach algebra with unit $1_{\mathfrak{R}}$. Let $x = (x_1, \dots, x_d) \in \mathfrak{R}^d$. Then*

$$\sigma_{\mathfrak{R}}(x) = \sigma_{\mathfrak{R}}(1_{\mathfrak{R}}; x).$$

Proof: Immediate from Theorem 2.2.10 and definitions. $\diamond$

Proposition 2.2.12 *Let $x = (x_1, \dots, x_d) \in \mathfrak{R}^d$ be a d-tuple of pairwise commuting elements and let $\mathfrak{B}$ be a commutative subalgebra of $\mathfrak{R}$ containing $1_{\mathfrak{R}}, x_1, \dots, x_d$. Then*

$$\sigma_{\mathfrak{B}}(x) = \sigma_{\mathfrak{R}}(x).$$

Proof: Clearly, $\sigma_{\mathfrak{R}}(x) \subseteq \sigma_{\mathfrak{B}}(x)$. For the reverse inclusion, we note that $\sigma_{\mathfrak{B}}(1_{\mathfrak{R}}; x) \subseteq \sigma_{\mathfrak{R}}(x)$ and apply Lemma 2.2.11. $\diamond$

Let $\mathfrak{D}$ be a unital complex Banach algebra with unit $1_{\mathfrak{D}}$. For the remainder of this section we fix $x = (x_1, \dots, x_d) \in \mathfrak{R}^d$. Let E denote the linear span of $1_{\mathfrak{R}}$ and $x_1, \dots, x_d$. Let $\phi : E \to \mathfrak{D}$ be a linear mapping satisifying $\|\phi\| = 1$ and $\phi(1_{\mathfrak{R}}) = 1_{\mathfrak{D}}$. We write $\phi(x)$ for the d-tuple $(\phi(x_1), \dots, \phi(x_d))$. The following lemma will prove important in the sequel.

Lemma 2.2.13 ([2], Lemma 5.3) *The inclusion,*

$$V_{\mathfrak{D}}(\phi(x)) \subseteq V_{\mathfrak{R}}(x)$$

holds with equality if ϕ is an isometry.

Proof: Let $f \in D(\mathfrak{D}, 1)$. We have $\|f \circ \phi\| \leq \|f\|\|\phi\| = 1$ and equality follows from the fact $(f \circ \phi)(1_{\mathfrak{R}}) = 1$. Hence, the Hahn-Banach theorem applies and there exists a functional in $D(\mathfrak{R}, 1)$ that agrees with $f \circ \phi$ on E. If $g \in D(\mathfrak{R}, 1)$ and ϕ is an isometry then $g \circ \phi^{-1} \in D(\mathfrak{D}, 1)$. $\diamond$

Corollary 2.2.14 *Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$ and suppose that $\mathfrak{D}$ is a subalgebra of $\mathfrak{R}$ endowed with the norm of $\mathfrak{R}$ and containing $1_{\mathfrak{R}}, x_1, \ldots, x_d$. Then*

$$V_{\mathfrak{D}}(x) = V_{\mathfrak{R}}(x).$$

Corollary 2.2.15 *Suppose that $\mathfrak{R}$ is a C^*-algebra. Then*

$$\operatorname{co} \sigma_{\mathfrak{D}}(\phi(x)) \subseteq V_{\mathfrak{R}}(x) \tag{2.1}$$

with equality if ϕ is an isometry.

Proof: By Theorem 2.2.8

$$\operatorname{co} \sigma_{\mathfrak{D}}(\phi(x)) = \bigcap_{p \in N} V_{\mathfrak{D}}^{(p)}(\phi(x))$$

and by Lemma 2.2.13, $V_{\mathfrak{D}}^{(p)}(\phi(x)) \subseteq V_{\mathfrak{R}}^{(p)}(x)$ with equality if ϕ is an isometry. Since $\bigcap_{p \in N} V_{\mathfrak{R}}^{(p)}(x) \subseteq V_{\mathfrak{R}}(x)$, the result follows. ◇

The next theorem shows that when $\mathfrak{R}$ is a C^*-algebra and $x = (x_1, \ldots, x_d)$ is a d-tuple of commuting elements, $V_{\mathfrak{R}}(x)$ coincides with the numerical range of x with respect to the closed commutative C^*-subalgebra of $\mathfrak{R}$ generated by $1_{\mathfrak{R}}, x_1, \ldots, x_d$.

Theorem 2.2.16 ([2], Corollary 5.5) *Suppose that $\mathfrak{R}$ is a unital C^*-algebra. Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$ be a d-tuple of pairwise commuting elements. Then*

$$\operatorname{co} \sigma_{\mathfrak{D}}(\phi(x)) \subseteq V_{\mathfrak{D}}(\phi(x)) \subseteq V_{\mathfrak{R}}(x) = \operatorname{co} \sigma_{\mathfrak{B}}(x) = \operatorname{co} \sigma_{\mathfrak{R}}(x)$$

where $\phi(x) := (\phi(x_1), \ldots, \phi(x_d))$ and $\mathfrak{B}$ is any commutative C^*-subalgebra of $\mathfrak{R}$ containing $1_{\mathfrak{R}}, x_1, \ldots, x_d$.

Proof: By Theorem 2.2.3 and Theorem 2.1.4 we have the first inclusion; the second inclusion follows from Lemma 2.2.13; by Corollary 2.2.14 we have $V_{\mathfrak{R}}(x) = V_{\mathfrak{B}}(x)$ and by Corollary 2.2.8 it follows that $V_{\mathfrak{R}}(x) = \operatorname{co} \sigma_{\mathfrak{R}}(x)$. By Proposition 2.2.12, $\sigma_{\mathfrak{B}}(x) = \sigma_{\mathfrak{R}}(x)$. This completes the proof. ◇

2.3 Operating Algebras

In the following, let E be any vector subspace of $\mathfrak{R}$ and let $\mathfrak{D}$ be a commutative complex unital Banach algebra with unit $1_{\mathfrak{D}}$. Let $\phi : E \to \mathfrak{D}$ be a one-to-one linear mapping and denote by $\langle \phi(E) \rangle$ the subalgebra of $\mathfrak{D}$ generated by $\phi(E)$. Let $\mathcal{S}_n$ be the symmetric group of order n.

Definition 2.3.1 ([2]) *A commutative complex unital Banach $\mathfrak{D}$ is called an* operating algebra with respect to E and $\mathfrak{R}$ *if there exists a one-to-one linear mapping $\phi : E \to \mathfrak{D}$ such that $\langle \phi(E) \rangle$ is dense in $\mathfrak{D}$ and a continuous linear mapping $Sym_{\mathfrak{R}}^{(\mathfrak{D},\phi)} : \mathfrak{D} \to \mathfrak{R}$ determined uniquely by the conditions $Sym_{\mathfrak{R}}^{(\mathfrak{D},\phi)}(1_{\mathfrak{D}}) = 1_{\mathfrak{R}}$ and*

$$Sym_{\mathfrak{R}}^{(\mathfrak{D},\phi)}(y_1, \dots y_n) := \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \phi^{-1} y_{\sigma(1)} \cdots \phi^{-1} y_{\sigma(n)} \tag{2.1}$$

for all $y_1, \dots, y_n \in \langle \phi(E) \rangle$, $n \in \mathbb{N}$.

Let $S(E)$ be the symmetric algebra of E and let $\hat{} : E \to S(E)$ be the canonical inclusion mapping. It is well-known that any $\alpha \in S(E)$ can be written in the form,

$$\alpha = a + \sum_{i=1}^{n} \hat{x}_{j_1}^{(i)} \cdots \hat{x}_{j_i}^{(i)}, \tag{2.2}$$

where $a \in \mathbb{C}$, $n, j_1, \dots, j_n \in \mathbb{N}$ and $x_k^{(j)} \in E$.

It can be checked [35] that the formula

$$\|\alpha\|_{S(E)} := \inf \ \{|a| + \sum_{i=1}^{n} \|x_{j_1}^{(i)}\|_{\mathfrak{R}} \cdots \|x_{j_i}^{(i)}\|_{\mathfrak{R}}\}, \tag{2.3}$$

where the infimum is taken over all representations of α of the form (2.2) defines an algebra norm on $S(E)$. The completion of $S(E)$ with respect to this norm is a commutative Banach algebra which we denote $\bar{S}(E)$. For simplicity of notation, we write $Sym_{\mathfrak{R}}^{(E)}$ for $Sym_{\mathfrak{R}}^{(\bar{S}(E),\hat{})}$.

Lemma 2.3.2 *The complete symmetric algebra $\bar{S}(E)$ is an operating algebra with respect to E and $\mathfrak{R}$.*

Proof: As the right-hand-side of (2.1) is symmetric and multilinear, the map $Sym_{\mathfrak{R}}^{(E)}$ is defined on $S(E)$ with $Sym_{\mathfrak{R}}^{(E)}a = a$ for all $a \in \mathbb{C}$. Since $\|Sym_{\mathfrak{R}}^{(E)}\alpha\|_{\mathfrak{R}} \leq \|\alpha\|_{\bar{S}(V)}$, $Sym_{\mathfrak{R}}^{(E)}$ has a unique continuous extension to $\bar{S}(V)$. $\diamond$

Proposition 2.3.3 ([2], Proposition 3.3) *Every operating algebra $\mathfrak{D}$ with respect to E and $\mathfrak{R}$ is isometrically isomorphic to the completion of a quotient of $S(E)$ endowed with some algebra norm.*

Proof: Suppose that $\mathfrak{D}$ admits a mapping $Sym_{\mathfrak{R}}^{(\mathfrak{D},\phi)}$ as above. By the universal property of the symmetric algebra, there exists an algebra homomorphism $h_{\mathfrak{D}} : S(E) \to \mathfrak{D}$ such that $h_{\mathfrak{D}}(\hat{x}) = \phi(x)$. Setting $\rho_{\mathfrak{D}}(\alpha) := \|h_{\mathfrak{D}}\|$ for all $\alpha \in S(E)$, we have a submultiplicative seminorm on $S(E)$ and an induced algebra norm on the quotient $S(E)/\rho^{-1}(0)$, which is isomorphic to $\langle\phi(E)\rangle$ since $\rho^{-1}(0) = \ker h$, and it follows that the completion of $S(E)/\rho^{-1}(0)$ is isometrically isomorphic to $\mathfrak{D}$.

Conversely, since $T := Sym_{\mathfrak{R}}^{(S(E),\hat{\ })}$ is symmetric and multilinear it is well-defined on $S(E)$. If ρ is a submultiplicative seminorm on $S(E)$ with respect to which T is continuous, then we have a unique continuous mapping on the completion of $S(E)/\rho^{-1}(0)$ with respect to ρ. $\diamond$

Lemma 2.3.4 ([2], Lemma 5.1) *Suppose that $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$ is a d-tuple of pairwise commuting elements and let $\mathfrak{D}$ be an operating algebra. Then*

$$\sigma_{\mathfrak{R}}(x) \subseteq \sigma_{\mathfrak{D}}(\phi(x)).$$

Proof: Suppose that $\lambda \in \mathbb{R}^d - \sigma_{\mathfrak{D}}(\phi(x))$. Then there exist $u_1, \ldots, u_d \in \mathfrak{D}$ such that,

$$(\phi(x_1) - \lambda_1 1_{\mathfrak{D}})u_1 + \cdots + (\phi(x_d) - \lambda_d 1_{\mathfrak{D}})u_d = 1_{\mathfrak{D}}. \tag{2.4}$$

As $Sym_{\mathfrak{R}}^{(\mathfrak{D},\phi)}$ is a unital homomorphism, applying it to both sides of (2.4) gives $\lambda \in \mathbb{R}^d \setminus \sigma_{\mathfrak{R}}(x)$. $\diamond$

In the setting of operating algebras we obtain the following strengthening of Theorem 2.2.16:

Theorem 2.3.5 ([2], Theorem 5.7) *Suppose that* $\mathfrak{R}$ *is a* C^**-algebra. Let* $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$ *is a d-tuple of pairwise commuting elements. Let* $\mathfrak{D}$ *be an operating algebra. Then*

$$\operatorname{co}\sigma_{\mathfrak{D}}(\phi(x)) = V_{\mathfrak{D}}(\phi(x)) = V_{\mathfrak{R}}(x) = \sigma_{\mathfrak{B}}(x) = \operatorname{co}\sigma_{\mathfrak{R}}(x)$$

where $\mathfrak{B}$ *is an arbitrary* C^**-subalgebra of* $\mathfrak{R}$ *containing* $1_{\mathfrak{R}}, x_1, \ldots, x_d$.

Proof: This follows immediately from Lemma 2.3.4, Theorem 2.2.16 and the fact that $\sigma_{\mathfrak{B}}(x) \subseteq \sigma_{\mathfrak{R}}(x)$. $\diamond$

Chapter 3

The Algebra of Operants

The Nelson algebra of operants [35] is a generalisation of the symmetric algebra of a vector space to the context of commutative Banach algebras that incorporates spectral theory in a canonical fashion. It was originally introduced as part of framework facilitating a rigorous interpretation of Feynman's operational calculus [19]. We give a mathematical motivation of its construction with the observations below.

In Section 3.1 we describe the theory of the Weyl functional calculus for a d-tuple of hermitian elements of a Banach algebra. This formulation is more general than the original expositions [41, 3, 35] which concern self-adjoint operators on a Hilbert space. We make use here of some results of the previous chapter. In Chapter 4 we will give an explicit formula for the Weyl calculus for hermitian matrices which will used in a fundamental way in the sequel.

In Section 3.2, Nelson's algebra of operants is introduced. We construct it with the view that its spectrum should coincide with the support of the Weyl calculus of its generators, and that it must, in a sense to be made more precise, be canonical among all operating algebras having this property (Theorem 3.2.4). This feature will be used in Chapter 6 to derive the orbit correspondence for compact Lie groups in a manner which is independent of Kirillov's character formula.

For a more general approach, using Clifford analysis, to the Weyl calculus for a d-tuple $A = (A_1, \ldots, A_d)$ of possibly unbounded operators on a Hilbert space subject to the condition that $\xi_1 A_1 + \cdots \xi_d A_d$ is real for all $\xi \in \mathbb{R}^d$, there is the seminal paper of Jefferies et. al. [25] and the monograph of Jefferies [23].

3.1 The Weyl Functional Calculus

Let $\mathfrak{R}$ and $\mathfrak{D}$ be Banach algebras with units $1_{\mathfrak{R}}$ and $1_{\mathfrak{D}}$ respectively. Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$ be a hermitian element and let E be the vector subspace of $\mathfrak{R}$ spanned by $1_{\mathfrak{R}}, x_1, \ldots, x_d$. Let $\phi : E \to \mathfrak{D}$ be a linear mapping with $\|\phi\| = 1$ and $\phi(1_{\mathfrak{R}}) = 1_{\mathfrak{D}}$. By Lemma 2.2.13, $\phi(x)$ is a hermitian element of $\mathfrak{D}$. Hence, for any function $f : \mathbb{R}^d \to \mathbb{C}$ with integrable Fourier transform $\hat{f}$, the following $\mathfrak{D}$-valued Bochner integral converges:

$$f_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(x) := \int_{\mathbb{R}^d} \hat{f}(\lambda) e^{i\lambda \cdot \phi(x)}\, d\lambda.$$

Here the Fourier transform $\hat{f}$ and the normalisation of Lebesgue measure dx are chosen exactly so that when $\mathfrak{R} = \mathfrak{D} = \mathbb{C}$ and ϕ is the identity mapping id then $f(x) := f_{\mathbb{C}}^{(\mathbb{C},\,id)}(x)$ has the usual meaning.

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space [22] of rapidly decreasing functions on $\mathbb{R}^d$. The mapping $W_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(x)$ given by $\langle W_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(x), f\rangle := f_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(x)$ for all $f \in \mathcal{S}(\mathbb{R}^d)$ is continuous and linear on $\mathcal{S}(\mathbb{R}^d)$, and hence defines a $\mathfrak{D}$-valued tempered distribution.

Definition 3.1.1 *The $\mathfrak{D}$-valued tempered distribution $W_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(x)$ is called the* Weyl calculus *of x with respect to $\mathfrak{D}$ and ϕ.*

Let id be the identity mapping on $\mathfrak{R}$. We write $W_{\mathfrak{R}}(x)$ for $W_{\mathfrak{R}}^{(\mathfrak{R},\,id)}(x)$.

The following lemma is proved in [41] using the Trotter product formula, for bounded self-adjoint operators on a Hilbert space. As every C^*-algebra is isometrically isomorphic to a closed subalgebra of bounded operators on some Hilbert space, we are able to state this lemma in slightly more generality.

Lemma 3.1.2 ([41], p.92) *Suppose that $\mathfrak{D}$ is a unital C^*-algebra and let x_1, x_2 be hermitian elements of $\mathfrak{D}$. Then*

$$\|\exp(x_1 + ix_2)\| \leq \|\exp x_1\|.$$

For $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$, let $\Im z := (\Im z_1, \ldots, \Im z_d)$ and $\Re z := (\Re z_1, \ldots, \Re z_d)$. The Hahn-Banach theorem ensures that there are sufficiently many linear functionals to obtain the following vector-valued extension of the usual Paley-Wiener theorem [22].

Theorem 3.1.3 (Paley-Wiener) *An $\mathfrak{R}$-valued tempered distribution u on $\mathbb{R}^d$ has compact support if and only if u is the Fourier transform of an analytic function $e : \mathbb{C}^d \to \mathfrak{R}$ for which there exists constants $C \geq 0$, $s \geq 0$ such that $\|e(\zeta)\|_{\mathfrak{R}} \leq C(1 + |\zeta|)^s e^{r|\Im\zeta|}$ for all $\zeta \in \mathbb{C}^d$.*

Proof: See [22]. $\diamond$

Theorem 3.1.4 *Suppose that $\mathfrak{D}$ is a unital C^*-algebra. Then the distribution $W_{\mathfrak{R}}^{(\mathfrak{D},\phi)}(x)$ has compact support.*

Proof: We verify that the entire function $e(z) := \exp(i\langle z, \phi(x)\rangle)$, for $z \in \mathbb{C}^d$ satisfies the estimate required by Theorem 3.1.3. We have,

$$\begin{aligned}
\|\exp i\langle z, \phi(x)\rangle\| &= \|\exp\left(\langle \Im z,\, \phi(x)\rangle - i\langle \Re z,\, \phi(x)\rangle\right)\| \\
&\leq \|\exp\langle \Im z,\, \phi(x)\rangle\| \\
&\leq \exp\|\langle \Im z,\, \phi(x)\rangle\| \\
&\leq \exp|\Im z|\|\phi(x)\| \\
&\leq \exp|\Im z|\|x\|
\end{aligned}$$

$\diamond$

Let $C^\infty(\mathbb{R}^d)$ be the space of infinitely differentiable $\mathbb{C}$-valued functions on $\mathbb{R}^d$.

Corollary 3.1.5 *Suppose that $\mathfrak{D}$ is a unital C^*-algebra. Then $W_{\mathfrak{R}}^{(\mathfrak{D},\phi)}(x)$ extends uniquely to a distribution on $C^\infty(\mathbb{R}^d)$.*

3.1.1 A Characterisation

The following theorem gives a characterisation of the Weyl calculus distribution by its values on polynomials.

Theorem 3.1.6 ([3], Theorem 2.8) *Suppose that $\mathfrak{R}$ is a unital C^*-algebra and let the monomial $p\colon \mathbb{R}^d \to \mathbb{R}$ be given by $p(\lambda_1, \dots, \lambda_d) := \lambda_1^{k_1} \cdots \lambda_d^{k_d}$ for $k_1, \dots, k_d \in \mathbb{N}$. Then $W_{\mathfrak{R}}(x)$ is the unique $\mathfrak{R}$-valued distribution such that*

$$\langle W_{\mathfrak{R}}(x),\, p\rangle = \frac{k_1! \cdots k_d!}{k!} \sum_{\pi} x_{\pi(1)} \cdots x_{\pi(k)}$$

where $k = k_1 + \cdots + k_d$ and π runs over the set of all maps from $\{1, \dots, k\}$ into $\{1, \dots, d\}$ that assume the value j exactly k_j times for $j = 1, \dots, d$.

We prove Theorem 3.1.6 after recalling some well-known facts from distribution theory, which we prove in the vector-valued setting.

Lemma 3.1.7 ([3], Lemma 2.6) *Suppose that $f \equiv 1$ on $\mathbb{R}^d$ and let u be a compactly supported $\mathfrak{D}$-valued distribution on $\mathbb{R}^d$. Then $\langle u,\, f\rangle = \hat{u}(0)$ where $\hat{u}$ denotes the Fourier transform of u.*

Proof: Let $\theta \in C_c^\infty(\mathbb{R}^d)$ be identically equal to 1 in a neighbourhood of zero. For $\epsilon > 0$ define $\theta_\epsilon(x) := \theta(\epsilon x)$. Since $\hat{u}$ is an analytic function,

$$\begin{aligned}\langle u,\, \theta_\epsilon\rangle &= \langle \hat{u},\, \check{\theta}_\epsilon\rangle \\ &= \int_{\mathbb{R}^d} \hat{u}(\xi)\hat{\theta}_\epsilon(-\xi)\, d\xi \\ &= \frac{1}{\epsilon^n} \int_{\mathbb{R}^d} \hat{u}(\xi)\hat{\theta}(-\frac{\xi}{\epsilon})d\xi.\end{aligned}$$

As

$$\frac{1}{\epsilon^n} \int_{\mathbb{R}^d} \hat{\theta}(-\frac{\xi}{\epsilon})d\xi = \theta(0) = 1$$

for all $\epsilon > 0$, it follows that

$$\begin{aligned}\langle u,\, f\rangle &= \lim_{\epsilon\to 0}\langle u,\, \theta_\epsilon\rangle \\ &= \hat{u}(0).\end{aligned}$$

$\diamond$

Let p be a polynomial in d variables over $\mathbb{C}$. Define the differential operators $\frac{\partial}{\partial\xi} := (\frac{\partial}{\partial\xi_1}, \ldots, \frac{\partial}{\partial\xi_d})$ and $p(i\frac{\partial}{\partial\xi}) := p(i\frac{\partial}{\partial\xi_1}, \ldots, i\frac{\partial}{\partial\xi_d})$.

Theorem 3.1.8 ([3], Lemma 2.7) *Let u be a $\mathfrak{D}$-valued distribution with compact support on $\mathbb{R}^d$. Then*

$$\langle u,\, p\rangle = p(i\frac{\partial}{\partial\xi})\hat{u}(0).$$

Proof: Let $f \equiv 1$ on $\operatorname{supp} u$.Then

$$\langle u,\, p\rangle = \langle u,\, pf\rangle = \langle pu,\, f\rangle = (pu)\hat{}(0) = p(i\frac{\partial}{\partial\xi})\hat{u}(0).$$

$\diamond$

We are now ready to prove Theorem 3.1.6.

Proof of Theorem 3.1.6: By Theorem 3.1.8,

$$\begin{aligned}\langle W_{\mathfrak{R}}(x),\, p\rangle &= \left(p(i\frac{\partial}{\partial\xi})\exp(-i\xi\cdot x)\right)(0) \\ &= \frac{1}{k!}\left(\frac{\partial^{k_1+\cdots+k_d}}{\partial\xi_1^{k_1}\cdots\partial\xi_d^{k_d}}(\xi_1 x_1 + \cdots \xi_d x_d)^k\right) \\ &= \frac{k_1!\cdots k_d!}{k!}\sum_\pi x_{\pi(1)}\cdots x_{\pi(k)}\end{aligned}$$

as required. $\diamond$

The symmetry properties of operating algebras (Section 2.3) come into play in the next definition.

Corollary 3.1.9 *Suppose that $\mathfrak{D}$ is a commutative unital C^*-algebra which is also an operating algebra with respect to E and $\mathfrak{R}$, ϕ is one-to-one, and that $\langle \phi(E) \rangle$ is dense in $\mathfrak{D}$. Then*

$$\langle W_{\mathfrak{R}}(x),\, f\rangle = Sym_{\mathfrak{R}}^{(\mathfrak{D},\phi)} \langle W_{\mathfrak{R}}^{(\mathfrak{D},\phi)}(x),\, f\rangle$$

for all $f \in C^\infty(\mathbb{R}^d)$.

Proof: By Theorem 3.1.6, if $p(\lambda_1, \ldots, \lambda_d) := \lambda^{k_1} \cdots \lambda^{k_d}$ then

$$\begin{aligned} \langle W_{\mathfrak{R}}^{(\mathfrak{D},\phi)}(x),\, p\rangle &= \left(p(i\frac{\partial}{\partial \xi}) \exp(-i\xi \cdot \phi)(x) \right)(0) \\ &= \frac{1}{k!} \frac{\partial^{k_1 + \cdots k_d}}{\partial \xi^{k_1} \cdots \partial \xi^{k_d}} (\xi_1 \phi(x_1) + \cdots \xi_d \phi(x_d))^k \\ &= \phi(x_1)^{k_1} \cdots \phi(x_d)^{k_d} \end{aligned}$$

Hence

$$\begin{aligned} Sym_{\mathfrak{R}}^{(\mathfrak{D},\phi)} \langle W_{\mathfrak{R}}^{(\mathfrak{D},\phi)}(x),\, p\rangle &= Sym_{\mathfrak{R}}^{(\mathfrak{D},\phi)} \phi(x_1)^{k_1} \cdots \phi(x_d)^{k_d} \\ &= \langle W_{\mathfrak{R}}(x),\, p\rangle. \end{aligned}$$

The last equality follows from Theorem 3.1.6. Hence, by linearity, the distributions $W_{\mathfrak{R}}(x)$ and $Sym_{\mathfrak{R}}^{(\mathfrak{D},\phi)} \circ W_{\mathfrak{R}}^{(\mathfrak{D},\phi)}(x)$ agree for all polynomials and hence for all $f \in C^\infty(\mathbb{R}^d)$. This completes the proof. $\diamond$

Corollary 3.1.10 *Suppose that $f \in C^\infty(\mathbb{R}^d)$ and for some $g \in C^\infty(\mathbb{R}^d)$, $d' < d$, $f(\xi_1, \ldots, \xi_d) = g(\xi_1, \ldots, \xi_{d'})$ for all $\xi_1, \ldots, \xi_d \in \mathbb{R}$. Then*

$$\langle W_{\mathfrak{R}}(x),\, f\rangle = \langle W_{\mathfrak{R}}(x'),\, g\rangle$$

where $x' = (x_1, \ldots, x_{d'})$.

3.1.2 Spectral Properties

Recall that the **Gelfand spectrum** $\sigma(\mathfrak{D})$ of the commutative Banach algebra $\mathfrak{D}$ is the set of continuous non-zero multiplicative linear functionals on $\mathfrak{D}$, endowed with the weak* topology. By Theorem 2.2.3 these functionals are necessarily real-valued on hermitian elements.

The argument for the following proposition comes from the proof of [35, Theorem 8].

Proposition 3.1.11 ("Spectral Mapping") *Let $\mathfrak{D}$ be a commutative Banach algebra. Let $\psi \in \sigma(\mathfrak{D})$. Then for any $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$\psi(\langle W_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(x),\, f\rangle) = (f \circ \psi \circ \phi)(x).$$

Proof: By Fourier inversion and the properties of the Bochner integral, since $\psi \in \sigma(\mathfrak{D})$,

$$\begin{aligned}
(f \circ \psi \circ \phi)(x) &= \int_{\mathbb{R}^d} \hat{f}(\lambda) e^{i\lambda\cdot(\psi\circ\phi)(x)}\, d\lambda \\
&= \int_{\mathbb{R}^d} \hat{f}(\lambda)\psi(e^{i\lambda\cdot\phi(x)}) d\lambda \\
&= \psi\left(\int_{\mathbb{R}^d} \hat{f}(\lambda) e^{i\lambda\cdot\phi(x)} d\lambda\right) \\
&= \psi(\langle W_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(x),\, f\rangle).
\end{aligned}$$

$\diamond$

Lemma 3.1.12 *Let $f_1,\, f_2 \in \mathcal{S}(\mathbb{R}^d)$. Then*

$$\langle W_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(x),\, f_1 f_2\rangle = \langle W_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(x),\, f_1\rangle \langle W_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(x),\, f_2\rangle.$$

Proof: Since $\mathfrak{D}$ is commutative,

$$\begin{aligned}
\langle W_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(x) f_1,\, f_2\rangle &= \int (f_1 f_2)\hat{\ }(\lambda) e^{i\lambda\cdot\phi(x)}\, d\lambda \\
&= \int \hat{f}_1 * \hat{f}_2(\lambda) e^{i\lambda\cdot\phi(x)}\, d\lambda \\
&= \int\int \hat{f}_1(\lambda - t)\hat{f}_2(t) e^{i(\lambda-t)\cdot\phi(x)} e^{it\cdot\phi(x)}\, d\lambda dt \\
&= \int \hat{f}_1(\lambda) e^{i\lambda\cdot\phi(x)}\, d\lambda \int \hat{f}_2(\lambda) e^{i\lambda\cdot\phi(x)}\, d\lambda \\
&= \langle W_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(x),\, f_1\rangle \langle W_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(x),\, f_2\rangle
\end{aligned}$$

which completes the proof. $\diamond$

Let $\Omega \subseteq \mathbb{R}^d$ be an open set and write $W_{\mathfrak{R}}^{(\mathfrak{D},\phi)}(x)|_\Omega$ for the restriction of the distribution $W_{\mathfrak{R}}^{(\mathfrak{D},\phi)}(x)$ to the space $C^\infty(\Omega)$ of test functions supported on Ω. The proof of the next theorem is based on an argument in the proof of [35, Theorem 8].

Theorem 3.1.13 *Suppose that $\mathfrak{D}$ is a unital commutative C^*-algebra. Let Ω be an open subset of $\mathbb{R}^d$. Then*

$$\operatorname{supp} W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x)|_\Omega = \sigma_{\mathfrak{F}}(\phi(x)) \cap \Omega$$

where $\mathfrak{F}$ is any closed subalgebra of $\mathfrak{D}$ containing $1_{\mathfrak{D}}, \phi(x_1), \ldots, \phi(x_d)$.

Proof: We identify the spectrum $\sigma_{\mathfrak{F}}(\phi(x))$ of $\phi(x)$ in $\mathfrak{F}$ with the set $\{(\psi \circ \phi)(x) | \, \psi \in \sigma(\mathfrak{F})\}$ and recall that $\sigma_{\mathfrak{F}}(\phi(x))$ is a compact subset of $\mathbb{R}^d$.

Hence if $\lambda \in \sigma_{\mathfrak{F}}(\phi(x))$ then $(\psi \circ \phi)(\phi(x)) = \lambda$ for some $\psi \in \sigma(\mathfrak{F})$. If $\lambda \notin \operatorname{supp} W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x)$ then there exists $f \in \mathcal{S}(\mathbb{R}^d)$ with $\langle W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x),\, f\rangle = 0$ such that,

$$0 \neq f(\lambda) = (f \circ \psi \circ \phi)(x) = \psi(\langle W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x),\, f\rangle) = \psi(0) = 0,$$

which is a contradiction. Therefore $\sigma_{\mathfrak{F}}(\phi(x)) \subseteq \operatorname{supp} W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x)$ and it follows that $\sigma_{\mathfrak{F}}(\phi(x)) \cap \Omega \subseteq \operatorname{supp} W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x)|_\Omega$.

For the reverse inclusion, suppose that $\lambda \in \operatorname{supp} W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x)$ but $\lambda \notin \sigma_{\mathfrak{F}}(\phi(x))$. Then since $\sigma_{\mathfrak{F}}(\phi(x))$ is compact we can find $f \in \mathcal{S}(\mathbb{R}^d)$ such that $f(\lambda) \neq 0$, $\langle W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x),\, f\rangle \neq 0$ and f vanishes on $\sigma_{\mathfrak{F}}(\phi(x))$. Then for all $\psi \in \sigma(\mathfrak{F})$,

$$\psi(\langle W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x),\, f\rangle) = (f \circ \psi \circ \phi)(x) = 0$$

from which it follows that $\langle W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x),\, f\rangle$ is in the radical of $\mathfrak{F}$ and that

$$\lim_{n\to\infty} \|\langle W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x),\, f\rangle^n\|^{\frac{1}{n}} = 0.$$

We show that $\langle W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x),\, f\rangle = 0$. Choose $g \in \mathcal{S}(\mathbb{R}^d)$ such that $g(\sigma_{\mathfrak{F}}(\phi(x))) =$

$\{0\}$ and $g(\operatorname{supp} f) = \{1\}$. Then for all $m > 0$,

$$\begin{aligned} \|\langle W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x),\, f\rangle\| &= \|\langle W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x),\, g^m f\rangle\| \\ &\leq \|\langle W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x),\, g\rangle\|^m \|\langle W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x),\, f\rangle\| \end{aligned}$$

and repeating the argument above with g in place of f we have that $\langle W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x),\, g\rangle$ is in the radical of $\mathfrak{F}$ and hence that $\|\langle W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x),\, g\rangle\|^m < 1$ for some m. This implies that $\langle W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x),\, f\rangle = 0$ which is a contradiction. Hence $\operatorname{supp} W_{\mathfrak{R}}^{(\mathfrak{F},\phi)}(x)|_\Omega \subseteq \sigma_{\mathfrak{F}}(\phi(x)) \cap \Omega$ and the result follows. ◇

Lemma 3.1.14 *Let $x = (x_1, \ldots, x_d)$ be a d-tuple of hermitian elements of $\mathfrak{R}$. All derivatives of the entire function* $\exp i\xi \cdot x$ *are bounded as functions of ξ, for $\xi \in \mathbb{R}^d$.*

Proof: See Lemma on p.94 of [41]. ◇

Corollary 3.1.15 *Let $p : \mathbb{R}^d \to \mathbb{C}$ be a polynomial. Then*

$$\langle W_{\mathfrak{R}}^{(\mathfrak{D},\phi)}(x),\, fp\rangle = \langle W_{\mathfrak{R}}^{(\mathfrak{D},\phi)}(x),\, f\rangle \langle W_{\mathfrak{R}}^{(\mathfrak{D},\phi)}(x),\, p\rangle$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Proof: By Lemma 3.1.14 we can apply integration by parts to obtain,

$$\begin{aligned} \langle W_{\mathfrak{R}}^{(\mathfrak{D},\phi)}(x),\, fp\rangle &= \int_{\mathbb{R}^d} (fp)\hat{}(\xi) e^{i\xi\cdot\phi(x)}\, d\xi \\ &= \int_{\mathbb{R}^d} p\left(i\frac{\partial}{\partial\xi}\right) \hat{f}(\xi) e^{i\xi\cdot\phi(x)}\, d\xi \\ &= \int_{\mathbb{R}^d} \hat{f}(\xi) \left[p\left(-i\frac{\partial}{\partial\xi}\right) e^{i\xi\cdot\phi(x)}\right] d\xi \\ &= \langle W_{\mathfrak{R}}^{(\mathfrak{D},\phi)}(x),\, f\rangle \langle W_{\mathfrak{R}}^{(\mathfrak{D},\phi)}(x),\, p\rangle \end{aligned}$$

where the last equality follows by the same argument as in the proof of Corollary 3.1.9 ◇

Alternatively, Corollary 3.1.15 can be seen as a consequence of Lemma 3.1.12 and the continuity of $W_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(x)$.

Let $I := I(\mathfrak{D},\,\phi) = \{y_1 \in \mathfrak{D} \,|\, Sym_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(y_1 y_2) = 0 \text{ for all } y_2 \in \mathfrak{D}\}$. Then I is a closed ideal of $\mathfrak{D}$. Set $\mathfrak{D}' := \mathfrak{D}/I$ and let $\phi' : E \to \mathfrak{D}$ be the map induced by ϕ.

Lemma 3.1.16 *If $\mathfrak{D}$ is an operating algebra with respect to E and $\mathfrak{R}$ then $\mathfrak{D}'$ is also an operating algebra with respect to E and $\mathfrak{R}$.*

Proof: Clearly $\phi'(1_{\mathfrak{R}}) = 1_{\mathfrak{D}'}$ and $\langle \phi'(E) \rangle$ is dense in $\mathfrak{D}'$. Since $Sym_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}$ vanishes on I, the map $Sym_{\mathfrak{R}}^{(\mathfrak{D}',\,\phi)}$ is well-defined and has all the properties required by Definition 2.3.1. $\diamond$

The following theorem is a consequence of general facts about non-analytic functional calculi of operators [2, Theorem 5.10]; our proof is based on an argument of Nelson [35, Theorem 8], from which the restriction property (3.1) can be more easily observed.

Theorem 3.1.17 ([2], Theorem 5.10; [35], Theorem 8) *Suppose that $\mathfrak{D}$ is an unital commutative C^*-algebra and an operating algebra with respect to E and $\mathfrak{R}$. Let Ω be an open subset of $\mathbb{R}^d$. Then*

$$\operatorname{supp} W_{\mathfrak{R}}^{(\mathfrak{D}',\,\phi')}(x)|_{\Omega} = \operatorname{supp} W_{\mathfrak{R}}(x)|_{\Omega} = \sigma_{\mathfrak{D}'}(\phi'(x)) \cap \Omega. \tag{3.1}$$

Proof: Since $\|\phi'\| = 1$, the distribution $W_{\mathfrak{R}}^{(\mathfrak{D}',\,\phi)}(x)$ is well-defined. The equality $\operatorname{supp} W_{\mathfrak{R}}^{(\mathfrak{D}',\,\phi')}(x)|_{\Omega} = \sigma_{\mathfrak{D}'}(\phi'(x)) \cap \Omega$ follows from Theorem 3.1.13. Let $\lambda \in \operatorname{supp} W_{\mathfrak{R}}(x)|_{\Omega}$. Then for every neighbourhood U of λ with $U \subseteq \Omega$ there exists $f \in \mathcal{S}(\Omega)$ such that $\operatorname{supp} f \subseteq U$ and $\langle W_{\mathfrak{R}}(x),\, f \rangle \neq 0$. Since $W_{\mathfrak{R}} \equiv Sym_{\mathfrak{R}}^{(\mathfrak{D}',\,\phi')} \circ W_{\mathfrak{R}}^{(\mathfrak{D}',\,\phi')}$ by Corollary 3.1.9, it follows that $\langle W_{\mathfrak{R}}^{(\mathfrak{D}',\,\phi')}(x),\, f \rangle \neq 0$. Hence $\operatorname{supp} W_{\mathfrak{R}}(x)|_{\Omega} \subseteq \operatorname{supp} W_{\mathfrak{R}}^{(\mathfrak{D}',\,\phi')}(x)|_{\Omega}$.

Suppose that $\lambda \in \sigma_{\mathfrak{D}'}(\phi'(x)) \cap \Omega$ but $\lambda \notin \operatorname{supp} W_{\mathfrak{R}}(x)|_{\Omega}$. Then there exists $f \in \mathcal{S}(\Omega)$ such that $f(\lambda) \neq 0$ and $\langle W_{\mathfrak{R}}(x),\, fp \rangle = 0$ for all d-variable polynomials $p:\mathbb{R}^d\to\mathbb{C}$. By Corollary 3.1.15,

$$\begin{aligned}\int_{\mathbb{R}^d}(fp)\hat{}(\lambda)e^{i\lambda\cdot x}\,d\lambda &= Sym_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}\int_{\mathbb{R}^d}(fp)\hat{}(\lambda)e^{i\lambda\cdot\phi(x)}\,d\lambda\\ &= Sym_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(\langle W_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(x),\,f\rangle\langle W_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(x),\,p\rangle)\\ &= 0\end{aligned}$$

and it follows that $Sym_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(\langle W_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(x),\,f\rangle\,y) = 0$ for all $y\in\mathfrak{D}$. Thus $\langle W_{\mathfrak{R}}^{(\mathfrak{D}',\,\phi')}(x),\,f\rangle = 0$. Since $\lambda\in\sigma_{\mathfrak{D}'}(\phi'(x))$ we can find $\psi\in\sigma(\mathfrak{D}')$ such that

$$\begin{aligned}(\psi\circ W_{\mathfrak{R}}^{(\mathfrak{D}',\,\phi')}(x))f &= (f\circ\psi\circ\phi')(x)\\ &= f(\lambda)\\ &\neq 0\end{aligned}$$

which implies that $\langle W_{\mathfrak{R}}^{(\mathfrak{D}',\,\phi')}(x),\,f\rangle\neq 0$, a contradiction. Therefore, $\sigma_{\mathfrak{D}'}(\phi'(x))\cap\Omega\subseteq\operatorname{supp}W_{\mathfrak{R}}(x)|_{\Omega}$. This completes the proof. $\diamond$

3.2 A Canonical Quotient Operating Algebra

Suppose that V is a normed vector subspace of $\mathfrak{R}$ containing $1_{\mathfrak{R}}$. As in Section (2.3), denote by $S(V)$ the symmetric algebra of V and by $\bar{S}(V)$ be the completion of $S(V)$ with respect to norm (2.3). Let $\hat{}:V\to\bar{S}(V)$ be the canonical inclusion mapping.

Definition 3.2.1 *Let $\mathcal{I}=\mathcal{I}(V):=\{\alpha\in\bar{S}(V)\,|\,Sym_{\mathfrak{R}}^{(V)}(\alpha\beta)=0$ for all $\beta\in\bar{S}(V)\}$. The quotient $\mathcal{A}=\mathcal{A}(V):=\bar{S}(V)/\mathcal{I}$ is called the Nelson algebra of* operants *over V.*

Let $\tilde{}:V\to\mathcal{A}$ be the canonical projection.

Lemma 3.2.2 $\tilde{1}_{\mathfrak{R}}=1_{\mathcal{A}}$.

Proof: The unit in $\mathcal{A}$ is clearly the image of $1 \in \mathbb{C}$. However, for any $\alpha \in S(V)$,

$$\begin{aligned} Sym_{\mathfrak{R}}^{(V)}(1 - \hat{1}_{\mathfrak{R}})\alpha &= Sym_{\mathfrak{R}}^{(V)}(\alpha - \hat{1}_{\mathfrak{R}}\alpha) \\ &= Sym_{\mathfrak{R}}^{(V)}\alpha - Sym_{\mathfrak{R}}^{(V)}\hat{1}_{\mathfrak{R}}\alpha \\ &= 0. \end{aligned}$$

By continuity, $Sym_{\mathfrak{R}}^{(V)}(1 - \hat{1}_{\mathfrak{R}})\alpha = 0$ for all $\alpha \in \bar{S}(V)$. The result follows. $\diamond$

Lemma 3.2.3 *The maps* $\hat{}: E \to S(E)$ *and* $\tilde{}: E \to \mathcal{A}(E)$ *are isometries. Furthermore,* $\|\hat{}\| = \|\tilde{}\| = 1$.

Proof: By equation (2.3), $\|\hat{t}\|_{S(V)} \leq \|t\|_{\mathfrak{R}}$. However, since $\hat{t}$ is a homogeneous element of degree one in $S(V)$, $\hat{t}$ must have the form $\hat{t} = \lambda_1\hat{t}_1 + \cdots + \lambda_n\hat{t}_n$ for some choice of $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, not necessarily unique. By the triangle inequality, $\|t\|_{\mathfrak{R}} \leq |\lambda_1|\|t_1\|_{\mathfrak{R}} + \cdots + |\lambda_n|\|t_n\|_{\mathfrak{R}}$. Hence $\|\hat{t}\|_{S(V)} \geq \|t\|_{\mathfrak{R}}$ and it follows that $\|\hat{t}\|_{S(V)} = \|t\|_{\mathfrak{R}}$. Now $\|\tilde{t}\|_{\mathcal{A}} = \inf_{i \in \mathcal{I}} \|\hat{t} + i\|_{\bar{S}(V)} \leq \|\hat{t}\|_{S(V)} + \inf_{i \in \mathcal{I}} \|i\|_{\bar{S}(V)} = \|t\|_{\mathfrak{R}}$. Also, $\|t\|_{\mathfrak{R}} = \|Sym_{\mathfrak{R}}^{(V)}\tilde{t}\|_{\mathfrak{R}} \leq \|\tilde{t}\|_{\mathcal{A}}$. Therefore, $\|\tilde{t}\|_{\mathfrak{R}} = \|\tilde{t}\|_{\mathcal{A}}$.

We have,

$$\|\tilde{}\| = \sup\{\|\tilde{t}\|_{\mathcal{A}} : t \in V, \|t\|_{\mathfrak{R}} = 1\} = \sup\{\|\tilde{t}\|_{\mathfrak{R}} : t \in V, \|t\|_{\mathfrak{R}} = 1\} = 1.$$

Similarly, $\|\hat{}\| = 1$. This completes the proof. $\diamond$

We thus have the following

Theorem 3.2.4 ([35], Theorem 8) *Suppose that* $\mathfrak{R}$ *is a unital* C^**-algebra. Let* Ω *be an open subset of* $\mathbb{R}^d$. *Then* $\mathcal{A}(E)$ *is the freest quotient operating algebra with respect to* E *and* $\mathfrak{R}$ *such that*

$$\operatorname{supp} W_{\mathfrak{R}}^{(\bar{S}(E),\hat{})}(x)|_{\Omega} = \operatorname{supp} W_{\mathfrak{R}}(x)|_{\Omega} = \sigma_{\mathcal{A}}(\tilde{x}) \cap \Omega. \tag{3.1}$$

Proof: By Lemma 2.3.2, $\bar{S}(E)$ is an operating algebra with respect to E and $\mathfrak{R}$, hence by Lemma 3.1.16 $\mathcal{A}$ is an operating algebra with respect to E and

$\mathfrak{R}$. Since $\|\tilde{}\| = 1$, the distribution $W_{\mathfrak{R}}^{(\mathcal{A}(E),\tilde{})}(x)$ is defined and equality (3.1) follows from Theorem 3.1.17. As by Proposition 2.3.3 all operating algebras with respect to E and $\mathfrak{R}$ are completions of quotients of $S(E)$ for some norm, it follows that $\mathcal{A}$ is the freest $(E, \mathfrak{R})$-operating algebra for which (3.1) holds. $\diamond$

Example 3.2.5 *Let* $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *Set* $E := Span_{\mathbb{C}}\{I, A\}$. *Then* $S(E) \cong \mathbb{C}[A]$. *Since* $A^2 - I = 0$ *and* $Sym := Sym_{L(\mathbb{C}^2)}^{(E)}$ *is a homomorphism we have* $(\hat{A}^2 - \hat{I})\bar{S}(E) \subseteq \mathcal{I}(E)$. *On the other hand, if* $c_0\hat{I} + c_1\hat{A} + \cdots + c_n\hat{A}^n \in \mathcal{I}(E)$ *then*

$$\begin{aligned} c_0 I + c_1 A + \cdots + c_n A^n &= Sym(c_0\hat{I} + c_1\hat{A} + \cdots c_n\hat{A}^n) \\ &= 0 \end{aligned}$$

and it follows that $\mathcal{I}(E) = (1 - \hat{I})\bar{S}(E) + (\hat{A}^2 - \hat{I})\bar{S}(E)$ *by continuity.*

Suppose that $\phi \in \sigma(\bar{S}(E))$. *Then in particular* $\phi(\mathcal{I}(E)) = \{0\}$, *or equivalently* $\phi(1 - \hat{I}) = \phi(\hat{A}^2 - I) = 0$; *hence* $\phi(\hat{I}) = 1$ *and* $\phi(A) = \{-1, 1\}$, *and it follows that* $\sigma(\hat{A}) = \{-1, 1\}$.

However, for $f \in \mathcal{S}(\mathbb{R})$ *we have*

$$\begin{aligned} \langle W_{L(\mathbb{C}^2)}(A), f\rangle &= \int_{\mathbb{R}} \hat{f}(\xi)e^{i\xi A}\, d\xi \\ &= \begin{pmatrix} f(1) & 0 \\ 0 & f(-1) \end{pmatrix} \end{aligned}$$

and it follows that $\operatorname{supp} W_{L(\mathbb{C}^2)}(A) = \{-1, 1\}$. *Hence we have shown that* $\sigma(A) = \operatorname{supp} W_{L(\mathbb{C}^2)}(A)$.

Let V be a finite dimensional complex normed vector space.

Lemma 3.2.6 ([35], Theorem 1) *The spectrum* $\sigma(\bar{S}(V))$ *of the complete symmetric algebra* $\bar{S}(V)$ *of* V *is homeomorphic to a subset of the unit ball in* V^*.

Proof: Observing that the compact set $\sigma(S(V))$ and the unit ball in V^* are Hausdorff spaces, it suffices to exhibit a continuous bijection between them. Define $\psi : \sigma(S(V)) \rightarrow V^*$ by $\psi(\phi)(v) := \phi(\hat{v})$ for all $v \in V$. Since $\|v\| = \|\hat{v}\|$ for all $v \in V$ and $\|\phi\| \leq 1$ we have that $\|\psi(\phi)\| \leq 1$ for all $\phi \in \sigma(S(V))$ and hence that ψ maps into the unit ball in V^*. Now suppose that $\psi(\phi) = \psi(\phi_1)$. Then $\phi(\tilde{A}) = \phi_1(\tilde{A})$ for all $A \in V$. Therefore, ϕ and ϕ_1 agree on $S_0(V)$. However, since $\|\phi\|$, $\|\phi_1\| \leq 1$, it follows that $\phi = \phi_1$. Thus ψ is a bijection. Suppose that $\phi_n \rightarrow \phi$ in $\sigma(S(V))$. Then,

$$\begin{aligned}
\|\psi\phi_n - \psi\phi\| &= \sup_{\|A\|=1} \|(\psi\phi_n - \psi\phi)(A)\| \\
&= \sup_{\|A\|=1} \|\phi_n(\tilde{A}) - \phi(\tilde{A})\| \\
&\leq \|\phi_n - \phi\|
\end{aligned}$$

and it follows that $\psi\phi_n \rightarrow \psi\phi$. Therefore ψ is continuous; this completes the proof. $\diamond$

Corollary 3.2.7 *The spectrum $\sigma(\mathcal{A})$ of the algebra $\mathcal{A}$ of operants over V is homeomorphic to a closed subset of the unit ball in V^*.*

In the remainder of this chapter we determine the spectrum $\sigma_{\mathcal{A}}(\tilde{x})$ in the case $\mathfrak{R} = V = L(\mathcal{H})$ is the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. The following "lower bound" is due to E. Nelson.

Theorem 3.2.8 ([35], Theorem 5) *Let $A = (A_1 \ldots, A_d)$ be a d-tuple of bounded linear operators on a Hilbert space $\mathcal{H}$ and define for each $u \in \mathcal{H}$ such that $\|u\| = 1$, $E_u \in L(\mathcal{H})$ given by $E_u(v) := (u, v)u$ for all $v \in \mathcal{H}$. Then*

$$\{(Au, u) : \|u\| = 1,\, E_u \in Span_{\mathbb{C}}\{A_1, \ldots, A_d\}\} \subseteq \sigma_{\mathcal{A}}(\tilde{A}) \tag{3.2}$$

where $(Au, u) := ((A_1u, u), \ldots, (A_du, u))$.

Proof: Fix $u \in \mathcal{H}$ with $\|u\| = 1$. We note that $E^2 = E$ and by the Cauchy-Schwartz inequality $\|E\| = \sup_{\|v\|=1} \|Ev\| = \sup_{\|v\|=1} |(u, v)| \leq 1$ from which it follows that $\|E\| = 1$, since $\|E\| = |(u, v)| = 1$.

We show that the map $\psi : \bar{S}(L(\mathcal{H})) \to \mathbb{C}$ given by

$$\psi(\alpha) = \lim_{m\to\infty} (u, T(\tilde{E}^m \alpha)u) \tag{3.3}$$

for all $\alpha \in \bar{S}(L(\mathcal{H}))$ is well-defined, vanishes on $\mathcal{I}$ and consequently induces a unique functional in $\sigma(\mathcal{A})$ sending $\tilde{A}$ to (Au, u) for all $A \in L(\mathcal{H})$.

Let $A_1, \ldots, A_m \in L(\mathcal{H})$. Of the $(m+n)!$ terms in the expansion of $T\tilde{E}^m \tilde{A}_1 \cdots \tilde{A}_n$ exactly $m!n!C_n^{m-n}$ have at least one E before and after each A_i; therefore, the proportion of terms not of this form can be made arbitrarily small for m sufficiently large and we have,

$$\begin{aligned}\lim_{n\to\infty} T\tilde{E}^m \tilde{A}_1 \cdots \tilde{A}_n &:= T(EA_1E)\tilde{} \cdots (EA_nE)\tilde{} \\ &= (u, A_1u) \cdots (u, A_nu)E.\end{aligned}$$

Since the linear span of terms of the form $\tilde{A}_1 \cdots \tilde{A}_n$ is dense in $\bar{S}(L(\mathcal{H}))$ and $\|T\tilde{E}^m\alpha\| \leq \|\tilde{E}^m\alpha\| \leq \|\alpha\|$ for each $m > 0$ and $\alpha \in S(L(\mathcal{H}))$, it follows that ψ is well-defined. Clearly if $\alpha \in \mathcal{I}$ then $T\tilde{E}^m\alpha = 0$, so ψ factors through to $\mathcal{A}$. $\diamond$

Chapter 4

A Formula for the Weyl Calculus

Let $\mathcal{H}$ be an n-dimensional Hilbert space and let $A := (A_1, \dots, A_d)$ be a d-tuple of bounded operators on $\mathcal{H}$. In this chapter we determine the Weyl calculus $W_{L(\mathcal{H})}(A)$ of A, under special circumstances.

In Section 4.1 we verify Nelson's explicit formula (Theorem 4.1.1) for $W_{L(\mathcal{H})}(A)$ when the d-tuple A consists of hermitian matrices, by-passing the intricate argument involving recurrence relations and induction that appears in [35]. This will be used in fundamental way in the sequel. We show in Section 4.2 that this formula is a special case of a more general expression due to B. Jefferies [23], valid for any d-tuple A of arbitrary complex matrices. It would be interesting to see how the results of Chapter 6 concerning unitary representations of compact Lie groups may generalise to non-unitary representations using Jefferies' formula.

4.1 Nelson's Formula for Hermitian Matrices

Define $\Sigma_n := \{u \in \mathcal{H} : \|u\| = 1\}$ and let ν denote the unitarily invariant probability measure on Σ_n. Define the *joint numerical range* map,

$$W_A : \Sigma_n \to \mathbb{R}^d : u \longmapsto (\langle A_1 u, u\rangle, \dots, \langle A_d u, u\rangle)$$

and set $\mu_A := \nu \circ W_A^{-1}$. Choose a basis of $\mathcal{H}$ so that the A_j are represented by matrices. Let $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}$ be the operators of partial differentiation on $\mathbb{R}^d$. Define $A \cdot \frac{\partial}{\partial x} = A_1 \frac{\partial}{\partial x_1} + \cdots + A_d \frac{\partial}{\partial x_d}$. For any matrix differential operator M acting on the space of distributions on $C_c^\infty(\mathbb{R}^d)$ we denote by $\phi_j(M)$ the sum of the principal minors of M of order j.

Theorem 4.1.1 ([35], Theorem 9) *Let $A := (A_1, \ldots, A_d)$ be a d-tuple of bounded self-adjoint operators on a n-dimensional Hilbert space $\mathcal{H}$. The Weyl calculus for A is given by*

$$W_{L(\mathcal{H})}(A) = \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \sum_{m=0}^{j} (-1)^{j+k+m+1} \binom{j}{m} \frac{(n-1)!}{(n-1-j+m)!} \times \left(A \cdot \frac{\partial}{\partial x} \right)^k \phi_{n-j-k-1}(A \cdot \frac{\partial}{\partial x})(\frac{\partial}{\partial x} \cdot x)^m \mu_A. \tag{4.1}$$

Corollary 4.1.2 *Suppose that $A := (A_1, \ldots, A_d)$ is a basis of the vector space of bounded self-adjoint operators on $\mathcal{H}$. Then,*

$$\operatorname{supp} W_{L(\mathcal{H})}(A) = \{(\langle A_1 u, u\rangle, \ldots, \langle A_d u, u\rangle) \ : \ u \in \mathcal{H}, \ \|u\| = 1\}.$$

Proof: By Theorem 4.1.1,

$$\operatorname{supp} W_{L(\mathcal{H})}(A) \subseteq \{(\langle A_1 u, u\rangle, \ldots, \langle A_d u, u\rangle) \ : \ u \in \mathcal{H}, \ \|u\| = 1\}.$$

The reverse inclusion follows from Theorems 3.2.8 and Theorem 3.2.4. $\diamond$

We present below our version of E. Nelson's proof of Theorem 4.1.1.

As a preliminary step, we derive a closed form for the exponential map of $U(n)$, $\exp : \mathfrak{u}(n) \longrightarrow U(n)$. Let,

$$c_{n-1}(M) := \int_{\Sigma_n} \exp\langle Mu, u\rangle \, d\nu(u)$$

and

$$c_k(M) := \sum_{j=0}^{n-k-1} (-1)^{j+k} \phi_{n-j-k-1}(M) \frac{\partial^j}{\partial t^j} t^{n-1} c_{n-1}(tM)|_{t=1}. \tag{4.2}$$

We have the following

Lemma 4.1.3 *Let $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, where the $\lambda_i \in \mathbb{R}$. Then*

$$e^{iD} = \sum_{k=0}^{n-1} c_k(iD)(iD)^k.$$

Corollary 4.1.4 *Let M be an $n \times n$ hermitian matrix. Then*

$$e^{iM} = \sum_{k=0}^{n-1} c_j(iM)(iM)^k.$$

Proof: Let $M = U^* D U$ be the unitary decomposition of M. Then,

$$\begin{aligned} e^{iM} &= U^* e^{iD} U \\ &= U^* \left(\sum_{k=0}^{n-1} c_k(iD)(iD)^k \right) U \\ &= \sum_{k=0}^{n-1} c_k(iD)(iM)^k \\ &= \sum_{k=0}^{n-1} c_k(iM)(iM)^k \end{aligned}$$

where the last equality is an immediate consequence of our definitions and the unitary invariance of the inner product $\langle\, , \rangle$. ◇

We now devote the remainder of this section to the proof of the Lemma 4.1.3.

Lemma 4.1.5 *Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Then*

$$e^{\lambda_n} \int_0^1 \int_0^{1-w_{n-1}} \cdots \int_0^{1-w_{n-1}-\cdots-w_2} \exp(\sum_{j=1}^{n-1} (\lambda_j - \lambda_n) w_j)\; dw_1 \cdots dw_{n-1} = \sum_{j=1}^{n} \frac{e^{\lambda_j}}{\prod_{j \neq k} (\lambda_j - \lambda_k)}. \quad (4.3)$$

Proof: Observe that,

$$e^{\lambda_2} \int_0^1 e^{(\lambda_1 - \lambda_2) w_1}\, dw_1 = \frac{e^{\lambda_1}}{\lambda_1 - \lambda_2} + \frac{e^{\lambda_2}}{\lambda_2 - \lambda_1}$$

Hence (4.3) is true when $n = 2$. Assume that the identity holds for $n > 2$. Then

$$
\begin{aligned}
& e^{\lambda_{n+1}} \int_0^1 \int_0^{1-w_n} \cdots \int_0^{1-w_n-w_{n-2}-\cdots-w_2} \exp(\sum_{j=1}^{n} (\lambda_j - \lambda_{n+1}) w_j) \; dw_1 \cdots dw_n \\
&= \frac{e^{\lambda_{n+1}}}{\lambda_1 - \lambda_{n+1}} \int_0^1 \int_0^{1-w_n} \cdots \int_0^{1-w_n-\cdots-w_3} \exp\left(\sum_{j=1}^{n} (\lambda_j - \lambda_{n+1}) w_j\right) |_0^{1-w_n-\cdots w_2} dw_2 \cdots dw_n \\
&= \frac{e^{\lambda_{n+1}}}{\lambda_1 - \lambda_{n+1}} \int_0^1 \int_0^{1-w_n} \cdots \int_0^{1-w_n-\cdots-w_3} \exp(\sum_{j=2}^{n} (\lambda_j - \lambda_1) w_j) \\
&\qquad\qquad - \exp(\sum_{j=2}^{n} (\lambda_j - \lambda_{n+1}) w_j) \; dw_2 \cdots dw_n \\
&= \frac{1}{\lambda_1 - \lambda_{n+1}} \sum_{j=1}^{n} \frac{e^{\lambda_j}}{\prod_{j \neq k < n+1} (\lambda_j - \lambda_k)} - \frac{1}{\lambda_1 - \lambda_{n+1}} \sum_{j=2}^{n+1} \frac{e^{\lambda_j}}{\prod_{j \neq k > 1} (\lambda_j - \lambda_k)} \\
&= \frac{e^{\lambda_1}}{\prod_{j \neq 1} (\lambda_1 - \lambda_j)} + \frac{e^{\lambda_{n+1}}}{\prod_{k \neq n+1} (\lambda_{n+1} - \lambda_k)} + \\
&\qquad \sum_{j=2}^{n} \frac{e^{\lambda_j}}{\lambda_1 - \lambda_{n+1}} \left(\frac{1}{\prod_{j \neq k, 1 < k < n+1} (\lambda_j - \lambda_k)} \right) \left(\frac{1}{\lambda_j - \lambda_1} - \frac{1}{\lambda_j - \lambda_{n+1}} \right) \\
&= \sum_{j=1}^{n+1} \frac{e^{\lambda_j}}{\prod_{j \neq k} (\lambda_j - \lambda_k)}
\end{aligned}
$$

and hence the identity (4.3) follows by induction. ◇

Lemma 4.1.6 *Let D be diagonal matrix with distinct entries $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Then*

$$
c_{n-1}(D) = \sum_{k=1}^{n} \frac{e^{\lambda_k}}{\prod_{j \neq k} (\lambda_k - \lambda_j)}.
$$

Proof: Let $\Delta_n = \{(w_1, \ldots, w_n) \in \mathbb{R}^n | w_i \geq 0, \sum w_i = 1\}$.

$$
\begin{aligned}
c_{n-1}(D) &= \int_{\Sigma_n} \exp\left(\sum_{j=1}^{n} \lambda_j |u_j|^2\right) \, d\nu(u) \\
&= \int_{\mathbb{C}^n} \chi_{\Sigma_n}(u) \exp\left(\sum_{j=1}^{n} \lambda_j |u_j|^2\right) \, du_1 \cdots du_n \\
&= \int_{\mathbb{R}_+^n} \int_{\mathbb{T}^n} \chi_{\mathbb{S}^{n-1}}(|u|) \exp\left(\sum_{j=1}^{n} \lambda_j |u_j|^2\right) \, |u_1| d|u_1| d\theta_1 \cdots |u_n| d|u_n| d\theta_n
\end{aligned}
$$

$$= \frac{(2\pi)^n}{2^n} \int_{\mathbb{R}_+^n} \chi_{\mathbb{S}^{n-1}}(|u|) \exp\left(\sum_{j=1}^n \lambda_j |u_j|^2\right) d|u_1|^2 \cdots d|u_n|^2$$

$$\begin{aligned}
&= \int_{\mathbb{R}_+^n} \chi_{\Delta_n}(w) \exp\left(\sum_{j=1}^n \lambda_j w_j\right) dw_1 \cdots dw_n \\
&= e^{\lambda_n} \int_0^1 \int_0^{1-w_{n-1}} \cdots \int_0^{1-w_{n-1}-\cdots-w_2} \exp\left(\sum_{j=1}^n (\lambda_j - \lambda_n) w_j\right) dw_1 \cdots dw_{n-1} \\
&= \sum_{k=0}^n \frac{e^{\lambda_k}}{\prod_{j\neq k}(\lambda_k - \lambda_j)}
\end{aligned}$$

For a proof of the last equality, see the appendix. ◇

We are now ready to prove Lemma 4.1.3.

Proof of Lemma 4.1.3: First assume that the λ_i are distinct. Then by Cramer's rule the system of linear equations,

$$\begin{aligned}
e^{\lambda_1} &= g_0 + g_1\lambda_1 + \cdots + g_{n-1}\lambda_1^{n-1} \\
&\vdots \\
e^{\lambda_n} &= g_0 + g_1\lambda_n + \cdots + g_{n-1}\lambda_n^{n-1}
\end{aligned}$$

for $(g_0, \ldots, g_{n-1}) \in \mathbb{R}^n$, has a unique solution given by

$$g_k(\lambda_1, \ldots, \lambda_n) = \frac{(-1)^{n-k-1}}{\prod_{j\neq k}(\lambda_k - \lambda_j)} \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-1} & \lambda_1^{k+1} & \cdots & \lambda_1^{n-1} & e^{\lambda_1} \\ \vdots & & & & & & & \\ 1 & \lambda_n & \cdots & \lambda_n^{k-1} & \lambda_n^{k+1} & \cdots & \lambda_n^{n-1} & e^{\lambda_n} \end{vmatrix}$$

for $k = 0, \ldots, n-1$. It follows that

$$e^{iD} = \sum_{k=0}^{n-1} g_k(iD)(iD)^k.$$

However

$$
\begin{aligned}
g_{n-1}(D) &= |\ 1\ \lambda_i \cdots\ \lambda_i^{n-2}\ e^{\lambda_i}|/|\ 1\ \lambda_i \cdots\ \lambda_i^{n-2}\ \lambda_i^{n-1}| \\
&= \sum_{k=1}^{n}(-1)^{n+k-2}e^{\lambda_k} \prod_{i<j,i,j\neq k}(\lambda_j-\lambda_i)\prod_{i<j}(\lambda_j-\lambda_i)^{-1} \\
&= \sum_{k=1}^{n}(-1)^{n+k-2}e^{\lambda_k} \prod_{j=k+1}^{n}(\lambda_j-\lambda_k)\prod_{j=1}^{k-1}(\lambda_k-\lambda_j)^{-1} \\
&= \sum_{k=1}^{n}e^{\lambda_k} \prod_{j=k+1}^{n}(\lambda_j-\lambda_k)\prod_{j=1}^{k-1}(\lambda_k-\lambda_j)^{-1} \\
&= \sum_{k=1}^{n}e^{\lambda_k} \prod_{j\neq k}(\lambda_k-\lambda_j)^{-1} \\
&= c_{n-1}(D).
\end{aligned}
$$

Hence, to prove that $g_k(D)=c_k(D)$ for $1\le k\le n-1$ it suffices to show that the g_k satisfy the same recurrence (4.2) as the c_k. To this end, observe that on expanding the numerator of $t^{n-1}g_{n-1}(t\lambda_1,\ldots,t\lambda_n)$ along the last column we get that the coefficient of $(-1)^{k+j}e^{\lambda_j}V_n^{-1}$, where V_n is the Vandermonde determinant in $\lambda_1\ldots,\lambda_n$, in the right hand side of (4.2) is given by,

$$
\left(\sum_{l=0}^{n-k-1}(-1)^l\phi_{n-k-l-1}(D)\lambda_j^l\right)
\begin{vmatrix}
1 & \lambda_1 & \cdots & \lambda_1^{n-2} \\
\vdots & & & \\
\widehat{1} & \widehat{\lambda}_j & \cdots & \widehat{\lambda}_j^{n-2} \\
\vdots & & & \\
1 & \lambda_n & \cdots & \lambda_n^{n-2}
\end{vmatrix}
$$

$$
= \left(\sum_{\substack{l_1<\cdots<l_{n-k-1},\\ l_\nu\neq j}}\lambda_{l_1}\cdots\lambda_{l_{n-k-1}}\right)
\begin{vmatrix}
1 & \lambda_1 & \cdots & \lambda_1^{n-2} \\
\vdots & & & \\
\widehat{1} & \widehat{\lambda}_j & \cdots & \widehat{\lambda}_j^{n-2} \\
\vdots & & & \\
1 & \lambda_n & \cdots & \lambda_n^{n-2}
\end{vmatrix}
$$

$$= \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-1} & \lambda_1^{k+1} & \cdots & \lambda_1^{n-1} \\ \vdots & & & & & & \\ \widehat{1} & \widehat{\lambda}_j & \cdots & \widehat{\lambda}_j^{k-1} & \widehat{\lambda}_j^{k+1} & \cdots & \widehat{\lambda}_j^{n-1} \\ \vdots & & & & & & \\ 1 & \lambda_n & \cdots & \lambda_n^{k-1} & \lambda_n^{k+1} & \cdots & \lambda_n^{n-1} \end{vmatrix}. \tag{4.4}$$

The last equality is a well known identity (see [36, p.93]).

Now (4.4) is the coefficient of $(-1)^{k+j}e^{\lambda_j}V_n^{-1}$ in the expansion of $g_k(\lambda_1, \ldots, \lambda_n)$ along the last column and it follows that $c_k(D) = g_k(D)$ for all k when D has distinct eigenvalues.

Now assume that D has possibly repeated eigenvalues. Since $\sum \lambda_j |u_j|^2$ remains bounded as u ranges over Σ_n it follows from the dominated convergence theorem that the integral,

$$c_{n-1}(M) = \int_{\Sigma_n} \exp(\sum \lambda_j |u_j|^2)\, d\nu(u),$$

is continuous in $\lambda = (\lambda_1, \ldots, \lambda_n)$. Similarly, since

$$\begin{aligned} & \frac{\partial^j}{\partial t^j} \int_{\Sigma_n} \exp(\sum t\lambda_j |u_j|^2)\, d\nu(u) \\ = & \int_{\Sigma_n} \frac{\partial^j}{\partial t^j} \exp(\sum t\lambda_j |u_j|^2)\, d\nu(u) \\ = & \int_{\Sigma_n} \left(\sum_{j=0}^{n} \lambda_j |u_j|^2 \right)^j \exp(\sum t\lambda_j |u_j|^2)\, d\nu(u) \\ \leq & \; M \int_{\Sigma_n} \exp(\sum t\lambda_j |u_j|^2)\, d\nu(u) \end{aligned}$$

it follows that the c_k $(1 \leq k \leq n-1)$ are continuous in D. Hence, the lemma follows upon approximating D by diagonal matrices with distinct eigenvalues and taking limits of the identity that we have already established. ◇

We now prove Theorem 4.1.1.

Proof of Theorem 4.1.1: Observe that,

$$\begin{aligned} c_{n-1}(\xi \cdot A) &= \int_{\Sigma_n} e^{\langle i\xi\cdot Au,u\rangle}\, d\nu(u) \\ &= \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\, d\mu_A(x) \\ &= \check{\mu}_A(\xi). \end{aligned}$$

Applying the Leibniz rule for the derivative of a product, when $k < n-1$,

$$\begin{aligned} c_k(\xi \cdot A) &= \sum_{j=0}^{n-k-1} (-1)^{j+k} \phi_{n-j-k-1}(\xi \cdot A) \frac{\partial^j}{\partial t^j} t^{n-1} c_{n-1}(t\xi \cdot A)|_{t=1} \\ &= \sum_{j=0}^{n-k-1} (-1)^{j+k} \phi_{n-j-k-1}(\xi \cdot A) \sum_{m=0}^{j} \begin{pmatrix} j \\ m \end{pmatrix} \frac{(n-1)!}{(n-j+m-1)!} \frac{\partial^m}{\partial t^m} c_{n-1}(t\xi \cdot A)|_{t=1} \\ &= \sum_{j=0}^{n-k-1} (-1)^{j+k} \phi_{n-j-k-1}(\xi \cdot A) \sum_{m=0}^{j} \begin{pmatrix} j \\ m \end{pmatrix} \frac{(n-1)!}{(n-j+m-1)!} c_{n-1}^{(m)}(\xi \cdot A)(\xi \cdot A)^m \\ &= \sum_{j=0}^{n-k-1} (-1)^{j+k} \phi_{n-j-k-1}(\xi \cdot A) \sum_{m=0}^{j} \begin{pmatrix} j \\ m \end{pmatrix} \frac{(n-1)!}{(n-j+m-1)!} \check{\mu}_A^{(m)}(\xi)\xi^m \\ &= \sum_{j=0}^{n-k-1} (-1)^{j+k} \phi_{n-j-k-1}(\xi \cdot A) \sum_{m=0}^{j} \begin{pmatrix} j \\ m \end{pmatrix} \frac{(n-1)!}{(n-1-j+m)!} (\xi \cdot \frac{\partial}{\partial \xi})^m \check{\mu}_A(\xi) \end{aligned}$$

Since $(\xi_j f(\xi))\hat{\ }(x) = i\, \frac{\partial \hat{f}}{\partial x_j}(x)$ for all $f \in \mathcal{S}(\mathbb{R}^d)$ it follows that,

$$(\phi_k(i\xi \cdot A)\check{\mu}_A(\xi))\hat{\ } = -\phi_k(A \cdot \frac{\partial}{\partial x})\mu_A.$$

Furthermore,

$$[(\xi \cdot \frac{\partial}{\partial \xi})\check{\mu}_A(\xi)]\hat{\ } = -\left(\frac{\partial}{\partial x} \cdot x\right)\mu_A.$$

By Corollary 4.1.4

$$e^{i\xi\cdot A} = c_0(i\xi \cdot A) + c_1(i\xi \cdot A)(i\xi \cdot A) + \cdots + c_{n-1}(i\xi \cdot A)(i\xi \cdot A)^{n-1}$$

for all $\xi \in \mathbb{R}^d$. Therefore,

$$
\begin{aligned}
(e^{i\xi\cdot A})\hat{} &= \sum_{k=0}^{n-1} \left(A \cdot \frac{\partial}{\partial x}\right)^k [c_k(i\xi \cdot A)]\hat{} \\
&= \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \sum_{m=0}^{j} (-1)^{j+k+m+1} \begin{pmatrix} j \\ m \end{pmatrix} \frac{(n-1)!}{(n-1-j+m)!} \left(A \cdot \frac{\partial}{\partial x}\right)^k \phi_{n-j-k-1} \\
&\qquad (A \cdot \frac{\partial}{\partial x})(\frac{\partial}{\partial x} \cdot x)^m \mu_A.
\end{aligned}
$$

◇

Example 4.1.7 Let $d = 3$ and set

$$
A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.
$$

These are called the “Pauli spin matrices” and iA_1, iA_2, iA_3 define a basis for the Lie algebra $\mathfrak{su}(2)$ (see Chapter 5). We verify Theroem 4.1.1 in this case. For all $\xi \in \mathbb{R}^3$, let $|\xi| := \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$. Since $\frac{1}{2}(A_j A_k + A_k A_j) = \delta_{jk} I$ for $j, k = 1, 2, 3$ where δ_{ij} is the Kronecker delta and I is the 2×2 identity matrix, we have $(\xi \cdot A)^2 = |\xi|^2 I$ for all $\xi \in \mathbb{R}^3$ and by power-series expansion,

$$
\exp(i\xi \cdot A) = \cos|\xi| \, \mathrm{I} + \frac{\sin|\xi|}{|\xi|} i\xi \cdot A.
$$

Now,

$$
\begin{aligned}
\left(\xi \cdot \frac{\partial}{\partial \xi}\right) \frac{\sin|\xi|}{|\xi|} &= |\xi| \frac{d}{d|\xi|} \left(\frac{\sin|\xi|}{|\xi|}\right) \\
&= \frac{|\xi| \cos|\xi| - \sin|\xi|}{|\xi|} \\
&= \cos|\xi| - \frac{\sin|\xi|}{|\xi|}.
\end{aligned}
$$

Taking Fourier transforms of both sides we obtain

$$
(\cos|\xi|)\hat{} = \nu - \left(\frac{\partial}{\partial \xi} \cdot \xi\right) \nu
$$

where ν is the unitarily invariant probability measure on the unit sphere $S^2 \subseteq \mathbb{R}^3$. Thus,

$$W_{L(\mathbb{C}^2)}(A) = (\exp(i\xi \cdot A))\hat{} = \nu\, I - \left(\frac{\partial}{\partial x} \cdot x\right) \nu I - \left(\sigma \cdot \frac{\partial}{\partial x}\right) \nu.$$

On the other hand, for $u = (u_1, u_2) \in \mathbb{C}^2$ we have

$$\langle A_1 u, u\rangle = 2\Re(\bar{u}_1 u_2),$$

$$\langle A_2 u, u\rangle = 2\Im(\bar{u}_1 u_2),$$

and

$$\langle A_3 u, u\rangle = |u_1|^2 - |u_2|^2.$$

When $|u_1|^2 + |u_2|^2 = 1$,

$$\begin{aligned} |W_A(u)|^2 &= \sum_{k=1}^{3} \langle A_k u, u\rangle^2 \\ &= (|u_1|^2 + |u_2|^2)^2 \\ &= 1. \end{aligned}$$

Therefore $W_A(\Sigma_2) \subseteq S^2$ and by rotation invariance we have $W_A(\Sigma_2) = S^2$. Hence, $\operatorname{supp} \mu_A = S^2$ and by the definition of μ_A and its invariance under rotations, it follows that μ_A agrees with ν, the uniform density probability measure on S^2.

4.2 The Jefferies Formula for Arbitrary Matrices

Let M be an $n \times n$ matrix over $\mathbb{C}$. Denote by p_M the characteristic polynomial of M, defined by $p_M(z) := \det(M - zI)$ for all $z \in \mathbb{C}$.

By the Riesz functional calculus [13], we have the representation

$$e^{iM} = \frac{1}{2\pi i} \int_C e^{i\zeta} (\zeta I - M)^{-1}\, d\zeta \tag{4.1}$$

where C is any simple closed curve in $\mathbb{C}$ which surrounds the set of eigenvalues of M, which we denote by $\sigma(M)$. Following [24] (see also [23]), we determine an expression for the resolvent $(\zeta I - M)^{-1}$ of the form

$$(\zeta I - M)^{-1} = \alpha_0(M, \zeta) + \alpha_1(M, \zeta)M + \cdots + \alpha_{n-1}(M, \zeta)M^{n-1}$$

for all $\zeta \in \mathbb{C} \setminus \sigma(M)$, to thereby obtain a more general version of Theorem 4.1.1, valid for d-tuples of arbitrary complex matrices.

Let $\mathcal{H}$ be an n-dimensional Hilbert space and $A = (A_1, \ldots, A_d)$ be a d-tuple of bounded operators on $\mathcal{H}$. We choose a basis of $\mathcal{H}$ so that the A_j are represented by matrices. Define a tempered, complex-valued distribution T_A given by the Fourier transform of the uniformly bounded function

$$\xi \longmapsto \frac{i^n(n-1)!}{(2\pi)^{d+1}} \int_{C(\xi)} \frac{e^{iz}}{p_{\langle A,\, \xi\rangle}(z)}\, dz, \qquad \xi \in \mathbb{R}^d \tag{4.2}$$

where for every $\xi \in \mathbb{R}^d$, $C(\xi)$ is a simple closed curve containing $\sigma(\langle A, \xi\rangle)$ in its interior.

Theorem 4.2.1 ([23], Theorem 5.10) *Let $A = (A_1, \ldots, A_d)$ be a d-tuple of bounded operators on an n-dimensional Hilbert space $\mathcal{H}$ satisfying the spectral condition*

$$\sigma(\langle A, \xi\rangle) \subseteq \mathbb{R} \tag{4.3}$$

for all $\xi \in \mathbb{R}^d$. Then there exist numbers $C > 0$ and $r > 0$ such that

$$\|e^{i\langle A,\, \xi\rangle}\| \leq C(1 + |\zeta|)^{n-1} e^{r|\mathrm{Im}\,\zeta|}$$

for all $\zeta \in \mathbb{C}^n$.

We have the following

Corollary 4.2.2 ([23], Corollary 5.11) *Let $A = (A_1, \ldots, A_d)$ be a d-tuple of bounded operators on an n-dimensional Hilbert space $\mathcal{H}$ and satisfying spectral condition (4.3). Then Weyl calculus for A is defined, has compact support* *and is given by the expression*

$$W_{L(\mathcal{H})}(A) = \sum_{k=0}^{n-1}\sum_{j=0}^{n-k-1}\sum_{m=0}^{j}(-1)^{j+k+m+1}\begin{pmatrix} j \\ m \end{pmatrix}\frac{1}{(n-1-j+m)!}\times$$
$$\left(A\cdot\frac{\partial}{\partial x}\right)^k \phi_{n-j-k-1}(A\cdot\frac{\partial}{\partial x})(\frac{\partial}{\partial x}\cdot x)^m T_A. \tag{4.4}$$

Proof: If $f \in \mathcal{S}(\mathbb{R}^d)$ then we have also $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$ [22] and hence by Theorem 4.2.1 and the bound (4.2.1), $\|e^{i\langle A,\,\xi\rangle}\|_{L(\mathcal{H})}|\hat{f}| \le C(1+|\xi|)^{-n-1}$. As

$$\int_{\mathbb{R}}(1+|\xi|)^{-n-1}\,d\xi\,<\,\infty,$$

it follows that the tempered distribution $W_{L(\mathcal{H})}(A)$ is defined. By the Paley-Wiener theorem, since $W_{L(\mathcal{H})}(A)$ is the Fourier transform of the analytic function $e(\zeta) := e^{i\langle A,\,\xi\rangle}$ ($\zeta \in \mathbb{C}^d$), it is compactly supported.

For each $\xi \in \mathbb{R}^d$, let $a_0(\langle A,\,\xi\rangle),\ldots,a_n(\langle A,\,\xi\rangle)$ be the coefficients of the characteristic polynomial of $\langle A,\,\xi\rangle$. The following identity is easily verified by the Cayley-Hamilton theorem,

$$p_{\langle A,\,\xi\rangle}(\zeta)I = (\zeta I - M)\sum_{k=0}^{n-1}\left(\sum_{j=0}^{n-k-1}a_{j+k+1}(\langle A,\,\xi\rangle)\zeta^j\right)\langle A,\,\xi\rangle^k.$$

By (4.1) it follows that

$$e^{i\langle A,\,\xi\rangle} = \frac{1}{2\pi i}\sum_{k=0}^{n-1}\left(\sum_{j=0}^{n-k-1}a_{j+k+1}(\langle A,\,\xi\rangle)\int_C\frac{e^{i\zeta}\zeta^j}{p_{\langle A,\,\xi\rangle}}\,d\zeta\right)(\langle A,\,\xi\rangle)^k$$

where C is any simple closed curve in $\mathbb{C}$ containing $\sigma(\langle A,\,\xi\rangle)$ in its interior. If $t \in \mathbb{R}$ is chosen so that C also contains $t\sigma(\langle A,\,\xi\rangle)$,

$$\begin{aligned}\int_C\frac{e^{it\zeta}(i\zeta)^j}{p_{\langle A,\,\xi\rangle}(\zeta)}\,d\zeta &= \frac{\partial^j}{\partial t^j}\int_C\frac{e^{it\zeta}}{p_{\langle A,\,\xi\rangle}(\zeta)}\,d\zeta \\ &= \frac{\partial^j}{\partial t^j}t^{n-1}\int_C\frac{e^{i\zeta}}{p_{t\langle A,\,\xi\rangle}(\zeta)}\,d\zeta\end{aligned}$$

and using Leibniz's formula for the differentiation of products we obtain

$$\begin{aligned} e^{i\langle A,\,\xi\rangle} &= \frac{1}{2\pi i}\sum_{k=0}^{n-1}\langle A,\,\xi\rangle^k \sum_{j=0}^{n-k-1} i^{-j} a_{j+k+1}(\langle A,\,\xi\rangle) \\ &\qquad\qquad \times \sum_{m=0}^{j}\binom{j}{m}\frac{(n-1)!}{(n-1-j+m)!}\left[\frac{\partial^m}{\partial t^m}\int_C \frac{e^{iz}}{p_{t\langle A,\,\xi\rangle}(z)}\,dz\right]_{t=1}. \end{aligned} \tag{4.5}$$

The formula (4.4) now follows on taking Fourier transforms of both sides of (4.5), noting that $\frac{\partial}{\partial t}\check{T}_A(t\xi)|_{t=1} = (\xi\cdot\frac{\partial}{\partial\xi})\check{T}_A(\xi)$, and $a_s(\langle A,\,\xi\rangle) = (-1)^s\phi_{n-s}(\langle A,\,\xi\rangle)$ for $s = 0, 1, \ldots, n-1$. $\diamond$

In [23], Corollary. 5.11, it is also shown that if the d-tuple $A = (A_1, \ldots, A_d)$ satisfies

$$\sigma(\langle A,\,\xi\rangle) \subseteq \mathbb{R} \tag{4.6}$$

for all $\xi \in \mathbb{R}^d$, then the distribution W_A has compact support contained in the rectangle $[-\|A_1\|,\,\|A_1\|] \times \cdots \times [-\|A_d\|,\,\|A_d\|]$.

Observe that if M is a diagonal matrix with distinct entries $\lambda_1, \ldots, \lambda_n$ and C is a simple closed curve in $\mathbb{C}$ surrounding $\sigma(M)$,

$$\begin{aligned} \int_C \frac{e^{i\zeta}}{p_M(\zeta)}\,dz &= \int_C \frac{e^{i\zeta}}{(\lambda_1-\zeta)\cdots(\lambda_n-\zeta)}\,d\zeta \\ &= \sum_{k=1}^{n}\frac{1}{\prod_{j\neq k}(\lambda_k-\lambda_j)}\int_C \frac{e^{i\zeta}}{\zeta-\lambda_k}\,d\zeta \\ &= \sum_{k=1}^{n}\frac{e^{i\lambda_k}}{\prod_{j\neq k}(\lambda_k-\lambda_j)} \end{aligned}$$

and hence by Lemma 4.1.6 the two formulas (4.1) and (4.4) agree in the case of hermitian matrices.

Let ν be the unitarily invariant probability measure on the unit sphere Σ in $\mathbb{C}^n$. More generally we have

Proposition 4.2.3 ([24]) *Let M be a normal $n\times n$ matrix. Let U be a simply connected open subset of $\mathbb{C}$ containing $\operatorname{co}\sigma(M)$. Let $f : U \to \mathbb{C}$ be an analytic* *function. Then for any simple closed curve C in U containing $\sigma(M)$ in its interior,*

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{p_M(\zeta)}\,d\zeta = \frac{(-1)^n}{(n-1)!}\int_\Sigma f^{(n-1)}(\langle Mu,\, u\rangle)\,d\nu(u). \tag{4.7}$$

Chapter 5

Lie Groups and Lie Algebras

In this chapter, we present a streamlined exposition of basic Lie theory, concentrating only upon those results which will be used in the sequel. All of these facts and their proofs can found in the standard references [39] or [42].

5.1 Basic Structures

Definition 5.1.1 *A topological group G is called a* (real) Lie group *if it is endowed with the structure of a real analytic manifold for which the group operations are morphisms of analytic manifolds.*

It was shown in 1952 by Gleason, Montgomery and Zippen that every locally Euclidean topological group admits a unique Lie group structure compatible with its topological structure [32].

Example 5.1.2 *The group $GL(n, \mathbb{R})$ of invertible $n \times n$ matrices real matrices inherits the Euclidean topology from its canonical embedding in $\mathbb{R}^{n^2}$; the determinant defines a continuous function from $GL(n, \mathbb{R})$ into $\mathbb{R}$; hence, $GL(n, \mathbb{R})$ is an open set and in particular an analytic manifold. Since the group operations are given component-wise by rational functions of the matrix entries, it follows that $GL(n, R)$ is a Lie group. In a similar fashion, it can be shown that for any finite dimensional real or complex vector space V, the group $GL(V)$ of* *automorphisms of V can be endowed with a Lie group structiure.*

Many more examples of Lie groups are furnished by the following classical

Theorem 5.1.3 (Cartan) *Every closed subgroup of a Lie group is a Lie group in a unique way.*

In what follows, we assume that G is a Lie group with identity element e. For each $g \in G$ we define the conjugation mapping $c_g : G \to G$ given by $c_g(h) := ghg^{-1}$ for all $h \in G$. Since c_g is analytic and fixes e, the tangent map dc_g of c_g at the identity e is an automorphism of the tangent space T_eG of G at e. Since T_eG is a finite-dimensional vector space, we can make the following

Definition 5.1.4 *The mapping $Ad : G \to GL(T_eG)$ given by $Ad(g) := dc_g$ for all $g \in G$ is called the* adjoint representation *of G on T_eG.*

Definition 5.1.5 *A complex-valued function f on T_eG is G-*invariant *or Ad-*invariant *if it is stable under the the adjoint representation Ad, i.e. $f(Ad(g)X) = f(X)$ for all $g \in G$ and all $X \in T_eG$.*

Since $GL(T_eG)$ is open, the tangent map dAd of Ad at the identity e is a linear mapping from T_eG into the space $End\, T_eG$ of endomorphisms of T_eG and we denote it by ad.

Definition 5.1.6 *The mapping $ad : T_eG \to End\, T_eG$ defined by $ad := dAd$ is called the* adjoint representation *of T_eG on $End\, T_eG$.*

Equipped with the **Lie bracket** $[\,\cdot\,,\,\cdot\,]$ given by

$$[X, Y] := ad(X)Y$$

for all $X, Y \in T_eG$, the vector space T_eG inherits the structure of a non-associative algebra over $\mathbb{R}$.

Definition 5.1.7 *The pair $(T_eG, [\,\cdot\,,\,\cdot\,])$ is called the* Lie algebra *of G.*

In the sequel, we denote the Lie algebra of G by $\mathfrak{g}$. The proposition below shows that the association of a Lie algebra $\mathfrak{g}$ to a given Lie group G is functorial.

Proposition 5.1.8 *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Suppose that $\phi : G \to H$ is a morphism of Lie groups. Then the tangent map $d\phi : \mathfrak{g} \to \mathfrak{h}$ of ϕ at the identity $e \in G$ is a morphism of Lie algebras.*

It follows by Proposition 5.1.8 that when H is a Lie subgroup of G then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.

Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. We write $\mathfrak{gl}(n, \mathbb{F})$ for the Lie algebra of $GL(n, \mathbb{F})$ and we denote by $Mat_n(\mathbb{F})$ the vector space of $n \times n$ matrices with entries in $\mathbb{F}$.

Example 5.1.9 *Suppose that G is a Lie subgroup of $GL(n, \mathbb{F})$. Let $X \in \mathfrak{g}$ and let $\alpha : \mathbb{R} \to G$ be any curve in G corresponding to X, i.e. $\alpha(0) = I$ and $\alpha'(0) = X$. Then for each $g \in G$, by the Leibniz rule,*

$$Ad(g)X = \frac{d}{dt} g\alpha(t)g^{-1}|_{t=0} = g\alpha'(0)g^{-1} = gXg^{-1},$$

independently of α, and for all $X, Y \in Mat_n \mathbb{F}$,

$$\begin{aligned}[X, Y] \;=\; ad(X)Y \;&=\; \frac{d}{dt} Ad(\alpha(t)Y\alpha(t)^{-1}|_{t=0} \\ &=\; \alpha'(0)Y + Y\frac{d}{dt}\alpha(t)^{-1}|_{t=0} = XY - YX. \qquad (5.1)\end{aligned}$$

If $X \in Mat_n\mathbb{F}$, we write X^T for the matrix transpose of X. We denote by $\bar{X}^T$ the conjugate transpose X^* of X.

Example 5.1.10 *Let $U(n)$ denote the group of unitary matrices of order n. Since each column of a unitary matrix has unit norm, it follows that $U(n)$ is homeomorphic to a subset of the product of n copies of S^{2n-1}. In particular, the group $U(n)$ is closed in $GL(n, \mathbb{C})$ and hence by Theorem 5.1.3 inherits a Lie group structure. We denote by $\mathfrak{u}(n)$ the Lie algebra of $U(n)$. If α is a curve in G corresponding to the matrix $X \in \mathfrak{u}(n)$, we differentiate the equation $\alpha(t)\alpha(t)^* = I$ at $t = 0$ to obtain,*

$$X^* + X = 0. \qquad (5.2)$$

Hence, the Lie algebra $\mathfrak{u}(n)$ contains all skew-hermitian matrices. Conversely, suppose that X satisfies equation (5.2). Define $\alpha(t) := e^{tX}$ for all $t \in \mathbb{R}$. Then $\alpha(0) = I$ and $\alpha'(0) = x$. By (5.2) we have,

$$(tX^*)^n = (-tX)^n$$

for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Hence,

$$e^{tX^*} = e^{-tX}$$

and since $e^{tX^*} = (e^{tX})^*$, it follows that $(e^{tX})^* e^{tX} = I$. Thus, the Lie algebra $\mathfrak{u}(n)$ is the vector space of skew-hermitian matrices equipped with the Lie bracket (5.1).

Example 5.1.11 *Let $SU(n)$ denote the subgroup of $U(n)$ consisting of those matrices with determinant unity; this is an open connected component of $U(n)$ and hence a Lie group. In view of Example 5.1.10, using the identity*

$$\det e^A = e^{Tr\,A},$$

we determine that the Lie algebra $\mathfrak{su}(n)$ of $SU(n)$ is the vector space of traceless skew-hermitian matrices equipped with the Lie bracket (5.1).

Definition 5.1.12 *A* one-parameter subgroup *of G is a morphism of analytic manifolds from the additive Lie group $\mathbb{R}$ into G.*

The following theorem is proved using the theory of differential equations.

Theorem 5.1.13 *Suppose that $X \in \mathfrak{g}$. Then there is a unique one-parameter subgroup $\alpha : \mathbb{R} \to G$ with $\alpha(0) = e$ and $\alpha'(0) = X$.*

We denote by α_X the one-parameter subgroup corresponding to X.

Definition 5.1.14 *The mapping* $\exp : \mathfrak{g} \to G$ *defined by*

$$\exp X := \alpha_X(1)$$

for all $X \in \mathfrak{g}$ is called the exponential map *of G.*

Example 5.1.15 *Suppose that G is a Lie subgroup of $GL(n, \mathbb{C})$. Then for any $X \in \mathfrak{g}$, $\alpha_X(1) := e^{tX}$ $(t \in \mathbb{R})$ is a one-parameter subgroup of G. Hence, by uniqueness, it follows that the map* exp *is just the ordinary matrix exponential.*

The following proposition plays an important role in the next chapter.

Theorem 5.1.16 *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. The following diagram is commutative,*

$$\begin{tikzcd} G \arrow[r, "\phi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "d\phi"] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

where $d\phi$ denotes the tangent map of ϕ at the identity $e \in G$.

Theorem 5.1.17 *The exponential map* $\exp : \mathfrak{g} \to G$ *is a local diffeomorphism at* $0 \in \mathfrak{g}$.

Theorem 5.1.18 *Suppose that G is a connected compact Lie group. Then the exponential map* exp *is a surjection.*

Definition 5.1.19 *The j-*function *of G is given by*

$$j(X) := \det^{1/2}\left(\frac{\sinh ad(X/2)}{ad(X/2)}\right).$$

We next develop a theory of integration on the Lie group G.

Definition 5.1.20 *The* left-translation *of G by the element $g \in G$ is the map $L_g : G \to G$ given by $L_g(h) := gh$ for all $h \in G$.*

A right-translation of G is defined similarly.

Theorem 5.1.21 *There exists a positive measure dg on G which is invariant under left-translations. Moreover, the measure dg is unique up to multiplication by a complex number.*

The measure dg is called **Haar measure** on G.

Theorem 5.1.22 *The total of mass of Haar measure dg is finite if and only if G is compact.*

If G is compact, we assume that Haar measure dg is normalised to have total mass unity.

Theorem 5.1.23 *Suppose that G is compact. Then the Haar measure dg is invariant under left-translations, right-translations and inversion, i.e.,*

$$\int_G f(hg)\,dg = \int_G f(gh)\,dg = \int_G f(g^{-1})\,dg = \int_G f(g)\,dg$$

for all $h \in G$ and any Borel measurable function f on G.

Let dX be Lebesgue measure on $\mathfrak{g}$ and let $U \subseteq \mathfrak{g}$ be a neighbourhood of $0 \in \mathfrak{g}$ on which the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism. For any complex-valued, Borel measurable function f on G with compact support in $\exp U$ we have

$$\int_G f(g)\,dg = \int_U f(\exp X) J(X)\,dX$$

where J is the Jacobian determinant of exp.

Theorem 5.1.24 *For all $X \in \mathfrak{g}$,*

$$J(X) = |j(X)|^2 = \det\left(\frac{1 - e^{-ad\,X}}{ad\,X}\right).$$

The quantity j will play an important role in the sequel.

5.2 Representations

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the group of bounded linear operators on $\mathcal{H}$ with bounded inverse, endowed with the norm topology.

Definition 5.2.1 *A* representation $(\pi, \mathcal{H})$ *of G is a pair consisting of a Hilbert space $\mathcal{H}$ and a homomorphism $\pi : G \to \mathcal{B}(\mathcal{H})$ with the property that the mapping of $G \times \mathcal{H}$ into $\mathcal{H}$ given by $(g, v) \mapsto \pi(g)v$ is continuous.*

The representation $(\pi,\, \mathcal{H})$ of G is called **irreducible** if there are no proper non-zero closed subspaces of $\mathcal{H}$ which are invariant under the action of the set $\pi(G)$ of operators; it is **unitary** if $\pi(g)$ is a unitary operator for each $g \in G$.

We will often denote the representation $(\pi,\, \mathcal{H})$ simply by π. The **dimension** $\dim \pi$ of π is the dimension of $\mathcal{H}$. If for all $g \in G$, $\pi(g)$ is the identity mapping on $\mathcal{H}$ then π is called the **defining** representation on $\mathcal{H}$.

Lemma 5.2.2 *Let $(\pi,\, \mathcal{H})$ be a finite dimensional representation of G. Then the mapping $\pi : G \to GL(\mathcal{H})$ is a morphism of Lie groups.*

Theorem 5.2.3 *Suppose that G is compact and π is an irreducible representation of G. Then π is finite dimensional.*

Definition 5.2.4 *An* intertwining operator *for two finite dimensional representations $(\pi,\, \mathcal{H})$ and $(\pi',\, \mathcal{H}')$ of G is a linear mapping $T : \mathcal{H} \to \mathcal{H}'$ such that $T \circ \pi = \pi' \circ T$.*

Two representations are **equivalent** if there is a bijective intertwining operator between them. If the representation $(\pi,\, \mathcal{H})$ is finite-dimensional then it is equivalent to the representation $(\pi',\, \mathcal{H}')$ if for each $g \in G$ there exist bases of $\mathcal{H}$ and $\mathcal{H}'$ with respect to which the operators $\pi(g)$ and $\pi'(g)$ have identical matrix representations.

Lemma 5.2.5 (Schur) *Let $(\pi,\, \mathcal{H})$ and $(\pi',\, \mathcal{H})$ be two irreducible representations of G. Then they are equivalent if their set of intertwining operators coincides with the set of scalar multiples of the identity; otherwise, they have no non-zero intertwining operators.*

Let $\langle \cdot, \cdot \rangle$ denote the inner product on $\mathcal{H}$. Then whenever G is compact we can define a new inner product on $\mathcal{H}$ by,

$$(v,\, v') := \int_G \langle gv,\, gv' \rangle \, dg$$

for all $u, v \in \mathcal{H}$. Since dg is invariant under right-translations, $(gu,\, gv) = (u,\, v)$ for every $g \in G$.

In essence we have proved the following

Theorem 5.2.6 *Suppose that G is compact and that $(\pi, \mathcal{H})$ is a representation of G. Then there exists an inner product on $\mathcal{H}$ with respect to which π is unitary.*

Theorem 5.2.7 *Let $(\pi, \mathcal{H})$ be a finite dimensional unitary representation of G. Then there is a finite direct sum decomposition of Hilbert spaces,*

$$\mathcal{H} = \bigoplus \mathcal{H}_i$$

so that for each i, the representation $(\pi|_{\mathcal{H}_i}, \mathcal{H}_i)$ is irreducible.

Corollary 5.2.8 *Suppose that G is compact. Then every finite dimensional representation of G admits a decomposition as a finite direct sum of irreducible representations of G.*

Definition 5.2.9 *The* character χ_π *of a finite dimensional representation $(\pi, \mathcal{H})$ of G is a complex-valued function on G given by $\chi_\pi(g) := Tr\, \pi(g)$ for all $g \in G$.*

The next theorem shows that the character separates non-equivalent representation.

Theorem 5.2.10 *Suppose that G is compact. Then two finite-dimensional representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ of G are equivalent if and only if $\chi_\pi = \chi_{\pi'}$.*

Definition 5.2.11 *A* matrix coefficient $f = f^{\pi}_{u,v}$ *of a finite-dimensional unitary representation $(\pi, \mathcal{H})$ of G is a complex-valued function on G of the form $f(g) = \langle \pi(g)u, w\rangle$ for some $u, v \in \mathcal{H}$.*

We denote by $C(G)_\pi$ the linear span of the matrix coefficients of π.

Theorem 5.2.12 *Suppose that G is compact and let π and π' be equivalent irreducible representations of G. Then $C(G)_\pi = C(G)_{\pi'}$.*

Theorem 5.2.13 (Schur Orthogonality) *Suppose that G is compact. Let $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ be a unitary irreducible representation of G. Then for any $u, v \in \mathcal{H}$ and $u', v' \in \mathcal{H}$,*

$$\int_G f^{\pi}_{u,v}(g)\overline{f^{\pi'}_{u',v'}(g)}\,dg = \begin{cases} (\dim \pi)^{-1}\langle u,\, u'\rangle\langle v,\, v'\rangle, & \pi \sim \pi' \\ 0, & \pi \nsim \pi'. \end{cases}$$

Denote by $\widehat{G}$ the set of equivalence classes of finite-dimensional irreducible representations of G.

Theorem 5.2.14 *Suppose that G is compact. The space $L^2(G, dg)$ decomposes as the Hilbert sum,*

$$L^2(G) = \widehat{\bigoplus}_{[\pi]\in\widehat{G}} C(G)_\pi.$$

5.3 Compact Lie Groups

Unless indicated otherwise, in this section we assume that G is compact and connected.

5.3.1 Tori

Let $\mathbb{T}$ denote the torus $\mathbb{R}/\mathbb{Z}$.

Theorem 5.3.1 *Suppose that G is commutative. Then G is Lie group isomorphic to $\mathbb{T}^k$.*

Remark: When G is commutative but not necessarily connected it can be shown that there is a Lie group isomorphism between G and $\mathbb{T}^k \times F$ for some finite abelian group F and some $k \in \mathbb{N}$.

The Lie algebra $\mathfrak{g}$ is **commutative** if $[\mathfrak{g},\, \mathfrak{g}] = 0$.

Definition 5.3.2 *A* maximal torus *T in G is a maximal connected commutative Lie subgroup of G.*

Maximal tori in compact Lie groups exist by virtue of the following

Theorem 5.3.3 *Suppose that T is a connected subgroup of G with Lie algebra $\mathfrak{t}$. Then T is a maximal torus if and only if $\mathfrak{t}$ a maximal commutative subalgebra of $\mathfrak{g}$.*

Theorem 5.3.4 *Suppose that T and T' are maximal tori in G with Cartan subalgebras $\mathfrak{t}$ and $\mathfrak{t}'$ respectively. Then for some $g \in G$, $gTg^{-1} = T'$ and $Ad\,(g)\mathfrak{t} = \mathfrak{t}'$. Furthermore, for every element $g \in G$ there exists a maximal torus T in G with $g \in T$.*

It follows from Theorem 5.3.4 that $G = \bigcup_{g \in G} gTg^{-1}$ and $\mathfrak{g} = Ad(G)\mathfrak{t}$, where T is a maximal torus of G with Lie algebra $\mathfrak{t}$.

Example 5.3.5 *Suppose that $G = U(n)$. We can choose a maximal torus*

$$T := \{diag(e^{i\theta_1}, \ldots, e^{i\theta_n}) \,:\, \theta_i \in \mathbb{R}\}$$

with corresponding Cartan subalgebra

$$\mathfrak{t} := \{diag(i\theta_1, \ldots, i\theta_n) \,:\, \theta_i \in \mathbb{R}\}.$$

Then Theorem 5.3.4 is the familiar fact that every hermitian matrix is unitarily equivalent to a diagonal matrix.

5.3.2 Roots and Weights

We denote by $\mathfrak{g}_{\mathbb{C}}$ the complexification of the Lie algebra $\mathfrak{g}$, i.e. the vector space $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ equipped with the $\mathbb{C}$-linear extension of the Lie bracket $[\,,\,]$. For any representation π of G, we identify the map $d\pi$ with its $\mathbb{C}$-linear extension to $\mathfrak{g}_{\mathbb{C}}$. Let V be a vector space. We write V^* for the vector space dual of V. In the following, $\mathfrak{t}$ is a fixed Cartan subalgebra of $\mathfrak{g}$.

Definition 5.3.6 *A* weight *of a finite dimensional representation $(\pi, \mathcal{H})$ of G is a functional $\lambda \in \mathfrak{t}^*_{\mathbb{C}}$ for which there is a non-zero vector $v \in \mathcal{H}$ such that*

$$d\pi(H)v = \lambda(H)v \tag{5.1}$$

for all $H \in \mathfrak{t}$. The subspace of $\mathcal{H}$ consisting of all vectors v for which equation (5.1) holds is the weight space $\mathcal{H}_\lambda$ *of λ.*

We denote the set of weights of π by $\Lambda(\pi) = \Lambda(\pi, \mathfrak{t})$.

Theorem 5.3.7 *Let $(\pi, \mathcal{H})$ be a finite dimensional representation of G. Then $\Lambda(\pi)$ is a finite subset of $i\mathfrak{t}^*$ and there is the weight space decomposition,*

$$\mathcal{H} = \bigoplus_{\lambda \in \Lambda(\pi)} \mathcal{H}_\lambda.$$

Furthermore, $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$ are orthogonal whenever $\lambda \neq \mu$.

The **multiplicity** of the weight λ in $\mathcal{H}$ is the dimension of $\mathcal{H}_\lambda$.

Definition 5.3.8 *The* Killing form *of $\mathfrak{g}$ is the bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by $B(X, Y) := Tr\,(ad\,X \circ ad\,Y)$ for all $X, Y \in \mathfrak{g}$.*

It can be shown that the Killing form B is the unique bilinear form on $\mathfrak{g}$, up to multiplication by a scalar.

Definition 5.3.9 *The Lie algebra $\mathfrak{g}$ is called* simple *if it is not commutative and it contains no proper non-zero ideals; it is* semisimple *if it is the direct sum of simple ideals. The Lie group G is simple (resp. semisimple) if its Lie algebra $\mathfrak{g}$ is simple (resp. semisimple).*

Theorem 5.3.10 *The Killing form B is negative definite if and only if the Lie algebra $\mathfrak{g}$ is compact and semisimple.*

The conjugation map $\tau : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ is given by $\tau(X \otimes z) := X \otimes \bar{z}$. The form,

$$(X, Y) := -B(X, \tau Y)$$

defines a positive definite inner product on $\mathfrak{g}_{\mathbb{C}}$. We also use the notation $(\,,\,)$ for the dual inner product on $\mathfrak{g}_{\mathbb{C}}^*$.

For the remainder of this section, we assume that G is compact, connected and semisimple.

Definition 5.3.11 *A* root *(resp.* root space*) of $\mathfrak{g}$ with respect to $\mathfrak{t}$ is a non-zero weight (resp. weight space) of the adjoint representation.*

We denote the set of roots of $\mathfrak{g}$ by $R = R(\mathfrak{g}, \mathfrak{t})$ and we write $\mathfrak{g}_\alpha$ for the root space corresponding to $\alpha \in R$.

Corollary 5.3.12 *The root space decomposition*

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

is orthogonal with respect to the inner product (,).

Lemma 5.3.13 *If $\alpha \in R$ then $-\alpha \in R$. Furthermore, $\mathfrak{g}_0 = \mathfrak{t}_{\mathbb{C}}$ and for all non-zero roots $\alpha \in R$,* $\dim \mathfrak{g}_\alpha = 1$.

For $\alpha \in R$, we define a reflection $s_\alpha \in End(i\mathfrak{t}^*)$ given by

$$s_\alpha(\lambda) = \lambda - 2\frac{(\lambda,\, \alpha)}{(\alpha,\, \alpha)}\alpha$$

for all $\lambda \in i\mathfrak{t}^*$.

Definition 5.3.14 *The* Weyl group *of $(\mathfrak{g}, \mathfrak{t})$ is the subgroup W of $GL(i\mathfrak{t}^*)$ generated by the reflections s_α for all $\alpha \in R$.*

Theorem 5.3.15 *The Weyl group W is finite.*

Theorem 5.3.16 *Let $(\pi, \mathcal{H})$ be an irreducible representation of G. Then the Weyl group W permutes the set $\Lambda(\pi)$ of weights and moreover,* $\dim\mathcal{H}_\lambda = \dim \mathcal{H}_{w(\lambda)}$ *for all $\lambda \in \Lambda(\pi)$ and all $w \in W$.*

Since every root $\alpha \in R$ is non-zero, the set $i\mathfrak{t} \setminus \bigcup_{\alpha \in R} \ker \alpha$ is a complement of hyperplanes and consists of finitely many connected components, the **Weyl chambers** all of them convex. We chose one of them, the **positive Weyl chamber**, and denote it by $\mathcal{C}$. The hyperplanes $\ker \alpha$, are the **walls** of the Weyl chambers.

We say a root $\alpha \in R$ is **positive** if $\alpha(H) > 0$ for all $H \in \mathcal{C}$. We denote the set of positive roots by R^+.

Definition 5.3.17 *Let* $(\pi, \mathcal{H})$ *be a finite dimensional representation of* G*. A* highest weight vector *for* π *is a non-zero vector* $v \in \mathcal{H}$ *such that* $d\pi(\mathfrak{t}_{\mathbb{C}}) \subseteq \mathbb{C}v$ *and* $d\pi(X)v = 0$ *for all* $X \in \mathfrak{g}_\alpha$ *and all* $\alpha \in R^+$*.*

Theorem 5.3.18 *Let* $(\pi, \mathcal{H})$ *be an irreducible representation of* G*. Then there exists a unique weight* $\lambda \in \Lambda(\pi)$ *such that* $\mathcal{H}_\lambda = \mathbb{C}v$ *where* v *is a highest weight vector for* π*.*

Definition 5.3.19 *The* highest weight $\lambda \in \Lambda(\pi)$ *of a representation* π *of* G *is the weight* λ *of the highest vector* v *of* π*.*

Lemma 5.3.20 *Two irreducible representations of* G *have identical highest weights if and only if they are equivalent.*

Theorem 5.3.21 *The set weights* $\Lambda(\pi)$ *is contained in the convex hull* $\operatorname{co}(W\cdot\lambda)$ *of the Weyl orbit* $W \cdot \lambda$ *of the highest-weight* λ*.*

Definition 5.3.22 *The* weight lattice *of* $\mathfrak{g}$ *with respect to* $\mathfrak{t}$ *is the set*

$$P := \{\lambda \in i\mathfrak{t}^* \,:\, 2\frac{(\lambda,\, \alpha)}{(\alpha,\, \alpha)} \in \mathbb{Z}, \text{ for all } \alpha \in R\}.$$

A weight $\lambda \in P$ is **dominant** if it belongs to the closure of the positive Weyl chamber $\mathcal{C}$. We denote the set of dominant weights by Λ^+.

Theorem 5.3.23 *The assignment of the highest weight of an equivalence class of irreducible representations of* G *is a bijection of* $\widehat{G}$ *onto* Λ^+*.*

Theorem 5.3.24 *There exists a subset* Π *of* R^+ *which is a basis for* $i\mathfrak{t}^*$ *and is such that every root in* R^+ *can be expressed as a positive integral combination of the roots in* Π

The set $\Pi = \Pi(\mathfrak{g},\, \mathfrak{t})$ is called the set of **simple roots** of $(\mathfrak{g},\, \mathfrak{t})$.

Theorem 5.3.25 *Let* π *be an irreducible representation of* G *with highest weight* λ_0*. For every* $\lambda \in \Lambda(\pi)$ *we have*

$$\lambda = \lambda_0 - \sum_{\alpha_i \in \Pi} n_i \alpha_i \tag{5.2}$$

for some choice of positive integers n_i.

By Theorem 5.3.25, we can define a partial order on the set P by $\lambda \succeq \lambda'$ if and only if $\lambda - \lambda'$ is a positive integral combination of simple roots.

5.3.3 Weyl's Formulas

The sign function sgn on W is defined so that $\mathrm{sgn}(w)$ equals 1 for every $w \in W$ which is a product of an even number of reflections and -1 otherwise.

We define the functional $\delta \in i\mathfrak{t}^*$ by,

$$\delta := \frac{1}{2} \sum_{\alpha \in R^+} \alpha.$$

Definition 5.3.26 *The* Weyl denominator *function* $\Delta : \mathfrak{g} \to \mathbb{C}$ *is given by*

$$\Delta(X) := \sum_{w \in W} \mathrm{sgn}(w) e^{w\delta(X)}$$

for all $X \in \mathfrak{t}$.

Define $\Phi^+ := \{\alpha \in \mathfrak{t}^* \,:\, i\alpha \in R^+\}$. The function Δ can be expressed in the following alternative form,

Theorem 5.3.27 *For all* $H \in \mathfrak{t}$,

$$\Delta(H) := \prod_{\alpha \in \Phi^+} 2 \sin(\alpha(H)/2).$$

Definition 5.3.28 *An element* $X \in \mathfrak{g}$ *is* regular *if* $\Delta(X) \neq 0$.

We denote by $\mathfrak{g}^{reg}$ the set of regular elements of $\mathfrak{g}$.

Theorem 5.3.29 *The set* $\mathfrak{g}^{reg}$ *is open and dense in* $\mathfrak{g}$. *Furthermore,* $\mathfrak{g}^{reg} = \bigcup_{g \in G} Ad(g)\mathfrak{t}^{reg}$.

The following decomposition of Lebesgue measure dX on $\mathfrak{g}$ will be useful in the next chapter.

Theorem 5.3.30 (Weyl integration formula) *Let f be a complex-valued Borel measurable function on $\mathfrak{g}$. Then*

$$\int_{\mathfrak{g}} f(X)\,dX = \frac{1}{|W|}\int_G \int_{\mathfrak{t}} f(Ad(g)H) \prod_{\alpha\in\Phi^+} \alpha(H)^2\,dH dg,$$

where $|W|$ denotes the order of the Weyl group W.

Theorem 5.3.31 (Weyl character formula) *Let π be an irreducible representation of G with highest weight λ. The character χ_π is given by*

$$\chi_\pi(\exp H) = \frac{\sum_{w\in W} \mathrm{sgn}(w) e^{w(\lambda+\delta)(H)}}{\Delta(H)}$$

for all $H \in \mathfrak{t}^{reg}$.

Corollary 5.3.32 (Weyl dimension formula) *Let π be an irreducible representation of G with highest weight λ. Then the dimension of π is given by*

$$\dim \pi = \prod_{\alpha\in R^+} \frac{(\lambda+\delta,\, \alpha)}{(\delta,\, \alpha)}.$$

5.3.4 Kirillov Character Formula

Definition 5.3.33 *The* coadjoint representation $Ad^* : G \to GL(\mathfrak{g}^*)$ *of a Lie group G is given for each $g \in G$ by $\langle Ad^*(g)(\lambda),\, X\rangle = \langle \lambda,\, Ad(g^{-1})X\rangle$ for all $X \in \mathfrak{g}$ and all $\lambda \in \mathfrak{g}^*$.*

We define the mapping $ad^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$ by $ad^* := dAd^*$, i.e. $\langle ad^*(X)\xi,\, \cdot\,\rangle = -\langle \xi\,, ad(\,\cdot\,)X\rangle$ for all $X \in \mathfrak{g}$ and all $\xi \in \mathfrak{g}^*$. We denote by $\mathcal{O}_\lambda$ the orbit of the functional $\lambda \in \mathfrak{g}^*$ under the coadjoint representation Ad^*, i.e. $\mathcal{O}_\lambda := \{Ad^*(g)\lambda \mid g \in G\}$. We call $\mathcal{O}_\lambda$ the **coadjoint orbit** of λ.

By Theorem 5.3.4, every coadjoint orbit of G is the orbit of some functional $\lambda \in \mathfrak{t}^*$.

A proof of the following theorem is in [9].

Theorem 5.3.34 *The intersection of $\mathcal{O}_\lambda$ with the dual of the Cartan subalgebra $\mathfrak{t}^*$ is the Weyl group orbit $W \cdot \lambda$ of λ.*

Definition 5.3.35 *A* symplectic manifold *is a pair (M, ω) consisting of a smooth manifold M and a closed, non-degenerate 2-form ω.*

Symplectic manifolds are necessarily even-dimensional.

Definition 5.3.36 *Let G be a Lie group and let $\mathcal{O}$ be a coadjoint orbit of G. The* Kirillov-Kostant-Souriau form *$\omega_{\mathcal{O}}$ on $\mathcal{O}$ is given by $\omega_{\mathcal{O}}(\xi)(ad^*(X)\xi, ad^*(Y)\xi) := \langle \xi, [X, Y]\rangle$ for all $X, Y \in \mathfrak{g}$ and all $\xi \in \mathfrak{g}^*$.*

Theorem 5.3.37 *Let G be a Lie group and let $\mathcal{O}$ be a coadjoint orbit of G. The form $\omega_{\mathcal{O}}$ induces a symplectic structure on $\mathcal{O}$.*

Definition 5.3.38 *Let G be a Lie group and let the dimension of the orbit $\mathcal{O}$ be $2d$. The* Liouville measure *$\mu_{\mathcal{O}}$ is given by $\mu_{\mathcal{O}} := \frac{1}{d!}\omega_{\mathcal{O}} \wedge \cdots \wedge \omega_{\mathcal{O}}$ (d times).*

For $X \in \mathfrak{g}$ define,

$$F_\lambda(X) := \int_{\mathcal{O}_\lambda} e^{if(X)}\, d\mu_{\mathcal{O}_\lambda}(f).$$

An element $\lambda \in \mathfrak{t}^*$ is **regular** if $i\lambda$ does not lie on a wall of a Weyl chamber in $i\mathfrak{t}^*$. By G-invariance, the function F_λ is clearly determined by its values on $\mathfrak{t}$.

Theorem 5.3.39 (Harish-Chandra) *Let $\lambda \in \mathfrak{t}^*$ be a regular element. Then*

$$F_\lambda(H) = \prod_{\alpha \in R_+} \frac{1}{\alpha(H)} \sum_{w \in W} \operatorname{sgn}(w) e^{iw\lambda(H)} \tag{5.3}$$

for all regular $H \in \mathfrak{t}^*$.

Proof: See [7], Corollary 7.25. ◇

The volume $\operatorname{vol}(\mathcal{O}_\lambda)$ of the symplectic manifold $(\mathcal{O}_\lambda, \omega_{\mathcal{O}_\lambda})$ is the limit of $F_\lambda(H)$ as H tends 0 and can be computed when λ is regular.

Theorem 5.3.40 *Let $\lambda \in \mathfrak{t}^*$ be a regular element. Then*

$$\operatorname{vol}(\mathcal{O}_\lambda) = \prod_{\alpha \in R^+} \frac{(i\lambda, \alpha)}{(\delta, \alpha)}.$$

Proof: See [7], Proposition 7.26. $\diamond$

Hence, for a representation of G of highest weight $i\lambda$ we have $\dim \pi = \operatorname{vol} \mathcal{O}_{\lambda+\delta}$.

Theorem 5.3.41 *The j-function is a G-invariant function and is given by,*

$$j(H) = \prod_{\alpha \in R_+} \frac{2\sin(\alpha(H)/2)}{\alpha(H)}$$

for all $H \in \mathfrak{t}$.

Proof: See ([21], Chap 5, Theorem 1.10). $\diamond$

The following corollary follows immediately from Theroem 5.3.27 and Theorem 5.3.39,

Corollary 5.3.42 $F_\delta = j$.

By the Weyl character formula, we have the following remarkable generalisation,

Theorem 5.3.43 (Kirillov, [28]) *Let π be a unitary irreducible representation of G with highest weight $i\lambda$. Then*

$$j(X)\chi_\pi(\exp X) = \int_{\mathcal{O}_{\lambda+\delta}} e^{i\beta(X)}\, d\mu_{\lambda+\delta}(\beta) \tag{5.4}$$

for all $X \in \mathfrak{g}$.

In the next chapter, will shall prove (5.4) independently of the Weyl character formula.

Chapter 6

Convexity and Irreducible Representations

Let G be a real, semisimple, compact and connected Lie group with Lie algebra $\mathfrak{g}$. Let $X := (X_1, \ldots, X_d)$ be any basis of $\mathfrak{g}$ which is orthogonal with respect to the Killing form and such that for some $0 < l \leq d$, $X' := (X_1, \ldots, X_l)$ is a basis of a Cartan subalgebra of $\mathfrak{g}$ which we fix and denote by $\mathfrak{t}$. Let W be the Weyl group. Let $(\pi, \mathcal{H})$ be a unitary irreducible representation of G of highest weight λ. We write $d\pi$ for the Lie derivative of π. In this chapter we study the support of the Weyl calculus of $\frac{1}{i}d\pi(X) := (\frac{1}{i}d\pi(X_1), \ldots, \frac{1}{i}d\pi(X_d))$.

The main result of Section 6.1 (Theorem 6.1.16) is that the convex hull of the support of the Weyl calculus of $\frac{1}{i}d\pi(X)$ is the convex hull of the coadjoint orbit through λ and characterises the representation π.

The results of Section 6.2 interpret the δ-shift in the representation theory of compact Lie groups in terms of convex geometry. We also show that the convex hull of the support of the Fourier transform of the product of the j-function and the pull-back of an arbitrary matrix coefficient of the π equals the convex hull of the coadjoint orbit through $\lambda + \delta$, and that the singular support is a finite union of orbits, also characterising the representation. This in particular gives a new demonstration of the orbit correspondence for compact Lie groups [28, 17].

An earlier version of these results appears in [15].

6.1 Spectra and Matrix Coefficients

Let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. Let $\lambda_1, \ldots, \lambda_n \in \mathfrak{t}^*$ be the weights of π. Denote by $p : \mathfrak{g}^* \to \mathfrak{t}^*$ the canonical projection of $\mathfrak{g}^*$ onto $\mathfrak{t}^*$, i.e. $p(\xi) := \xi|_{\mathfrak{t}^*}$ for all $\xi \in \mathfrak{g}^*$.

We write $W_\pi(X)$ for $W_{L(\mathcal{H})}(\frac{1}{i}d\pi(X))$. Let $I_\pi := \{\frac{1}{i}\langle d\pi(\,\cdot\,)u,\, u\rangle \mid u \in \mathcal{H},\, \|u\| = 1\}$. The set I_π is known as the **moment set** of π in the literature and has been studied for a variety of classes of Lie groups [43, 44, 4, 34, 38, 31, 6, 1].

Theorem 6.1.1 ([4, 44]) *Let π be an irreducible representation of G of highest weight λ. If the set $\{\lambda - w\lambda \,:\, w \in W\}$ does not contain a root then $I_\pi = \operatorname{co}\mathcal{O}_\lambda$; otherwise, I_π is not convex and $I_\pi \subsetneq \mathcal{O}_\lambda$.*

When $G = SU(n)$ and $(\pi,\, \mathcal{H})$ is the defining representation, the moment set I_π is a single coadjoint orbit by the invariance properties of the inner product of $\mathcal{H}$.

By Theorem 4.1.1 we have

Theorem 6.1.2 $\operatorname{supp} W_\pi(X) \subseteq I_\pi$.

We have as a consequence of Corollary 4.1.2,

Theorem 6.1.3 *Suppose that π is the defining representation. Then*

$$\operatorname{supp} W_\pi(X) = I_\pi.$$

Lemma 6.1.4 $p(I_\pi) = \operatorname{co}\{\lambda_1, \ldots, \lambda_n\}$.

Proof: For all $H \in \mathfrak{t}$ and $v \in \mathcal{H}$ with $\|u\| = 1$,

$$\langle\, d\pi(H)v,\, v\rangle = \sum_{j=1}^{n} \langle\, i\lambda_j(H)v_j,\, v_j\rangle,$$

where $v_1, \ldots, v_n$ are the components of v in the orthogonal decomposition of $\mathcal{H}$ into weight spaces. Since $\sum\langle v_j,\, v_j\rangle = 1$ we have $p(I_\pi) \subseteq \operatorname{co}\{\lambda_1, \ldots, \lambda_n\}$.

Let $t_1\lambda_1 + \cdots + t_n\lambda_n$ be a convex combination of the weights λ_i. For $j = 1, \ldots, n$ let v_j be a weight vector corresponding to λ_j with $\|v_j\| = 1$ and set $v = \sqrt{t_1}v_1 + \cdots + \sqrt{t_n}v_n$. Then $\|v\| = 1$ and $\langle d\pi(\,\cdot\,)v,\, v\rangle = t_1\lambda_1 + \cdots + t_n\lambda_n$. This shows that $\mathrm{co}\,\{\lambda_1, \ldots, \lambda_n\} \subseteq p(I_\pi)$. Hence $p(I_\pi) = \mathrm{co}\,\{\lambda_1, \ldots, \lambda_n\}$ as required. $\diamond$

Corollary 6.1.5 $\mathrm{co}\,\mathrm{supp}\, W_{\pi|_T}(X') = \mathrm{co}\,(W \cdot \lambda)$.

Proof: Clearly $I_{\pi|_T} = p(I_\pi)$. In view of Lemma 6.1.4, the result now follows from Theorem 2.3.5 and Theorem 2.1.2, and the fact that $\lambda_1, \ldots, \lambda_n \in \mathrm{co}\,(W \cdot \lambda)$. $\diamond$

Lemma 6.1.6 $\mathcal{O}_\lambda \subseteq I_\pi$.

Proof: Since I_π is invariant under Ad^* and $\mathcal{O}_\lambda \cap \mathfrak{t}^* = W \cdot \lambda$, the statement follows from the containment $W \cdot \lambda \subseteq I_\pi$. $\diamond$

For any $\xi \in \mathfrak{g}^*$ and $g \in G$ we write $g \cdot \xi$ for $Ad^*(g)\xi$.

Lemma 6.1.7 $G \cdot \mathrm{co}\,\{\lambda_1, \ldots, \lambda_n\} = \mathrm{co}\,\mathcal{O}_\lambda$

Proof: Define $D := \mathrm{co}\,\{\lambda_1, \ldots, \lambda_n\}$. We have

$$G \cdot D = G \cdot \mathrm{co}\, W \cdot \lambda \subseteq \mathrm{co}\, G \cdot \lambda = \mathrm{co}\,\mathcal{O}_\lambda.$$

Conversely, let e_λ be a highest weight vector with $\|e_\lambda\| = 1$. Then $\langle d\pi(\,\cdot\,)e_\lambda,\, e_\lambda\rangle = \lambda(\,\cdot\,)$ and

$$p(\mathrm{co}\,\mathcal{O}_\lambda) = \mathrm{co}\, p(\mathcal{O}_\lambda) \subseteq \mathrm{co}\, p(I_\pi) = \mathrm{co}\, D = D.$$

Since $\mathfrak{t}^* \cap \mathrm{co}\ (\mathcal{O}_\lambda) = p(\mathrm{co}\,\mathcal{O}_\lambda)$, we have $\mathrm{co}\,\mathcal{O}_\lambda \subseteq G \cdot D$ and the result follows. $\diamond$

By Lemma 2.1.3, the numerical range $V_{L(\mathcal{H})}(\frac{1}{i}d\pi(X))$ does not depend on the choice of orthogonal basis X; we write V_π for $V_{L(\mathcal{H})}(\frac{1}{i}d\pi(X))$.

Theorem 6.1.8 $V_\pi = \mathrm{co}\,\mathcal{O}_\lambda$.

Proof: We have $I_\pi = G \cdot (I_\pi \cap \mathfrak{t}^*) \subseteq G \cdot p(I_\pi) = \operatorname{co} \mathcal{O}_\lambda$ where the last equality follows from Lemma 6.1.4 and Lemma 6.1.7. By Lemma 6.1.6 we have $\operatorname{co} \mathcal{O}_\lambda \subseteq \operatorname{co} I_\pi$. Hence $\operatorname{co} I_\pi = \operatorname{co} \mathcal{O}_\lambda$ and the result follows from Theorem 2.1.2. $\diamond$

For all $Y \in \mathfrak{g}$ and $g \in G$ we write $g \cdot Y$ for $Ad(g)Y$.

Lemma 6.1.9 $\operatorname{supp} W_\pi(X) = G \cdot \operatorname{supp} W_\pi(X)$.

Proof: Suppose that $\xi \in \mathfrak{g}^* \setminus \operatorname{supp} W_\pi(X)$. Then there exists $\phi \in \mathcal{S}(\mathfrak{g}^*)$ such that ϕ is nonvanishing in a neighbourhood of ξ and $\langle W_\pi(X), \phi \rangle = 0$. Let $g \in G$. Set $\tilde{\phi} := \phi \circ Ad^*(g^{-1})$. Then $\tilde{\phi} \in \mathcal{S}(\mathfrak{g}^*)$ and $\tilde{\phi}$ is nonzero near $g \cdot \xi$. Now,

$$\begin{aligned}
\langle W_\pi(X), \tilde{\phi} \rangle &= \int_{\mathfrak{g}} \hat{\phi}(g \cdot X) e^{d\pi(X)} \, dX \\
&= \int_{\mathfrak{g}} \hat{\phi}(X) e^{d\pi(g^{-1} \cdot X)} \, dX \\
&= \pi(g^{-1}) \int_{\mathfrak{g}} \hat{\phi}(X) e^{d\pi(X)} \, dX \pi(g) \\
&= \pi(g^{-1}) \langle W_\pi(X), \phi \rangle \pi(g) \\
&= 0.
\end{aligned}$$

Hence $g \cdot \xi \in \mathfrak{g}^* \setminus \operatorname{supp} W_\pi(X)$ and it follows that $G \cdot \operatorname{supp} W_\pi(X) \subseteq \operatorname{supp} W_\pi(X)$. The reverse inclusion is clear. This completes the proof. $\diamond$

Similarly we have

Lemma 6.1.10 $\operatorname{sing\,supp} W_\pi(X) = G \cdot \operatorname{sing\,supp} W_\pi(X)$.

Theorem 6.1.11 $\operatorname{supp} W_\pi(X) = G \cdot \operatorname{supp} W_{\pi|_T}(X')$.

Proof: Let $\epsilon > 0$. Define $\mathfrak{g}_\epsilon := \mathfrak{t} \oplus (-\epsilon, \epsilon) X_{d'+1} \oplus (-\epsilon, \epsilon) X_d$ where $(-\epsilon, \epsilon)$ denotes the open interval $\{x \in \mathbb{R} : -\epsilon < x < \epsilon\}$. By Theorem 3.2.4

$$\mathfrak{g}_\epsilon^* \cap \operatorname{supp} W_\pi(X) = \operatorname{supp} W_\pi(X)|_{\mathfrak{g}_\epsilon^*}. \tag{6.1}$$

Taking limits of both sides of (6.1) as ϵ approaches 0 and applying and Lemma 3.1.10 we have

$$\mathfrak{t}^* \cap \operatorname{supp} W_\pi(X) = \operatorname{supp} W_\pi(X)|_{\mathfrak{t}^*} = \operatorname{supp} W_{\pi|_T}(X')$$

and the result follows. $\diamond$

Let A be a subset of $\mathbb{R}^d$. The set of extremal points of A, denoted $\operatorname{Ext} A$, is the collection of points in A that are not contained in any open interval the end-points of which lie in A.

Lemma 6.1.12 ([4], Lemme 15) $\operatorname{Ext} \operatorname{co} \{\lambda_1 \ldots, \lambda_n\} = W \cdot \lambda$.

Corollary 6.1.13 $\operatorname{Ext} \operatorname{co} \mathcal{O}_\lambda = \mathcal{O}_\lambda$.

Proof: We have

$$\begin{aligned} \mathfrak{t}^* \cap \operatorname{co} \mathcal{O}_\lambda &\subseteq p(\operatorname{co} \mathcal{O}_\lambda) \\ &= \operatorname{co} p(\mathcal{O}_\lambda) \\ &\subseteq \operatorname{co} p(I_\lambda) \\ &= \operatorname{co} \{\lambda_1 \ldots, \lambda_n\} \end{aligned}$$

and the result follows from Lemma 6.1.12 on observing that $\operatorname{Ext} \operatorname{co} \mathcal{O}_\lambda$ is invariant under the coadjoint action. $\diamond$

Let p be a nonzero polynomial in d variables with complex coefficients and define the associated differential operator $p(i\frac{\partial}{\partial \xi}) := p(i\frac{\partial}{\partial \xi_1}, \ldots, i\frac{\partial}{\partial \xi_d})$. We will need the following well-known theorem from distribution theory.

Theorem 6.1.14 ([22], Theorem 7.3.9) *Let u be a scalar-valued distribution on $\mathbb{R}^d$ with compact support. Then*

$$\operatorname{co} \operatorname{sing} \operatorname{supp} u = \operatorname{co} \operatorname{sing} \operatorname{supp} p(i\frac{\partial}{\partial \xi})u. \tag{6.2}$$

Corollary 6.1.15 $\mathcal{O}_\lambda \subseteq \operatorname{sing} \operatorname{supp} W_\pi(X)$.

Proof: Clear from Lemma 6.1.12 and Theorem 4.1.1. $\diamond$

We give three proofs of the next result, utilising the various techniques of the previous chapters.

Theorem 6.1.16 $\operatorname{co}\operatorname{supp} W_\pi(X) = \operatorname{co}\mathcal{O}_\lambda$.

Proof: By Corollary 6.1.11

$$\begin{aligned}
\operatorname{co}\operatorname{supp} W_\pi(X) &= \operatorname{co}\, G\cdot \operatorname{supp} W_{\pi|_T}(X') \\
&\supseteq G\cdot \operatorname{co}\operatorname{supp} W_{\pi|_T}(X') \\
&= G\cdot V_{\pi|_T}
\end{aligned}$$

where the last equality follows from Theorem 2.3.5. By Theorem 6.1.8, $V_{\pi|_T} = \operatorname{co}\{\lambda_1,\ldots,\lambda_n\}$ and we have $\operatorname{co}\operatorname{supp} W_\pi(X) \supseteq \operatorname{co}\mathcal{O}_\lambda$ by Lemma 6.1.7. By Theorem 2.2.3, $\operatorname{co}\operatorname{supp} W_\pi(X) \subseteq V_\pi$ and the reverse inclusion follows. This completes the proof.

Second proof. Let $E := Span_{\mathbb{C}}\{I, \frac{1}{i}d\pi(X_1),\ldots,\frac{1}{i}d\pi(X_d)\}$. By Lemma 3.2.3, the map $\tilde{}\colon E\to\mathcal{A}(E)$ is an isometry and hence

$$\operatorname{co}\sigma_{\mathcal{A}(E)}(\frac{1}{i}d\pi(X)) = V_\pi$$

by Corollary 2.2.15. By Theorem 3.2.4 and Theorem 6.1.8 we have the required result.

Third proof. By Theorem 4.1.1 and Theorem 6.1.8, $\operatorname{co}\operatorname{supp} W_\pi(X) \subseteq \operatorname{co}\mathcal{O}_\lambda$. By Corollary 6.1.15, $\operatorname{co}\mathcal{O}_\lambda \subseteq \operatorname{co}\operatorname{supp} W_\pi(X)$. Hence $\operatorname{co}\operatorname{supp} W_\pi(X) = \operatorname{co}\mathcal{O}_\lambda$. $\diamond$

6.2 Convexity and the δ-shift

As in Section 5.3.2, let $\Lambda(\pi)$ denote the set of weights $\lambda_1,\ldots,\lambda_n$ of the representation π and let Δ be the set of roots of G. The following proposition is a special case of Lemma 7.1 of [40].

Proposition 6.2.1 ([40], Lemma 7.1) *Let $S \subseteq \Lambda(\pi)$. If $\lambda_{i_1} - \lambda_{i_2} \notin \Delta$ whenever $\lambda_{i_1}, \lambda_{i_2} \in S$ then*

$$S \subseteq I_\pi \cap \mathfrak{t}^*.$$

For $\eta \in \mathfrak{g}^*$, let μ_η denote Liouville measure on $\mathcal{O}_\eta$. We write μ_π for $\mu_{\frac{1}{i}d\pi(X)}$, the canonical measure on I_π, as defined in Section 4.1.

Corollary 6.2.2 *Let π be a unitary irreducible representation of G of highest weight λ. Then $\operatorname{supp}\mu_\delta * \mu_\pi$ is not convex if and only if $\mathcal{O}_\lambda \subsetneq \operatorname{co}\mathcal{O}_\delta$, and μ_δ is the unique G-invariant measure up to normalisation which convolved with μ_π yields a measure with convex support for all but minimal finite number of λ in the highest weight lattice. Furthermore, if $\operatorname{supp}\mu_\delta * \mu_\pi$ is convex then $\operatorname{supp}\mu_\delta * \mu_\pi = \operatorname{co}\mathcal{O}_{\lambda+\delta}$.*

Proof: Since μ_δ and μ_π are positive measures, we have that $\operatorname{supp}\mu_\delta * \mu_\pi = \operatorname{supp}\mu_\delta + \operatorname{supp}\mu_\pi$. Since

$$\begin{aligned} \operatorname{Ext}(\mathcal{O}_\delta + I_\pi) &\subseteq \operatorname{Ext}(\operatorname{Ext}\operatorname{co}\mathcal{O}_\delta + \operatorname{Ext}\operatorname{co} I_\pi) \\ &= \operatorname{Ext}(\mathcal{O}_\delta + \mathcal{O}_\lambda) \\ &= \mathcal{O}_{\lambda+\delta} \end{aligned}$$

it follows by $Ad^*(G)$-invariance that whenever $\mathcal{O}_\delta + I_\pi$ is convex then necessarily $\mathcal{O}_\delta + I_\pi = \operatorname{co}\mathcal{O}_{\lambda+\delta}$.

Suppose that $\mathcal{O}_\lambda \subsetneq \operatorname{co}\mathcal{O}_\delta$. Then since $\operatorname{Ext}\operatorname{co} I_\pi = \mathcal{O}_\lambda$, it follows that for any $\eta \in I_\pi$ and $g \in G$

$$(g \cdot \delta - \eta,\, g \cdot \delta - \eta) \geq (\delta,\, \delta) - (\lambda,\, \lambda) > 0.$$

Hence, $0 \notin \mathcal{O}_\delta + I_\pi$ and in particular $\mathcal{O}_\delta + I_\pi$ is not convex.

On the other hand, suppose that $\operatorname{co}\mathcal{O}_\delta = \mathcal{O}_\lambda$ or $\mathcal{O}_\lambda \not\subset \operatorname{co}\mathcal{O}_\delta$. If $w \in W$ denotes the reflection through the hyperplane orthogonal to the root $\alpha \in R$ then $\delta - w\delta = \alpha$. In view of Proposition 6.2.1, implies that $\mathcal{O}_\delta + I_\pi$ must be convex.

$\diamond$

By Corollary 6.2.2, the notorious δ-shift in representation theory can be interpreted as a geometric procedure that "minimises" the non-convexity of the support of the measure μ_π.

Recall that the singular support $\operatorname{sing\,supp} u$ of a distribution u is the complement of the largest open set on which u corresponds to a smooth function.

Corollary 6.2.3 $\operatorname{sing\,supp} \mu_\delta * \mu_\pi = \bigcup_{w \in W} \mathcal{O}_{\lambda + w\delta}$.

Proof: Clearly, for all $w \in W$, the points $W \cdot (\lambda + w\delta)$ are singularities of $\mu_\delta * \mu_\pi$. By Proposition 6.2.1, the convolution $\mu_\delta|_{\mathfrak{t}^*} * \mu_\pi|_{\mathfrak{t}^*}$ is smooth at all other points. Hence, the result follows by $Ad^*(G)$-invariance and the fact that $(\mu_\delta * \mu_\pi)|_{\mathfrak{t}^*} = \mu_\delta|_{\mathfrak{t}^*} * \mu_\pi|_{\mathfrak{t}^*}$ $\diamond$

Identifying the Lie algebra $\mathfrak{g}$ with $\mathbb{R}^{\dim G}$, let $\hat{j}$ denote the distributional Fourier transform of the j-function. By Theorem 4.1.1 and commutation relations, the following is immediate,

Theorem 6.2.4 *The entries of $\hat{j} * W_\pi(X)$ are polynomials in derivatives in the root, toral and radial directions applied to the measure $\mu_\delta * \mu_\pi$.*

Theorem 6.2.4 and some of its consequences will be discussed in more detail in the following chapter.

Theorem 6.2.5 $\operatorname{sing\,supp} \hat{j} * W_\pi(X) = \bigcup_{w \in W} \mathcal{O}_{\lambda + w\delta}$.

Proof: By Theorem 4.1.1 and Proposition 6.2.1, $\operatorname{sing\,supp} \hat{j} * W_\pi(X) \subseteq \cup_{w \in W} \mathcal{O}_{\lambda + w\delta}$. By Corollary 6.1.15, $\mathcal{O}_\lambda \subseteq \operatorname{supp} W_\pi(X) \subseteq \operatorname{co} \mathcal{O}_\lambda$. Since $\operatorname{Ext}(\operatorname{co} \mathcal{O}_\lambda) = \mathcal{O}_\lambda$, the reverse inclusion follows. $\diamond$

Theorem 6.2.6 ([22], Theorem 4.3.3) *Let u_1, u_2 be compactly supported distributions on $\mathbb{R}^d$. Then*

$$\operatorname{co\,supp} u_1 * u_2 = \operatorname{co\,supp} u_1 + \operatorname{co\,supp} u_2. \tag{6.1}$$

Lemma 6.2.7 $\operatorname{co}\mathcal{O}_{\lambda+\delta} = \operatorname{co}\mathcal{O}_\lambda + \operatorname{co}\mathcal{O}_\delta$.

Proof: We have,

$$\begin{aligned}\operatorname{Ext}(\operatorname{co}\mathcal{O}_\lambda + \operatorname{co}O_\delta) &\subseteq \operatorname{Ext}(\operatorname{Ext}\operatorname{co}\mathcal{O}_\lambda + \operatorname{Ext}\operatorname{co}\mathcal{O}_\delta) \\ &= \operatorname{Ext}(\mathcal{O}_\lambda + \mathcal{O}_\delta) \\ &= \mathcal{O}_{\lambda+\delta}.\end{aligned}$$

Therefore, by the Krein-Milman theorem, $\operatorname{co}\mathcal{O}_{\lambda+\delta} = \operatorname{co}\mathcal{O}_\lambda = \overline{\operatorname{co}\mathcal{O}_\delta}$. Since the coadjoint orbits of G are compact and the convex hull of a closed set is closed, the result follows. ◇

Theorem 6.2.8 $\operatorname{co}\operatorname{supp}\hat{j} * W_\pi(X) = \operatorname{co}\mathcal{O}_{\lambda+\delta}$.

Proof: This is immediate from Theorem 6.2.6 and Theorem 6.1.16.

Second proof. Since $\operatorname{supp}\mu_\delta * \mu_\pi \subseteq \operatorname{co}\mathcal{O}_{\lambda+\delta}$ we have $\operatorname{supp}\hat{j} * W_\pi(X) \subseteq \operatorname{co}\mathcal{O}_{\lambda+\delta}$ by Theorem 4.1.1. By Corollary 6.1.15, $\mathcal{O}_\lambda \subseteq \operatorname{supp} W_\pi(X)$ and it follows that $\mathcal{O}_{\lambda+\delta} \subseteq \operatorname{supp}\hat{j} * W_\pi(X)$ which establishes the reverse inclusion. ◇

Insofar as $\operatorname{Ext}\operatorname{co}\mathcal{O}_{\lambda+\delta} = \mathcal{O}_{\lambda+\delta}$, Theorem 6.2.8 yields what can be thought of as a procedure inverse to geometric quantisation, but which is independent of the Kirillov formula.

Chapter 7

A Non-Commutative Kirillov-type Formula

According to the Kirillov character formula, the Fourier transform of the product of the j-function character of an irreducible representation π of a compact Lie group G coincides with the Liouville measure of the coadjoint orbit through $\lambda + \delta$, where λ is the highest weight of π.

In this chapter, we show that this formula admits a noncommutative extension in the sense that the Fourier transform of the product of j and an arbitrary matrix coefficient of π is a distribution having the form of a linear combination of derivatives in root, toral and radial directions of the Lie algebra $\mathfrak{g}$ applied to a measure supported inside the convex hull the coadjoint orbit through $\lambda + \rho$, with singularities lying inside a finite union of coadjoint orbits (Theorem 7.3.1). These results strengthen and generalise work of Cazzaniga [12] concerning $SU(2)$.

7.1 The $SU(2)$ Case

The group $SU(2)$ consists of all matrices g of the form $g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ where $\alpha, \beta \in \mathbb{C}$ and satisfy the condition $|\alpha|^2 + |\beta|^2 = 1$. The Lie algebra $\mathfrak{su}(2)$

consists of all traceless skew-hermitian matrices of order 2 and we fix for it a basis $i\sigma = (i\sigma_1, i\sigma_2, i\sigma_3)$ given by,

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In particular, the $\mathbb{C}$-span of σ_1 determines a Cartan subalgebra of $\mathfrak{su}(2)$ which we fix and denote by $\mathfrak{t}$. The roots are given by $\alpha_j(i\xi\sigma_1) := 2(-1)^{j-1}\xi$ for all $\xi \in \mathbb{R}$ and $j = 1, 2$. The Weyl group is the symmetric group $\mathcal{S}_2$ of order 2.

By means of the basis σ, we identify $\mathfrak{su}(2)$ with $\mathbb{R}^3$ and $\mathfrak{t}$ with $\mathbb{R}$. The directional derivatives $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$ in the coordinate directions of $\mathbb{R}^3$ correspond to operators of differentiation in the root directions and the single toral direction respectively of the Lie algebra $\mathfrak{su}(2)$. We denote the derivative in the radial direction of $\mathbb{R}^3$ by $\frac{\partial}{\partial r}$, i.e. $r\frac{\partial}{\partial r} = x \cdot \frac{\partial}{\partial x}$ where $r = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$. For $X \in \mathfrak{su}(2)$, let $\|X\|$ denote the euclidean norm of X as a vector in $\mathbb{R}^3$.

We identify $\mathfrak{su}(2)$ with the dual $\mathfrak{su}(2)^*$ by means of the form $-\mathrm{Tr}(X\,\cdot)$. Observing that

$$\|X\|^2 = -\frac{1}{2}\mathrm{Tr}(X^2) = \det X\,, \tag{7.1}$$

it follows from the invariance of det that the coadjoint orbits of $SU(2)$ are spheres.

For a non-negative integer $\lambda \in \mathbb{N}$, denote by H_λ the vector space of single-variable complex polynomials f of degree at most λ, i.e. $f(z) = \sum_{j=0}^{\lambda} a_j z^j$ for $a_1, \ldots, a_\lambda \in \mathbb{C}$ and define the mapping $\pi^{(\lambda)} : SU(2) \to End(H_\lambda)$ given by

$$(\pi^{(\lambda)}(g)f)(z) := (\beta z + \bar{\alpha})^\lambda f\left(\frac{\alpha z - \bar{\beta}}{\beta z + \bar{\alpha}}\right).$$

We have the following

Theorem 7.1.1 *The set of mappings $\{\pi^{(\lambda)} : \lambda \in \mathbb{N}\}$ is a complete set of representatives for the equivalence classes of unitary irreducible representations of $SU(2)$. Furthermore, the representation $\pi^{(\lambda)}$ has highest weight λ.*

By Theorem 5.3.41, the j-function of $SU(2)$ is given by

$$j(X) = \frac{\sin\|X\|}{\|X\|}$$

and the Fourier transform $\hat{j}$ coincides with the unitarily invariant probability measure ν on the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$. Let $\mathcal{B}$ denote the convolution algebra generated by the measure ν. We denote by B_λ the ball in $\mathbb{R}^3$ of radius λ centred at the origin, i.e. $B_\lambda := \{X \in \mathbb{R}^3 : \|X\| \leq \lambda\}$, and we write ∂B_λ for its boundary.

Theorem 7.1.2 (Cazzaniga) *For each $\lambda \in \mathbb{N}$, the matrix-entries of the distribution $\hat{j} * W_{\pi(\lambda)}(\sigma)$ are polynomial in the differential operators $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$ and $\frac{\partial}{\partial r}$ applied to members of the algebra $\mathcal{B}$. Moreover,*

$$\operatorname{supp} \hat{j} * W_{\pi(\lambda)}(\sigma) \subseteq B_{\lambda+1}$$

and

$$\operatorname{sing\,supp} \hat{j} * W_{\pi(\lambda)}(\sigma) \subseteq \bigcup_{j=0}^{\lfloor \frac{\lambda+1}{2} \rfloor} \partial B_{(\lambda+1-2j)}.$$

Proof: See [12]. $\diamond$

By the results of the previous chapter, we have the following strengthening of Theorem 7.1.2.

Theorem 7.1.3 *For each $\lambda \in \mathbb{N}$, the matrix entries of the distribution $\hat{j} * W_{\pi(\lambda)}(\sigma)$ are polynomial in the differential operators $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$ and $\frac{\partial}{\partial r}$ applied to the measure $\mu_\delta * \mu_{\pi(\lambda)}$. Moreover,*

$$\operatorname{co\,supp} \hat{j} * W_{\pi(\lambda)}(\sigma) = B_{\lambda+1} \tag{7.2}$$

and

$$\operatorname{sing\,supp} \hat{j} * W_{\pi(\lambda)}(\sigma) = \partial B_{\lambda-1} \cup \partial B_{\lambda+1}. \tag{7.3}$$

Proof: The first statement follows directly from Theorem 4.1.1 and the easily verified fact that

$$\frac{\partial}{\partial x} \cdot x = r\frac{\partial}{\partial r} + 3I.$$

Choosing a normalisation of of the metric on $\mathfrak{su}(2)^*$ so that the length of δ is unity, we can replace $r\frac{\partial}{\partial r}\mu_\delta$ with $\frac{\partial}{\partial r}\mu_\delta$. The equalities (7.2) and (7.3) follow from Theorem 6.2.8 and Theorem 6.2.5 respectively. $\diamond$

7.2 The $SU(n)$ Case

The Lie algebra $\mathfrak{su}(n)$ consists of traceless skew-hermitian matrices of order n. Let E_{ij} denote the $n \times n$ matrix with 1 in the (i, j)-th position. We fix a basis $i\sigma = (i\sigma_1, \ldots, i\sigma_{n^2-1})$ of $\mathfrak{su}(n)$ given by $\sigma_j = E_{jj} - E_{nn}$ for $1 \leq j \leq n-1$, $\sigma_{n+j+k-2} = iE_{jk} - iE_{kj}$ for $1 \leq j \neq k \leq n-1$ and $\sigma_{(\frac{n}{2}+1)(n-1)+j+k-1} = E_{jk} + E_{kj}$ for $1 \leq j \neq k \leq n-1$. The $\mathbb{C}$-span of $\sigma_1, \ldots, \sigma_{n-1}$ determines a Cartan subalgebra which we denote by $\mathfrak{t}$. By means of this basis, we identify $\mathfrak{t}$ with $\mathbb{R}^{n-1}$. Then the roots are given by $\alpha_{jk}(\xi) = \xi_j - \xi_k$ for $\xi \in \mathbb{R}^{n-1}$ and $j \neq k$. We also identify $\mathfrak{su}(n)$ with $\mathbb{R}^{n^2-1}$ and denote by $\|X\|$ the euclidean norm of the coordinates of $X \in \mathfrak{su}(n)$. The choice of positive Weyl chamber $\mathcal{C} := \{\xi \in \mathfrak{t} : \xi_j - \xi_{j+1}, 1 \leq j < n\}$ corresponds to the choice of positive roots $\{\alpha_{jk} : j < k\}$. The sum of the positive roots is given by $2\delta = \sum_{j=1}^{n}(n-2j+1)\xi_j$ and the set of dominant weights P consists of $(n-1)$-tuples $(\xi_1, \ldots, \xi_{n-1})$ satisfying $\xi_{j+1} > \xi_j$ for all $1 \leq j < n$. The Weyl group is $\mathcal{S}_n$.

The derivatives $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n^2-1}}$ in coordinate directions of $\mathbb{R}^{n^2-1}$ correspond to operators of differentiation in the root and toral directions of $\mathfrak{su}(n)$. We denote the radial derivative by $\frac{\partial}{\partial r}$.

As in the $SU(2)$ case above, the dual $\mathfrak{su}(n)^*$ can be identified with $\mathfrak{su}(n)$ by the trace form $-Tr(X\cdot)$. We write $\mathcal{O}_\lambda$ for the coadjoint orbit through $\lambda \in \mathfrak{su}(n)^*$. We have

Theorem 7.2.1 *Let π be a unitary irreducible representation of $SU(n)$ of highest weight λ. The matrix entries of the distribution $\hat{j} * W_\pi(X)$ lie in the vector space spanned by the measure $\mu_\delta * \mu_\pi$ and the differential operators $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n^2-1}}$ and $r\frac{\partial}{\partial r}$. Moreover,*

$$\operatorname{co\,supp} \hat{j} * W_\pi(\sigma) = \mathcal{O}_{\lambda+\delta} \tag{7.1}$$

and

$$\operatorname{sing\,supp} \hat{j} * W_\pi(\sigma) = \bigcup_{w \in \mathcal{S}_n} \mathcal{O}_{\lambda+w\delta}. \tag{7.2}$$

Proof: The first statement follows directly from Theorem 4.1.1 and the fact that
$$\frac{\partial}{\partial x}\cdot x = r\frac{\partial}{\partial r} + (n^2-1)I.$$
The equalities (7.1) and (7.2) follow from Theorem 6.2.8 and Theorem 6.2.5 respectively. ◇

7.3 A Formula for Compact Lie Groups

Let G be a semisimple connected compact Lie group and let X be any basis of the Lie algebra $\mathfrak{g}$ of G which is orthogonal with respect to the Killing form. We can thereby identify $\mathfrak{g}$ with a Euclidean space and consider the coordinate derivatives in that space as derivatives in the directions of the roots and toral directions.

If $\dim\mathfrak{g} = d$ then commutation relations imply that
$$\frac{\partial}{\partial x}\cdot x = r\frac{\partial}{\partial r} + dI \tag{7.1}$$
where $\frac{\partial}{\partial r}$ denotes the radial derivative in $\mathbb{R}^n$ and for the right choice of metric we have the following generalisation of the above result,

Theorem 7.3.1 *Let π be a unitary irreducible representation of compact semisimple connected Lie group G with highest weight λ. The matrix entries of the distribution $\hat{j} * W_\pi(X)$ are polynomial in the differential operators $\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_d}$ and $r\frac{\partial}{\partial r}$ applied to the measure $\mu_\delta * \mu_\pi$. Moreover,*
$$\operatorname{co}\operatorname{supp}\hat{j} * W_\pi(X) = \operatorname{co}\mathcal{O}_{\lambda+\delta} \tag{7.2}$$
and
$$\operatorname{sing\,supp}\hat{j} * W_\pi(X) = \bigcup_{w\in W}\mathcal{O}_{\lambda+w\delta}. \tag{7.3}$$

Proof: The first statement follows directly from Theorem 4.1.1 and 7.1.
$$\frac{\partial}{\partial x}\cdot x = r\frac{\partial}{\partial r} + dI.$$

The equalities (7.2) and (7.3) follow from Theorem 6.2.8 and Theorem 6.2.5 respectively. $\diamond$

Since the measure μ_π is uniform, the entries of $\hat{j} * W_\pi(X)$ are also expressible in form $D_{ij} \cdot \mu_\delta * \mu_\lambda$ where D_{ij} is a polynomial in the derivatives $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n^2-1}}$ and $\frac{\partial}{\partial r}$, of order at most that determined by Nelson's formula as in the theorem above, and μ_λ is the Liouville measure on the orbit $\mathcal{O}_\lambda$. It is an intersting open problem to determine D as well as $\operatorname{supp} D \cdot \mu_\delta * \mu_\lambda$.

For $\lambda \in \mathfrak{t}^*$, denote by e_λ the point-mass at λ and let μ_λ^p represent the orthogonal projection of μ_λ onto $\mathfrak{t}^*$. We write $\mathfrak{t}_+^*$ for the image of the positive Weyl chamber. We have the following theorem,

Theorem 7.3.2 ([16], Theorem 3.4) *Let $\lambda, \lambda' \in \mathfrak{t}^*$ be regular elements. Then*

$$\mu_\lambda * \mu_{\lambda'} = \int_{\mathfrak{t}_+^*} \phi(\lambda, \lambda', \lambda'') \mu_{\lambda''} d\lambda''$$

where

$$\phi(\lambda, \lambda', \lambda'') = \sum_{w \in W} \operatorname{sgn} e_{w\lambda} * \mu_{\lambda'}^p(\lambda'')$$

for all $\lambda'' \in \mathfrak{t}_+^*$.

Corollary 7.3.3 ([16], Corollary 3.5) *Let $\lambda, \lambda' \in \mathfrak{t}^*$ be regular elements and suppose that $\lambda + \operatorname{co} W\lambda \subseteq \mathfrak{t}_+^*$. Then $\phi(\lambda, \lambda', \lambda'') = e_\lambda * \mu_{\lambda'}^p(\lambda'')$ for all $\lambda'' \in \mathfrak{t}_+^*$.*

Corollary 7.3.3 can give upper bounds for $\operatorname{supp} D \cdot \mu_\delta * \mu_\lambda$ when λ is large. This is the subject of a future work.

The expression of Theorem 4.1.1 is non-unique; in another direction, we have two alternatives expressions discussed below. Let M be an $n \times n$ hermitian matrix and denote by $P_j(M)$ the homogoneous component of degree j in the expansion of $\det(I - A)$. Let Σ_n denote the unit sphere in $\mathbb{C}^n$ and write ν for the unitary invariant probability measure on Σ_n.

Theorem 7.3.4 ([18]) *Let A be an $n \times n$ hermitian matrix. For $1 \leq j, k \leq n$,*

$$(e^A)_{jk} = \frac{1}{\gamma(r)} \sum_{j=0}^{r} P_j(A) \frac{d^{n-j}}{ds^{r-j}} \left(s^n \int_{\Sigma_n} u_j \bar{u}_k e^{s\langle Au, \bar{u} \rangle} \, d\nu(u) \right) \Bigg|_{s=1}.$$

Let σ be a basis of the vector space of $n \times n$ hermitian matrices as above. Then if A is such a matrix we write e write $A = \frac{1}{2}\xi \cdot \sigma$. We also have,

Theorem 7.3.5 ([30]) *Let A be an $n \times n$ hermitian matrix. Then*

$$e^{iA} = \frac{1}{n} K(\xi) I - i \frac{\partial}{\partial \xi} K(\xi) \cdot \lambda$$

where $K(\xi) := \operatorname{Tr} e^{iA}$.

Let $C_j(\xi) := \operatorname{Tr} A^j$. The following formula also appears in [30]:

$$A^j = \frac{C_j}{n} I + \frac{1}{j} \frac{\partial}{\partial \xi} C_{j+1}(\xi) \cdot \lambda. \tag{7.4}$$

for $j = 1, \ldots, n-1$.

Substituting equation (7.4) into the proof of formula (4.1) facilitates the extraction of explicit forms for the matrix coefficients of the Weyl calculus without having to calculuate high-powers of hermitian matrices. Expressions for the C_j are computed in [30] for $n = 2, 3, 4$, but it is not clear how the methods involved in these computations can be generalised.

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