

Existence and uniqueness of solutions to non-linear first order dynamic equations on time scales

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Publication Date:

2009

DOI:

<https://doi.org/10.26190/unsworks/14894>

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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO
NON—LINEAR FIRST ORDER DYNAMIC EQUATIONS
ON TIME SCALES

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Date of Completion 12/03/2010



**Existence and uniqueness of solutions to
non-linear first order dynamic equations on time scales**

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2009

Submitted in accordance with the requirements for the award of
Doctor of Philosophy

Acknowledgements

For the accomplishment of this research, my deepest gratitude goes to my supervisor, Dr Chris Tisdell. He offered an interesting idea of investigation in a novel field of mathematics and provided timeless and continuous academic guidance with all possible financial, moral and social support to carry out the project.

I would also like to thank the School of Mathematics and Statistics for all academic, financial and administrative support I enjoyed during this project.

My special thanks go to my colleagues: James Pettigrew, for his guidance and help in the software I used to compile this project; and Nathan Pearce, for proof reading of my thesis.

I am particularly thankful to my family for all the support they provided me during the writing of this thesis.

Lastly, I am indebted to my (late) father who always encouraged me to be a doctor in a field of my choice.

Atiya H. Zaidi

2009

Dedication

I dedicate this work to the great mathematician, scientist and social reformer of the late 8th century

Jaffer Bin Muhammad AL-SADIQ

the sixth progeny of prophet Mohammad(S). Developing an academy founded by his father and grandfather, Al-Sadiq laid the foundations of a modern university and research centre that is believed to have produced over 4000 graduates and researchers in natural science, social science and mathematics. The resulting increased awareness in the society was taken by the rulers of his time as a threat to their political power. Gradually, Al-Sadiq and a large number of his fellows and students were killed and the university was closed down by the Abbaside ruler, Abu Jaffer Al-Mansoor and his followers. The remaining students moved to other areas. Many did not reveal their identities with Al-Sadiq and his academy to save their lives yet continued their work.

One of his most prominent students, Jabir-bin-Hayan, a famous chemistry scholar also known as Geber, is called the 'father of chemistry'. Another scholar named Muhammad bin Musa Al-Khwarizmi, a Persian mathematician, astronomer and geographer, is known as the 'father of algebra'. He is also believed to have been a student at the academy of Al-Sadiq.

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Abstract

The theory of dynamic equations on time scales provides an important bridge between the fields of differential and difference equations. It is particularly useful in describing phenomena that possess a hybrid continuous–discrete behaviour in their growth, like many temperate–zone insect populations and crops. A dynamic equation on a time scale is a generalised ‘two-in-one’ model, it serves as a differential equation for purely continuous domains and as a difference equation for purely discrete ones.

The field of “dynamic equations on time scales” is about 20 years old. As such, much of the basic (yet very important) linear theory has been established, however the non-linear extensions are yet to be fully developed. This thesis aims to fill this gap by providing the foundational framework of non-linear results from which further lines of inquiry can be launched.

This thesis answers several important questions regarding the qualitative and quantitative properties of solutions to non-linear dynamic equations on time scales. Namely,

- (i) When do solutions exist?
- (ii) If solutions exist, then are they unique?
- (iii) How can such solutions be closely approximated?
- (iv) How can we explicitly solve certain problems to extract their solutions?

The methods employed to address the above questions include: dynamic inequalities; iterative techniques and the method of successive approximations; and the fixed point approaches of Banach and Schauder.

ATIYA ZAIDI
27-04-2010

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Chapter 1

Introduction

1.1 Historical development of the theory

The study of dynamic equations on time scales was initiated by S. Hilger (1988) when he introduced the concept and calculus of *measure chains* in his attempt to unify mathematical analyses of continuous and discrete dynamics, see [45]. Dynamic equations describe the ideas of continuous and discrete mathematics through a single type of equation that could be equally useful to model processes defined on continuous domains, discrete domains or both at the same time. This means that dynamic equations present a hybrid framework of continuous and discrete dynamic modelling. This hybrid framework serves as a differential equation when the time scale is connected and as a difference equation when the time scale is the set of integers. For this reason, the equations involved are termed as dynamic equations on time scales.

There has been wide research in the field of dynamic equations on time scales. A systematic development of dynamic systems on time scales initiated by Hilger [45] was considered later in a monograph [54] and a few articles comprising time scale inequalities [1], exponential functions on time scales [21], boundary value problems [7] and linear dynamic equations [20], revealing several lines of further investigation. A more comprehensive collection of ideas including the time scale calculus, the linear and some non-linear theory and the dynamic inequalities was presented in [19]. This book has been the major source of literature in this work. There has been further advancement related to dynamic equations on time scales gathered in [22]. These advancements have been on: nabla dynamic equations [10]; lower and upper solutions

to boundary value problems [8]; self-adjoint equations with mixed derivatives [61]; measure theory [18]; positive solutions of boundary value problems [9], [13]; higher order dynamic equations [36]; boundary value problems on infinite intervals [4]; symplectic systems [33], [48], [47]; oscillatory and non-oscillatory dynamic equations on time scales [23], [37], [89]; and impulsive dynamic equations [15], [29], [39].

1.2 Applications

The above advancements suggest some open areas for a unified analytic approach. For instance: dynamic modelling of ecological systems [27], [85]; analysis of various complex dynamic models regarding social and economic systems [68], [62]; dynamic modelling using neural networks for natural and artificial intelligent systems [28], [49], [56], [58], [57], [76]; modelling of phenomena undergoing pulsations or time-delay effects [53], [73]; modelling describing behaviours regarding changes in population dynamics [74], [77], [82]; and dynamic models regarding epidemic diseases and their outcomes and control [90].

The above ideas open new horizons for analysis and modelling of various types of phenomena having hybrid structure and attracted researchers from all over the world to take interest into this relatively young area of applied mathematics. As a result, analogues of many ideas from differential and difference calculus have been developed into the time scale setting. Naturally, this led to the development of mathematical modelling involving dynamic equations on time scales as a more generalised and flexible mode of understanding physical phenomena through a single type of equation.

1.3 Contribution of this work

So far researchers in the field of dynamic equations on time scales have developed a profound linear theory. This work contributes to the non-linear theory by establishing properties of solutions to non-linear dynamic equations on time scales. The properties explored include: non-multiplicity of solutions; successive approximations of solutions via classical methods; and existence and uniqueness of solutions using fixed point theory, as well as the method of upper and lower solutions within the

time scale setting. In addition, methods are provided to solve non-linear dynamic equations on time scales by separation of variables and by substitution. Thus, the current work will be of interest for theoretical and applied mathematicians as well as for graduate students having some background in ordinary differential equations and functional analysis.

1.3.1 Publications arising from this work

A number of results herein have been published in [82] and [83]. Another set of results [87] has been submitted for publication.

1.4 Literature review

Since this work examines properties of solutions to non-linear first order dynamic equations on time scales, several pieces of literature regarding solutions of first and second order initial and boundary value problems have been studied. These include books and articles on existence and uniqueness of solutions using fixed point methods, such as [2], [32], [34], [43], [44], [78]; using degree theory or maximum principles [65], [81]; using dynamic inequalities [79]; and using approximation methods [38], [52]. In addition, properties like multiplicity/non-multiplicity of solutions [80] and boundedness and uniqueness of solutions to dynamic equations on time scales [66] have also been studied. Considering [45], [19] and [22] as primary sources, investigation has been carried out for existence and uniqueness of solutions to first order non-linear (delta) dynamic initial value problems using fixed point methods and the classical method of successive approximations and for (nabla) dynamic equations using the method of upper and lower solutions.

1.5 The main idea

This work concerns first order dynamic equations of the type

$$\mathbf{x}^\Delta = \mathbf{f}(t, \mathbf{x}), \tag{1.5.1}$$

$$\mathbf{x}^\Delta = \mathbf{f}(t, \mathbf{x}^\sigma), \tag{1.5.2}$$

and of the type

$$x^\nabla = f(t, x). \quad (1.5.3)$$

The symbols Δ and ∇ used above carry the ‘feature’ of a derivative in the time scale setting. The only difference between these two is of the forward and backward movement of a point in the time scale. Thus, results involving functions allowing advancement to the ‘right’ of a point in the time scale will be discussed in terms of a dynamic equation of the type (1.5.1) or (1.5.2) called ‘delta equations’. Results are produced considering these IVPs as n -dimensional systems for $n \geq 1$.

On the other hand, results involving functions carrying the feature of backward movement will be produced discussing (1.5.3) called a ‘nabla equation’, which is a scalar dynamic IVP as it involves scalar functions x and f . See notation on page 145.

While equations (1.5.1) and (1.5.2) look similar, they are actually different. For example: if

$$\mathbf{f}(t, \mathbf{x}) := t + \mathbf{x},$$

then, using the Simple Useful Formula in Theorem A.3.2(4), we obtain

$$\begin{aligned} \mathbf{x}^\Delta &= \mathbf{f}(t, \mathbf{x}^\sigma) = t + \mathbf{x}^\sigma \\ &= t + \mu \mathbf{x}^\Delta + \mathbf{x} \\ &= \mu \mathbf{x}^\Delta + \mathbf{f}(t, \mathbf{x}) \\ &= \mu \mathbf{f}(t, \mathbf{x}^\sigma) + \mathbf{f}(t, \mathbf{x}). \end{aligned}$$

This yields

$$\mathbf{f}(t, \mathbf{x}^\sigma) = \frac{\mathbf{f}(t, \mathbf{x})}{1 - \mu}.$$

Here μ is the graininess function defined in (A.1.1), such that $\mu \neq 1$.

1.6 Development and organisation

This thesis is organised in the following manner.

In Chapter 2, we discuss non-multiplicity of solutions to dynamic initial value problems involving delta equations. Our results in this chapter use a generalised

uniform Lipschitz condition and Gronwall's inequality in the context of time scales. The ideas presented in the chapter will be used in many other results in the following chapters. Results are also presented using other conditions in the absence of the Lipschitz condition. Examples have been provided to reinforce results. The non-multiplicity of solutions leads to further investigations like existence, uniqueness and smoothness of solutions. These properties have been explored and discussed in the following chapters.

In Chapter 3, we use classical methods of constructing approximate solutions to the IVPs involving delta equations, within a subset of a time scale interval as well as in the entire interval, following the Picard Lindelöf approach. The results have been established considering the initial approximation as a continuous function of time. We have used the Lipschitz condition and the Weierstrass test as important tools to prove our results in this chapter. Existence and uniqueness results have also been presented considering our IVP as the limit of a sequence of IVPs, with the help of Arzela–Ascoli theorem, while the Lipschitz condition is not used. The results have been reinforced with examples and extended to higher order dynamic equations.

In Chapter 4, the existence and uniqueness of solutions of the IVPs involving delta equations has been established using the Banach's fixed point theorem. We prove these results by defining new ways of measuring distances using the exponential functions and, hence, constructing Banach spaces on the time scale platform. Doing this, we have been able to eliminate some previously established restrictions on the existence of unique solutions to our dynamic IVPs. We also discuss a special case of our results in this chapter, where they have been applied within certain balls using the local Banach theory. Moreover, The unique solutions established through our results have been shown to be Lipschitz continuous with respect to the initial value. The results in this chapter have also been extended to higher order equations.

Chapter 5 consists of existence results for *nabla* initial value problems. To work with these equations, the time scale calculus regarding the *nabla* derivative and the *nabla* integral has been discussed and the IVPs have been redefined. The results in this chapter are obtained using the method of lower and upper solutions and provide a way of 'locating' solutions to the given IVPs within the range defined by lower and upper solutions. The Arzela–Ascoli theorem and Schauder's fixed point theorem

have been used as the main tools to prove our results in this chapter.

In Chapter 6, we present explicit solutions to some non-linear delta equations employing methods of separation of variables and solution by substitution. The ideas provide novel ways of solving non-linear dynamic equations extending ideas from the theory of differential equations.

The thesis is concluded with some open problems and questions for further research.

An explanation of notation along with some preliminary aspects of time scale calculus used in this work is provided in Appendix A at the end of the thesis. The time scale calculus includes definitions and properties of forward and backward movement functions in the time scale, continuity, delta and nabla differentiation, delta and nabla integration, the chain rule and the special functions. The above concepts have been explained with the help of ample examples using a variety of time scales. Results regarding unique solutions of linear dynamic equations have been presented and non-linear forms of dynamic equations have been introduced.

Chapter 2

The non–multiplicity of solutions

2.1 Introduction

In this chapter, we explore the “non–multiplicity” of solutions to first order dynamic initial value problems on time scales. We discuss conditions under which these IVPs have either one solution or no solution at all.

The multiplicity and non–multiplicity of solutions to dynamic initial value problems is mathematically interesting for both theoretical and practical purposes. The initial condition and the function f play a vital role in determining whether a given IVP has at least one solution, at most one solution, exactly one solution or no solution. For example, consider the dynamic IVP

$$x^\Delta = \frac{x-1}{t}, \quad \text{for all } t \in (0, \infty)_{\mathbb{T}}; \quad (2.1.1)$$

$$x(0) = 1. \quad (2.1.2)$$

Note that there are infinite number of solutions $x(t) = 1 + ct$, where c is an arbitrary constant, to the above IVP for all $t \in (0, \infty)_{\mathbb{T}}$. However, if we change the initial condition to $x(0) = 0$, then there is no solution. On the other hand, an initial condition $x(1) = 1$ would result in only one solution corresponding to $c = 0$.

Similarly, if we change the function f to $\frac{x+1}{t}$, for all $t \in (0, \infty)_{\mathbb{T}}$, then (2.1.1),

(2.1.2) takes the form

$$x^\Delta = \frac{x+1}{t}, \quad \text{for all } t \in (0, \infty)_{\mathbb{T}}; \quad (2.1.3)$$

$$x(0) = 1. \quad (2.1.4)$$

Solutions to (2.1.3) would be of the form $x(t) = -1 + ct$, where c is an arbitrary constant, and would not satisfy (2.1.4).

Thus, a change in the initial condition or the function f can change the multiplicity of solutions to an IVP.

In real life problems, it may not be possible to change the initial or prevailing states of a problem modelled by a dynamic equation. If we know *a priori* that a mathematical formulation of a physical system is an initial value problem that has either one solution or no solution, then the ‘existence’ of a solution to the system would guarantee its uniqueness. In this way, the property ‘one solution or no solution’ of an initial value problem forms a basic stepping stone to explore further properties. This property is termed as the *non-multiplicity* of solutions.

Most of the ideas in our results have gained inspiration from the non-linear theory of ordinary differential equations. Analogues of these ideas have been transformed to the time scale setting and add to the non-linear theory of dynamic equations on time scales.

2.1.1 The main objective

Let $t_0, a \in \mathbb{T}$ and $a > t_0$. Let $[t_0, t_0 + a]_{\mathbb{T}}$ be an arbitrary interval in \mathbb{T} and $D \subseteq \mathbb{R}^n$. Consider a right-Hilger-continuous (see Definition A.2.2), possibly non-linear function $\mathbf{f} : [t_0, t_0 + a]_{\mathbb{T}}^\kappa \times D \rightarrow \mathbb{R}^n$. That is, \mathbf{f} maps elements of $[t_0, t_0 + a]_{\mathbb{T}}^\kappa \times D$ to \mathbb{R}^n for $n \geq 1$.

Let \mathbf{x}_0 be a point of \mathbb{R}^n . Consider the initial value problems

$$\mathbf{x}^\Delta = \mathbf{f}(t, \mathbf{x}), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa; \quad (2.1.5)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0, \quad (2.1.6)$$

and

$$\mathbf{x}^\Delta = \mathbf{f}(t, \mathbf{x}^\sigma), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa; \quad (2.1.7)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0. \quad (2.1.8)$$

The main aim of this chapter is to answer the question:

Under what conditions do the systems (2.1.5), (2.1.6) and (2.1.7), (2.1.8) of dynamic IVPs have, at most, one solution? That is, when do these IVPs have either one solution, or no solution at all?

Some of our results in this chapter concern a so called *scalar* dynamic IVP which is a special case of (2.1.5), (2.1.6) or (2.1.7), (2.1.8) considering \mathbb{R}^n for $n = 1$. Thus, for a right-Hilger-continuous scalar function $f : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D \rightarrow \mathbb{R}$, the scalar dynamic IVP of the first type will be

$$x^{\Delta} = f(t, x), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}; \quad (2.1.9)$$

$$x(t_0) = x_0, \quad (2.1.10)$$

and of the second type will be of the similar form with x replaced by x^{σ} in the right hand side of (2.1.9).

2.1.2 What we mean by a solution

The following definitions describe solutions to the dynamic IVPs (2.1.5), (2.1.6) and (2.1.7), (2.1.8).

Definition 2.1.1 *Let $D \subseteq \mathbb{R}^n$. A solution of (2.1.5), (2.1.6) on $[t_0, t_0 + a]_{\mathbb{T}}$ is a function $\mathbf{x} : [t_0, t_0 + a]_{\mathbb{T}} \rightarrow \mathbb{R}^n$ such that: the points $(t, \mathbf{x}(t)) \in [t_0, t_0 + a]_{\mathbb{T}} \times D$; $\mathbf{x}(t)$ is delta differentiable with $\mathbf{x}^{\Delta}(t) = \mathbf{f}(t, \mathbf{x}(t))$ for each $t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}$; and $\mathbf{x}(t_0) = \mathbf{x}_0$.*

□

Definition 2.1.2 *Let $D \subseteq \mathbb{R}^n$. A solution of (2.1.7), (2.1.8) on $[t_0, t_0 + a]_{\mathbb{T}}$ is a function $\mathbf{x} : [t_0, t_0 + a]_{\mathbb{T}} \rightarrow \mathbb{R}^n$ such that: the points $(t, \mathbf{x}(t)) \in [t_0, t_0 + a]_{\mathbb{T}} \times D$; $\mathbf{x}(t)$ is delta differentiable with $\mathbf{x}^{\Delta}(t) = \mathbf{f}(t, \mathbf{x}^{\sigma}(t))$ for each $t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}$; and $\mathbf{x}(t_0) = \mathbf{x}_0$.*

□

The following preliminary lemmas give equivalence of the dynamic IVPs (2.1.5), (2.1.6) and (2.1.7), (2.1.8) with delta integral equations of the form (A.7.5) and (A.7.6) respectively. Delta integral equations are more convenient to work with.

Lemma 2.1.3 Consider the dynamic IVP (2.1.5), (2.1.6). Let $\mathbf{f} : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D \rightarrow \mathbb{R}^n$ be a right-Hilger-continuous function. A function \mathbf{x} solves (2.1.5), (2.1.6) on $[t_0, t_0 + a]_{\mathbb{T}}$ if and only if it solves the delta integral equation

$$\mathbf{x}(t) = \int_{t_0}^t \mathbf{f}(s, \mathbf{x}(s)) \Delta s + \mathbf{x}_0, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \quad (2.1.11)$$

Proof: Let \mathbf{x} be a solution of (2.1.5), (2.1.6) on $[t_0, t_0 + a]_{\mathbb{T}}$. Then \mathbf{x} is delta differentiable on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa}$ by Definition 2.1.1, and so is continuous on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa}$. Moreover, \mathbf{x} satisfies

$$\mathbf{x}^{\Delta}(t) = \mathbf{f}(t, \mathbf{x}(t)), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}. \quad (2.1.12)$$

We delta integrate both sides of (2.1.12) over $[t_0, t]_{\mathbb{T}}$ obtaining

$$\int_{t_0}^t \mathbf{x}^{\Delta}(s) \Delta s = \int_{t_0}^t \mathbf{f}(s, \mathbf{x}(s)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}.$$

The right hand side of the above equation is well defined as, by Theorem A.5.2, right-Hilger-continuous functions are always delta integrable. Delta integrating the left hand side, we obtain

$$\mathbf{x}(t) - \mathbf{x}(t_0) = \int_{t_0}^t \mathbf{f}(s, \mathbf{x}(s)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}.$$

Using (2.1.6), we obtain

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(s, \mathbf{x}(s)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}.$$

Hence, \mathbf{x} is a solution to (2.1.11).

Conversely, let \mathbf{x} satisfy (2.1.11). Then by delta differentiating (2.1.11), we obtain

$$\mathbf{x}^{\Delta}(t) = \mathbf{f}(t, \mathbf{x}(t)), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}. \quad (2.1.13)$$

It is also evident from (2.1.11) that

$$\mathbf{x}(t_0) = \mathbf{x}_0.$$

Hence \mathbf{x} satisfies (2.1.6). Moreover, since \mathbf{x} is continuous on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa}$ and \mathbf{f} is right-Hilger-continuous on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$, \mathbf{x}^{Δ} is rd-continuous on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa}$ such that $\mathbf{x}(t) \in D$. Thus, \mathbf{x} is a solution of (2.1.5), (2.1.6) such that the points $(t, \mathbf{x}(t)) \in [t_0, t_0 + a]_{\mathbb{T}} \times D$.

Lemma 2.1.4 *Consider the dynamic equations (2.1.7), (2.1.8). Let $\mathbf{f} : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D \rightarrow \mathbb{R}^n$ be a right-Hilger-continuous function. Then a function \mathbf{x} solves (2.1.7), (2.1.8) on $[t_0, t_0 + a]_{\mathbb{T}}$ if and only if it solves the delta integral equation*

$$\mathbf{x}(t) = \int_{t_0}^t \mathbf{f}(s, \mathbf{x}^{\sigma}(s)) \Delta s + \mathbf{x}_0, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \quad (2.1.14)$$

Proof: The proof is similar to that of Lemma 2.1.3 and is omitted. □

Remark 2.1.5 *We note that Lemma 2.1.3 and Lemma 2.1.4 also hold for \mathbf{f} being continuous, as all continuous functions are right-Hilger-continuous and are delta integrable (see Theorem A.5.2).*

2.1.3 Approach and organisation

The techniques employed to answer the question in Subsection 2.1.1 involve the introduction and formulation of appropriate dynamic inequalities. The inequalities are extensions and refinements from the theory of ordinary differential equations to the more general time scale environment.

Many results in this chapter are proved through applications of the Lipschitz condition and Gronwall's inequality. In some results, we use modifications of the Lipschitz condition that are formed using ideas from ordinary differential equations with suitable transformations according to the requirements of the time scale calculus.

In this section, we establish foundational definitions and lemmas regarding solutions to the IVP (2.1.5), (2.1.6), the IVP (2.1.7), (2.1.8) and the IVP (2.1.9), (2.1.10). Our results in this chapter are organised in the following manner.

In Section 2.2, we extend the ideas of Lipschitz for non-multiplicity results from ordinary differential equations to the generalised time scale platform using the well-known Lipschitz condition [19, Definition 8.14(iv)] and Gronwall's inequality [19, Theorem 6.4]. We also develop non-multiplicity results using modifications of Lipschitz condition.

In Section 2.3, we establish a Peano criterion [2, p.10] on an arbitrary time scale \mathbb{T} to obtain non-multiplicity results for (2.1.9), (2.1.10) for cases where the Lipschitz criteria do not work.

2.2 Lipschitz criteria on \mathbb{T}

The following definition will be referred to as the *uniform Lipschitz condition* for \mathbf{f} on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$. The uniform Lipschitz condition will be a fundamental tool to establish results regarding existence and/or uniqueness of solutions to dynamic initial value problems in this work. The idea comes from [19, Definition 8.14(iv)], [16, p.151] and [51, Definition 8.8].

Definition 2.2.1 The uniform Lipschitz condition

Let $D \subseteq \mathbb{R}^n$ and $\mathbf{f} : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D \rightarrow \mathbb{R}^n$. If there exists a constant $L > 0$ such that

$$\|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\| \leq L \|\mathbf{p} - \mathbf{q}\|, \text{ for all } (t, \mathbf{p}), (t, \mathbf{q}) \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D, \quad (2.2.1)$$

then we say \mathbf{f} satisfies a uniform Lipschitz condition on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$.

□

If \mathbf{f} satisfies the uniform Lipschitz condition (2.2.1) on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$ then \mathbf{f} is said to be *Lipschitz continuous* on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$. Any value of L satisfying (2.2.1) is called a Lipschitz constant for \mathbf{f} on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$.

Classically, the Lipschitz constant L in (2.2.1) is independent of \mathbf{x} and t but may, in general, depend on the domain $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$ [42, p.6]. It is not easy, in general, to identify if a function satisfies the Lipschitz continuity in a given domain. However, if $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$ is convex and \mathbf{f} is a smooth function on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$, then the following theorem [17, p.22], [30, p.248], [2, Lemma 3.2.1] is helpful to identify if a given function satisfies a Lipschitz condition on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$.

Theorem 2.2.2 Let $b > 0$. Let $t_0 \in \mathbb{T}$ and $\mathbf{x}_0 \in \mathbb{R}^n$. Consider a function \mathbf{f} defined on a rectangle of the type

$$R^{\kappa} := \{(t, \mathbf{p}) \in \mathbb{T}^{\kappa} \times \mathbb{R}^n : t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}, \|\mathbf{p} - \mathbf{x}_0\| \leq b\}, \quad (2.2.2)$$

or on an infinite strip of the type

$$S^{\kappa} := \{(t, \mathbf{p}) \in \mathbb{T}^{\kappa} \times \mathbb{R}^n : t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}, \|\mathbf{p}\| < \infty\}. \quad (2.2.3)$$

If $\frac{\partial \mathbf{f}(t, \mathbf{p})}{\partial p_i}$ exists for all $i = 1, 2, \dots$, and is continuous on R^{κ} (or S^{κ}), and there is a constant $K > 0$ such that for all $(t, \mathbf{p}) \in R^{\kappa}$ (or S^{κ}), we have

$$\left\| \frac{\partial \mathbf{f}(t, \mathbf{p})}{\partial p_i} \right\| \leq K, \quad \text{for all } i = 1, 2, \dots, \quad (2.2.4)$$

then \mathbf{f} satisfies a Lipschitz condition on R^κ (or S^κ) with Lipschitz constant $K = L$.

Proof: For a proof see [2, Lemma 3.2.1].

□

Remark 2.2.3 Note in the above theorem that $\frac{\partial \mathbf{f}(t, \mathbf{p})}{\partial p_i}$ is the slope of the tangent line at any point (t, \mathbf{p}) in R^κ or S^κ in the direction of the i -th component of \mathbf{p} . Therefore, if the rate of change of $\mathbf{f}(t, \mathbf{p})$ is bounded at all points (t, \mathbf{p}) and the line joining any two points $(t, \mathbf{p}), (t, \mathbf{q})$ can not have a slope steeper than a certain positive number K , then \mathbf{f} remains within $\pm K(\mathbf{p} - \mathbf{x}_0)$ for R^κ and within $\pm K\mathbf{p}$ for S^κ .

□

Remark 2.2.4 Also note that the inequality (2.2.4) is a sufficient condition for the Lipschitz condition (2.2.1) to hold for all \mathbf{f} that have bounded partial derivatives with respect to the second argument on R^κ or S^κ .

□

The following is a corollary of Gronwall's inequality [19, Corollary 6.6] in the time scale setting. Historically, Gronwall's inequality has been widely used as a tool to prove existence and uniqueness of solutions to initial value problems. We use this result to prove non-multiplicity of solutions to the IVPs (2.1.5), (2.1.6) and (2.1.7), (2.1.8) in this chapter and in many other results in latter chapters.

Corollary 2.2.5 Let $t_1 \in \mathbb{T}$ and $z \in C_{rd}(\mathbb{T})$. Let $l : \mathbb{T} \rightarrow (0, \infty)$ with $l \in \mathcal{R}^+$. If z and l satisfy

$$z(t) \leq \int_{t_1}^t l(s)z(s) \Delta s \quad \text{for all } t \in \mathbb{T},$$

then

$$z(t) \leq 0, \quad \text{for all } t \in \mathbb{T}. \quad (2.2.5)$$

□

Now we present our first result regarding non-multiplicity of solutions to the dynamic IVP (2.1.5), (2.1.6), using the above corollary, when the rate of change in \mathbf{f} is bounded by a positive constant. The result is an extension of Lipschitz's classical non-multiplicity result from ordinary differential equations, see [2, Theorem 1.2.4], [25, Theorem 3.4] and [16, p.152], to the time scale setting.

Theorem 2.2.6 *Let $D \subseteq \mathbb{R}^n$ and $\mathbf{f} : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D \rightarrow \mathbb{R}^n$ be a right-Hilger-continuous function. If there exists a constant $L > 0$ such that*

$$\|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\| \leq L \|\mathbf{p} - \mathbf{q}\|, \text{ for all } (t, \mathbf{p}), (t, \mathbf{q}) \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D; \quad (2.2.6)$$

then the IVP (2.1.5), (2.1.6) has, at most, one solution, \mathbf{x} , such that $\mathbf{x}(t) \in D$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

Proof: Let \mathbf{x}, \mathbf{y} be two solutions of (2.1.5), (2.1.6) with $\mathbf{x}(t) \in D$ and $\mathbf{y}(t) \in D$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. We show that $\mathbf{x} \equiv \mathbf{y}$ on $[t_0, t_0 + a]_{\mathbb{T}}$.

By Lemma 2.1.3, we have

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(s, \mathbf{x}(s)) \Delta s, & \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}} \\ \text{and } \mathbf{y}(t) &= \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(s, \mathbf{y}(s)) \Delta s, & \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \end{aligned}$$

Then for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$, we have

$$\begin{aligned} \|\mathbf{x}(t) - \mathbf{y}(t)\| &\leq \int_{t_0}^t \|\mathbf{f}(s, \mathbf{x}(s)) - \mathbf{f}(s, \mathbf{y}(s))\| \Delta s \\ &\leq L \int_{t_0}^t \|\mathbf{x}(s) - \mathbf{y}(s)\| \Delta s, \end{aligned}$$

where we have used (2.2.6) in the last step. Applying Corollary 2.2.5, taking $l(t) := L$ and $z(t) := \|\mathbf{x}(t) - \mathbf{y}(t)\|$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$, we obtain

$$\|\mathbf{x}(t) - \mathbf{y}(t)\| \leq 0, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}.$$

But $\|\mathbf{x}(t) - \mathbf{y}(t)\|$ is non-negative for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. Thus, $\mathbf{x}(t) = \mathbf{y}(t)$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

□

Note that the result in [83, Theorem 3.2] is a special case of the above result where $l(t) := L$, a constant function for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

Corollary 2.2.7 *The above theorem also holds if \mathbf{f} has continuous partial derivatives with respect to the second argument and there exists $K > 0$ such that $\left\| \frac{\partial \mathbf{f}(t, \mathbf{p})}{\partial \mathbf{p}} \right\| \leq K$. In that case, \mathbf{f} satisfies the Lipschitz condition 2.2.6 on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$ with $L := K$ by Theorem 2.2.2.*

□

Example 2.2.8 *Let $D := \{\mathbf{p} \in \mathbb{R}^2 : \|\mathbf{p}\| \leq 2\}$, where $\mathbf{p} = (p_1, p_2)$. Consider the IVP*

$$\begin{aligned} \mathbf{x}^\Delta = \mathbf{f}(t, \mathbf{x}) &= (1 + x_1^2, t^2 + x_2), \quad \text{for all } t \in [0, 1]_{\mathbb{T}}^{\kappa}; \\ \mathbf{x}(0) &= (1, 0). \end{aligned}$$

We claim that this dynamic IVP has, at most, one solution, \mathbf{x} , such that $\|\mathbf{x}(t)\| \leq 2$ for all $t \in [0, 1]_{\mathbb{T}}$.

Proof: We show that $\mathbf{f}(t, \mathbf{p}) := (1 + p_1^2, t^2 + p_2)$ satisfies the conditions of Theorem 2.2.6 for all $(t, p) \in [0, 1]_{\mathbb{T}}^{\kappa} \times D$.

Note that for $\mathbf{p} = (p_1, p_2) \in D$, we have $p_1^2 + p_2^2 \leq 4$. Thus, $|p_1| \leq 2$ and $|p_2| \leq 2$.

1. \mathbf{f} is right-Hilger-continuous on $[0, 1]_{\mathbb{T}}^{\kappa} \times D$: We note that the composition function $g(t) := (1 + (x_1(t))^2, t^2 + x_2(t))$ is rd-continuous for all $t \in [0, 1]_{\mathbb{T}}$. Thus, our \mathbf{f} is right-Hilger-continuous on $[0, 1]_{\mathbb{T}}^{\kappa} \times D$;
2. \mathbf{f} is Lipschitz continuous on $[0, 1]_{\mathbb{T}}^{\kappa} \times D$: We note that for all $t \in [0, 1]_{\mathbb{T}}^{\kappa}$ and $(p_1, p_2) \in D$, we have

$$\left\| \frac{\partial \mathbf{f}}{\partial p_1} \right\| = \|(2p_1, 0)\| = |2p_1| \leq 4.$$

Similarly, we obtain

$$\left\| \frac{\partial \mathbf{f}}{\partial p_2} \right\| = \|(0, 1)\| = 1.$$

Thus, employing Corollary 2.2.7, we have (2.2.6) holding for $L = 4$.

Hence, all conditions of Theorem 2.2.6 are satisfied and we conclude that our example has, at most, one solution, $\mathbf{x}(t) \in D$, for all $t \in [0, 1]_{\mathbb{T}}$.

□

Our next result concerns the scalar dynamic IVP (2.1.9), (2.1.10) when a right-Hilger-continuous scalar function f satisfies a one-sided Lipschitz condition defined as follows.

Definition 2.2.9 Let $D \subseteq \mathbb{R}$ and $f : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D \rightarrow \mathbb{R}$ be right-Hilger-continuous. If there exists $L > 0$ such that, for all $p > q$, the inequality

$$f(t, p) - f(t, q) \leq L(p - q), \quad \text{for all } (t, p), (t, q) \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D \quad (2.2.7)$$

holds, then f is said to satisfy a one-sided Lipschitz condition on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$.

□

In the light of the above definition, we can establish a corollary of Theorem 2.2.2 to obtain a sufficient condition for a function f to satisfy the one-sided Lipschitz condition 2.2.7 on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$.

Corollary 2.2.10 Let $a, b > 0$. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}$. Consider a function f defined either on R^{κ} (or S^{κ}). If $\frac{\partial f(t, p)}{\partial p_i}$ exists for all $i = 1, 2, \dots$, and is continuous on R^{κ} (or S^{κ}), and there is a constant $K > 0$ such that for all $(t, p) \in R^{\kappa}$ (or S^{κ}), we have

$$\frac{\partial f(t, p)}{\partial p_i} \leq K, \quad \text{for all } i = 1, 2, \dots, \quad (2.2.8)$$

then f satisfies the one-sided Lipschitz condition 2.2.7 on R^{κ} (or S^{κ}) with Lipschitz constant $K = L$.

Proof: The proof is similar to that of [2, Lemma 3.2.1] except that $\frac{\partial f(t, p)}{\partial p}$ is considered bounded above by $L = K$.

□

Remark 2.2.11 The above theorem shows that if the rate of change of $f(t, p)$ is bounded above at all points (t, p) and the line joining any two points $(t, p), (t, q)$ can not have a slope steeper than a certain positive number K , then f remains below the line $K(p - x_0)$ for R^{κ} and below the line Kp for S^{κ} .

□

Our next theorem is a time scale extension of [2, Theorem 1.2.5] and is a non-multiplicity result when f satisfies a one-sided Lipschitz condition on $[t_0, t_0 + a]_{\mathbb{T}} \times D$.

Theorem 2.2.12 *Let $D \subset \mathbb{R}$ and $f : [t_0, t_0 + a]_{\mathbb{T}} \times D \rightarrow \mathbb{R}$ be right-Hilger-continuous. If there exists a constant $L > 0$ such that the inequality (2.2.7) holds for $p > q$, then the IVP (2.1.9), (2.1.10) has, at most, one solution, x , with $x(t) \in D$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.*

Proof: Let $t_1, t_2 \in (t_0, t_0 + a]_{\mathbb{T}}$ and $t_2 > t_1$. There are two cases to consider. In both cases, our argument is of the proof by contradiction style.

Case 1: Without loss of generality we assume solutions x, y with $x(t) \in D$ and $y(t) \in D$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$, that satisfy

$$x(t) = y(t), \quad \text{for all } t \in [t_0, t_1]_{\mathbb{T}} \subset [t_0, t_2]_{\mathbb{T}}, \quad (2.2.9)$$

$$\text{and} \quad x(t) < y(t), \quad \text{for all } t \in (t_1, t_2]_{\mathbb{T}}. \quad (2.2.10)$$

Therefore, for $t \in (t_1, t_2]_{\mathbb{T}}$, we have from Lemma 2.1.3

$$\begin{aligned} y(t) - x(t) &= \int_{t_1}^t (f(s, y(s)) - f(s, x(s))) \Delta s \\ &\leq L \int_{t_1}^t (y(s) - x(s)) \Delta s, \end{aligned}$$

where we have used (2.2.7) in the last step. Applying Corollary 2.2.5, taking $l(t) := L$ and $z(t) := y(t) - x(t)$, we obtain

$$y(t) - x(t) \leq 0, \quad \text{for all } t \in (t_1, t_2]_{\mathbb{T}},$$

which is a contradiction to (2.2.10).

Case 2: If, x, y satisfy $x(t) \in D$ and $y(t) \in D$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$, with

$$x(t) = y(t), \quad \text{for all } t \in [t_0, t_1]_{\mathbb{T}} \subset [t_0, t_2]_{\mathbb{T}},$$

$$\text{and} \quad x(t) > y(t), \quad \text{for all } t \in (t_1, t_2]_{\mathbb{T}}, \quad (2.2.11)$$

then, using (2.2.7), we have, for all $t \in (t_1, t_2]_{\mathbb{T}}$,

$$\begin{aligned} x(t) - y(t) &= \int_{t_1}^t (f(s, x(s)) - f(s, y(s))) \Delta s \\ &\leq L \int_{t_1}^t (x(s) - y(s)) \Delta s. \end{aligned}$$

Again applying Corollary 2.2.5, taking $l(t) := L$ and $z(t) := x(t) - y(t)$, we have

$$x(t) - y(t) \leq 0, \quad \text{for all } t \in (t_1, t_2]_{\mathbb{T}},$$

which is a contradiction to (2.2.11). Hence, $x(t) - y(t) = 0$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

Thus, $x(t) = y(t)$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

□

Note that the above result is more flexible than Theorem 2.2.6 which requires an upper as well as a lower bound on the change in f for the existence of non-multiple solutions to (2.1.9), (2.1.10).

Corollary 2.2.13 *The above theorem also holds if f has continuous partial derivatives with respect to the second argument and there exists $K > 0$ such that $\frac{\partial f(t, p)}{\partial p} \leq K$. In that case, f satisfies the one-sided Lipschitz condition on S^κ with $L := K$ by Corollary 2.2.10.*

□

Remark 2.2.14 *The above corollary weakens the condition of Corollary 2.2.7 for a smooth function f on R^κ or S^κ and the existence of an upper bound on $\frac{\partial f}{\partial p}$ is sufficient for (2.2.7) to exist on $[t_0, t_0 + a]_{\mathbb{T}}^\kappa \times D$.*

□

Our further results concern the non-multiplicity of solutions to the dynamic IVPs (2.1.5), (2.1.6) and (2.1.7), (2.1.8) which Theorem 2.2.6 or Corollary 2.2.10 do not directly apply to.

In the next result, we consider a positive constant L such that $-2L \in \mathcal{R}^+$. That is, $1 - 2\mu L > 0$. This is possible if we can choose L large enough such that the step size, $\mu(t)$, can be made smaller than $\frac{1}{2L}$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$ or vice versa. We employ this condition to prove the non-multiplicity of solutions to the vector dynamic IVP (2.1.7), (2.1.8), within a domain $D \subseteq \mathbb{R}^n$ by constructing a modified one-sided Lipschitz condition. This result has gained inspiration from [2, Theorem 3.2.2] for ordinary differential equations.

Theorem 2.2.15 *Let $D \subseteq \mathbb{R}^n$ and $\mathbf{f} : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D \rightarrow \mathbb{R}^n$ be right-Hilger-continuous. If there exists a positive constant $L > 0$ with $-2L \in \mathcal{R}^+$ such that \mathbf{f} satisfies the condition*

$$\langle \mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q}), \mathbf{p} - \mathbf{q} \rangle \leq L \|\mathbf{p} - \mathbf{q}\|^2, \quad (2.2.12)$$

$$\text{for all } (t, \mathbf{p}), (t, \mathbf{q}) \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D,$$

then the IVP (2.1.7), (2.1.8) has, at most, one solution \mathbf{x} , with $\mathbf{x}(t) \in D$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

Proof: Let \mathbf{x} and \mathbf{y} be two solutions of (2.1.7), (2.1.8), with $\mathbf{x}(t) \in D$ and $\mathbf{y}(t) \in D$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. Consider

$$v(t) := \|\mathbf{x}(t) - \mathbf{y}(t)\|^2, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}.$$

We show that $v \equiv 0$ on $[t_0, t_0 + a]_{\mathbb{T}}$ and so \mathbf{x} and \mathbf{y} are the same function.

Using the product rule (Theorem A.3.5(3)) and the simple useful formula (SUF, Theorem A.3.2(4)) for all $t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}$, we have

$$\begin{aligned} v^{\Delta}(t) &= \langle \mathbf{x}^{\Delta}(t) - \mathbf{y}^{\Delta}(t), \mathbf{x}(t) - \mathbf{y}(t) \rangle + \langle \mathbf{x}^{\sigma}(t) - \mathbf{y}^{\sigma}(t), \mathbf{x}^{\Delta}(t) - \mathbf{y}^{\Delta}(t) \rangle \\ &= \langle \mathbf{x}^{\Delta}(t) - \mathbf{y}^{\Delta}(t), \mathbf{x}^{\sigma}(t) - \mu(t)\mathbf{x}^{\Delta}(t) - \mathbf{y}^{\sigma}(t) + \mu(t)\mathbf{y}^{\Delta}(t) \rangle \\ &\quad + \langle \mathbf{x}^{\sigma}(t) - \mathbf{y}^{\sigma}(t), \mathbf{x}^{\Delta}(t) - \mathbf{y}^{\Delta}(t) \rangle \\ &= 2\langle \mathbf{x}^{\Delta}(t) - \mathbf{y}^{\Delta}(t), \mathbf{x}^{\sigma}(t) - \mathbf{y}^{\sigma}(t) \rangle + \langle \mathbf{x}^{\Delta}(t) - \mathbf{y}^{\Delta}(t), -\mu(t)(\mathbf{x}^{\Delta}(t) - \mathbf{y}^{\Delta}(t)) \rangle \\ &= 2\langle \mathbf{x}^{\Delta}(t) - \mathbf{y}^{\Delta}(t), \mathbf{x}^{\sigma}(t) - \mathbf{y}^{\sigma}(t) \rangle - \mu(t)\|\mathbf{x}^{\Delta}(t) - \mathbf{y}^{\Delta}(t)\|^2 \\ &\leq 2\langle \mathbf{x}^{\Delta}(t) - \mathbf{y}^{\Delta}(t), \mathbf{x}^{\sigma}(t) - \mathbf{y}^{\sigma}(t) \rangle \\ &= 2\langle \mathbf{f}(t, \mathbf{x}^{\sigma}(t)) - \mathbf{f}(t, \mathbf{y}^{\sigma}(t)), \mathbf{x}^{\sigma}(t) - \mathbf{y}^{\sigma}(t) \rangle \\ &\leq 2L\|\mathbf{x}^{\sigma}(t) - \mathbf{y}^{\sigma}(t)\|^2 \\ &= 2Lv^{\sigma}(t), \end{aligned}$$

where we have used (2.2.12) in the second last step.

Thus,

$$v^{\Delta}(t) \leq 2Lv^{\sigma}(t), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}.$$

Rearranging, we obtain

$$v^{\Delta}(t) - 2Lv^{\sigma}(t) \leq 0, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}. \quad (2.2.13)$$

Since $-2L \in \mathcal{R}^+$, we use $e_{-2L}(t, t_0)$ as an integrating factor in (2.2.13). Thus, we obtain

$$v^\Delta(t)e_{-2L}(t, t_0) - 2Le_{-2L}(t, t_0)v^\sigma(t) \leq 0, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa.$$

Using the product rule (Theorem A.3.5(3)) again, we have

$$[v(t)e_{-2L}(t, t_0)]^\Delta \leq 0, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa. \quad (2.2.14)$$

We note that $v(t)e_{-2L}(t, t_0)$ is non-increasing for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. Since $v(t_0) = 0$ and $e_{-2L}(t, t_0) > 0$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$, we have $v(t) \leq 0$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. But v is non-negative on $[t_0, t_0 + a]_{\mathbb{T}}$. Thus $v(t) = 0$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. Hence, $\mathbf{x}(t) = \mathbf{y}(t)$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. □

From the above results we note that $\langle \mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q}), \mathbf{p} - \mathbf{q} \rangle$ is the product of variation in \mathbf{f} with respect to \mathbf{p} and variation in \mathbf{p} itself and (2.2.12) provides an upper bound on this product for non-multiple solutions of (2.1.7), (2.1.8). From another result (see [79, Theorem 2.5] considering $R = 0$ and $M = 1$ in the boundary condition (3) in which case it becomes an initial condition) the non-multiplicity of (2.1.5), (2.1.6) is ensured for a negligibly small μ if the above product is strictly positive. In that case, the IVP (2.1.5), (2.1.6) and (2.1.7), (2.1.8) can be treated as ODEs. Thus, for sufficiently large L (or $\mathbb{T} = \mathbb{R}$), the non-multiple solutions of (2.1.5), (2.1.6) and (2.1.7), (2.1.8) exist for $0 < \langle \mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q}), \mathbf{p} - \mathbf{q} \rangle \leq L\|\mathbf{p} - \mathbf{q}\|^2$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

Also, for sufficiently small μ , the above result restricts the non-multiplicity of solutions of (2.1.7), (2.1.8) to small variations in \mathbf{p} producing small variations in \mathbf{f} no matter how large L is, in a restricted domain.

The following example illustrates Theorem 2.2.15.

Example 2.2.16 Let $D := [-1, 1]$ and $L_1 > 0$ be a constant. Then $L_1 \in \mathcal{R}^+$.

Consider the scalar dynamic IVP

$$x^\Delta = f(t, x^\sigma) := -L_1 x^\sigma + \frac{1}{e_{L_1}(t, 0)}, \quad \text{for all } t \in [0, 1]_{\mathbb{T}}^\kappa; \quad (2.2.15)$$

$$x(0) = 0. \quad (2.2.16)$$

We claim that this dynamic IVP has, at most, one solution, x , such that $x(t) \in D$ for all $t \in [0, 1]_{\mathbb{T}}$.

Proof: We prove our claim by showing that $f(t, p) := -L_1 p + \frac{1}{e_{L_1}(t, 0)}$ satisfies the conditions of Theorem 2.2.15 for $L := L_1$, for all $(t, p) \in [0, 1]_{\mathbb{T}}^{\kappa} \times D$.

- (1) f is right-Hilger-continuous on $[0, 1]_{\mathbb{T}}^{\kappa} \times D$: We note that for all $t \in [0, 1]_{\mathbb{T}}^{\kappa}$, the functions $\frac{1}{e_{L_1}(t, 0)}$ and $x^{\sigma}(t)$ are rd-continuous. Therefore, the composition function $g(t) := -L_1 x^{\sigma}(t) + \frac{1}{e_{L_1}(t, 0)}$ is rd-continuous for all $t \in [0, 1]_{\mathbb{T}}$. Thus our f is right-Hilger-continuous on $[0, 1]_{\mathbb{T}}^{\kappa} \times D$;
- (2) f satisfies (2.2.12) on $[0, 1]_{\mathbb{T}}^{\kappa} \times D$: We note that for all $t \in [0, 1]_{\mathbb{T}}^{\kappa}$ and $p, q \in [-1, 1]$, we have

$$\begin{aligned} (f(t, p) - f(t, q))(p - q) &= -L_1 (p - q)^2 \\ &\leq L_1 \|p - q\|^2. \end{aligned}$$

Hence (2.2.12) holds for f . Thus, (2.2.15), (2.2.16) satisfies all conditions of Theorem 2.2.15 and so, has, at most, one solution, x , with $x(t) \in D$ for all $t \in [0, 1]_{\mathbb{T}}^{\kappa}$. In fact, (2.2.15), (2.2.16) is linear and so, by Theorem A.7.7, has a unique solution given by

$$x(t) = \frac{t}{e_{L_1}(t, 0)}, \quad \text{for all } t \in [0, 1]_{\mathbb{T}}.$$

□

Corollary 2.2.17 Let $D \subseteq \mathbb{R}$ and $f : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D \rightarrow \mathbb{R}$ be right-Hilger-continuous. If f satisfies

$$(f(t, p) - f(t, q))(p - q) \leq 0, \text{ for all } (t, p), (t, q) \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D, \quad (2.2.17)$$

then the IVP (2.1.9), (2.1.10) has, at most, one solution x with $x(t) \in D$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

Proof: If (2.2.17) holds then Theorem 2.2.15 holds for $L = 0$.

□

The next corollary concerns the non-multiplicity of solutions to the scalar dynamic IVP

$$x^\Delta = f(t, x^\sigma), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}; \quad (2.2.18)$$

$$x(t_0) = x_0, \quad (2.2.19)$$

using Theorem 2.2.15.

Corollary 2.2.18 *Let $D \subseteq \mathbb{R}$ and $f : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D \rightarrow \mathbb{R}$ be right-Hilger-continuous. If f satisfies*

$$(f(t, p) - f(t, q))(p - q) \leq 0, \text{ for all } (t, p), (t, q) \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D, \quad (2.2.20)$$

then the IVP (2.2.18), (2.2.19) has, at most, one solution x with $x(t) \in D$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

Proof: If (2.2.20) holds then Theorem 2.2.15 holds for (2.2.18), (2.2.19) for $L = 0$.

□

Note that the above two corollaries hold only for sufficiently large μ .

The following example illustrates Corollary 2.2.18.

Example 2.2.19 *Let $D := (0, \infty)$ and $f : [0, 1]_{\mathbb{T}}^{\kappa} \times (0, \infty) \rightarrow \mathbb{R}$. Consider the dynamic IVP*

$$\begin{aligned} x^\Delta &= f(t, x^\sigma) = \frac{t^3}{(x^\sigma)^2}, & \text{for all } t \in [0, 1]_{\mathbb{T}}^{\kappa}; \\ x(0) &= 1. \end{aligned} \quad (2.2.21)$$

We claim that this IVP has, at most, one solution x such that $x(t) > 0$ for all $t \in [0, 1]_{\mathbb{T}}$.

Proof: We prove our claim by showing that Corollary 2.2.18 holds for $f(t, u) := \frac{t^3}{u^2}$ for all $(t, u) \in [0, 1]_{\mathbb{T}}^{\kappa} \times (0, \infty)$.

- (a) *f is right-Hilger-continuous on $[0, 1]_{\mathbb{T}}^{\kappa} \times (0, \infty)$:* We note that the composition function $k(t) := \frac{t^3}{(x^\sigma(t))^2}$ is rd-continuous for all $t \in [0, 1]_{\mathbb{T}}$. So our f is right-Hilger-continuous on $[0, 1]_{\mathbb{T}}^{\kappa} \times (0, \infty)$;

(b) f satisfies (2.2.20) on $[0, 1]_{\mathbb{T}}^{\kappa} \times (0, \infty)$: We note that for all $(t, u), (t, v) \in [0, 1]_{\mathbb{T}}^{\kappa} \times (0, \infty)$, we have

$$\begin{aligned} (f(t, u) - f(t, v))(u - v) &= t^3 \left(\frac{1}{u^2} - \frac{1}{v^2} \right) (u - v) \\ &= t^3 \left(\frac{v^2 - u^2}{u^2 v^2} \right) (u - v) \\ &= -t^3 \frac{(v + u)(u - v)^2}{u^2 v^2} \\ &\leq 0. \end{aligned}$$

Hence, (2.2.20) holds for f and f satisfies all conditions of Corollary 2.2.18. Thus the given IVP has, at most, one solution, x , with $x(t) \in (0, \infty)$ for all $t \in [0, 1]_{\mathbb{T}}$. □

We observe from (2.2.20) and from the above example that for any p, q with $p \geq q$, the inequality (2.2.20) yields

$$f(t, p) - f(t, q) \leq 0, \quad \text{for all } (t, p), (t, q) \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D.$$

Thus, f will be non-increasing in the second argument on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$.

In the next section, we show that, for non-increasing functions on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$, the non-multiplicity of solutions to (2.1.9), (2.1.10) may hold without the Lipschitz condition holding on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$.

The next theorem concerns the non-multiplicity of solutions to the IVP (2.1.7), (2.1.8) within a domain $D \subseteq \mathbb{R}^n$. Here \mathbf{f} , which is a vector valued function, assumes a restriction that apparently depends on the graininess function μ . However, we will prove in the following theorem that this dependence is removable. We note that this is a more generalised result for non-multiplicity of solutions to (2.1.7), (2.1.8) than Theorem 2.2.15 and the condition that ‘ L ’ be large is no more necessary.

Theorem 2.2.20 *Let $D \subseteq \mathbb{R}^n$ and let $\mathbf{f} : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D \rightarrow \mathbb{R}^n$ be a right-Hilger-continuous function, with $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$. If there exist positive constants L, β, γ , such that $\beta = \gamma L$ for $\gamma \geq 2$, such that \mathbf{f} satisfies*

$$\begin{aligned} \|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\| &\leq \frac{L}{1 + \mu(t) \beta} \|\mathbf{p} - \mathbf{q}\|, \\ &\text{for all } (t, \mathbf{p}), (t, \mathbf{q}) \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D, \end{aligned} \tag{2.2.22}$$

then the IVP (2.1.7), (2.1.8) has, at most, one solution, \mathbf{x} , with $\mathbf{x}(t) \in D$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

Proof: Consider $\mathbf{x}(t)$ and $\mathbf{y}(t)$ as two solutions of (2.1.7), (2.1.8) with $\mathbf{x}(t) \in D$ and $\mathbf{y}(t) \in D$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. Let

$$\omega(t) := \|\mathbf{x}(t) - \mathbf{y}(t)\|^2, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}.$$

We show that $\omega \equiv 0$ on $[t_0, t_0 + a]_{\mathbb{T}}$, and so $\mathbf{x}(t) = \mathbf{y}(t)$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

Using the product rule, Theorem A.3.5(3), and the identity (4) of Theorem A.3.2 for all $t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}$, we have

$$\begin{aligned} \omega^\Delta(t) &= \langle \mathbf{x}^\Delta(t) - \mathbf{y}^\Delta(t), \mathbf{x}(t) - \mathbf{y}(t) \rangle + \langle \mathbf{x}^\sigma(t) - \mathbf{y}^\sigma(t), \mathbf{x}^\Delta(t) - \mathbf{y}^\Delta(t) \rangle \\ &= \langle \mathbf{x}^\Delta(t) - \mathbf{y}^\Delta(t), \mathbf{x}^\sigma(t) - \mu(t)\mathbf{x}^\Delta(t) - \mathbf{y}^\sigma(t) + \mu(t)\mathbf{y}^\Delta(t) \rangle \\ &\quad + \langle \mathbf{x}^\sigma(t) - \mathbf{y}^\sigma(t), \mathbf{x}^\Delta(t) - \mathbf{y}^\Delta(t) \rangle \\ &= 2\langle \mathbf{x}^\Delta(t) - \mathbf{y}^\Delta(t), \mathbf{x}^\sigma(t) - \mathbf{y}^\sigma(t) \rangle + \langle \mathbf{x}^\Delta(t) - \mathbf{y}^\Delta(t), -\mu(t)(\mathbf{x}^\Delta(t) - \mathbf{y}^\Delta(t)) \rangle \\ &= 2\langle \mathbf{x}^\Delta(t) - \mathbf{y}^\Delta(t), \mathbf{x}^\sigma(t) - \mathbf{y}^\sigma(t) \rangle - \mu(t)\|\mathbf{x}^\Delta(t) - \mathbf{y}^\Delta(t)\|^2 \\ &\leq 2\langle \mathbf{x}^\Delta(t) - \mathbf{y}^\Delta(t), \mathbf{x}^\sigma(t) - \mathbf{y}^\sigma(t) \rangle \\ &= 2\langle \mathbf{f}(t, \mathbf{x}^\sigma(t)) - \mathbf{f}(t, \mathbf{y}^\sigma(t)), \mathbf{x}^\sigma(t) - \mathbf{y}^\sigma(t) \rangle \\ &\leq \frac{2L}{1 + \mu(t)\beta} \|\mathbf{x}^\sigma(t) - \mathbf{y}^\sigma(t)\|^2, \end{aligned} \tag{2.2.23}$$

where we used (2.2.22) in the last step. We also note that for $\beta > 0$, we have

$$\frac{1}{1 + \mu(t)\beta} = \frac{e_\beta(t, t_0)}{e_\beta^\sigma(t, t_0)}, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \tag{2.2.24}$$

Thus, for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$, the inequality (2.2.23) takes the form

$$\begin{aligned} \omega^\Delta(t) &\leq \frac{2L e_\beta(t, t_0)}{e_\beta^\sigma(t, t_0)} \|\mathbf{x}^\sigma(t) - \mathbf{y}^\sigma(t)\|^2 \\ &= \frac{2L e_\beta(t, t_0)}{e_\beta^\sigma(t, t_0)} \omega^\sigma(t). \end{aligned} \tag{2.2.25}$$

Since $\beta = \gamma L$ and $\gamma \geq 2$, the inequality (2.2.25) can be written as

$$\omega^\Delta(t) \leq \frac{\beta e_\beta(t, t_0)}{e_\beta^\sigma(t, t_0)} \omega^\sigma(t) \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}.$$

Rearranging, we obtain

$$\omega^\Delta(t) - \frac{\beta e_\beta(t, t_0)}{e_\beta^\sigma(t, t_0)} \omega^\sigma(t) \leq 0, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}.$$

Since $e_\beta(t, t_0) > 0$, we further obtain

$$\frac{\omega^\Delta(t)}{e_\beta(t, t_0)} - \frac{\beta}{e_\beta^\sigma(t, t_0)} \omega^\sigma(t) \leq 0, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa.$$

Again using the product rule (Theorem A.3.5(3.)), the above inequality reduces to

$$\left[\frac{\omega(t)}{e_\beta(t, t_0)} \right]^\Delta \leq 0, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa.$$

Thus $\frac{\omega(t)}{e_\beta(t, t_0)}$ is non-increasing in $[t_0, t_0 + a]_{\mathbb{T}}$. Hence $\omega(t) \leq 0$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

But $\omega(t_0) = 0$, and ω is non-negative on $[t_0, t_0 + a]_{\mathbb{T}}$. Hence, $\omega \equiv 0$ on $[t_0, t_0 + a]_{\mathbb{T}}$.

This means that $\mathbf{x}(t) = \mathbf{y}(t)$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

□

Example 2.2.21 Let $D = [1, \infty)$. Let $L = 1$ and $\beta = 3$. Consider the dynamic IVP

$$x^\Delta = f(t, x) = \frac{e_3(t, 0) \ln x}{e_3^\sigma(t, 0)}, \quad t \in [0, 1]_{\mathbb{T}}; \quad (2.2.26)$$

$$x(0) = 1. \quad (2.2.27)$$

We claim that the above IVP has, at most, one solution x such that $x(t) \in [1, \infty)$ for all $t \in [0, 1]_{\mathbb{T}}$.

Proof: We show that f satisfies all conditions of Theorem 2.2.20.

(a) f is right-Hilger-continuous on $[0, 1]_{\mathbb{T}}^\kappa \times D$: We note that $e_3(t, 0)$ and $e_3^\sigma(t, 0)$ are rd-continuous for all $t \in [0, 1]_{\mathbb{T}}$. Thus, the composition function $h(t) := \frac{e_3(t, 0) \ln x(t)}{e_3^\sigma(t, 0)}$ will be rd-continuous for all $t \in [0, 1]_{\mathbb{T}}$. Therefore, our f is right-Hilger-continuous in $[0, 1]_{\mathbb{T}}^\kappa \times D$;

(b) f satisfies condition (2.2.22) on $[0, 1]_{\mathbb{T}}^\kappa \times D$: We first show that for all $p \in [1, \infty)$, the function

$$r(p) := \ln p$$

is Lipschitz continuous with Lipschitz constant $L = 1$. Note that for $p \geq 1$

$$\left| \frac{\partial r}{\partial p} \right| = \left| \frac{1}{p} \right| \leq 1.$$

Hence, by Theorem 2.2.2, r satisfies a Lipschitz condition on $[1, \infty)$ with Lipschitz constant $L = 1$. Thus, we have

$$|\ln p - \ln q| \leq |p - q| \quad \text{for all } p, q \in [1, \infty). \quad (2.2.28)$$

Next, we show that the condition (2.2.22) is satisfied for all $(t, p), (t, q) \in [0, 1]_{\mathbb{T}}^{\kappa} \times D$. Note that

$$|f(t, p) - f(t, q)| = \frac{L e_3(t, 0)}{e_3^{\sigma}(t, 0)} |\ln p - \ln q|, \quad \text{for all } (t, p), (t, q) \in [0, 1]_{\mathbb{T}}^{\kappa} \times D.$$

From (2.2.24), we obtain,

$$|f(t, p) - f(t, q)| = \frac{1}{1 + 3\mu(t)} |\ln p - \ln q|, \quad \text{for all } (t, p), (t, q) \in [0, 1]_{\mathbb{T}}^{\kappa} \times D.$$

Thus, using (2.2.26), and (2.2.28), we obtain

$$|f(t, p) - f(t, q)| \leq \frac{1}{1 + 3\mu(t)} |p - q|, \quad \text{for all } (t, p), (t, q) \in [0, 1]_{\mathbb{T}}^{\kappa} \times D.$$

Hence f satisfies (2.2.22) for all $t \in [0, 1]_{\mathbb{T}}$.

We note that f satisfies all conditions of Theorem 2.2.20. Therefore, the given IVP has, at most, one solution, x , with $x(t) \in [1, \infty)$ for all $t \in [0, 1]_{\mathbb{T}}$.

□

2.3 Peano criterion on \mathbb{T}

This section comprises a result regarding the non-multiplicity of solutions to the IVP (2.1.9), (2.1.10) in the absence of the Lipschitz condition (2.2.6) or any of its modifications defined in the previous section.

It has been discussed above that the non-multiplicity of (2.1.9), (2.1.10) is ensured if a right-Hilger-continuous function $f : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D \rightarrow \mathbb{R}$ satisfies (2.2.20), in which case, f will be non-increasing on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$. In the following result, we prove the converse using classical method. That is, we prove that the non-multiplicity of solutions to (2.1.9), (2.1.10) holds for every right-Hilger-continuous function f that is non-increasing in the second argument on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$. The result is an extension of [2, Theorem 1.3.1] to time scales.

Theorem 2.3.1 *Let $D \subseteq \mathbb{R}$ and $f : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D \rightarrow \mathbb{R}$ be a right-Hilger-continuous function. If, for all $p \leq q$, f satisfies the inequality*

$$f(t, p) \geq f(t, q), \quad \text{for all } (t, p), (t, q) \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D; \quad (2.3.1)$$

then the dynamic IVP (2.1.9), (2.1.10) has, at most, one solution, x , such that $x(t) \in D$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

Proof: Let $t_1, t_2 \in (t_0, t_0 + a]_{\mathbb{T}}$ with $t_2 > t_1$.

Without loss of generality we assume x and y as two solutions of (2.1.9), (2.1.10) with $x(t) \in D$ and $y(t) \in D$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. Let

$$r(t) := x(t) - y(t), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \quad (2.3.2)$$

We note that $r(t_0) = 0$. Assume $r(t) \neq 0$ for all $t \in (t_0, t_0 + a]_{\mathbb{T}}$. We consider two cases. In each case we use proof by contradiction.

Case 1: Assume

$$r(t) = 0, \quad \text{for all } t \in [t_0, t_1]_{\mathbb{T}} \subset [t_0, t_2]_{\mathbb{T}}, \quad (2.3.3)$$

$$\text{and} \quad r(t) < 0, \quad \text{for all } t \in (t_1, t_2]_{\mathbb{T}}. \quad (2.3.4)$$

So, $x(t) < y(t)$ for all $t \in (t_1, t_2]_{\mathbb{T}}$, and since f is non-increasing in the second argument on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$, we have

$$f(t, x(t)) \geq f(t, y(t)), \quad \text{for all } t \in [t_0, t_2]_{\mathbb{T}}. \quad (2.3.5)$$

Thus, for all $t \in [t_0, t_2]_{\mathbb{T}}^{\kappa}$, we have

$$\begin{aligned} r^{\Delta}(t) &= x^{\Delta}(t) - y^{\Delta}(t), \\ &= f(t, x(t)) - f(t, y(t)) \\ &\geq 0, \end{aligned}$$

and so r is non-decreasing in $[t_0, t_2]_{\mathbb{T}}$. Combining this with (2.3.3), we note that $r \geq 0$ on $[t_0, t_2]_{\mathbb{T}}$ and this contradicts (2.3.4).

Case 2: Now, assume

$$r(t) = 0, \quad \text{for all } t \in [t_0, t_1]_{\mathbb{T}} \subset [t_0, t_2]_{\mathbb{T}}, \quad (2.3.6)$$

$$\text{and} \quad r(t) > 0, \quad \text{for all } t \in (t_1, t_2]_{\mathbb{T}}. \quad (2.3.7)$$

Then, $x(t) > y(t)$ for all $t \in (t_1, t_2]_{\mathbb{T}}$, and since f is non-increasing in its second variable on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$, we have

$$f(t, x(t)) \leq f(t, y(t)), \quad \text{for all } t \in [t_0, t_2]_{\mathbb{T}}. \quad (2.3.8)$$

This yields, for all $t \in [t_0, t_2]_{\mathbb{T}}^{\kappa}$

$$\begin{aligned} r^{\Delta}(t) &= x^{\Delta}(t) - y^{\Delta}(t), \\ &= f(t, x(t)) - f(t, y(t)) \\ &\leq 0, \end{aligned}$$

and so r is non-increasing in $[t_0, t_2]_{\mathbb{T}}$. Combining this with (2.3.6), we note that $r \leq 0$ on $[t_0, t_2]_{\mathbb{T}}$ and this contradicts (2.3.7).

Thus, $r(t) = 0$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. Hence, $x(t) = y(t)$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

□

The above result can also be obtained from Theorem 2.2.12 provided $L = 0$.

Now we consider an example to illustrate the above theorem.

Example 2.3.2 Let $D = [0, \infty)$. Consider the IVP

$$\begin{aligned} x^{\Delta} &= t - x^{2/3}, \quad \text{for all } t \in [0, 1]_{\mathbb{T}}^{\kappa}; \\ x(0) &= 0. \end{aligned}$$

We claim that this IVP has, at most, one solution, x , such that $x(t) \in [0, \infty)$ for all $t \in [0, 1]_{\mathbb{T}}$.

Proof: We show that f satisfies the conditions of Theorem 2.3.1 on $[0, 1]_{\mathbb{T}}^{\kappa} \times D$.

(i) f is right-Hilger-continuous on $[0, 1]_{\mathbb{T}}^{\kappa} \times D$: We note that the composition function $l(t) := t - (x(t))^{2/3}$ is rd-continuous for all $t \in [0, 1]_{\mathbb{T}}$. Thus, f is right-Hilger-continuous in $[0, 1]_{\mathbb{T}}^{\kappa} \times D$;

(ii) f is non-increasing in $[0, 1]_{\mathbb{T}}^{\kappa} \times D$: Note that for all $p \leq q$ we have $-p^{2/3} \geq -q^{2/3}$. Therefore, for all $(t, p), (t, q) \in [0, 1]_{\mathbb{T}}^{\kappa} \times D$, we have

$$f(t, p) = t - p^{3/2} \geq t - q^{3/2} = f(t, q).$$

Hence f is non-increasing in p on $[0, 1]_{\mathbb{T}}^{\kappa} \times D$.

We note that the given IVP satisfies both conditions of Theorem 2.3.1 and so, has, at most, one solution $x(t) \in [-1, 0]$ for all $t \in [0, 1]_{\mathbb{T}}$.

□

We note that Theorem 2.3.1 provides a sufficient condition for the non-multiplicity of solutions to the scalar dynamic IVP (2.1.9), (2.1.10) on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times D$. In the next chapter we will extend the above result to prove that the system (2.1.5), (2.1.6) has a unique solution when the Lipschitz condition (2.2.6) is and is not satisfied.

In this chapter, we presented results that identified conditions under which the systems (2.1.5), (2.1.6) and (2.1.7), (2.1.8) or the scalar IVPs (2.1.9), (2.1.10) and (2.2.18), (2.2.19) have either one solution or no solution at all. In the next chapter, we extend our discussion to existence of solutions to the above IVPs using the classical method of constructing successive approximations converging to a unique limit.

Chapter 3

Successive approximation of solutions

3.1 Introduction

In this chapter, we explore the existence and uniqueness of solutions to first order non-linear dynamic initial value problems using classical methods. Our approach involves constructing sequences of functions that converge to a unique solution to the problem under consideration.

The method of successive approximations is a powerful tool for gaining existence and computation of solutions to initial and boundary value problems. This method is explicitly developed, *for the first time*, in the time scale setting in this work and is used to prove several new existence theorems. The results are extended to n -th order dynamic equations. We also provide some interesting examples illustrating the new results.

Liouville and Picard's work on the method of successive approximation has been a key to analyse and establish the existence of unique solutions to non-linear initial and boundary value problems for ODEs and dates back to the nineteenth century [60, p.444]. Generally speaking, the method attempts to solve an equation of the kind

$$\mathbf{x} = \mathbf{F}(\mathbf{x}),$$

where \mathbf{F} is a continuous function. The approximation procedure starts from an initial

value \mathbf{x}_0 and then employing the successive iterations as a sequence of functions defined by

$$\mathbf{x}_{n+1} := \mathbf{F}(\mathbf{x}_n), \quad \text{for } n = 0, 1, 2, \dots$$

A set of assumptions is then developed to assert that \mathbf{x}_n converges to some function Φ . In addition, Φ is proved to be the unique solution to the given equation, with a small error estimated for $\|\mathbf{x}_n - \Phi\|$ for all $n \geq 0$.

We consider a first order non-linear delta IVP and use the above method to establish a set of iterations that successively converge to a function, Φ , and prove that Φ is the unique solution of the IVP. Traditionally, the method involves taking the initial approximation to be a constant which is usually the initial value. In this chapter, we develop a generalised method of successive approximations in which the initial approximation is a continuous function of t , where $t \in \mathbb{T}$.

3.1.1 The main objective

Let $[t_0, t_0 + a]_{\mathbb{T}}$ be a closed and bounded interval in \mathbb{T} and \mathbf{x}_0 be a point in \mathbb{R}^n . Consider the rectangle

$$R^\kappa = \{(t, \mathbf{p}) \in \mathbb{T}^\kappa \times \mathbb{R}^n : t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa, \|\mathbf{p} - \mathbf{x}_0\| \leq b\} \quad (3.1.1)$$

and a right-Hilger-continuous function $\mathbf{f} : R^\kappa \rightarrow \mathbb{R}^n$.

In this chapter we explore the existence and uniqueness of solutions to the dynamic initial value problem

$$\mathbf{x}^\Delta = \mathbf{f}(t, \mathbf{x}), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa; \quad (3.1.2)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0 \quad (3.1.3)$$

using the method of successive approximations.

In contrast to the question of “non-multiplicity of solutions” answered in Chapter 2, this chapter answers the following questions:

1. Under what conditions does the dynamic IVP (3.1.2), (3.1.3) have a unique solution?
2. Under what conditions can we closely approximate that solution?
3. Can we always construct sequences that converge to a unique solution?

3.1.2 Methodology and organisation

The methodology to answer the above questions involves the introduction and formulation of the method of successive approximations (originally from ordinary differential equations) to the time scale setting forming an analogue of the Picard–Lindelöf theorem [6, Theorem 8.1], [30, pp. 200-205], [51, Theorem 8.13], [24, pp.314-325], [69, pp. 48-50].

To apply the method of successive approximations, we will use the Lipschitz condition (see Definition 2.2.1) along with the Weierstrass test [26, p.266], [72, p.600]. The Lipschitz condition has been an important tool to determine the existence of solutions as unique limit of iterative procedures.

We note from Theorem 2.2.2 that the Lipschitz condition holds for functions having continuous partial derivatives in a given domain. However, we observed in Example 2.2.19 and Example 2.3.2 that the partial derivatives of non-increasing functions were not defined at 0. The uniqueness of a solution, if it exists, is, however, guaranteed by Theorem 2.3.1. Indeed, functions whose partial derivatives are not defined at a certain point may have infinitely many solutions through that point. This is further illustrated in the following example.

Example 3.1.1 *Let $a, b \in \mathbb{T}$ with $b > a$. Define*

$$U^\kappa := \{(t, p) \in \mathbb{T}^\kappa \times \mathbb{R} : t \in [a, b]_{\mathbb{T}}^\kappa, |p| < \infty\}.$$

Consider the initial value problem

$$y^\Delta = 3y^{2/3} \quad \text{for all } t \in [a, b]_{\mathbb{T}}^\kappa; \quad (3.1.4)$$

$$y(0) = 0. \quad (3.1.5)$$

Then we note that $f(t, y) = 3y^{2/3}$ is right-Hilger-continuous everywhere in U^κ for all $t \in [a, b]_{\mathbb{T}}$. However, its partial derivative

$$\frac{\partial f}{\partial y} = \frac{2}{y^{1/3}}$$

is not defined at $y = 0$. Thus we cannot apply Theorem 2.2.2 to identify a Lipschitz constant for $f(t, y)$ for all $(t, y) \in U^\kappa$.

Now consider a right-dense point $t = k \geq 0$ and define

$$\phi^k(t) := \begin{cases} 0; & -\infty < t \leq k, \\ (t - k)^3; & k \leq t < \infty. \end{cases}$$

Then $\phi^k(t)$ satisfies $y^\Delta = 3y^{2/3}$ for all $t \in (-\infty, \infty)$ for all $k \geq 0$. In addition, $y \equiv 0$ is a solution to the above problem.

□

The above example suggests that in the absence of continuous partial derivatives of f , a dynamic IVP may have a solution.

In this chapter, we present existence results by constructing successive approximations that converge to a unique solution of (3.1.2), (3.1.3), using Lipschitz continuity as a sufficient condition for our proof. Then we develop an interesting example to show that in the absence of the Lipschitz condition the successive iterations may not converge to a unique limit, but a solution may exist. Moreover, we present another result that ensures the existence of a solution to (3.1.2), (3.1.3) as a unique limit of uniformly convergent sequences without using the Lipschitz condition.

This chapter is organised as follows. The next section, Section 3.2, explains the main characteristics of the Picard–Lindelöf theorem on the time scale platform and a few preliminaries for the main results.

In Section 3.3, we establish a Picard–Lindelöf theorem on \mathbb{T} locally. That is, we construct iterations that converge to a unique solution of (3.1.2), (3.1.3) within a small rectangle. We reinforce our findings with interesting examples and prove that Picard theorem provides a sufficient condition for the convergence of successive approximations to a unique solution.

In Section 3.4, we extend our results so that the Picard iterations globally converge to a unique solution of (3.1.2), (3.1.3) on an infinite strip.

In Section 3.5, we present a special case of local existence of solutions within an n -sphere considering the initial value lying within another smaller n -sphere.

In Section 3.6, we develop Peano's existence theorem in the time scale setting using the method of successive approximations to ensure the existence of at least one solution of (3.1.2), (3.1.3) that lies within a small rectangle.

Finally, in Section 3.7, we extend our results to higher order dynamic equations.

Most of our results in Section 3.2, Section 3.3 and Section 3.4 have been published, see [83, pp.66–79, 84].

3.2 Picard–Lindelöf Theorem on \mathbb{T}

In this section, we construct an analogue of the Picard–Lindelöf theorem [6, Theorem 8.1], [30, pp.200–205], [51, Theorem 8.13], [24, pp.314–325], [69, pp. 48–50] on the platform of the time scale calculus.

Let α be a point in $[t_0, t_0 + a]_{\mathbb{T}}$ to be made explicit a little later such that $t_0 < \alpha \leq a$. We prove that (3.1.2), (3.1.3) has a unique solution in a closed neighbourhood of t_0 within a subinterval $[t_0, t_0 + \alpha]_{\mathbb{T}} \subseteq [t_0, t_0 + a]_{\mathbb{T}}$ as well as over the entire interval $[t_0, t_0 + a]_{\mathbb{T}}$.

Define

$$R := \{(t, \mathbf{p}) \in \mathbb{T} \times \mathbb{R}^n : t \in [t_0, t_0 + a]_{\mathbb{T}}, \|\mathbf{p} - \mathbf{x}_0\| \leq b\}. \quad (3.2.1)$$

Note that R is an extension of R^κ , as it contains all points $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

We construct successive approximations of solutions to (3.1.2), (3.1.3) in a right neighbourhood of the point $(t_0, \mathbf{x}_0) \in R$ and show that these approximations converge to a unique limit which is the solution to (3.1.2), (3.1.3) on $[t_0, t_0 + a]_{\mathbb{T}}$.

Let Φ be a solution of the IVP (3.1.2), (3.1.3). Then, by Definition 2.1.1, Φ is delta differentiable on $[t_0, t_0 + a]_{\mathbb{T}}^\kappa$ and the points $(t, \Phi(t))$ are in R for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. Since \mathbf{f} is right–Hilger–continuous on R^κ , it follows from Lemma 2.1.3 that

$$\Phi(t) = x_0 + \int_{t_0}^t \mathbf{f}(s, \Phi(s)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \quad (3.2.2)$$

We consider a sequence of functions $\Phi_0, \Phi_1, \Phi_2, \dots$ such that Φ_k is defined on $[t_0, t_0 + a]_{\mathbb{T}}$ for all $k = 1, 2, \dots$. Let Φ_0 be a continuous function on $[t_0, t_0 + a]_{\mathbb{T}}$. Proceeding in an inductive manner, we define the $(k + 1)$ th iteration, for each $k = 0, 1, 2, \dots$, as follows:

$$\Phi_{k+1}(t) := \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(s, \Phi_k(s)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \quad (3.2.3)$$

The right–Hilger–continuity of \mathbf{f} on R^κ implies that \mathbf{f} is bounded on R^κ . Let $M > 0$ be a constant that bounds \mathbf{f} on R^κ . Then we have

$$\|\mathbf{f}(t, \mathbf{p})\| \leq M, \quad \text{for all } (t, \mathbf{p}) \in R^\kappa. \quad (3.2.4)$$

Furthermore, let

$$\alpha := \min \left\{ a, \frac{b}{M} \right\}. \quad (3.2.5)$$

We establish successive approximations to solutions of (3.1.2), (3.1.3) on $[t_0, t_0 + \alpha]_{\mathbb{T}} \subseteq [t_0, t_0 + a]_{\mathbb{T}}$. The above choice of α is appropriate for this purpose in the sense that $\alpha \leq a$ and for a solution Φ of (3.1.2), (3.1.3) to lie within the region R for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$, we should have

$$\|\Phi(t) - \mathbf{x}_0\| \leq \int_{t_0}^t \|\mathbf{f}(s, \Phi(s))\| \Delta s \leq M(t - t_0) \leq b,$$

which is satisfied if

$$\alpha = t - t_0 \leq b/M, \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}}. \quad (3.2.6)$$

Now if the sequence $\{\Phi_k\}$ converges uniformly to a continuous function Φ on $[t_0, t_0 + \alpha]_{\mathbb{T}}$ such that the point $(t, \Phi_k(t)) \in R$ for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$, then we may expect that as $k \rightarrow \infty$, Φ would be our desired solution. In this way, $\Phi_1, \Phi_2, \Phi_3, \dots, \Phi_k, \dots$ as defined in (3.2.3) would be successive approximations to (3.2.2).

Hence, we show the following:

- (a) Each Φ_k exists as a continuous function on $[t_0, t_0 + \alpha]_{\mathbb{T}}$ such that the graph of $(t, \Phi_k(t))$ lies in R for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$;
- (b) Φ_k converges to Φ uniformly on $[t_0, t_0 + \alpha]_{\mathbb{T}}$ and there exists an error bound on $\|\Phi_k - \Phi\|$ on $[t_0, t_0 + \alpha]_{\mathbb{T}}$. That is, for each $k = 0, 1, 2, \dots$, there is a positive constant ε_k such that

$$\|\Phi_k(t) - \Phi(t)\| < \varepsilon_k, \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}};$$

- (c) Φ is the unique solution to (3.1.2), (3.1.3) on $[t_0, t_0 + \alpha]_{\mathbb{T}}$.

We address the above points in two steps: existence of Φ_k as continuous functions on $[t_0, t_0 + \alpha]_{\mathbb{T}}$, which responds to (a); and the approximation of Φ_k to a unique solution, Φ , of (3.1.2), (3.1.3) with a small error, which covers (b) and (c).

3.2.1 Existence of successive approximations as continuous functions

In this section, we present our first result which assures that each of the Φ_1, Φ_2, \dots is well defined and continuous on the interval $[t_0, t_0 + \alpha]_{\mathbb{T}}$.

Lemma 3.2.1 *Let $\mathbf{f} : R^\kappa \rightarrow \mathbb{R}^n$ be right-Hilger-continuous. If Φ_0 is continuous on $[t_0, t_0 + \alpha]_{\mathbb{T}}$ such that*

$$\|\Phi_0(t) - \mathbf{x}_0\| \leq b, \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}}, \quad (3.2.7)$$

then the successive approximations, Φ_k , defined in (3.2.3) exist as continuous functions on $[t_0, t_0 + \alpha]_{\mathbb{T}}$ such that the points $(t, \Phi_k(t)) \in R$, for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$.

Proof: Since \mathbf{f} is right-Hilger-continuous on R^κ , we note from (3.2.3) that each Φ_k is well defined on $[t_0, t_0 + \alpha]_{\mathbb{T}}$. We show that each Φ_k is continuous on $[t_0, t_0 + \alpha]_{\mathbb{T}}$ and satisfies

$$\|\Phi_k(t) - \mathbf{x}_0\| \leq b, \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}}, \quad (3.2.8)$$

so that the graph of $(t, \Phi_k(t))$ lies in R for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$.

We begin with the initial approximation Φ_0 . By assumption, Φ_0 exists as a continuous function on $[t_0, t_0 + \alpha]_{\mathbb{T}}$ and satisfies (3.2.7). Thus, the point $(t, \Phi_0(t)) \in R$ for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$.

It follows from (3.2.3) that the next iteration will be

$$\Phi_1(t) := \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(s, \Phi_0(s)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}}. \quad (3.2.9)$$

Let us define the function

$$\mathbf{F}_0(t) := \mathbf{f}(t, \Phi_0(t)), \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}}.$$

Since \mathbf{f} is right-Hilger-continuous on R^κ , we have \mathbf{F}_0 rd-continuous on $[t_0, t_0 + \alpha]_{\mathbb{T}}^\kappa$ and, hence, on $[t_0, t_0 + \alpha]_{\mathbb{T}}^\kappa$. Thus, we can write

$$\Phi_1(t) = \mathbf{x}_0 + \int_{t_0}^t \mathbf{F}_0(s) \Delta s, \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}}.$$

Hence Φ_1 is continuous on $[t_0, t_0 + \alpha]_{\mathbb{T}}$.

We also note that using (3.2.4), we can re-write (3.2.9), for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$, as

$$\begin{aligned}\|\Phi_1(t) - \mathbf{x}_0\| &\leq \int_{t_0}^t \|\mathbf{f}(s, \Phi_0(s))\| \Delta s \\ &\leq M(t - t_0) \\ &\leq b,\end{aligned}$$

where we used (3.2.6) in the second last step. Hence (3.2.8) is satisfied for Φ_1 and the point $(t, \Phi_1(t)) \in R$ for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$.

We assume that the assertion is true for $\Phi_2, \Phi_3, \dots, \Phi_k$, and, by induction, show that it holds for Φ_{k+1} .

Since Φ_k are continuous on $[t_0, t_0 + \alpha]_{\mathbb{T}}$ and the points $(t, \Phi_k(t)) \in R$ for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$, the function

$$\mathbf{F}_k(t) := \mathbf{f}(t, \Phi_k(t))$$

exists and is rd-continuous for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}^{\kappa}$. Thus, the function Φ_{k+1} defined by

$$\Phi_{k+1}(t) := \mathbf{x}_0 + \int_{t_0}^t \mathbf{F}_k(s) \Delta s$$

exists as a continuous function for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$. Thus, for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$, we have

$$\begin{aligned}\|\Phi_{k+1}(t) - \mathbf{x}_0\| &\leq \int_{t_0}^t \|\mathbf{f}(s, \Phi_k(s))\| \Delta s \\ &\leq M(t - t_0) \\ &\leq b.\end{aligned}$$

Hence Φ_{k+1} also satisfies (3.2.8) such that the point $(t, \Phi_{k+1}(t)) \in R$ for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$.

Thus, by induction, each Φ_k exists as a continuous function on $[t_0, t_0 + \alpha]_{\mathbb{T}}$ and the points $(t, \Phi_k(t)) \in R$ for all $[t_0, t_0 + \alpha]_{\mathbb{T}}$.

□

In the next section, we show that the successive approximations Φ_k converge on $[t_0, t_0 + \alpha]_{\mathbb{T}}$ to a unique solution Φ defined in (3.2.2) and an error bound exists for each Φ_k on $[t_0, t_0 + \alpha]_{\mathbb{T}}$.

3.3 Local existence of solutions

We now present sufficient conditions for the existence of a unique solution, Φ , to the system (3.1.2), (3.1.3) on the interval $[t_0, t_0 + \alpha]_{\mathbb{T}} \subseteq [t_0, t_0 + a]_{\mathbb{T}}$. For this reason, this theorem is termed as the *local existence theorem*. To prove this result, we need the following lemma due to Lin and Xiang [59, Theorem 3.2] and the *Weierstrass test* [26, p.266], [72, p.600].

Lemma 3.3.1 *Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous and non-decreasing function. If $t_1, t_2 \in \mathbb{T}$ with $t_1 \leq t_2$ then*

$$\int_{t_1}^{t_2} h(t) \Delta t \leq \int_{t_1}^{t_2} h(t) dt. \quad (3.3.1)$$

□

The Weierstrass test is a theorem which gives a sufficient condition for the uniform convergence of a series of functions by comparing it with an appropriate series of non-negative constants.

Theorem 3.3.2 Weierstrass test

Suppose $\{g_i\}$ is a sequence of real-valued functions defined on a set A , and that there exists a sequence of non-negative constants K_i such that

$$|g_i(x)| \leq K_i \quad \text{for all } i \geq 1 \quad \text{and all } x \in A.$$

Suppose further that the series $\sum_{i=1}^{\infty} K_i$ converges. Then, the series $\sum_{i=1}^{\infty} g_i(x)$ converges uniformly on A .

□

The following theorem gives sufficient conditions for the existence of a unique solution to (3.1.2), (3.1.3).

Theorem 3.3.3 The local existence theorem

Consider the rectangle R^κ and let $\mathbf{f} : R^\kappa \rightarrow \mathbb{R}^n$ be a right-Hilger-continuous function. If:

(a) Φ_0 is continuous on $[t_0, t_0 + \alpha]_{\mathbb{T}}$ such that

$$\|\Phi_0(t) - \mathbf{x}_0\| \leq b, \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}};$$

(b) there exists $L > 0$ such that \mathbf{f} satisfies

$$\|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\| \leq L \|\mathbf{p} - \mathbf{q}\|, \quad \text{for all } (t, \mathbf{p}), (t, \mathbf{q}) \in R^\kappa, \quad (3.3.2)$$

then the sequence $\{\Phi_k\}$ generated by (3.2.3) converges uniformly on the compact interval

$$[t_0, t_0 + \alpha]_{\mathbb{T}} = \left[t_0, t_0 + \min \left\{ a, \frac{b}{M} \right\} \right]_{\mathbb{T}}$$

to the unique solution Φ of the IVP (3.1.2), (3.1.3). Furthermore, the following error estimate holds for all $k = 0, 1, \dots$,

$$\|\Phi_k(t) - \Phi(t)\| \leq N e^{L\alpha} \epsilon_k, \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}}, \quad (3.3.3)$$

where $\max_{t \in [t_0, t_0 + \alpha]_{\mathbb{T}}} \|\Phi_1(t) - \Phi_0(t)\| = N$.

Proof: Let $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$. We write Φ_k as

$$\begin{aligned} \Phi_k(t) &= \Phi_0(t) + (\Phi_1(t) - \Phi_0(t)) + (\Phi_2(t) - \Phi_1(t)) + \dots + (\Phi_k(t) - \Phi_{k-1}(t)) \\ &\leq \Phi_0(t) + \sum_{i=1}^k \|\Phi_i(t) - \Phi_{i-1}(t)\| \\ &\leq \Phi_0(t) + \sum_{i=1}^{\infty} \|\Phi_i(t) - \Phi_{i-1}(t)\|. \end{aligned} \quad (3.3.4)$$

That is, $\Phi_k(t)$ is a partial sum of the series $\sum_{i=1}^{\infty} \|\Phi_i(t) - \Phi_{i-1}(t)\|$ for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$. Hence, if we show that the right hand side of (3.3.4) converges absolutely and uniformly in the interval $[t_0, t_0 + \alpha]_{\mathbb{T}}$ to some function Φ , then Φ will be the uniform limit of $\{\Phi_k(t)\}$ for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$, for all $k = 1, 2, \dots$. We estimate the terms $\|\Phi_i(t) - \Phi_{i-1}(t)\|$ in (3.3.4) for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$.

We split the proof into parts to explain various elements explicitly.

(a) *Uniform convergence of Φ_k on $[t_0, t_0 + \alpha]_{\mathbb{T}}$:*

We know from our assumption on Φ_0 and Lemma 3.2.1 that each Φ_k exist as continuous functions such that the points $(t, \Phi_k(t)) \in R$ for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$. Thus, Φ_k are bounded on $[t_0, t_0 + \alpha]_{\mathbb{T}}$ for all $k = 0, 1, 2, \dots$. Let $N > 0$ such that

$$\max_{t \in [t_0, t_0 + \alpha]_{\mathbb{T}}} \|\Phi_1(t) - \Phi_0(t)\| = N. \quad (3.3.5)$$

Using (3.2.3), (3.3.2) and (3.3.5), we can write for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$,

$$\begin{aligned}
\|\Phi_2(t) - \Phi_1(t)\| &\leq \int_{t_0}^t \|\mathbf{f}(s, \Phi_1(s)) - \mathbf{f}(s, \Phi_0(s))\| \Delta s \\
&\leq L \int_{t_0}^t \|\Phi_1(s) - \Phi_0(s)\| \Delta s \\
&\leq L N \int_{t_0}^t \Delta s \\
&= L N (t - t_0).
\end{aligned}$$

We will prove by induction on i that for all $i = 1, 2, \dots$, the inequality

$$\|\Phi_i(t) - \Phi_{i-1}(t)\| \leq N \frac{[L(t - t_0)]^{i-1}}{(i-1)!}, \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}} \quad (3.3.6)$$

holds.

We have shown that (3.3.6) is true for $i = 1, 2$. We assume that it also holds for some $i = m \geq 1$ and show that it holds for $i = m + 1$ by induction. It follows from (3.2.3), (3.3.2) and our assumption that, for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$,

$$\begin{aligned}
\|\Phi_{m+1}(t) - \Phi_m(t)\| &\leq \int_{t_0}^t \|\mathbf{f}(s, \Phi_m(s)) - \mathbf{f}(s, \Phi_{m-1}(s))\| \Delta s \\
&\leq L \int_{t_0}^t \|\Phi_m(s) - \Phi_{m-1}(s)\| \Delta s \\
&\leq L \int_{t_0}^t N \frac{[L(s - t_0)]^{m-1}}{(m-1)!} \Delta s \\
&\leq L \int_{t_0}^t N \frac{[L(s - t_0)]^{m-1}}{(m-1)!} ds \\
&= \frac{N L^m}{(m-1)!} \int_{t_0}^t (s - t_0)^{m-1} ds \\
&= \frac{N L^m}{(m-1)!} \frac{(t - t_0)^m}{m} \\
&= \frac{N [L(t - t_0)]^m}{m!},
\end{aligned}$$

where we have used (3.3.6) and Lemma 3.3.1 in the third and fourth steps respectively. Thus the inequality (3.3.6) holds for all $i \geq 1$.

Next, we show that the series $\sum_{i=1}^{\infty} \|\Phi_i(t) - \Phi_{i-1}(t)\|$ converges uniformly for all $[t_0, t_0 + \alpha]_{\mathbb{T}}$. We note from (3.3.6) that for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$, we have

$$\begin{aligned} \sum_{i=1}^{\infty} \|\Phi_i(t) - \Phi_{i-1}(t)\| &\leq N \sum_{i=1}^{\infty} \frac{[L(t - t_0)]^{i-1}}{(i-1)!} \\ &\leq N \sum_{i=0}^{\infty} \frac{[L(t - t_0)]^i}{i!} \\ &= N \sum_{i=0}^{\infty} \frac{[L\alpha]^i}{i!}, \end{aligned} \quad (3.3.7)$$

which converges to $N e^{L\alpha}$. Hence, by the Weierstrass test, $\sum_{i=1}^{\infty} \|(\Phi_i(t) - \Phi_{i-1}(t))\|$ converges uniformly for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$. Consequently, each Φ_k converges uniformly on $[t_0, t_0 + \alpha]_{\mathbb{T}}$. Thus, there exists a function, Φ , on $[t_0, t_0 + \alpha]_{\mathbb{T}}$ such that

$$\lim_{k \rightarrow \infty} \Phi_k(t) = \Phi(t), \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}}. \quad (3.3.8)$$

(b) *The error estimate for $\|\Phi_k - \Phi\|$:*

We note from (3.3.4) that

$$\Phi_k(t) = \Phi_0(t) + \sum_{i=1}^k (\Phi_i(t) - \Phi_{i-1}(t)), \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}}.$$

Therefore, as $k \rightarrow \infty$, we have

$$\Phi(t) = \Phi_0(t) + \sum_{i=1}^{\infty} (\Phi_i(t) - \Phi_{i-1}(t)), \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}}. \quad (3.3.9)$$

Thus, for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$, we have

$$\begin{aligned} \|\Phi_k(t) - \Phi(t)\| &\leq \sum_{i=k+1}^{\infty} \|(\Phi_i(t) - \Phi_{i-1}(t))\| \\ &\leq \sum_{i=k+1}^{\infty} \frac{N[L\alpha]^i}{i!} \\ &= N \sum_{i=1}^{\infty} \frac{[L\alpha]^{i+k}}{(i+k)!}. \end{aligned}$$

Using the identity $i!k! \leq (i+k)!$, the above inequality reduces to

$$\begin{aligned} \|\Phi_k(t) - \Phi(t)\| &\leq N \sum_{i=1}^{\infty} \frac{[L\alpha]^{i+k}}{i!k!} \\ &= N \frac{[L\alpha]^k}{k!} \sum_{i=1}^{\infty} \frac{[L\alpha]^i}{i!} \\ &\leq N \frac{[L\alpha]^k}{k!} \sum_{i=0}^{\infty} \frac{[L\alpha]^i}{i!} \\ &= N e^{L\alpha} \epsilon_k, \end{aligned}$$

where

$$\epsilon_k := \frac{[L\alpha]^k}{k!}, \quad \text{for all } k = 1, 2, \dots. \quad (3.3.10)$$

As the right hand side of (3.3.10) is convergent, so is the left hand side. Hence, for all $k = 1, 2, \dots$, we have

$$\|\Phi_k(t) - \Phi(t)\| \leq Ne^{L\alpha}\epsilon_k, \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}}. \quad (3.3.11)$$

Thus, Φ_k satisfy (3.3.3), which gives an error bound on $\|\Phi_k(t) - \Phi(t)\|$ for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$.

(c) *The limit function Φ is a solution:*

To show that Φ is a solution to the IVP (3.1.2), (3.1.3), we show that:

- (i) Φ is continuous on $[t_0, t_0 + \alpha]_{\mathbb{T}}$;
- (ii) the point $(t, \Phi(t)) \in R$ for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$;
- (iii) Φ satisfies (3.2.2) on $[t_0, t_0 + \alpha]_{\mathbb{T}}$. That is

$$\Phi(t) = \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(s, \Phi(s)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}}. \quad (3.3.12)$$

For (i), we note that for $t_1, t_2 \in [t_0, t_0 + \alpha]_{\mathbb{T}}$, we have from (3.2.3)

$$\begin{aligned} \|\Phi_{k+1}(t_2) - \Phi_{k+1}(t_1)\| &= \left\| \int_{t_1}^{t_2} \mathbf{f}(s, \Phi_k(s)) \Delta s \right\| \\ &\leq M|t_2 - t_1|. \end{aligned}$$

Thus, letting $k \rightarrow \infty$, we obtain from (3.3.8),

$$\|\Phi(t_2) - \Phi(t_1)\| \leq M|t_2 - t_1|, \quad \text{for all } t_1, t_2 \in [t_0, t_0 + \alpha]_{\mathbb{T}}.$$

Replacing t_2 with t in the above inequality, where $t \in (t_1 - \delta, t_1 + \delta)_{\mathbb{T}}$ for some $\delta > 0$, we note that for each $\epsilon > 0$ and $\delta := \delta(\epsilon) = \frac{\epsilon}{M}$, we have

$$\|\Phi(t) - \Phi(t_1)\| \leq \epsilon \quad \text{whenever } t \in (t_1 - \delta, t_1 + \delta)_{\mathbb{T}}.$$

Hence, Φ is continuous on $[t_0, t_0 + \alpha]_{\mathbb{T}}$.

For (ii), we show that the graph of $(t, \Phi(t))$ lies in R for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$. For this we let $k \rightarrow \infty$ in (3.2.8). This yields

$$\|\Phi(t) - \mathbf{x}_0\| \leq M|t - t_0| \leq b, \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}}.$$

Hence the point $(t, \Phi(t)) \in R$ for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$.

For (iii), we show that $\Phi(t)$ satisfies (3.3.12) for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$. For this we show that our approximations (3.2.3) converge uniformly to (3.3.12) within $[t_0, t_0 + \alpha]_{\mathbb{T}}$. We have seen in (3.3.8) that the left hand side of (3.2.3) converges to the left hand side of (3.3.12) in $[t_0, t_0 + \alpha]_{\mathbb{T}}$. Therefore, we only need to prove the convergence for the right hand sides of these equations. Thus, we show that as $k \rightarrow \infty$,

$$\int_{t_0}^t \mathbf{f}(s, \Phi_k(s)) \Delta s \rightarrow \int_{t_0}^t f(s, \Phi(s)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}}. \quad (3.3.13)$$

We note that for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$, we have

$$\begin{aligned} \left| \int_{t_0}^t \mathbf{f}(s, \Phi_k(s)) \Delta s - \int_{t_0}^t \mathbf{f}(s, \Phi(s)) \Delta s \right| &\leq \int_{t_0}^t \|\mathbf{f}(s, \Phi_k(s)) - \mathbf{f}(s, \Phi(s))\| \Delta s \\ &\leq L \int_{t_0}^t \|\Phi_k(s) - \Phi(s)\| \Delta s \\ &\leq L \int_{t_0}^t N e^{L\alpha} \epsilon_k \Delta s \\ &\leq L N e^{L\alpha} \epsilon_k (t - t_0), \end{aligned}$$

where we have used (3.3.11) in the second last step. We further note from (3.3.10) that $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$. Therefore, as $k \rightarrow \infty$, we obtain

$$\left| \int_{t_0}^t \mathbf{f}(s, \Phi_k(s)) \Delta s - \int_{t_0}^t \mathbf{f}(s, \Phi(s)) \Delta s \right| \rightarrow 0, \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}}.$$

Hence

$$\int_{t_0}^t \mathbf{f}(s, \Phi_k(s)) \Delta s \rightarrow \int_{t_0}^t \mathbf{f}(s, \Phi(s)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}}.$$

(d) Φ is unique

We assume that $\Psi(t)$ is another solution of (3.1.2), (3.1.3) such that the point

$(t, \Psi(t)) \in R$ for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$. Then from (3.3.12), we obtain for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$,

$$\begin{aligned} \|\Phi(t) - \Psi(t)\| &\leq \int_{t_0}^t \|\mathbf{f}(s, \Phi(s)) - f(s, \Psi(s))\| \Delta s \\ &\leq L \int_{t_0}^t \|\Phi(s) - \Psi(s)\| \Delta s, \end{aligned}$$

where we have used (3.3.2) in the last step. Applying Corollary 2.2.5 of Gronwall's inequality taking $z(t) := \|\Phi(t) - \Psi(t)\|$ and $L(t) := L$ with $g(t) = 0$ for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$, we obtain

$$\|\Phi(t) - \Psi(t)\| = 0, \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}}.$$

Thus $\Phi(t) = \Psi(t)$ for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$. This completes the proof. □

Remark 3.3.4 *In Theorem 3.3.3 we note that the simplest choice of the initial approximation Φ_0 would be a constant function. Given the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, where $\mathbf{x}_0 \in \mathbb{R}^n$ is fixed, one can choose $\Phi_0 = \mathbf{x}_0$ as a special case of the initial approximation. The corresponding result will be a special case of Theorem 3.3.3. The result can be viewed in [83, pp.67–72].* □

Corollary 3.3.5 *Theorem 3.3.3 also holds if \mathbf{f} has continuous partial derivatives with respect to the second argument on R^κ and there exists $K > 0$ such that $\left\| \frac{\partial \mathbf{f}}{\partial p} \right\| \leq K$. In that case, by Theorem 2.2.2, \mathbf{f} satisfies (4.4.1) for $L := K$.* □

We now present a few examples to illustrate Theorem 3.3.3.

The following example considers the initial approximation $\Phi_0(t)$ to be a quadratic function of t . The successive iterations, converging to a unique solution, are then generated from (3.2.3). A small error is also estimated between the 5th iteration and the solution.

Example 3.3.6 Consider the rectangle

$$R := \{(t, p) \in \mathbb{T} \times \mathbb{R} : t \in [0, 1]_{\mathbb{T}}, |p| \leq 1\}. \quad (3.3.14)$$

We consider the scalar initial value problem

$$x^\Delta = f(t, x) = t x + \cos x, \quad \text{for all } t \in [0, a]_{\mathbb{T}}^\kappa; \quad (3.3.15)$$

$$x(0) = 0. \quad (3.3.16)$$

Assume the initial approximation to be $\phi_0(t) = t^2$ for all $t \in [0, 1]_{\mathbb{T}}$. We claim that, for some $\alpha \in \mathbb{T}$ such that $0 < \alpha \leq 1$, the sequence $\{\phi_k\}$ generated by the Picard iterative scheme (3.2.3) converges on an interval $[0, \alpha]_{\mathbb{T}}$, to the unique solution, ϕ , of (3.3.15), (3.3.16) such that the point $(t, \phi(t)) \in R$ for all $t \in [0, \alpha]_{\mathbb{T}}$.

Proof: We show that the given IVP satisfies all conditions of Theorem 3.3.3. Consider

$$R^\kappa := \{(t, p) \in \mathbb{T}^\kappa \times \mathbb{R} : t \in [0, 1]_{\mathbb{T}}^\kappa, |p| \leq 1\}.$$

We show the following:

- (a) f is right-Hilger-continuous on R^κ : Since t is rd-continuous on $[0, 1]_{\mathbb{T}}$, the composition function $g(t) := t x(t) + \cos x(t)$ is rd-continuous for all $t \in [0, 1]_{\mathbb{T}}^\kappa$. Hence our f is right-Hilger-continuous on R^κ .

- (b) f is bounded on R^κ : Note that for all $(t, p) \in R^\kappa$,

$$\begin{aligned} |f(t, p)| &= |t p + \cos p| \\ &\leq |t p| + |\cos p| \\ &\leq |t| |p| + 1 \\ &\leq 2. \end{aligned}$$

Thus, f is bounded by $M := 2$.

- (c) f is Lipschitz continuous on R^κ : We also note that for all $(t, p) \in R^\kappa$ we have

$$\begin{aligned} \left| \frac{\partial f}{\partial p} \right| &= |t - \sin p| \\ &\leq |t| + |\sin p| \\ &\leq 2. \end{aligned}$$

Hence, by Corollary 3.3.5, our f is Lipschitz continuous with Lipschitz constant $L := 2$, which satisfies condition (b) of Theorem 3.3.3. We also note that, with $M = 2$ and for $[0, \alpha]_{\mathbb{T}} \subseteq [0, 1]_{\mathbb{T}}$, we obtain from (3.2.5)

$$\alpha = \min \left\{ 1, \frac{1}{2} \right\} = \frac{1}{2}. \quad (3.3.17)$$

Thus, the maximum interval of convergence for $\{\phi_k\}$ is $[0, 1/2]_{\mathbb{T}}$.

- (d) ϕ_k is convergent on $[0, 1/2]_{\mathbb{T}}$: We note that $\phi_0(t) = t^2$ is continuous, and so is rd-continuous for all $t \in [0, 1]_{\mathbb{T}}$ and hence for all $t \in [0, 1/2]_{\mathbb{T}}$. Thus, for all $t \in [0, 1/2]_{\mathbb{T}}$ we have, by assumption

$$|\phi_0(t) - x_0| = |\phi_0(t)| = |t^2| \leq 1/4 < 1,$$

which satisfies condition (a) of Theorem 3.3.3.

Hence, applying Theorem 3.3.3, the successive approximations $\phi_k(t)$ given by

$$\begin{aligned} \phi_{k+1}(t) &:= x_0 + \int_0^t f(s, \phi_k(s)) \Delta s \\ &= \int_0^t (s \phi_k(s) + \cos(\phi_k(s))) \Delta s \end{aligned}$$

converge uniformly to a unique solution $\phi(t)$ for all t in the optimal interval of convergence $[0, 1/2]_{\mathbb{T}}$.

It is easy to note that for all $t \in [0, 1/2]_{\mathbb{T}}$, we have

$$\begin{aligned} \phi_1(t) &= \int_0^t (s \phi_0(s) + \cos(\phi_0(s))) \Delta s \\ &= \int_0^t (s^3 + \cos(s^2)) \Delta s \\ &\leq \int_0^t (s^3 + \cos(s^2)) ds \end{aligned}$$

where we have used Lemma 3.3.1 in the last step. We note that for all $t \in [0, 1/2]_{\mathbb{T}}$, the above inequality further reduces to

$$\begin{aligned} \phi_1(t) &\leq \int_0^t (s^3 + 1) ds \\ &= \frac{t^4}{4} + t \\ &\leq \frac{33}{64}. \end{aligned}$$

Thus, for all $t \in [0, 1/2]_{\mathbb{T}}$, we have

$$\begin{aligned} |\phi_1(t) - \phi_0(t)| &\leq \left| \frac{33}{64} - t^2 \right| \\ &\leq \frac{33}{64} + \frac{1}{4} \\ &= \frac{49}{64}. \end{aligned}$$

Hence, $N = 49/64$ and so, as $k \rightarrow \infty$, the error estimate between the k th approximation and the solution will be

$$|\phi_k(t) - \phi(t)| \leq \frac{49}{64} e^{L\alpha} \epsilon_k, \quad \text{for all } t \in [0, 1/2]_{\mathbb{T}},$$

where

$$\epsilon_k := \frac{[L\alpha]^k}{k!}.$$

Since $a = 1$, $\alpha = 1/2$ and $L = 2$, we note that for $k = 5$, we obtain

$$\epsilon_5 = \frac{1}{5!} = .008.$$

Hence, the error estimate between the fifth approximation and the solution will be

$$|\phi_5(t) - \phi(t)| \leq \frac{49}{64} e(.008) = .02, \quad \text{for all } t \in [0, 1/2]_{\mathbb{T}}.$$

□

Our next example takes the initial approximation ϕ_0 to be linear, with successive iterations developed from (3.2.3). The example shows convergence of these iterations to a unique solution with very small error estimate between the 10th iteration and the solution.

Example 3.3.7 *Let the rectangle R be defined by*

$$R := \{(t, p) \in \mathbb{T} \times \mathbb{R} : t \in [0, 1]_{\mathbb{T}}, |p - 1| \leq 1\}. \quad (3.3.18)$$

Consider the initial value problem using the Riccati equation

$$x^\Delta = t + x^2, \quad \text{for all } t \in [0, 1]_{\mathbb{T}}; \quad (3.3.19)$$

$$x(0) = 1. \quad (3.3.20)$$

Choosing the first approximation to be

$$\phi_0(t) := t + 1 \quad \text{for all } t \in [0, 1]_{\mathbb{T}}, \quad (3.3.21)$$

we claim that the sequence of functions ϕ_k generated by (3.2.3) converges on the interval $[0, 1/5]_{\mathbb{T}}$ to the unique solution ϕ of the IVP (3.3.19), (3.3.20) such that the point $(t, \phi(t)) \in R$ for all $t \in [0, 1/5]_{\mathbb{T}}$.

Proof: We prove that the given IVP satisfies the conditions of Theorem (3.2.3). We note from (3.3.18) that

$$R^\kappa := \{(t, p) \in \mathbb{T}^\kappa \times \mathbb{R} : t \in [0, 1]_{\mathbb{T}}^\kappa, |p - 1| \leq 1\}.$$

We prove the following:

(i) f is right-Hilger-continuous on R^κ : We note that the composition function $k(t) := t + (p(t))^2$ is rd-continuous for all $t \in [0, 1]_{\mathbb{T}}$. Thus our f is right-Hilger-continuous on R^κ ;

(ii) f is bounded on R^κ : We note that $p \leq 2$. Therefore, for all $t \in [0, 1]_{\mathbb{T}}^\kappa$, we have

$$|f(t, p)| = |t + p^2| \leq |t| + |p^2| \leq 1 + 4 = 5.$$

Thus, f is bounded by $M = 5$ for all $t \in [0, 1]_{\mathbb{T}}$. Thus, for an interval $[0, \alpha]_{\mathbb{T}} \subseteq [0, 1]_{\mathbb{T}}$, we have $\alpha \leq \frac{b}{M} = \frac{1}{5}$;

(iii) f is Lipschitz-continuous on R^κ : Note that for $p \leq 2$, we have

$$\left| \frac{\partial f(t, p)}{\partial p} \right| = |2p| \leq 4, \quad \text{for all } t \in [0, 1]_{\mathbb{T}}^\kappa.$$

Thus, by Corollary 3.3.5, f is Lipschitz continuous with Lipschitz constant $L = 4$;

(iv) ϕ_k is convergent on $[0, 1/5]_{\mathbb{T}}$: We note that $\phi_0(t) = t + 1$ is continuous for all $t \in [0, 1/5]_{\mathbb{T}}$ and

$$|\phi_0(t) - x_0| = |t + 1 - 1| = |t| \leq 1/5 < 1, \quad \text{for all } t \in [0, 1/5]_{\mathbb{T}}.$$

Thus, by Theorem 3.3.3, the successive approximations given by

$$\begin{aligned} \phi_{k+1}(t) &:= x_0 + \int_0^t f(s, \phi_k(s)) \Delta s \\ &= 1 + \int_0^t (s + \phi_k^2(s)) \Delta s \end{aligned} \tag{3.3.22}$$

converge uniformly to a unique solution, and have the optimal interval of convergence as $[0, 1/5]_{\mathbb{T}}$. Thus, for all $t \in [0, 1/5]_{\mathbb{T}}$, the next approximation will be

$$\begin{aligned}\phi_1(t) &= 1 + \int_0^t (s + \phi_0^2(s)) \Delta s \\ &= 1 + \int_0^t (s^2 + 3s + 1) \Delta s, \\ &\leq 1 + t + \frac{3}{2}t^2 + \frac{t^3}{3},\end{aligned}$$

where we have used Lemma 3.3.1 in the second last step. Thus, for all $t \in [0, 1/5]_{\mathbb{T}}$,

$$\begin{aligned}|\phi_1(t) - \phi_0(t)| &= \left| 1 + t + \frac{3}{2}t^2 + \frac{t^3}{3} - 1 - t \right| \\ &\leq \left| \frac{3}{2}t^2 \right| + \left| \frac{t^3}{3} \right| \\ &\leq .08.\end{aligned}$$

Thus, $N = .08$ for $\alpha = 1/5$. Hence, with $L = 4$ and choosing $k = 10$, we obtain

$$\epsilon_{10} = \frac{(4/5)^{10}}{10!} = 3 \times 10^{-8}.$$

Therefore, for all $t \in [0, 1/5]_{\mathbb{T}}$, the error estimate between the 10th approximation and the solution will be

$$|\phi_{10}(t) - \phi(t)| \leq Ne^{L\alpha}\epsilon_{10} \leq 6 \times 10^{-9}.$$

□

In the next section, we further discuss the convergence of successive approximations to unique solutions in the light of an example that has been discussed by researchers for the ODE case (see [63, pp. 628–632], [69, pp 51–52]). Of interest is the relationship between the convergence of Picard iterations and uniqueness of solutions. We discuss the case in the time scale setting.

3.3.1 Convergence of successive approximations and uniqueness of solution

In this section, we consider a time scale transformation of [69, pp. 51–52]. We search answers to the following questions regarding our local existence results in the previous section:

- Is the right–Hilger–continuity of f sufficient to ensure that the sequence (or subsequences) of successive approximations actually converge to a solution?
- If the successive approximations do not converge, can we still have a solution of a dynamic IVP?
- Can we always construct sequences (or subsequences) by successive approximation that converge to a unique solution of a dynamic IVP?

The answers to the first and the third question are ‘no’ and for the second question is ‘yes’. To see how, we look at the following example.

Example 3.3.8 Consider a continuously delta differentiable function $\theta : [0, 1]_{\mathbb{T}} \rightarrow [0, \infty)$ such that

$$\theta(0) = 0; \quad (3.3.23)$$

$$\theta(t) > 0, \quad \text{for all } t \in (0, 1]_{\mathbb{T}}; \quad (3.3.24)$$

$$\text{and } \theta^{\Delta}(t) > 0, \quad \text{for all } t \in (0, 1]_{\mathbb{T}}^{\kappa}. \quad (3.3.25)$$

Let $f : (0, 1]_{\mathbb{T}}^{\kappa} \rightarrow \mathbb{R}$ be defined by

$$f(t, p) := \begin{cases} 0, & \text{for all } t = 0, -\infty < p < \infty; \\ \theta^{\Delta}(t), & \text{for all } t \in (0, 1]_{\mathbb{T}}^{\kappa}, p \leq 0; \\ \theta^{\Delta}(t) - \frac{\theta^{\Delta}(t)}{\theta(t)}p, & \text{for all } t \in (0, 1]_{\mathbb{T}}^{\kappa}, 0 < p \leq \theta(t); \\ 0, & \text{for all } t \in (0, 1]_{\mathbb{T}}^{\kappa}, p > \theta(t). \end{cases} \quad (3.3.26)$$

Consider the scalar dynamic IVP

$$x^{\Delta} = f(t, x), \quad \text{for all } t \in (0, 1]_{\mathbb{T}}^{\kappa}; \quad (3.3.27)$$

$$x(0) = 0. \quad (3.3.28)$$

We claim that the successive approximations defined by

$$\phi_0(t) = 0; \quad (3.3.29)$$

$$\phi_{k+1}(t) := \int_0^t f(s, \phi_k(s)) \Delta s \quad (3.3.30)$$

do not converge to a unique limit for $0 < t \leq 1$.

Proof: We note that if $x(t) := \frac{\theta(t)}{2}$, then from (3.3.23), we have $x(0) = 0$. Furthermore, we note from (3.3.26) that

$$f(t, x) = f\left(t, \frac{\theta(t)}{2}\right) = \frac{\theta^\Delta(t)}{2} = x^\Delta(t), \quad \text{for all } t \in (0, 1]_{\mathbb{T}}^\kappa.$$

Thus, $x(t) := \frac{\theta(t)}{2}$ is a solution to (3.3.27), (3.3.28).

Next we note that θ is delta differentiable and for any continuous function p , it follows from (3.3.26) that our f is right-Hilger-continuous on $(0, 1]_{\mathbb{T}}^\kappa \times \mathbb{R}$. We show that the sequence (or subsequences) of successive approximations do not converge to the above solution. Note that, from (3.3.29), that for all $t \in (0, 1]_{\mathbb{T}}$, the first iteration will be of the form,

$$\begin{aligned} \phi_1(t) &= \int_0^t f(s, \phi_0(s)) \Delta s \\ &= \int_0^t f(s, 0) \Delta s \\ &= \int_0^t \theta^\Delta(s) \Delta s \\ &= \theta(t). \end{aligned}$$

The second iteration will, therefore, be, for all $t \in (0, 1]_{\mathbb{T}}$

$$\begin{aligned} \phi_2(t) &= \int_0^t f(s, \phi_1(s)) \Delta s \\ &= \int_0^t f(s, \theta(s)) \Delta s \\ &= \int_0^t \theta^\Delta(s) - \frac{\theta^\Delta(s)}{\theta(s)} \theta(s) \Delta s \\ &= 0. \end{aligned}$$

Hence, $\phi_3(t) = \phi_1(t) = \theta(t)$, for all $t \in (0, 1]_{\mathbb{T}}$. In this way, we will have

$$\phi_{2k}(t) = 0 \quad \text{and} \quad \phi_{2k+1}(t) = \theta(t), \quad \text{for all } t \in (0, 1]_{\mathbb{T}}. \quad (3.3.31)$$

Thus, ϕ_k does not converge to a unique limit in $(0, 1]_{\mathbb{T}}$ in general. We further note from (3.3.31) that $f(t, 0) \neq 0$ and $f(t, \theta(t)) \neq \theta^\Delta(t)$ for all $t \in (0, 1]_{\mathbb{T}}$. Hence, we conclude that

- the two subsequences converge, but not to the *same* limit;
- neither of the limits of the subsequences are solutions to our problem.

The above example also shows that

- the right–Hilger–continuity of f alone is not sufficient to ensure that the sequence (or subsequences) of the successive approximations actually converge to a solution.
- the Picard–Lindelöf theorem provides only a sufficient condition for the successive approximations to converge to a unique solution.

3.4 Global existence of solutions

The result presented in Theorem 3.3.3 restricts the domain of solution for points that lie within a small interval containing t_0 in $[t_0, t_0 + a]_{\mathbb{T}}$. For this reason, we call it the local existence theorem. If a function $\Phi(t)$ solves the initial value problem (3.1.2), (3.1.3) for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$, then we say that the solution exists non–locally. In this section, we present a result that guarantees convergence of iterations (3.2.3) to a non–local solution of the system (3.1.2), (3.1.3) and we call it the *global existence theorem*.

In our next result, the global existence theorem, we show that if \mathbf{f} satisfies a Lipschitz condition on the infinite strip

$$S^\kappa := \{(t, \mathbf{p}) \in \mathbb{T}^\kappa \times \mathbb{R}^n : t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa, \|\mathbf{p}\| < \infty\},$$

then the solution exists in the entire interval $[t_0, t_0 + a]_{\mathbb{T}}$.

Theorem 3.4.1 (The global existence theorem)

Let \mathbf{f} be a right–Hilger–continuous function on S^κ . If:

(a) there exists $L > 0$ such that \mathbf{f} satisfies

$$\|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\| \leq L\|\mathbf{p} - \mathbf{q}\|, \quad \text{for all } (t, \mathbf{p}), (t, \mathbf{q}) \in R^\kappa; \quad (3.4.1)$$

(b) Φ_0 is continuous on $[t_0, t_0 + a]_{\mathbb{T}}$,

then the sequence $\{\Phi_k(t)\}$ generated by the Picard iterative scheme (3.2.3) exists in the entire interval $[t_0, t_0 + a]_{\mathbb{T}}$ and converges to the unique solution Φ of the IVP (3.1.2), (3.1.3), on $[t_0, t_0 + a]_{\mathbb{T}}$, with the error estimate

$$\|\Phi_k(t) - \Phi(t)\| \leq Ne^{La}\epsilon_k, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}, \quad (3.4.2)$$

where $N = \max_{t \in [t_0, t_0+a]_{\mathbb{T}}} \|\Phi_1(t) - \Phi_0(t)\|$.

Proof: We refer to the proof of Theorem 3.3.3 and follow similar steps.

1. *Uniform convergence of Φ_k on $[t_0, t_0 + a]_{\mathbb{T}}$:*

We consider the successive approximations defined in (3.2.3) for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. By assumption $\Phi_0(t)$ exists as a continuous function for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$ and is, therefore, bounded for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. Since \mathbf{f} is right-Hilger-continuous on S^κ , $\mathbf{f}(t, \Phi_0(t))$ is bounded for all $t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa$. Thus, there exists $M_1 > 0$ such that

$$\|\mathbf{f}(t, \Phi_0(t))\| \leq M_1, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa. \quad (3.4.3)$$

Thus, using (3.2.3), we obtain, for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$,

$$\begin{aligned} \|\Phi_1(t) - \mathbf{x}_0\| &\leq \int_{t_0}^t \|\mathbf{f}(s, \Phi_{k-1}(s))\| \Delta s \\ &\leq M_1(t - t_0) \\ &= M_1 a. \end{aligned} \quad (3.4.4)$$

Consider the successive approximations defined in (3.2.3). Then, by induction, as proved in Lemma 3.2.1, each Φ_k exists as a continuous function on $[t_0, t_0 + a]_{\mathbb{T}}$. Also, for $k = 2, 3, \dots$, an expression for $\Phi_k(t)$ can be written as in (3.3.4). Thus, for all $[t_0, t_0 + a]_{\mathbb{T}}^\kappa$, we can write

$$\Phi_k(t) \leq \Phi_0(t) + \sum_{i=1}^{\infty} \|\Phi_i(t) - \Phi_{i-1}(t)\|. \quad (3.4.5)$$

Using (3.4.4), we have, for all $[t_0, t_0 + a]_{\mathbb{T}}^\kappa$,

$$\|\Phi_k(t) - \mathbf{x}_0\| \leq \|\Phi_0(t) - \mathbf{x}_0\| + \sum_{i=1}^{\infty} \|\Phi_i(t) - \Phi_{i-1}(t)\|. \quad (3.4.6)$$

Since Φ_0 and Φ_1 are continuous on $[t_0, t_0 + a]_{\mathbb{T}}$, the assumption

$$\|\Phi_1(t) - \Phi_0(t)\| \leq N, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}} \quad (3.4.7)$$

is well-defined. Hence, we can write (3.4.6) as

$$\begin{aligned} \|\Phi_k(t) - \mathbf{x}_0\| &\leq \|\Phi_0(t) - \Phi_1(t)\| + \|\Phi_1(t) - \mathbf{x}_0\| + \sum_{i=1}^{\infty} \|\Phi_i(t) - \Phi_{i-1}(t)\| \\ &\leq N + M_1(t - t_0) + \sum_{i=1}^{\infty} \|\Phi_i(t) - \Phi_{i-1}(t)\|. \end{aligned} \quad (3.4.8)$$

By induction, as obtained in the proof of Theorem 3.3.3, the error inequality (3.3.6) also holds for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$ and so, for all $i \geq 1$, we have

$$\|\Phi_i(t) - \Phi_{i-1}(t)\| \leq N \frac{[L(t - t_0)]^{i-1}}{(i-1)!} \text{ for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \quad (3.4.9)$$

Thus, the series $\sum_{i=1}^{\infty} \|\Phi_i(t) - \Phi_{i-1}(t)\|$ converges to Ne^{La} . In this way, the right hand side of (3.4.5) is convergent. Therefore, by Weierstrass test, the left hand side is also convergent. Hence, there exists a function Φ such that

$$\Phi_k(t) \rightarrow \Phi(t), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \quad (3.4.10)$$

Moreover, using (3.4.9), we can re-write (3.4.8), for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$, as,

$$\begin{aligned} \|\Phi_k(t) - \mathbf{x}_0\| &\leq N + M_1 a + N \sum_{i=1}^{\infty} \|\Phi_i(t) - \Phi_{i-1}(t)\| \\ &\leq N + M_1 a + N \sum_{i=1}^{\infty} \frac{(La)^{i-1}}{(i-1)!} \\ &= N + M_1 a + N \sum_{i=0}^{\infty} \frac{(La)^i}{(i)!} \\ &\leq N + M_1 a + Ne^{La} \\ &= M_1 a + N(e^{La} + 1). \end{aligned}$$

Define $b := M_1 a + N(e^{La} + 1)$. Then, we obtain

$$\|\Phi_k(t) - \mathbf{x}_0\| \leq b, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \quad (3.4.11)$$

Thus, the points $(t, \Phi_k(t)) \in R$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

2. *The error estimate on $\|\Phi_k - \Phi\|$:*

The error estimate for $\|\Phi_k(t) - \Phi(t)\|$ is given by the inequality (3.4.9) using (3.2.3) for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. It follows that, for $\epsilon_k := \frac{[La]^k}{k!}$ where $k \geq 1$, we have, for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$, we have

$$\|\Phi_k(t) - \Phi(t)\| \leq Ne^{La} \epsilon_k,$$

which gives an error bound on $\|\Phi_k(t) - \Phi(t)\|$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

3. *The limit function Φ is the unique solution:*

We show that: Φ is continuous on $[t_0, t_0 + a]_{\mathbb{T}}$; the graph of $(t, \Phi(t))$ lies within R for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$; Φ satisfies (3.2.2); and Φ is unique.

The continuity of Φ is the same as proved in part(c(i)) of the proof for Theorem 3.3.3, with α replaced with a . It follows from (3.4.10) and (3.4.11) that as $k \rightarrow \infty$, we have

$$\|\Phi(t) - \mathbf{x}_0\| \leq b, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}.$$

Thus, the point $(t, \Phi(t)) \in R$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

We also note from (3.4.10) that the left hand side of (3.2.3) converges to the left hand side of (3.2.2). The proof for the right hand sides is the same as shown in the proof (c(iii)) of Theorem 3.3.3, replacing α by a . Hence for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$, we have

$$\int_{t_0}^t \mathbf{f}(s, \Phi_k(s)) \Delta s \rightarrow \int_{t_0}^t f(s, \Phi(s)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}.$$

The uniqueness of solution also follows in the same way as proved in (d) of Theorem 3.3.3.

This completes the proof and a unique solution exists for IVP (3.1.2), (3.1.3) in the entire interval $[t_0, t_0 + a]_{\mathbb{T}}$.

□

The above theorem is a generalisation of the global existence theorem [83, Theorem 4.13] in which $\Phi_0(t)$ was a scalar and was taken to be the initial value x_0 . Thus, the bound b on $\|\Phi_k(t) - \mathbf{x}_0\|$ in the above theorem includes an additional term $N + Ma$.

Corollary 3.4.2 *Theorem 3.4.1 also holds if \mathbf{f} has continuous partial derivatives with respect to the second argument on S^κ and there exists $K > 0$ such that $\left\| \frac{\partial \mathbf{f}}{\partial p} \right\| \leq K$. In that case, by Theorem 2.2.2, \mathbf{f} satisfies (4.4.1) for $L := K$.*

□

The following example is an extension of Example 3.3.7, illustrating the existence of solution to the IVPs (3.3.19), (3.3.20) in the entire interval $[0, 1]$.

Example 3.4.3 We re-consider Example 3.3.7 and the Riccati initial value problem (3.3.19), (3.3.20). Define the infinite strip

$$S := \{(t, p) \in \mathbb{T} \times \mathbb{R} : t \in [0, 1]_{\mathbb{T}}, |p| < \infty\}, \quad (3.4.12)$$

We have the first approximation to be a continuous function $\phi_0(t) := t + 1$ for all $t \in [0, 1]_{\mathbb{T}}$. We claim that the sequence of functions ϕ_k generated by the Picard iterative scheme (3.2.3) converges to the unique solution ϕ of (3.3.19), (3.3.20) such that the point $(t, \phi(t)) \in R$ for all $t \in [0, 1]_{\mathbb{T}}$, where

$$R := \{(t, p) \in \mathbb{T} \times \mathbb{R} : t \in [0, 1]_{\mathbb{T}}, |p - 1| \leq 9\}. \quad (3.4.13)$$

Proof: We show that (3.3.19), (3.3.20) satisfies all conditions of Theorem 3.4.1. Note that in this case

$$S^\kappa := \{(t, p) \in \mathbb{T}^\kappa \times \mathbb{R} : t \in [0, 1]_{\mathbb{T}}^\kappa, |p| < \infty\}.$$

We show the following:

- (i) f is right-Hilger-continuous on S^κ : We proved the right-Hilger-continuity of f in R^κ in Example 3.3.7. By the same arguments f is right-Hilger-continuous on S^κ ;
- (ii) f is Lipschitz continuous on S^κ : Note that, for all $t \in [0, 1]_{\mathbb{T}}$, we have

$$\begin{aligned} |f(t, \phi_0(t))| &= |t + \phi_0^2(t)| \\ &= |t + (t + 1)^2| \\ &\leq |t^2| + 3|t| + 1 \\ &\leq 5. \end{aligned}$$

Thus, $M_1 = 5$.

We also note from Example 3.3.7 that

$$\phi_1(t) \leq 1 + t + \frac{3}{2}t^2 + \frac{t^3}{3}, \quad \text{for all } t \in [0, 1]_{\mathbb{T}}.$$

Therefore, for all $t \in [0, 1]_{\mathbb{T}}$, we have

$$|\phi_1(t) - \phi_0(t)| \leq \left| \frac{3}{2}t^2 \right| + \left| \frac{t^3}{3} \right| \leq \frac{11}{6}.$$

Thus, $N = 11/6$.

Using (3.4.11) and the values of a, M_1, N , we note that the minimum integral value of b for which $L > 0$ is $b = 9$. Thus, choosing $b = 9$, we obtain a Lipschitz constant $L = .16$ for f .

(iii) Φ_k are convergent on $[0, 1]_{\mathbb{T}}$: By Theorem 3.4.1, the successive approximations given by

$$\begin{aligned}\phi_{k+1}(t) &:= x_0 + \int_0^t f(s, \phi_k(s)) \Delta s \\ &= 1 + \int_0^t (s + \phi_k^2(s)) \Delta s,\end{aligned}$$

converge to a unique solution ϕ for all $t \in [0, 1]_{\mathbb{T}}$ such that the point $(t, \phi(t)) \in R$. Hence, with $M_1 = 5, L = .16$ and $N = 11/6$, if we choose $k = 5$, we obtain

$$\epsilon_5 = \frac{[.16]^5}{5!} = 1 \times 10^{-6}.$$

Therefore, the error estimate between the 5–th approximation and the solution will be

$$|\phi_5(t) - \phi(t)| \leq \frac{11}{6} e^{.16} (1 \times 10^{-6}) = 3 \times 10^{-7}, \quad \text{for all } t \in [0, 1]_{\mathbb{T}}.$$

□

In the next section, we present a time scale transformation of a theorem of Keller [50, Chapter 1] from ordinary differential equations, applying ideas from Theorem 3.4.1.

3.5 Keller's existence theorem on \mathbb{T}

In this section, we present another existence result employing the method of successive approximations using ideas from [50, Chapter 1]. We consider a dynamic initial value problem with the initial value in an n –sphere of radius r for some $r > 0$. We also consider a right–Hilger–continuous function \mathbf{f} defined on a larger sphere.

Let $r, M > 0$ and t a point in an arbitrary compact interval $[t_0, t_0 + a]_{\mathbb{T}} \subseteq \mathbb{T}$. Let $\mathbf{A}_0 \in \mathbb{R}^n$ and define

$$N_r(\mathbf{A}_0) := \{\mathbf{p} : \|\mathbf{p} - \mathbf{A}_0\| \leq r\} \tag{3.5.1}$$

and

$$\mathbf{P}_{r,M}(\mathbf{A}_0) := \{(t, \mathbf{p}) : t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \text{ and } \|\mathbf{p} - \mathbf{A}_0\| \leq r + M(t - t_0)\}. \quad (3.5.2)$$

Consider the vector dynamic IVP

$$\mathbf{x}^\Delta = \mathbf{f}(t, \mathbf{x}), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}; \quad (3.5.3)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0. \quad (3.5.4)$$

Then the following theorem guarantees a unique solution to the IVP (3.5.3), (3.5.4) in $\mathbf{P}_{r,M}(\mathbf{A}_0)$.

Theorem 3.5.1 *Let $\mathbf{A}_0 \in \mathbb{R}^n$ and $\mathbf{x}_0 \in N_r(\mathbf{A}_0)$. Consider positive constants r, M, N such that (3.5.1) and (3.5.2) hold. Let $\mathbf{f} : \mathbf{P}_{r,M}(\mathbf{A}_0) \rightarrow \mathbb{R}^n$ be a right-Hilger-continuous function. If:*

(i) \mathbf{f} satisfies

$$\|\mathbf{f}(t, \mathbf{p})\| \leq M, \quad \text{for all } (t, \mathbf{p}) \in \mathbf{P}_{r,M}(\mathbf{A}_0); \quad (3.5.5)$$

(ii) there exists a constant $L > 0$ such that

$$\|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\| \leq L \|\mathbf{p} - \mathbf{q}\|, \quad \text{for all } (t, \mathbf{p}), (t, \mathbf{q}) \in \mathbf{P}_{r,M}(\mathbf{A}_0); \quad (3.5.6)$$

(iii) the initial approximation Φ_0 is continuous on $[t_0, t_0 + a]_{\mathbb{T}}$ such that

$$\|\Phi_0(t) - \mathbf{A}_0\| \leq r + M(t - t_0), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}, \quad (3.5.7)$$

then the successive approximations defined by

$$\Phi_{k+1}(t; \mathbf{x}_0) := \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(s, \Phi_k(s; \mathbf{x}_0)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}} \quad (3.5.8)$$

converge uniformly on $[t_0, t_0 + a]_{\mathbb{T}}$ to the unique solution, $\mathbf{x} := \mathbf{x}(t, \mathbf{x}_0)$, of the dynamic IVP (3.5.3), (3.5.4) such that the point $(t, \mathbf{x}(t, \mathbf{x}_0)) \in \mathbf{P}_{r,M}(\mathbf{A}_0)$, with the error estimate

$$\|\Phi_k(t; \mathbf{x}_0) - \Phi_0(t; \mathbf{x}_0)\| \leq Q e^{L(t-t_0)} \epsilon_k, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}, \quad (3.5.9)$$

where $Q := \max_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \|\Phi_1(t; \mathbf{x}_0) - \Phi_0(t; \mathbf{x}_0)\|$.

Proof: As before, we divide the proof into smaller sections for the purpose of clarity.

We show the following:

(a) $\{\Phi_k\}$ is uniformly convergent on $[t_0, t_0 + a]_{\mathbb{T}}$:

We first show that Φ_k are uniformly continuous on $[t_0, t_0 + a]_{\mathbb{T}}$ for all $k \geq 0$ such that the points $(t, \Phi_k(t; \mathbf{x}_0)) \in \mathbf{P}_{r,M}(\mathbf{A}_0)$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. Next we show that the sequence $\{\Phi_k\}$ converges to the unique solution Φ in $\mathbf{P}_{r,M}(\mathbf{A}_0)$. Note that \mathbf{f} is right-Hilger-continuous on $\mathbf{P}_{r,M}(\mathbf{A}_0)$ and, by assumption, Φ_0 is continuous on $[t_0, t_0 + a]_{\mathbb{T}}$ such that (3.5.7) holds. It follows by Lemma 3.2.1 that Φ_k are continuous on $[t_0, t_0 + a]_{\mathbb{T}}$ for all $k \geq 0$.

Next, we show that each Φ_k satisfies

$$\|\Phi_k(t; \mathbf{x}_0) - \mathbf{A}_0\| \leq r + M(t - t_0), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}, \quad (3.5.10)$$

so that the points $(t, \Phi_k(t; \mathbf{x}_0)) \in \mathbf{P}_{r,M}(\mathbf{A}_0)$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. We prove this by induction on k . We note from (3.5.7) that (3.5.10) holds for $k = 0$. Next we assume that (3.5.10) holds for some $k = i > 0$, so that

$$\|\Phi_i(t; \mathbf{x}_0) - \mathbf{A}_0\| \leq r + M(t - t_0), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}.$$

Using (3.5.1), (3.5.5) and (3.5.8), we obtain, for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$,

$$\begin{aligned} \|\Phi_{i+1}(t; \mathbf{x}_0) - \mathbf{A}_0\| &\leq \|\mathbf{x}_0 - \mathbf{A}_0\| + \int_{t_0}^t \|\mathbf{f}(s, \Phi_i(s; \mathbf{x}_0))\| \Delta s \\ &\leq r + M(t - t_0). \end{aligned}$$

Hence (3.5.10) is true for $i = k + 1$, and so, holds in general. Thus, the points $(t, \Phi_k(t; \mathbf{x}_0)) \in \mathbf{P}_{r,M}(\mathbf{A}_0)$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

Finally, it will be sufficient to show that, for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$, the estimate

$$\|\Phi_k(t; \mathbf{x}_0) - \Phi_{k-1}(t; \mathbf{x}_0)\| \leq Q \frac{[L(t - t_0)]^{k-1}}{(k-1)!} \quad (3.5.11)$$

holds. We prove the above inequality by induction on k . Note that, from (iii), (3.5.11) holds for $k = 1$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. For $k = 2$, we note from (3.5.8) that, for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$,

$$\begin{aligned} \|\Phi_2(t; \mathbf{x}_0) - \Phi_1(t; \mathbf{x}_0)\| &\leq \int_{t_0}^t \|\mathbf{f}(s, \Phi_1(s; \mathbf{x}_0)) - \mathbf{f}(s, \Phi_0(s; \mathbf{x}_0))\| \Delta s \\ &\leq L \int_{t_0}^t \|\Phi_1(s; \mathbf{x}_0) - \Phi_0(s; \mathbf{x}_0)\| \Delta s \\ &\leq QL(t - t_0), \end{aligned}$$

where we have used (3.5.6) in the second last step. Thus, (3.5.11) is true for $k = 1, 2$. We assume that (3.5.11) holds for some $k = i \geq 1$ and note that using Lemma 3.3.1, (3.5.6) and (3.5.11), we obtain

$$\begin{aligned}
\|\Phi_{i+1}(t; \mathbf{x}_0) - \Phi_i(t; \mathbf{x}_0)\| &\leq \int_{t_0}^t \|\mathbf{f}(s, \Phi_i(s; \mathbf{x}_0)) - \mathbf{f}(s, \Phi_{i-1}(s; \mathbf{x}_0))\| \Delta s \\
&\leq L \int_{t_0}^t \|\Phi_i(s; \mathbf{x}_0) - \Phi_{i-1}(s; \mathbf{x}_0)\| \Delta s \\
&\leq QL \int_{t_0}^t \frac{[L(s - t_0)]^{i-1}}{(i-1)!} \Delta s \\
&\leq QL \int_{t_0}^t \frac{[L(s - t_0)]^{i-1}}{(i-1)!} ds \\
&\leq Q \frac{[L(t - t_0)]^i}{i!}.
\end{aligned}$$

Thus the inequality (3.5.11) holds for all i . We also note that, for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$, the series

$$\begin{aligned}
\sum_{i=1}^{\infty} \|\Phi_i(t; \mathbf{x}_0) - \Phi_{i-1}(t; \mathbf{x}_0)\| &\leq Q \sum_{i=1}^{\infty} \frac{[L(s - t_0)]^i}{i!} \\
&\leq Q \sum_{i=0}^{\infty} \frac{[L(s - t_0)]^i}{i!}. \tag{3.5.12}
\end{aligned}$$

We note that the right hand side of (3.5.12) is convergent and so the left hand side is also convergent applying the Weierstrass test. Therefore, we conclude that Φ_k converges uniformly on $[t_0, t_0 + a]_{\mathbb{T}}$ to some function Φ . The error estimate for $\|\Phi_k - \Phi\|$ is obtained in the same way as in the proof (part (3)) of Theorem 3.4.1. Thus, for all $k \geq 0$, there exists $\epsilon_k := \frac{[L(t - t_0)]^k}{k!}$ such that

$$\|\Phi_k(t; \mathbf{x}_0) - \Phi(t; \mathbf{x}_0)\| \leq Q e^{[L(t-t_0)]} \epsilon_k, \text{ for all } t \in [t_0, t_0 + a]_{\mathbb{T}}, \tag{3.5.13}$$

and so,

$$\Phi_k(t; \mathbf{x}_0) \rightarrow \Phi(t; \mathbf{x}_0) \text{ as } k \rightarrow \infty, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \tag{3.5.14}$$

(b) *The limit function Φ is a solution to (3.5.3), (3.5.4):*

We note that since \mathbf{f} is right-Hilger-continuous on $\mathbf{P}_{r,M}(\mathbf{A}_0)$, we have Φ a solution of (3.5.3), (3.5.4) if and only if Φ satisfies the delta integral equation

$$\Phi(t; \mathbf{x}_0) = \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(s; \Phi(s; \mathbf{x}_0)), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \tag{3.5.15}$$

We note that $\Phi(t; \mathbf{x}_0)$ is a continuous function for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$, such that the point $(t, \Phi(t; \mathbf{x}_0)) \in \mathbf{P}_{r,M}(\mathbf{A}_0)$. The proof being the same as in proof(c)(i) of Theorem 3.3.3 and is omitted.

Next we show that (3.5.8) converges to (3.5.15). We have already proved in (3.5.14) that the left hand side of (3.5.8) converges to the left hand side of (3.5.15). Therefore, we only need to prove that

$$\int_{t_0}^t \mathbf{f}(s, \Phi_k(s; \mathbf{x}_0)) \Delta s \rightarrow \int_{t_0}^t f(s, \Phi(s; \mathbf{x}_0)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}.$$

Note that, for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$, we obtain

$$\begin{aligned} & \left| \int_{t_0}^t \mathbf{f}(s, \Phi_k(s; \mathbf{x}_0)) \Delta s - \int_{t_0}^t \mathbf{f}(s, \Phi(s; \mathbf{x}_0)) \Delta s \right| \\ & \leq \int_{t_0}^t \|\mathbf{f}(s, \Phi_k(s; \mathbf{x}_0)) - \mathbf{f}(s, \Phi(s; \mathbf{x}_0))\| \Delta s \\ & \leq L \int_{t_0}^t \|\Phi_k(s; \mathbf{x}_0) - \Phi(s; \mathbf{x}_0)\| \Delta s \\ & \leq LQe^{L(t-t_0)} \int_{t_0}^t \epsilon_k \Delta s \\ & \leq LQe^{L(t-t_0)} \epsilon_k (t - t_0), \end{aligned}$$

where we have used (3.5.13) in the second last step. Since $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$, we have,

$$\left| \int_{t_0}^t \mathbf{f}(s, \Phi_k(s; \mathbf{x}_0)) \Delta s - \int_{t_0}^t f(s, \Phi(s; \mathbf{x}_0)) \Delta s \right| \rightarrow 0 \text{ for all } t \in [t_0, t_0 + a]_{\mathbb{T}}.$$

Hence

$$\int_{t_0}^t \mathbf{f}(s, \Phi_k(s; \mathbf{x}_0)) \Delta s \rightarrow \int_{t_0}^t f(s, \Phi(s; \mathbf{x}_0)) \Delta s \text{ for all } t \in [t_0, t_0 + a]_{\mathbb{T}}.$$

(c) Φ is unique

The proof is exactly the same as in part (d) of the proof of Theorem 3.3.3 and is therefore omitted.

□

Corollary 3.5.2 *Theorem 3.5.1 also holds if \mathbf{f} has continuous partial derivatives with respect to the second argument on $P_{r,M}(\mathbf{A}_0)$ and there exists $K > 0$ such that $\left\| \frac{\partial \mathbf{f}}{\partial p} \right\| \leq K$. In that case, \mathbf{f} satisfies (4.4.1) for $L := K$ by Theorem 2.2.2.*

□

3.6 Peano's existence theorem on \mathbb{T}

In previous sections, we discussed the existence of solution to the dynamic IVP (3.1.2), (3.1.3) as the unique limit of successive approximations to the solution when a right–Hilger–continuous function \mathbf{f} satisfied a Lipschitz condition on a given compact domain R^κ . In this section, we prove that, in the absence of the Lipschitz condition, the IVP (3.1.2), (3.1.3) has a solution which is the uniform limit of a sequence of solutions to the system of IVPs

$$\mathbf{x}^\Delta = \mathbf{f}_k(t, \mathbf{x}_k), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa; \quad (3.6.1)$$

$$\mathbf{x}(t_k) = \mathbf{x}_k, \quad (3.6.2)$$

where $k \geq 1$ such that

$$t_k \rightarrow t_0 \in \mathbb{T}, \text{ and } \mathbf{x}_k \rightarrow \mathbf{x}_0 \quad \text{as } k \rightarrow \infty. \quad (3.6.3)$$

More precisely, we consider a sequence of right–Hilger–continuous functions $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_k$ defined on R^κ such that the uniform limit

$$\lim_{k \rightarrow \infty} \mathbf{f}_k(t, \mathbf{p}) = \mathbf{f}(t, \mathbf{p}), \quad \text{for all } (t, \mathbf{p}) \in R^\kappa \quad (3.6.4)$$

exists. Denote by $\{\Phi_k(t)\}$ a sequence of continuous functions and $\{\Phi_{k(i)}(t)\}$ as a subsequence for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. We show in our next result that the above subsequence has a uniform limit, Φ , which is a solution to the limit problem

$$\mathbf{x}^\Delta = \mathbf{f}(t, \mathbf{x}), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa; \quad (3.6.5)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0. \quad (3.6.6)$$

We further show that if Φ is unique then Φ will be the uniform limit of the sequence $\{\Phi_k(t)\}$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

Definition 3.6.1 [51, p.346]

Let t be a point in $[t_0, t_0 + a]_{\mathbb{T}} \subseteq \mathbb{T}$. A family of functions \mathbf{x}_i defined on $[t_0, t_0 + a]_{\mathbb{T}}$ is said to be equicontinuous if, for every $\epsilon > 0$, there exists $\delta := \delta(\epsilon) > 0$ such that for all $i = 1, 2, 3, \dots$,

$$\|\mathbf{x}_i(t) - \mathbf{x}_i(s)\| \leq \epsilon, \text{ whenever } s \in (t - \delta, t + \delta)_{\mathbb{T}} \text{ for all } t, s \in [t_0, t_0 + a]_{\mathbb{T}}.$$

Remark 3.6.2 [42, p.3]

Let $\mathbf{x} : S \subseteq \mathbb{T} \rightarrow \mathbb{R}^n$ be arbitrary. Consider a family of functions \mathbf{x}_i uniformly Lipschitz continuous on S . That is, there exists $L > 0$ such that for all $i \geq 1$, we have

$$\|\mathbf{x}_i(t) - \mathbf{x}_i(s)\| \leq L|t - s|, \quad \text{for all } t, s \in S.$$

Then \mathbf{x}_i are equicontinuous on S .

□

Lemma 3.6.3 [42, p.3]

Consider a compact set E and a family of continuous functions \mathbf{x}_i such that \mathbf{x}_i are uniformly convergent on E . Then \mathbf{x}_i are uniformly bounded and equicontinuous in E .

□

The next theorem, called the Arzela–Ascoli Theorem (sometimes written as Ascoli–Arzela Theorem) [31], [51, Theorem 8.26], [16, p.178], will be useful in some important results in this and the next chapter to establish uniform convergence of compact maps.

Theorem 3.6.4 Arzela–Ascoli Theorem

Consider a family of uniformly bounded and equicontinuous functions $\mathbf{x}_1, \mathbf{x}_2, \dots$ defined on a compact set $E_n \subseteq \mathbb{R}^n$. Then, there exists a subsequence $\mathbf{x}_{i(1)}, \mathbf{x}_{i(2)}, \dots$ that converges uniformly on E_n for all $i = 1, 2, \dots$.

Remark 3.6.5 [42, p.4]

If the uniformly convergent subsequences $\mathbf{x}_{i(1)}, \mathbf{x}_{i(2)}, \dots$ in Theorem (3.6.4) converge to the same limit, Φ , then

$$\mathbf{x}_i \rightarrow \Phi, \quad \text{for all } i = 1, 2, \dots$$

□

Theorem 3.6.6 Consider a sequence $\{\mathbf{f}_k\}$ of right–Hilger–continuous functions $\mathbf{f}_k : \mathbb{R}^\kappa \rightarrow \mathbb{R}^n$ such that (3.6.4) holds. We show the following:

(a) If, for all $k = 1, 2, \dots$, Φ_k is a solution to (3.6.1), (3.6.2) on $[t_0, t_0 + a]_{\mathbb{T}}$, then the subsequence $\Phi_{k(1)}, \Phi_{k(2)}, \dots$ is uniformly convergent on $[t_0, t_0 + a]_{\mathbb{T}}$;

(b) if we denote the limit of this uniformly convergent subsequence by

$$\Phi(t) = \lim_{k \rightarrow \infty} \Phi_{k(i)}(t), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}, \quad (3.6.7)$$

then Φ will be a solution to (3.6.5), (3.6.6) on $[t_0, t_0 + a]_{\mathbb{T}}$;

(c) if $\Phi(t)$ is the unique solution of (3.6.5), (3.6.6) for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$, then

$$\Phi(t) = \lim_{k \rightarrow \infty} \Phi_k(t), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \quad (3.6.8)$$

Proof: (a) We show that the subsequence $\{\Phi_{k(i)}\}$ is uniformly convergent to Φ on $[t_0, t_0 + a]_{\mathbb{T}}$:

We note that $\{\mathbf{f}_k\}$ is a sequence of right-Hilger-continuous functions, converging to a limit \mathbf{f} uniformly on the compact set R^κ . Hence, \mathbf{f}_k are uniformly bounded on R^κ . Thus, there exists $K > 0$ such that for all $k \geq 1$

$$\|\mathbf{f}_k(t, \mathbf{p})\| \leq K, \quad \text{for all } (t, \mathbf{p}) \in R^\kappa. \quad (3.6.9)$$

Since $\Phi_k(t)$ is a solution to (3.6.1), (3.6.2) for all $k \geq 1$, we have from (3.6.9)

$$\|\Phi_k^\Delta(t)\| \leq K, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa.$$

Hence, by Theorem 2.2.2, Φ_k are Lipschitz continuous with Lipschitz constant K for all $k \geq 1$, and by Remark 3.6.2, Φ_k are equicontinuous on $[t_0, t_0 + a]_{\mathbb{T}}$. Thus, for every $\epsilon > 0$, there exists $\delta := \frac{\epsilon}{K}$ such that for all $t, s \in [t_0, t_0 + a]_{\mathbb{T}}$, we have

$$\|\Phi_k(t) - \Phi_k(s)\| \leq \epsilon, \quad \text{whenever } |t - s| \leq \delta.$$

Therefore, by Theorem 3.6.4, there exists a subsequence $\Phi_{k(i)}$ which is uniformly convergent on $[t_0, t_0 + a]_{\mathbb{T}}$ for all $k \geq 1$. Hence, by our assumption on Φ , we have

$$\Phi_{k(i)}(t) \rightarrow \Phi(t) \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}} \quad (3.6.10)$$

(b) We show that Φ is a solution to (3.6.5), (3.6.6):

By Lemma 2.1.3, Φ will be a solution to (3.6.5), (3.6.6) if and only if it solves the delta integral equation

$$\Phi(t) = \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(s, \Phi(s)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \quad (3.6.11)$$

Since Φ_k is a solution to (3.6.1), (3.6.2), $\Phi_{k(i)}$ will be a solution to (3.6.1), (3.6.2) corresponding to the $k(i)$ -th equation. Thus, by Lemma 2.1.3, for all $k, i \geq 1$, $\Phi_{k(i)}$ satisfies the delta integral equation

$$\Phi_{k(i)}(t) = \mathbf{x}_{k(i)} + \int_{t_0}^t \mathbf{f}_{k(i)}(s, \Phi(s)) \Delta s, \text{ for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \quad (3.6.12)$$

Since R^κ is compact and $\mathbf{f}_k \rightarrow \mathbf{f}$ uniformly on R^κ , we have

$$\mathbf{f}_{k(i)}(t, \mathbf{x}) \rightarrow \mathbf{f}(t, \mathbf{x}), \quad \text{for all } (t, \mathbf{x}) \in R^\kappa.$$

Similarly, from (3.6.3) and (3.6.2), we have $t_{k(i)} \rightarrow t_0$ and $\mathbf{x}(t_{k(i)}) =: \mathbf{x}_{k(i)} \rightarrow \mathbf{x}_0$ for all $k(i) \geq 1$. Thus, we have the right hand side of (3.6.12) convergent to the right hand side of (3.6.11) and by (3.6.10), the left hand side of (3.6.12) also converges to the left hand side of (3.6.11). Thus, Φ satisfies (3.6.11). By Remark (3.6.5), $\Phi_k \rightarrow \Phi$ and $\Phi(t)$ is a solution to (3.6.8) for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

(c) Φ is the unique solution of (3.6.5), (3.6.6):

Note that, from (b) above, $\Phi(t)$ is a solution to (3.6.8) for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$ and so, is a continuous function. Also, Φ_k is a solution to (3.6.1), (3.6.2) on $[t_0, t_0 + a]_{\mathbb{T}}$ and the point $(t, \Phi_k(t)) \in R$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. Thus, as $k \rightarrow \infty$, the graph $(t, \Phi(t)) \in R$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. It is already proved that Φ satisfies (3.6.11). Thus, Φ is continuous on $[t_0, t_0 + a]_{\mathbb{T}}$. The uniqueness of Φ is proved in the same way as in the proof (d) of Theorem 3.3.3 and is omitted. This completes the proof. \square

Now, we present the main result of this section. This result is an extension of [42, pp.10-11] to the time scale setting and guarantees the existence of solutions to (3.6.5), (3.6.6) in the absence of Lipschitz condition. In this way, the result is more flexible for the existence of a solution to (3.6.5), (3.6.6) than Theorem 3.3.3. However, it does not guarantee the uniqueness of solutions.

Theorem 3.6.7 Peano's existence theorem on \mathbb{T}

Let $\mathbf{f} : R^\kappa \rightarrow \mathbb{R}^n$ be a right-Hilger-continuous function. If there exists $t_0 < \alpha < a$ such that

$$\alpha := \min \left\{ a, \frac{b}{M} \right\}, \quad (3.6.13)$$

then the IVP (3.1.2), (3.1.3) has at least one solution in the interval $[t_0, t_0 + \alpha]_{\mathbb{T}} \subseteq [t_0, t_0 + a]_{\mathbb{T}}$.

Proof: We first approximate \mathbf{f} uniformly on R^κ by a sequence of right-Hilger continuous functions $\{\mathbf{f}_k\}$ such that $\mathbf{f}_k : R^\kappa \rightarrow \mathbb{R}^n$, for each $k = 1, 2, \dots$. Let $\psi(s)$ be a real-valued smooth function defined for all $s \geq 0$ such that $\psi(s) > 0$ for $0 \leq s < 1$; and $\psi(s) = 0$ for $s \geq 1$. Then, by [42, p.6], there exists a constant $c > 0$ depending on $\psi(s)$ and the dimension n , such that for every $\epsilon > 0$, we have for all $i = 1, 2, \dots, n$

$$c \epsilon^{-n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \psi(\epsilon^{-2} \|\mathbf{x} - \mathbf{p}\|^2) dp_i = 1, \text{ for all } (t, \mathbf{p}) \in R^\kappa,$$

where $\|\mathbf{x}\| = (\sum |x^i|^2)^{1/2}$. So if, for all $i = 1, 2, \dots, n$, we define

$$\begin{aligned} \mathbf{f}_k(t, \mathbf{x}) &:= c \epsilon^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathbf{f}(t, \mathbf{p}) \psi(\epsilon^{-2} \|\mathbf{x} - \mathbf{p}\|^2) dp_i, \\ &\text{for all } (t, \mathbf{p}) \in R^\kappa, \end{aligned} \quad (3.6.14)$$

then for each $i = 1, 2, \dots, n$ we have

$$\begin{aligned} \mathbf{f}_k(t, \mathbf{x}) &:= c \epsilon^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathbf{f}(t, \mathbf{x} - \mathbf{p}) \psi(\epsilon^{-2} \|\mathbf{p}\|^2) dp_i, \\ &\text{for all } (t, \mathbf{p}) \in R^\kappa. \end{aligned} \quad (3.6.15)$$

We note from [42, p.6] that as $\epsilon \rightarrow 0$,

$$\mathbf{f}_k \rightarrow \mathbf{f}, \quad \text{uniformly on } R^\kappa. \quad (3.6.16)$$

Thus, there exists $M > 0$ such that $\|\mathbf{f}_k\| \leq M$, for all $k = 1, 2, \dots$. Furthermore, $\{\mathbf{f}_k\}$ has continuous partial derivatives of all orders with respect to x_1, x_2, \dots, x_n , and is uniformly Lipschitz continuous.

Hence, $\{\mathbf{f}_k\}$ satisfies the conditions of Theorem 3.3.3. Thus, Φ_k will be solutions to the family of dynamic IVP (3.6.1), (3.6.2) in the compact interval

$$[t_0, t_0 + \alpha] = \left[t_0, t_0 + \min \left\{ a, \frac{b}{M} \right\} \right]_{\mathbb{T}}$$

for all $k \geq 1$. Moreover, Φ_k are equicontinuous and for all $k \geq 1$,

$$\|\Phi_k(t) - \mathbf{x}_0\| \leq b, \quad \text{for all } t \in [t_0, t_0 + \alpha]_{\mathbb{T}}. \quad (3.6.17)$$

Furthermore, since $\mathbf{f}_k \rightarrow \mathbf{f}$ on R^κ by (3.6.16), Theorem 3.6.6 applies, and so we have $\Phi_k(t) \rightarrow \Phi(t)$ uniformly for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$ and $\Phi(t)$ is a solution to (3.6.5), (3.6.6) for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$ and so for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$.

Next, we show that the point $(t, \Phi(t)) \in R$. This is evident by the convergence of Φ_k in (3.6.17). Hence, as $k \rightarrow \infty$, we obtain

$$\|\Phi(t) - \mathbf{x}_0\| \leq b, \quad \text{for all } t \in [t_0, t_0 + \alpha].$$

Thus, the graph of the point $(t, \Phi(t))$ lies entirely in R and the theorem is proved. □

3.7 Higher order equations

This section extends the ideas in Theorem 3.3.3 to equations of order n following [30, pp.258-260].

Let \mathbb{T} be an arbitrary time scale. Consider a set of continuous functions $\mathbf{x} : [t_0, t_0 + a]_{\mathbb{T}} \rightarrow \mathbb{R}^n$. That is $\mathbf{x} = (x_1, x_2, \dots, x_n)$, such that

$$x_1 := x; \ x_2 := x^\Delta; \ x_3 := x^{\Delta^2}; \dots; \ x_{n-1} := x^{\Delta^{n-2}}; \ x_n := x^{\Delta^{n-1}}. \quad (3.7.1)$$

Here x^Δ is the generalised derivative of x . If we delta-differentiate the above system of equations, we obtain a set of first order dynamic equations

$$\begin{aligned} x_1^\Delta &= x_2; \\ x_2^\Delta &= x_3; \\ &\vdots \\ x_{n-1}^\Delta &= x_n; \\ x_n^\Delta &= f_n(t, x_1, x_2, \dots, x_n) = f_n(t, \mathbf{x}). \end{aligned} \quad (3.7.2)$$

Consider

$$\begin{aligned} R^\kappa &= \{(t, \mathbf{p}) \in \mathbb{T}^\kappa \times \mathbb{R} : t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa, \ \|\mathbf{p} - \mathbf{A}\| \leq b\} \\ &\text{for all } a, b > 0, \text{ and } t_0, t_0 + a \in \mathbb{T}. \end{aligned} \quad (3.7.3)$$

Let $\mathbf{f} : R^\kappa \rightarrow \mathbb{R}^n$ be a right-Hilger-continuous function of $1 + n$ variables.

We consider the system of n -th order dynamic equations

$$x^{\Delta^n} = \mathbf{f}(t, x, x^\Delta, \dots, x^{\Delta^{n-1}}), \text{ for all } t \in \mathbb{T}^\kappa; \quad (3.7.4)$$

$$(x^\Delta)^i(t_0) = A_i, \quad (3.7.5)$$

where $i = 1, 2, \dots$. Under the assumption (3.7.1) and (3.7.2) this system can be reduced to

$$\mathbf{x}^\Delta = \mathbf{f}(t, \mathbf{x}), \quad \text{for all } t \in \mathbb{T}^\kappa; \quad (3.7.6)$$

$$\mathbf{x}(t_0) = \mathbf{A}, \quad (3.7.7)$$

where $\mathbf{A} = (A_1, A_2, \dots, A_n)$.

If the system (3.7.4), (3.7.5) has a solution $\Phi(t)$ for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}} \subseteq [t_0, t_0 + a]_{\mathbb{T}}$, then

$$\Phi = (\phi_1, \phi_2, \dots, \phi_n) := (\phi, \phi^\Delta, \phi^{\Delta^2}, \dots, \phi^{\Delta^{n-1}}), \quad (3.7.8)$$

where $\phi, \phi^\Delta, \phi^{\Delta^2}, \dots, \phi^{\Delta^{n-1}}$ are the respective solutions for the system (3.7.2). The following result guarantees the existence of a unique solution of (3.7.4), (3.7.5) in R^κ .

Theorem 3.7.1 *Let f_n defined in (3.7.2) be a right-Hilger-continuous function on R^κ . If f_n satisfies the conditions that:*

(a) *there exists $L_1 > 0$ such that*

$$\|f_n(t, \mathbf{x}) - f_n(t, \mathbf{y})\| \leq L_1 \|\mathbf{x} - \mathbf{y}\| \text{ for all } (t, \mathbf{x}), (t, \mathbf{y}) \in R^\kappa; \quad (3.7.9)$$

(b) *the initial approximation, $\Phi_0(t) = (\phi_0(t), \phi_0^\Delta(t), \dots, \phi_0^{\Delta^{n-1}}(t))$, where $\phi_0^{\Delta^k}(t)$ are the initial approximations to $\phi_1, \phi_2, \dots, \phi_n$ for $k = 0, 1, \dots, n-1$, is continuous for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$ such that*

$$\|\Phi_0(t) - \mathbf{A}\| \leq b, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}, \quad (3.7.10)$$

then the sequence $\{\Phi_k(t)\}$ generated by the Picard iterative scheme (3.2.3) converges on the compact interval

$$[t_0, t_0 + \alpha]_{\mathbb{T}} = \left[t_0, t_0 + \min \left\{ a, \frac{b}{M} \right\} \right]_{\mathbb{T}}$$

to the unique solution $\Phi(t)$ of n -th order IVP (3.7.6), (3.7.7) for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$.

The error estimate

$$\|\Phi_k(t) - \Phi(t)\| \leq N e^{L\alpha} \epsilon_k, \quad \text{for } k = 0, 1, 2, \dots \quad (3.7.11)$$

also holds for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$, where $N = \max_{t \in [t_0, t_0 + \alpha]_{\mathbb{T}}} \|\Phi_1(t) - \Phi_0(t)\|$.

Proof: We show that \mathbf{f} satisfies the conditions of Theorem 3.3.3.

(i) \mathbf{f} is bounded on R^κ :

Since f is right-Hilger-continuous on R^κ , there exists $K > 0$ such that

$$\|f_n(t, \mathbf{p})\| \leq K, \quad \text{for all } (t, \mathbf{p}) \in R^\kappa. \quad (3.7.12)$$

Using the above inequality along with (3.7.2) for all $(t, \mathbf{p}) \in R^\kappa$, we obtain for all $t \in [t_0, t_0 + \alpha]_{\mathbb{T}}$,

$$\begin{aligned} \|\mathbf{f}(t, \mathbf{p})\| &= \|\mathbf{f}(t, p, p^\Delta, \dots, p^{\Delta^{n-1}})\| \\ &= \|\mathbf{p}^\Delta\| \\ &= (|p_1^\Delta|^2 + |p_2^\Delta|^2 + \dots + |p_n^\Delta|^2)^{1/2} \\ &= (|p_2|^2 + |p_3|^2 + \dots + \|f_n(t, \mathbf{p})\|^2)^{1/2}, \\ &\leq (|p_1|^2 + |p_2|^2 + \dots + |p_n|^2 + \|f_n(t, \mathbf{p})\|^2)^{1/2} \\ &\leq \|\mathbf{p}\| + \|f_n(t, \mathbf{p})\| \\ &\leq \|\mathbf{A}\| + b + K, \end{aligned}$$

where we used (3.7.3) in the last step. Thus, for $M := \|\mathbf{A}\| + b + K$, we have $\|\mathbf{f}(t, \mathbf{x})\| \leq M$ for all $t \in [t_0, t_0 + \alpha]$ such that the point $(t, \mathbf{x}) \in R^\kappa$. Thus, our \mathbf{f} is bounded on R^κ .

(ii) \mathbf{f} is Lipschitz continuous on R^κ :

We note from (3.7.4), (3.7.5) that for all $(t, \mathbf{p}), (t, \mathbf{q}) \in R^\kappa$

$$\begin{aligned} &\|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\| \\ &= \|\mathbf{p}^\Delta - \mathbf{q}^\Delta\| \\ &= \|(p_1^\Delta, p_2^\Delta, \dots, p_n^\Delta) - (q_1^\Delta, q_2^\Delta, \dots, q_n^\Delta)\| \\ &= \|(p_2, p_3, \dots, f_n(t, \mathbf{p})) - (q_2, q_3, \dots, f_n(t, \mathbf{q}))\| \\ &= [(p_2 - q_2)^2 + (p_3 - q_3)^2 + \dots + (f_n(t, \mathbf{p}) - f_n(t, \mathbf{q}))^2]^{1/2} \\ &\leq [(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2 + \dots + (f_n(t, \mathbf{p}) - f_n(t, \mathbf{q}))^2]^{1/2} \\ &\leq [(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2 + \dots \\ &\quad + L[(p_1 - q_1)^2 + (p_2 - q_2)^2 + \dots + (p_n - q_n)^2]]^{1/2} \\ &= [(1 + L_1)((p_1 - q_1)^2 + (p_2 - q_2)^2 + \dots + (p_n - q_n)^2)]^{1/2} \\ &= [1 + L_1]^{1/2} \|\mathbf{p} - \mathbf{q}\|. \end{aligned}$$

Hence \mathbf{f} satisfies the Lipschitz condition (3.3.2), with Lipschitz constant $L := [1 + L_1]^{1/2}$.

□

Corollary 3.7.2 *Theorem 3.7.1 also holds if \mathbf{f} has continuous partial derivatives with respect to the second argument on R^κ and there exists $K > 0$ such that $\left\| \frac{\partial \mathbf{f}}{\partial p} \right\| \leq K$. In that case, by Theorem 2.2.2, \mathbf{f} satisfies (4.4.1) for $L := K$.*

□

The proofs for the continuity and convergence of Φ_k , for $k = 1, \dots, n$ follow from the continuity and convergence of $\phi_1, \phi_2, \dots, \phi_n$ on $[t_0, t_0 + \alpha]_{\mathbb{T}}$. The uniqueness of the solution Φ defined in (3.7.8) follows from the uniqueness of its components. Hence \mathbf{f} satisfies all conditions of Theorem 3.3.3, and the dynamic IVP (3.7.6), (3.7.7) has a unique solution $\Phi(t)$ such that the point $(t, \Phi(t)) \in R$ for all $t \in [t_0, t_0 + \alpha]$.

In this chapter, we presented results regarding existence of solutions to the systems (2.1.5), (2.1.6) and the scalar IVP (2.1.9), (2.1.10) as the unique limit of successive approximations to the above IVPs. In the next chapter, we use analytical approach to extend our results in this chapter to the entire space \mathbb{R}^n using Banach's fixed point theory.

Chapter 4

Existence results using Banach's fixed point theory

4.1 Introduction

This chapter comprises more results on the existence and uniqueness properties of solutions to first order non-linear dynamic initial value problems. The results in this chapter are obtained with a more modern approach in contrast to the classical methods used in Chapter 4. Instead, we use analytical methods and construct a weighted Banach space in the time scale setting using the exponential function and establish existence and uniqueness results using Banach's fixed point theorem. Furthermore, we establish Lipschitz continuity of solutions with respect to the initial state. We also establish local version of Banach's principle in the time scale setting. Finally, we extend our results to a generalised Banach space as well as to dynamic equations of higher order. Major results in this chapter have been published in [82].

The Banach fixed point theorem (also known as the contraction mapping theorem) has been widely used as an important tool to determine the existence and uniqueness of solutions of initial value problems defined on complete metric spaces. It uses the property that contractive maps in metric spaces have fixed points and guarantees the uniqueness of those fixed points. Moreover, it provides an iterative technique to accurately obtain those fixed points of contractive maps [41, p.9].

We construct certain metrics and norms that are suitable to the time scale setting

and establish a Banach space with respect to these metrics and norms. Using these metrics and norms we define contractive maps on our Banach space that yield fixed points as solutions to our IVPs. The next section classifies the concerned IVPs and their domains that will be later used to construct the required Banach space.

4.1.1 The main objective

Consider the time scale interval $[t_0, t_0 + a]_{\mathbb{T}}$, where $t_0 \in \mathbb{T}$ and $a > 0$. Consider a right-Hilger-continuous function $\mathbf{f} : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. Let $\mathbf{x}_0 \in \mathbb{R}^n$.

Our results in this chapter concern the IVPs

$$\mathbf{x}^{\Delta} = \mathbf{f}(t, \mathbf{x}), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}; \quad (4.1.1)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0; \quad (4.1.2)$$

and

$$\mathbf{x}^{\Delta} = \mathbf{f}(t, \mathbf{x}^{\sigma}), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}; \quad (4.1.3)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0. \quad (4.1.4)$$

The main aim of this chapter is to answer the questions:

1. Under what conditions do the dynamic IVPs (4.1.1), (4.1.2) and (4.1.3), (4.1.4) have a unique solution by applying Banach's fixed-point theory?
2. Can we extend the above results to a generalised Banach space?
3. What is the behaviour of solutions to the above IVPs with respect to their initial state?

Consider the space $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ of all continuous functions on $[t_0, t_0 + a]_{\mathbb{T}}$. Our results show that the IVPs (4.1.1), (4.1.2) and (4.1.3), (4.1.4) have unique solutions in $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ and also within certain balls of $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$. We further prove that these solutions are smooth with respect to their initial state. We apply our ideas to a generalised Banach space and to systems of higher order.

4.1.2 Methodology and organisation

Through the application of a novel definition of measuring distance in normed and metric spaces on the time scale platform, we obtain a significant range of qualitative

information about the solutions to (4.1.1), (4.1.2) and (4.1.3), (4.1.4). We apply Banach's principle to prove the existence and uniqueness of solutions of the above vector dynamic IVPs. The analysis takes place in the setting of a complete metric space. These new results significantly improve those of Hilger [45, Theorem 5.5] and also provide nice estimates on the rate of convergence of "approximating iterations" to the solution of the above IVPs using Banach's Theorem.

Our work in this chapter is organised as follows. In the next section, we review the definition of a contractive map and Banach's fixed-point theorem.

In Section 4.3, we introduce a novel "weighted" metric and "weighted" norm derived from the usual sup-metric and sup-norm. This new metric has been constructed in the time scale setting using the exponential function $e_p(t, t_0)$, where p is a regressive function and $t \in [t_0, t_0 + a]_{\mathbb{T}}$. This establishes a sufficient background to construct a new Banach space to apply Banach's Theorem for the existence and uniqueness of solutions to the dynamic IVPs (4.1.1), (4.1.2) and (4.1.3), (4.1.4). The construction of a new metric and norm has enabled us to use the Lipschitz condition without any other conditions imposed on the Lipschitz constant [45, Theorem 5.5].

In Section 4.4, we establish existence and uniqueness results for the above IVPs and illustrate the results with examples.

In Section 4.5, we extend our results to a generalised Banach space.

In Section 4.6, we present results about local existence of unique solutions for the above IVPs.

In Section 4.7, we establish Lipschitz continuity of solutions to the IVPs (4.1.1), (4.1.2) and (4.1.3), (4.1.4) within certain balls.

Finally, in Section 4.8, we extend our results to higher order dynamic equations on time scales.

4.2 The Banach fixed point theorem

We begin with the definition of a contractive map in a metric space [41, p.9].

Definition 4.2.1 Contractive map

Let (X, d) be a complete metric space. A map $F : X \rightarrow X$ is called contractive if

there exists a constant $0 < \alpha < 1$ such that

$$d(F(x), F(y)) \leq \alpha d(x, y), \quad \text{for all } x, y \in X. \quad (4.2.1)$$

The number α is called the contraction constant for F in (X, d) .

□

For any given $x \in X$, we define the iterated function sequence $\{F^i(x)\}$ recursively by:

$$F^0(x) := x; \quad (4.2.2)$$

$$\text{and} \quad F^{i+1}(x) := F[F^i(x)]. \quad (4.2.3)$$

We now present Banach's fixed-point theorem (without proof, see [34, Theorem 1.1], [40, Theorem 2.1], [51, Theorem 7.5]) which ensures the existence of a fixed point of F and the convergence of the sequence $\{F^i\}$ in (X, d) to that fixed point.

Theorem 4.2.2 Banach's fixed point theorem

Let (X, d) be a complete metric space and $F : X \rightarrow X$ be a contractive map. Then there exists a unique fixed point u of F in X . Moreover, for any $x \in X$ the iterated sequence $\{F^i(x)\}$ converges to the fixed point u , that is,

$$F^i(x) \rightarrow u, \quad \text{for all } x \in X.$$

□

Our focus is on the contraction condition (4.2.1) on F in Banach's fixed-point theorem. We begin with an arbitrary $x \in X$ and using Banach's Theorem, establish an 'error' estimate between the i th iteration F^i and the fixed point u , for all $i \geq 1$, as

$$d(F^i x, u) \leq \frac{\alpha^i}{1 - \alpha} d(x, Fx), \quad \text{for all } x \in X, \quad (4.2.4)$$

which depends on the contraction constant α and the initial displacement $d(x, Fx)$.

We further note that a map may be contractive under one particular definition of metric and not with respect to a different metric. Consider the following example.

Example 4.2.3 Let (M, d) be the metric space with the usual Euclidean metric

$$d(x, y) := |x - y|, \quad \text{for all } x, y \in M \subset \mathbb{R}$$

and (N, d^*) be another metric space with metric

$$d^*(u, v) := \frac{|u - v|}{|u + v|}, \quad \text{for all } u, v \in N \subset \mathbb{R},$$

where $|u + v| \neq 0$. Let T be a map defined on both (M, d) and (N, d^*) by $T(p) := p^2$.

We note that if $0 < |x + y| < 1$ then for all $x, y \in M$, we have

$$d(Tx, Ty) = |Tx - Ty| = |x^2 - y^2| < |x - y| = d(x, y).$$

Thus T is a contractive map in M for all x, y such that $0 < |x + y| < 1$. On the other hand, for all $u, v \in N$ such that $|u + v| > 0$, we have

$$d^*(Tu, Tv) = \frac{|Tu - Tv|}{|Tu + Tv|} = \frac{|u^2 - v^2|}{u^2 + v^2} > \frac{|u^2 - v^2|}{(u + v)^2} = \frac{|u - v|}{|u + v|} = d^*(u, v).$$

Hence T is not a contractive map in N .

□

The above example suggests that if we construct a suitable metric defined on X then the contraction condition may exist even for maximal class of F on X with minimum conditions imposed. Therefore, we establish our results in this chapter on the basis of a suitably defined metric and a norm to construct a Banach space in the time scale setting that offers a suitable platform for the existence of solutions to the dynamic IVPs (4.1.1), (4.1.2) and (4.1.3), (4.1.4).

4.3 Construction of a Banach space in \mathbb{T}

In this section, we first introduce a novel metric (and norm) in the time scale setting using a suitable exponential function (see Definition A.6.3). This metric and norm are named as the ' β -metric' and the ' β -norm' respectively. These are defined below along with the well-known sup-metric and sup-norm. This is followed by construction of a Banach space using the so called β -metric and β -norm.

Definition 4.3.1 Let $\|\cdot\|$ denote the Euclidean norm on \mathbb{R}^n . Let $\beta > 0$ be a constant. We couple the space of all continuous functions $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ with the β -metric, $d_\beta(\mathbf{x}, \mathbf{y})$, defined as

$$d_\beta(\mathbf{x}, \mathbf{y}) := \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \frac{\|\mathbf{x}(t) - \mathbf{y}(t)\|}{e_\beta(t, t_0)}, \quad (4.3.1)$$

for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$ and $\mathbf{x}, \mathbf{y} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$;

or the sup-metric, $d_0(\mathbf{x}, \mathbf{y})$ defined as

$$d_0(\mathbf{x}, \mathbf{y}) := \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \|\mathbf{x}(t) - \mathbf{y}(t)\|, \quad (4.3.2)$$

for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$ and $\mathbf{x}, \mathbf{y} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$.

We will also consider $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ coupled with the β -norm, $\|\cdot\|_\beta$, defined as

$$\|\mathbf{x}\|_\beta := \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \frac{\|\mathbf{x}(t)\|}{e_\beta(t, t_0)}, \quad (4.3.3)$$

for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$ and $\mathbf{x} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$;

or the sup-norm, $\|\mathbf{x}\|_0$, defined as

$$\|\mathbf{x}\|_0 := \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \|\mathbf{x}(t)\|, \quad (4.3.4)$$

for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$ and $\mathbf{x} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$.

□

The above definitions of d_β and $\|\cdot\|_\beta$ are new generalisations of Bielecki's metric and norm [[34], pp. 25-26], [[35], pp. 153-155] in the time-scale environment. The following Lemma describes some important properties of d_β and $\|\cdot\|_\beta$.

Lemma 4.3.2 If $\beta > 0$ is a constant then:

- (1) $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ is a vector (linear) space over \mathbb{R} ;
- (2) $\|\cdot\|_\beta$ is a norm and is equivalent to the sup-norm $\|\cdot\|_0$;
- (3) $(C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n), \|\cdot\|_\beta)$ is a Banach space;
- (4) $(C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n), d_\beta)$ is a metric space.

Proof: (1) We show that $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ is a vector space over \mathbb{R} . Note that elements of $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ are continuous functions. Hence $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ is closed under addition and scalar multiplication. Therefore, $1\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$. The commutative and associative laws also hold with respect to addition for continuous functions. The zero vector, $\mathbf{0}$, exists as the additive identity and for all $\mathbf{x} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$, $-\mathbf{x}$ will be the additive inverse. Furthermore, distributive laws hold for scalar multiplication over vector addition, and for vector multiplication over scalar addition for continuous functions for the scalar field \mathbb{R} . That is, for all $\mathbf{u}, \mathbf{v} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ and $\lambda, \nu \in \mathbb{R}$, we have

$$\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$$

$$(\lambda + \nu)\mathbf{u} = \lambda\mathbf{u} + \nu\mathbf{u}$$

$$\lambda(\nu\mathbf{u}) = (\lambda\nu)\mathbf{u}.$$

Hence, $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ is a vector space over \mathbb{R} .

(2) We show that $\|\cdot\|_{\beta}$ is a norm and is equivalent to the sup-norm $\|\cdot\|_0$. We note that $\beta \in C_{rd}([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ as any constant function is always rd-continuous. Since $\mu > 0$, we have $1 + \mu(t)\beta > 0$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. Therefore, $\beta \in \mathcal{R}^+$ (see Definition A.6.1). Thus, $e_{\beta}(t, t_0) > 0$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$ (see Theorem A.6.4(9)). It follows that for each $\mathbf{x}, \mathbf{y} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ we have

(a) $\|\mathbf{x}\|_{\beta} \geq 0$ and $\|\mathbf{x}\|_{\beta} = 0$ if and only if $\mathbf{x} = 0$.

(b) for $\lambda \in \mathbb{R}$ and $\mathbf{x} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$,

$$\begin{aligned} \|\lambda\mathbf{x}\|_{\beta} &= \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \frac{\|\lambda\mathbf{x}\|}{e_{\beta}(t, t_0)} \\ &= |\lambda| \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \frac{\|\mathbf{x}\|}{e_{\beta}(t, t_0)} \\ &= |\lambda| \|\mathbf{x}\|_{\beta}, \end{aligned}$$

and for all $\mathbf{x}, \mathbf{y} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$,

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\|_{\beta} &= \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \frac{\|\mathbf{x} + \mathbf{y}\|}{e_{\beta}(t, t_0)} \\ &\leq \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \frac{\|\mathbf{x}\|}{e_{\beta}(t, t_0)} + \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \frac{\|\mathbf{y}\|}{e_{\beta}(t, t_0)} \\ &= \|\mathbf{x}\|_{\beta} + \|\mathbf{y}\|_{\beta}. \end{aligned}$$

Thus $\|\cdot\|_\beta$ is a norm and $(C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n), \|\cdot\|_\beta)$ is a normed space.

Next we show that the β -norm, $\|\cdot\|_\beta$, is equivalent to the sup-norm, $\|\cdot\|_0$. For this, we show that there exist positive constants k and K such that

$$k\|x_0\| \leq \|x\|_\beta \leq K\|x\|_0. \quad (4.3.5)$$

Since $\beta > 0$ we have $e_\beta(t_0 + a, t_0) > 1$. Hence, choosing $k = 1/e_\beta(t_0 + a, t_0)$ and $K = 1$, we obtain

$$\frac{\|x_0\|}{e_\beta(t_0 + a, t_0)} \leq \|x\|_\beta \leq \|x\|_0.$$

Hence the β -norm and the sup-norm are equivalent.

(3) We show that $(C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n), \|\cdot\|_\beta)$ is a Banach space. For this, we show that $(C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n), \|\cdot\|_\beta)$ is complete by showing that every Cauchy sequence in $(C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n), \|\cdot\|_\beta)$ converges to a function in $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$. Let $\mathbf{x}_i(t)$ be a Cauchy sequence in $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$. This means that for every $\epsilon > 0$ there is a positive integer N_ϵ such that

$$\frac{\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\|}{e_\beta(t, t_0)} < \epsilon, \quad \text{for all } i, j > N_\epsilon, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}.$$

It follows that the sequence $\mathbf{x}_i(t)$ is uniformly convergent for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. Since \mathbf{x}_i is continuous for all i , it converges to a continuous function in $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$. Hence taking $\mathbf{x}_j \rightarrow \mathbf{x}$ as $j \rightarrow \infty$, we obtain

$$\lim_{j \rightarrow \infty} \frac{\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\|}{e_\beta(t, t_0)} = \frac{\|\mathbf{x}_i(t) - \mathbf{x}(t)\|}{e_\beta(t, t_0)} < \epsilon, \quad \text{for all } i > N_\epsilon, \quad \text{for } t \in [t_0, t_0 + a]_{\mathbb{T}}.$$

Therefore, \mathbf{x}_i converges to a point in $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$. Since \mathbf{x}_i is a Cauchy sequence, $(C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n), \|\cdot\|_\beta)$ is a complete normed vector space and, hence, a Banach space by [51, Theorem 7.4].

(4) Finally, we show that $(C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n), d_\beta)$ is a metric space. This is trivial, as by (4.3.1) and (4.3.3), d_β is the metric induced by $\|\cdot\|_\beta$. Hence $(C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n), d_\beta)$ is a complete metric space.

□

4.4 Existence and uniqueness of solutions

This section consists of results about the existence of unique solutions to the dynamic IVPs of the type (4.1.1), (4.1.2) and (4.1.3), (4.1.4) using ideas from Section 4.2. We note that a right-Hilger-continuous function \mathbf{f} is always delta integrable (see Theorem A.5.2) and so, from Lemma 2.1.3, a solution of the form (2.1.11) is well-defined for the IVP (4.1.1), (4.1.2). In fact, we will prove that given a Lipschitz condition on \mathbf{f} , such a solution always exists and is unique.

Let $C_{rd}([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ be the space of all rd-continuous functions on $[t_0, t_0 + a]_{\mathbb{T}}$. The following result concerns the dynamic IVP (4.1.1), (4.1.2).

Theorem 4.4.1 *Let $\mathbf{f} : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be right-Hilger-continuous. If there exists a positive constant L such that*

$$\|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\| \leq L \|\mathbf{p} - \mathbf{q}\|, \quad \text{for all } (t, \mathbf{p}), (t, \mathbf{q}) \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times \mathbb{R}^n, \quad (4.4.1)$$

then the dynamic IVP (4.1.1), (4.1.2) has a unique solution $\mathbf{x} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$. In addition, if a sequence of functions \mathbf{y}_i is defined inductively by choosing any $\mathbf{y}_0 \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ and setting

$$\mathbf{y}_{i+1}(t) := \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(s, \mathbf{y}_i(s)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}, \quad (4.4.2)$$

then the sequence \mathbf{y}_i converges uniformly on $[t_0, t_0 + a]_{\mathbb{T}}$ to the unique solution \mathbf{x} of (4.1.1), (4.1.2). Furthermore, $\mathbf{x}^{\Delta} \in C_{rd}([t_0, t_0 + a]_{\mathbb{T}}^{\kappa}; \mathbb{R}^n)$.

Proof: We note that (4.4.2) is well defined as \mathbf{f} is right-Hilger-continuous. Let $L > 0$ be the constant defined in (4.4.1). Define $\beta := L\gamma$ where $\gamma > 1$ is an arbitrary constant. Consider the complete metric space $(C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n), d_{\beta})$. Let

$$[\mathbf{F}\mathbf{y}](t) := \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(s, \mathbf{y}(s)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \quad (4.4.3)$$

Since \mathbf{f} is right-Hilger-continuous on $[t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times \mathbb{R}^n$, we have $[\mathbf{F}\mathbf{y}] \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ for every $\mathbf{y} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$. Further, $[\mathbf{F}\mathbf{y}](t_0) = \mathbf{x}_0 \in \mathbb{R}^n$. Hence,

$$\mathbf{F} : C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n) \rightarrow C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n).$$

According to Lemma 2.1.3, fixed points of \mathbf{F} will be solutions to the dynamic IVP (4.1.1), (4.1.2). We prove that there exists a unique, continuous function \mathbf{x} such that

$\mathbf{F}\mathbf{x} = \mathbf{x}$. To do this, we show that \mathbf{F} is a contractive map with contraction constant $\alpha = 1/\gamma < 1$ so that Banach's Theorem applies.

Let $\mathbf{p}, \mathbf{q} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$. Using (4.3.1), we note that

$$\begin{aligned} d_{\beta}(\mathbf{F}\mathbf{p}, \mathbf{F}\mathbf{q}) &:= \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \frac{\|\mathbf{F}\mathbf{p}(t) - \mathbf{F}\mathbf{q}(t)\|}{e_{\beta}(t, t_0)} \\ &\leq \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \left[\frac{1}{e_{\beta}(t, t_0)} \int_{t_0}^t \|\mathbf{f}(s, \mathbf{p}(s)) - \mathbf{f}(s, \mathbf{q}(s))\| \Delta s \right] \\ &\leq \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \left[\frac{1}{e_{\beta}(t, t_0)} \int_{t_0}^t L \|\mathbf{p}(s) - \mathbf{q}(s)\| \Delta s \right], \end{aligned}$$

where we used (4.4.1) in the last step. We can rewrite the above inequality as

$$d_{\beta}(\mathbf{F}\mathbf{p}, \mathbf{F}\mathbf{q}) \leq \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \left[\frac{1}{e_{\beta}(t, t_0)} \int_{t_0}^t L e_{\beta}(s, t_0) \sup_{s \in [t_0, t_0 + a]_{\mathbb{T}}} \frac{\|\mathbf{p}(s) - \mathbf{q}(s)\|}{e_{\beta}(s, t_0)} \Delta s \right].$$

Again using (4.3.1) and employing Theorem A.6.4(7) with $L/\beta = 1/\gamma = \alpha < 1$, we obtain

$$\begin{aligned} d_{\beta}(\mathbf{F}\mathbf{p}, \mathbf{F}\mathbf{q}) &\leq \frac{d_{\beta}(\mathbf{p}, \mathbf{q})}{\gamma} \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \left[\frac{1}{e_{\beta}(t, t_0)} \int_{t_0}^t \beta e_{\beta}(s, t_0) \Delta s \right] \\ &= \frac{d_{\beta}(\mathbf{p}, \mathbf{q})}{\gamma} \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \left[\frac{1}{e_{\beta}(t, t_0)} (e_{\beta}(t, t_0) - 1) \right] \\ &= \frac{d_{\beta}(\mathbf{p}, \mathbf{q})}{\gamma} \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \left[1 - \frac{1}{e_{\beta}(t, t_0)} \right] \\ &= \frac{d_{\beta}(\mathbf{p}, \mathbf{q})}{\gamma} \left[1 - \frac{1}{e_{\beta}(t_0 + a, t_0)} \right] \\ &< \alpha d_{\beta}(\mathbf{p}, \mathbf{q}), \end{aligned}$$

where $0 < \alpha < 1$. Thus, \mathbf{F} satisfies (4.2.1) and is a contractive map and so, Banach's Theorem applies. Therefore, there exists a unique fixed point \mathbf{x} of \mathbf{F} in $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$. Banach's Theorem also yields that the sequence \mathbf{y}_i defined in (4.4.2) converges uniformly in the β -norm, $\|\cdot\|_{\beta}$, and so, also in the sup-norm, $\|\cdot\|_0$, to that fixed point \mathbf{x} in $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$. This completes the proof. □

Corollary 4.4.2 *Theorem 4.4.1 also holds if \mathbf{f} has continuous partial derivatives with respect to the second argument on $[t_0, t_0 + a]_{\mathbb{T}} \times \mathbb{R}^n$ and there exists $K > 0$ such that $\left\| \frac{\partial \mathbf{f}}{\partial \mathbf{p}} \right\| \leq K$. In that case, \mathbf{f} satisfies (4.4.1) for $L := K$.*

Proof: The proof is the same as for Theorem 2.2.2 and is omitted.

□

Remark 4.4.3 *Theorem 4.4.1 has two important outcomes:*

1. *it eliminates the condition*

$$La < 1$$

imposed in [45, Theorem 5.5] for the existence of a unique solution of dynamic IVP of the type (4.1.1), (4.1.2). The use of β -metric, d_β , in the proof of Theorem 4.4.1 demonstrates that this condition is not needed;

2. *it gives a nice estimate on the rate of convergence of iterates (4.4.2) using (4.2.4). This means that if $\mathbf{x}, \mathbf{y}_0 \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$, then, for $\beta := \gamma L$ with $\gamma > 1$, we have, from (4.2.4)*

$$d_\beta(\mathbf{F}^i \mathbf{y}_0, \mathbf{x}) \leq \frac{\gamma^{-i}}{1 - \gamma^{-1}} d_\beta(\mathbf{y}_0, \mathbf{F} \mathbf{y}_0).$$

Using the definition of d_β in (4.3.1) with the fact that $e_\beta(t, t_0) > 0$ for all t , we note that

$$\begin{aligned} \frac{1}{e_\beta(t_0 + a, t_0)} \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \|\mathbf{F}^i \mathbf{y}_0 - \mathbf{x}\| &\leq \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \frac{\|\mathbf{F}^i \mathbf{y}_0 - \mathbf{x}\|}{e_\beta(t, t_0)} \\ &\leq \frac{\gamma^{-i}}{1 - \gamma^{-1}} \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \|\mathbf{y}_0 - \mathbf{F} \mathbf{y}_0\|. \end{aligned}$$

Using (4.3.4), we obtain

$$\|\mathbf{F}^i \mathbf{y}_0 - \mathbf{x}\|_0 \leq e_\beta(t_0 + a, t_0) \frac{\gamma^{-i}}{1 - \gamma^{-1}} \|\mathbf{y}_0 - \mathbf{F} \mathbf{y}_0\|_0.$$

If we choose $\gamma := i/La$, then the rate of convergence may be given by

$$\|\mathbf{F}^i \mathbf{y}_0 - \mathbf{x}\|_0 \leq e_{\frac{i}{a}}(t_0 + a, t_0) \left(\frac{La}{i} \right)^i \frac{i}{i - La} \|\mathbf{y}_0 - \mathbf{F} \mathbf{y}_0\|_0;$$

3. *there is no need for “extension” of a solution, as the result guarantees existence over the entire interval $[t_0, t_0 + a]_{\mathbb{T}}$.*

□

The following example demonstrates Theorem 4.4.1 with the help of a scalar dynamic IVP.

Example 4.4.4 Consider the scalar dynamic IVP

$$x^\Delta = 2[x^2 + 5]^{1/2} + t, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa; \quad (4.4.4)$$

$$x(t_0) = x_0. \quad (4.4.5)$$

We claim that this dynamic IVP has a unique solution, x , such that $x \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R})$.

Proof: We prove that the given IVP satisfies the conditions of Theorem 4.4.1. Note that

$$f(t, p) = 2[p^2 + 5]^{1/2} + t, \quad \text{for all } (t, p) \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa \times \mathbb{R}.$$

We observe that

(i) *The function f is right-Hilger-continuous on $[t_0, t_0 + a]_{\mathbb{T}}^\kappa \times \mathbb{R}$:* We note that the composition function $g(t) := 2[(x(t))^2 + 5]^{1/2} + t$ will be rd-continuous for all $t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa$. Hence, f is right-Hilger-continuous on $[t_0, t_0 + a]_{\mathbb{T}}^\kappa \times \mathbb{R}$.

(ii) *f is Lipschitz continuous on $[t_0, t_0 + a]_{\mathbb{T}}^\kappa \times \mathbb{R}$:* We note that for all $t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa$, we have

$$\left| \frac{\partial f(t, p)}{\partial p} \right| = \left| \frac{2p}{[p^2 + 5]^{1/2}} \right| \leq 2,$$

where we used $\left| \frac{p}{[p^2 + 5]^{1/2}} \right| \leq 1$ above. Hence, applying Corollary 4.4.2, we note that f is Lipschitz continuous in the second argument on $[t_0, t_0 + a]_{\mathbb{T}}^\kappa \times \mathbb{R}$ with Lipschitz constant $L = 2$.

From (i) and (ii) above, we note that all conditions of Theorem 4.4.1 are satisfied. Thus, the dynamic IVP (4.4.4), (4.4.5) has a unique solution, x , such that $x \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R})$. □

Example 4.4.5 Consider the strip

$$S^\kappa := \{(t, p) : t \in [-1, 1]_{\mathbb{T}}^\kappa, |p| < \infty\}$$

and the scalar dynamic IVP

$$x^\Delta = t + 2 \cos x, \quad \text{for all } t \in [-1, 1]_{\mathbb{T}}^\kappa; \quad (4.4.6)$$

$$x(-1) = \frac{\pi}{2}. \quad (4.4.7)$$

We claim that this dynamic IVP has a unique solution, x , with domain $[-1, 1]_{\mathbb{T}}$.

Proof: We show that the given IVP satisfies the conditions of Theorem 4.4.1. Note that $f(t, p) = t + 2 \cos p$ for all $(t, p) \in S^\kappa$.

(i) *The function f is right-Hilger-continuous on S^κ :* Note that the composition function $k(t) := t + 2 \cos x(t)$ will be rd-continuous for all $t \in [-1, 1]_{\mathbb{T}}$. Hence, our f will be right-Hilger-continuous on S^κ .

(ii) *f is Lipschitz continuous on S^κ :* We also note that for all $(t, p) \in S^\kappa$, we have

$$\left| \frac{\partial f(t, p)}{\partial p} \right| = |-2 \sin p| \leq 2.$$

Hence, by Theorem 2.2.2, f is Lipschitz continuous in the second argument on S^κ with Lipschitz constant $L = 2$.

Thus, all conditions of Theorem 4.4.1 are satisfied and the dynamic IVP (4.4.6), (4.4.7) has a unique solution, x , such that $\text{dom } x = [-1, 1]_{\mathbb{T}}$

□

Example 4.4.6 Let $\mathbf{x} = (x_1, x_2)$ and $a > 0$. Let $\mathbf{f} : [0, a]_{\mathbb{T}}^\kappa \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Consider the dynamic IVP

$$\mathbf{x}^\Delta = (t + 2x_1, t - x_2), \quad \text{for all } t \in [0, a]_{\mathbb{T}}^\kappa; \quad (4.4.8)$$

$$\mathbf{x}(0) = \mathbf{0} = (0, 0). \quad (4.4.9)$$

We claim that the above dynamic IVP has a unique solution, \mathbf{x} , such that $\mathbf{x} \in C([0, 1]_{\mathbb{T}}; \mathbb{R}^2)$.

Proof: We show that the IVP (4.4.8), (4.4.9) satisfies the conditions of Theorem 4.4.1. Note that

$$\mathbf{f}(t, \mathbf{p}) = (t + 2p_1, t - p_2), \quad \text{for all } (t, p_1), (t, p_2) \in [0, a]_{\mathbb{T}}^\kappa \times \mathbb{R}.$$

(i) *The function \mathbf{f} is right-Hilger-continuous on $[0, 1]_{\mathbb{T}}^\kappa \times \mathbb{R}^2$:* Note that the composition functions $g(t) := t + 2x_1(t)$ and $k(t) := t - x_2(t)$ will be rd-continuous for all $t \in [0, 1]_{\mathbb{T}}$. Therefore, our \mathbf{f} is right-Hilger-continuous on $[0, a]_{\mathbb{T}}^\kappa \times \mathbb{R}^2$.

(ii) \mathbf{f} is Lipschitz continuous on $[0, a]_{\mathbb{T}}^{\kappa} \times \mathbb{R}^2$: We also note that for $\mathbf{p} = (p_1, p_2)$, we have, for all $t \in [0, a]_{\mathbb{T}}^{\kappa}$,

$$\left\| \frac{\partial \mathbf{f}(t, \mathbf{p})}{\partial p_1} \right\| = \|(2, 0)\| = 2,$$

and

$$\left\| \frac{\partial \mathbf{f}(t, \mathbf{p})}{\partial p_2} \right\| = \|(0, -1)\| = 1.$$

Hence, by Corollary 4.4.2, \mathbf{f} satisfies a uniform Lipschitz condition on $[0, a]_{\mathbb{T}}^{\kappa} \times \mathbb{R}^2$, with Lipschitz constant $L = 2$.

Thus all conditions of Theorem 4.4.1 are satisfied and the dynamic IVP (4.4.8), (4.4.9) has a unique solution, $\mathbf{x} \in C([0, a]_{\mathbb{T}}; \mathbb{R}^2)$.

□

Our next theorem concerns the existence of solutions to the dynamic IVP (4.1.3), (4.1.4) using Banach's fixed-point theorem. We note from Lemma 2.1.4 that a solution of the form (2.1.14) is well-defined for (4.1.3), (4.1.4). We define a modified Lipschitz condition for \mathbf{f} that guarantees a unique solution to (4.1.3), (4.1.4) using (2.1.14).

Theorem 4.4.7 *Let $\mathbf{f} : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a right-Hilger-continuous function. Let $L > 0$ be a constant. If there exists $\gamma > 1$ with $\beta := L\gamma$ such that \mathbf{f} satisfies*

$$(1 + \mu(t)\beta)\|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\| \leq L\|\mathbf{p} - \mathbf{q}\|, \quad (4.4.10)$$

$$\text{for all } (t, \mathbf{p}), (t, \mathbf{q}) \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times \mathbb{R}^n,$$

then the dynamic IVP (4.1.3), (4.1.4) has a unique solution, \mathbf{x} , such that $\mathbf{x} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$. In addition, if a sequence of functions \mathbf{z}_i is defined inductively by choosing any $\mathbf{z}_0 \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ and setting

$$\mathbf{z}_{i+1}(t) := \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(s, \mathbf{z}_i^{\sigma}(s)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}, \quad (4.4.11)$$

then the sequence \mathbf{z}_i converges uniformly on $[t_0, t_0 + a]_{\mathbb{T}}$ to the unique solution \mathbf{x} of (4.1.3), (4.1.4). Furthermore, $\mathbf{x}^{\Delta} \in C_{rd}([t_0, t_0 + a]_{\mathbb{T}}^{\kappa}; \mathbb{R}^n)$.

Proof: We note that (4.4.11) is well-defined, as \mathbf{f} is right-Hilger-continuous. Consider the complete metric space $(C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n), d_\beta)$. Let $L > 0$ be the constant defined in (4.4.10) such that $\beta := L\gamma$, where $\gamma > 1$ is an arbitrary constant. Define, for all $\mathbf{z} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$,

$$[\mathbf{F}\mathbf{z}](t) := \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(s, \mathbf{z}^\sigma(s)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \quad (4.4.12)$$

Since \mathbf{f} is right-Hilger-continuous on $[t_0, t_0 + a]_{\mathbb{T}}^\kappa \times \mathbb{R}^n$, we have $[\mathbf{F}\mathbf{z}] \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ for all $\mathbf{z} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$. Further, $[\mathbf{F}\mathbf{z}](t_0) = \mathbf{x}_0$. Hence,

$$\mathbf{F} : C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n) \rightarrow C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n).$$

Thus, according to Lemma 2.1.4, fixed points of \mathbf{F} will be solutions to the dynamic IVP (4.1.3), (4.1.4). We prove that there exists a unique, continuous function \mathbf{x} such that $\mathbf{F}\mathbf{x} = \mathbf{x}$. To do this, we show that \mathbf{F} is a contractive map with contraction constant $\alpha = 1/\gamma < 1$ so that Banach's Theorem applies.

Let $\mathbf{p}, \mathbf{q} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$. Employing (4.3.1), we have

$$\begin{aligned} d_\beta(\mathbf{F}\mathbf{p}, \mathbf{F}\mathbf{q}) &:= \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \frac{\|\mathbf{F}\mathbf{p}(t) - \mathbf{F}\mathbf{q}(t)\|}{e_\beta(t, t_0)} \\ &\leq \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \left[\frac{1}{e_\beta(t, t_0)} \int_{t_0}^t \|\mathbf{f}(s, \mathbf{p}^\sigma(s)) - \mathbf{f}(s, \mathbf{q}^\sigma(s))\| \Delta s \right] \\ &\leq \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \left[\frac{1}{e_\beta(t, t_0)} \int_{t_0}^t \frac{L}{1 + \mu\beta} \|\mathbf{p}^\sigma(s) - \mathbf{q}^\sigma(s)\| \Delta s \right], \end{aligned}$$

where we used (4.4.10) in the last step. Moreover, we note from Theorem A.6.4(2) that

$$\frac{1}{1 + \mu(t)\beta} = \frac{e_\beta(t, t_0)}{e_\beta^\sigma(t, t_0)}, \quad \text{for all } t \in \mathbb{T}. \quad (4.4.13)$$

Using this property of the exponential function, $e_\beta(t, t_0)$ in this case, and the assumption $\beta = \gamma L$, our further computations take the form

$$\begin{aligned} d_\beta(\mathbf{F}\mathbf{p}, \mathbf{F}\mathbf{q}) &\leq \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \left[\frac{1}{e_\beta(t, t_0)} \int_{t_0}^t \frac{L}{e_\beta^\sigma(s, t_0)} \|\mathbf{p}^\sigma(s) - \mathbf{q}^\sigma(s)\| \Delta s \right] \\ &\leq \frac{1}{\gamma} \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \left[\frac{1}{e_\beta(t, t_0)} \int_{t_0}^t \beta e_\beta(s, t_0) \sup_{s \in [t_0, t_0 + a]_{\mathbb{T}}} \frac{\|\mathbf{p}^\sigma(s) - \mathbf{q}^\sigma(s)\|}{e_\beta^\sigma(s, t_0)} \Delta s \right] \\ &= \frac{1}{\gamma} d_\beta(\mathbf{p}, \mathbf{q}) \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \left[\frac{1}{e_\beta(t, t_0)} \int_{t_0}^t e_\beta^\Delta(s, t_0) \Delta s \right], \end{aligned}$$

where we used (4.3.1) and Theorem A.6.4(7) in the last step. Thus, we obtain

$$\begin{aligned} d_\beta(\mathbf{F}\mathbf{p}, \mathbf{F}\mathbf{q}) &\leq \frac{1}{\gamma} d_\beta(\mathbf{p}, \mathbf{q}) \sup_{t \in [t_0, t_0+a]_{\mathbb{T}}} \left[\frac{1}{e_\beta(t, t_0)} (e_\beta(t, t_0) - 1) \right] \\ &= \frac{1}{\gamma} d_\beta(\mathbf{p}, \mathbf{q}) \left[1 - \frac{1}{e_\beta(t_0 + a, t_0)} \right] \\ &< \frac{1}{\gamma} d_\beta(\mathbf{p}, \mathbf{q}), \end{aligned}$$

where $1/\gamma = \alpha < 1$. Thus, our \mathbf{F} satisfies (4.2.1) and is a contractive map. Thus, Banach's Theorem applies and there exists a unique fixed point \mathbf{x} of \mathbf{F} in $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$. Banach's Theorem also yields that the sequence \mathbf{z}_i defined in (4.4.11) converges uniformly to \mathbf{x} in the β -norm, $\|\cdot\|_\beta$, and in the sup-norm, $\|\cdot\|_0$, to that fixed point \mathbf{x} . This completes the proof. □

Example 4.4.8 Consider the dynamic IVP

$$x^\Delta = \frac{1}{1 + 2\mu(t)} \sin x^\sigma, \quad \text{for all } t \in [0, 1]_{\mathbb{T}}^\kappa; \quad (4.4.14)$$

$$x(0) = 0. \quad (4.4.15)$$

We claim that the above dynamic IVP has a unique solution $x \in C([0, 1]_{\mathbb{T}}; \mathbb{R})$.

Proof: We show that the IVP (4.4.14), (4.4.15) satisfies the conditions of Theorem 4.4.7. Note that

$$f(t, p) = \frac{1}{1 + 2\mu(t)} \sin p, \quad \text{for all } (t, p) \in [0, 1]_{\mathbb{T}}^\kappa \times \mathbb{R}.$$

(i) *The function f is right-Hilger-continuous on $[0, 1]_{\mathbb{T}}^\kappa \times \mathbb{R}$:* We note that the composition function $g(t) := \frac{1}{1 + 2\mu(t)} \sin x^\sigma(t)$ will be rd-continuous for all $t \in [0, 1]_{\mathbb{T}}$. Hence, our f will be right-Hilger-continuous on $[0, 1]_{\mathbb{T}}^\kappa \times \mathbb{R}$.

(ii) *f satisfies (4.4.10):* Note that for all $t \in [0, 1]_{\mathbb{T}}^\kappa$, we have

$$|f(t, p) - f(t, q)| = \left| \frac{1}{1 + 2\mu(t)} \right| |\sin p - \sin q|. \quad (4.4.16)$$

Note that $\sin x$ has continuous partial derivatives on \mathbb{R} , which are bounded by $L_1 = 1$. Thus, $\sin x$ is Lipschitz continuous by Theorem 2.2.2 with Lipschitz constant $L_1 = 1$ and we can re-write (4.4.16), for all $t \in [0, 1]_{\mathbb{T}}$, as

$$|f(t, p) - f(t, q)| \leq \left| \frac{1}{1 + 2\mu(t)} \right| |p - q|.$$

Hence, all conditions of Theorem 4.4.7 are satisfied and the given dynamic IVP has a unique solution, $x \in C([0, 1]_{\mathbb{T}}; \mathbb{R})$.

□

4.5 Generalisations

In this section, we reconsider Theorem 4.4.1 and Theorem 4.4.7 and show that the results hold for a generalised Banach space.

Let X be a Banach space. Consider a function \mathbf{f} defined on $\mathbb{T}^\kappa \times X$. Then the following definition describes the right-Hilger-continuity of $\mathbf{f} : \mathbb{T}^\kappa \times X \rightarrow X$.

Definition 4.5.1 *Consider an arbitrary time scale \mathbb{T} . A function $\mathbf{f} : \mathbb{T}^\kappa \times X \rightarrow X$ having the property that \mathbf{f} is continuous at each (t, \mathbf{x}) where t is right-dense; and the limits*

$$\lim_{(s, \mathbf{y}) \rightarrow (t^-, \mathbf{x})} \mathbf{f}(s, \mathbf{y}) \quad \text{and} \quad \lim_{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(t, \mathbf{y})$$

both exist and are finite at each (t, \mathbf{x}) where t is left-dense, is said to be right-Hilger-continuous on $\mathbb{T}^\kappa \times X$.

□

In the following results of this section, $\|\cdot\|_X$ represents the norm associated with the Banach space X . The next definition ([19, Definition 8.14]) describes the Lipschitz continuity of \mathbf{f} on $[t_0, t_0 + a]_{\mathbb{T}}^\kappa \times X$.

Definition 4.5.2 *Let $\mathbf{f} : [t_0, t_0 + a]_{\mathbb{T}}^\kappa \times X \rightarrow X$. If there exists a constant $L > 0$ such that*

$$\|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\|_X \leq L \|\mathbf{p} - \mathbf{q}\|_X, \tag{4.5.1}$$

$$\text{for all } (t, \mathbf{p}), (t, \mathbf{q}) \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa \times X,$$

then we say \mathbf{f} satisfies a uniform Lipschitz condition on $[t_0, t_0 + a]_{\mathbb{T}}^\kappa \times X$. The smallest value of L satisfying (4.5.1) is called a Lipschitz constant for \mathbf{f} on $[t_0, t_0 + a]_{\mathbb{T}}^\kappa \times X$.

□

Let \mathbf{x}_0 be a point of X . Consider a right–Hilger–continuous non–linear function $\mathbf{f} : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times X \rightarrow X$ and the generalised initial value problems

$$\mathbf{x}^{\Delta} = \mathbf{f}(t, \mathbf{x}), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}; \quad (4.5.2)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0, \quad (4.5.3)$$

and

$$\mathbf{x}^{\Delta} = \mathbf{f}(t, \mathbf{x}^{\sigma}), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}; \quad (4.5.4)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0. \quad (4.5.5)$$

The following definitions describe solutions of the generalised IVPs (4.5.2), (4.5.3) and (4.5.4), (4.5.5).

Definition 4.5.3 *A solution of (4.5.2), (4.5.3) is a function $\mathbf{x} : [t_0, t_0 + a]_{\mathbb{T}} \rightarrow X$ such that: the points $(t, \mathbf{x}(t)) \in [t_0, t_0 + a]_{\mathbb{T}} \times X$; $\mathbf{x}(t)$ is delta differentiable with $\mathbf{x}^{\Delta}(t) = \mathbf{f}(t, \mathbf{x}(t))$ for each $t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}$; and $\mathbf{x}(t_0) = \mathbf{x}_0$.*

□

Definition 4.5.4 *A solution of (4.5.4), (4.5.5) is a function $\mathbf{x} : [t_0, t_0 + a]_{\mathbb{T}} \rightarrow X$ such that: the points $(t, \mathbf{x}(t)) \in [t_0, t_0 + a]_{\mathbb{T}} \times X$; $\mathbf{x}(t)$ is delta differentiable with $\mathbf{x}^{\Delta}(t) = \mathbf{f}(t, \mathbf{x}^{\sigma}(t))$ for each $t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}$; and $\mathbf{x}(t_0) = \mathbf{x}_0$.*

□

The following two lemmas establish the equivalence of the dynamic IVPs (4.5.2), (4.5.3) and (4.5.4), (4.5.5) as delta integral equations in X . The proofs being similar to Lemma 2.1.3 and Lemma 2.1.4 have been omitted.

Lemma 4.5.5 *Consider the dynamic IVP (4.5.2), (4.5.3). Let $\mathbf{f} : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times X \rightarrow X$ be a right–Hilger–continuous function. Then a function \mathbf{x} solves (4.5.2), (4.5.3) if and only if it satisfies*

$$\mathbf{x}(t) = \int_{t_0}^t \mathbf{f}(s, \mathbf{x}(s)) \Delta s + \mathbf{x}_0, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \quad (4.5.6)$$

□

Lemma 4.5.6 Consider the dynamic equations (4.5.4), (4.5.5). Let $\mathbf{f} : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times X \rightarrow X$ be a right-Hilger-continuous function. Then a function \mathbf{x} solves (4.5.4), (4.5.5) if and only if it satisfies the delta integral equation

$$\mathbf{x}(t) = \int_{t_0}^t \mathbf{f}(s, \mathbf{x}^\sigma(s)) \Delta s + \mathbf{x}_0, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \quad (4.5.7)$$

□

Remark 4.5.7 Lemma 4.5.5 and lemma 4.5.6 also hold for \mathbf{f} being continuous, as all continuous functions are right-Hilger-continuous and are delta integrable by Theorem A.5.2.

□

The following theorem is an extension of Theorem 4.4.1 to the generalised Banach space X . It also partially extends ideas in [19, Theorem 8.16] in the sense that it proves the existence of a unique solution on the (entire) compact interval $[t_0, t_0 + a]_{\mathbb{T}}$. However, the result does not consider the other half of the interval $[t_0 - a, t_0]_{\mathbb{T}}$ for which the Lipschitz continuity of \mathbf{f} is not a sufficient condition ([19, p.323]).

Theorem 4.5.8 Let $\mathbf{f} : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times X \rightarrow X$ be right-Hilger-continuous. If there exists a positive constant L such that

$$\|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\|_X \leq L \|\mathbf{p} - \mathbf{q}\|_X, \quad (4.5.8)$$

$$\text{for all } (t, \mathbf{p}), (t, \mathbf{q}) \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times X,$$

then the dynamic IVP (4.5.2), (4.5.3) has a unique solution $\mathbf{x} \in C([t_0, t_0 + a]_{\mathbb{T}}; X)$. In addition, if a sequence of functions \mathbf{y}_i is defined inductively by choosing any $\mathbf{y}_0 \in C([t_0, t_0 + a]_{\mathbb{T}}; X)$ and setting

$$\mathbf{y}_{i+1}(t) := \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(s, \mathbf{y}_i(s)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}, \quad (4.5.9)$$

then the sequence \mathbf{y}_i converges uniformly on $[t_0, t_0 + a]_{\mathbb{T}}$ to the unique solution \mathbf{x} of (4.5.2), (4.5.3). Furthermore, $\mathbf{x}^\Delta \in C_{rd}([t_0, t_0 + a]_{\mathbb{T}}^{\kappa}; X)$.

Proof: Since X is a Banach space, X is a complete metric space. As proved in Theorem 4.4.1, the map \mathbf{F} defined by (4.4.3) will be contractive in $C([t_0, t_0 + a]_{\mathbb{T}}; X)$.

Thus, Banach Contraction Principle holds for \mathbf{F} [41, Theorem 1.1] and \mathbf{F} has a unique fixed point \mathbf{x} in $C([t_0, t_0 + a]_{\mathbb{T}}; X)$ such that, for all $\mathbf{y} \in C([t_0, t_0 + a]_{\mathbb{T}}; X)$ and $i \geq 1$, the sequence \mathbf{y}_i defined by

$$\mathbf{y}_{i+1}(t) = \int_{t_0}^t \mathbf{f}(s, \mathbf{y}_i(s)) \Delta s + \mathbf{x}_0, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}} \quad (4.5.10)$$

converges uniformly in the β -norm, $\|\cdot\|_{\beta}$, and the sup-norm, $\|\cdot\|_0$, to that fixed point \mathbf{x} in $C([t_0, t_0 + a]_{\mathbb{T}}; X)$ and \mathbf{x} will be a solution to (4.1.1), (4.1.2) by Lemma 4.5.5.

□

Theorem 4.5.9 *Let $\mathbf{f} : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times X \rightarrow X$ be a right-Hilger-continuous function. Let $L > 0$ be a constant. If there exists $\gamma > 1$ with $\beta := L\gamma$ such that \mathbf{f} satisfies*

$$(1 + \mu(t)\beta)\|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\|_X \leq L\|\mathbf{p} - \mathbf{q}\|_X, \quad (4.5.11)$$

$$\text{for all } (t, \mathbf{p}), (t, \mathbf{q}) \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times X,$$

then the dynamic IVP (4.5.4), (4.5.5) has a unique solution, \mathbf{x} , such that $\mathbf{x} \in C([t_0, t_0 + a]_{\mathbb{T}}; X)$. In addition, if a sequence of functions \mathbf{z}_i is defined inductively by choosing any $\mathbf{z}_0 \in C([t_0, t_0 + a]_{\mathbb{T}}; X)$ and setting

$$\mathbf{z}_{i+1}(t) := \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(s, \mathbf{z}_i^{\sigma}(s)) \Delta s, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}, \quad (4.5.12)$$

then the sequence \mathbf{z}_i converges uniformly on $[t_0, t_0 + a]_{\mathbb{T}}$ to the unique solution \mathbf{x} of (4.5.4), (4.5.5). Furthermore, $\mathbf{x}^{\Delta} \in C_{rd}([t_0, t_0 + a]_{\mathbb{T}}^{\kappa}; X)$.

Proof: The proof is the same as for Theorem 4.5.8, as from Theorem 4.4.7, \mathbf{F} defined by (4.4.12) will be contractive in $C([t_0, t_0 + a]_{\mathbb{T}}; X)$ and, so, Banach Contraction Principle holds for \mathbf{F} by [41, Theorem 1.1].

□

We note that Theorem 4.5.8 and Theorem 4.5.9 ensure the existence and uniqueness of solutions to the first order non-linear IVPs of the form (4.5.2), (4.5.3) and (4.5.4), (4.5.5) in the entire span of Banach spaces of continuous functions defined on $[t_0, t_0 + a]_{\mathbb{T}}$. In this way, these results are stronger as compared to Theorem 4.4.1 and Theorem 4.4.7 which confine the solutions to the space \mathbb{R}^n .

In the next section, we take a reverse approach and explore existence and uniqueness of solutions within smaller subsets of X employing the local version of Banach's fixed point theory.

4.6 Local Banach theory

The Banach principle introduced the ideas of unique fixed points of contractive maps in metric spaces no matter how large they are. However, not all maps are contractive for an entire space but they may be contractive within a small subset usually considered as a ball in a metric space. Such maps are called locally contractive maps [34, pp.10–11]. In order that locally contractive maps can be utilised for having fixed points within a ball in a metric space, there exists a local version of the Banach theorem presented as the following corollary [34, Corollary 1.2].

Corollary 4.6.1 *Let (X, d) be a complete metric space containing an open ball having centre x_0 and radius r . That is, there exists*

$$\mathcal{B}_r(x_0) := \{x \in X : d(x, x_0) < r\} \subseteq X. \quad (4.6.1)$$

Let $F : \mathcal{B}_r(x_0) \rightarrow X$ be a contractive map with a positive number $\alpha < 1$ as the contraction constant. If

$$d(Fx_0, x_0) < (1 - \alpha)r, \quad (4.6.2)$$

then F has a unique fixed point in $\mathcal{B}_r(x_0)$.

□

The next results concerns the existence and uniqueness of solutions to dynamic equations (4.1.1), (4.1.2) within certain balls using the local Banach corollary.

Theorem 4.6.2 *Let $M > 0$ and define*

$$R^\kappa := \{(t, \mathbf{p}) : t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa \text{ and } \|\mathbf{p} - \mathbf{x}_0\| \leq M\}.$$

Consider a right-Hilger-continuous function $\mathbf{f} : R^\kappa \rightarrow \mathbb{R}^n$. If:

1. there exists a positive constant L such that

$$\|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\| \leq L\|\mathbf{p} - \mathbf{q}\|, \text{ for all } (t, \mathbf{p}), (t, \mathbf{q}) \in R^\kappa; \quad (4.6.3)$$

2. the inequality

$$\int_{t_0}^{t_0+a} \|\mathbf{f}(s, \mathbf{x}_0)\| \Delta s < \frac{M}{[e_L(t_0 + a, t_0)]^2} \quad (4.6.4)$$

holds,

then the dynamic IVP (4.1.1), (4.1.2) has at least one solution \mathbf{x} on $[t_0, t_0 + a]_{\mathbb{T}}$, with a unique solution satisfying

$$d_L(\mathbf{x}, \mathbf{x}_0) < \frac{M}{e_L(t_0 + a, t_0)}. \quad (4.6.5)$$

Proof: Choose $R > 0$ such that

$$R e_L(t_0 + a, t_0) = M. \quad (4.6.6)$$

Let d_L satisfy (4.3.1) for $\beta = L$. Consider the complete metric space $(C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n), d_L)$ and an open ball $\mathcal{B}_R(\mathbf{x}_0) \subset C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ defined by

$$\mathcal{B}_R(\mathbf{x}_0) := \{\mathbf{x} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n) : d_L(\mathbf{x}, \mathbf{x}_0) < R\}, \quad (4.6.7)$$

with an operator $\mathbf{F} : \mathcal{B}_R(\mathbf{x}_0) \rightarrow C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ defined by

$$[\mathbf{F}\mathbf{x}](t) := \int_{t_0}^t \mathbf{f}(s, \mathbf{x}(s)) \Delta s + \mathbf{x}_0, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \quad (4.6.8)$$

We show that \mathbf{F} is a contractive map with a contraction constant $\alpha < 1$. We also show that \mathbf{F} satisfies (4.6.2) and, so, by Corollary 4.6.1, has a unique fixed point in $\mathcal{B}_R(\mathbf{x}_0)$.

Let $\mathbf{u}, \mathbf{v} \in \mathcal{B}_R(\mathbf{x}_0)$. Then, using (4.3.1) together with (4.6.3), we can write

$$\begin{aligned}
d_L(\mathbf{Fu}, \mathbf{Fv}) &:= \sup_{t \in [t_0, t_0+a]_{\mathbb{T}}} \frac{\|\mathbf{Fu}(t) - \mathbf{Fv}(t)\|}{e_L(t, t_0)} \\
&\leq \sup_{t \in [t_0, t_0+a]_{\mathbb{T}}} \left[\frac{1}{e_L(t, t_0)} \int_{t_0}^t \|\mathbf{f}(s, \mathbf{u}(s)) - \mathbf{f}(s, \mathbf{v}(s))\| \Delta s \right] \\
&\leq \sup_{t \in [t_0, t_0+a]_{\mathbb{T}}} \left[\frac{1}{e_L(t, t_0)} \int_{t_0}^t L \|\mathbf{u}(s) - \mathbf{v}(s)\| \Delta s \right] \\
&\leq \sup_{t \in [t_0, t_0+a]_{\mathbb{T}}} \left[\frac{1}{e_L(t, t_0)} \int_{t_0}^t L e_L(s, t_0) \sup_{s \in [t_0, t_0+a]_{\mathbb{T}}} \frac{\|\mathbf{u}(s) - \mathbf{v}(s)\|}{e_L(s, t_0)} \Delta s \right] \\
&= d_L(\mathbf{u}, \mathbf{v}) \sup_{t \in [t_0, t_0+a]_{\mathbb{T}}} \left[\frac{1}{e_L(t, t_0)} \int_{t_0}^t L e_L(s, t_0) \Delta s \right] \\
&= d_L(\mathbf{u}, \mathbf{v}) \sup_{t \in [t_0, t_0+a]_{\mathbb{T}}} \left[\frac{1}{e_L(t, t_0)} (e_L(t, t_0) - 1) \right],
\end{aligned}$$

where we used Theorem A.6.4(7) in the last step. Thus, our computations reduce to

$$\begin{aligned}
d_L(\mathbf{Fu}, \mathbf{Fv}) &\leq d_L(\mathbf{u}, \mathbf{v}) \sup_{t \in [t_0, t_0+a]_{\mathbb{T}}} \left[1 - \frac{1}{e_L(t, t_0)} \right] \\
&= d_L(\mathbf{u}, \mathbf{v}) \left[1 - \frac{1}{e_L(t_0 + a, t_0)} \right].
\end{aligned}$$

Letting

$$\alpha := 1 - \frac{1}{e_L(t_0 + a, t_0)}, \quad (4.6.9)$$

we note

$$d_L(\mathbf{Fu}, \mathbf{Fv}) < \alpha d_L(\mathbf{u}, \mathbf{v})$$

and so \mathbf{F} is a contractive map. We further note that $e_L(t_0 + a, t_0) > 0$. Using (4.6.4), (4.6.6) and (4.6.8), we obtain, for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$,

$$\begin{aligned}
d_L(\mathbf{F}(\mathbf{x}_0), \mathbf{x}_0) &= \sup_{t \in [t_0, t_0+a]_{\mathbb{T}}} \frac{\|\mathbf{F}(\mathbf{x}_0) - \mathbf{x}_0\|}{e_L(t, t_0)} \\
&= \sup_{t \in [t_0, t_0+a]_{\mathbb{T}}} \left[\frac{1}{e_L(t, t_0)} \int_{t_0}^{t_0+a} \|\mathbf{f}(s, \mathbf{x}_0)\| \Delta s \right] \\
&< \frac{M}{[e_L(t_0 + a, t_0)]^2} \sup_{t \in [t_0, t_0+a]_{\mathbb{T}}} \left[\frac{1}{e_L(t, t_0)} \right] \\
&= \frac{M}{[e_L(t_0 + a, t_0)]^2} \left[\frac{1}{e_L(t_0 + a, t_0)} \right] \\
&\leq \frac{M}{[e_L(t_0 + a, t_0)]^2} \\
&= \frac{R}{[e_L(t_0 + a, t_0)]} \\
&= (1 - \alpha)R,
\end{aligned}$$

where we used (4.6.9) in the last step. Hence, all conditions of the local Banach corollary are satisfied. Thus, \mathbf{F} has a unique fixed point $\mathbf{x} \in \mathcal{B}_R(\mathbf{x}_0)$. Therefore the dynamic IVP (4.1.1), (4.1.2) has a unique solution in $\mathcal{B}_R(\mathbf{x}_0)$.

□

We now present an example to illustrate Theorem 4.6.2.

Example 4.6.3 *Consider*

$$R^\kappa := \{(t, p) : t \in [0, 1/2]_{\mathbb{T}}^\kappa, |p| \leq 1\}.$$

Consider the scalar dynamic IVP

$$x^\Delta = f(t, x) = x^2 + t + \sigma(t), \quad \text{for all } t \in [0, 1/2]_{\mathbb{T}}^\kappa; \quad (4.6.10)$$

$$x(0) = 0. \quad (4.6.11)$$

We claim that, for $[e_2(1/2, 0)]^2 < 4$, the above dynamic IVP has a unique solution, x , such that $|x(t)| < \frac{1}{e_2(1/2, 0)}$ for all $t \in [0, 1/2]_{\mathbb{T}}$.

Proof: We show that the given IVP satisfies the conditions of Theorem 4.6.2.

(i) *The function f is right-Hilger-continuous on R^κ :* We note that the composition function $k(t) := (x(t))^2 + t + \sigma(t)$ is rd-continuous for all $t \in [0, 1/2]_{\mathbb{T}}$ and so, our f is right-Hilger-continuous on R^κ .

(ii) *f is Lipschitz continuous on R^κ :* We note that for all $(t, p) \in R^\kappa$ we have

$$\left| \frac{\partial f(t, p)}{\partial p} \right| = |2p| \leq 2.$$

Thus, by Theorem 2.2.2, f satisfies a Lipschitz condition on R^κ with Lipschitz constant $L = 2$.

(iii) *f satisfies (4.6.4):* Note that, for all $t \in [0, 1/2]_{\mathbb{T}}$, we obtain using [19, Example 1.25]

$$\begin{aligned} \int_0^{1/2} |f(s, 0)| \Delta s &= \int_0^{1/2} (s + \sigma(s)) \Delta s \\ &= 1/4, \end{aligned}$$

which satisfies (4.6.4) for $[e_2(1/2, 0)]^2 < 4$. So, by Theorem 4.6.2, the given IVP has a unique solution x such that $|x(t)| < \frac{1}{e_2(1/2, 0)}$ for all $t \in [0, 1/2]_{\mathbb{T}}$.

4.7 Lipschitz continuity of solutions

In this section, we prove results about the smoothness of solutions to the dynamic IVPs (4.1.1), (4.1.2) and (4.1.3), (4.1.4), brought forward under the conditions of Theorem 4.4.1, with respect to their initial state.

Let $\mathbf{x}(t; \mathbf{A})$ denote the solution of (4.1.1), (4.1.2) with $\mathbf{x}_0 = \mathbf{A}$. We show that $\mathbf{x}(t; \mathbf{A})$ is Lipschitz continuous in \mathbf{A} . That is, there exists a function $K(t) > 0$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$ such that for all $\mathbf{A}, \mathbf{B} \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}^n)$ satisfying (4.1.2), we have

$$\|\mathbf{x}(t; \mathbf{A}) - \mathbf{x}(t; \mathbf{B})\| \leq K(t)\|\mathbf{A} - \mathbf{B}\|, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \quad (4.7.1)$$

Theorem 4.7.1 *Let the condition of Theorem 4.4.1 hold. If $\mathbf{x}(t; \mathbf{A})$ is the unique solution to the IVP (4.1.1), (4.1.2), then $\mathbf{x}(t; \mathbf{A})$ is Lipschitz continuous in \mathbf{A} , for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. More explicitly, if there exist \mathbf{A}, \mathbf{B} satisfying (4.1.2) then*

$$\|\mathbf{x}(t; \mathbf{A}) - \mathbf{x}(t; \mathbf{B})\| \leq e_L(t, t_0)\|\mathbf{A} - \mathbf{B}\|, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}. \quad (4.7.2)$$

Proof: Let $\mathbf{x}(t; \mathbf{A}), \mathbf{x}(t; \mathbf{B})$ be solutions to (4.1.1), (4.1.2) corresponding to $\mathbf{x}_0 = \mathbf{A}$ and $\mathbf{x}_0 = \mathbf{B}$, respectively. Then, from Lemma 2.1.3, we obtain for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$,

$$\begin{aligned} \|\mathbf{x}(t; \mathbf{A}) - \mathbf{x}(t; \mathbf{B})\| &\leq \|\mathbf{A} - \mathbf{B}\| + \int_{t_0}^t \|\mathbf{f}(s; \mathbf{x}(s; \mathbf{A})) - \mathbf{f}(s; \mathbf{x}(s; \mathbf{B}))\| \Delta s, \\ &\leq \|\mathbf{A} - \mathbf{B}\| + L \int_{t_0}^t \|\mathbf{x}(s; \mathbf{A}) - \mathbf{x}(s; \mathbf{B})\| \Delta s, \end{aligned} \quad (4.7.3)$$

where we used (4.4.1) above. Let us define, for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^n$,

$$r(t) := \int_{t_0}^t \|\mathbf{x}(s; \mathbf{A}) - \mathbf{x}(s; \mathbf{B})\| \Delta s, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}.$$

Then from inequality (4.7.3), we obtain

$$r^\Delta(t) - Lr(t) \leq \|\mathbf{A} - \mathbf{B}\|, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa. \quad (4.7.4)$$

Since $L > 0$, we have $1 + \mu L > 0$, where μ is the graininess function on $[t_0, t_0 + a]_{\mathbb{T}}$.

Hence $L \in \mathcal{R}^+$, and $e_L(t, t_0) > 0$ (Theorem A.6.4(9)) for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

Simplifying (4.7.4) by taking $e_L^\sigma(t, t_0)$ as the integrating factor, we obtain

$$\frac{r^\Delta(t)e_L(t, t_0) - Le_L(t, t_0)r(t)}{e_L(t, t_0)e_L^\sigma(t, t_0)} \leq \frac{\|\mathbf{A} - \mathbf{B}\|}{e_L^\sigma(t, t_0)}, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa.$$

Applying [19, Theorem 1.20(v), Theorem 2.35], we have, for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^n$,

$$\left[\frac{r(t)}{e_L(t, t_0)} \right]^\Delta \leq \frac{\|\mathbf{A} - \mathbf{B}\|}{e_L^\sigma(t, t_0)}, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^\kappa.$$

An integration from t_0 to t on both sides yields, for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$,

$$\begin{aligned} \frac{r(t)}{e_L(t, t_0)} &\leq \|\mathbf{A} - \mathbf{B}\| \int_{t_0}^t \frac{1}{e_L^\sigma(s, t_0)} \Delta s \\ &= \frac{\|\mathbf{A} - \mathbf{B}\|}{-L} \left[\frac{1}{e_L(t, t_0)} - 1 \right] \Delta s, \\ &= \frac{1 - e_L(t, t_0)}{-Le_L(t, t_0)} \|\mathbf{A} - \mathbf{B}\|, \end{aligned}$$

where we have used Theorem A.6.4(8) in the second last step. Therefore, from (4.7.4), we obtain, for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$,

$$\|\mathbf{x}(t; \mathbf{A}) - \mathbf{x}(t; \mathbf{B})\| - L \left(\frac{e_L(t, t_0) - 1}{L} \right) \|\mathbf{A} - \mathbf{B}\| \leq \|\mathbf{A} - \mathbf{B}\|,$$

which gives (4.7.2). □

Theorem 4.7.1 shows that the change in the solution $\mathbf{x}(t; \mathbf{A})$ of (4.1.1), (4.1.2) with respect to the initial state is bounded by a continuously differentiable function $K(t) := e_L(t, t_0)$. That is, the solution stays between the lines $\pm e_L(t, a)(\mathbf{A} - \mathbf{B})$ for all $t \in [t_0, t_0 + a]$, and so, is stable with respect to the initial state.

The above result partially strengthens the ideas in [54, Theorem 2.6.1], however, it removes the restriction $L\mu < 1$ and adds smoothness to the solution with respect to the initial state.

The next theorem concerns the Lipschitz continuity of solutions to the dynamic IVP (4.1.3), (4.1.4). We show that the unique solution of these IVPs satisfies (4.7.1), with Lipschitz constant $K := e_\beta(t_0 + a, t_0)$ under the β -norm defined in (4.3.3).

Theorem 4.7.2 *Consider the dynamic IVP (4.1.3), (4.1.4) and let the condition of Theorem 4.4.7 hold. If $\mathbf{x}(t; \mathbf{A})$ is the unique solution to (4.1.3), (4.1.4), then $\mathbf{x}(t; \mathbf{A})$ is Lipschitz continuous with respect to \mathbf{A} , under the β -norm, with Lipschitz constant $e_\beta(t_0 + a, t_0)$. That is, for any positive constant β such that $\beta := L\gamma$ for $\gamma \geq 1$, and for all \mathbf{A}, \mathbf{B} satisfying (4.1.4),*

$$\|\mathbf{x}(t; \mathbf{A}) - \mathbf{x}(t; \mathbf{B})\|_\beta \leq e_\beta(t_0 + a, t_0) \|\mathbf{A} - \mathbf{B}\|_\beta, \quad (4.7.5)$$

for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

Proof: Since $\mathbf{x}(t; \mathbf{A}), \mathbf{x}(t; \mathbf{B})$ solve (4.1.3), (4.1.4) for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$, we can write, from Lemma 2.1.2 and (4.3.3),

$$\begin{aligned}
& \|\mathbf{x}(t; \mathbf{A}) - \mathbf{x}(t; \mathbf{B})\|_{\beta} \\
&= \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \frac{\|\mathbf{x}(t; \mathbf{A}) - \mathbf{x}(t; \mathbf{B})\|}{e_{\beta}(t, t_0)} \\
&\leq \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \frac{1}{e_{\beta}(t, t_0)} \left[\int_{t_0}^t \|\mathbf{f}(s, \mathbf{x}^{\sigma}(s; \mathbf{A})) - \mathbf{f}(s, \mathbf{x}^{\sigma}(s; \mathbf{B}))\| \Delta s + \|\mathbf{A} - \mathbf{B}\| \right] \\
&\leq \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \frac{1}{e_{\beta}(t, t_0)} \left[\int_{t_0}^t \frac{L}{1 + \mu(s)\beta} \|\mathbf{x}^{\sigma}(s; \mathbf{A}) - \mathbf{x}^{\sigma}(s; \mathbf{B})\| \Delta s \right] + \|\mathbf{A} - \mathbf{B}\|_{\beta} \\
&= \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \frac{1}{e_{\beta}(t, t_0)} \left[\int_{t_0}^t L e_{\beta}(s, t_0) \frac{\|\mathbf{x}^{\sigma}(s; \mathbf{A}) - \mathbf{x}^{\sigma}(s; \mathbf{B})\|}{e_{\beta}^{\sigma}(s, t_0)} \Delta s \right] + \|\mathbf{A} - \mathbf{B}\|_{\beta},
\end{aligned}$$

where we used the identity (4.4.13) in the last step. Further using (4.3.3) and Theorem A.6.4(7) with $\beta = L\gamma$, the above computations take the form

$$\begin{aligned}
& \|\mathbf{x}(t; \mathbf{A}) - \mathbf{x}(t; \mathbf{B})\|_{\beta} \\
&\leq \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \frac{1}{e_{\beta}(t, t_0)} \left[\int_{t_0}^t L e_{\beta}(s, t_0) \sup_{s \in [t_0, t_0 + a]_{\mathbb{T}}} \frac{\|\mathbf{x}^{\sigma}(s; \mathbf{A}) - \mathbf{x}^{\sigma}(s; \mathbf{B})\|}{e_{\beta}^{\sigma}(s, t_0)} \Delta s \right] + \|\mathbf{A} - \mathbf{B}\|_{\beta} \\
&= \frac{1}{\gamma} \|\mathbf{x}(t; \mathbf{A}) - \mathbf{x}(t; \mathbf{B})\|_{\beta} \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \left[\frac{1}{e_{\beta}(t, t_0)} \int_{t_0}^t e_{\beta}^{\Delta}(s, t_0) \Delta s \right] + \|\mathbf{A} - \mathbf{B}\|_{\beta}, \\
&= \frac{1}{\gamma} \|\mathbf{x}(t; \mathbf{A}) - \mathbf{x}(t; \mathbf{B})\|_{\beta} \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \left[1 - \frac{1}{e_{\beta}(t, t_0)} \right] + \|\mathbf{A} - \mathbf{B}\|_{\beta} \\
&\leq \|\mathbf{x}(t; \mathbf{A}) - \mathbf{x}(t; \mathbf{B})\|_{\beta} \left[1 - \frac{1}{e_{\beta}(t_0 + a, t_0)} \right] + \|\mathbf{A} - \mathbf{B}\|_{\beta},
\end{aligned}$$

where we used $\gamma \geq 1$ in the last step. A rearrangement of the above inequality yields

$$\|\mathbf{x}(t; \mathbf{A}) - \mathbf{x}(t; \mathbf{B})\|_{\beta} < e_{\beta}(t_0 + a, t_0) \|\mathbf{A} - \mathbf{B}\|_{\beta}, \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}.$$

Hence $\mathbf{x}(t; \mathbf{A})$ is Lipschitz continuous in \mathbf{A} in the β -norm, with Lipschitz constant $e_{\beta}(t_0 + a, t_0)$. Thus, a variation in $\mathbf{x}(t; \mathbf{A})$ is bounded by the lines $\pm e_{\beta}(t_0 + a, t_0)$. This completes the proof. □

In our next result, we consider the n -sphere $N_r(\mathbf{A}_0)$ defined in (3.5.1) and the $n + 1$ -sphere $\mathbf{P}_{r,M}(\mathbf{A}_0)$ defined in (3.5.2). We assume that $\mathbf{A} \in N_r(\mathbf{A}_0)$. Then the following theorem guarantees a unique solution to the dynamic IVP (4.1.1), (4.1.2) in $\mathbf{P}_{r,M}(\mathbf{A}_0)$. Moreover, the result also ensures the existence of continuous partial

derivatives of the solution with respect to the initial value $\mathbf{x}_0 = \mathbf{A}$ in the n -sphere $N_r(\mathbf{A}_0)$.

Theorem 4.7.3 *Consider the vector $\mathbf{A}_0 \in \mathbb{R}^n$ and positive constants r, M such that (3.5.1) and (3.5.2) hold. Let $\mathbf{f} : \mathbf{P}_{r,M}(\mathbf{A}_0) \rightarrow \mathbb{R}^n$ be right-Hilger-continuous. If $\frac{\partial \mathbf{f}(t, \mathbf{p})}{\partial \mathbf{x}}$ exist and are continuous for all $(t, \mathbf{p}) \in \mathbf{P}_{r,M}(\mathbf{A}_0)$, then the dynamic IVP (4.1.1), (4.1.2) has a unique solution $\mathbf{x}(t; \mathbf{A})$ for all $(t, \mathbf{x}) \in \mathbf{P}_{r,M}(\mathbf{A}_0)$. Furthermore, the partial derivatives of the solution, $\frac{\partial \mathbf{x}(t; \mathbf{A})}{\partial \mathbf{A}}$, are also continuous in \mathbf{A} for all $\mathbf{A} \in N_r(\mathbf{A}_0)$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.*

Proof: We know from Theorem 4.4.3, that a unique solution to the system (3.1.2), (3.1.3), which is the same as (4.1.1), (4.1.2) exists in $\mathbf{P}_{r,M}(\mathbf{A}_0)$. Let us call this solution $\mathbf{x}(t; \mathbf{A})$. So, we only show that this solution $\mathbf{x}(t; \mathbf{A})$ has continuous partial derivatives with respect to the initial value \mathbf{A} for all $(t, \mathbf{x}) \in \mathbf{P}_{r,M}(\mathbf{A}_0)$. That is, we show that $\frac{\partial \mathbf{x}(t; \mathbf{A})}{\partial \mathbf{A}}$ exists and is continuous in \mathbf{A} for all $\mathbf{A} \in N_r(\mathbf{A}_0)$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$.

We note, from Theorem 4.7.1, that $\mathbf{x}(t; \mathbf{A})$ satisfies a uniform Lipschitz condition in \mathbf{A} . Thus, from [42, p.3], for every $\epsilon > 0$, we can define a $\delta = \delta(\epsilon) := \frac{\epsilon}{e_L(t, t_0)}$ for which

$$\|\mathbf{x}(t; \mathbf{A}) - \mathbf{x}(t; \mathbf{B})\| \leq \epsilon, \quad \text{whenever } \|\mathbf{A} - \mathbf{B}\| \leq \delta.$$

Let $\mathbf{q} := (0, 0, \dots, 0, 1, 0, \dots, 0)^T \in \mathbb{R}^n$ be the k -th unit vector. Then for an arbitrarily small $\delta > 0$ we define $\mathbf{B} = \mathbf{A} + \delta \mathbf{q}$. Using Taylor's theorem [75, p.624], we obtain

$$\begin{aligned} \left| \frac{\partial \mathbf{x}(t; \mathbf{A})}{\partial \mathbf{A}} \right| &= \lim_{\delta \rightarrow 0} \frac{\|\mathbf{x}(t; \mathbf{A} + \delta \mathbf{q}) - \mathbf{x}(t; \mathbf{A})\|}{\delta} \\ &\leq \lim_{\delta \rightarrow 0} \frac{\epsilon}{\delta}, \\ &= e_L(t, t_0), \end{aligned}$$

whenever $\|\mathbf{A} - \mathbf{B}\| \leq \delta$.

Thus $\frac{\partial \mathbf{x}(t; \mathbf{A})}{\partial \mathbf{A}}$ exist and are rd-continuous for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$ for all $\mathbf{A} \in N_r(\mathbf{A}_0)$. This completes the proof. □

4.8 Dynamic equations of higher order

The ideas of Section 4.3 can be extended to higher-order equations. Consider \mathbf{x} to be a continuously differentiable vector function of order n , where $n = 1, 2, \dots$. That is, $\mathbf{x} = (x_1, x_2, \dots, x_n)$. We define the k -th derivative of x as

$$x^{\Delta^k} := [x^{\Delta^{k-1}}]^{\Delta}, \quad \text{for all } k = 1, 2, \dots, n,$$

and define x_1, x_2, \dots, x_n as follows:

$$x_1 := x; \quad x_2 := x^{\Delta}; \quad x_3 := x^{\Delta^2}; \dots; \quad x_{n-1} := x^{\Delta^{n-2}}; \quad x_n := x^{\Delta^{n-1}}. \quad (4.8.1)$$

If we delta-differentiate the above system of equations, we obtain a set of first order dynamic equations

$$\begin{aligned} x_1^{\Delta} &= x_2; \\ x_2^{\Delta} &= x_3; \\ &\vdots \end{aligned} \quad (4.8.2)$$

$$\begin{aligned} x_{n-1}^{\Delta} &= x_n; \\ x_n^{\Delta} &= x^{\Delta^n} = \mathbf{f}(t, x_1, x_2, \dots, x_n) = \mathbf{f}(t, \mathbf{x}). \end{aligned} \quad (4.8.3)$$

Now consider the dynamic initial value problem

$$\mathbf{x}^{\Delta^k} = \mathbf{f}(t, x, x^{\Delta}, x^{\Delta^2}, \dots, x^{\Delta^{k-1}}), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}; \quad (4.8.4)$$

$$x(a) = A_1; x^{\Delta}(a) = A_2; \dots; x^{\Delta^{k-1}}(a) = A_k, \quad (4.8.5)$$

where $A_k \in \mathbb{R}$. Then this system can be written as

$$\mathbf{x}^{\Delta} = \mathbf{f}(t, \mathbf{x}), \quad \text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}; \quad (4.8.6)$$

$$\mathbf{x}(a) = \mathbf{A}, \quad \text{where } \mathbf{A} = (A_1, \dots, A_n), \quad (4.8.7)$$

which is the same as (4.1.1), (4.1.2). Any continuous and n -times delta differentiable function \mathbf{x} satisfying (4.1.1), (4.1.2) will be a solution to the dynamic IVPs (4.8.4), (4.8.5).

Similarly, if we define $\sigma^1(b) := \sigma(b)$; and $\sigma^{k+1}(b) := \sigma(\sigma^k(b))$ for all $k = 1, \dots, n$, then any rd-continuous function x that is n times delta differentiable on a time scale

interval $[a, \sigma^n(b)]_{\mathbb{T}}$ and satisfies the dynamic IVP

$$\mathbf{x}^{\Delta^k} = \mathbf{f}(t, x^\sigma, x^\Delta, x^{\Delta^2}, \dots, x^{\Delta^{k-1}}), \text{ for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}; \quad (4.8.8)$$

$$x(a) = A_1; x^\Delta(a) = A_2; \dots; x^{\Delta^{k-1}}(a) = A_k \quad (4.8.9)$$

will be a solution to this equation.

We now present our results for the higher order dynamic IVPs (4.8.4), (4.8.5) as follows.

Theorem 4.8.1 *Let $\mathbf{f} : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a right-Hilger-continuous function, and $L^* > 0$ be a fixed number. If, for all $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$, \mathbf{f} satisfies*

$$\|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\|^2 \leq L_1[(p_1 - q_1)^2 + \dots + (p_n - q_n)^2], \quad (4.8.10)$$

$$\text{for all } t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa},$$

then the IVP (4.8.4), (4.8.5) has a unique solution.

Proof: We show that \mathbf{f} satisfies the Lipschitz condition (4.4.1) and Theorem 4.4.1 applies.

Consider $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$, then for all $t \in [t_0, t_0 + a]_{\mathbb{T}}^{\kappa}$, we can write from (4.8.4)

$$\begin{aligned} & \|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\| \\ &= \|\mathbf{p}^\Delta - \mathbf{q}^\Delta\| \\ &= \|(p_1^\Delta, p_2^\Delta, \dots, p_n^\Delta) - (q_1^\Delta, q_2^\Delta, \dots, q_n^\Delta)\| \\ &= \|(p_2, p_3, \dots, \mathbf{f}(t, p_1, p_2, \dots, p_n)) - (q_2, q_3, \dots, \mathbf{f}(t, q_1, q_2, \dots, q_n))\|, \\ &= [(p_2 - q_2)^2 + (p_3 - q_3)^2 + \dots + \|\mathbf{f}(t, p_1, p_2, \dots, p_n) - \mathbf{f}(t, q_1, q_2, \dots, q_n)\|^2]^{1/2} \\ &\leq [(p_1 - q_1)^2 + (p_2 - q_2)^2 + \dots + (p_n - q_n)^2 + \|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\|^2]^{1/2} \\ &\leq [(p_1 - q_1)^2 + (p_2 - q_2)^2 + \dots + (p_n - q_n)^2 \\ &\quad + L_1[(p_1 - q_1)^2 + (p_2 - q_2)^2 + \dots + (p_n - q_n)^2]^{1/2}]^{1/2} \\ &= [(1 + L_1)((p_1 - q_1)^2 + (p_2 - q_2)^2 + \dots + (p_n - q_n)^2)]^{1/2} \\ &= [1 + L_1]^{1/2} \|\mathbf{p} - \mathbf{q}\|. \end{aligned}$$

Hence, \mathbf{f} satisfies (4.4.1) with Lipschitz constant $L := [1 + L_1]^{1/2}$. Thus, by Theorem 4.4.1, the IVP (4.8.6), (4.8.7) has a unique solution. In other words, the IVP (4.8.4), (4.8.5) has a unique solution.

The next theorem concerns the uniqueness of solution to the dynamic IVP (4.8.8), (4.8.9).

Theorem 4.8.2 *Let $\mathbf{f} : [t_0, t_0 + a]_{\mathbb{T}}^{\kappa} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a right-Hilger-continuous function, and $L_2 > 0$ be a fixed number. If there exists $\beta = \gamma L_2$, where $\gamma \geq 1$, such that, for all $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$, \mathbf{f} satisfies*

$$[\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})]^2 \leq \frac{L_2}{(1 + \mu(t)\beta)^2} [(p_1 - q_1)^2 + \cdots + (p_n - q_n)^2], \quad (4.8.11)$$

for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$,

then the IVP (4.8.8), (4.8.9) has a unique solution.

Proof: We show that \mathbf{f} satisfies (4.4.10) and Theorem 4.4.7 applies. In this case, we define the components x_1, \dots, x_n of \mathbf{x} as

$$x_1 := x^\sigma; \quad x_2 := x^\Delta; \quad x_3 := x^{\Delta^2}; \cdots; \quad x_{n-1} := x^{\Delta^{n-2}}; \quad x_n := x^{\Delta^{n-1}}, \quad (4.8.12)$$

and delta-differentiate the above equations taking $x^{\sigma^\Delta} := (x \circ \sigma)^\Delta$ (see Theorem A.3.11). Thus, we obtain

$$\begin{aligned} x_1^\Delta &= x^{\sigma^\Delta}; \\ x_2^\Delta &= x_3; \\ &\vdots \end{aligned} \quad (4.8.13)$$

$$\begin{aligned} x_{n-1}^\Delta &= x_n; \\ x_n^\Delta &= x^{\Delta^n} = \mathbf{f}(t, x_1, x_2, \dots, x_n) = \mathbf{f}(t, \mathbf{x}). \end{aligned} \quad (4.8.14)$$

Then for any $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ having components as in (4.8.12), we have from (4.8.14),

$$\begin{aligned} \|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\| &= \|(p_1^{\sigma^\Delta}, p_3, \dots, \mathbf{f}(t, p_1, p_2, \dots, p_n)) - (q_1^{\sigma^\Delta}, q_3, \dots, \mathbf{f}(t, q_1, q_2, \dots, q_n))\|, \\ &= [(p_1^{\sigma^\Delta} - q_1^{\sigma^\Delta})^2 + (p_3 - q_3)^2 + \cdots + (\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q}))^2]^{1/2} \\ &\leq [(p_1^{\sigma^\Delta} - q_1^{\sigma^\Delta})^2 + (p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2 + \cdots + (p_n - q_n)^2 \\ &\quad + (\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q}))^2]^{1/2} \\ &\leq [(p_1^{\sigma^\Delta} - q_1^{\sigma^\Delta})^2 + (p_1 - q_1)^2 + (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2 \\ &\quad + \frac{L_2}{(1 + \mu(t)\beta)^2} [(p_1 - q_1)^2 + (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2]]^{1/2}, \end{aligned}$$

where we used (4.8.11) above. Thus, our further computations take the form

$$\begin{aligned}
\|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\| &= [(p_1^{\sigma^\Delta} - q_1^{\sigma^\Delta})^2 + \\
&\quad \left(1 + \frac{L_2}{(1 + \mu(t)\beta)^2}\right) [(p_1 - q_1)^2 + (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2]]^{1/2} \\
&\leq \left| (p_1^{\sigma^\Delta} - q_1^{\sigma^\Delta}) + \right. \\
&\quad \left. \frac{((1 + \mu(t)\beta)^2 + L_2)^{1/2}}{1 + \mu(t)\beta} [(p_1 - q_1)^2 + (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2]^{1/2} \right| \\
&\leq |p_1^{\sigma^\Delta} - q_1^{\sigma^\Delta}| + \\
&\quad \left| \frac{((1 + \mu(t)\beta)^2 + L_2)^{1/2}}{1 + \mu(t)\beta} \right| [(p_1 - q_1)^2 + (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2]^{1/2} \\
&= |p_1^{\sigma^\Delta} - q_1^{\sigma^\Delta}| + \left| \frac{((1 + \mu(t)\beta)^2 + L_2)^{1/2}}{1 + \mu(t)\beta} \right| \|\mathbf{p} - \mathbf{q}\|
\end{aligned}$$

Let

$$\alpha := |p_1^{\sigma^\Delta} - q_1^{\sigma^\Delta}|.$$

Then the above inequality becomes

$$\begin{aligned}
\|\mathbf{f}(t, \mathbf{p}) - \mathbf{f}(t, \mathbf{q})\| &\leq \alpha + \left| \frac{((1 + \mu(t)\beta)^2 + L_2)^{1/2}}{1 + \mu(t)\beta} \right| \|\mathbf{p} - \mathbf{q}\| \\
&\leq \alpha \|\mathbf{p} - \mathbf{q}\| + \left| \frac{((1 + \mu(t)\beta)^2 + L_2)^{1/2}}{1 + \mu(t)\beta} \right| \|\mathbf{p} - \mathbf{q}\| \\
&= \frac{\alpha |1 + \mu(t)\beta| + |((1 + \mu(t)\beta)^2 + L_2)^{1/2}|}{|1 + \mu(t)\beta|} \|\mathbf{p} - \mathbf{q}\|.
\end{aligned}$$

Hence, \mathbf{f} satisfies (4.4.10) with

$$L := \alpha |1 + \mu(t)\beta| + |((1 + \mu(t)\beta)^2 + L_2)^{1/2}|,$$

where $\alpha = |p_1^{\sigma^\Delta} - q_1^{\sigma^\Delta}|$. Thus, by Theorem 4.3.6, the IVP (4.8.8), (4.8.9) has a unique solution. □

In this chapter, we presented results regarding existence of solutions to the systems (2.1.5), (2.1.6) and (2.1.7), (2.1.8) and also to the scalar IVP (2.1.9), (2.1.10) using Banach's fixed point theory and its applications. In the next chapter, we replace the above IVPs by another scalar IVP involving nabla equations and explore existence of solutions within a defined location, using the method of lower and upper solutions, employing Schauder's approach from ordinary differential equations.

Chapter 5

Existence results using lower and upper solutions

5.1 Introduction

So far we have examined the existence and uniqueness of solutions to first order non-linear dynamic initial value problems involving delta equations. We used methods from classical analysis, such as successive approximations, and employed ideas from fixed point theory, such as Banach's fixed point theorem. In this chapter we consider the nabla equation

$$x^\nabla = f(t, x), \quad \text{for all } t \in [0, a]_{\kappa, \mathbb{T}},$$

subject to an initial condition and examine: the existence and uniqueness of solutions to the above initial value problem employing Schauder's fixed point theorem; restriction of solutions within known regions defined by $[0, a]_{\mathbb{T}}$, an upper solution, u , and a lower solution, l , on $[0, a]_{\mathbb{T}}$. We also establish successive approximations of solutions via lower and upper solutions to an initial value problem involving the above nabla equation.

It had been shown in [12] and [11] that the existence results involving lower and upper solutions for boundary value problems on time scales can be proved with less restrictions using nabla equations than using delta equations. By a similar argument, we prove our results using nabla equations to allow the solution to assume maximal values at the right end point of a given interval of existence, $[l, u]$, using the

maximum principle. In this way, our existence results are different both in context and methodology from results proved in Chapter 3 and Chapter 4 for first order dynamic IVPs.

Our method of employing lower and upper solutions using the maximum principle to obtain existence and uniqueness of solutions to the IVP (5.1.1), (5.1.2) also make our results in this chapter different in context and methodology from those proved in [54, Theorem 4.1.2].

The results follow some notions of La Salle [55] extended to the time scale setting. In this way, our results exhibit a broader span of modelling a system described as a first order initial value problem, no matter if the system has a discrete or a continuous domain or a hybrid of both.

5.1.1 The main objective

We consider a left–Hilger–continuous non–linear function (see Definition A.2.6) $f : [0, a]_{\kappa, \mathbb{T}} \times [l, u] \subset \mathbb{R}^2 \rightarrow \mathbb{R}$, where l, u are continuous on $[0, a]_{\mathbb{T}} = [0, a] \cap \mathbb{T}$ for an arbitrary time scale \mathbb{T} .

Consider the scalar initial value problem

$$x^\nabla = f(t, x), \quad \text{for all } t \in [0, a]_{\kappa, \mathbb{T}}; \quad (5.1.1)$$

$$x(0) = 0. \quad (5.1.2)$$

Here x^∇ is the ‘nabla’ derivative of x introduced in [13, p.77].

The main aim of this chapter is to answer the following questions:

1. Under what conditions does the dynamic IVP (5.1.1), (5.1.2) have a solution?
2. Under what conditions does (5.1.1), (5.1.2) have solutions lying within the interval $[l, u]$, where l, u are known to be (respectively) the lower and upper solutions to the IVP (5.1.1), (5.1.2)?
3. Under what conditions do l and u approximate solutions to (5.1.1), (5.1.2) with an error estimate on the i –th approximation?

Our results show that given u, l the upper and lower solutions to the IVP (5.1.1), (5.1.2) the IVP has at least one solution which is bounded above by u and is bounded below by l . We apply our ideas to establish non–negative solutions to (5.1.1), (5.1.2).

5.1.2 Methodology and organisation

Our results in this chapter use the method of lower and upper solutions. The motivation for using upper and lower solutions in our results was developed due to the wide use of this method to establish existence results for a variety of first and second order initial and boundary value problems, see [3], [7], [8], [11], [12], [13], [19], [22], [86] and the references therein. We use this method to determine: existence of solutions to the IVP (5.1.1), (5.1.2); and establishing successive approximations converging to a solution of the above IVP.

This chapter is organised in the following manner. In Section 5.2, we define lower and upper solutions to the dynamic IVP (5.1.1), (5.1.2) and establish the existence and uniqueness of solutions to (5.1.1), (5.1.2) within the lower and upper solutions to (5.1.1), (5.1.2).

In Section 5.3, we show that $l(t), u(t)$ are zero approximations to solutions of (5.1.1), (5.1.2) established in Section 5.2, for all $t \in [0, a]_{\mathbb{T}}$. We also prove that an upper bound exists on the error of the i -th approximation on $[0, a]_{\mathbb{T}}$ which approaches to zero for a unique solution.

5.2 Existence results

We prove that the dynamic IVP (5.1.1), (5.1.2) has a solution on $[0, a]_{\mathbb{T}}$ that lies within the interval $[l, u]$, where $l(t), u(t)$ act respectively as lower and upper solutions to (5.1.1), (5.1.2) for all $t \in [0, a]_{\mathbb{T}}$, using Schauder's fixed point theorem.

We begin with some preliminary ideas that will be used to prove the main results.

Definition 5.2.1 Lower and upper solutions

Let l, u be nabla differentiable functions on $[0, a]_{\kappa, \mathbb{T}}$. We call l a lower solution to (5.1.1), (5.1.2) on $[0, a]_{\mathbb{T}}$ if

$$l^{\nabla}(t) \leq f(t, l(t)), \quad \text{for all } t \in [0, a]_{\kappa, \mathbb{T}}; \quad (5.2.1)$$

$$l(0) = 0. \quad (5.2.2)$$

Similarly, we call u an upper solution to (5.1.1), (5.1.2) on $[0, a]_{\mathbb{T}}$ if

$$u^{\nabla}(t) \geq f(t, u(t)), \quad \text{for all } t \in [0, a]_{\kappa, \mathbb{T}}; \quad (5.2.3)$$

$$u(0) = 0. \quad (5.2.4)$$

Definition 5.2.2 Let $D \subseteq \mathbb{R}$. A solution of (5.1.1), (5.1.2) is a function $x : [0, a]_{\kappa, \mathbb{T}} \rightarrow \mathbb{R}$ such that: the points $(t, x(t)) \in [0, a]_{\kappa, \mathbb{T}} \times D$; $x(t)$ is nabla differentiable with $\mathbf{x}^\nabla(t) = f(t, x(t))$ for each $t \in [0, a]_{\kappa, \mathbb{T}}$; and $x(0) = 0$.

□

All ld-continuous functions are nabla integrable [19, Theorem 8.45]. The following lemma establishes equivalence of the IVP (5.1.1), (5.1.2) as nabla integral equations. The result is nabla-equivalent of Lemma 2.1.3 for the ‘delta’ case. Therefore, the proof is omitted.

Lemma 5.2.3 Let $D \subseteq \mathbb{R}$. Consider the dynamic IVP (5.1.1), (5.1.2). Let $f : [0, a]_{\kappa, \mathbb{T}} \times D \rightarrow \mathbb{R}$ be a left-Hilger-continuous function. Then a function x solves (5.1.1), (5.1.2) if and only if it satisfies the nabla integral equation

$$x(t) = \int_0^t f(s, x(s)) \nabla s, \quad \text{for all } t \in [0, a]_{\mathbb{T}}. \quad (5.2.5)$$

□

The following definition and the next two theorems are the keys to our proof for the existence of solutions to (5.1.1), (5.1.2).

Definition 5.2.4 [88, p.54] Let U, V be Banach spaces and $F : A \subseteq U \rightarrow V$. We say F is compact on A if:

- F is continuous on A ;
- for every bounded set B of A , $F(B)$ is relatively compact in V .

□

The next theorem is another form of the Arzela–Ascoli theorem [64, Theorem 1.3] stated in Chapter 3, see Theorem 3.6.4. This form is more suitable for our results in this chapter.

Theorem 5.2.5 Arzela–Ascoli theorem on \mathbb{T}

Let $D \subseteq C([a, b]_{\mathbb{T}}; \mathbb{R})$. Then D is relatively compact if and only if it is bounded and equicontinuous.

Theorem 5.2.6 Schauder's Fixed Point Theorem

Let X be a normed linear space and D be a closed, bounded and convex subset of X . If $F : D \rightarrow D$ is a compact map then F has at least one fixed point.

□

Define an infinite strip

$$S_\kappa := \{(t, p) : t \in [0, a]_{\kappa, \mathbb{T}}, \text{ and } -\infty < p < \infty\}.$$

Let $g : S_\kappa \rightarrow \mathbb{R}$ be a left-Hilger-continuous function. Our next theorem concerns the existence of solutions to the initial value problem

$$x^\nabla = g(t, x), \quad \text{for all } t \in [0, a]_{\kappa, \mathbb{T}}; \quad (5.2.6)$$

$$x(0) = 0 \quad (5.2.7)$$

in S_κ . We prove this result by using Schauder's Theorem.

Theorem 5.2.7 Consider the initial value problem (5.2.6), (5.2.7) with g left-Hilger-continuous on S_κ . If g is uniformly bounded on S_κ then (5.2.6), (5.2.7) has at least one solution, x , such that the point $(t, x(t))$ lies in the infinite strip

$$S := \{(t, p) : t \in [0, a]_{\mathbb{T}}, \text{ and } -\infty < p < \infty\}.$$

Proof: From Lemma 2.1.3, a solution of (5.2.6), (5.2.7) is given by

$$x(t) := \int_0^t g(s, x(s)) \nabla s, \quad \text{for all } t \in [0, a]_{\mathbb{T}}. \quad (5.2.8)$$

Since g is uniformly bounded on S_κ , there exists $M > 0$ such that

$$|g(t, p)| \leq M, \quad \text{for all } (t, p) \in S_\kappa. \quad (5.2.9)$$

Define $K := Ma$ and consider the Banach space $(C([0, a]_{\mathbb{T}}; \mathbb{R}), |\cdot|_0)$ [82, Lemma 3.3].

Let $D \subset C([0, a]_{\mathbb{T}}; \mathbb{R})$ defined by

$$D := \{x \in C([0, a]_{\mathbb{T}}; \mathbb{R}); |x|_0 \leq K\}.$$

Then D is closed, bounded and convex. We show that a compact map $F : D \rightarrow D$ exists and Schauder's Theorem applies.

Define

$$[Fx](t) := \int_0^t g(s, x(s)) \nabla s, \quad \text{for all } t \in [0, a]_{\mathbb{T}}. \quad (5.2.10)$$

Note that F is well defined on $C([0, a]_{\mathbb{T}}; \mathbb{R})$ as g is left-Hilger-continuous on S_κ .

We show that $F : D \rightarrow D$ is a compact map. For this, we show that the following properties hold for F :

(i) F is continuous on D ;

(ii) for every bounded subset B of D , $F(B)$ is relatively compact in $C([0, a]_{\mathbb{T}}; \mathbb{R})$,

and verify Definition 5.2.4.

To show that F is continuous on D , we define

$$B_K(0) := \{p \in \mathbb{R} : |p| \leq K\}.$$

Note that $B_K(0)$ is closed and bounded and hence compact in \mathbb{R} . Therefore, g is bounded and uniformly left-Hilger-continuous on $[0, a]_{\kappa, \mathbb{T}} \times B_K(0)$. Thus, for a given $\epsilon_1 > 0$ there exists a $\delta_1 = \delta_1(\epsilon_1)$ such that for $(t, x_1), (t, x_2) \in [0, a]_{\kappa, \mathbb{T}} \times B_K(0)$, we have

$$|g(t, x_1) - g(t, x_2)| < \epsilon_1 \quad \text{whenever } |x_1 - x_2| < \delta_1. \quad (5.2.11)$$

Let x_i be a convergent sequence in D with $x_i \rightarrow x$ for all i . Then for every $\delta_1 > 0$ there exists $N > 0$ such that

$$|x_i - x| < \delta_1, \quad \text{for all } i \geq N.$$

We show that the sequence $F_i := Fx_i$ is uniformly convergent in \mathbb{R} . Let $\epsilon_0 := \epsilon_1 a$.

We note that

$$\begin{aligned} |Fx_i - Fx|_0 &= \sup_{t \in [0, a]_{\mathbb{T}}} |Fx_i(t) - Fx(t)| \\ &\leq \sup_{t \in [0, a]_{\mathbb{T}}} \left| \int_0^t (g(s, x_i(s)) - g(s, x(s))) \nabla s \right| \\ &\leq \sup_{t \in [0, a]_{\mathbb{T}}} \left| \int_0^t |g(s, x_i(s)) - g(s, x(s))| \nabla s \right| \\ &< \epsilon_1 a \quad \text{whenever } |x_i - x| < \delta_1 \\ &= \epsilon_0, \end{aligned}$$

for all $i \geq N$. Thus F_i are uniformly convergent on D and hence are uniformly continuous on D .

We show that $F : D \rightarrow D$: Note that for all $x \in D$, we have

$$\begin{aligned}
 |Fx|_0 &:= \sup_{t \in [0, a]_{\mathbb{T}}} |Fx(t)| \\
 &\leq \sup_{t \in [0, a]_{\mathbb{T}}} \int_0^t |g(s, x(s))| \nabla s \\
 &\leq Ma \\
 &= K.
 \end{aligned} \tag{5.2.12}$$

Thus, F is in D .

Next, we show that for every bounded subset B of D , $F(B)$ is relatively compact in $C[0, a]_{\mathbb{T}}$ using the Arzela–Ascoli Theorem.

Let B be an arbitrary bounded subset of D . Assume $x \in B$. Then we note from (5.2.12) that we have $|Fx|_0 \leq K$ for all $t \in [0, a]_{\mathbb{T}}$. Thus F is uniformly bounded on B .

We also note that for any given $\epsilon > 0$ we can define $\delta := \frac{\epsilon}{M}$ and for $t_1, t_2 \in [0, a]_{\mathbb{T}}$, we obtain

$$\begin{aligned}
 |[Fx](t_1) - [Fx](t_2)| &= \left| \int_{t_1}^{t_2} g(s, x(s)) \nabla s \right| \\
 &\leq \left| \int_{t_1}^{t_2} |g(s, x(s))| \nabla s \right| \\
 &\leq M |t_1 - t_2| \\
 &< \epsilon
 \end{aligned}$$

whenever $|t_1 - t_2| < \delta$. Hence, F is equicontinuous. By the Arzela–Ascoli Theorem, $F(B)$ is relatively compact in $C([a, b]_{\mathbb{T}}; \mathbb{R})$.

From (i) and (ii) above, we note that $F : D \rightarrow D$ is a compact map. We also note that F satisfies the conditions of Schauder’s Theorem and, so, has at least one fixed point in D given by (5.2.8). Hence, (5.2.6), (5.2.7) has at least one solution, x , such the point $(t, x(t)) \in S$.

□

The above result ensures existence of a solution to (5.2.6), (5.2.7) when the function g is bounded in an infinite domain S_κ and considers this as a sufficient condition for

the existence of a solution to the above IVP in the infinite domain S_κ . However, the result does not ensure the existence if the domain is restricted.

In the next result, we strengthen the above condition by restricting the solution to (5.2.6), (5.2.7) within a lower and an upper solution to (5.1.1), (5.1.2). Hence we prove the existence of a solution to (5.1.1), (5.1.2) within the region

$$R_\kappa := \{(t, p) : t \in [0, a]_{\kappa, \mathbb{T}}, \text{ and } l(t) \leq p \leq u(t)\},$$

where l, u are, respectively, lower and upper solutions to (5.1.1), (5.1.2). To prove this, we define a modified function g in terms of f in (5.1.1) and prove that g is uniformly bounded and use Theorem 5.2.7. We also prove that the solution, x , to the IVP (5.2.6), (5.2.7) satisfies $l(t) \leq x(t) \leq u(t)$ for all $t \in [0, a]_{\mathbb{T}}$, so that x must also be a solution to the original unmodified problem (5.1.1), (5.1.2).

Theorem 5.2.8 *Let $f : R_\kappa \rightarrow \mathbb{R}$ be a left-Hilger-continuous function. If l, u are, respectively, lower and upper solutions to (5.1.1), (5.1.2), then the IVP (5.1.1), (5.1.2) has at least one solution, x , such that $l(t) \leq x(t) \leq u(t)$ for all $t \in [0, a]_{\mathbb{T}}$.*

Proof: Consider the IVP (5.2.6), (5.2.7), where $g(t, p)$ is defined on S_κ such that for all $t \in [0, a]_{\kappa, \mathbb{T}}$,

$$g(t, p) := \begin{cases} f(t, l(t)) + \frac{l(t) - p}{1 + (l(t) - p)^2}, & \text{when } p < l(t); \\ f(t, p), & \text{when } l(t) \leq p \leq u(t); \\ f(t, u(t)) - \frac{p - u(t)}{1 + (p - u(t))^2}, & \text{when } p > u(t). \end{cases} \quad (5.2.13)$$

We first show that g is left-Hilger-continuous and uniformly bounded on S_κ and Theorem 5.2.7 applies.

Note that f is left-Hilger-continuous on the compact region R_κ and so it is bounded on R_κ . Thus, there exists $M_1 > 0$ such that $|f(t, p)| \leq M_1$ for all $(t, p) \in R_\kappa$. We also note that for $l(t) > p \in \mathbb{R}$, we have

$$\left| \frac{l(t) - p}{1 + (l(t) - p)^2} \right| < 1, \quad \text{for all } t \in [0, a]_{\mathbb{T}},$$

and so

$$f(t, l) + \left| \frac{l(t) - p}{1 + (l(t) - p)^2} \right| < 1 + M_1, \quad \text{for all } t \in [0, a]_{\kappa, \mathbb{T}}.$$

Define $M := 1 + M_1$. Then from (5.2.13), we obtain

$$|g(t, p)| \leq M, \quad \text{for all } (t, p) \in S_\kappa. \quad (5.2.14)$$

Hence g is uniformly bounded on S_κ . In addition, the left-Hilger-continuity of f on R_κ and the ld-continuity of l, u, p on $[0, a]_\mathbb{T}$ shows that the right hand side of (5.2.13) is left-Hilger-continuous on S_κ and, so, we have g left-Hilger-continuous on S_κ . By Theorem 5.2.7, the modified IVP (5.2.6), (5.2.7) has a solution, x , such that the graph $(t, x(t)) \in S$ for all $t \in [0, a]_\mathbb{T}$.

Next, we prove that $l(t) \leq x(t) \leq u(t)$ for all $t \in [0, a]_\mathbb{T}$. We split the inequality $l(t) \leq x(t) \leq u(t)$ into two parts and first show that

$$l(t) \leq x(t), \quad \text{for all } t \in [0, a]_\mathbb{T}, \quad (5.2.15)$$

using the contradiction method.

Let $r(t) := l(t) - x(t)$ for all $t \in [0, a]_\mathbb{T}$. Assume there exists a point $t_1 \in [0, a]_\mathbb{T}$ such that $l(t_1) > x(t_1)$. Note that $t_1 \neq 0$ as $x(0) = 0 = l(0)$ from (5.1.2) and (5.2.2). Without loss of generality, we may assume that

$$r(t_1) = \max_{t \in [0, a]_\mathbb{T}} r(t) > 0. \quad (5.2.16)$$

Thus, $r(t)$ is non-decreasing at $t = t_1$ and, so, $r^\nabla(t_1) \geq 0$.

On the other hand, since $x(t_1) < l(t_1)$ we note that using (5.2.6), (5.2.13) and (5.2.1), we obtain

$$\begin{aligned} 0 \leq r^\nabla(t_1) &= l^\nabla(t_1) - x^\nabla(t_1) \\ &= l^\nabla(t_1) - g(t_1, x(t_1)) \\ &= l^\nabla(t_1) - f(t_1, l(t_1)) - \frac{l(t_1) - x(t_1)}{1 + (l(t_1) - x(t_1))^2} \\ &< l^\nabla(t_1) - f(t_1, l(t_1)) \\ &\leq 0, \end{aligned}$$

which is a contradiction. Hence $l(t) \leq x(t)$ for all $t \in [0, a]_\mathbb{T}$.

It is very similar to show that $u(t) \geq x(t)$ for all $t \in [0, a]_\mathbb{T}$ as in the above case. We omit the details.

Thus, we have $l(t) \leq x(t) \leq u(t)$ for all $t \in [0, a]_{\mathbb{T}}$. Hence, from (5.2.13), $x(t)$ is a solution to (5.1.1), (5.1.2) for all $t \in [0, a]_{\mathbb{T}}$. This completes the proof.

□

The following example illustrates the above theorem.

Example 5.2.9 Consider the Riccati initial value problem

$$x^\nabla(t) = f(t, x) := x^2 - t, \quad \text{for all } t \in [0, 1]_{\kappa, \mathbb{T}}; \quad (5.2.17)$$

$$x(0) = 0. \quad (5.2.18)$$

We claim that there exists at least one solution, x , to the above IVP such that $-t \leq x(t) \leq t$ for all $t \in [0, 1]_{\mathbb{T}}$.

Proof: We note that the right hand side of (5.2.17) is a composition of a continuous function t and a continuous function x^2 and hence, is continuous on $[0, 1]_{\mathbb{T}} \times \mathbb{R}$. So our f is left-Hilger-continuous on $[0, 1]_{\kappa, \mathbb{T}} \times \mathbb{R}$.

Let us define

$$l(t) := -t, \quad \text{for all } t \in [0, 1]_{\mathbb{T}}.$$

Then we note that $l(0) = 0$ and $l^\nabla(t) = -1$ for all $t \in [0, 1]_{\mathbb{T}}$. We further note that for all $t \in [0, \rho(1)]_{\mathbb{T}}$, we have

$$\begin{aligned} f(t, l(t)) &= t^2 - t \\ &\geq -1 \\ &= l^\nabla(t). \end{aligned}$$

Thus, our l satisfies (5.2.1), (5.2.2) and is a lower solution to (5.2.17), (5.2.18).

In a similar way, the function $u(t) := t$ is an upper solution to (5.2.17), (5.2.18) for all $t \in [0, 1]_{\mathbb{T}}$.

By Theorem 5.2.8, there is at least one solution, x , to (5.2.17), (5.2.18) such that $-t \leq x(t) \leq t$ for all $t \in [0, 1]_{\mathbb{T}}$.

□

Our next result gives a sufficient condition for uniqueness of solution to (5.1.1), (5.1.2). We show that the solution, x , of the above IVP established in Theorem 5.2.8 is the only solution satisfying $l(t) \leq x(t) \leq u(t)$ for all $t \in [0, a]_{\mathbb{T}}$.

Theorem 5.2.10 *Let f be left-Hilger-continuous on R_κ . Assume l, u are, respectively, lower and upper solutions of (5.1.1), (5.1.2). If there exists $L > 0$ such that f satisfies*

$$|f(t, p) - f(t, q)| \leq L|p - q|, \quad \text{for all } (t, p), (t, q) \in R_\kappa, \quad (5.2.19)$$

then the solution x of (5.1.1), (5.1.2) brought forward under the conditions of Theorem 5.2.8 is the unique solution satisfying $l(t) \leq x(t) \leq u(t)$ for all $t \in [0, a]_{\mathbb{T}}$.

Proof: Let x, y be two solutions to (5.1.1), (5.1.2). Then, using (5.2.5), we obtain for all $t \in [0, a]_{\mathbb{T}}$,

$$\begin{aligned} |x(t) - y(t)| &\leq \int_0^t |f(s, x(s)) - f(s, y(s))| \nabla s \\ &\leq L \int_0^t |x(s) - y(s)| \nabla s, \end{aligned} \quad (5.2.20)$$

where we employed (5.2.19) in the last step.

Define

$$r(t) := |x(t) - y(t)|, \quad \text{for all } t \in [0, a]_{\mathbb{T}}.$$

Note that, $L > 0$ and so $L \in \mathcal{L}^+$ [12, p.225]. Applying Gronwall's inequality concerning nabla derivatives [12, Theorem 2.7] (taking $f(t) = 0$ and $p(t) = L$) to (5.2.20), we obtain

$$r(t) \leq 0, \quad \text{for all } t \in [0, a]_{\mathbb{T}}.$$

But $r(t) = |x(t) - y(t)|$ and so, is non-negative for all $t \in [0, a]_{\mathbb{T}}$. Thus, $x(t) = y(t)$ for all $t \in [0, a]_{\mathbb{T}}$.

□

The next theorem is nabla equivalent of Theorem 2.2.2 for a scalar function f and provides a sufficient condition for f to satisfy (5.2.19). The proof is, therefore, omitted.

Theorem 5.2.11 *Let $b > 0$. Consider a function f defined on a rectangle of the type*

$$R_\kappa := \{(t, p) \in \mathbb{T}_\kappa \times \mathbb{R} : t \in [0, a]_{\kappa, \mathbb{T}}, |p| \leq b\}, \quad (5.2.21)$$

or on an infinite strip of the type

$$S_\kappa := \{(t, p) \in \mathbb{T}_\kappa \times \mathbb{R} : t \in [0, a]_{\kappa, \mathbb{T}}, |p| < \infty\}. \quad (5.2.22)$$

If $\frac{\partial f(t, p)}{\partial p_i}$ exists for all $i = 1, 2, \dots$, and is continuous on R_κ (or S_κ), and there is a constant $K > 0$ such that for all $(t, p) \in R_\kappa$ (or S_κ), we have

$$\left\| \frac{\partial f(t, p)}{\partial p_i} \right\| \leq K, \quad \text{for all } i = 1, 2, \dots, \quad (5.2.23)$$

then f satisfies a Lipschitz condition on R_κ (or S_κ) with Lipschitz constant $K = L$.

□

The following example illustrates Theorem 5.2.10 using Theorem 5.2.11.

Example 5.2.12 Consider (5.2.17), (5.2.18). We claim that x is the unique solution of (5.2.17), (5.2.18) such that $-t \leq x(t) \leq t$ for all $t \in [0, 1]_{\mathbb{T}}$.

Proof. We note from (5.2.17) that $f(t, p) = p^2 - t$ for all $t \in [0, 1]_{\kappa, \mathbb{T}}$. Thus, for all $t \in [0, 1]_{\kappa, \mathbb{T}}$, we can write

$$\left| \frac{\partial f}{\partial p} \right| = |2p| \leq 2.$$

Thus, f has bounded partial derivatives in $[0, 1]_{\kappa, \mathbb{T}} \times [-t, t]$ and, by Theorem 5.2.11, satisfies (5.2.19) with $L = 2$. Therefore, the solution x of (5.2.17), (5.2.18) established in Example 5.2.9 is unique by Theorem 5.2.10.

□

The next corollary establishes existence of a unique, non-negative and bounded solution of the IVP (5.1.1), (5.1.2) on $[0, a]_{\mathbb{T}}$.

Corollary 5.2.13 Let $f : R_\kappa \rightarrow \mathbb{R}$ be a left-Hilger-continuous function satisfying (5.2.19). Let l, u be lower and upper solutions to (5.1.1), (5.1.2). If $l(t) = 0$ for all $t \in [0, a]_{\mathbb{T}}$, then the IVP (5.1.1), (5.1.2) has a unique, bounded and non-negative solution, $x(t)$, for all $t \in [0, a]_{\mathbb{T}}$.

Proof: The proof follows from Theorem 5.2.10, as $0 \leq x(t) \leq u(t)$ for all $t \in [0, a]_{\mathbb{T}}$.

The following example illustrates the above corollary.

Example 5.2.14 Consider the dynamic initial value problem

$$x^\nabla(t) = f(t, x) := \rho(t) + x^3, \quad \text{for all } t \in [0, 1]_{\kappa, \mathbb{T}}; \quad (5.2.24)$$

$$x(0) = 0. \quad (5.2.25)$$

We claim that the above IVP has a unique solution x such that $0 \leq x(t) \leq 1$ for all $t \in [0, 1]_{\mathbb{T}}$.

Proof: Note that $f(t, p) = \rho(t) + p^3$ for all $(t, p) \in [0, 1]_{\kappa, \mathbb{T}} \times \mathbb{R}$. Since $\rho(t)$ and p^3 are everywhere ld-continuous functions and so is their composition, our f is left-Hilger-continuous on $[0, 1]_{\kappa, \mathbb{T}} \times \mathbb{R}$. We define

$$l(t) := 0, \quad \text{and} \quad u(t) := t^2, \quad \text{for all } t \in [0, 1]_{\mathbb{T}}.$$

Then we note that $l(t) \leq u(t)$ for all $t \in [0, a]_{\mathbb{T}}$ with $l(0) = 0 = u(0)$.

It is evident that l satisfies (5.2.1) and so, is a lower solution to (5.2.24), (5.2.25). We also note that, for all $t \in [0, 1]_{\mathbb{T}}$

$$\begin{aligned} f(t, u(t)) &= \rho(t) + t^6 \\ &\leq \rho(t) + t \\ &= u^\nabla(t). \end{aligned}$$

Thus, our u satisfies (5.2.3) and is an upper solution to (5.2.24), (5.2.25). By Theorem 5.2.8, there exists a solution, x , to (5.2.24), (5.2.25) such that $0 \leq x(t) \leq t^2 \leq 1$, for all $t \in [0, 1]_{\mathbb{T}}$.

Moreover, for all $t \in [0, 1]_{\mathbb{T}}$, we have

$$\left| \frac{\partial f}{\partial p} \right| = |3p^2| \leq 3t^4 \leq 3.$$

Thus, f has bounded partial derivatives in $[0, 1]_{\mathbb{T}} \times [0, 1]$ and satisfies (5.2.19) for $L = 3$ by Theorem 5.2.11. From Corollary 5.2.13, x is the unique solution to (5.2.24), (5.2.25) such that $0 \leq x(t) \leq 1$ for all $t \in [0, 1]_{\mathbb{T}}$.

□

5.3 Approximation results

In this section, we establish conditions under which lower and upper solutions to (5.1.1), (5.1.2) approximate the existing solutions of (5.1.1), (5.1.2). We also establish error estimates on the i th approximation.

Let $f : R_\kappa \rightarrow \mathbb{R}$ be left-Hilger-continuous. Define $F : C([0, a]_\mathbb{T}; \mathbb{R}) \rightarrow C([0, a]_\mathbb{T}; \mathbb{R})$ by

$$[Fp](t) = \int_0^t f(s, p(s)) \nabla s, \quad \text{for all } t \in [0, a]_\mathbb{T}.$$

Then F is well-defined on $C([0, a]_\mathbb{T}; \mathbb{R})$. Under the conditions of Theorem 5.2.8, a fixed point x of F will be a solution to (5.1.1), (5.1.2) such that $l(t) \leq x(t) \leq u(t)$ for all $t \in [0, a]_\mathbb{T}$, where l, u are, respectively, lower and upper solutions of (5.1.1), (5.1.2).

Consider an iterative scheme defined as

$$[F^0 p](t) := [Fp](t) = \int_0^t f(s, p(s)) \nabla s, \quad \text{for all } t \in [0, a]_\mathbb{T}; \quad (5.3.1)$$

$$F^i := F[F^{i-1}], \quad \text{for all } i \geq 1. \quad (5.3.2)$$

It had been shown in [83, pp.78-79] that, in general, the continuity of a function f alone is not sufficient for a sequence or subsequences of successive approximations to converge to a solution on a compact rectangle. In our next result, we show that the successive approximations defined in (5.3.1), (5.3.2) provide a sequence of functions that converge to a solution to (5.1.1), (5.1.2).

We assume f to be non-decreasing on R_κ and prove that if x is a solution to (5.1.1), (5.1.2) such that $l(t) \leq x(t) \leq u(t)$ for all $t \in [0, a]_\mathbb{T}$, then $l(t)$ and $u(t)$ approximate $x(t)$ for all $t \in [0, a]_\mathbb{T}$. We also show that an upper bound on the error of the i th approximation will be $[F^i u](t) - [F^i l](t)$ for all $t \in [0, a]_\mathbb{T}$.

The next definition describes zero approximation to the solution of (5.1.1), (5.1.2) (see [55, p.724] for the ODE case).

Definition 5.3.1 *Let x be a solution to (5.1.1), (5.1.2) and $y : \mathbb{T} \rightarrow \mathbb{R}$ be a ld-continuous function. We call $y(t)$ a zero approximation to $x(t)$ for all $t \in [0, a]_\mathbb{T}$ if, $\{F^i y\}$ converge uniformly to x on $[0, a]_\mathbb{T}$.*

Theorem 5.3.2 *Let $f : R_\kappa \rightarrow \mathbb{R}$ be left-Hilger-continuous and l, u are lower and upper solutions to (5.1.1), (5.1.2). If f is non-decreasing in the second variable on R_κ , that is, for $p \leq q$, we have*

$$f(t, p) \leq f(t, q), \quad \text{for all } (t, p), (t, q) \in R_\kappa; \quad (5.3.3)$$

then $l(t)$ and $u(t)$ will be the zero approximations to a solution x of (5.1.1), (5.1.2) for all $t \in [0, a]_\mathbb{T}$.

Moreover, for $m, n \geq 0$, the sequence F^i given by (5.3.1), (5.3.2) satisfies

$$[F^m l](t) \leq [F^{m+1} l](t) \leq [F^{n+1} u](t) \leq [F^n u](t), \quad \text{for all } t \in [0, a]_\mathbb{T}. \quad (5.3.4)$$

Proof: We show that $l(t), u(t)$ satisfy Definition 5.3.1 and (5.3.4) holds for all $t \in [0, a]_\mathbb{T}$.

We note from (5.3.1) that for $p = u$, we obtain for all $t \in [0, a]_{\kappa, \mathbb{T}}$

$$\begin{aligned} [Fu](t) &= \int_0^t f(s, u(s)) \nabla s \\ &\leq \int_0^t u^\nabla(s) \nabla s \\ &= u(t). \end{aligned} \quad (5.3.5)$$

Similarly, for $p = l$, we obtain

$$l(t) \leq [Fl](t) \quad \text{for all } t \in [0, a]_\mathbb{T}. \quad (5.3.6)$$

Since f is non-decreasing in the second argument and is left-Hilger-continuous on R_κ , it follows from (5.3.6), (5.3.3) and (5.3.1) that, for all $t \in [0, a]_\mathbb{T}$

$$\begin{aligned} [F^0 l](t) &= [Fl](t) = \int_0^t f(s, l(s)) \nabla s \\ &\leq \int_0^t f(s, [Fl](s)) \nabla s \\ &= [F^1 l](t). \end{aligned} \quad (5.3.7)$$

Proceeding in this way, we obtain

$$[Fl](t) \leq [F^1 l](t) \leq [F^2 l](t) \leq [F^3 l](t) \leq \cdots, \quad \text{for all } t \in [0, a]_\mathbb{T}. \quad (5.3.8)$$

Note that, the sequence $\{F^i l\}$ is non-decreasing.

In a similar way, using (5.3.3), (5.3.1) and (5.3.5), we obtain

$$[Fu](t) \geq [F^1u](t) \geq [F^2u](t) \geq \cdots, \quad \text{for all } t \in [0, a]_{\mathbb{T}}. \quad (5.3.9)$$

Now since $l(t) \leq u(t)$ for all $t \in [0, a]_{\kappa}$, we can write using (5.3.8) and (5.3.9) that for all $t \in [0, a]_{\mathbb{T}}$

$$[F^n l](t) \leq [F^{n+1} l](t) \leq [F^{n+1} u](t) \leq [F^n u](t). \quad (5.3.10)$$

We further note that

$$[Fl](0) = 0 = [Fu](0). \quad (5.3.11)$$

We show that the sequence $\{F^i l\}$ converges uniformly to the fixed point x ($x(0) = 0$).

Define

$$r(t) := [Fu](t) - [Fl](t), \quad \text{for all } t \in [0, a]_{\mathbb{T}}.$$

Note that, $r(t) \geq 0$ for all $t \in [0, a]_{\mathbb{T}}$. Since f is non-decreasing in the second argument on R_{κ} , it follows from (5.3.1) that

$$r^{\nabla}(t) = f(t, u(t)) - f(t, l(t)) \geq 0, \quad \text{for all } t \in [0, a]_{\mathbb{T}}.$$

It is clear from (5.3.10) that for $n > m \geq 0$, we have

$$[F^m l](t) \leq [F^n l](t) \leq [F^n u](t),$$

and for $n < m$, we have

$$[F^m l](t) \leq [F^m u](t) \leq [F^n u](t).$$

Hence for any $m, n \geq 0$, we have the inequality

$$[F^m l](t) \leq [F^{m+1} l](t) \leq [F^{n+1} u](t) \leq [F^n u](t), \text{ for all } t \in [0, a]_{\mathbb{T}}.$$

The boundedness and equicontinuity of each $F^i l$ can be established in the same way as in Theorem 5.2.7.

Hence, as $i \rightarrow \infty$, $\{F^i l\}$ converges uniformly on $[0, a]_{\mathbb{T}}$ to a fixed point x . Similarly, $\{F^i u\}$ converges uniformly on $[0, a]_{\mathbb{T}}$ to a fixed point x . Thus $l(t)$ and $u(t)$ are

zero approximations to $x(t)$ with $r^i(t) := [F^i u](t) - [F^i l](t)$ as an upper bound on the error of the i -th approximation for all $t \in [0, a]_{\mathbb{T}}$.

If the solution is unique, then $r^i(t) \rightarrow 0$ for all $i \geq 1$ for all $t \in [0, a]_{\mathbb{T}}$. This completes the proof.

□

Example 5.3.3 Consider the dynamic IVP

$$x^\nabla(t) = f(t, x) := x^3 - t, \quad \text{for all } t \in [0, 1]_{\kappa, \mathbb{T}}; \quad (5.3.12)$$

$$x(0) = 0. \quad (5.3.13)$$

We claim that $l(t) = -t$ and $u(t) = t$ are zero approximations for (5.2.17), (5.2.18) for all $t \in [0, 1]_{\mathbb{T}}$. Moreover, for all $t \in [0, 1]_T$, the sequence F^i given by

$$\begin{aligned} F^0(t) &:= [Fx](t) = \int_0^t (x^3 - s) \nabla s \\ F^i &:= F[F^{i-1}], \quad \text{for all } i \geq 1. \end{aligned}$$

satisfies (5.3.4) for any $m, n \geq 0$.

Proof We note that $f(t, p) = p^3 - t$ for all $(t, p) \in [0, 1]_{\kappa, \mathbb{T}} \times \mathbb{R}$. Since t and p^3 are everywhere ld-continuous functions and so is their composition, our f is left-Hilger-continuous on $[0, 1]_{\kappa, \mathbb{T}} \times \mathbb{R}$. We further note that $l(0) = 0 = u(0)$ and for all $t \in [0, 1]_{\kappa, \mathbb{T}}$

$$\begin{aligned} f(t, l(t)) &= -t(t^2 + 1) \\ &\geq -1 \\ &= l^\nabla(t). \end{aligned}$$

Thus, l satisfies (5.2.1) and so, is a lower solution to (5.3.12), (5.3.13). In a similar way, we have u satisfying (5.2.3) and so, is an upper solution to (5.3.12), (5.3.13). By Theorem 5.2.8, there exists a solution, x , to (5.3.12), (5.3.13) such that $-t \leq x(t) \leq t$, for all $t \in [0, 1]_{\mathbb{T}}$.

Next, we note that for $p \leq q$, we have

$$f(t, p) = p^3 - t \leq q^3 - t = f(t, q), \quad \text{for all } (t, p), (t, q) \in [0, 1]_{\kappa, \mathbb{T}} \times [-t, t].$$

Thus, f is non-decreasing with respect to the second argument on $[0, 1]_{\mathbb{T}, \kappa} \times [-t, t]$ and so, by Theorem 5.3.2, the functions $-t$ and t are zero approximations to the solution x of (5.2.17), (5.2.18). We further note that for $x = l$, we have for all $t \in [0, 1]_{\mathbb{T}}$

$$\begin{aligned} [Fl](t) &= \int_0^t -(s^3 + s) \nabla s \\ &\geq -t \\ &= l(t) \end{aligned}$$

This leads to (5.3.7) and then to (5.3.8). We obtain (5.3.9) in a similar way. Thus, (5.3.4) holds for any $m, n \geq 0$.

□

Chapter 6

Some explicit solutions

6.1 Introduction

In this chapter, we present some techniques for extracting explicit solutions for various types of first order non-linear dynamic initial value problems. We do this by developing the separation of variables approach and extracting solutions by substitution. The separation of variables for dynamic equations on time scales is developed with the help of the chain rule defined in Theorem A.3.11.

Consider a point $t_0 \in \mathbb{T}$ where \mathbb{T} is an arbitrary time scale and fix $x_0 \in \mathbb{R}$. Assume x is delta differentiable on \mathbb{T}^κ and $f : \mathbb{T}^\kappa \times \mathbb{R} \rightarrow \mathbb{R}$ is a right-Hilger-continuous function.

We consider the first order scalar dynamic equations of the types

$$x^\Delta = f(t, x), \tag{6.1.1}$$

$$x^\Delta = f(t, x^\sigma), \tag{6.1.2}$$

$$x^\Delta = f(t, x, x^\sigma), \tag{6.1.3}$$

for all $t \in \mathbb{T}^\kappa$, subject to an initial condition $x(t_0) = x_0$.

To obtain explicit solutions for dynamic initial value problems of the above types, we manipulate ideas from ordinary differential equations into the time scale setting.

This chapter is organised in the following manner.

In Section 6.2, we introduce *separation of variables* approach in the time scale setting. We do this using the chain rule defined in Theorem A.3.11 for the generalised dynamic equation (6.1.3). The method involves splitting the right hand side

of (6.1.3) into a quotient of a function of t and a function of x, x^σ under certain conditions. The resultant equation comes out to be a separable equation. Examples including various types of dynamic IVPs solved using the separation of variables method have been provided.

Section 6.3 includes solutions by *substitution* manipulating ideas from ordinary differential equations and transforming them into the time scale setting. Examples are provided to reinforce the results.

6.2 The separation of variable approach

In this section, we give the definition of a separable dynamic equation and methods for its solution. The chain rule given by (A.3.7) will be the key tool to separate the variables in dynamic equations on time scales.

The following definition and result has been published in [82, p.3521].

Let \mathbb{T} be any arbitrary time scale and $f : \mathbb{T}^\kappa \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be right-Hilger-continuous. If we can split the right hand side of (6.1.3) as a quotient of a rd-continuous function $g(t)$ and a continuous function $h(x, x^\sigma)$ then we can define (6.1.3) as a separable equation as follows.

Definition 6.2.1 *Let $g : \mathbb{T} \rightarrow \mathbb{R}$ be rd-continuous and $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function. An equation of the form (6.1.3) will be called separable if we can write*

$$x^\Delta = f(t, x, x^\sigma) = \frac{g(t)}{h(x, x^\sigma)}, \quad \text{for all } t \in \mathbb{T}^\kappa. \quad (6.2.1)$$

The next theorem provides a method to solve an equation of the form (6.2.1) with the help of the chain rule in Theorem A.3.7.

□

Theorem 6.2.2 *Consider the initial value problem*

$$x^\Delta = f(t, x, x^\sigma) = \frac{g(t)}{h(x, x^\sigma)}, \quad \text{for all } t \in \mathbb{T}^\kappa; \quad (6.2.2)$$

$$x(t_0) = x_0. \quad (6.2.3)$$

If there exists a continuously differentiable real valued function H , such that

$$\int_0^1 H'[x + k(x^\sigma - x)] dk = h(x, x^\sigma), \quad \text{for all } t \in \mathbb{T}, \quad (6.2.4)$$

then, the IVP (6.2.2), (6.2.3) has an implicit solution, given by

$$H(x(t)) = \int_{t_0}^t g(s) \Delta s + H(x_0), \quad \text{for all } t \in \mathbb{T}. \quad (6.2.5)$$

The solution can be explicitly obtained if H is globally one-to-one.

Proof: Consider (6.2.2). We separate the variables and obtain

$$h(x(t), x^\sigma(t))x^\Delta(t) = g(t), \quad \text{for all } t \in \mathbb{T}^\kappa. \quad (6.2.6)$$

Using (6.2.4), we replace h in the above expression and write

$$\int_0^1 H'[x(t) + k(x^\sigma(t) - x(t))] dk \cdot x^\Delta(t) = g(t), \quad \text{for all } t \in \mathbb{T}^\kappa, \quad (6.2.7)$$

which yields

$$\int_0^1 H'[x(t) + k\mu(t)] dk \cdot x^\Delta(t) = g(t), \quad \text{for all } t \in \mathbb{T}^\kappa.$$

By the chain rule (A.3.7) we obtain

$$[H(x(t))]^\Delta = g(t), \quad \text{for all } t \in \mathbb{T}^\kappa.$$

Finally, taking the delta integral of both sides in the above expression and incorporating (6.2.3) we obtain (6.2.5).

□

The above result is illustrated by the following examples.

Example 6.2.3 Let α be a positive constant. Then $\alpha \in \mathcal{R}^+$, see (A.6.2).

Consider the dynamic IVP

$$x^\Delta = \frac{\sin_\alpha(t, 0)}{x^2 + xx^\sigma + (x^\sigma)^2}, \quad \text{for all } t \in \mathbb{T}; \quad (6.2.8)$$

$$x(0) = 1. \quad (6.2.9)$$

We claim that this IVP is separable and has solution

$$x(t) = \left[\frac{-1}{\alpha} (\cos_\alpha(t, 0) - 1) + 1 \right]^{1/3}, \quad \text{for all } t \in \mathbb{T}. \quad (6.2.10)$$

Proof Consider

$$g(t) := \sin_\alpha(t, 0), \quad \text{and} \quad h(v, v^\sigma) = v^2 + vv^\sigma + (v^\sigma)^2.$$

Then, by assumption, our g and h are well defined. Choose

$$H(v) = v^3, \quad \text{and so} \quad H'(v) = 3v^2.$$

We note that

$$\begin{aligned} \int_0^1 H'[x + k(x^\sigma - x)] dk &= \int_0^1 3[x + k(x^\sigma - x)]^2 dk \\ &= 3 \left[x^2 + x(x^\sigma - x) + \frac{1}{3}(x^\sigma - x)^2 \right] \\ &= (x^\sigma)^2 + xx^\sigma + x^2 \\ &= h(x, x^\sigma). \end{aligned}$$

Hence, the given dynamic IVP is separable and, by Theorem 6.2.2, has solution given by (6.2.5). Thus, for all $t \in \mathbb{T}$, we obtain

$$\begin{aligned} H(x(t)) = (x(t))^3 &= \int_0^t \sin_\alpha(s, 0) \Delta s + 1 \\ &= \frac{-1}{\alpha} (\cos_\alpha(t, 0) - 1) + 1. \end{aligned}$$

Since x^3 is one-to-one, the above equation yields (6.2.10).

□

Example 6.2.4 Let $p \in \mathcal{R}^+$. Consider the dynamic IVP

$$\begin{aligned} x^\Delta &= \frac{e_p(t, 0)}{x + x^\sigma}, \quad \text{for all } t \in [0, \infty)_{\mathbb{T}}; \\ x(0) &= 1. \end{aligned}$$

We claim that this IVP is separable and has solution, x , given by

$$(x(t))^2 = \frac{1}{p}(e_p(t, 0) - 1) + 1, \quad \text{for all } t \in [0, \infty)_{\mathbb{T}}. \quad (6.2.11)$$

Proof: Define $H(v) = v^2$. Then, we have $H'(v) = 2v = h(v)$, for all v . We also note that

$$\begin{aligned} \int_0^1 H'[x + k(x^\sigma - x)] dk &= \int_0^1 2[x + k(x^\sigma - x)] dk \\ &= x + x^\sigma \\ &= h(x, x^\sigma). \end{aligned}$$

Hence, the given dynamic IVP is separable and, by Theorem 6.2.2, has solution given by (6.2.5). Thus, for all $t \in [0, \infty)_{\mathbb{T}}$, we have

$$H(x(t)) = x^2(t) = \int_0^t e_p(s, 0) \Delta s + 1, \quad \text{for all } t \in [0, \infty)_{\mathbb{T}}.$$

This gives

$$(x(t))^2 = \frac{1}{p}(e_p(t, 0) - 1) + 1, \quad \text{for all } t \in [0, \infty)_{\mathbb{T}}.$$

□

6.3 Solution by substitution

In this section, we solve some dynamic equations by reducing them to linear equations by appropriate substitutions. The following theorem provides a generalised method to solve a dynamic equation by substitution. A special case is followed thereafter.

Theorem 6.3.1 *Let $p \in \mathcal{R}$. Let $x : \mathbb{T}^\kappa \rightarrow \mathbb{R}$ be delta differentiable and $g : \mathbb{R} \rightarrow \mathbb{R}$ be continuously differentiable. If $f : \mathbb{T} \rightarrow \mathbb{R}$ is rd-continuous, then a dynamic equation of any of the forms*

$$\left\{ \int_0^1 g'[x + k\mu(t)x^\Delta] dk \right\} x^\Delta + p(t)g(x) = f(t), \quad (6.3.1)$$

for all $t \in \mathbb{T}^\kappa$;

$$\left\{ \int_0^1 g'[x + k\mu(t)x^\Delta] dk \right\} x^\Delta + p(t)g(x^\sigma) = f(t), \quad (6.3.2)$$

for all $t \in \mathbb{T}^\kappa$

can be solved as a linear dynamic equation.

Proof: From the chain rule, we note that the delta derivative of the composition function $g(x)$ can be obtained as

$$[g(x(t))]^\Delta = (g \circ x)^\Delta(t) = x^\Delta(t) \int_0^1 g'[x(t) + k\mu(t)x^\Delta(t)] dk, \quad \text{for all } t \in \mathbb{T}^\kappa.$$

Thus, the above dynamic equations (6.3.1) and (6.3.2) can be written as

$$[g(x)]^\Delta + p(t)g(x) = f(t), \quad \text{for all } t \in \mathbb{T}^\kappa; \quad (6.3.3)$$

$$\text{and} \quad [g(x)]^\Delta + p(t)g(x^\sigma) = f(t), \quad \text{for all } t \in \mathbb{T}^\kappa. \quad (6.3.4)$$

If we substitute $u = g(x)$ above, then these equations would result in linear equations in u of the form

$$\begin{aligned} u^\Delta + p(t)u &= f(t), & \text{for all } t \in \mathbb{T}^\kappa \\ \text{or } u^\Delta + p(t)u^\sigma &= f(t), & \text{for all } t \in \mathbb{T}^\kappa. \end{aligned}$$

The above equations are linear by Definition A.7.5 and can be solved using (A.7.14) and (A.7.15).

A backward substitution into (6.3.3) (or (6.3.4)) would yield the solution of (6.3.1) (or (6.3.2)).

□

The following corollary is a special case of Theorem 6.3.1.

Corollary 6.3.2 *Let $p \in \mathcal{R}$. Let $x : \mathbb{T}^\kappa \rightarrow \mathbb{R}$ be delta differentiable and $g : \mathbb{R} \rightarrow \mathbb{R}$ be continuously differentiable. If $f : \mathbb{T} \rightarrow \mathbb{R}$ is rd-continuous, then a dynamic equation of the form*

$$\left\{ \sum_{k=0}^{n-1} x^k (x^\sigma)^{n-1-k} \right\} x^\Delta + p(t)x^n = f(t), \quad \text{for all } t \in \mathbb{T}^\kappa \quad (6.3.5)$$

can be solved as a linear dynamic equation.

Proof: Note that, for all $n = 1, 2, \dots$, we obtain from [19, p.337]

$$(x^n)^\Delta = x^\Delta \sum_{k=0}^{n-1} x^k (x^\sigma)^{n-1-k}.$$

Thus, (6.3.5) would take the form

$$(x^n)^\Delta + p(t)x^n = f(t), \quad \text{for all } t \in \mathbb{T}^\kappa. \quad (6.3.6)$$

A substitution $u = x^n$ in the above equation would then result in a linear equation in u and can be solved using (A.7.14). A backward substitution into (6.3.6) would yield a solution for x . In this way, any dynamic equation of the form (6.3.5) can be solved by the above substitution.

□

Example 6.3.3 Consider the initial value problem

$$x^\Delta = \frac{2x^2 + t}{x + x^\sigma}, \quad \text{for all } t \in \mathbb{T}^\kappa; \quad (6.3.7)$$

$$x(0) = 1. \quad (6.3.8)$$

We claim that the above system has an implicit solution

$$(x(t))^2 = \frac{5}{4}e_2(t, 0) - \frac{t}{2} - \frac{1}{4}, \quad \text{for all } t \in \mathbb{T}. \quad (6.3.9)$$

Proof: We note that (6.3.7) can be written as

$$(x + x^\sigma)x^\Delta = 2x^2 + t, \quad \text{for all } t \in \mathbb{T}^\kappa,$$

which can further be reduced to

$$(x^2)^\Delta - 2x^2 = t, \quad \text{for all } t \in \mathbb{T}^\kappa.$$

Using Corollary 6.3.2, we substitute $x^2 = u$ and note that (6.3.7), (6.3.8) takes the form

$$u^\Delta = 2u + t, \quad \text{for all } t \in \mathbb{T}^\kappa \quad (6.3.10)$$

$$u(0) = 1. \quad (6.3.11)$$

From (A.7.10) we note that (6.3.10) is linear. Since $1 + 2\mu > 0$, we have $2 \in \mathcal{R}^+$.

Thus, the solution of the IVP (6.3.10), (6.3.11) will be given by (A.7.14) as

$$u(t) = e_2(t, 0) + \int_0^t s e_2(t, \sigma(s)) \Delta s.$$

We further note that using Theorem A.6.4(8), the above solution can be simplified, for all $t \in \mathbb{T}$, as follows.

$$\begin{aligned} u(t) &= e_2(t, 0) + \int_0^t \frac{s}{e_2^\sigma(s, t)} \Delta s \\ &= e_2(t, 0) - \frac{1}{2} \int_0^t s \left[\frac{-2}{e_2(s, t)} \right]^\Delta \Delta s \\ &= e_2(t, 0) - \frac{1}{2} \left[\frac{t}{e_2(t, t)} - 0 - \int_0^t \frac{1}{e_2^\sigma(s, t)} \Delta s \right], \end{aligned}$$

where we used (A.5.4) in the last step. This yields, for all $t \in \mathbb{T}$

$$\begin{aligned}
 u(t) &= e_2(t, 0) - \frac{1}{2} \left[t + \frac{1}{2} \int_0^t \frac{-2}{e_2^\sigma(s, t)} \Delta s \right] \\
 &= e_2(t, 0) - \frac{1}{2} \left[t + \frac{1}{2} \left(\frac{1}{e_2(t, t)} - \frac{1}{e_2(0, t)} \right) \right] \\
 &= e_2(t, 0) - \frac{t}{2} - \frac{1}{4} + \frac{1}{4} e_2(t, 0) \\
 &= \frac{5}{4} e_2(t, 0) - \frac{t}{2} - \frac{1}{4}.
 \end{aligned}$$

Since x^2 is not globally one-to-one in \mathbb{R} , the backward substitution $u = x^2$ in the above equation yields the implicit solution

$$(x(t))^2 = \frac{5}{4} e_2(t, 0) - \frac{t}{2} - \frac{1}{4}, \quad \text{for all } t \in \mathbb{T},$$

by Theorem 6.2.2.

□

Chapter 7

Conclusions and open problems

This work presented a series of results regarding non-multiplicity, existence, uniqueness and successive approximations to solutions of first order dynamic equations on time scales that can model non-linear phenomena of a hybrid stop-start nature.

The field of dynamic equations on time scales was introduced in 1988 [45] and has gained a lot of attention in recent years, particularly, in the non-linear theory. Most investigations have been on boundary value problems on time scales while many areas of initial value problems have yet to be discovered.

Our results considered initial value problems, mostly with fixed initial conditions. This can be further extended considering periodic initial conditions and also with conditions that are continuous functions of t . We presented such a case in Chapter 4 regarding successive approximations to solutions of vector dynamic IVPs.

Extending Roger's ideas from ordinary differential equations [70, pp.609-611] to the time scale setting, we can establish the non-multiplicity of solutions to the scalar dynamic IVP

$$x^\Delta = f(t, x), \quad \text{for all } t \in [0, a]_{\mathbb{T}}; \quad (7.0.1)$$

$$x(0) = 0, \quad (7.0.2)$$

Let $D \subseteq \mathbb{R}$. Define

$$R^\kappa := \{(t, u) : t \in (0, a]_{\mathbb{T}} \text{ and } u \in D\}.$$

Then the following conjecture can be investigated.

Proposition 7.0.4 *Let $f : R^\kappa \rightarrow \mathbb{R}$ be right-Hilger-continuous. If there exists a positively regressive function $p(t) := \frac{1}{t\sigma(t)}$ for all $t \in (0, a]_{\mathbb{T}}$ such that f satisfies the conditions*

$$(a) \quad |f(t, u) - f(t, v)| \leq p(t)|u - v|, \text{ for all } (t, u), (t, v) \in R^\kappa;$$

(b) $f(t, x) = o(p(t)e_p(t, 0))$ for all $t \in (0, a]_{\mathbb{T}}$. That is, for a given $0 < \epsilon < 1/2$ we have

$$f(t, x) = \epsilon p(t)e_p(t, 0) \quad \text{for all } t \in (0, a]_{\mathbb{T}};$$

then the IVP (7.0.1), (7.0.2) has, at most, one solution $x : [0, a]_{\mathbb{T}} \rightarrow \mathbb{R}$, with $x(t) \in D$ for all $t \in [0, a]_{\mathbb{T}}$.

It will be interesting to investigate existence of solutions of singular and non-singular initial value problems on time scales. Singular initial value problems have important applications in industry that display a hybrid structure [84].

The Banach space constructed in Chapter 3 also provides a platform to investigate solutions of initial value problems extended to the entire neighbourhood $[t_0 - a, t_0 + a]_{\mathbb{T}}$ of a point $t_0 \in \mathbb{T}$ or within a smaller interval $[t_0 - \alpha, t_0 + \alpha]_{\mathbb{T}} \subseteq [t_0 - a, t_0 + a]_{\mathbb{T}}$.

Appendix A

Notation and fundamentals

This appendix explains notation used in this work, basic ideas and definitions of time scale calculus and some preliminary results.

A.1 Notation

Throughout these pages, we will follow the notation given below.

- \mathbb{T} denotes an arbitrary time scale, which is a closed, non-empty subset of \mathbb{R} .
- $[a, b]_{\mathbb{T}}$ – an interval in \mathbb{T} with $b > a$.
- $\mathbf{f} = (f_1, f_2, \dots, f_n)$ – an n -dimensional vector function.
- $\mathbf{x} = (x_1, x_2, \dots, x_n)$ – an n -dimensional vector function.
- f – a scalar function.
- x – a scalar function.
- All other bold faced letters refer to an element in \mathbb{R}^n , otherwise they refer the element to be in \mathbb{R} .
- $\mathbf{x}^\sigma = \mathbf{x} \circ \sigma$.
- \mathbf{x}^Δ – the delta derivative of \mathbf{x} .
- x^∇ – the nabla derivative of x .

- $\langle \mathbf{a}, \mathbf{b} \rangle$, where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ – the usual Euclidean inner product on \mathbb{R}^n .
- $\| \cdot \|$ – the Euclidean norm on \mathbb{R}^n .
- For any $t_0 \in \mathbb{T}$, we will write $e_1(\cdot, t_0) = e(\cdot, t_0)$.

A.2 Basic time scale calculus

Definition A.1.1 The time scale

A time scale, denoted by \mathbb{T} , may be any non-empty closed subset of \mathbb{R} . For example,

$$\mathbb{R}, \mathbb{Z}^+, [-1, 0] \cup [1, 2] \quad \text{and} \quad \{x \in \mathbb{R} : |x| \leq 1\} \cup \{n/2 : n \in \mathbb{N}\}$$

are examples of time scales.

□

Graphically, we can think of the points on a time scale to be as shown in Figure A.1.



Figure A.1:

An arbitrary time scale may or may not be connected. The notion of connectivity of points in a time scale is described in terms of the forward and backward jump operators defined as follows:

Definition A.1.2 The forward and backward jump operators

Let \mathbb{T} be an arbitrary time scale and t be a point in \mathbb{T} . The forward jump operator, $\sigma(t) : \mathbb{T} \rightarrow \mathbb{T}$, is defined as

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \text{for all } t \in \mathbb{T}.$$

In a similar way, we define the backward jump operator, $\rho(t) : \mathbb{T} \rightarrow \mathbb{T}$, as

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \quad \text{for all } t \in \mathbb{T}.$$

Thus, in Figure A.1, we note that $\sigma(A) = A$ while $\sigma(B) = C$. Similarly $\rho(F) = E = \rho(E)$. In this way, the forward and backward (or right and left) jump operators declare whether a point in a time scale is discrete and give the direction of discreteness of the point. The following table describes the left- and right-discreteness of a point t in an arbitrary time scale \mathbb{T} .

Point	Description
right-dense	$\sigma(t) = t$
right-scattered	$\sigma(t) > t$
left-dense	$\rho(t) = t$
left-scattered	$\rho(t) < t$
isolated	$\rho(t) < t < \sigma(t)$
dense	$\rho(t) = t = \sigma(t)$

The ‘step size’ at each point of a time scale is given by the forward graininess function, $\mu(t)$, or the backward graininess function, $\nu(t)$, defined respectively as

$$\mu(t) := \sigma(t) - t \quad \text{for all } t \in \mathbb{T}; \quad (\text{A.1.1})$$

$$\nu(t) := t - \rho(t) \quad \text{for all } t \in \mathbb{T}. \quad (\text{A.1.2})$$

If \mathbb{T} has a left scattered maximum value m_1 , then we define $\mathbb{T}^\kappa := \mathbb{T} \setminus m_1$, otherwise $\mathbb{T}^\kappa := \mathbb{T}$. The right scattered minimum of \mathbb{T} is defined in a similar way and if we denote it by m_2 then we define $\mathbb{T}_\kappa := \mathbb{T} \setminus m_2$, otherwise $\mathbb{T}_\kappa := \mathbb{T}$.

We consider some examples of time scales with corresponding values of $\sigma(t)$ and $\mu(t)$ accompanied by their graphs.

Example A.1.3 *Define*

$$\mathbb{T} := \left\{ \frac{n}{2} : n \in \mathbb{N}_0 \right\}.$$

Then, we have $\sigma(t) = \frac{n+1}{2} = t + \frac{1}{2}$, for all $t \in \mathbb{T}$. Hence, $\mu(t) = 1/2$, for all $t \in \mathbb{T}$ and the graph of \mathbb{T} is linear as shown in the following figure.

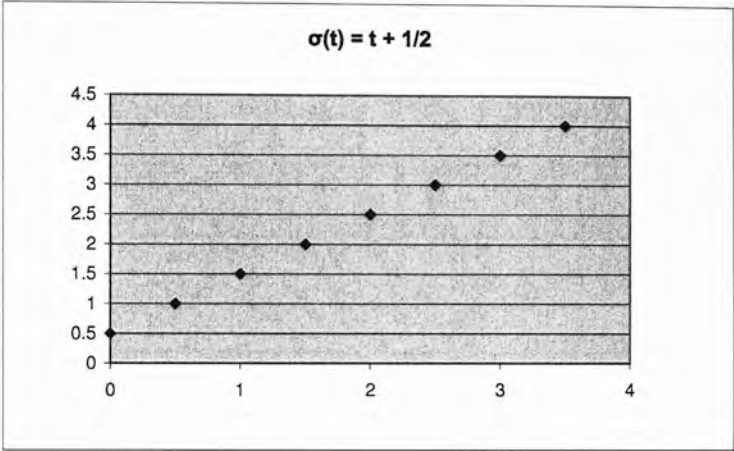


Figure A.2: The graph of $\mathbb{T} = n/2$

Example A.1.4 Define

$$\mathbb{T} := \{\sqrt{n} : n \in \mathbb{N}_0\}.$$

Then, we have $\sigma(t) = \sqrt{n+1} = \sqrt{t^2+1}$, for all $t \in \mathbb{T}$. Hence, $\mu(t) = \sqrt{t^2+1} - t$, for all $t \in \mathbb{T}$ and the graph of \mathbb{T} is non-linear as shown below.

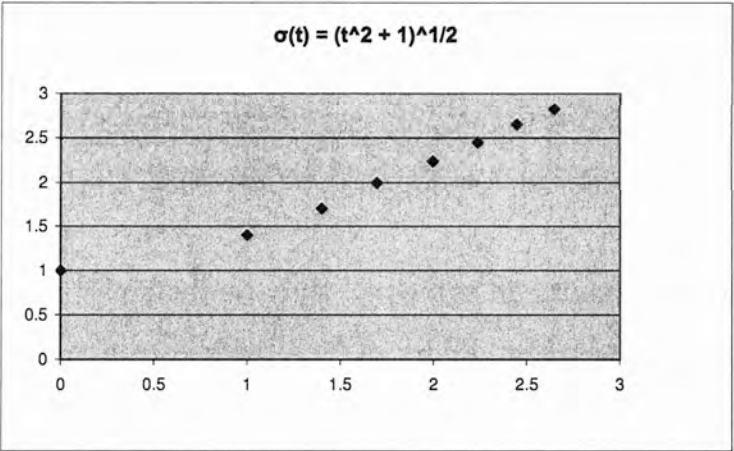


Figure A.3: The graph of $\mathbb{T} = \sqrt{n}$

Example A.1.5 Define

$$\mathbb{T} := \{q^n : n \in \mathbb{Z}, q > 1\} \cup \{0\}.$$

Then, we have $\sigma(t) = q^{n+1} = qt$, for all $t \in \mathbb{T}$. Hence, $\mu(t) = (q - 1)t$, for all $t \in \mathbb{T}$. The graph below shows \mathbb{T} taking $q = 2$.

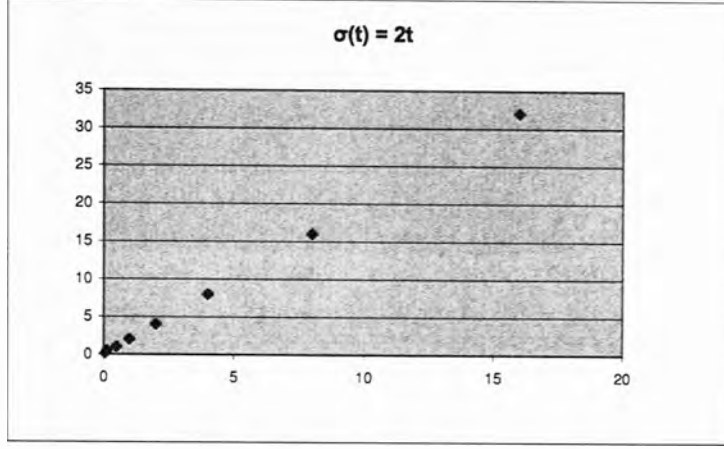


Figure A.4: The graph of $\mathbb{T} = q^n$ for $q = 2$

Example A.1.6 *Define*

$$\mathbb{T} = P_{a,b} := \bigcup_{k=0}^{\infty} [k(a+b), k(a+b) + a].$$

Then, for all $t \in \mathbb{T}$, we have

$$\sigma(t) = \begin{cases} t & \text{if } t \in \bigcup_{k=0}^{\infty} [k(a+b), k(a+b) + a) \\ t + b & \text{if } t \in \bigcup_{k=0}^{\infty} k(a+b) + a. \end{cases}$$

Thus, for all $t \in \mathbb{T}$, we obtain

$$\mu(t) = \begin{cases} 0 & \text{if } t \in \bigcup_{k=0}^{\infty} [k(a+b), k(a+b) + a) \\ b & \text{if } t \in \bigcup_{k=0}^{\infty} k(a+b) + a. \end{cases}$$

The graph of \mathbb{T} taking $a = 1$ and $b = 2$ is given below.

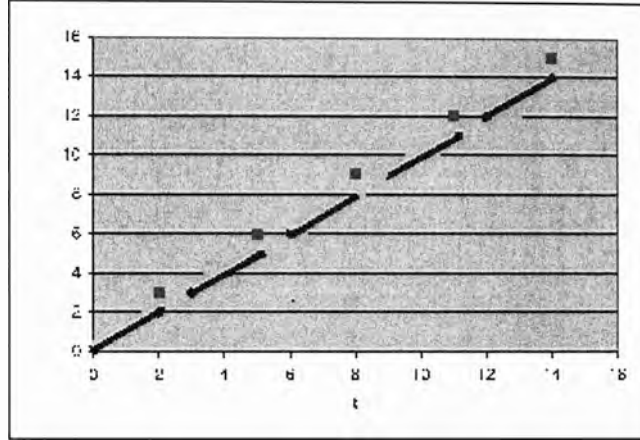


Figure A.5: The graph of $\mathbb{T} = P_{1,2}$

Example A.1.7 *Define*

$$\mathbb{T} := \left\{ 2^{1/n} : n \in \mathbb{N} \right\} \cup \{1\}.$$

Then we have

$$\sigma(t) = 2^{\frac{1}{n+1}} = t^{\frac{n}{n+1}}, \quad \text{for all } t \in \mathbb{T}.$$

Hence, the step size becomes

$$\mu(t) = t^{\frac{n}{n+1}} - t = t^{\frac{n}{n+1}} \left(1 - t^{\frac{1}{n+1}} \right), \quad \text{for all } t \in \mathbb{T}.$$

The graph of \mathbb{T} taking $n = 2$ is as follows:

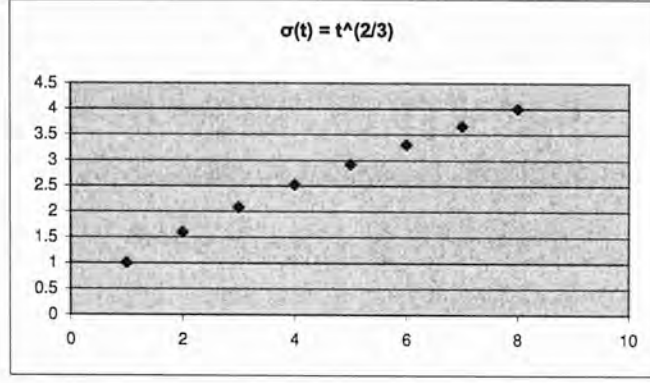


Figure A.6: The graph of $\mathbb{T} = 2^{1/n}$ for $n = 2$

Example A.1.8 Consider the Cantor set $C := \bigcap_{k=0}^{\infty} C_k$, where $C_0 = [0, 1]$ and each of the remaining C_i where $i = 1 \dots k$ is obtained successively by removing the open middle third interval of C_{i-1} . Hence $C_i \subset C_{i-1}$ for all i . Thus

$$C_1 = [0, 1/3] \cup [2/3, 1]$$

$$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$

In this way, the Cantor set being the intersection of all C_k is non-empty and closed.

Let $\mathbb{T} = C$. Note that for C_1 there are three subdivisions of the interval $[0, 1]$ and each dividing point t is the sum of $c_1/3$, where $c_1 \in \{0, 1, 2\}$. For C_2 there are nine subdivisions of $[0, 1]$ and each dividing point t is the sum of $\frac{c_1}{3} + \frac{c_2}{9}$, where $c_1, c_2 \in \{0, 1, 2\}$.

In this way, for each C_i , there are 3^i subdivisions of $[0, 1]$ and each dividing point t can be expressed as the ternary expansion [14, p.22] [71, p.40]

$$t = \sum_{i=1}^{\infty} \frac{c_i}{3^i}, \quad \text{where } c_i \in \{0, 1, 2\} \text{ for all } i.$$

Thus, if t is the 'inner' end point of an interval in \mathbb{T} (the end points except 0, 1), then t is either right-scattered or left-scattered.

In the earlier case, for each C_i we have

$$\sigma(t) = t + \frac{1}{3^i}, \quad \text{for all } t \in C_i,$$

and in the latter case we have

$$\rho(t) = t - \frac{1}{3^i}, \quad \text{for all } t \in C_i,$$

and so $\mu(t) = \frac{1}{3^i} = \nu(t)$ for all $t \in C_i$. Note that for all other t in C_i we have $\rho(t) = t = \sigma(t)$. Hence there are no isolated points in \mathbb{T} .

□

A.2 Continuity in time scales

Continuity of a function at a point $t \in \mathbb{T}$ depends on the appearance of t as ‘right-dense’ ($t = \sigma(t)$) or ‘left-dense’ ($t = \rho(t)$). Thus, for any $t \in \mathbb{T}$, a right-dense-continuous (usually written as rd-continuous) function is defined as follows.

Definition A.2.1 The right-dense continuity

Assume $\mathbf{k} : \mathbb{T} \rightarrow \mathbb{R}^n$. We define \mathbf{k} as right-dense continuous or rd-continuous if

$$\lim_{s \rightarrow t^+} \mathbf{k}(s) = \mathbf{k}(t), \quad \text{for all } t \in \mathbb{T}$$

where t is right-dense and

$$\lim_{s \rightarrow t} \mathbf{k}(s)$$

exists and is finite for all $t \in \mathbb{T}$ where t is left-dense. The set of all rd-continuous function on \mathbb{T} is denoted by $C_{rd}(\mathbb{T}; \mathbb{R}^n)$.

□

The next definition due to Hilger [45, p.39] describes the so-called *right-Hilger-continuous* function $\mathbf{f}(t, \mathbf{x})$ where the ordered n -pair $(t, \mathbf{x}) \in \mathbb{T} \times \mathbb{R}^n$. This is a more generalised definition and we have introduced this particular term for functions of several variables, to avoid confusion with rd-continuous functions of one variable.

Definition A.2.2 The right-Hilger-continuity

Consider an arbitrary time scale \mathbb{T} . A function $\mathbf{f} : \mathbb{T}^\kappa \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ having the property that \mathbf{f} is continuous at each (t, \mathbf{x}) where t is right-dense; and the limits

$$\lim_{(s, \mathbf{y}) \rightarrow (t^-, \mathbf{x})} \mathbf{f}(s, \mathbf{y}) \quad \text{and} \quad \lim_{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(t, \mathbf{y})$$

both exist and are finite at each (t, \mathbf{x}) where t is left-dense, is said to be right-Hilger-continuous on $\mathbb{T}^\kappa \times \mathbb{R}^n$.

□

Remark A.2.3 It should be noted that we will write \mathbf{f} defined above as rd-continuous if it is a function of t only, that is, $\mathbf{f}(t, \mathbf{x}) = \mathbf{g}(t)$ for all $t \in \mathbb{T}$.

□

Remark A.2.4 It can also be seen that a right-Hilger-continuous function \mathbf{f} will be continuous if $\mathbf{f}(t, \mathbf{x}) = \mathbf{h}(\mathbf{x})$ for all $t \in \mathbb{T}$.

□

Definition A.2.5 The left-dense continuity

Let \mathbb{T} be an arbitrary time scale. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be ld-continuous if it is continuous at each $t \in \mathbb{T}$ that is left dense and $\lim_{s \rightarrow t^+} f(s)$ exists and is finite at each $t \in \mathbb{T}$ that is right dense.

□

From the above definitions, we note that all continuous functions are rd- and ld-continuous [19, Theorem 1.60, Definition 8.43].

As for right-Hilger-continuous functions, the term ‘left-Hilger-continuous’ is used in equivalence with the term ‘ld-continuous’ for a function of two or more variables, the first of which should be from an arbitrary time scale.

Definition A.2.6 The left-Hilger-continuity

Let \mathbb{T} be an arbitrary time scale. A mapping $f : [a, b]_{\kappa, \mathbb{T}} \times \mathbb{R} \rightarrow \mathbb{R}$ is called left-Hilger continuous if: f is continuous at each (t, x) where t is left-dense; and the limits

$$\lim_{(s, y) \rightarrow (t^+, x)} f(s, y) \quad \text{and} \quad \lim_{y \rightarrow x} f(t, y)$$

both exist and are finite at each (t, x) where t is right-dense.

□

Remark A.2.7 *Remarks A.2.3 and A.2.4 also hold for left-Hilger-continuous and ld-continuous functions.*

□

Example A.2.8 *The following functions are left-Hilger-continuous:*

- (a) Consider $f(t, p) := tp^2$, where $t \in \mathbb{T}_\kappa$ and $p \in \mathbb{R}$. Note that the composition function $t(x(t))^2$ will be ld-continuous on $\mathbb{T}_\kappa \times \mathbb{R}$. Therefore, by definition, f is left-Hilger-continuous on $\mathbb{T}_\kappa \times \mathbb{R}$;
- (b) Consider $f(t, p) := t^2 \ln p$, where $t \in \mathbb{T}_\kappa$ and $p \in [0, k]$ where $k > 0$ is a continuous function. Note that the composition function $t^2 \ln x(t)$ will be ld-continuous on \mathbb{T}_κ . Therefore, $f(t, p)$ is left-Hilger-continuous for all $(t, p) \in \mathbb{T}_\kappa \times [0, k]$;
- (c) Consider $f(t, p) := \frac{1}{1+t} e^p$, where $t \in (-1, 0]_{\mathbb{T}}$ and $p \in [0, \infty)$. Note that the composition function $\frac{1}{1+t} \exp[x(t)]$ will be ld-continuous for all $t \in (-1, 0]_{\kappa, \mathbb{T}}$. Therefore, our f is left-Hilger-continuous on $(-1, 0]_{\kappa, \mathbb{T}} \times [0, \infty)$;
- (d) Consider $f(t, p) := \rho(t) + p$, where $t \in [0, 1]_{\mathbb{T}}$ and $p \in \mathbb{R}$. Then the composition function $\rho(t) + x(t)$ will be ld-continuous for all $t \in [0, 1]_{\kappa, \mathbb{T}}$. Therefore, our f is left-Hilger-continuous on $[0, 1]_{\kappa, \mathbb{T}} \times \mathbb{R}$.

□

A.3 The delta and nabla derivatives

Definition A.3.1 Fix $t \in \mathbb{T}^\kappa$, and let $\mathbf{x} : \mathbb{T} \rightarrow \mathbb{R}^n$. Define α to be the vector (if it exists) with the property that given $\epsilon > 0$ there is a neighbourhood U of t , that is, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$, such that

$$|(\mathbf{x}(\sigma(t)) - \mathbf{x}(s)) - \alpha(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s|, \text{ for all } s \in U. \quad (\text{A.3.1})$$

We call α the delta derivative of $\mathbf{x}(t)$ and denote it by $\mathbf{x}^\Delta(t)$ for all $t \in \mathbb{T}^\kappa$. If $\mathbf{x}^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$ then we say that \mathbf{x} is delta differentiable on \mathbb{T} .

□

Theorem A.3.2 Assume that $\mathbf{k} : \mathbb{T} \rightarrow \mathbb{R}^n$ and let $t \in \mathbb{T}^\kappa$.

1. If \mathbf{k} is delta differentiable at t then \mathbf{k} is continuous at t .
2. If \mathbf{k} is continuous at t and t is right-scattered then \mathbf{k} is delta differentiable at t with

$$\mathbf{k}^\Delta(t) = \frac{\mathbf{k}^\sigma(t) - \mathbf{k}(t)}{\sigma(t) - t}, \quad \text{for all } t \in \mathbb{T}^\kappa,$$

where $\mathbf{k}^\sigma = \mathbf{k} \circ \sigma$.

3. If \mathbf{k} is delta differentiable and t is right-dense then

$$\mathbf{k}^\Delta(t) = \lim_{s \rightarrow t} \frac{\mathbf{k}(t) - \mathbf{k}(s)}{t - s}, \quad \text{for all } t \in \mathbb{T}.$$

4. If \mathbf{k} is delta differentiable at t then

$$\mathbf{k}^\sigma(t) = \mathbf{k}(t) + \mu(t)\mathbf{k}^\Delta(t), \quad \text{for all } t \in \mathbb{T}^\kappa.$$

The above identity is often called the Simple Useful Formula and will be referred to as *SUF* in this work.

□

We illustrate the above ideas in two simple examples.

Example A.3.3 Consider an arbitrary time scale \mathbb{T} and define $x(t) := t$ for all $t \in \mathbb{T}$. We claim that $x^\Delta(t) = 1$ for all $t \in \mathbb{T}^\kappa$.

Proof: If t is right-scattered, then using Theorem A.3.2(2), we note that

$$x^\Delta(t) = \frac{\sigma(t) - t}{\sigma(t) - t} = 1, \quad \text{for all } t \in \mathbb{T}^\kappa. \quad (\text{A.3.2})$$

On the other hand, if t is right-dense, then using Theorem A.3.2(3), we again obtain $x^\Delta(t) = 1$ for all $t \in \mathbb{T}$.

□

Example A.3.4 Consider $\mathbb{T} := \{\sqrt{n} : n \in \mathbb{N}_0\}$. We claim that

$$[\sigma(t)]^\Delta = (\sqrt{\sigma(t)^2 + 1} - \sqrt{t^2 + 1})(\sqrt{t^2 + 1} + t), \quad \text{for all } t \in \mathbb{T}^\kappa.$$

Proof: Using Example A.1.4 and applying Theorem A.3.2(2), we obtain the delta derivative of x for all $t \in \mathbb{T}^\kappa$ as follows:

$$\begin{aligned}
[\sigma(t)]^\Delta &= \frac{\sigma(\sigma(t)) - \sigma(t)}{\mu(t)} \\
&= \frac{\sqrt{\sigma(t)^2 + 1} - \sqrt{t^2 + 1}}{\sqrt{t^2 + 1} - t} \\
&= \frac{\sqrt{\sigma(t)^2 + 1} - \sqrt{t^2 + 1}}{\sqrt{t^2 + 1} - t} \times \frac{\sqrt{t^2 + 1} + t}{\sqrt{t^2 + 1} + t} \\
&= (\sqrt{\sigma(t)^2 + 1} - \sqrt{t^2 + 1})(\sqrt{t^2 + 1} + t).
\end{aligned}$$

□

The next theorem [19, Theorem 1.20] provides some basic identities to obtain the delta derivatives of some delta differentiable maps defined on \mathbb{T}^κ . These identities have played a fundamental lipconinArole in our computations in this work and will be referred frequently.

Theorem A.3.5 *Let \mathbf{x}, \mathbf{y} be defined on an arbitrary time scale \mathbb{T} . If \mathbf{x}, \mathbf{y} are delta differentiable on \mathbb{T}^κ , then the following identities hold for \mathbf{x}, \mathbf{y} :*

- (1) *The sum rule: $\mathbf{x} + \mathbf{y}$ is delta differentiable on \mathbb{T}^κ such that*

$$(\mathbf{x} + \mathbf{y})^\Delta(t) = \mathbf{x}^\Delta(t) + \mathbf{y}^\Delta(t), \quad \text{for all } t \in \mathbb{T}^\kappa.$$

- (2) *The scaler multiplication rule: for any constant λ , $\lambda\mathbf{x}$ is delta differentiable on \mathbb{T}^κ such that*

$$(\lambda\mathbf{x})^\Delta(t) = \lambda\mathbf{x}^\Delta(t), \quad \text{for all } t \in \mathbb{T}^\kappa.$$

- (3) *The product rule: \mathbf{xy} is delta differentiable on \mathbb{T}^κ such that for all $t \in \mathbb{T}^\kappa$,*

$$\begin{aligned}
(\mathbf{xy})^\Delta(t) &= \mathbf{x}^\Delta(t)\mathbf{y}(t) + \mathbf{x}^\sigma(t)\mathbf{y}^\Delta(t) \\
&= \mathbf{x}(t)\mathbf{y}^\Delta(t) + \mathbf{x}^\Delta(t)\mathbf{y}^\sigma(t).
\end{aligned}$$

- (4) *The inverse rule: Define $1/\mathbf{x}$ as $(1/x_1, 1/x_2, \dots, 1/x_n)$ where $x_i \neq 0$ for all $i = 1 \dots n$. Then $1/\mathbf{x}$ is delta differentiable on \mathbb{T}^κ such that*

$$\left(\frac{1}{\mathbf{x}}\right)^\Delta(t) = \frac{-\mathbf{x}^\Delta(t)}{\langle \mathbf{x}(t)\mathbf{x}^\sigma(t) \rangle}, \quad \text{for all } t \in \mathbb{T}^\kappa,$$

provided $\langle \mathbf{x}(t)\mathbf{x}^\sigma(t) \rangle \neq 0$ for all $t \in \mathbb{T}^\kappa$. Here $\langle \mathbf{x}(t)\mathbf{x}^\sigma(t) \rangle$ is the usual Euclidean inner product of \mathbf{x} and \mathbf{x}^σ on \mathbb{R}^n .

(5) *The quotient rule: \mathbf{x}/\mathbf{y} is delta differentiable on \mathbb{T}^κ such that*

$$\left(\frac{\mathbf{x}}{\mathbf{y}}\right)^\Delta(t) = \frac{\mathbf{x}^\Delta(t)\mathbf{y}(t) - \mathbf{x}(t)\mathbf{y}^\Delta(t)}{\langle \mathbf{y}(t)\mathbf{y}^\sigma(t) \rangle}, \quad \text{for all } t \in \mathbb{T}^\kappa$$

provided $\langle \mathbf{y}(t)\mathbf{y}^\sigma(t) \rangle \neq 0$ for all $t \in \mathbb{T}^\kappa$.

□

The above theorem provides the most powerful tools to simplify delta differentiation. To show this, we consider the following examples.

Example A.3.6 *Let $x : \mathbb{T} \rightarrow \mathbb{R}$ be defined by $x(t) := t^3$ and $y : \mathbb{T} \rightarrow \mathbb{R}$ be defined by $y(t) := 1/t^2$ for all $t \in \mathbb{T} \setminus \{0\}$. We claim that*

$$x^\Delta(t) = t^2 + t \sigma(t) + \sigma(t)^2, \quad \text{for all } t \in \mathbb{T}^\kappa$$

and

$$y^\Delta(t) = -\frac{t + \sigma(t)}{(t\sigma(t))^2}, \quad \text{for all } t \in \mathbb{T}^\kappa \setminus \{0\}.$$

Proof: Applying the product rule (Theorem A.3.5 (3)) and [19, Example 1.25], we obtain for all $t \in \mathbb{T}^\kappa$

$$\begin{aligned} x^\Delta(t) &= t^2 + (t + \sigma(t))\sigma(t) \\ &= t^2 + t \sigma(t) + \sigma(t)^2. \end{aligned} \tag{A.3.3}$$

Similarly, applying the inverse rule (Theorem A.3.5 (4)), and using [19, Example 1.25] we obtain for all $t \in \mathbb{T}^\kappa \setminus \{0\}$

$$\begin{aligned} y^\Delta(t) &= \left(\frac{1}{t^2}\right)^\Delta \\ &= \frac{-(t^2)^\Delta}{t^2\sigma(t)^2} \\ &= -\frac{(t + \sigma(t))}{(t\sigma(t))^2}. \end{aligned} \tag{A.3.4}$$

□

We also note from the above example how for $\mathbb{T} = \mathbb{R}$ the delta derivative behaves as the ordinary derivative, as for $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$ and so (A.3.3) yields $x'(t) = 3t^2$ and (A.3.4) yields $y'(t) = -2/t^3$ for all $t \in \mathbb{T} \setminus \{0\}$.

Example A.3.7 Consider the time scale defined in Example A.1.3, that is, $\mathbb{T} := \left\{ \frac{n}{2} : n \in \mathbb{N}_0 \right\}$ with $\sigma(t) = t + 1/2$. Define $x : \mathbb{T} \rightarrow \mathbb{R}$ as

$$x(t) := t\sigma(t), \quad \text{for all } t \in \mathbb{T}.$$

Then using the product rule in Theorem A.3.5(3), we obtain for all $t \in \mathbb{T}^\kappa$

$$\begin{aligned} x^\Delta(t) &= (t\sigma(t))^\Delta \\ &= \sigma(t)(1 + \sigma(t)^\Delta) \\ &= 2\sigma(t) \\ &= 2t + 1. \end{aligned}$$

□

We can obtain results for the delta derivative of the generalised function $x(t) = (t - a)^n$ [19, Theorem 1.24], where a is a constant and $n \in \mathbb{N}$. This is shown in the following theorem.

Theorem A.3.8 Let \mathbb{T} be a generalised time scale and $t \in \mathbb{T}$. Let a be a constant. If there exist $x, y : \mathbb{T} \rightarrow \mathbb{R}$ such that $x(t) := (t - a)^n$ and $y(t) := (t - a)^{-m}$ for some $n, m \in \mathbb{N}$ then

$$x^\Delta(t) = \sum_{k=0}^{n-1} (t - a)^k (\sigma(t) - a)^{n-1-k}, \quad \text{for all } t \in \mathbb{T}^\kappa; \quad (\text{A.3.5})$$

$$y^\Delta(t) = - \sum_{k=0}^{m-1} (t - a)^{-m+k} (\sigma(t) - a)^{-1-k}, \quad \text{for all } t \in \mathbb{T}^\kappa, \quad (\text{A.3.6})$$

provided $(t - a)(\sigma(t) - a) \neq 0$ for all $t \in \mathbb{T}$.

□

Definition A.3.9 The nabla derivative

Let $x : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}_\kappa$. Define λ to be the number (if it exists) with the property that given $\epsilon > 0$ there is a neighbourhood N of t with

$$|[x(\rho(t)) - x(s)] - \lambda[\rho(t) - s]| \leq \epsilon|\rho(t) - s|, \quad \text{for all } s \in N.$$

We call λ the nabla-derivative of $x(t)$ and denote it by $x^\nabla(t)$ for all $t \in \mathbb{T}_\kappa$. If $x^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$ then we say that x is nabla differentiable on \mathbb{T} .

The nabla-derivative has similar properties to delta-derivative for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. In the former case, we have $x^\nabla(t) = x'(t)$ and for the latter case, we have $x^\nabla(t) = x(t) - x(t-1)$. For a generalised time scale, the properties of nabla-derivative can be seen through the following theorem.

Theorem A.3.10 *Let \mathbb{T} be an arbitrary time scale and a function $h : \mathbb{T}_\kappa \rightarrow \mathbb{R}$. Then the following hold for all $t \in \mathbb{T}_\kappa$:*

1. *if h is nabla differentiable at t , then h is continuous at t ;*
2. *if h is continuous at t and t is left-scattered, then h is nabla differentiable at t and*

$$h^\nabla(t) := \frac{h(t) - h(\rho(t))}{\nu(t)};$$

3. *If t is left-dense, then h is nabla differentiable at t defined by*

$$h^\nabla(t) := \lim_{s \rightarrow t} \frac{h(t) - h(s)}{t - s},$$

provided the limit on the right hand side exists and is finite;

4. *if h is nabla differentiable at t then*

$$h^\rho(t) := h(t) - \nu(t)h^\nabla(t).$$

□

For more details on nabla-derivatives, see [13, pp.77–81] and [19, Chapter 1, Chapter 8].

The next theorem [67, Theorem 1], [19, Theorem 1.90] presents a generalised form of the chain rule in the time scale setting.

Theorem A.3.11 *Let $x : \mathbb{T}^\kappa \rightarrow \mathbb{R}$ be a delta differentiable function. If there exists a continuously differentiable function $y : \mathbb{R} \rightarrow \mathbb{R}$ then $y \circ x : \mathbb{T} \rightarrow \mathbb{R}$ will be delta differentiable defined by*

$$(y \circ x)^\Delta(t) = [y(x(t))]^\Delta := x^\Delta(t) \int_0^1 y'[x(t) + k\mu(t)x^\Delta(t)] dk, \quad (\text{A.3.7})$$

for all $t \in \mathbb{T}^\kappa$.

□

A.4 The delta and nabla integrals

In Section 2.4, we described that the derivative of a function defined on a time scale is called delta-derivative. Likewise, the anti-derivative of a function in the time scale setting is termed as the *delta integral* and the process is called *delta-integration*.

Definition A.5.1 *Let \mathbb{T} be an arbitrary time scale and $\mathbf{j} : \mathbb{T}^\kappa \rightarrow \mathbb{R}^n$. If $\mathbf{J}^\Delta(t) = \mathbf{j}(t)$ then we define the delta integral of \mathbf{j} by*

$$\int_a^t \mathbf{j}(s) \Delta s = \mathbf{J}(t) - \mathbf{J}(a). \quad (\text{A.5.1})$$

□

If $\mathbb{T} = \mathbb{R}$ then

$$\int_a^t \mathbf{j}(s) \Delta s = \int_a^t \mathbf{j}(s) ds,$$

while if $\mathbb{T} = \mathbb{Z}$ then

$$\int_a^t \mathbf{j}(s) \Delta s = \sum_{s=a}^{t-1} \mathbf{j}(s).$$

It should be noted that all rd-continuous functions are delta integrable [19, Theorem 1.74].

Let $C(\mathbb{T}; \mathbb{R}^n)$ denote the space of all continuous functions on \mathbb{T} . The following theorem describes the existence of an anti-derivative of a right-Hilger-continuous function defined on $\mathbb{T}^\kappa \times \mathbb{R}^n$, where \mathbb{T} an arbitrary time scale.

Theorem A.5.2 *Let $\mathbf{f} : \mathbb{T}^\kappa \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $t_0 \in \mathbb{T}$. If \mathbf{f} is right-Hilger-continuous on $\mathbb{T}^\kappa \times \mathbb{R}^n$ then there exists a function $\mathbf{F} : C(\mathbb{T}; \mathbb{R}^n) \rightarrow C(\mathbb{T}; \mathbb{R}^n)$ called the delta integral of \mathbf{f} such that*

$$[\mathbf{F}\mathbf{x}](t) := \int_{t_0}^t \mathbf{f}(s, \mathbf{x}(s)) \Delta s, \quad \text{for all } t \in \mathbb{T}. \quad (\text{A.5.2})$$

□

The next theorem provides important identities for delta-integration of right-Hilger-continuous functions and will be frequently used in our work. This is a more generalised extension of [19, Theorem 1.77 (i) – (iv), (vii)] in the light of Remark A.2.3.

Theorem A.5.3 *Let $a, b, c \in \mathbb{T}$ and $\lambda \in \mathbb{R}$. If \mathbf{f}, \mathbf{g} are right-Hilger-continuous on $\mathbb{T} \times \mathbb{R}^n$, then the following identities hold for all $(t, \mathbf{x}) \in \mathbb{T} \times \mathbb{R}^n$:*

$$(i) \int_a^b [\mathbf{f}(t, \mathbf{x}) + \mathbf{g}(t, \mathbf{x})] \Delta t = \int_a^b \mathbf{f}(t, \mathbf{x}) \Delta t + \int_a^b \mathbf{g}(t, \mathbf{x}) \Delta t;$$

$$(ii) \int_a^b [\lambda \mathbf{f}](t, \mathbf{x}) \Delta t = \lambda \int_a^b \mathbf{f}(t, \mathbf{x}) \Delta t;$$

$$(iii) \int_a^b \mathbf{f}(t, \mathbf{x}) \Delta t = - \int_b^a \mathbf{f}(t, \mathbf{x}) \Delta t;$$

$$(iv) \int_a^b \mathbf{f}(t, \mathbf{x}) \Delta t = \int_a^c \mathbf{f}(t, \mathbf{x}) \Delta t + \int_c^b \mathbf{f}(t, \mathbf{x}) \Delta t;$$

$$(v) \int_a^a \mathbf{f}(t, \mathbf{x}) \Delta t = 0;$$

(vi) *if, for all $(t, \mathbf{x}) \in [a, b]_{\mathbb{T}} \times \mathbb{R}^n$, we have $\|\mathbf{f}(t, \mathbf{x})\| \leq \|\mathbf{g}(t, \mathbf{x})\|$, then*

$$\left\| \int_a^b \mathbf{f}(t, \mathbf{x}) \Delta t \right\| \leq \int_a^b \mathbf{g}(t, \mathbf{x}) \Delta t, \quad \text{for all } (t, \mathbf{x}) \in [a, b]_{\mathbb{T}} \times \mathbb{R}^n;$$

(vii) *if, for all $(t, \mathbf{x}) \in [a, b]_{\mathbb{T}} \times \mathbb{R}$, we have $\mathbf{f}(t, \mathbf{x}) \geq 0$, then*

$$\int_a^b \mathbf{f}(t, \mathbf{x}) \Delta t \geq 0, \quad \text{for all } (t, \mathbf{x}) \in [a, b]_{\mathbb{T}} \times \mathbb{R}^n.$$

Proof: The proof is similar to that of [19, Theorem 1.77 (i)–(iv), (vii)] and is, therefore, sketched only for part (i).

We note that \mathbf{f}, \mathbf{g} are right-Hilger-continuous on $\mathbb{T} \times \mathbb{R}^n$. Thus, by Theorem A.5.2, \mathbf{f}, \mathbf{g} possess antiderivatives \mathbf{F}, \mathbf{G} defined by (A.5.2). By the sum rule, Theorem A.3.5(1), we obtain for all $t \in \mathbb{T}^\kappa$

$$\begin{aligned} [\mathbf{F} + \mathbf{G}](\mathbf{x})^\Delta(t) &= [\mathbf{F}\mathbf{x}]^\Delta(t) + [\mathbf{G}\mathbf{x}]^\Delta(t) \\ &= \mathbf{f}(t, \mathbf{x}(t)) + \mathbf{g}(t, \mathbf{x}(t)). \end{aligned}$$

Thus, $\mathbf{F} + \mathbf{G}$ will be an antiderivative of $\mathbf{f} + \mathbf{g}$. Therefore, for all $t \in \mathbb{T}$, we obtain

$$\begin{aligned} \int_a^b [\mathbf{f}(t, \mathbf{x}(t)) + \mathbf{g}(t, \mathbf{x}(t))] \Delta t &= [\mathbf{F} + \mathbf{G}](\mathbf{x})(b) - [\mathbf{F} + \mathbf{G}](\mathbf{x})(a) \\ &= [\mathbf{F}\mathbf{x}](b) - [\mathbf{F}\mathbf{x}](a) + [\mathbf{G}\mathbf{x}](b) - [\mathbf{G}\mathbf{x}](a) \\ &= \int_a^b \mathbf{f}(t, \mathbf{x}) \Delta t + \int_a^b \mathbf{g}(t, \mathbf{x}) \Delta t. \end{aligned}$$

□

Example A.5.4 Let \mathbb{T} be an arbitrary time scale. Consider the functions $u(t) := t + \sigma(t)$ and $v(t) := \frac{t + \sigma(t)}{(t\sigma(t))^2}$, for all $t \in \mathbb{T} \setminus \{0\}$. We claim that u, v are delta integrable and find their anti-derivatives.

Proof: We note that the composition functions $t + \sigma(t)$ and $\frac{t + \sigma(t)}{(t\sigma(t))^2}$ are rd-continuous for all $t \in \mathbb{T} \setminus \{0\}$. So our u, v are delta integrable [19, Theorem 1.74].

Fix $t_0 \in \mathbb{T}$. Then using (A.5.1) and [19, Example 1.25], we note that for all $t \in \mathbb{T}$,

$$\begin{aligned} \int_{t_0}^t u(s) \Delta s &= \int_{t_0}^t s + \sigma(s) \Delta s \\ &= t^2 - t_0^2. \end{aligned}$$

Similarly, since $t\sigma(t) \neq 0$ for all $t \in \mathbb{T} \setminus \{0\}$, using Example A.3.6, we obtain for all $t \in \mathbb{T} \setminus \{0\}$

$$\begin{aligned} \int_{t_0}^t v(s) \Delta s &= \int_{t_0}^t \frac{s + \sigma(s)}{(s\sigma(s))^2} \Delta s \\ &= \frac{1}{t^2} - \frac{1}{t_0^2} \\ &= \frac{t_0^2 - t^2}{(t t_0)^2}. \end{aligned}$$

□

The following theorem provides two important identities, integration by parts, that hold for any rd-continuous functions k, r defined on \mathbb{T} , where $a, b \in \mathbb{T}$. [19, Theorem 1.77]

Theorem A.5.5 Define $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$, where \mathbb{T} is an arbitrary time scale. Let k, r be rd-continuous on \mathbb{T} . Then the following identities hold.

$$\int_a^b k^\sigma(t) r^\Delta(t) \Delta t = [kr](b) - [kr](a) - \int_a^b k^\Delta(t) r(t) \Delta t; \quad (\text{A.5.3})$$

$$\int_a^b k(t) r^\Delta(t) \Delta t = [kr](b) - [kr](a) - \int_a^b k^\Delta(t) r^\sigma(t) \Delta t. \quad (\text{A.5.4})$$

□

The improper integral of a right-Hilger-continuous function is defined as follows. The idea is due to [19, p.30].

Definition A.5.6 Let $t_0 \in \mathbb{T}$ with $\sup \mathbb{T} = \infty$. If \mathbf{f} is a right-Hilger-continuous function defined on $[t_0, \infty) \times \mathbb{R}^n$, then the improper integral of \mathbf{f} will be

$$\int_{t_0}^{\infty} \mathbf{f}(t, \mathbf{x}) \Delta t := \lim_{b \rightarrow \infty} \int_{t_0}^b \mathbf{f}(t, \mathbf{x}) \Delta t, \quad \text{for all } t \in [t_0, \infty)_{\mathbb{T}},$$

provided the limit on the right hand side exists. In that case, we say that the improper integral converges. If the limit does not exist then we say that the improper integral diverges.

□

The next theorem [19, Theorem 1.79] will be useful in our results in this work.

Theorem A.5.7 Assume $a, b \in \mathbb{T}$. If k is rd-continuous on \mathbb{T} , then the following identities describe equivalence of the delta integral of k :

- For $\mathbb{T} = \mathbb{R}$,

$$\int_a^b k(t) \Delta t = \int_a^b k(t) dt, \quad \text{for all } t \in \mathbb{T};$$

- For $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$ consisting of isolated points, we obtain for all $t \in \mathbb{T}$

$$\int_a^b k(t) \Delta t = \begin{cases} \sum_{t \in [a, b)_{\mathbb{T}}} \mu(t) k(t), & \text{if } a < b; \\ 0, & \text{if } a = b; \\ \sum_{t \in [b, a)_{\mathbb{T}}} \mu(t) k(t), & \text{if } a > b. \end{cases}$$

□

The nabla integral can be defined in a similar way as delta integral.

Definition A.5.8 The nabla anti-derivative

Let $h : \mathbb{T} \rightarrow \mathbb{R}$. A function $H : \mathbb{T} \rightarrow \mathbb{R}$ will be a nabla anti-derivative of h if $H^{\nabla}(t) = h(t)$ holds for all $t \in \mathbb{T}_{\kappa}$. Let $t_0 \in \mathbb{T}$ with $t_0 < t$ then the Cauchy nabla integral of h is defined as

$$\int_{t_0}^t h(s) \nabla s := H(t) - H(t_0), \quad \text{for all } t \in \mathbb{T}.$$

□

A.6 Special functions

The following functions provide a deeper understanding of the time scale calculus and are called special functions.

Definition A.6.1 [19, p.58] Consider a function $p : \mathbb{T} \rightarrow \mathbb{R}$. If p is rd-continuous on \mathbb{T} and $1 + p(t)\mu(t) \neq 0$ for all $t \in \mathbb{T}$, then p is called a regressive function.

□

Here μ is the graininess function defined in (A.1.1). The set of all regressive functions on \mathbb{T} is defined as

$$\mathcal{R} := \{p \in C_{rd}(\mathbb{T}; \mathbb{R}) : 1 + p(t)\mu(t) \neq 0, \text{ for all } t \in \mathbb{T}\}. \quad (\text{A.6.1})$$

The inequality $1 + p(t)\mu(t) \neq 0$ yields two possibilities, which form a partitioning of \mathcal{R} into two sets. The set of positively regressive functions, \mathcal{R}^+ , and the set of negatively regressive functions, \mathcal{R}^- . These are defined as:

$$\mathcal{R}^+ := \{p \in C_{rd}(\mathbb{T}; \mathbb{R}) : 1 + p(t)\mu(t) > 0, \text{ for all } t \in \mathbb{T}\}, \quad (\text{A.6.2})$$

and

$$\mathcal{R}^- := \{p \in C_{rd}(\mathbb{T}; \mathbb{R}) : 1 + p(t)\mu(t) < 0, \text{ for all } t \in \mathbb{T}\}. \quad (\text{A.6.3})$$

The elements of \mathcal{R} are closed under the operations \oplus and \ominus and hold the following properties. [19, pp. 58–59]

Theorem A.6.2 let $p, q : \mathbb{T} \rightarrow \mathbb{R}$. If $p, q \in \mathcal{R}$, then the following properties hold for p, q :

$$(a) \quad p \oplus q = p + q + \mu pq;$$

$$(b) \quad \ominus p = -\frac{p}{1 + \mu p};$$

$$(c) \quad p \ominus q = p \oplus (\ominus q) = \frac{p - q}{1 + \mu q};$$

$$(d) \quad p \ominus p = 0;$$

$$(e) \quad \ominus(\ominus p) = p;$$

$$(f) \quad \ominus(p \ominus q) = q \ominus p;$$

$$(g) \quad \ominus(p \oplus q) = (\ominus p) \oplus (\ominus q).$$

□

Definition A.6.3 Fix $t_0 \in \mathbb{T}$ and assume $p \in \mathcal{R}$. The exponential function denoted by $e_p(\cdot, t_0)$ [45], [46] is defined as

$$e_p(t, t_0) := \begin{cases} \exp \left(\int_{t_0}^t p(s) \, ds \right), & \text{for } t \in \mathbb{T}, \mu = 0; \\ \exp \left(\int_{t_0}^t \frac{\text{Log}(1 + \mu(s)p(s))}{\mu(s)} \Delta s \right), & \text{for } t \in \mathbb{T}, \mu > 0, \end{cases} \quad (\text{A.6.4})$$

where Log is the principal logarithm function.

□

We also note that $e_p(t, t_0)$ is an increasing function if $p(t) > 0$ for all $t \in \mathbb{T}$ and is a decreasing function if $p(t) < 0$ for all $t \in \mathbb{T}$ [46, Theorem 7]. Further properties of the exponential function [46], [19, Theorem 2.35, Theorem 2.36] are shown in the following theorem and will be used in further work.

Theorem A.6.4 Let $p, q : \mathbb{T} \rightarrow \mathbb{R}$. If $p, q \in \mathcal{R}$, then the following properties hold for all $t, s, r \in \mathbb{T}$:

$$(1) \quad e_0(t, s) = 1 = e_p(t, t);$$

$$(2) \quad e_p^\sigma(t, s) = e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s);$$

$$(3) \quad \frac{1}{e_p(t, s)} = e_{\ominus p}(t, s) = e_p(s, t);$$

$$(4) \quad e_p(t, s)e_p(s, r) = e_p(t, r);$$

$$(5) \quad e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s);$$

$$(6) \quad \frac{e_p(t, s)}{e_q(t, s)} = e_{p \ominus q}(t, s);$$

$$(7) \quad e_p^\Delta(t, s) = p(t)e_p(t, s);$$

$$(8) \quad \left[\frac{1}{e_p(t, s)} \right]^\Delta = -\frac{p(t)}{e_p^\sigma(t, s)};$$

(9) If $p \in \mathcal{R}^+$ then $e_p(t, t_0) > 0$ for all $t \in \mathbb{T}$.

□

The following definitions and theorem due to [19, Definition 3.25, Lemma 3.26] will be useful in many examples of our results.

Definition A.6.5 Let $t_0 \in \mathbb{T}$ and $p \in \mathcal{R}$. If $e_p(t, t_0)$ is the exponential function, then the trigonometric functions in the time scale setting are defined as follows:

$$\begin{aligned}\cos_p(t, t_0) &:= \frac{e_{ip}(t, t_0) + e_{-ip}(t, t_0)}{2}; \\ \sin_p(t, t_0) &:= \frac{e_{ip}(t, t_0) - e_{-ip}(t, t_0)}{2},\end{aligned}$$

where $i = \sqrt{-1}$.

□

Theorem A.6.6 Fix $t_0 \in \mathbb{T}$ and p be rd-continuous. If $-\mu p^2 \in \mathcal{R}$, then the following properties hold for all $t \in \mathbb{T}^\kappa$

$$(iv) \quad \sin_p^\Delta(t, t_0) = p(t) \cos_p(t, t_0);$$

$$(v) \quad \cos_p^\Delta(t, t_0) = -p(t) \sin_p(t, t_0);$$

□

A.7 Dynamic equations on time scales

From the above sections, we are familiar with the idea of right- and left-Hilger-continuity of a function $\mathbf{f} = \mathbf{f}(t, \mathbf{x})$ where $\mathbf{f} : \mathbb{T}^\kappa \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Definition A.7.1 Let $\mathbf{f} : \mathbb{T}^\kappa \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ be right-Hilger-continuous. A delta differential equation of the form

$$\mathbf{x}^\Delta = \mathbf{f}(t, \mathbf{x}, \mathbf{x}^\sigma), \quad \text{for all } t \in \mathbb{T}^\kappa \tag{A.7.1}$$

is a generalised first order delta equation on time scales.

□

A solution of (A.7.1) will be a delta differentiable function \mathbf{y} that satisfies (A.7.1).

For a right-Hilger-continuous function $\mathbf{f} : \mathbb{T}^\kappa \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ a dynamic equation of any of the forms

$$\mathbf{x}^\Delta = \mathbf{f}(t, \mathbf{x}), \quad \text{for all } t \in \mathbb{T}^\kappa; \quad (\text{A.7.2})$$

$$\mathbf{x}^\Delta = \mathbf{f}(t, \mathbf{x}^\sigma), \quad \text{for all } t \in \mathbb{T}^\kappa, \quad (\text{A.7.3})$$

is also a first order delta equation on time scales and is a special case of (A.7.1).

Definition A.7.2 Let $\mathbf{f} : \mathbb{T}^\kappa \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ be a right-Hilger-continuous function. An equation of the form

$$\mathbf{x}(t) = \int \mathbf{f}(s, \mathbf{x}(s), \mathbf{x}^\sigma) \Delta s, \quad \text{for all } t \in \mathbb{T} \quad (\text{A.7.4})$$

is called the generalised delta integral equation.

□

A solution to (A.7.4) will be a rd-continuous function \mathbf{y} that satisfies (A.7.4).

The delta integral equations corresponding to (A.7.2) and (A.7.3) will, therefore, be of the form

$$\mathbf{x}(t) = \int \mathbf{f}(s, \mathbf{x}(s)) \Delta s, \quad \text{for all } t \in \mathbb{T} \quad (\text{A.7.5})$$

and

$$\mathbf{x}(t) = \int \mathbf{f}(s, \mathbf{x}^\sigma) \Delta s, \quad \text{for all } t \in \mathbb{T}. \quad (\text{A.7.6})$$

Definition A.7.3 Let $f : \mathbb{T}_\kappa \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be left-Hilger-continuous. A nabla differential equation of the form

$$x^\nabla = f(t, x, x^\rho), \quad \text{for all } t \in \mathbb{T}_\kappa \quad (\text{A.7.7})$$

is a generalised first order scalar nabla equation on time scales.

□

A solution of (A.7.7) will be a nabla differentiable function y that satisfies (A.7.7).

For a left-Hilger-continuous function $f : \mathbb{T}_\kappa \times \mathbb{R} \rightarrow \mathbb{R}$ a dynamic equation of the form

$$x^\nabla = f(t, x), \quad \text{for all } t \in \mathbb{T}_\kappa; \quad (\text{A.7.8})$$

is also a first order nabla equation on time scales and is a special case of (A.7.7).

Definition A.7.4 Let $f : \mathbb{T}_\kappa \times \mathbb{R} \rightarrow \mathbb{R}$ be a left-Hilger-continuous function. A dynamic equation of the form

$$x(t) = \int f(s, x(s)) \nabla s, \quad \text{for all } t \in \mathbb{T} \quad (\text{A.7.9})$$

is called a nabla integral equation.

□

A solution to (A.7.9) will be a ld-continuous function y that satisfies (A.7.9).

A.7.1 Linear dynamic equations

In this section, we present some ideas about the linear first order dynamic equations on time scales. These ideas will be used in many examples constructed in this work.

Definition A.7.5 Let t be a point in an arbitrary time scale \mathbb{T} and $\mathbf{x} : \mathbb{T} \rightarrow \mathbb{R}^n$ be delta differentiable. Moreover, let $p : \mathbb{T} \rightarrow \mathbb{R}$ and $\mathbf{k} : \mathbb{T} \rightarrow \mathbb{R}^n$. Consider the first order dynamic IVPs of the form

$$\mathbf{x}^\Delta = p(t)\mathbf{x} + \mathbf{k}(t), \quad \text{for all } t \in \mathbb{T}^\kappa; \quad (\text{A.7.10})$$

$$\mathbf{x}(t_0) = \mathbf{x}_0, \quad (\text{A.7.11})$$

and

$$\mathbf{x}^\Delta = -p(t)\mathbf{x}^\sigma + \mathbf{k}(t), \quad \text{for all } t \in \mathbb{T}^\kappa; \quad (\text{A.7.12})$$

$$\mathbf{x}(t_0) = \mathbf{x}_0. \quad (\text{A.7.13})$$

The dynamic equations of the type (A.7.10) and (A.7.12) are called linear dynamic equations and so the above IVPs are called linear dynamic IVPs.

□

The linear dynamic equations (A.7.10) and (A.7.12) will be non-homogeneous if $\mathbf{k}(t) \neq 0$ for some $t \in \mathbb{T}$. Otherwise they will be homogeneous linear dynamic equations. The following theorems [19, p.77] provide solutions to the above linear first order initial value problems.

Theorem A.7.6 Consider the dynamic IVP (A.7.10), (A.7.11). If $p \in \mathcal{R}$, then the unique solution to (A.7.10), (A.7.11) will be given by

$$\mathbf{x}(t) = \mathbf{x}_0 e_p(t, t_0) + \int_{t_0}^t \mathbf{k}(s) e_p(t, \sigma(s)) \Delta s, \quad \text{for all } t \in \mathbb{T}. \quad (\text{A.7.14})$$

□

Theorem A.7.7 Consider the dynamic IVP (A.7.12), (A.7.13). If $p \in \mathcal{R}$, then the unique solution to (A.7.12), (A.7.13) will be given by

$$\mathbf{x}(t) = \mathbf{x}_0 e_p(t_0, t) + \int_{t_0}^t \mathbf{k}(s) e_p(s, t) \Delta s, \quad \text{for all } t \in \mathbb{T}. \quad (\text{A.7.15})$$

□

The following remarks indicate two important properties, the delta derivatives of two exponential functions, and are used extensively in this work. For details, see [19] [5].

Remark A.7.8 The exponential function $e_p(t, t_0)$ is the unique solution to the linear dynamic IVP

$$\begin{aligned} x^\Delta &= p(t)x; & \text{for all } t \in \mathbb{T}, \\ x(t_0) &= 1. \end{aligned}$$

□

Remark A.7.9 In the light of Theorem A.6.2, we note that $e_{\ominus p}(t, t_0)$ will be the unique solution of the dynamic IVP

$$\begin{aligned} x^\Delta &= -p(t)x^\sigma; & \text{for all } t \in \mathbb{T}, \\ x(t_0) &= 1. \end{aligned}$$

□

A.7.2 The non-linear dynamic equations

The dynamic equations of the form (A.7.1), (A.7.2), (A.7.3) are said to be non-linear if \mathbf{f} is a non-linear function in the second argument on $\mathbb{T}^\kappa \times \mathbb{R}^{2n}$ (for (A.7.1)) or on $\mathbb{T}^\kappa \times \mathbb{R}^n$ (for (A.7.2) and (A.7.3)).

Bibliography

- [1] R. Agarwal, M. Bohner, and A. Peterson, *Inequalities on time scales: a survey*, Math. Inequal. Appl. **4** (2001), no. 4, 535–557.
- [2] R. P. Agarwal and V Lakshmikantham, *Uniqueness and non-uniqueness criteria for ordinary differential equations*, World Scientific, Singapore, 1993.
- [3] R. P. Agarwal, Donal O'Regan, V. Lakshmikantham, and S. Leela, *Existence of positive solutions for singular initial and boundary value problems via the classical upper and lower solution approach*, Nonlinear Anal. **50** (2002), no. 2, Ser. A: Theory Methods, 215–222.
- [4] Ravi Agarwal, Martin Bohner, and Donal O'Regan, *Boundary value problems on infinite intervals: a topological approach*, Advances in dynamic equations on time scales, Birkhäuser Boston, Boston, MA, 2003, pp. 275–291.
- [5] Ravi Agarwal, Martin Bohner, Donal O'Regan, and Allan Peterson, *Dynamic equations on time scales: a survey*, J. Comput. Appl. Math. **141** (2002), no. 1-2, 1–26, Dynamic equations on time scales.
- [6] Ravi P. Agarwal and Donal O'Regan, *An introduction to ordinary differential equations*, Universitext, Springer, New York, 2008.
- [7] Elvan Akin, *Boundary value problems for a differential equation on a measure chain*, Panamer. Math. J. **10** (2000), no. 3, 17–30.
- [8] Elvan Akin-Bohner, Ferhan Merdivenci Atıcı, and Billûr Kaymakçalan, *Lower and upper solutions of boundary value problems*, Advances in dynamic equations on time scales, Birkhäuser Boston, Boston, MA, 2003, pp. 165–188.

- [9] Douglas Anderson, Richard Avery, John Davis, Johnny Henderson, and William Yin, *Positive solutions of boundary value problems*, Advances in dynamic equations on time scales, Birkhäuser Boston, Boston, MA, 2003, pp. 189–249.
- [10] Douglas Anderson, John Bullock, Lynn Erbe, Allan Peterson, and HoaiNam Tran, *Nabla dynamic equations*, Advances in dynamic equations on time scales, Birkhäuser Boston, Boston, MA, 2003, pp. 47–83.
- [11] F. M. Atici and D. C. Biles, *First- and second-order dynamic equations with impulse*, Adv. Difference Equ. (2005), no. 2, 119–132.
- [12] F. Merdivenci Atici and D. C. Biles, *First order dynamic inclusions on time scales*, J. Math. Anal. Appl. **292** (2004), no. 1, 222–237.
- [13] F. Merdivenci Atici and G. Sh. Guseinov, *On Green's functions and positive solutions for boundary value problems on time scales*, J. Comput. Appl. Math. **141** (2002), no. 1-2, 75–99, Dynamic equations on time scales.
- [14] R. Beals, *Analysis: An introduction*, Cambridge University press, New York, 2004.
- [15] M. Benchohra, J. Henderson, S. K. Ntouyas, and A. Ouahab, *On first order impulsive dynamic equations on time scales*, J. Difference Equ. Appl. **10** (2004), no. 6, 541–548.
- [16] Garrett Birkhoff and Gian-Carlo Rota, *Ordinary differential equations*, Second edition, Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto, Ont.-London, 1969.
- [17] ———, *Ordinary differential equations*, Second edition, Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto, Ont.-London, 1969.
- [18] Martin Bohner and Gusein Guseinov, *Riemann and Lebesgue integration*, Advances in dynamic equations on time scales, Birkhäuser Boston, Boston, MA, 2003, pp. 117–163.
- [19] Martin Bohner and Allan Peterson, *Dynamic equations on time scales*, Birkhäuser Boston Inc., Boston, MA, 2001, An introduction with applications.

- [20] ———, *First and second order linear dynamic equations on time scales*, J. Differ. Equations Appl. **7** (2001), no. 6, 767–792, On the occasion of the 60th birthday of Calvin Ahlbrandt.
- [21] ———, *A survey of exponential function on times scales*, Cubo Mat. Educ. **3** (2001), no. 2, 285–301.
- [22] Martin Bohner and Allan Peterson (eds.), *Advances in dynamic equations on time scales*, Birkhäuser Boston Inc., Boston, MA, 2003.
- [23] Martin Bohner and Christopher C. Tisdell, *Oscillation and nonoscillation of forced second order dynamic equations*, Pacific J. Math. **230** (2007), no. 1, 59–71.
- [24] F. Brauer and J. A. Nohel, *Ordinary differential equations*, W. A. Benjamin, Inc, Sydney, 1973, A first course.
- [25] Fred. Brauer and A Nohel, John, *The qualitative theory of ordinary differential equations: An introduction*, University Mathematics Series, W. A. Benjamin, Inc, New York, 1969.
- [26] R. Creighton Buck, *Advanced calculus*, third ed., McGraw-Hill Book Co., New York, 1978, With the collaboration of Ellen F. Buck, International Series in Pure and Applied Mathematics.
- [27] G. J. Butler, H. I. Freedman, and P. Waltman, *Persistence in dynamical systems models of three-population ecological communities*, Differential equations: qualitative theory, Vol. I, II (Szeged, 1984), Colloq. Math. Soc. János Bolyai, vol. 47, North-Holland, Amsterdam, 1987, pp. 167–178.
- [28] Oscar Castillo and Patricia Melin, *Neural networks, fuzzy logic and genetic algorithms for intelligent control of non-linear dynamical systems*, Differential Equations Dynam. Systems **12** (2004), no. 1-2, 223–258.
- [29] Yong-Kui Chang and Wan-Tong Li, *Existence results for impulsive dynamic equations on time scales with nonlocal initial conditions*, Math. Comput. Modelling **43** (2006), no. 3-4, 377–384.

- [30] Earl A. Coddington, *An introduction to ordinary differential equations*, Prentice-Hall Mathematics Series, Prentice-Hall Inc., Englewood Cliffs, N.J., 1961.
- [31] E. T. Copson, *Metric spaces*, Cambridge Tracts in Mathematics and Mathematical Physics, No.57, Cambridge University Press, London, 1968.
- [32] Qiuyi Dai and Christopher C. Tisdell, *Existence of solutions to first-order dynamic boundary value problems*, Int. J. Difference Equ. **1** (2006), no. 1, 1–17.
- [33] Ondřej Došlý, Stefan Hilger, and Roman Hilscher, *Symplectic dynamic systems*, Advances in dynamic equations on time scales, Birkhäuser Boston, Boston, MA, 2003, pp. 293–334.
- [34] James Dugundji and Andrzej Granas, *Fixed point theory. I*, Monografie Matematyczne [Mathematical Monographs], vol. 61, Państwowe Wydawnictwo Naukowe (PWN), Warsaw, 1982.
- [35] R. E. Edwards, *Functional analysis. Theory and applications*, Holt, Rinehart and Winston, New York, 1965.
- [36] Paul Elloe, *Disconjugacy and higher order dynamic equations*, Advances in dynamic equations on time scales, Birkhäuser Boston, Boston, MA, 2003, pp. 251–274.
- [37] L. Erbe, A. Peterson, and S. H. Saker, *Oscillation criteria for a forced second-order nonlinear dynamic equation*, J. Difference Equ. Appl. **14** (2008), no. 10-11, 997–1009.
- [38] L. Erbe, A. Peterson, and Christopher C. Tisdell, *Basic existence, uniqueness and approximation results for positive solutions to nonlinear dynamic equations on time scales*, Nonlinear Anal. **69** (2008), no. 7, 2303–2317.
- [39] Fengjie Geng, Yancong Xu, and Deming Zhu, *Periodic boundary value problems for first-order impulsive dynamic equations on time scales*, Nonlinear Anal. **69** (2008), no. 11, 4074–4087.

- [40] Kazimierz Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge Studies in Advanced Mathematics, vol. 28, Cambridge University Press, Cambridge, 1990.
- [41] Andrzej Granas and James Dugundji, *Fixed point theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.
- [42] Philip Hartman, *Ordinary differential equations*, John Wiley & Sons Inc., New York, 1964.
- [43] Johnny Henderson, Allan Peterson, and Christopher C. Tisdell, *On the existence and uniqueness of solutions to boundary value problems on time scales*, Adv. Difference Equ. (2004), no. 2, 93–109.
- [44] Johnny Henderson and Christopher C. Tisdell, *Topological transversality and boundary value problems on time scales*, J. Math. Anal. Appl. **289** (2004), no. 1, 110–125.
- [45] Stefan Hilger, *Analysis on measure chains—a unified approach to continuous and discrete calculus*, Results Math. **18** (1990), no. 1-2, 18–56.
- [46] ———, *Differential and difference calculus—unified!*, Proceedings of the Second World Congress of Nonlinear Analysts, Part 5 (Athens, 1996), vol. 30, 1997, pp. 2683–2694.
- [47] Roman Hilscher and Vera Zeidan, *Time scale symplectic systems without normality*, J. Differential Equations **230** (2006).
- [48] ———, *Applications of time scale symplectic systems without normality*, J. Math. Anal. Appl. **340** (2008), no. 1, 451–465.
- [49] S. Jagannathan, F. L. Lewis, and O. Pastravanu, *Discrete-time model reference adaptive control of nonlinear dynamical systems using neural networks*, Internat. J. Control **64** (1996), no. 2, 217–239.
- [50] Herbert B. Keller, *Numerical methods for two-point boundary-value problems*, Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto, Ont.-London, 1968.

- [51] W. Kelly and A. Peterson, *The theory of differential equations classical and qualitative*, Pearson Education, Inc., Upper Saddle River, NJ, 07458, 2004.
- [52] Rahmat Ali Khan, Juan J. Nieto, and V. Otero-Espinar, *Existence and approximation of solution of three-point boundary value problems on time scales*, J. Difference Equ. Appl. **14** (2008), no. 7, 723–736.
- [53] V. Lakshmikantham, D. D. Bařnov, and P. S. Simeonov, *Theory of impulsive differential equations*, Series in Modern Applied Mathematics, vol. 6, World Scientific Publishing Co. Inc., Teaneck, NJ, 1989.
- [54] V. Lakshmikantham, S. Sivasundaram, and B. Kaymakcalan, *Dynamic systems on measure chains*, Mathematics and its Applications, vol. 370, Kluwer Academic Publishers Group, Dordrecht, 1996.
- [55] J. LaSalle, *Uniqueness theorems and successive approximations*, Ann. of Math. (2) **50** (1949), 722–730.
- [56] Tao Li, Lei Guo, Chong Lin, and Changyin Sun, *New results on global asymptotic stability analysis for neural networks with time-varying delays*, Nonlinear Anal. Real World Appl. **10** (2009), no. 1, 554–562.
- [57] Ali Liaqat and Masahide Kimoto, *Applying neural networks to reconstruct nonlinear dynamical systems from observations*, World Conference on 21st Century Mathematics 2004, Sch. Math. Sci. G. C. Univ., Lahore, 2004, pp. 130–139.
- [58] Bingwen Liu, *New convergence behavior of solutions to shunting inhibitory cellular neural networks with unbounded delays and time-varying coefficients*, Appl. Math. Model. **33** (2009), no. 1, 54–60.
- [59] Hongbo Liu and X. Xiang, *A class of the first order impulsive dynamic equations on time scales*, Nonlinear Anal. **69** (2008), no. 9, 2803–2811.
- [60] Jean Mawhin, *Boundary value problems for nonlinear ordinary differential equations: from successive approximations to topology*, Development of mathematics 1900–1950 (Luxembourg, 1992), Birkhäuser, Basel, 1994, pp. 443–477.

- [61] Kirsten Messer, *Second order self-adjoint equations with mixed derivatives*, Advances in dynamic equations on time scales, Birkhäuser Boston, Boston, MA, 2003, pp. 85–116.
- [62] Ihar A. Miklashevich and Viatcheslav V. Barkaline, *Mathematical representations of the dynamics of social system. I. General description*, Chaos Solitons Fractals **23** (2005), no. 1, 195–206.
- [63] Max Müller, *Über das Fundamentaltheorem in der Theorie der gewöhnlichen Differentialgleichungen*, Math. Z. **26** (1927), no. 1, 619–645.
- [64] Donal O'Regan, *Existence theory for nonlinear ordinary differential equations*, Mathematics and its Applications, vol. 398, Kluwer Academic Publishers Group, Dordrecht, 1997.
- [65] A. C. Peterson, Y. N. Raffoul, and C. C. Tisdell, *Three point boundary value problems on time scales*, J. Difference Equ. Appl. **10** (2004), no. 9, 843–849.
- [66] Allan C. Peterson and Christopher C. Tisdell, *Boundedness and uniqueness of solutions to dynamic equations on time scales*, J. Difference Equ. Appl. **10** (2004), no. 13-15, 1295–1306.
- [67] Christian Pötzsche, *Chain rule and invariance principle on measure chains*, J. Comput. Appl. Math. **141** (2002), Dynamic equations on time scales.
- [68] T. Puu (ed.), *Complex dynamics in economic and social systems*, Amsterdam, Elsevier Science Publishers B.V., 1996, Special issue dedicated to Benoit B. Mandelbrot, Chaos Solitons Fractals **7** (1996), no. 12.
- [69] William T. Reid, *Ordinary differential equations*, John Wiley & Sons Inc., New York, 1971.
- [70] Thomas Rogers, *On Nagumo's condition*, Canad. Math. Bull. **15** (1972), 609–611.
- [71] H. L. Royden, *Real analysis*, third ed., Macmillan Publishing Company, New York, 1988.

- [72] Hans Sagan, *Advanced calculus of real-valued functions of a real variable and vector-valued functions of a vector variable*, Houghton Mifflin Co., Boston, Mass., 1974.
- [73] S. H. Saker, *Oscillation and global attractivity of impulsive periodic delay respiratory dynamics model*, Chinese Ann. Math. Ser. B **26** (2005), no. 4, 511–522.
- [74] Peter E. Smouse, *Mathematical models for continuous culture growth dynamics of mixed populations subsisting on a heterogeneous resource base. II. Predation and trophic structure*, Theoret. Population Biol. **20** (1981), no. 2, 127–149.
- [75] J. Stewart, *Calculus*, Brooks/Cole Publishing Company, Pacific Grove, CA, 1991.
- [76] Huajin Tang, Kay Chen Tan, and Zhang Yi, *Neural networks: computational models and applications*, Studies in Computational Intelligence, vol. 53, Springer, Berlin, 2007.
- [77] Sanyi Tang and Lansun Chen, *Density-dependent birth rate, birth pulses and their population dynamic consequences*, J. Math. Biol. **44** (2002), no. 2, 185–199.
- [78] Yu Tian and Weigao Ge, *Existence and uniqueness results for nonlinear first-order three-point boundary value problems on time scales*, Nonlinear Anal. **69** (2008), no. 9, 2833–2842.
- [79] Christopher C. Tisdell, *Existence and uniqueness to nonlinear dynamic boundary value problems*, Cubo **8** (2006), no. 3, 11–24.
- [80] Christopher C. Tisdell, Pavel Drábek, and Johnny Henderson, *Multiple solutions to dynamic equations on time scales*, Comm. Appl. Nonlinear Anal. **11** (2004), no. 4, 25–42.
- [81] Christopher C. Tisdell and H. B. Thompson, *On the existence of solutions to boundary value problems on time scales*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **12** (2005), no. 5, 595–606.

- [82] Christopher C. Tisdell and Atiya Zaidi, *Basic qualitative and quantitative results for solutions to nonlinear, dynamic equations on time scales with an application to economic modelling*, Nonlinear Anal. **68** (2008), no. 11, 3504–3524.
- [83] Christopher C. Tisdell and Atiya H. Zaidi, *Successive approximations to solutions of dynamic equations on time scales*, Comm. Appl. Nonlinear Anal. **16** (2009), no. 1, 61–87.
- [84] Peiguang Wang and Jing Zhang, *Monotone iterative technique for initial-value problems of nonlinear singular discrete systems*, J. Comput. Appl. Math. **221** (2008), no. 1, 158–164.
- [85] Yun Hua Wu, Xiao Feng Yuan, and Shi Ze Liu, *Hopf bifurcations in a class of three-dimensional ecological dynamical systems*, Acta Math. Sci. (Chinese) **18** (1998), no. 1, 70–77.
- [86] Bao Qiang Yan, Donal O'Regan, and Ravi P. Agarwal, *Positive solutions to singular boundary value problems with sign changing nonlinearities on the half-line via upper and lower solutions*, Acta Math. Sin. (Engl. Ser.) **23** (2007), no. 8, 1447–1456.
- [87] Atiya H. Zaidi, *Existence and convergence results for solutions to dynamic initial value problems using lower and upper solutions*, (A pre-print).
- [88] Eberhard Zeidler, *Nonlinear functional analysis and its applications. I*, Springer-Verlag, New York, 1986, Fixed-point theorems, Translated from the German by Peter R. Wadsack.
- [89] B. G. Zhang and Xinghua Deng, *Oscillation of delay differential equations on time scales*, Math. Comput. Modelling **36** (2002), no. 11–13, 1307–1318.
- [90] Hong Zhang, Lansun Chen, and Juan J. Nieto, *A delayed epidemic model with stage-structure and pulses for pest management strategy*, Nonlinear Anal. Real World Appl. **9** (2008), no. 4, 1714–1726.

