Implied volatility: general properties and asymptotics

Author:
Roper, Michael Paul Veran

Publication Date:
2009

DOI:
https://doi.org/10.26190/unsworks/21796

Implied Volatility: General Properties and Asymptotics

October 14, 2009

A thesis presented to

The University of New South Wales

in fulfilment of the thesis requirement
for the degree of

Doctor of Philosophy

by

MICHAEL PAUL VERAN ROPER

For Gail

PLEASE TYPE

THE UNIVERSITY OF NEW SOUTH WALES
Thesis/Dissertation Sheet

Surname or Family name: Roper

First name: Michael

Other name/s: Paul Veran

Abbreviation for degree as given in the University calendar:

School: Mathematics and Statistics

Faculty: Science

Title: Implied Volatility: General Properties and Asymptotics

Abstract 350 words maximum: (PLEASE TYPE)

This thesis investigates implied volatility in general classes of stock price models.

To begin with, we take a very general view. We find that implied volatility is always, everywhere, and for every expiry well-defined only if the stock price is a non-negative martingale. We also derive sufficient and close to necessary conditions for an implied volatility surface to be free from static arbitrage. In this context, free from static arbitrage means that the call price surface generated by the implied volatility surface is free from static arbitrage.

We also investigate the small time to expiry behaviour of implied volatility. We do this in almost complete generality, assuming only that the call price surface is non-decreasing and right continuous in time to expiry and that the call surface satisfies the no-arbitrage bounds $(S-K)^+ \leq C(K, \tau) \leq S$. We used S to denote the current stock price, K to be a option strike price, τ denotes time to expiry, and $C(K, \tau)$ the price of the K strike option expiring in τ time units. Under these weak assumptions, we obtain exact asymptotic formulae relating the call price surface and the implied volatility surface close to expiry.

We apply our general asymptotic formulae to determining the small time to expiry behaviour of implied volatility in a variety of models. We consider exponential Lévy models, obtaining new and somewhat surprising results. We then investigate the behaviour close to expiry of stochastic volatility models in the at-the-money case. Our results generalise what is already known and by a novel method of proof. In the not at-the-money case, we consider local volatility models using classical results of Varadhan. In obtaining the asymptotics for local volatility models, we use a representation of the European call as an integral over time to expiry. We devote an entire chapter to representations of the European call option; a key role is played by local time and the argument of Klebaner. A novel alternative that is especially useful in the local volatility case is also presented.

COPYRIGHT STATEMENT

'I hereby grant the University of New South Wales or its agents the right to archive and to make available my thesis or dissertation in whole or part in the University libraries in all forms of media, now or here after known, subject to the provisions of the Copyright Act 1968. I retain all proprietary rights, such as patent rights. I also retain the right to use in future works (such as articles or books) all or part of this thesis or dissertation.
I also authorise University Microfilms to use the 350 word abstract of my thesis in Dissertation Abstract International (this is applicable to doctoral theses only).
I have either used no substantial portions of copyright material in my thesis or I have obtained permission to use copyright material; where permission has not been granted I have applied/will apply for a partial restriction of the digital copy of my thesis or dissertation.'

Signed ....

Date ....14/10/09..........................

AUTHENTICITY STATEMENT

'I certify that the Library deposit digital copy is a direct equivalent of the final officially approved version of my thesis. No emendation of content has occurred and if there are any minor variations in formatting, they are the result of the conversion to digital format.'

Signed ....

Date ....14/10/09..........................

ORIGINALITY STATEMENT

'I hereby declare that this submission is my own work and to the best of my knowledge it contains no materials previously published or written by another person, or substantial proportions of material which have been accepted for the award of any other degree or diploma at UNSW or any other educational institution, except where due acknowledgement is made in the thesis. Any contribution made to the research by others, with whom I have worked at UNSW or elsewhere, is explicitly acknowledged in the thesis. I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in the project's design and conception or in style, presentation and linguistic expression is acknowledged.'

Signed ....

Date ....14/10/09....................................

Publications

Chapter 7, entitled "On the Relationship Between the Call Price Surface and the Implied Volatility Surface Close to Expiry", is a reorganised and edited version of the following paper:

M. Roper and M. Rutkowski, *On the Relationship Between the Call Price Surface and the Implied Volatility Surface Close to Expiry*, International Journal of Theoretical and Applied Finance, Vol. 12, No. 4 (2009), 427–441.

In particular, the following changes to the paper have been made:

1. It has been reorganised so that it has the same structure as the other chapters of the thesis.
2. Remark 7.3.3 has been added to this thesis.
3. "Log-moneyness", x, which is given by $\ln(S/K)$ in the paper is changed to $\ln(K/S)$ in the thesis for the sake of consistency.
4. A proof of the claim in Remark 6.1 in the paper is presented in Proposition A.7.1 starting on page 147 of the Appendix to this thesis. See also Remark 7.3.9 in this thesis.

Abstract

This thesis investigates implied volatility in general classes of stock price models. To begin with, we take a very general view. We find that implied volatility is always, everywhere, and for every expiry well-defined only if the stock price is a non-negative martingale. We also derive sufficient and close to necessary conditions for an implied volatility surface to be free from static arbitrage. In this context, free from static arbitrage means that the call price surface generated by the implied volatility surface is free from static arbitrage.

We also investigate the small time to expiry behaviour of implied volatility. We do this in almost complete generality, assuming only that the call price surface is non-decreasing and right continuous in time to expiry and that the call surface satisfies the no-arbitrage bounds $(S-K)^+ \leq C(K,\tau) \leq S$. We used S to denote the current stock price, K to be a option strike price, τ denotes time to expiry, and $C(K,\tau)$ the price of the K strike option expiring in τ time units. Under these weak assumptions, we obtain exact asymptotic formulae relating the call price surface and the implied volatility surface close to expiry.

We apply our general asymptotic formulae to determining the small time to expiry behaviour of implied volatility in a variety of models. We consider exponential Lévy models, obtaining new and somewhat surprising results. We then investigate the behaviour close to expiry of stochastic volatility models in the at-the-money case. Our results generalise what is already known and by a novel method of proof. In the not at-the-money case, we consider local volatility models using classical results of Varadhan. In obtaining the asymptotics for local volatility models, we use a representation of the European call as an integral over time to expiry. We devote an entire chapter to representations of the European call option; a key role is played by local time and the argument of Klebaner (see

(Kle02)). A novel alternative that is especially useful in the local volatility case is also presented.

Acknowledgements

I would like to take this opportunity to thank just a few of the people who have helped me write this thesis.

By far the greatest thanks must go to my supervisor, Assoc. Prof. Ben Goldys, for providing me with an interesting and challenging thesis topic and for always being happy to find time in his busy schedule to meet with me to answer my questions, no matter how trivial.

I am also especially grateful to my co-supervisor Assoc. Prof. Marek Rutkowski. My thesis would be much the poorer were it not for his enthusiastic help, interest, reading, commenting, and advice.

Thanks are due to Dr. Donna Salopek, Dr. Ilya Pavlyukevich, and Prof. Anna Chojnowska-Michalik for providing helpful criticism.

My fiancée Gail, my brother, Hudson, and my parents Duncan and Sonya have provided me with the greatest love, support, and encouragement.

I owe my friends and fellow students Ben Kaehler, Nathan Pearce, Jan Baldeaux, Kenneth Chan, and Greg Londish for their help and friendship over the time I have been doing my PhD.

I owe a special debt of gratitude to Nathan Pearce for his help with Latex. Finally, I give thanks to the Lord Jesus Christ for helping me along the road.

Contents

Chapter 1

Introduction

We start with the classical Black-Scholes model for the price of a stock

$$dS_t = \mu S_t \, \mathrm{d}t + \sigma S_t \, \mathrm{d}W_t, \tag{1.1}$$

where μ is the drift of the stock price process and σ is its volatility. Under the assumptions that the model (1.1) holds and that both μ and σ are constants, the price of a European call option on the stock is given by the Black-Scholes call pricing formula

$$C^{BS}(K, \sigma, \tau; S_t, t, r) = S_t \Phi\left(d_1\right) - K e^{-r\tau} \Phi\left(d_2\right),$$

where $K > 0$ is the strike price of the call, $t + \tau$ is the maturity time of the call, S_t is the current stock price, σ is the Black-Scholes volatility of the stock, r is the risk-neutral interest rate, Φ is the standard Normal cumulative distribution function, and

$$d_1 = \frac{\ln(S_t/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

and

$$d_2 = d_1 - \sigma\sqrt{\tau}.$$

It was the genius of Black and Scholes to show that the call option pricing equation should not include the stock drift μ but instead the so called risk-neutral interest rate, r.

Today there is a reasonably liquid market in European calls. We still concern ourselves with the Black-Scholes setup, however, because market participants relate the prices of European options to their Black-Scholes price. Indeed, since C^{BS} is strictly monotonic in σ and ("usually") the observed call price is bounded above by S_t and below by $(S_t - K)^+$, one can solve

$$C^{BS}(K, \sigma, \tau; S_t, t) = C^{obs}(K, \tau; S_t, t) \tag{1.2}$$

for σ – now termed the implied volatility. The C^{obs} term is, of course, the observed market price of the European call. It is well known that the market implied volatilities of the calls vary over time, time to maturity, and strike price. This is contrary to the Black-Scholes setup: if the Black-Scholes model was correct, then the implied volatility surface would be a constant for all times, maturities, and strikes.

As we have noted, implied volatility may vary over K, τ and t. We shall denote the implied volatility – the solution to Equation (1.2) – as $\Sigma_t(K, \tau)$ which we term the implied volatility surface at time t. (We are assuming that the risk free interest rate and dividend yield of the stock are zero.)

There are two main proposed solutions to the manifest failure of the Black-Scholes model. Firstly, one may develop a more complicated stock price model that includes stochastic volatility and/or jumps. A problem with these models is that they are non-trivial to calibrate to the market, for example because the volatility process is unobserved. Secondly, and gaining in popularity, are market models of the entire (stochastic) implied volatility surface. There is a growing literature on the construction of such market models of implied volatility: these have been termed Stochastic Implied Volatility (SIV) models (see (AFV01), (BFG07), (BGKW01), (BGvdHW02), (LSC98), (Sch99), (SW08), and (Wis08)). SIV models are "market models" in the sense that the implied volatility surface is modelled directly. They therefore share some features of market models of the term structure of interest rates (see, for example, (BGM97)). SIV modelling is more difficult, however, because consistency over strikes and maturities is required while term structure models need concern themselves only with consistency over maturities. A well functioning SIV model should facilitate the model calibration and also, according to Schönbucher (see (Sch99)), give a SIV model a large degree of flexibility. In this thesis, we solve some open questions in the theory of SIV modelling.

In addition to SIV modelling, the study of implied volatility is of interest in mathematical finance for a number of other reasons. Firstly, it is very widely used by practitioners in options markets. As observed by Lee (see (Lee05)), implied volatility provides a useful, industry standard "language" to describe the state of the options market and allows options to be "compared across different strikes, maturities, and underlyings". For example, under a non-negative martingale model for the stock price, the implied volatility of a European call and put with the same strike and expiry have the same implied volatility by put-call parity (see (FPS00)).

In this thesis, we will mostly be concerned with the instantaneous properties of the call price and implied volatility surfaces. In these cases, we may just write $C(K,\tau)$ and $\Sigma(K,\tau)$ for the call and implied volatility surfaces respectively.

In what follows, we describe the content of this thesis and the main results obtained in each chapter. Chapter 2 presents some notation and terminology used throughout the thesis.

Chapter 3 addresses various preliminary issues such as the mathematical and probabilistic setups. It should be noted, however, that each chapter has a somewhat different setup.

Chapter 4 details a number of reasonably well-known properties of non-negative local martingales that come in useful in later chapters.

Chapter 5 presents our first contribution. If we take as given a filtered probability space with filtration $\mathbb{F}$ and suppose that a non-negative, $\mathbb{F}$-adapted process X has $\mathbb{E}\left(X_t\right) < \infty$ for all $t \geq 0$, then X is a martingale if and only if for every $0 \leq u \leq t$

$$(\forall K > 0) \quad (X_u - K)^+ \leq \mathbb{E}\left((X_t - K)^+ | \mathscr{F}_u\right) \leq X_u, \quad \mathbb{P}\text{-a.s.}$$

We use this to show that the largest class of non-negative local martingales in which implied volatility makes sense is the class of non-negative martingales. This result is of practical significance for developing stochastic volatility models because there are many stochastic volatility models in the literature that are strictly local martingales (see (AP07) and (LM06)). Moreover, some work in the literature on small time to maturity asymptotics of implied volatility allows for local martingale models.

Chapter 6 presents the solution to an open problem in SIV modelling (see (SIV.3) on p. 270 of (MR05)). The problem is to describe conditions on a candidate initial "implied volatility surface" for it to be arbitrage free, in the sense that the corresponding candidate "call price surface" is free from static arbitrage. We term a call price surface, write it $C(K,\tau)$ with $K,\tau \geq 0$, free from static arbitrage if there exists a non-negative martingale model such that $C(K,\tau) = \mathbb{E}\left((X_\tau - K)^+\right)$ for every $K,\tau \geq 0$. In this chapter, we give a set of sufficient conditions on a candidate "implied volatility surface" for it to be "arbitrage free" and show that these conditions are close to being necessary. Conditions of this type are sometimes termed "static properties" (see (HHK07)). Aside from some technical conditions on the implied volatility surface, the sufficient conditions we presented have all appeared in the literature as necessary conditions for the surface to be free from static arbitrage. As an application of these static conditions we check a number of proposed implied volatility smile parameterisations for the arbitrage free property. None of those tested were free from arbitrage. The most difficult property to satisfy appears to be a condition of Durrleman (see (Dur03)) which ensures that the corresponding call price surface for fixed maturity is convex in the strike. As Lee (see (Lee05)) observes, there are a number of other applications of knowing these static conditions. Lee notes that in the first place, if the implied volatility surfaces generated by a model differ markedly from those that are observed in the market, then this is an indication of model misspecification. Secondly, Lee notes that, if there are analytical expressions approximating some regions of the implied volatility, for instance small times to maturities, then this may facilitate model calibration to the implied volatility smile. Finally, Lee remarks that if a model violates any of the necessary conditions on the implied volatility surface, then this can also indicate model misspecification.

In Chapter 7, we relate the call price surface and the implied volatility surface close to maturity. In particular, we derive exact asymptotic formulae for the implied volatility at a fixed strike and the call option price at that same strike in the small time to expiry limit. There are two markedly different asymptotic formulae: one for the at-the-money case $S_t = K$, and the other for the not at-the-money case $S_t \neq K$. We also show that there exist arbitrage free models for the stock price in which the implied volatility (at fixed strike) does not converge to any finite or infinite value as time to expiry goes to zero. With respect to the asymptotic formulae derived in Chapter 7, it should be noted that they are "model free" in the sense that they are not derived using any particular postulated model. Instead, it is assumed that at some fixed time, for convenience zero, and for some fixed positive stock price, S_0, the call pricing function at any fixed $K > 0$ satisfies the three conditions:

(1) $(S_0 - K)^+ \leq C(K, \tau) \leq S_0$.

(2) $\tau \mapsto C(K, \tau)$ is non-decreasing.

(3) $\lim_{\tau \to 0^+} C(K, \tau) = (S_0 - K)^+$.

The first two conditions hold for any non-negative martingale. We can establish the third under some weak technical assumptions. The results of this chapter make the study of implied volatility close to expiry much easier by translating it into the study of certain, rather simple functions of the call option price. Most of the rest of this thesis applies this result to obtaining small time to expiry asymptotics of implied volatility for various classes of models. From a practical point of view, the use of asymptotic methods is well justified since the expiry time of traded options is typically small. Finally, the study of small expiry asymptotics is also of some theoretical interest since the implied volatility surface close to expiry for models with jumps is markedly different to those without jumps. Another contribution of this chapter is to SIV modelling: we establish a condition under which $\sqrt{\tau}\Sigma(K,\tau)$ goes to zero as $\tau \to 0^+$; here $\Sigma(K,\tau)$ is the implied volatility at strike K and time to expiry τ.

In Chapter 8, we present a study of the small time to expiry behaviour of implied volatility in exponential Lévy models. By an exponential Lévy model, we mean a non-negative martingale S with representation

$$S_t = S_0 e^{X_t}, \quad S_0 > 0, t \geq 0$$

where X is a Lévy process constrained so that S is a martingale. We obtain the somewhat surprising result that in almost all models of exponential Lévy type that are of interest in finance

$$\lim_{\tau \to 0^+} \Sigma_t(K,\tau) = \begin{cases} \infty, & \text{if } S_t \neq K \\ \sigma, & \text{if } S_t = K, \end{cases}$$

where σ^2 is the Gaussian part of the characteristic triplet of X and may be zero. The at-the-money limit holds for any (martingale) exponential Lévy model. The range of Ks for which the not at-the-money limit holds depends on how the Lévy measure is distributed on $\mathbb{R}$. For most models used in finance, however, the Lévy measure has a density which is strictly positive for all $x \in \mathbb{R} \setminus \{0\}$ and the not at-the-money limit is indeed infinity. Now, combine the results of this chapter on the limit of implied volatility as expiry goes to zero, with the example in Chapter 7 of a model with non-convergent implied volatility. With these different models we see that implied volatility can converge to any non-negative real number, infinity, or not converge at all.

Chapter 9 is devoted to the small time to expiry asymptotics of implied volatility in the at-the-money case. The class of models considered are those that may be represented by

$$S_\tau = S_0 + \int_0^\tau \sigma_u S_u \, \mathrm{d}W_u, \quad S_0 > 0, \quad \tau \geq 0,$$

where S_0 is a positive, finite constant. It has been known, since at least 1998 (see (LSC98)), that in diffusion type models the at-the-money implied volatility converges to the instantaneous spot volatility as time to expiry goes to zero. That is

$$\lim_{\tau \to 0^+} \Sigma_t(S_t, \tau) = \sigma_t.$$

The first rigorous proof in the literature was Durrleman's paper (Dur08). In (Dur08), Durrleman considered a model with a volatility process bounded above and below by strictly positive constants and also including a jump component. Our analysis in this chapter does not allow for jumps, but we are able to significantly improve Durrleman's conditions on the volatility process.

The major contribution of Chapter 10 to this thesis is the representation of the call price as an integral over time to maturity. This is a key component in our derivation of the small time behaviour of implied volatilities in diffusion type models in the final chapter. The representation we use for a one-dimensional, non-negative martingale diffusion, say

$$\mathrm{d}S_t = \sigma(S_t, t) \, \mathrm{d}W_t,$$

is

$$\mathbb{E}\left((S_\tau - K)^+\right) = (S_0 - K)^+ + \frac{1}{2}\int_0^\tau \sigma^2(K, u) p(0, S, u, K) \, \mathrm{d}u,$$

where p is the transition density for the model. The other representation we obtain in Chapter 10 is

$$\begin{aligned} \mathbb{E}\left((S_\tau - K)^+\right) &= (S_0 - K)^+ + \frac{1}{2}\mathbb{E}\left(L_\tau^K\right) \\ &= (S_0 - K)^+ + \frac{1}{2}\int_0^\tau \mathbb{E}\left(\sigma_t^2 | S_t = K\right) f(t, K) \, \mathrm{d}t. \end{aligned} \tag{1.3}$$

The first representation is a result of Madan and Yor (see (MY06) or Theorem 10.1.3). We will derive the second result using Klebaner's local time argument (see (Kle02)). It is assumed throughout this chapter that S arises from a diffusion process model and L^K is the local time of S at K. Using a result of Madan and Yor (see (MY06)), we extend the class of martingales, S, for which Equation (1.3) is satisfied (see (Kle02)), has been shown to hold. Klebaner required S to be in $\mathscr{H}^1$. (Recall that a continuous martingale, M, is in $\mathscr{H}^1$ when $\mathbb{E}\left(\sqrt{\langle M\rangle_\infty}\right) < \infty$ (see (Pro04)). The other representations of $\mathbb{E}\left((S_\tau - K)^+\right)$ obtained in Chapter 10 use a novel smoothing method and is essential analytic. It allows us to easily handle diffusions with discontinuous coefficients. It is also shown that the formula remains unchanged, in the one-dimensional case, if the diffusion has a law with an atom at zero – such as the Constant Elasticity of Variance (CEV) model:

$$\mathrm{d}S_t = S_t^\beta \, \mathrm{d}W_t, \quad S_0 > 0$$

where $\beta \in (0,1)$ and an absorbing boundary condition is imposed at zero.
In Chapter 11, we investigate the close to expiry limit of implied volatility in a class of local volatility models. We obtain the same results as Berestycki *et al.* (see (BBF02)). However, Berestycki *et al.* use a more complex proof, while we use the well known behaviour of transition densities presented in (Var67a). Our method of obtaining the limiting at-the-money implied volatility is different to that of Berestycki *et al.*, but has the virtue of being simpler. We do not cover all the cases handled by Berestycki *et al.* in (BBF02).

Chapter 2

Notation and Terminology

Notation 2.0.1. $x \wedge y = \min(x, y)$ for $x, y \in \mathbb{R}$.

Notation 2.0.2. $x^+ = \max(x, 0)$ for $x \in \mathbb{R}$.

Notation 2.0.3. We take $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Notation 2.0.4. $\mathbb{1}_A(x) = 1$ if $x \in A$ and zero otherwise.

Notation 2.0.5. We generally denote partial differentiation as ∂_x instead of $\dfrac{\partial}{\partial x}$.

Notation 2.0.6. (Landau notation) Let $a \in [-\infty, \infty]$. Then $f(x) \sim g(x)$ means $f(x)/g(x) \to 1$ as $x \to a$, $f(x) = \mathrm{O}(g(x))$ as $x \to a$ means that there is a neighbourhood of a and a constant $\alpha > 0$ such that $|f(x)| \leq \alpha |g(x)|$ in this neighbourhood. Finally, $f(x) = \mathrm{o}(g(x))$ when $f(x)/g(x) \to 0$ as $x \to a$. Obvious adjustments are made for one sided limits. See (Olv97), for an introduction to the Landau notation.

Definition 2.0.7. Let

$$\begin{aligned} \phi : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \frac{\exp(-x^2/2)}{\sqrt{2\pi}} \end{aligned}$$

denote the standard normal density.

Definition 2.0.8. Let

$$\begin{aligned} \Phi : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \int_{-\infty}^{x} \phi(y)\,\mathrm{d}y, \end{aligned}$$

denote the standard normal cumulative distribution function.

Definition 2.0.9. Let erf denote the Error function. It is defined as

$$\begin{aligned} \mathrm{erf} : \mathbb{R} &\to \mathbb{R} \\ \theta &\mapsto \frac{2}{\sqrt{\pi}} \int_0^{\theta} \exp(-t^2)\,\mathrm{d}t \end{aligned}$$

See (Olv97) and Lemma A.5.1 starting on page 140 in the Appendix for the definition and properties of the Error function.

Definition 2.0.10. We use the Upper Incomplete Gamma function. It is defined by

$$\begin{aligned}\Gamma : \mathbb{R} \times (0, \infty) &\to \mathbb{R}\\ (a, z) &\mapsto \int_z^\infty t^{a-1} e^{-t} \, \mathrm{d}t.\end{aligned}$$

See (Olv97) and Lemma A.5.3 starting on page 140 in the Appendix for the definition and properties of the Upper Incomplete Gamma function.

Chapter 3

Preliminaries

In this chapter we describe the market and probabilistic setups for this thesis, we also define implied volatility.

3.1 Background

3.1.1 Setup

Suppose we have a filtered probability space, $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. We do not always assume that the filtration satisfies the usual conditions. By the "usual conditions" we mean that the filtration is right-continuous and $\mathscr{F}_0$ contains all the $\mathbb{P}$-negligible events in $\mathscr{F}$ (see (KS91), p. 10). We assume that the stock price process on this space is always (at least) a non-negative local martingale started at a strictly positive constant. However, we do not always assume that the stock price process is càdlàg, i.e. right-continuous on $[0, \infty)$ with finite left hand limits on $(0, \infty)$ (see (KS91), p. 4). Our motivation for these weak hypotheses comes from our desire to construct martingale processes given an instantaneous call price surface.

In our model of the financial market, we have zero interest rates, zero dividend yield, and no transaction costs.

In addition, we suppose that the probability measure is the pricing measure, so that the call price is given by the conditional expectation of the call payoff under this measure.

3.2 Stochastic Analysis

When we work with conditional expectations, we mean a version of the conditional expectation.

Definition 3.2.1. Let X be a process on the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. We say that X is integrable if $\mathbb{E}\left(|X_t|\right) < \infty$ for every $t \geq 0$.

Definition 3.2.2. Let X be a process on the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. We say that X is square integrable if $\mathbb{E}\left(X_t^2\right) < \infty$ for every $t \geq 0$.

3.3 Implied Volatility

In order to define implied volatility we need the following definitions and notation.

3.3.1 Definitions, Notation and Basic Facts

We define the Black-Scholes call pricing function as follows.

Definition 3.3.1. The Black-Scholes price of a European call option with strike K, time to expiry τ, and volatility σ is given by

$$C^{BS} : (0,\infty) \times [0,\infty) \times [0,\infty] \to [0,\infty)$$

$$(K,\tau,\sigma) \mapsto \begin{cases} s, & \text{if } \sigma\sqrt{\tau} = \infty \\ s\Phi\left(\dfrac{-\ln(K/S)}{\sigma\sqrt{\tau}} + \dfrac{1}{2}\sigma\sqrt{\tau}\right) & \\ \qquad -K\Phi\left(\dfrac{-\ln(K/s)}{\sigma\sqrt{\tau}} - \dfrac{1}{2}\sigma\sqrt{\tau}\right), & \text{if } 0 < \sigma\tau < \infty, \\ (s-K)^+, & \text{if } \sigma\sqrt{\tau} = 0, \end{cases}$$

where $s > 0$ is the current stock price.

We observe that in the definition of C^{BS} above, the case $K = 0$ is excluded. This is to avoid unnecessary technicalities. As a flow on we will work with a definition of implied volatility which is only well-defined for $K \in (0,\infty)$. We will "fill in" the missing K, i.e. $K = 0$, later on when required.

We are now in a position to define implied volatility. We will give two definitions; in a certain sense they are equivalent. One may refer to Lee (Lee05), (Lee04), or Musiela and Rutkowski (MR05).

When we work with an instantaneous snapshot of the call price surface and the stock price, then it is natural to think of the call price surface as a deterministic function and the stock price as a constant. Obviously, the call price surface must satisfy certain conditions, but these will be developed later on (see, in particular, Chapters 4 and 6). We call this the deterministic setup.

Probabilistically, the more natural viewpoint is to think of S as a process and the call price surface as a family of conditional expectations of a certain function of a non-negative (possibly local) martingale S. Indeed, at any fixed time $t \geq 0$ the call price surface is given by

$$(K, \tau) \mapsto \mathbb{E}\left((S_{t+\tau} - K)^+ | \mathscr{F}_t\right)$$

and the instantaneous stock price is just the $\mathscr{F}_t$-measurable random variable S_t. We term this the probabilistic setup.

First, we provide further details on the deterministic setup.

Deterministic Setup

Definition 3.3.2. Let $s \geq 0$ be fixed and the call price surface be the function

$$\begin{aligned} C : (0, \infty) \times [0, \infty) &\to \mathbb{R} \\ (K, \tau) &\mapsto C(K, \tau). \end{aligned}$$

Suppose that for all $K > 0$ and $\tau \geq 0$, $(s - K)^+ \leq C(K, \tau) < s$. Then, if $s > 0$, the *implied volatility* corresponding to C and s is the function

$$\Sigma : (0, \infty) \times [0, \infty) \to [0, \infty]$$

defined implicitly by

$$C(K, \tau) = C^{BS}(K, \tau, \Sigma),$$

for $K > 0$. It is well-defined since the Black-Scholes pricing function is strictly increasing with respect to the volatility. If $s = 0$, then the *implied volatility* corresponding to C and s is defined *explicitly* to be identically zero. This later case is not interesting since a non-negative local martingale is absorbed at zero if it ever hits zero (see 4.3.2).

Probabilistic Setup

Definition 3.3.3. Fix $t \geq 0$. If $S_t > 0$, then the time t *implied volatility surface* corresponding to the call price surface, $(\tau, K) \mapsto \mathbb{E}\left((S_{t+\tau} - K)^+ | \mathscr{F}_t\right)$, for $\tau \geq 0$ and $K > 0$ is given pointwise (in K and τ) as the solution of

$$\mathbb{E}\left((S_{t+\tau} - K)^+ \middle| \mathscr{F}_t\right) = C^{BS}(K, \tau, \Sigma_t(K, \tau)), \tag{3.1}$$

for Σ_t. If $S_t = 0$, then *implied volatility* is defined *explicitly* to be identically zero for every τ and K. We thus have that the implied volatility surface is a map

$$\begin{aligned}\Sigma : [0,\infty) \times (0,\infty) \times [0,\infty) \times \Omega &\to [0,\infty] \\ (t, K, \tau, \omega) &\mapsto \Sigma_t(K,\tau)(\omega),\end{aligned}$$

defined as the solution to Equation (3.1). We emphasise again that K must be strictly positive.

Remark 3.3.4. We will frequently write $\Sigma(K,\tau)$, i.e. omitting the arguments t and ω. There is no lack of clarity in omitting the time argument since we are mainly interested in the instantaneous implied volatility surface. The dependence on $\omega \in \Omega$ is clear from the context. As we are taking a version of the conditional expectation in the definition of Σ, it should be clear that we take a version of the implied volatility as well.

Chapter 4

Properties of Non-negative, Local Martingales

In this chapter, we collect a number of basic facts about non-negative, local martingales and also the conditional expectation of the call option payoff when the stock is such a process. It is expected that these facts are all well known, although we were unable to locate all of them in the literature. One may refer, however, to Cox and Hobson ((CH05)) for many of the properties. It should be noted that we are careful to identify which of our results require the local martingale to be right continuous or whether it is on a filtration satisfying the usual conditions. This is motivated by the fact that the martingales we construct in Chapter 6 need not satisfy these conditions.

The chapter is organised as follows. In Section 4.1, we describe the mathematical setup of the chapter. Section 4.2 provides some facts about a class of convex functions that are closely related to the call option price at fixed times. Section 4.3 gives the main results: some basic properties of non-negative local martingales and then properties of the conditional expectation of the call option payoff.

4.1 Background

4.1.1 Setup

We will use a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ on which we define our stock price process, S. It is a non-negative local martingale unless stated otherwise. In this chapter, unless specifically stated, we assume in general neither the usual conditions on the filtration nor that S is càdlàg. However, we assume that S is a non-negative process and that the initial value of S is a finite and non-negative constant. We will have a need for these weaker assumptions on S and its filtration in Chapter 6 when we look at arbitrage free surfaces.

4.2 Lemmas

The motivation behind the following Lemma is that the map

$$K \mapsto \mathbb{E}\left((S_{t+\tau} - K)^+ | \mathscr{F}_t\right),$$

where $t, \tau \geq 0$ are fixed, satisfies the properties (i)-(iv) of the following lemma for $K > 0$.

Lemma 4.2.1. *Let $f : (0, \infty) \to \mathbb{R}$ be a function that admits a continuous extension, call it g, to $[0, \infty)$. We consider the following conditions on f.*

(i) f is convex;

(ii) f is non-increasing;

(iii) $f(x) \to 0$ as $x \to \infty$; and

(iv) for some $a \in [0, \infty)$, $(a - x)^+ \leq f(x) \leq a$ for every $x > 0$.

Then

(i′) if f satisfies (i), then g is convex (on $[0, \infty)$);

(ii′) if f satisfies (ii), then g is non-increasing (on $[0, \infty)$);

(iii′) if f satisfies (iii), then $g(x) \to 0$ as $x \to \infty$;

(iv′) if f satisfies (iv), then $(g(0) - x)^+ = (a - x)^+ \leq g(x) \leq a = g(0)$ for every $x \geq 0$;

(v′) if f satisfies (iv), then $a = g(0) = \lim_{x \to 0^+} g(x)$;

(vi′) if f satisfies (i), (ii), and (iv), then the right-hand derivative of g, write it g'_+, exists, and is non-decreasing and right-continuous on $x \geq 0$;

(vii′) if f satisfies (i), (ii), and (iv), then for every $x \geq 0$, $-1 \leq g'_+(x) \leq 0$; and

(viii′) if f satisfies (i), (ii), and (iv), then for every $x > 0$, $-1 \leq \dfrac{g(x) - g(0)}{x} \leq g'_+(x) \leq 0$.

Proof. See the Appendix starting from page 142. ☐

4.3 Main Results

4.3.1 Process Properties

Proposition 4.3.1. *(Super-martingale property) Recall that S is a non-negative, local martingale. Then S is a super-martingale and, of course, $\mathbb{E}\left(S_t|\mathscr{F}_u\right) < \infty$ for all $0 \leq u \leq t$.*

Proof. Let (T_n) be a localising sequence for S and fix $0 \leq u \leq t$. Since S is non-negative we may apply the conditional version of Fatou's Lemma to get that

$$\begin{aligned}
\mathbb{E}\left(S_t|\mathscr{F}_u\right) &= \mathbb{E}\left(\liminf_{n\to\infty} S_t^{T_n}|\mathscr{F}_u\right) \\
&\leq \liminf_{n\to\infty} \mathbb{E}\left(S_t^{T_n}|\mathscr{F}_u\right) \\
&= \liminf_{n\to\infty} S_u \\
&= S_u \\
&< \infty
\end{aligned}$$

$\mathbb{P}$-a.s. The last step holds because S_u must be finite $\mathbb{P}$-a.s. since $\mathbb{E}\left(S_u\right) < \infty$ and S is non-negative. □

Corollary 4.3.2. *(Absorption at zero) Let S be a non-negative local martingale. If $S_t = 0$ for some $t \geq 0$, then $S_{t+\tau} = 0$ for all $\tau \geq 0$, $\mathbb{P}$-a.s.*

Proof. Let $S_t = 0$ and $t \geq 0$. From Proposition 4.3.1, S is a super-martingale, therefore $\mathbb{E}\left(S_{t+\tau}|\mathscr{F}_t\right) \leq S_t = 0$ for all $\tau \geq 0$, $\mathbb{P}$-a.s. Recall that S is non-negative, therefore $S_{t+\tau} = 0$ for all $\tau \geq 0$, $\mathbb{P}$-a.s. □

Remark 4.3.3. This property of zero absorption means that studying the properties of the implied volatility surface when the stock price is zero is entirely trivial.

4.3.2 Conditional Expectation Properties

We now recall some properties of the conditional expectation of the European call option payoff in non-negative local martingale models.

Proposition 4.3.4. *Let S be a non-negative local martingale. For fixed $t, \tau \geq 0$ let*

$$\begin{aligned} C : [0,\infty) &\to \mathbb{R} \\ K &\mapsto \mathbb{E}\left((S_{t+\tau} - K)^+)|\mathscr{F}_t\right). \end{aligned}$$

Then, for every $t, \tau \geq 0$

(M1) C is convex on $[0,\infty)$, $\mathbb{P}$-a.s.

(M2) C is non-increasing on $[0,\infty)$, $\mathbb{P}$-a.s.

(M3) If S is a martingale, then for each $K \geq 0$

$$\widehat{\tau} \mapsto \mathbb{E}\left((S_{t+\widehat{\tau}} - K)^+|\mathscr{F}_t\right) \text{ is non-decreasing,}$$

$\mathbb{P}$*-a.s.*

(M4) We have the "large strike" limit

$$\lim_{K\to\infty} C(K) = 0,$$

$\mathbb{P}$*-a.s.*

(M5) For each $K \geq 0$

$$(C(0) - K)^+ \leq C(K) \leq C(0)\ (\leq S_t),$$

and, in particular, when S is a martingale,

$$(S_t - K)^+ \leq C(K) \leq S_t, \tag{4.1}$$

both $\mathbb{P}$*-a.s.*

(M6) For each $K \geq 0$

$$\mathbb{E}\left((S_t - K)^+|\mathscr{F}_t\right) = (S_t - K)^+,$$

$\mathbb{P}$*-a.s.*

(M7) We have the "small strike" limit

$$\lim_{K\to 0^+} C(K) = C(0),$$

$\mathbb{P}$*-a.s.*

Proof.

The reader may fill in "$\mathbb{P}$-a.s." as is required, Fix $t \geq 0$ and $\tau \geq 0$.

(M1) For each $0 \leq \alpha \leq 1$ and $K_1, K_2 \geq 0$ we have that

$$\begin{aligned}&\mathbb{E}\left((S_{t+\tau} - (\alpha K_1 + (1-\alpha)K_2))^+ \middle| \mathscr{F}_t \right) \\ &\leq \alpha \mathbb{E}\left((S_{t+\tau} - K_1)^+ \middle| \mathscr{F}_t \right) + (1-\alpha)\mathbb{E}\left((S_{t+\tau} - K_2)^+ \middle| \mathscr{F}_t \right)\end{aligned}$$

by the convexity of $K \mapsto (x-K)^+$ and the linearity of the conditional expectation operator.

(M2) If $0 \leq K_1 \leq K_2$, then $(x-K_1)^+ \geq (x-K_2)^+$ for fixed $x \in \mathbb{R}$. It follows that $\mathbb{E}\left((S_{t+\tau} - K_1)^+ | \mathscr{F}_t\right) \geq \mathbb{E}\left((S_{t+\tau} - K_2)^+ | \mathscr{F}_t\right)$.

(M3) By assumption, S is a martingale. From the convexity of $x \mapsto (x-K)^+$ and the martingale property of S, $(S-K)^+$ is a sub-martingale. Now fix $0 \leq \tau_1 \leq \tau_2$. Observe that

$$(S_{t+\tau_1} - K)^+ \leq \mathbb{E}\left((S_{t+\tau_2} - K)^+ \middle| \mathscr{F}_{t+\tau_1} \right).$$

Then

$$\begin{aligned}\mathbb{E}\left((S_{t+\tau_1} - K)^+ \middle| \mathscr{F}_t \right) &\leq \mathbb{E}\left(\mathbb{E}\left((S_{t+\tau_2} - K)^+ \middle| \mathscr{F}_{t+\tau_1} \right) \middle| \mathscr{F}_t \right) \\ &= \mathbb{E}\left((S_{t+\tau_2} - K)^+ \middle| \mathscr{F}_t \right),\end{aligned}$$

using the tower law for conditional expectations.

(M4) By Proposition 4.3.1 and Lebesgue's Dominated Convergence theorem

$$\lim_{K\to\infty} \mathbb{E}\left((S_{t+\tau} - K)^+ \middle| \mathscr{F}_t \right) = \mathbb{E}\left(\lim_{K\to\infty} (S_{t+\tau} - K)^+ \middle| \mathscr{F}_t \right) = 0.$$

(M5) For $x \geq 0$ and $K \geq 0$, $(x-K)^+ \leq x^+ = x$. Therefore, since S is a super-martingale, by Proposition 4.3.1, we arrive at

$$\mathbb{E}\left((S_{t+\tau} - K)^+ | \mathscr{F}_t \right) \leq \mathbb{E}\left(S_{t+\tau} | \mathscr{F}_t \right) \leq S_t$$

and the upper bound is established. For the lower bound, use Jensen's inequality, which is applicable by Proposition 4.3.1. When S is a martingale, it remains only to evaluate the conditional expectation of $\mathbb{E}\left(S_{t+\tau} | \mathscr{F}_t\right)$ to get Equation (4.1).

(M6) Obvious.

(M7) By Dominated Convergence and Proposition 4.3.1,

$$\lim_{K\to 0^+} \mathbb{E}\left((S_{t+\tau}-K)^+ \middle| \mathscr{F}_t\right) = \mathbb{E}\left(\lim_{K\to 0^+} (S_{t+\tau}-K)^+ | \mathscr{F}_t\right) = \mathbb{E}\left(S_{t+\tau} | \mathscr{F}_t\right).$$

□

Corollary 4.3.5. *Fix t and $\tau \geq 0$. Let C be defined by*

$$C(K) = \mathbb{E}\left((S_{t+\tau}-K)^+ | \mathscr{F}_t\right)$$

as in Proposition 4.3.4. Let C'_+ denote the right-hand derivative of C (with respect to K). Then,

(1) For $K \geq 0$, C'_+ is well-defined, right-continuous, non-decreasing, and satisfies

$$-1 \leq C'_+(K) \leq 0, \quad \mathbb{P}\text{-a.s.}$$

(2) For $K > 0$,

$$-1 \leq \frac{C(K)-C(0)}{K} \leq C'_+(K) \quad \mathbb{P}\text{-a.s.}$$

Proof. Fix $t, \tau \geq 0$. By (M7), $C(\cdot)$ is right-continuous at zero. We use this without further comment in what follows. By the properties (M1), (M2) and (M5) of Proposition 4.3.4, we may apply (vii') and (vi') of Lemma 4.2.1 to get the first part of the claim. Again invoking (M1), (M2), and (M5) of Proposition 4.3.4, we may apply Lemma 4.2.1 (viii') to get the second part of the claim. □

In future chapters, we will need that

$$\lim_{\tau\to 0^+} \mathbb{E}\left((S_{t+\tau}-K)^+ | \mathscr{F}_t\right) = (S_t - K)^+, \quad \mathbb{P}-a.s.$$

when S is a non-negative martingale. Our proof of this requires that the filtration, to which S is adapted, is right-continuous. We also use that S is a càdlàg process. We now present a definition and a theorem that we need for our proof of Proposition 4.3.8 following.

Definition 4.3.6. (Class (DL)) Consider the class $\mathscr{U}_a$ of all stopping times T of the filtration $\mathbb{F}$ which satisfy $\mathbb{P}(T \leq a)$ for a given finite number $a > 0$. The right-continuous process X defined on this space is said to be of class DL, if the family $\{X_T\}_{T\in\mathscr{U}_a}$ is uniformly integrable of class, for every $0 < a < \infty$. See (KS91), p. 24.

Theorem 4.3.7 (Yeh, Theorem 8.22 in (Yeh95))**.** *On a right-continuous filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, every right-continuous, non-negative sub-martingale is of the class (DL).*

The following result is known (see Proposition 2.4 of (Dur08)). We present an original proof.

Proposition 4.3.8. *Let S be a càdlàg, non-negative martingale on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ with $\mathbb{F}$ satisfying the usual conditions.*
Then, for each $K, t, \tau_1 \geq 0$

$$\lim_{\tau_2 \to \tau_1^+} \mathbb{E}\left((S_{t+\tau_2} - K)^+ | \mathscr{F}_t\right) = \mathbb{E}\left((S_{t+\tau_1} - K)^+ | \mathscr{F}_t\right), \quad \mathbb{P}\text{-a.s.}$$

That is to say that $\tau \mapsto \mathbb{E}\left((S_{t+\tau} - K)^+ | \mathscr{F}_t\right)$ is right-continuous for each $t \geq 0$ and $K \geq 0$. In particular, with $\tau_1 = 0$, we have

$$\lim_{\tau \to 0^+} \mathbb{E}\left((S_{t+\tau} - K)^+ | \mathscr{F}_t\right) = (S_t - K)^+, \quad \mathbb{P}\text{-a.s.}$$

Proof. Fix $K, t, \tau_1 \geq 0$.
For simplicity of notation introduce a càdlàg process X defined by

$$X_{t'} = (S_{t'} - K)^+, \quad \forall t' \geq 0.$$

The convexity of $x \mapsto (x - K)^+$ and the fact that S is a martingale, implies that X is a sub-martingale. Obviously, X is a non-negative process. Therefore, X is of the class (DL) by Theorem 4.3.7.
Since X is of the class (DL) it holds, in particular, that $(X_{t'})_{t' \in [t, t+1]}$ is a uniformly integrable family of random variables. Using this and that X is càdlàg, we get

$$\lim_{\tau_2 \to \tau_1^+} \mathbb{E}\left(X_{t+\tau_2}\right) = \mathbb{E}\left(X_{t+\tau_1}\right),$$

which implies that

$$\lim_{\tau_2 \to \tau_1^+} \mathbb{E}\left(X_{t+\tau_2} - X_{t+\tau_1}\right) = 0.$$

Therefore

$$\lim_{\tau_2 \to \tau_1^+} \mathbb{E}\left(\mathbb{E}\left(X_{t+\tau_2} - X_{t+\tau_1} | \mathscr{F}_t\right)\right) = 0. \tag{4.2}$$

Now we need to interchange the limit and the first expectation in Equation (4.2) and this can be done by the Lebesgue Dominated Convergence theorem; the application of which we will now justify.

Note that for fixed $t \geq 0$,

$$\tau \mapsto \mathbb{E}\left(X_{t+\tau} | \mathscr{F}_t\right) \text{ is a non-negative, non-decreasing function, } \mathbb{P}\text{-a.s.,} \tag{4.3}$$

which is a consequence of the sub-martingale property of X. We have that

$$\mathbb{E}\left(X_{t+\tau_2} - X_{t+\tau_1} | \mathscr{F}_t\right) \geq 0, \quad \forall \tau_2 \geq \tau_1 \geq 0,$$

by (4.3) and the non-negativity of X. Also,

$$\begin{aligned}
\mathbb{E}\left(X_{t+\tau_2} - X_{t+\tau_1} | \mathscr{F}_t\right) &\leq \mathbb{E}\left(X_{t+\tau_2} | \mathscr{F}_t\right) \\
&= \mathbb{E}\left((S_{t+\tau_2} - K)^+ | \mathscr{F}_t\right) \\
&\leq \mathbb{E}\left(S^+_{t+\tau_2} | \mathscr{F}_t\right) \\
&= \mathbb{E}\left(S_{t+\tau_2} | \mathscr{F}_t\right) \\
&= S_t \\
&< \infty,
\end{aligned}$$

all $\mathbb{P}$-a.s.; we used (4.3). We have established that $\left|\mathbb{E}\left(X_{t+\tau_2} - X_{t+\tau_1} | \mathscr{F}_t\right)\right| \leq S_t$, $\mathbb{P}$-a.s. Clearly, S_t is integrable since S is a martingale. Note that there exists an $\mathbb{F}$-adapted process M^{K,τ_1} on our probability space defined by

$$M_t^{K,\tau_1} := \lim_{\tau_2 \to \tau_1^+} \mathbb{E}\left(X_{t+\tau_2} - X_{t+\tau_1} | \mathscr{F}_t\right), \quad t \geq 0,$$

$\mathbb{P}$-a.s. We take a version. The limit defining M^{K,τ_1} exists because of (4.3). Combining the foregoing, it is clear that Lebesgue Dominated Convergence indeed applies.

Using Equation (4.2) and Lebesgue Dominated Convergence, we get that

$$\begin{aligned}
0 &= \lim_{\tau_2 \to \tau_1^+} \mathbb{E}\left(\mathbb{E}\left(X_{t+\tau_2} - X_{t+\tau_1} | \mathscr{F}_t\right)\right) \\
&= \mathbb{E}\left(\lim_{\tau_2 \to \tau_1^+} \mathbb{E}\left(X_{t+\tau_2} - X_{t+\tau_1} | \mathscr{F}_t\right)\right) \\
&= \mathbb{E}\left(M_t^{K,\tau_1}\right).
\end{aligned}$$

However, M^{K,τ_1} is a non-negative process by (4.3). Therefore, the only way that we can have $\mathbb{E}\left(M_t^{K,\tau_1}\right) = 0$ is for $M_t^{K,\tau_1} = 0$, $\mathbb{P}$-a.s. Therefore,

$$\lim_{\tau_2 \to \tau_1^+} \mathbb{E}\left(X_{t+\tau_2} - X_{t+\tau_1} | \mathscr{F}_t\right) = 0, \quad \mathbb{P}\text{-a.s.}$$

so that

$$\lim_{\tau_2\to\tau_1^+} \mathbb{E}\left(X_{t+\tau_2}|\mathscr{F}_t\right) = \mathbb{E}\left(X_{t+\tau_1}|\mathscr{F}_t\right), \quad \mathbb{P}\text{-a.s.}$$

and finally

$$\lim_{\tau_2\to\tau_1^+} \mathbb{E}\left((S_{t+\tau_2}-K)^+|\mathscr{F}_t\right) = \mathbb{E}\left((S_{t+\tau_1}-K)^+|\mathscr{F}_t\right), \quad \mathbb{P}\text{-a.s.}$$

□

We now draw some conclusions about the extreme cases

$$\mathbb{E}\left((S_{t+\tau}-K)^+|\mathscr{F}_t\right) = (S_t-K)^+$$

and

$$\mathbb{E}\left((S_{t+\tau}-K)^+|\mathscr{F}_t\right) = S_t.$$

Proposition 4.3.9. *Let S be a non-negative martingale. Fix $\tau > 0$, $K > 0$ and $t \geq 0$.*

(Ai) Suppose that $S_t \leq K$, then $\mathbb{P}$-a.s. $\mathbb{E}\left((S_{t+\tau}-K)^+|\mathscr{F}_t\right) = (S_t-K)^+$ if and only if $\mathbb{P}\left(S_{t+\tau} > K|\mathscr{F}_t\right) = 0$.

(Aii) Suppose that $S_t > K$, then $\mathbb{P}$-a.s. $\mathbb{E}\left((S_{t+\tau}-K)^+|\mathscr{F}_t\right) = (S_t-K)^+$ if and only if $\mathbb{P}\left(S_{t+\tau} < K|\mathscr{F}_t\right) = 0$.

(Aiii) If $\mathbb{E}\left((S_{t+\tau}-K)^+|\mathscr{F}_t\right) = S_t$ then $\mathbb{P}\left(S_{t+\tau} = 0|\mathscr{F}_t\right) = 1$, $\mathbb{P}$-a.s.

Proof. Fix $\tau, K > 0$ and $t \geq 0$.

(Ai) Suppose that $S_t \leq K$. Then $\mathbb{E}\left((S_{t+\tau}-K)^+|\mathscr{F}_t\right) = (S_t-K)^+ (= 0)$,

$$\begin{aligned}
&\Leftrightarrow \quad \int_0^\infty (y-(y\wedge K))\,\mathbb{P}\left(S_{t+\tau}\in \mathrm{d}y|\mathscr{F}_t\right) = 0\\
&\Leftrightarrow \quad \int_0^\infty y\,\mathbb{P}\left(S_{t+\tau}\in \mathrm{d}y|\mathscr{F}_t\right) - \int_0^\infty (y\wedge K)\,\mathbb{P}\left(S_{t+\tau}\in \mathrm{d}y|\mathscr{F}_t\right) = 0\\
&\Leftrightarrow \quad \int_{(K,\infty)} y\,\mathbb{P}\left(S_{t+\tau}\in \mathrm{d}y|\mathscr{F}_t\right) = \int_{(K,\infty)} K\,\mathbb{P}\left(S_{t+\tau}\in \mathrm{d}y|\mathscr{F}_t\right)\\
&\Leftrightarrow \quad \mathbb{P}\left(S_{t+\tau} > K|\mathscr{F}_t\right) = 0.
\end{aligned}$$

(Aii) Suppose that $S_t > K$. Then $\mathbb{E}\left((S_{t+\tau}-K)^+|\mathscr{F}_t\right) = (S_t-K)^+$

$$\begin{aligned}
&\Leftrightarrow \int_0^\infty (y-(y\wedge K))\,\mathbb{P}\left(S_{t+\tau}\in \mathrm{d}y|\mathscr{F}_t\right) = \int_0^\infty y\,\mathbb{P}\left(S_{t+\tau}\in \mathrm{d}y|\mathscr{F}_t\right) - K\\
&\Leftrightarrow \int_0^\infty (y\wedge K)\,\mathbb{P}\left(S_{t+\tau}\in \mathrm{d}y|\mathscr{F}_t\right) = \int_0^\infty K\,\mathbb{P}\left(S_{t+\tau}\in \mathrm{d}y|\mathscr{F}_t\right)\\
&\Leftrightarrow \int_{[0,K)} y\,\mathbb{P}\left(S_{t+\tau}\in \mathrm{d}y|\mathscr{F}_t\right) = \int_{[0,K)} K\,\mathbb{P}\left(S_{t+\tau}\in \mathrm{d}y|\mathscr{F}_t\right)\\
&\Leftrightarrow \mathbb{P}\left(S_{t+\tau} < K|\mathscr{F}_t\right) = 0.
\end{aligned}$$

(Aiii) We have $\mathbb{E}\left((S_{t+\tau}-K)^+|\mathscr{F}_t\right) = S_t$

$$\begin{aligned}
&\Longrightarrow \quad \int_0^\infty (y-(y\wedge K))\,\mathbb{P}\left(S_{t+\tau}\in \mathrm{d}y|\mathscr{F}_t\right) = \int_0^\infty y\,\mathbb{P}\left(S_{t+\tau}\in \mathrm{d}y|\mathscr{F}_t\right)\\
&\Longrightarrow \quad \int_0^\infty (y\wedge K)\,\mathbb{P}\left(S_{t+\tau}\in \mathrm{d}y|\mathscr{F}_t\right) = 0\\
&\Longrightarrow \quad \mathbb{P}\left(S_{t+\tau}=0|\mathscr{F}_t\right) = 1.
\end{aligned}$$

□

4.4 Summary of Results and Conclusion

We have presented and proved a number of useful facts about non-negative local martingales and the expectation of the European call option payoff. The property of zero absorption means that studying the properties of the implied volatility surface when the stock price is zero is entirely trivial. Hence we will always assume that the (instantaneous) stock price is strictly positive when we study the implied volatility surface.

Chapter 5

Existence of Implied Volatility in Stock Price Models

In this chapter, we examine the issue of existence of implied volatility in stock price models. Taking our lead from the Fundamental Theorem of Asset Pricing, what we mean by "stock price model" is precisely a non-negative, local martingale. We show that implied volatility exists "always and everywhere" only if we assume a model of the stock price in which the stock is a non-negative martingale. In other words, in non-negative strictly local martingale models, implied volatility is necessarily ill-defined for certain strikes, maturities, and times. This result justifies the restriction of the class of stock price models analysed in later chapters of this thesis to non-negative martingale models. There does not exist in the literature an analysis of implied volatility in non-negative, strictly local martingale models of a stock.

It is well known that in (non-negative) martingale models of a stock price, the following bounds necessarily hold

$$(S_t - K)^+ \leq \mathbb{E}\left((S_{t+\tau} - K)^+ | \mathscr{F}_t\right) \leq S_t, \tag{5.1}$$

$\mathbb{P}$-a.s. and for every $t, \tau \geq 0$. (See (Mer73).) It is shown in (CH05) that these bounds fail when the stock is a strictly local (non-negative) martingale. It appears, however, to not be widely known that these bounds hold for all times, maturities, and strikes if and only if the stock price process is a non-negative martingale.

In (HW99), (HKLW02), (HL05), (BC05), (Obł07) the authors obtain approximations of implied volatility in local and stochastic volatility models. It is assumed without comment that implied volatility is well-defined in the class of models they consider. However, this is not always the case as we will illustrate in this chapter. It should be noted that in (BBF02), (BBF04), and (Dur08), the authors checked that the bounds[1] in Equation (5.1) hold on the call price, but no link to the martingale property is mentioned in either (BBF02) or (BBF04), which is to be expected since they use the partial differential equation method.

In Section 5.1, we give the mathematical setup for the chapter. Section 5.2 proves some results needed in the proofs of the results in Section 5.3. There are two main results in Section 5.3: the first expresses the martingale property in terms of the classical no arbitrage bounds on the call price; the second gives some properties of

[1] Actually, the tighter bounds $((S_t - K)^+, S_t)$ are established.

implied volatility in non-negative, strictly local martingale models. In Section 5.4, we show that the results of this chapter are not moot, by giving examples in the literature where the stock price process is a strictly local martingale but in which implied volatility can only be sensibly approximated for a subset of strikes. The characterisation of the martingale property in terms of the no-arbitrage bounds in Lemma 5.2.1 may be of independent interest. Section 5 concludes.

5.1 Background

5.1.1 Setup

We let S be a non-negative, local martingale on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. The filtration need not satisfy all the usual conditions. In particular, we do not require it to be right-continuous.

5.2 Lemmas

The main results of this chapter are largely consequences of the following characterisation of martingales in terms of the well-known arbitrage bounds on the price of the call option in Equation (5.1), i.e.

$$(S_t - K)^+ \leq \mathbb{E}\left((S_{t+\tau} - K)^+ | \mathscr{F}_t\right) \leq S_t.$$

Lemma 5.2.1. *Recall from Definition 3.2.1 that an integrable process is one for which* $\mathbb{E}(|X_t|) < \infty$ *for each* $t \geq 0$. *Let X be a non-negative process on a filtered probability space* $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ *with X $\mathbb{F}$-adapted. Then the following statements are equivalent*

(1) X is a martingale;

(2) X is an integrable process and for every $0 \leq u \leq t$

$$(\forall K > 0) \quad (X_u - K)^+ \leq \mathbb{E}\left((X_t - K)^+ \middle| \mathscr{F}_u\right) \leq X_u, \quad \mathbb{P}\text{-a.s.}; \tag{5.2}$$

and

(3) *X is a non-negative local martingale (hence a super-martingale) and for every $0 \leq u \leq t$*

$$(\forall K > 0) \quad (X_u - K)^+ \leq \mathbb{E}\left((X_t - K)^+ | \mathscr{F}_u\right) \leq X_u, \quad \mathbb{P}\text{-a.s.}$$

Proof. The reader is invited to include the rider $\mathbb{P}$-a.s. as is required. Fix $0 \leq u \leq t$ and $K \geq 0$. Assume first that (1) holds. Then:

(1 $\Longrightarrow$ 2) X is an integrable process as it is a martingale. The first inequality follows from Jensen's inequality combined with the martingale property and the second from the inequality $(X_t - K)^+ \leq X_t$ along with the martingale property.

(1 $\Longrightarrow$ 3) Any martingale is a local martingale. For the inequality use again Jensen's inequality along with the martingale property of X. The upper bound is trivial.

The other implications are proved as follows.

(2 $\Longrightarrow$ 1) From Equation (5.2), Dominated Convergence, and the non-negativity of X,

$$\begin{aligned}
X_u &= \lim_{K \to 0^+} (X_u - K)^+ \\
&\leq \lim_{K \to 0^+} \mathbb{E}\left((X_t - K)^+ | \mathscr{F}_u\right) \\
&= \mathbb{E}\left(\lim_{K \to 0^+} (X_t - K)^+ | \mathscr{F}_u\right) \\
&= \mathbb{E}\left(X_t | \mathscr{F}_u\right) \\
&\leq X_u.
\end{aligned}$$

(3 $\Longrightarrow$ 2) Since X is a non-negative local martingale, it is a super-martingale (see Proposition 4.3.1), so

$$\mathbb{E}\left((X_t - K)^+ \middle| \mathscr{F}_u\right) \leq \mathbb{E}\left(X_t \middle| \mathscr{F}_u\right) \leq X_u.$$

The lower bound in Equation (5.2) holds by assumption. $\square$

Relating the martingale property to implied volatility, we note the following well known result.

Lemma 5.2.2. *Let $s \geq 0$ and*

$$C : (0, \infty) \times [0, \infty) \to \mathbb{R}$$

be such that

$$(s-K)^+ \le C(K,\tau) \le s, \quad \forall K>0, \tau \ge 0.$$

Then the implied volatility corresponding to C and s is well defined for all $K > 0$ and $\tau \ge 0$.

Proof. The Black-Scholes function, C^{BS}, is strictly monotonic increasing in σ and ranges from $(s-K)^+$ when $\sigma\sqrt{\tau}=0$ to s when $\sigma\sqrt{\tau}=\infty$.

$\square$

5.3 Main Results

The first result of significance in this chapter is the following which relates implied volatility and the martingale property. The second result shows that implied volatility is necessarily ill-defined for some set of strikes, maturities, and times in strict local martingale models. The final result establishes some facts about the set of implied volatilities which are ill-defined in non-negative, strictly local martingale models of a stock.

Theorem 5.3.1. *Let S be a non-negative martingale. Fix $t \ge 0$. Define*

$$\begin{aligned} C_t : (0,\infty) \times [0,\infty) &\to \mathbb{R} \\ (K,\tau) &\mapsto \mathbb{E}\left((S_{t+\tau}-K)^+ \middle| \mathscr{F}_t \right). \end{aligned}$$

Then the implied volatility, Σ_t, corresponding to C_t and S_t is $\mathbb{P}$-a.s well-defined for every $\tau \ge 0$ and $K > 0$.

Proof. Since X is a non-negative martingale, we may apply Lemma 5.2.1 to get that

$$(X_t-K)^+ \le C_t(K,\tau) \le X_t,$$

for every $\tau, K \ge 0$ and so that implied volatility is clearly well-defined up to equivalence. $\square$

We have shown that implied volatilities are always well-defined when the stock price is a non-negative martingale. This changes when the stock price is a strictly local martingale. What can we say for the latter case? The answer to this question is of interest when one tries to obtain approximations to implied volatility in stochastic volatility models. This is because in many of the stochastic volatility models popular in finance the stock price is a strictly local martingales (see (AP07) and (LM06)).

First we give a general result.

Proposition 5.3.2. *Let S be a non-negative, strictly local martingale with $S_t > 0$. Then there exists a non-empty set of strikes, maturities and times such that implied volatility is ill-defined.*

Proof. To be well-defined for all strikes, maturities and times requires that call prices of all strikes, maturities and times lie in the range $(S_t - K)^+ \leq \mathbb{E}\left((S_{t+\tau} - K)^+ | \mathscr{F}_t\right) \leq S_t$. But then S must be a martingale, by Lemma 5.2.1. $\square$

Now we give some results about which implied volatilities will be ill-defined in strictly local martingale models.

Proposition 5.3.3. *Let S be a positive local martingale. Fix $t \geq 0$. To avoid trivialities we suppose that $S_t > 0$. Then*

(I) If $t + \tau > t$ is such that $\mathbb{E}\left(S_{t+\tau} | \mathscr{F}_t\right) < S_t$, then the set of strikes for maturity $t + \tau$ for which implied volatility is not well defined is of the form $[0, K^*)$ for some $K^* \in [S_t - \mathbb{E}\left(S_{t+\tau} | \mathscr{F}_t\right), S_t]$, $\mathbb{P}$-a.s.

(II) The implied volatility of out-of-the-money ($K > S_t$) and at-the-money ($K = S_t$) call options are well-defined for each time to expiry $\tau > 0$, $\mathbb{P}$-a.s.

Proof. Fix $t \geq 0$ and $\tau > 0$. To ease the notation let

$$\begin{aligned} C : (0, \infty) &\to \mathbb{R} \\ K &\mapsto \mathbb{E}\left((S_{t+\tau} - K)^+ | \mathscr{F}_t\right), \end{aligned}$$

and recall that implied volatility is not defined for $K = 0$,

(I) From Proposition 4.3.4, C is $\mathbb{P}$-a.s. non-increasing, continuous, non-negative, and bounded above by S_t. It is therefore clear that $C(K) = S_t - K$, for some $K \in [S_t - \mathbb{E}\left(S_{t+\tau} | \mathscr{F}_t\right), S_t]$. Moreover, if K^* is such that $C(K^*) = S_t - K^*$, then for every $\widehat{K} > K^*$, we have $(S_t - \widehat{K})^+ \leq C(\widehat{K}) \leq S_t$, $\mathbb{P}$-a.s. This follows from the non-negativity of C and the bounds $-1 \leq C'_+(\widetilde{K}) \leq 0$, where C'_+ is the right-hand derivative of C, and $\widetilde{K} > 0$.

(II) Follows from (I).

$\square$

Remark 5.3.4. At the present level of generality, we cannot say much more about the maturities, times and strikes at which implied volatility is not well-defined. This is because:

(1) for a super-martingale not to be a martingale only requires there to be a *single* time t at which there is a *single* maturity $t + \tau$ for which $\mathbb{E}\left(S_{t+\tau}|\mathscr{F}_t\right) < S_t$; and

(2) non-negative strictly local martingales, as opposed to martingales, do *not* in general have the property that, with K fixed, $\tau \mapsto \mathbb{E}\left((S_{t+\tau} - K)^+|\mathscr{F}_t\right)$ is non-decreasing.

5.4 Examples

The question of whether a stock price is a martingale as opposed to a strictly local martingale is not moot. Indeed, stochastic volatility models "often" produce strictly local martingale models of a stock (see (AP07) and (LM06), for example). There have been a number of papers and working papers produced in which small time asymptotics for implied volatilities in local and (diffusion type) stochastic volatility models are obtained. (see, for example, (HW99), (HKLW02), (HL05), (BC05), (Obł07), (BBF02), and (BBF04)).
In (HW99), Hagan and Woodward provide a detailed analysis of the CEV model:

$$dS_t = \alpha S_t^{\beta} dW_t, \quad S_0 > 0$$

where no restriction is placed on β and we may, in particular, choose $\beta > 1$. In this case, S is a strictly local martingale. See Cox and Hobson (CH05). Therefore there is a range of strikes (and a set of maturities) for which implied volatility is ill-defined.

In (HKLW02), (HL05), and (Obł07) various stochastic volatility models are examined; all of the expressions for the implied volatility were for small expires. They all allow for, or examine specifically, the log-normal model, i.e.

$$dS_t = \sigma_t S_t \, \mathrm{d}W_t^1,$$
$$d\sigma_t = \nu \sigma_t \, \mathrm{d}W_t^2,$$

where the Brownian motions W^1, W^2 are correlated: $\mathrm{d}\left\langle W^1, W^2 \right\rangle_t = \rho \, \mathrm{d}t$, $\rho \in (-1, 1)$, and $\nu, \sigma_0, S_0 > 0$. In this case, we know from the results of Musiela and Lions (see (LM06)), that when $\rho > 0$, S is a strictly local martingale; indeed $\mathbb{E}\left(S_t | S_0 = x, \sigma_0 = y\right) < S_0$ for all $t > 0$. It is then clear, from the time-homogeneous nature of the model that for every time $t \geq 0$ and every maturity $t + \tau > 0$ that implied volatility is ill-defined for all strikes small enough.

5.5 Summary of Results and Conclusion

In this chapter, we have shown that implied volatility is always and everywhere well-defined only in stock price models when the stock is a (non-negative) martingale. We have also shown that in local martingale models, the set of strikes which is ill-defined must be of the form $[0, K^*)$ where $K^* \in [S_t - \mathbb{E}\left(S_{t+\tau} | \mathscr{F}_t\right), S_t]$. We observed that some papers in the literature on small time estimates for implied volatility allow for non-negative, strictly local martingale models. In such cases, we know that there is a certain set of strikes, times, and maturities for which implied volatility is ill-defined.

Chapter 6

Arbitrage Free Surfaces

This chapter establishes sufficient and close to necessary conditions for an implied volatility surface to be free from "static arbitrage". Static arbitrage is best explained via call option price surfaces: A call option price surface $(K,\tau) \mapsto C(K,\tau)$ is free from static arbitrage if and only if there exists a non-negative martingale, say S, such that $C(K,\tau) = \mathbb{E}\left((S_\tau - K)^+\right)$ for every $K,\tau \geq 0$. We then say that an implied volatility surface is free from static arbitrage if the call price surface $C(K,\tau) = C^{BS}(K,\tau,\Sigma(K,\tau))$ is free from static arbitrage. The idea of static arbitrage is that if it is absent from a call price surface, then there can be no arbitrage opportunities "trading in the surface". In this chapter, we establish conditions for an implied volatility surface to be free from "static arbitrage". We first establish the conditions for the call price surface and then translate each of these conditions on the call price surface into conditions on the implied volatility surface. Determining this set of conditions is an important problem in stochastic implied volatility modelling since one needs to know how to specify an initial arbitrage free implied volatility surface. As noted by Musiela and Rutkowski (see pp. 270-271 of (MR05)), this is an open problem in stochastic implied volatility modelling.

Necessary and sufficient conditions on the call price surface for it to be free from static arbitrage are, under various conditions, quite well known. Buehler (see (Bue06)) considers the case of a finite family of strikes and maturities in the call price surface. Davis and Hobson (see (DH07)) do the same, but with a different focus. Carr and Madan (see (CM05)) allow for a countably infinite collection of strikes and, for each strike, a finite number of maturities. Madan and Yor (see (MY02)) present a number of ways to construct martingales whose marginal densities are known. Of course, it was first shown by Breeden and Litzenberger (see (BL78)) that the risk-neutral marginal law of the stock price at a maturity τ may be obtained from the market's quoted option prices at maturity τ (see p. 42 of (MR05)). Cox and Hobson ((CH05)) consider a call price surface to be arbitrage free if it may be matched by a local martingale. We do not use this approach because of the problem of implied volatility being ill-defined in strictly local martingale models. The setup closest to ours is that of Föllmer and Schied (see (FS04)). Indeed, we use an argument of theirs in our proof of Theorem 6.2.1.

Various necessary conditions on the implied volatility surface for it to be free from static arbitrage are known. However, no set of sufficient conditions on the implied volatility surface for it to be free from static arbitrage has been presented in the literature. We will now attribute to their author those necessary conditions that we we will later show are also sufficient. The conditions are given in terms of the time scaled implied volatility (in log-moneyness form) (see Definition 6.2.7) and they are numbered as they appear in Theorem 6.2.11.

(IV3) Durrleman's condition ensures that the constructed call price surface is convex in the strike. It was presented in (Dur03).

(IV4) Bounds on the slope in log-moneyness of the implied volatility surface were presented by Lee in (Lee05) and by Fouque *et al.* in (FPS00).

(IV5) The monotonicity in time to expiry of the implied volatility in log-moneyness form appears to be due to Durrleman (see (Dur03)).

(IV6) The large moneyness behaviour appears to be due to Lee (see (Lee04)).

(IV7) The value of implied volatility in log-moneyness form at expiry is due to Durrleman (see (Dur03)).

We note that Durrleman did not present any slope bounds in (Dur03). We have made two other corrections to his work. Lee presents the necessary bounds on the partial derivative of time-scaled implied volatility in log-moneyness form with respect to log-moneyness. In particular, he presents lower and upper bounds. The lower bound is attributed to Gatheral (Gat99). Actually, the lower bound need not be specified as a separate necessary condition. In Corollary 6.2.14, we show that the lower bound follows from some of the other necessary conditions. We use Durrleman's proof that (IV3) implies that the call surface constructed from the implied volatility surface using Theorem 6.2.1 is convex in the strike. We also use significant parts of his proof that the large moneyness condition (IV6) implies that the call surface constructed from the implied volatility surface using Theorem 6.2.1 has the correct large strike behaviour.

This chapter is organised as follows. In the first section, we give the mathematical setup and recall some results from the literature on the construction of martingales satisfying a given set of marginal laws. In the second section, we give necessary and sufficient conditions for a call price surface to be free of static arbitrage. We also present sufficient conditions for an implied volatility surface to be free of static arbitrage and show that they are necessary under certain (rather mild) technical conditions and examine their impact. In Section 6.3, we give examples of implied volatility smile parameterisations that have been presented in the literature and show, using our results, that they are not arbitrage-free.

6.1 Background

6.1.1 Setup

We assume throughout this chapter zero interest rates and dividend yield. We will also restrict ourselves to non-negative price processes, and that we are in a perfectly liquid market for European calls. That is, call option prices for all strikes $K > 0$ and maturities $\tau \geq 0$ are known in the market. We specifically exclude strike $K = 0$ for two reasons. Firstly, it becomes problematic when one deals with implied volatility parameterised in terms of log moneyness ($\ln(K/s)$, with s the stock price). Secondly, it turns out to be simple to always take the continuous extension of the call price surface to $[0,\infty) \times [0,\infty)$, so that there is no real loss of generality in supposing that $K > 0$.

Definition 6.1.1. A *call price surface parameterised by* s is a function

$$\begin{aligned} C : [0,\infty) \times [0,\infty) &\to \mathbb{R} \\ (K,\tau) &\mapsto C(K,\tau) \end{aligned}$$

along with a real number $s > 0$.

It will sometimes be convenient to simply refer to a call price surface parameterised by s as a call price surface or a call surface.
The next definition follows that presented in (CH05) except for our insistence on true martingales.

Definition 6.1.2. There is *no static arbitrage* in a call price surface C if there is a non-negative martingale X on some stochastic basis $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$

with $C(K,\tau) = \mathbb{E}\left((X_\tau - K)^+ \,|\, \mathscr{F}_0\right)$ for each $(K,\tau) \in [0,\infty) \times [0,\infty)$. If such a martingale and probability space exists, then we say that the call price surface is free of static arbitrage.

6.1.2 Auxiliary Facts

Definition 6.1.3 (Buehler, Definition 2 in (Bue06))**.** Let μ, ν be two measures defined on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Then μ is said to precede ν in the Balayage order if and only if

$$\int f \,\mathrm{d}\mu \leq \int f \,\mathrm{d}\nu$$

for all convex functions f. This is denoted by $\mu \preceq \nu$ for probability measures with finite expectation.

Lemma 6.1.4 (Föllmer and Schied, Corollary 2.63 in (FS04))**.** *Let μ, ν be two measures defined on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. We have $\mu \preceq \nu$ if and only if*

$$\int (x-K)^+ \,\mathrm{d}\mu \leq \int (x-K)^+ \,\mathrm{d}\nu$$

for all $K \in \mathbb{R}$.

Theorem 6.1.5 (Kellerer, Theorem 3 in (Kel72))**.** *Let $\mathcal{M} = (\mu_t)_{t \in \mathcal{T}}$ be a set of probability measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ with finite expectation at each $t \in \mathcal{T}$, where $\mathcal{T} \subseteq [0,\infty)$ is some Borel set. A Markov sub-martingale with marginal distributions μ_t exists if $\mathcal{M}$ is in Balayage order, that is*

$$\mu_s \preceq \mu_t$$

for all $s < t$ with $s, t \in \mathcal{T}$.

Corollary 6.1.6. *Let $\mathcal{M} = (\mu_t)_{t \in \mathcal{T}}$ be a set of probability measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ with finite* constant *expectation for all $t \in \mathcal{T}$, where $\mathcal{T} \subseteq [0,\infty)$ is some Borel set. A Markov* martingale *with marginal distributions μ_t exists if $\mathcal{M}$ is in Balayage order, that is*

$$\mu_s \preceq \mu_t$$

for all $s < t$ with $s, t \in \mathcal{T}$.

Proof. It is standard that a sub-martingale with constant, finite mean is a martingale. $\square$

6.2 Main Results

6.2.1 The Call Price Surface

In this section, we give necessary and sufficient conditions for a call price surface to be free of static arbitrage. The following, except for minor differences, is from Lemma 7.2.3 of Föllmer and Schied (see (FS04)). We emphasise that we are now allowing for $K = 0$.

Theorem 6.2.1. *Let $s > 0$ be a constant. Let $C : (0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfy the following conditions.*

(A1) (Convexity in K)

$$C(\cdot, \tau) \text{ is a convex function}, \quad \forall \tau > 0;$$

(A2) (K monotonicity)

$$C(\cdot, \tau) \textit{ is non-increasing}, \quad \forall \tau > 0;$$

(A3) (Monotonicity in τ)

$$\text{for each } K > 0, C(K, \cdot) \text{ is non-decreasing;}$$

(A4) (Large strike limit)

$$\lim_{K \to \infty} C(K, \tau) = 0, \quad \forall \tau > 0;$$

(A5) (Bounds)

$$(s - K)^+ \leq C(K, \tau) \leq s, \quad \forall K, \tau > 0; \textit{and}$$

(A6) (Expiry Value)

$$C(K, 0) = (s - K)^+, \quad \forall K > 0.$$

Then

(1) the function

$$\widehat{C} : [0,\infty) \times [0,\infty) \to \mathbb{R}$$
$$(K,\tau) \mapsto \begin{cases} s, & \text{if } K = 0 \\ C(K,\tau), & \text{if } K > 0 \end{cases}$$

satisfies assumptions (A1)-(A6) but with $K \geq 0$ instead of $K > 0$; and

(2) there exists a non-negative Markov martingale X with the property that

$$\widehat{C}(K,\tau) = \mathbb{E}\left((X_\tau - K)^+ | X_0 = s\right)$$

for all $K, \tau \geq 0$.

All of the listed conditions are necessary properties of $\widehat{C}$ for it to be the conditional expectation of the call option under the assumption that X is a (non-negative) martingale.

Proof.

(1) Note first that it is immediate from (A3) and the fact that $\widehat{C}(0,\cdot)$ is constant that (A3) is satisfied by $\widehat{C}(K,\cdot)$ for all $K \geq 0$. Also, note that $\widehat{C}(\cdot,\tau)$ is right-continuous at zero, by (A5), (A6), and the supposition that $\widehat{C}(0,\tau) = s$ for all $\tau \geq 0$.

<u>**Case 1:** $\tau = 0$</u>

Observe that $\widehat{C}(\cdot,0)$ is the continuous extension of $C(\cdot,0)$ to $[0,\infty)$. Now use Lemma 4.2.1.

<u>**Case 2:** $\tau > 0$</u>

Pick a $\tau > 0$. The function $\widehat{C}(\cdot,\tau)$ is the continuous extension of $C(\cdot,\tau)$ to $[0,\infty)$. Indeed, (A5) and the definition of $\widehat{C}$ give that

$$\lim_{K\to 0^+}(s-K)^+ = s^+ = s \leq \lim_{K\to 0^+} C(K,\tau) \leq s$$

so that $\lim_{K\to 0^+} C(K,\tau) = \lim_{K\to 0^+} \widehat{C}(K,\tau) = \widehat{C}(0,\tau) = s$.

By (A1), $C(\cdot,\tau)$ is convex, non-increasing by (A2), has $\lim_{K\to\infty} C(K,\tau) = 0$ by (A4), and satisfies the bounds $(s-K)^+ \leq C(K,\tau) \leq s$ for each $K > 0$, by (A5). We can therefore use all conclusions of Lemma 4.2.1.

(A1′) By Lemma 4.2.1(i′).

(A2′) By Lemma 4.2.1(ii′).

(A3′) Already proved.

(A4′) From Lemma 4.2.1(iii′).

(A5′) From Lemma 4.2.1(iv′).

(A6′) Not applicable.

(2) We first show that for each fixed $\tau \geq 0$ there exists a probability measure μ_τ on $(\mathbb{R}, \mathscr{B}(R))$ such that

$$s = \int x \, \mu_\tau(\mathrm{d}x), \quad \forall \tau \geq 0 \tag{6.1}$$

and

$$\widehat{C}(K, \tau) = \int (x - K)^+ \, \mu_\tau(\mathrm{d}x), \quad \forall K, \tau \geq 0. \tag{6.2}$$

We will let $\partial_K^+ \widehat{C}$ denote the right-handed partial derivative of $\widehat{C}$ with respect to K. We note that it exists, is right-continuous and is bounded above by zero and below by -1 at every $K \geq 0$, by Corollary 4.3.5.
Since $\widehat{C}(\cdot, \tau)$ is convex on $[0, \infty)$, it holds for every $\epsilon > 0$ that

$$\widehat{C}(K, \tau) = \widehat{C}(\epsilon, \tau) + \int_\epsilon^K \partial_K^+ \widehat{C}(k, \tau) \, \mathrm{d}k$$

(see Proposition A.4 of (FS04)). Taking the limit as $\epsilon \to 0^+$ and using the fact that

$$\lim_{K \to 0^+} \widehat{C}(K, \tau) = s,$$

we get that

$$\widehat{C}(K, \tau) = s + \int_0^K \partial_K^+ \widehat{C}(k, \tau) \, \mathrm{d}k.$$

Then (A4) yields

$$\lim_{K \to \infty} \int_0^K \partial_K^+ \widehat{C}(k, \tau) \, \mathrm{d}k = -s. \tag{6.3}$$

We may conclude that

$$\lim_{k \to \infty} \partial_K^+ \widehat{C}(k, \tau) = 0.$$

By Corollary 4.3.5, $\partial_K^+ \widehat{C}(\cdot, \tau)$ is well-defined, non-decreasing and right-continuous on $K \geq 0$. We also recall that $\partial_K^+ \widehat{C}(\cdot, \tau)$ is bounded above by 0 and below by -1. It follows that

$$F(k, \tau) = \begin{cases} 1 + \partial_K^+ \widehat{C}(K, \tau), & \text{if } K \geq 0 \\ 0, & \text{if } K < 0, \end{cases}$$

is a cumulative distribution function. Of course, F defines a probability measure μ_τ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

By Equation (6.3), we have

$$s = \int_0^\infty -\partial_K^+ \widehat{C}(k, \tau)\, \mathrm{d}k = \int_0^\infty \mu_\tau((k, \infty))\, \mathrm{d}k = \int_{\mathbb{R}} y\, \mu_\tau(\mathrm{d}y),$$

and we have proved Equation (6.1). Using this we have that

$$\begin{aligned} \widehat{C}(K, \tau) &= s - \int_0^K -\partial_K^+ \widehat{C}(k, \tau)\, \mathrm{d}k \\ &= s - \int_0^K \int_k^\infty \mu_\tau(\,\mathrm{d}y)\, \mathrm{d}k \\ &= s - \int_0^\infty \int_0^K \mathbb{1}_{\{k<y\}}\, \mathrm{d}k\, \mu_\tau(\mathrm{d}y) \\ &= s - \int_0^\infty (y \wedge K)\, \mu_\tau(\mathrm{d}y) \\ &= \int_0^\infty (y-K)^+\, \mu_\tau(\mathrm{d}y). \end{aligned}$$

So Equation (6.2) holds for each $\tau \geq 0$.

We now show that the family of measures $\mathscr{M} = (\mu_\tau)_{\tau \in [0,\infty)}$, with μ_τ defined as in the first part of the proof, is in Balayage order. Fix $K \geq 0$. Let $0 \leq \tau_1 < \tau_2 < \infty$. Using (A3) and the representation of $\widehat{C}(K, \tau)$ given in Equation (6.2), we have that

$$\int (y-K)^+\, \mu_{\tau_1}(\mathrm{d}y) = \widehat{C}(K, \tau_1) \leq \widehat{C}(K, \tau_2) = \int (y-K)^+\, \mu_{\tau_2}(\mathrm{d}y),$$

for all $K \geq 0$. By construction, every $\mu_\tau \in \mathscr{M}$ has support contained in $[0, \infty)$. We can now argue that for fixed $K < 0$

$$\begin{aligned} \int_{-\infty}^\infty (y-K)^+\, \mu_{\tau_1}(\mathrm{d}y) &= \int_0^\infty (y-K)^+\, \mu_{\tau_1}(\mathrm{d}y) \\ &= \int_0^\infty (y-K)\, \mu_{\tau_1}(\mathrm{d}y) \\ &= s - K \\ &= \int_{-\infty}^\infty (y-K)^+\, \mu_{\tau_2}(\mathrm{d}y). \end{aligned}$$

So by Lemma 6.1.4, we may conclude that $\mathscr{M}$ is in Balayage order. The existence of a Markov martingale with marginals $\mathscr{M}$ now follows from Corollary 6.1.6.

The necessity comes from Proposition 4.3.4. $\square$

Remark 6.2.2. It is to be noted that the constructed martingale of the previous theorem need not be càdlàg, nor is the filtration necessarily right-continuous.

6.2.2 The Implied Volatility Surface

We now turn our attention to the implied volatility surface and derive sufficient and close to necessary conditions for it to be free from static arbitrage.

Definition 6.2.3. Let

$$\begin{aligned} d_+ : \mathbb{R} \times (0,\infty) &\to \mathbb{R} \\ (u,v) &\mapsto -u/v + v/2 \end{aligned}$$

and

$$\begin{aligned} d_- : \mathbb{R} \times (0,\infty) &\to \mathbb{R} \\ (u,v) &\mapsto -u/v - v/2. \end{aligned}$$

Following Durrleman (Dur03), we will work with a "scaled Black-Scholes" function as it simplifies the calculations.

Definition 6.2.4. Let

$$\begin{aligned} B : \mathbb{R} \times [0,\infty) &\to \mathbb{R} \\ (x,\theta) &\mapsto \begin{cases} \lim_{\theta \to 0^+} B(x,\theta) = (1-\exp(x))^+, & \text{if } \theta = 0 \\ \Phi\left(d_+(x,\theta)\right) - \exp(x)\Phi\left(d_-(x,\theta)\right), & \text{if } \theta \in (0,\infty) \\ \lim_{\theta\to\infty} B(x,\theta) = 1, & \text{if } \theta = \infty \end{cases} \end{aligned}$$

Remark 6.2.5. It is clear that

$$(1-\exp(x))^+ \leq B(x,\theta) \leq 1$$

for every $x \in \mathbb{R}$ and $\theta \in [0,\infty)$.

Remark 6.2.6. The standard representation of the Black-Scholes call pricing function can be recovered from B as follows

$$C^{BS}(K,\tau,\sigma) = \begin{cases} S_t B(\ln(K/S_t), \sigma\sqrt{\tau}), & \text{if } K > 0, \\ S_t, & \text{if } K = 0, \end{cases}$$

where S_t is the current stock price and τ is the time to expiry.
Observe that B is conveniently written solely in terms of log-moneyness, i.e. $\ln(K/S_t)$, and time-scaled volatility $\sigma\sqrt{\tau}$.

We now introduce a variant of implied volatility which we have termed *time-scaled implied volatility in log-moneyness form*.

Definition 6.2.7. Time scaled implied volatility (in log-moneyness form) parameterised by s is a function defined by

$$\begin{aligned} \Xi : \mathbb{R} \times [0,\infty) &\to [0,\infty] \\ (x,\tau) &\mapsto \sqrt{\tau}\Sigma(s\exp(x),\tau), \end{aligned}$$

where Σ is implied volatility and $s > 0$ is the stock price. K enters via x which is log-moneyness, i.e. $x = \ln(K/s)$.

Remark 6.2.8. Durrleman (see (Dur03), uses something very close to time scaled implied volatility (in log-moneyness form).

Remark 6.2.9. A time scaled implied volatility surface (in log-moneyness form) parameterised by s defines a call price surface parameterised by s via

$$C(K,\tau) = \begin{cases} sB(\ln(K/s), \Xi(\ln(K/s),\tau)), & \text{if } K > 0 \\ s, & \text{if } K = 0, \end{cases}$$

and we see why there was no need to define implied volatility for $K = 0$.

Following Lee (Lee05), we will use the Mill's Ratio.

Definition 6.2.10. The function

$$\begin{aligned} R : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \frac{1-\Phi(x)}{\phi(x)}. \end{aligned}$$

is termed the Mill's Ratio, where we recall that Φ is the standard normal cumulative distribution function and ϕ is the standard normal density.

The main result of this section is the following.

Theorem 6.2.11. *Let $s > 0$ and $\Xi : \mathbb{R} \times [0, \infty) \to \mathbb{R}$.*

Let Ξ satisfy the following conditions

(IV1) (Smoothness) for every $\tau > 0$, $\Xi(\cdot, \tau)$ is twice differentiable.

(IV2) (Positivity) for every $x \in \mathbb{R}$ and $\tau > 0$,

$$\Xi(x, \tau) > 0.$$

(IV3) (Durrleman's Condition) for every $\tau > 0$ and $x \in \mathbb{R}$

$$0 \leq \left(1 - \frac{x\partial_x \Xi}{\Xi}\right)^2 - \frac{1}{4}\Xi^2 \left(\partial_x \Xi\right)^2 + \Xi \partial_{xx}^2 \Xi, \tag{6.4}$$

where we have written Ξ for $\Xi(x, \tau)$.

(IV4) (Slope bound) for every $\tau > 0$ and $x \in \mathbb{R}$

$$\partial_x \Xi(x, \tau) \leq R\left(-d_-(x, \Xi(x, \tau))\right).$$

(IV5) (Monotonicity in τ) for every $x \in \mathbb{R}$, $\Xi(x, \cdot)$ is non-decreasing.

(IV6) (Large moneyness behaviour) for every $\tau > 0$

$$\lim_{x \to \infty} d_+(x, \Xi(x, \tau)) = -\infty.$$

In particular, assuming (IV2), this condition is always satisfied when

$$\limsup_{x \to \infty} \frac{\Xi(x, \tau)}{\sqrt{2x}} \in [0, 1),$$

never satisfied when

$$\limsup_{x \to \infty} \frac{\Xi(x, \tau)}{\sqrt{2x}} > 1,$$

and may, or may not, be satisfied when

$$\limsup_{x \to \infty} \frac{\Xi(x, \tau)}{\sqrt{2x}} = 1.$$

(IV7) (Value at maturity) for every $x \in \mathbb{R}$,

$$\Xi(x, 0) = 0.$$

Then

$$
\begin{aligned}
\widetilde{C} : [0, \infty) \times [0, \infty) &\to \mathbb{R} \\
(K, \tau) &\mapsto \begin{cases} sB(\ln(K/s), \Xi(\ln(K/s), \tau)), & \text{if } K > 0 \\ s, & \text{if } K = 0 \end{cases}
\end{aligned}
$$

is a call price surface parameterised by s that is free of static arbitrage. There exists a non-negative Markov martingale X with the property that $\widetilde{C}(K, \tau) = \mathbb{E}\left((X_\tau - K)^+ | X_0 = s\right)$ *for all* $K, \tau \geq 0$*.*

Proof. It is sufficient to check that the conditions of Theorem 6.2.1 are satisfied by $\widetilde{C}$. We will then have the claimed existence of a martingale from the second part of Theorem 6.2.1.

We will use $x = \ln(K/s)$. In the following, we will sometimes omit the arguments of functions. For example we will write d_+ instead of $d_+(u, v)$ and d_+ instead of $d_+(u, v)$, etc. We will let $\partial_1 B(u, v)$ denote the partial derivative of B with respect to its first argument evaluated at $(u, v) \in \text{interior}(\text{dom}(B))$, $\partial^2_{12} B(u, v)$ denote the mixed partial derivative of B with respect to its first and second arguments evaluated at $(u, v) \in \text{interior}(\text{dom}\, B)$, and so on.

As in Durrleman (Dur03), we use

$$
\begin{aligned}
\partial_1 B(u, v) &= -\exp(u)\Phi\left(d_-\right) \\
\partial_2 B(u, v) &= \phi\left(d_+\right) \\
\partial^2_{11} B(u, v) &= -\exp(u)\Phi\left(d_-\right) + \frac{1}{v}\phi\left(d_+\right) \\
\partial^2_{12} B(u, v) &= \left(-\frac{u}{v^2} + \frac{1}{2}\right)\phi\left(d_+\right)
\end{aligned}
$$

and

$$
\partial^2_{22} B(u, v) = \left(\frac{u^2}{v^3} - \frac{v}{4}\right)\phi\left(d_+\right);
$$

which are trivially obtained using that $\exp(u)\phi(d_-(u, v)) = \phi(d_+(u, v))$ for $u, v \in \mathbb{R}$ with $v \neq 0$ and $\phi'(z) = -z\phi(z)$ for all $z \in \mathbb{R}$.

We now turn to the proof of the present claim.

Case 1: $\tau = 0$

It is enough to ensure that $\widehat{C}(K,0) = (s-K)^+$. This is an obvious consequence of the definition of the function B and in particular that $B(x,0) = (1-\exp(x))^+$.

<u>Case 2: $\tau > 0$</u>

(A1) **Convexity in** K: The following argument for (A1) is due to Durrleman (see (Dur03)).

Fix $(x,\tau) \in \text{interior}(\text{dom}\,\Xi)$ and let $K = s\exp(x)$. We use that $K > 0$ which holds by assumption on s. The positivity of Ξ and Equation (6.4), give that

$$0 \leq \frac{\phi(d_+)}{K^2\Xi}\left(\left(1-\frac{x}{\Xi}\partial_1\Xi\right)^2 - \frac{1}{4}\Xi^2\left(\partial_1\Xi\right)^2 + \Xi\partial_{11}^2\Xi\right),$$

where, here and following, Ξ and d_+ are evaluated at (x,τ) and (x,Ξ) respectively. For the sake of brevity, we will also write d_- for $d_-(x,\Xi)$. Now,

$$\begin{aligned}
&\frac{\phi(d_+)}{K^2\Xi}\left(\left(1-\frac{x}{\Xi}\partial_1\Xi\right)^2 - \frac{1}{4}\Xi^2\left(\partial_1\Xi\right)^2 + \Xi\partial_{11}^2\Xi\right) \\
&= \frac{\phi(d_+)}{K^2}\left(\frac{1}{\Xi} + \left(1-\frac{2x}{\Xi^2}\right)\partial_1\Xi + \left(\frac{x^2}{\Xi^3}-\frac{\Xi}{4}\right)(\partial_1\Xi)^2 + \partial_{11}^2\Xi - \partial_1\Xi\right) \\
&= \frac{1}{K^2}\left(\partial_{11}^2 B + 2\partial_1\Xi\partial_{12}^2 B + (\partial_1\Xi)^2\partial_{22}^2 B + (\partial_{11}^2\Xi - \partial_1\Xi)\partial_2 B - \partial_1 B\right),
\end{aligned}$$

where, here and following, B is evaluated at (x,Ξ),

$$\begin{aligned}
&= \left[\frac{1}{K^2}\partial_{11}^2 B + \frac{1}{K^2}\partial_{12}^2 B\partial_1\Xi\right] - \left[\frac{1}{K^2}\partial_1 B\right] + \left[\left(\frac{1}{K}\partial_{12}^2 B + \frac{1}{K}\partial_{22}^2 B\partial_1\Xi\right)\frac{\partial_1\Xi}{K}\right] \\
&\quad + \left[\left(\frac{1}{K^2}\left(-\partial_1\Xi + \partial_{11}^2\Xi\right)\right)\partial_2 B\right] \\
&= \left\{\left[\frac{1}{K}\left(\partial_{11}^2 B\partial_K x + \partial_{12}^2 B\partial_K\Xi\right)\right] - \left[\frac{1}{K^2}\partial_1 B\right]\right\} \\
&\quad + \left\{\left[\left(\partial_{12}^2 B\partial_K x + \partial_{22}^2 B\partial_K\Xi\right)\partial_K\Xi\right] + \left[\partial_{KK}^2\Xi\partial_2 B\right]\right\} \\
&= \{\partial_K(\partial_1 B\partial_K x)\} + \{\partial_K(\partial_2 B\partial_K\Xi)\} \\
&= \partial_{KK} B.
\end{aligned}$$

We may conclude that

$$\partial_{KK}\widetilde{C}(K,\tau) = s\partial_{KK}B(\ln(K/s),\Xi(\ln(K/s),\tau)) \geq 0$$

as required.

(A2) K **slope**:

Fix $(x, \tau) \in \text{interior}(\text{dom}\,\Xi)$. Write Ξ for $\Xi(x, \tau)$, B for $B(x, \Xi)$ and similarly for d_- and d_+. By assumption, we have that

$$\begin{aligned} \partial_x \Xi \leq R(-d_-) &= \frac{1 - \Phi(-d_-)}{\phi(-d_-)} \\ &= \frac{\Phi(d_-)}{\phi(d_-)} \\ &= \frac{\exp(x)\Phi(d_-)}{\exp(x)\phi(d_-)}. \end{aligned}$$

Then, since $\Xi(x, \tau) > 0$ for $(x, \tau) \in \text{interior}(\text{dom}(B))$ by (IV2), we have

$$\begin{aligned} \partial_x \Xi &\leq R(-d_-) \\ &= \frac{\exp(x)\Phi(d_-)}{\phi(d_+)} \\ &= \frac{-\partial_1 B}{\partial_2 B}. \end{aligned}$$

It follows that

$$\partial_K \widetilde{C}(K, \tau) = \frac{s}{K}\left(\partial_1 B + \partial_2 B \partial_x \Xi\right) \leq 0.$$

(A3) **Monotonicity in** τ:

B is a strictly increasing function of its second argument, since $\partial_2 B(u, v) = \phi(d_+(u, v)) > 0$. Since $\Xi(x, \tau)$ is non-decreasing in τ for each fixed $x \in \mathbb{R}$ by (IV5), it follows that $\widetilde{C}(K, \cdot) = sB(\ln(K/s), \Xi(\ln(K/s), \cdot))$ is non-decreasing for each fixed $K > 0$. The case $K = 0$ is an immediate consequence of the definition of $\widetilde{C}$.

(A4) **Large strike limit**:

Much of the argument for (A4) is taken from Durrleman (see (Dur03)). Fix $\tau > 0$. We must show that

$$\lim_{x \to \infty} sB(x, \Xi(x, \tau)) = s \lim_{x \to \infty} \left[\Phi(d_+(x, \Xi)) - \exp(x)\Phi(d_-(x, \Xi))\right] = 0.$$

We have that

$$\lim_{x \to \infty} d_+(x, \Xi(x, \tau)) = -\infty.$$

It follows from this that

$$\lim_{x \to \infty} \Phi(d_+(x, \Xi)) = 0.$$

It remains to show that

$$\lim_{x\to\infty} \exp(x)\Phi(d_-(x,\Xi)) = 0.$$

To see that this is the case, argue as follows. Let $D = (0,\infty)\times(0,\infty)$. As Durrleman (see (Dur03)) observed, we can use the Arithmetic-Geometric mean inequality and the fact that Ξ is strictly positive (by (IV2)), to get

$$d_-(x,\Xi(x,\tau)) = -\frac{x}{\Xi(x,\tau)} - \frac{\Xi(x,\tau)}{2} \le -\sqrt{2x}, \qquad \forall (x,\tau)\in D.$$

Now, $\Phi(\cdot)$ is increasing, so

$$0 \le \exp(x)\Phi(d_-) \le \exp(x)\Phi(-\sqrt{2x}), \qquad \forall (x,\tau)\in D.$$

By L'Hôpital's Rule, we have $\exp(x)\Phi(-\sqrt{2x}) \to 0$ as $x\to\infty$, so that

$$\lim_{x\to\infty} \exp(x)\Phi(d_-) = 0.$$

We have shown that

$$\lim_{x\to\infty} sB(x,\Xi(x,\tau)) = 0,$$

given that $d_+ \to \infty$.

Let

$$U := \limsup_{x\to\infty} \frac{\Xi(x,\tau)}{\sqrt{2x}} \in [0,1),$$

then $d_+ \to -\infty$ as $x\to\infty$.

We now show that this is sufficient for $d_+ \to -\infty$ as $x\to\infty$. Because $U\in[0,1)$, we have $(\forall\epsilon>0)$ $(\exists M>0)$ $(\forall x>M)$

$$\frac{\Xi(x,\tau)}{\sqrt{2x}} < U+\epsilon. \tag{6.5}$$

In particular, we may choose $\epsilon = \dfrac{1-U}{2}$, such that $0 < U+\epsilon = (1+U)/2 < 1$. From Equation (6.5), we have for all x large enough that

$$\frac{\sqrt{2x}}{\Xi(x,\tau)} > \frac{1}{U+\epsilon} \quad \text{and} \quad -\frac{\Xi(x,\tau)}{\sqrt{2x}} > -(U+\epsilon).$$

Therefore, since $0<U+\epsilon<1$,

$$\frac{\sqrt{2x}}{\Xi(x,\tau)} - \frac{\Xi(x,\tau)}{\sqrt{2x}} > \frac{1-(U+\epsilon)^2}{U+\epsilon} > 0,$$

for x large enough, by the choice of ϵ. It follows, using Durrleman's decomposition (see (Dur03)) of d_+, that

$$\begin{aligned}
\limsup_{x\to\infty} d_+(x, \Xi(x,\tau)) &= \limsup_{x\to\infty} -\frac{\sqrt{x}}{\sqrt{2}}\left(\frac{\sqrt{2x}}{\Xi(x,\tau)} - \frac{\Xi(x,\tau)}{\sqrt{2x}}\right) \\
&= -\infty \\
&= \lim_{x\to\infty} d_+(x, \Xi(x,\tau))
\end{aligned}$$

So $d_+ \to -\infty$ as $x \to \infty$.

We now show that it may or may not happen that $d_+ \to -\infty$ as $x \to \infty$ when $U = 1$. Let

$$\Xi(x,1) = \sqrt{2}(x^{\frac{1}{2}} - x^{\frac{1}{4}}), \quad x \text{ large enough.}$$

It satisfies $d_+(x,\Xi) \to -\infty$ as $x \to \infty$ and $\lim_{x\to\infty} \Xi(x,1)/\sqrt{2x} = 1$.

Now suppose that

$$\Xi(x,1) = \sqrt{2x}, \quad x \text{ large enough.}$$

Clearly, $\limsup_{x\to\infty} \Xi(x,1)/\sqrt{2x} = 1$. But,

$$d_+(x,\Xi) = -\frac{x}{\sqrt{2x}} + \frac{\sqrt{2x}}{2} = -\frac{\sqrt{x}}{\sqrt{2}} + \frac{\sqrt{x}}{\sqrt{2}} = 0 \nrightarrow -\infty.$$

We do not show here that $d_+ \nrightarrow -\infty$ when $U > 1$; see the proof of Theorem 6.2.15.

(A5) **Bounds**:

As already noted, we have for all $x \in \mathbb{R}$ and $\theta \in [0,\infty]$ that

$$(1 - \exp(x))^+ \leq B(x,\theta) \leq 1.$$

Multiplying through by s, which is assumed to be positive, and recalling that $x = \ln(K/s)$ we are done.

(A6) **Expiry value**:

Immediate.

$\square$

Remark 6.2.12. It may be shown by a direct argument that $\lim_{K \to 0^+} \widetilde{C}(K, \tau) = s$. Alternatively, we can use that $\widetilde{C}(K, \tau)$ satisfies

$$(s-K)^+ \leq \widetilde{C}(K,\tau) \leq s, \quad \forall \tau \geq 0, K > 0,$$

so that the continuous extension of $\widetilde{C}(\cdot, \cdot)$ to $[0, \infty) \times [0, \infty)$ is

$$\widetilde{C}(K,\tau) = \begin{cases} sB(\ln(K/s), \Xi(\ln(K/s), \tau)), & \text{if } K > 0 \\ s, & \text{if } K = 0. \end{cases}$$

Remark 6.2.13. In (Lee04), Lee assumes that the stock price is a non-negative martingale and derives that Ξ must satisfy

$$\limsup_{x \to -\infty} \frac{\Xi^2(x,\tau)}{|x|} \in [0, 2], \tag{6.6}$$

equivalently

$$\limsup_{x \to -\infty} \frac{\Xi(x,\tau)}{\sqrt{2\,|x|}} \in [0, 1],$$

for each $\tau \geq 0$. We do not require such a condition. However, we know that this property must be satisfied by any function Ξ satisfying all the conditions in Theorem 6.2.11. Indeed, we have shown that there exists a non-negative martingale matching the call surface $\widetilde{C}$: Lee's argument may then be applied and we can conclude that Equation (6.6) holds. It is difficult to see how we may directly derive Equation (6.6) from the analytic conditions that we have imposed.

In (Lee05), Lee presents Gatheral's bounds on the derivative of (time-scaled) implied volatility (in log-moneyness form).
We now show that the lower bound is a consequence of the other assumptions of Theorem 6.2.11, most importantly the convexity assumption.

Corollary 6.2.14. *Let Ξ and $\widetilde{C}$ be as in the Theorem 6.2.11 and suppose that (IV3), (IV4) hold as well as $\Xi \in (0, \infty)$. Then for each $x \in \mathbb{R}$ and $\tau > 0$*

$$-R(d_+(x, \Xi(x,\tau))) \leq \partial_x \Xi(x,\tau) \leq R(-d_-(x, \Xi(x,\tau))).$$

Proof. The upper bound holds by Assumption (IV4). We have shown that $(s - K)^+ \leq \widehat{C}(K,\tau) \leq s$, for each K, see (A5) in the proof of Theorem 6.2.11 For the lower bound we use Lemma 4.2.1 to get that for $K > 0$

$$\begin{aligned}
\partial_K \widetilde{C}(K,\tau) &= \frac{s}{K}\left(\partial_1 B + \partial_2 B \partial_x \Xi\right) \\
&\geq \frac{\widetilde{C}(K,\tau) - s}{K} \\
&= \frac{sB - s}{K} \\
&= \frac{s(B-1)}{K}.
\end{aligned}$$

Hence,

$$\partial_1 B + \partial_2 B \partial_x \Xi \geq B - 1,$$

so that

$$\begin{aligned}
\partial_x \Xi &\geq \frac{B - 1 - \partial_1 B}{\partial_2 B} \\
&= \frac{\Phi(d_+) - e^x \Phi(d_-) - 1 + e^x \Phi(d_-)}{\partial_2 B} \\
&= \frac{\Phi(d_+) - 1}{\phi(d_+)} \\
&= -\frac{1 - \Phi(d_+)}{\phi(d_+)} \\
&= -R(d_+).
\end{aligned}$$

□

We show that the stated conditions are necessary under the smoothness and positivity requirements to ensure that the resulting call surface is free from static arbitrage.

Theorem 6.2.15. *Let $s > 0$ and $\Xi : \mathbb{R} \times [0,\infty) \to \mathbb{R}$. Let Ξ satisfy the following conditions*

(1) (Smoothness) For every $\tau > 0$, $\Xi(\cdot,\tau)$ is twice differentiable.

(2) (Positivity) For every $x \in \mathbb{R}$ and $\tau > 0$,

$$\Xi(x,\tau) > 0.$$

Let

$$\widetilde{C} : [0,\infty) \times [0,\infty) \to \mathbb{R}$$
$$(K,\tau) \mapsto \begin{cases} sB(\ln(K/s), \Xi(\ln(K/s),\tau)), & \textit{if } K > 0, \\ s, & \textit{if } K = 0. \end{cases}$$

Then if Ξ violates any of the remaining conditions (IV3)-(IV7) of Theorem 6.2.11, $\widetilde{C}$ is not a call surface free from static arbitrage.

Proof. The reasoning of Theorem 6.2.11 may easily be reversed for each condition except for (IV6): the Large moneyness behaviour condition. We now show the necessity of this condition.

Fix $\tau > 0$. First note that by the positivity assumption it is necessarily the case that

$$\limsup_{x\to\infty} \frac{\Xi(x,\tau)}{\sqrt{2x}} \in [0,\infty], \quad \forall \tau > 0.$$

Let

$$U := \limsup_{x\to\infty} \frac{\Xi(x,\tau)}{\sqrt{2x}}.$$

We must show that if the large moneyness behaviour assumption on Ξ is not met, then either

$$\lim_{x\to\infty} B(x,\Xi(x,\tau)) \neq 0 \tag{6.7}$$

or

$$\lim_{x\to\infty} B(x,\Xi(x,\tau)) \text{ does not exist.} \tag{6.8}$$

But

$$B(x,\Xi(x,\tau)) = \Phi(d_+(x,\Xi)) - \exp(x)\Phi(d_-(x,\Xi))$$

and

$$\lim_{x\to\infty} \exp(x)\Phi(d_-(x,\Xi)) = 0,$$

see the argument in the proof of property (A4) in Theorem 6.2.11. Also Φ is a continuous function, so that Equation (6.7), respectively Equation (6.8), can be satisfied if and only if either

$$\lim_{x\to\infty} d_+(x,\Xi) \neq -\infty,$$

or.

$$\lim_{x\to\infty} d_+(x,\Xi) \text{ does not exist.}$$

□

Some remarks regarding the technical conditions on Ξ are in order. We first note the following simple result.

Proposition 6.2.16. *Suppose that $K > 0$, $\tau > 0$, and $S_t > 0$. Then it holds $\mathbb{P}$-a.s that*

(1) $\Sigma_t(K,\tau) = \infty$ if and only if $\mathbb{E}\left((S_{t+\tau} - K)^+ | \mathscr{F}_t\right) = S_t$, and

(2) $\Sigma_t(K,\tau) = 0$ if and only if $\mathbb{E}\left((S_{t+\tau} - K)^+ | \mathscr{F}_t\right) = (S_t - K)^+$.

Proof. Simple consequence of the definitions of the Black-Scholes function and the implied volatility function. See also Proposition 4.3.9. □

Remark 6.2.17. Suppose that $K, \tau, S_t > 0$. We have that $\Sigma(K,\tau) = \infty$ if and only if $\Xi(\ln(K/S_t),\tau) = \infty$. Also, $\Sigma(K,\tau) = 0$ if and only if $\Xi(\ln(K/S_t),\tau) = 0$.

Remark 6.2.18. $\Xi(\cdot,\tau)$ **is twice differentiable in its first argument**:
This condition is necessary for Durrleman's condition (i.e. (IV3) of Theorem 6.2.11) to be equivalent to the constructed call price surface, $\widetilde{C}$, being convex in K for each $\tau > 0$. This convexity condition on $\widetilde{C}$ is necessary, see Proposition 4.3.4, for $\widetilde{C}$ to be free from static arbitrage.

Remark 6.2.19. $\Xi(x,\tau) > 0$ **provided that** $\tau > 0$ **and** $x \in \mathbb{R}$:
Suppose that we had $\Xi(x',\tau') = 0$ for some $\tau' > 0$ and $x' \in \mathbb{R}$. In the constructed call price surface, $\widetilde{C}$, we would then have $\sqrt{\tau'}\Sigma(K',\tau') = 0$, where $K' = se^{x'} > 0$ with $s > 0$ being the current stock price. We would then have

$$\widetilde{C}(K',\tau') = sB(x',0) = (s - K')^+.$$

There are then two cases: $0 < s \leq K'$ and $s > K' > 0$. Let $\mathbb{P}$ be the probability measure of the (Markov) martingale process constructed from the call surface $\widetilde{C}$. Suppose first that $s \leq K'$, then $\widetilde{C}(K',\tau) = 0 (= (s-K')^+)$ is equivalent to $\mathbb{P}\left(S_{\tau'} > K'\right) = 0$, by Proposition 4.3.9 (Ai), so that if $\mathbb{P}\left(S_{\tau'} > K'\right) = 0$, then $\mathbb{P}\left(S_{\tau'} > \widehat{K}\right) = 0$ for all $\widehat{K} > K'$. Therefore for $\widehat{x} > x'$, we must have that $\Xi(\widehat{x},\tau) = 0$. Suppose now that $s > K'$, then $\widetilde{C}(K',\tau) = (s-K')^+ = s - K'$ if and

only if $\mathbb{P}\left(S_{\tau'} < K'\right) = 0$, by Proposition 4.3.9 (Aii), therefore for all $0 < K^* < K'$ we must have $\mathbb{P}\left(S_{\tau'} < K^*\right) = 0$ and so $\Xi(x^*, \tau) = 0$.
Such restrictions on the support of the stock price at future times may be an unrealistic modelling function.

Remark 6.2.20. $\Xi(x, \tau) = \infty$ **provided that** $\tau > 0$ **and** $x \in \mathbb{R}$:
Suppose that $\Xi(x', \tau') = \infty$. We have $K' = se^{x'} > 0$. Then $\widetilde{C}(K', \tau') = s$, the current stock price. From Proposition 4.3.9 (Aiii), this implies that $\mathbb{P}\left(S_{\tau'} = 0\right) = 1$. Again, this may be undesirable from a modelling point of view.

6.3 Examples: Parameterisations of the Implied Volatility Smile

In this section, we investigate whether or not some proposed parameterisations of the implied volatility smile are arbitrage-free or not. By the *implied volatility smile*, we mean the function $x \mapsto \Xi(x, \tau)$, i.e. time scaled implied volatility (in log-moneyness form) with time to expiry fixed. That is we fix τ and analyse the dependence of Ξ on x alone. It turns out that the parameterisations we examine are specified as functions of log-moneyness alone. In order to check whether or not our conditions on the implied volatility are satisfied, we consider the proposed parameterisations as $x \mapsto \Xi(x, 1)$.
The conditions that we need to check and, in particular, Durrleman's Condition are onerous to check algebraically. We therefore resort to graphical analysis. For each of the three parameterisations we plot $\Xi(\cdot, 1)$, $\partial_1\Xi(\cdot, 1)$, and $\mathscr{I}\Xi(\cdot, 1)$. The parameter choice is set out in the subsections below. The function $\mathscr{I}\Xi$ is given by

$$\mathscr{I}\Xi(x,\tau) := \left(1 - \frac{x\partial_x\Xi(x,\tau)}{\Xi(x,\tau)}\right)^2 - \frac{1}{4}\Xi(x,\tau)^2\left(\partial_x\Xi(x,\tau)\right)^2 + \Xi(x,\tau)\partial^2_{xx}\Xi(x,\tau),$$

where $\partial_x := \dfrac{\partial}{\partial x}$ and $\partial_{xx} := \dfrac{\partial^2}{\partial x^2}$. Observe that this is the right hand side of Durrleman's condition in Equation (6.4).
From Theorem 6.2.15, we know that the no arbitrage condition requires that at each $\tau > 0$ the slope of the implied volatility smile, i.e. $\partial_x\Xi$, satisfies

$$\partial_x\Xi \leq R\left(-d_-(x, \Xi(x,\tau))\right),$$

where R is the Mill's Ratio, see Definition 6.2.10. Another necessary condition is

$$\mathscr{I}\Xi \geq 0 \tag{6.9}$$

for all x and $\tau > 0$. We note that these conditions are non-linear, so we expect the properties of the parameterisations to be very sensitive to change. We therefore use the specific numerical values of the various parameters indicated by the author of the parameterisation.

We now present the graphical analysis of three proposed smile parameterisations.

6.3.1 Gatheral's "SVI" Parameterisation

In (Gat06) and (Gat04), Gatheral proposed the implied volatility smile parameterisation

$$\Xi^{\text{SVI}}(x,\tau) = \sqrt{\left|a + b(\rho(x-m) + \sqrt{(x-m)^2+\sigma^2})\right|};$$

the absolute value and the square root appear because Gatheral specified Ξ^2 instead of Ξ. We will refer to this parameterisation as the SVI parameterisation. For a fixed time to maturity, which we will take to be one time unit, Gatheral ((Gat04)) suggests the parameter values

$$a = 0.04 \quad b = 0.8 \quad \sigma = 0.1 \quad \rho = -0.4 \text{ and } \quad m = 0.$$

Figure 6.2 shows that $\partial_x \Xi^{\text{SVI}}$ is below the no-arbitrage upper bound at least for x close to zero. In Figure 6.3, we can see that $\mathscr{I}\Xi$ is negative for $x \in (-1, -0.5)$. Therefore, the inequality in Equation (6.9) is not satisfied for all x and hence the parameterisation is not arbitrage free. (Gatheral's condition that $b(1+|\rho|) \leq 4/\tau$ is clearly satisfied by the parameters we used.)

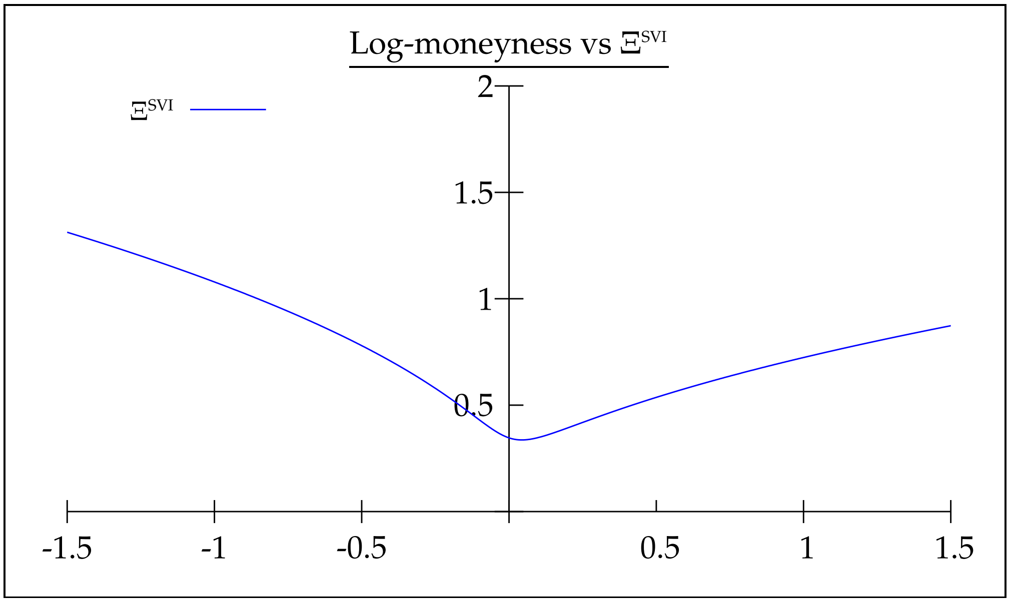

Figure 6.1: Plot of the implied volatility function Ξ^{SVI}

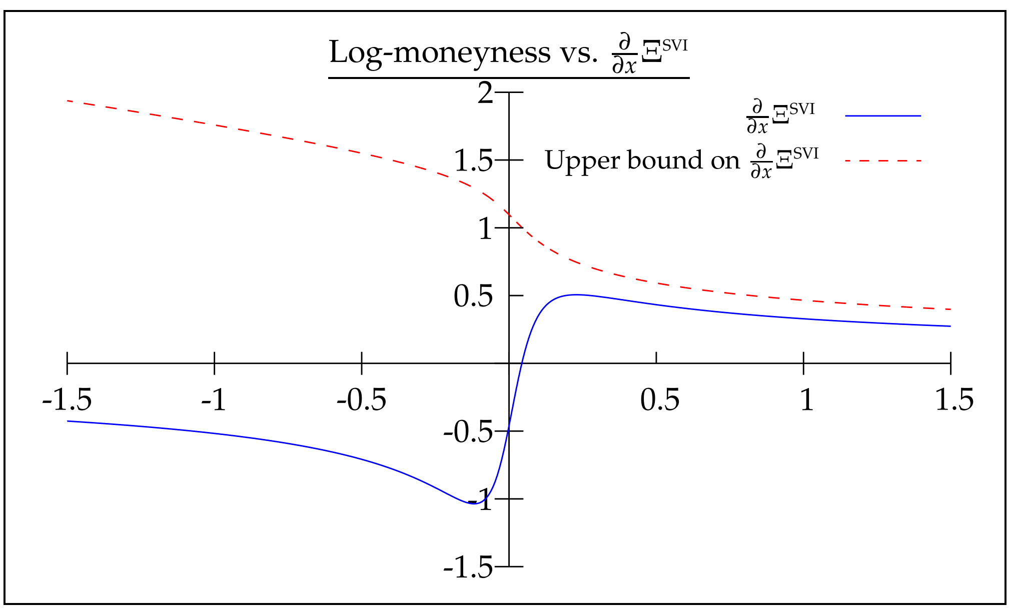

Figure 6.2: Plot of $\frac{\partial}{\partial x}\Xi^{\mathrm{SVI}}$ and the no arbitrage upper bound on $\frac{\partial}{\partial x}\Xi^{\mathrm{SVI}}$

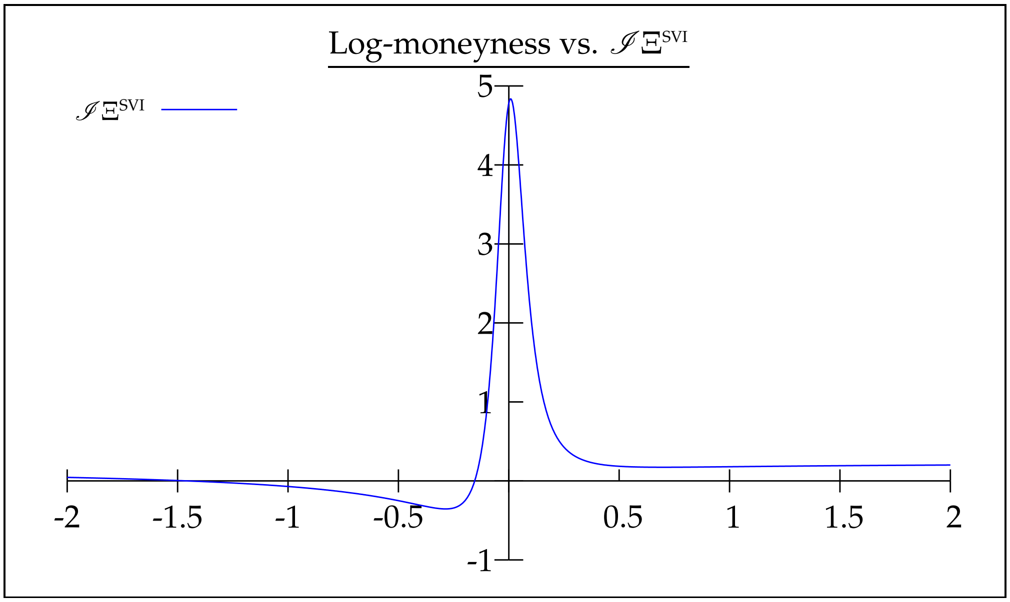

Figure 6.3: Plot of $\mathscr{I}\Xi^{\mathrm{SVI}}$

6.3.2 Avellaneda's "SABR" Parameterisation

In (Ave05), Avellaneda suggests the following parameterisation

$$\Xi^{\mathrm{SABR}}(x,1) = \frac{\kappa\,|x|}{\ln(\kappa\,|f(x)| + \sqrt{1+\kappa^2 f(x)^2})}$$

where

$$f(x) = \frac{1-\exp(-\beta x)}{\sigma_0 \beta}.$$

Avellaneda takes

$$\sigma_0 = 0.2 \quad \beta = -4.0 \quad \text{ and } \kappa = 0.5.$$

From Figure 6.5, the slope of Ξ^{SABR} with respect to log-moneyness is dominated by the no-arbitrage upper bound, at least in the region examined. From Figure 6.6, we can see that the condition in Equation (6.9) is not satisfied by Ξ^{SABR}. Therefore the parametersation is not arbitrage free as Avellaneda claims.

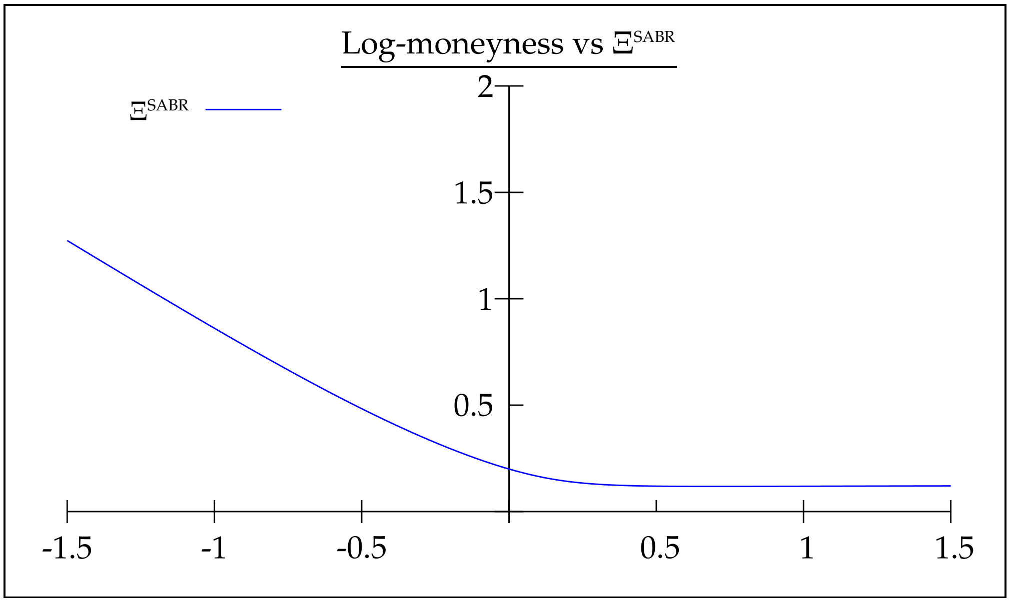

Figure 6.4: Plot of the implied volatility function Ξ^{SABR}

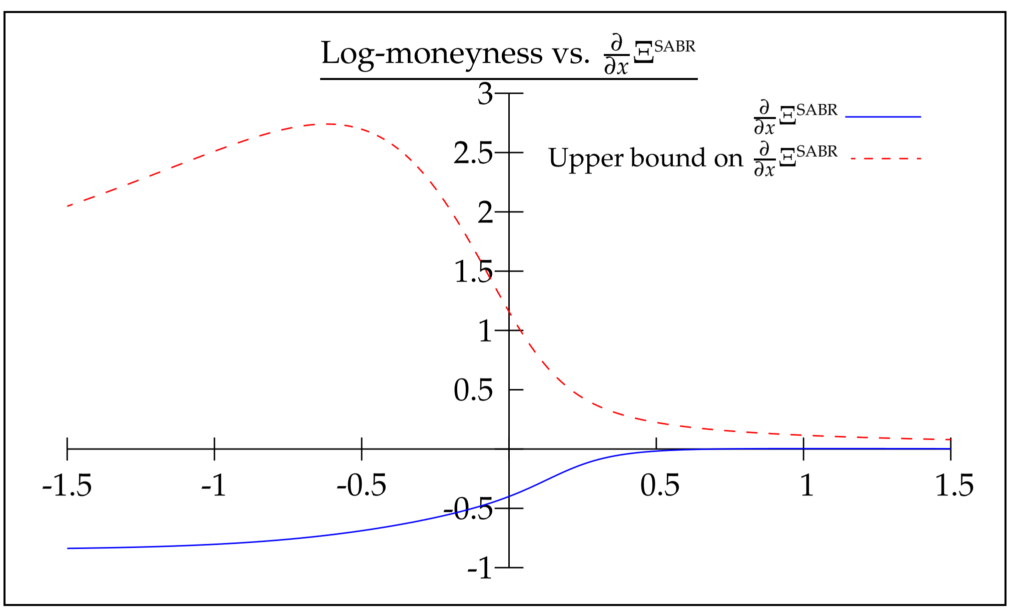

Figure 6.5: Plot of $\frac{\partial}{\partial x}\Xi^{\text{SABR}}$ and the no arbitrage upper bound on $\frac{\partial}{\partial x}\Xi^{\text{SABR}}$

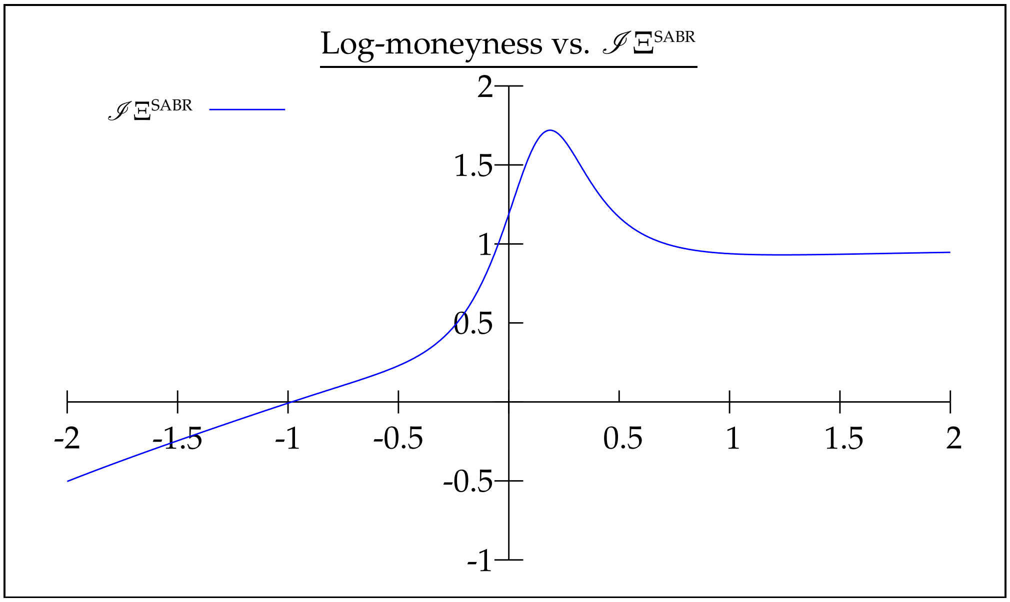

Figure 6.6: Plot of $\mathscr{I}\Xi^{\text{SABR}}$

6.3.3 Quadratic Parameterisation

We take, as suggested in (Ave05), a quadratic parameterisation for the smile curve, in particular,

$$\Xi(x,1) = 0.16 - 0.34x + 4.45x^2,$$

taking the coefficients as given in (Ave05). This is not really meant as a plausible candidate since it does not satisfy Lee's moment formula, i.e.

$$\lim_{x\to\infty} \frac{\Xi}{\sqrt{2x}} \in [0,1] \quad \text{and} \quad \lim_{x\to-\infty} \frac{\Xi}{\sqrt{2\,|x|}} \in [0,1].$$

However, it appears that close to zero this parameterisation is often used and the behaviour at large $|x|$ is specified by some other formula if it is required. From Figure 6.8, the slope of Ξ^{QUAD} with respect to log-moneyness is not dominated by the no-arbitrage upper bound in the region examined. In addition, from Figure 6.9, we can see that the condition (6.9) is not satisfied by Ξ^{QUAD}. Therefore the parametrisation is not arbitrage free, even quite close to zero log-moneyness.

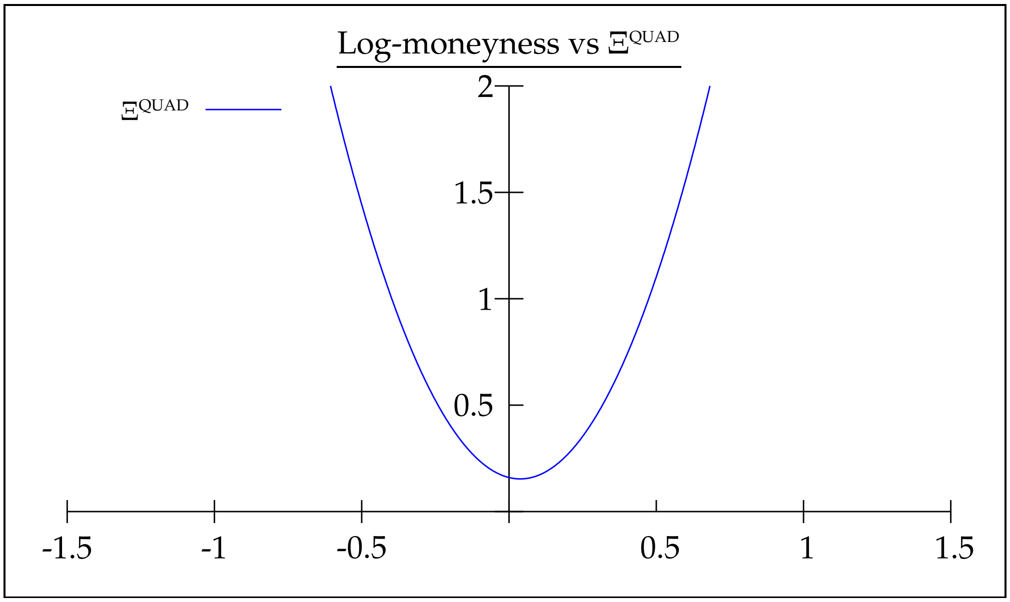

Figure 6.7: Plot of the implied volatility function Ξ^{QUAD}

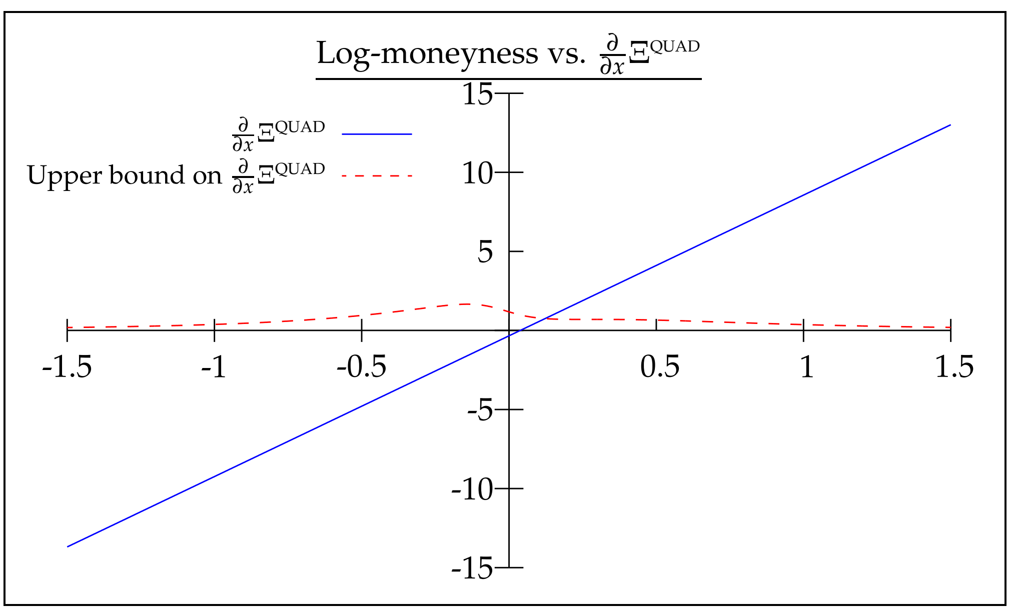

Figure 6.8: Plot of $\frac{\partial}{\partial x}\Xi^{\text{QUAD}}$ and the no arbitrage upper bound on $\frac{\partial}{\partial x}\Xi^{\text{QUAD}}$

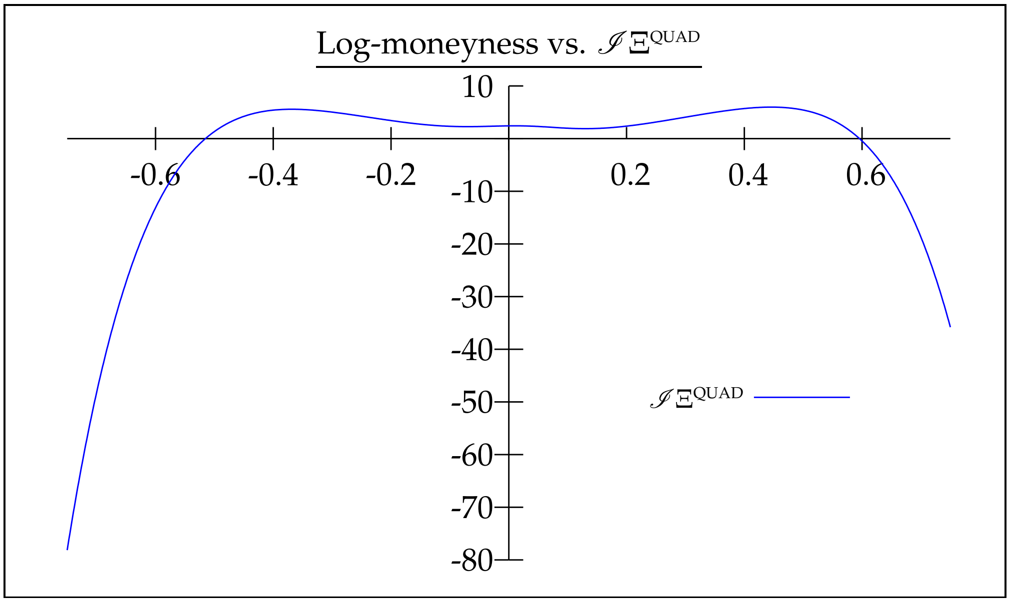

Figure 6.9: Plot of $\mathscr{I}\Xi^{\mathrm{QUAD}}$

6.4 Summary of Results and Conclusion

In this chapter, we have

(1) presented sufficient and necessary conditions for a call price surface to be free from static arbitrage;

(2) presented sufficient and close to necessary conditions for a time-scaled implied volatility surface in log-moneyness form to be free from static arbitrage, thereby providing a solution to an open problem in SIV modelling (see (a) on p. 271 and (SIV.3) on p. 270 of (MR05); and

(3) investigated whether or not some proposed parameterisations of the time-scaled implied volatility smile are free of static arbitrage.

Chapter 7

On the Relationship Between the Call Price Surface and the Implied Volatility Surface Close to Expiry

In this chapter – a reorganised version of the publication (RR), we examine the asymptotic behaviour of the call price surface and the associated Black-Scholes implied volatility surface in the small time to expiry limit under the condition of no arbitrage. Exact asymptotics are derived. In the process of obtaining this result, we obtain a number of interesting properties of the *time-scaled implied volatility*, i.e. the square root of the time to maturity times implied volatility (see Definition 7.3.1). It is worth stressing that these asymptotic results are universal, in the sense that they do not depend on a choice of a model for an underlying asset. We therefore term the results "model-free asymptotics" since they may be expressed in terms of the call pricing function, the stock price, the option time to expiry and the strike price. An incidental result is the small time to expiry asymptotics of the Black-Scholes call option price, which we present. Our other main result is a proof of the claim that implied volatility may fail to converge (to a finite or infinite limit) as time to expiry goes to zero, even in the simple case of the Black-Scholes model with time-dependent volatility. This observation emphasises the advantage of dealing with the time-scaled implied volatility, as opposed to the more standard concept of implied volatility.

A number of "model-free" asymptotic formulae for implied volatility have previously been presented. (See, for example, (BCS96), (BS88), (CN01), (Cha96), (CCIV06), (CM96) and (Li05).) The interested reader can review (Li05) for a good review of the extant literature. It should be acknowledged that the asymptotic behaviour in the at-the-money case (that is, for $K = S$) is well-established (see (BS88)). In the at-the-money case, it is easy to obtain an exact expression for the implied volatility in terms of the inverse error function. The asymptotic properties of the inverse error function are well known. We re-derive this result in the present chapter using our general method. The work (CCIV06) contains some asymptotic formulae involving the inverse error function for the the case $K \neq S$.

An entirely different approach to the study of small time to expiry asymptotics of implied volatility is presented in (BBF02) and (BBF04). In these papers, the stock price is driven by a diffusion model and a partial differential equation for the implied volatility is derived. The limiting behaviour of this equation is given in terms of various characteristic properties of the diffusion. We call these derived asymptotic formulae "model-dependent".

As already mentioned, this chapter is largely concerned with model-free asymptotics.

This chapter is organised as follows. In Section 7.1, we present the setup as well as necessary results from the literature. Section 7.2 presents the lemmas used in the proof of our main results, making some comments on their application. Proofs of the lemmas are relegated to the Appendix to Chapter 7 which starts on page 145. The main results are presented in Section 7.3. Section 7.4 concludes.

7.1 Background

7.1.1 Setup

For simplicity, we assume throughout that the risk-free interest rate and dividend yield are zero. Throughout we will be considering an instantaneous call price surface. In this context, we have found it convenient to simply assume that the surface satisfies certain properties which we call static no-arbitrage constraints (see (HHK07)). In fact, these properties can be established in a model where the underlying asset is a càdlàg martingale in a filtration satisfying the usual conditions (see Propositions 4.3.4 and 4.3.8). We prefer our "axiomatic" setup since it works well with the modern situation in which the call price surface may actually be taken to be completely specified by the market and not to come from a pre-specified model. Since we consider the market instantaneously, we have no need to consider the current time. Indeed, we may simply suppose that the instantaneous stock price, S, is a strictly positive constant. As for the call price, it is clearly sufficient to suppose that it is a given function of the strike price and time to expiry only.

Standing assumptions. We assume that for each $K > 0$ the call pricing function $C : (0, \infty) \times (0, \infty) \to [0, \infty)$:

(1) satisfies the no arbitrage bounds

$$(S - K)^+ \leq C(K, \tau) \leq S, \ \forall \quad \tau > 0; \tag{7.1}$$

(2) converges to the option payoff as time to expiry goes to zero, i.e.

$$\lim_{\tau\to 0^+} C(K,\tau) = (S-K)^+; \text{ and} \tag{7.2}$$

(3) is a non-decreasing function of time to expiry, i.e.

$$\tau \mapsto C(K,\tau) \text{ is non-decreasing.} \tag{7.3}$$

7.1.2 Auxiliary Facts

We require some results on certain special functions, in particular, the error function, upper incomplete Gamma function, and an unnamed function closely related to the Lambert W function (see (CGH[+]96) for the latter). We present the asymptotics we require and refer the reader to the Appendix for the proofs.

Lemma 7.1.1. *Recall that the error function is defined as*

$$\begin{aligned} \operatorname{erf} : \mathbb{R} &\to \mathbb{R} \\ \theta &\mapsto \frac{2}{\sqrt{\pi}} \int_0^\theta \exp(-t^2)\,\mathrm{d}t \end{aligned}$$

and

$$\operatorname{erf}(\theta) \sim \frac{2}{\sqrt{\pi}}\theta, \quad \theta \to 0^+.$$

Proof. See Lemma A.5.1 which starts on page 140 of the Appendix. □

Lemma 7.1.2. *Fix $\alpha \geq 0$. Let $\Lambda_\alpha(z) = y$ be the unique positive solution of*

$$y^\alpha e^y = z,$$

for z, y large enough. Then

$$\Lambda_\alpha(z) \sim \ln(z), \quad z \to \infty.$$

Proof. See Lemma A.5.2 on page 140 of the Appendix. □

Lemma 7.1.3. *For each fixed $a \in \mathbb{R}$,*

$$\Gamma(a,z) \sim z^{a-1}e^{-z}, \quad z \to \infty,$$

where

$$\begin{aligned} \Gamma : \mathbb{R} \times (0,\infty) &\to \mathbb{R} \\ (a,z) &\mapsto \int_z^\infty t^{a-1}e^{-t}\,\mathrm{d}t \end{aligned}$$

is termed the Upper Incomplete Gamma function.

Proof. See Lemma A.5.3 on page 140 of the Appendix. $\square$

7.2 Lemmas

The following result provides a representation of the Black-Scholes formula that is particularly convenient in the study of the small time properties of the Black-Scholes formula.

Lemma 7.2.1. *The Black-Scholes call pricing function, C^{BS} (cf. Definition 3.3.1 on page 15), admits the representation*

$$C^{BS}(K, \tau, \sigma) = (S - K)^+ + S \int_0^\theta \phi \left(-\frac{x}{v} + \frac{v}{2} \right) \mathrm{d}v, \tag{7.4}$$

where $x = \ln(K/S)$, $\theta = \sigma\sqrt{\tau}$ and ϕ is the standard normal density.

Proof. Given in the Appendix, starting from page 145. $\square$

Remark 7.2.2. Observe that the integrand of the integral in Equation (7.4), $\phi(-x/v + v/2)$, converges to 0 as $v \to 0^+$ if and only if $x \neq 0$. When $x = 0$ it converges to $(\sqrt{2\pi})^{-1}$. This is a consequence of the fact that the transition density of geometric Brownian motion started at S converges to the Dirac measure concentrated at S as time goes to zero. The fact that we obtain markedly different behaviour for at-the-money and not at-the-money implied volatility stems from this basic fact.

To simplify the development of the sequel we introduce the following function (see the statement of Lemma 7.2.1 for its origin), since it is useful to separate the temporal structure of the call option pricing function into the upper limit of integration.

Definition 7.2.3. Let F be defined by

$$\begin{aligned} F : \mathbb{R} \times [0, \infty] &\to [0, \infty) \\ (x, \theta) &\mapsto \int_0^\theta \phi \left(-\frac{x}{v} + \frac{v}{2} \right) \mathrm{d}v. \end{aligned}$$

We now develop some properties of the function F.

Lemma 7.2.4. *For every $x \in \mathbb{R}$, $F(x, \cdot)$ is continuous and strictly increasing. For every fixed non-negative x (non-positive x, respectively) $F(x, \cdot)$ ($\exp(-x)F(x, \cdot)$, respectively) is a strictly increasing, continuous cumulative distribution function.*

Proof. See the Appendix starting from page 146. □

We now obtain the asymptotics of $F(x, \theta)$ as $\theta \to 0^+$. We find that two distinct asymptotic behaviours occur depending on whether $x = 0$ or not. In financial parlance, the case $x = 0$ corresponds to an option being at-the-money. The term arises since $x = \ln(K/S) = 0$ if and only if $K = S$, that is, when the stock and strike prices coincide.

Lemma 7.2.5. *Let* $\operatorname{erf}(\cdot)$ *denote the error function defined in Lemma 7.1.1. Then, for $x = 0$,*

$$F(x, \theta) = \operatorname{erf}\left(\frac{\theta}{2\sqrt{2}}\right) \sim \frac{\theta}{\sqrt{2\pi}}, \quad \theta \to 0^+.$$

Proof. See the Appendix starting from 146. □

Lemma 7.2.6. *Let* $\Gamma(\cdot, \cdot)$ *denote the Upper Incomplete Gamma function defined in Lemma 7.1.3. For $x \neq 0$,*

$$F(x, \theta) \sim \frac{|x| \exp(x/2)}{4\sqrt{\pi}} \Gamma\left(-\frac{1}{2}, \frac{x^2}{2\theta^2}\right) \tag{7.5}$$

$$\sim \frac{\theta^3}{\sqrt{2\pi}x^2} \exp\left(-\frac{x^2 - \theta^2 x}{2\theta^2}\right), \tag{7.6}$$

both as $\theta \to 0^+$.

Proof. See the Appendix starting from page 147. □

7.3 Main Results

In this section, we work with the representation of the Black-Scholes formula

$$C^{BS}(K, \tau, \sigma) = (S - K)^+ + S \int_0^\theta \phi\left(-\frac{x}{v} + \frac{v}{2}\right) \mathrm{d}v, \tag{7.7}$$

where $x = \ln(K/S)$, $\theta = \sigma\sqrt{\tau}$ and ϕ is the standard normal density. This representation is derived in Lemma 7.2.1. We write it in terms of F as follows:

$$C^{BS}(K, \tau, \sigma) = (S - K)^+ + SF(\ln(K/s), \sigma\sqrt{\tau}).$$

7.3.1 Time Scaled Implied Volatility

In Equation (7.7), after fixing x we are left with the term $\theta = \sigma\sqrt{\tau}$. We therefore find it useful to develop some properties of the associated map $(K, \tau) \mapsto \Sigma(K, \tau)\sqrt{\tau}$.

Definition 7.3.1. We term the map $\Theta : (0, \infty) \times (0, \infty) \to [0, \infty]$ defined by

$$\Theta(K, \tau) = \Sigma(K, \tau)\sqrt{\tau}$$

the *time-scaled implied volatility*.

The next result yields some basic properties of the time-scaled implied volatility.

Proposition 7.3.2. *Suppose that $S > 0$. Let $0 \leq a < b \leq \infty$. Under Assumptions* (7.1), (7.2) *and* (7.3), *it holds for every $K > 0$ that:*

(1) $\tau \mapsto \Theta(K, \tau)$ *is non-decreasing on* $(0, \infty)$*;*

(2) $\lim_{\tau \to 0^+} \Theta(K, \tau) = 0$*;*

(3) $\lim_{\tau \to \infty} \Theta(K, \tau) = \infty$*, when* $C(K, \tau) \to S$ *as* $\tau \to \infty$*;*

(4) $\tau \mapsto \Theta(K, \tau)$ *is strictly increasing on* (a, b) *if and only if* $C(K, \cdot)$ *is strictly increasing on* (a, b)*; and*

(5) $\tau \mapsto \Theta(K, \tau)$ *is (right) continuous on* (a, b) *if and only if* $C(K, \cdot)$ *is (right) continuous on* (a, b).

Proof. Fix $K > 0$ and write x for $\ln(K/S)$. From Lemma 7.2.1, the definition of Θ and the fact that $S > 0$ it holds that

$$\frac{C(K, \tau) - (S - K)^+}{S} = F\left(x, \Theta(K, \tau)\right), \quad \forall\, \tau > 0. \tag{7.8}$$

From Lemma 7.2.4, $F(x, \cdot)$ is continuous and strictly increasing. It therefore has a uniquely defined, strictly increasing, continuous inverse which we denote by $F^{-1}(x, \cdot)$. It holds that

$$F^{-1}\left(x, \frac{C(K, \tau) - (S - K)^+}{S}\right) = \Theta(K, \tau), \quad \forall\, \tau > 0. \tag{7.9}$$

We now prove each part of the claim.

(1) $(C(K,\cdot) - (S-K)^+)/S$ is non-decreasing by Assumption (7.3). We have noted that $F^{-1}(x,\cdot)$ is continuous and strictly increasing. The first part of the claim follows using Equation (7.9).

(2) Using Assumption (7.2), we have that

$$\lim_{\tau \to 0^+} (C(K,\tau) - (S-K)^+)/S = 0.$$

Note now that $F^{-1}(x,0) = 0$ by Lemma 7.2.4. Take the limit of both sides of Equation (7.9) as $\tau \to 0^+$. Using that $F^{-1}(x,\cdot)$ is continuous, we are done.

(3) Considering the cases $x < 0$ and $x \geq 0$ separately, the claim follows using an argument similar to the second part of the claim. The condition that $C(K,\tau) \to S$ allows us to use that $F(x,\infty) = 1$ when $x \geq 0$ and $F(x,\infty) = K/S$ when $x \leq 0$, see Lemma 7.2.4.

(4) The forward implication follows from Equation (7.8) and the fact that $F(x,\cdot)$ and $F^{-1}(x,\cdot)$ are strictly increasing, see Lemma 7.2.4. The backward implication follows from Equation (7.9) and the fact that $F^{-1}(x,\cdot)$ is strictly increasing.

(5) Replace "strictly increasing" with "continuous" in the proof of the fourth part of the claim.

□

This result is useful as it allows us to think of $\Theta(K,\tau)$ as defining a natural time scale for the K strike call option. In particular, with K fixed, it allows us to replace limits of the form $\Theta(K,\tau) \to 0^+$ (respectively $\Theta(K,\tau) \to \infty$) with limits of the form $\tau \to 0^+$ (respectively $\tau \to \infty$). We will use this repeatedly in the next section where we consider the asymptotics of implied volatility.

Remark 7.3.3. Proposition 7.3.2, in particular, establishes the"feedback condition" $\sqrt{\tau}\Sigma(K,\tau) \to 0$ as $\tau \to 0^+$ for all $K > 0$, for any call pricing function that satisfies Assumptions (7.1), (7.2) and (7.3). For example, this will hold if the price is determined as the expectation of the call payoff, assuming zero interest rates and dividend yield and that the stock is a non-negative càdlàg martingale on a probability space with a right-continuous filtration. This condition is important in stochastic implied volatility modelling, see p. 270 of (MR05). Other authors have already shown this property using different arguments, see, for example, (ALV06) and (Dur05). It is a model assumption in (BGKW01).

7.3.2 Small Time to Expiry Asymptotics of the Black-Scholes Call Pricing Function

The asymptotic behaviour close to expiry of the Black-Scholes call price is a simple consequence of the preceding results. Note that the at-the-money asymptotics are well known (see (Li05) and references therein).

Proposition 7.3.4. *For $S, K > 0$ and $0 < \sigma < \infty$ we have*

$$C^{BS}(K,\tau,\sigma) - (S-K)^+ \sim \begin{cases} \dfrac{S\sigma\sqrt{\tau}}{\sqrt{2\pi}}, & \text{if } K = S, \\ \dfrac{\sqrt{SK}\sigma^3\tau^{3/2}}{\sqrt{2\pi}(\ln(K/S))^2}\exp\left(-\dfrac{(\ln(K/S))^2}{2\sigma^2\tau}\right), & \text{if } K \neq S, \end{cases}$$

as $\tau \to 0^+$.

Proof. It suffices to combine Lemma 7.2.5 on the asymptotics of F when $x = 0$, i.e. the at-the-money case, Lemma 7.2.6 on the not at-the-money case ($x \neq 0$), and our working representation of the Black-Scholes call pricing function derived in Lemma 7.2.1. □

7.3.3 Small Time to Expiry Asymptotics of Implied Volatility

We now obtain the small time to expiry asymptotics of the implied volatility itself. Note again that the at-the-money asymptotics are well known. Also, the answer is trivial when the call option price is "flat" close to expiry. Specifically, if there exists a $\delta > 0$ such that

$$C(K,\delta) = (S-K)^+,$$

then manifestly $\Sigma(K,\tau) = 0$ for every $\tau \in (0,\delta)$. This is an immediate consequence of the definition of the implied volatility, Lemma 7.2.1 and Assumption (7.3). Recall also our standing assumption that $S > 0$.

Theorem 7.3.5. *Suppose that Assumptions* (7.1), (7.2) *and* (7.3) *hold. If there exists a constant $\delta > 0$ such that, for every $\tau \in (0, \delta)$,*

$$C(K,\tau) > (S-K)^+,$$

then we have

$$\Sigma(K,\tau) \sim \begin{cases} \sqrt{2\pi}\,\dfrac{C(K,\tau)}{S\sqrt{\tau}}, & \text{if } K = S, \\ \dfrac{|\ln(K/S)|}{\sqrt{-2\tau\ln(C(K,\tau)-(S-K)^+)}}, & \text{if } K \neq S, \end{cases}$$

as $\tau \to 0^+$, where $K, S > 0$.

Proof. Since the map $C(K,\cdot)$ is non-decreasing by Assumption (7.3), we may suppose that $C(K,\tau) - (S-K)^+ > 0$ for all $\tau > 0$. Suppose first that $K = S$. We have that

$$C(S,\tau) = C^{BS}(S,\tau,\Sigma(S,\tau)) = SF(0,\Theta(S,\tau)), \quad \forall\, \tau > 0,$$

by definition of the implied volatility and Lemma 7.2.1. By Lemma 7.2.5, it therefore holds that

$$C(S,\tau)/S \sim \frac{\Sigma(S,\tau)\sqrt{\tau}}{\sqrt{2\pi}}, \quad \tau \to 0^+,$$

so that the result follows from Proposition 7.3.2.

Suppose now that $K \neq S$. Write x for $\ln(K/S)$. Arguing as before, we have that

$$\frac{C(K,\tau)-(S-K)^+}{S} = F(x,\Theta(K,\tau)), \quad \forall\, \tau > 0.$$

Using Proposition 7.3.2 and Lemma 7.2.6, it therefore holds that

$$\frac{C(K,\tau)-(S-K)^+}{S} \sim \frac{\Theta^3(K,\tau)}{\sqrt{2\pi}x^2}\exp\left(-\frac{x^2-\Theta^2(K,\tau)x}{2\Theta^2(K,\tau)}\right), \quad \tau \to 0^+. \tag{7.10}$$

It is convenient to now introduce the functions

$$\begin{aligned} A &: (K,\tau) \mapsto 4\sqrt{\pi}\,\frac{C(K,\tau)-(S-K)^+}{\sqrt{KS}\,|x|}, \\ G&(K,\tau) \mapsto \frac{x^2}{2\Theta^2(K,\tau)}, \end{aligned}$$

defined for $K, \tau > 0$. Note that A and G are both strictly positive. Some simple manipulations of Equation (7.10) give that

$$A(K,\tau) \sim (G(K,\tau))^{-3/2}\exp\left(-G(K,\tau)\right), \quad \tau \to 0^+. \tag{7.11}$$

From Equation (7.11), we have that there exists a function $g_K : (0, \infty) \to \mathbb{R}$, possibly depending on K, such that $g_K(\tau) = \mathrm{o}(1)$ as $\tau \to 0^+$ and

$$A(K,\tau)(1+g_K(\tau)) = (G(K,\tau))^{-3/2} \exp(-G(K,\tau)) \tag{7.12}$$

holds for all positive τ less than some $\delta' > 0$. We want to solve Equation (7.12) for G, at least asymptotically. Therefore, we let $\Lambda_{3/2}(z)$ be the value of y that is the unique positive solution of

$$y^{3/2} \exp(y) = z.$$

Taking z to be $(A(K,\tau)(1+g_K(\tau)))^{-1}$ and y to be $G(K,\tau)$ it is clear that

$$G(K,\tau) = \Lambda_{3/2}\left(\frac{1}{A(K,\tau)(1+g_K(\tau))}\right),$$

for $0 < \tau < \delta'$. Now it follows from Lemma A.5.2 that

$$\Lambda_{3/2}(z) \sim \ln(z), \quad z \to \infty.$$

Also, it follows from Assumption (7.2) and the positivity of A that

$$(A(K,\tau)(1+g_K(\tau)))^{-1} \to \infty, \quad \tau \to 0^+,$$

so that

$$G(K,\tau) \sim -\ln(A(K,\tau)(1+g_K(\tau))), \quad \tau \to 0^+.$$

Using that g_K is $\mathrm{o}(1)$ as $\tau \to 0^+$, we get that

$$G(K,\tau) \sim -\ln(A(K,\tau)(1+g_K(\tau))) \sim -\ln(A(K,\tau)), \quad \tau \to 0^+,$$

which is to say that

$$\frac{(\ln(K/S))^2}{2\Sigma^2(K,\tau)\tau} \sim -\ln\left(4\sqrt{\pi}\,\frac{C(K,\tau)-(S-K)^+}{\sqrt{KS}\,|x|}\right), \quad \tau \to 0^+.$$

It is therefore clear that

$$\Sigma^2(K,\tau)\tau \sim \frac{(\ln(K/S))^2}{-2\ln\left(4\sqrt{\pi}\frac{C(K,\tau)-(S-K)^+}{\sqrt{KS}\,|x|}\right)}, \qquad \tau \to 0^+,$$

from which

$$\Sigma^2(K,\tau)\tau \sim \frac{(\ln(K/S))^2}{-2\ln\left(C(K,\tau)-(S-K)^+\right)}, \qquad \tau \to 0^+,$$

so that we finally have that

$$\Sigma(K,\tau) \sim \frac{|\ln(K/S)|}{\sqrt{-2\tau\ln\left(C(K,\tau)-(S-K)^+\right)}}, \qquad \tau \to 0^+.$$

$\square$

The following result is a straightforward consequence of Theorem 7.3.5.

Corollary 7.3.6. *Suppose that Assumptions* (7.1), (7.2) *and* (7.3) *hold. If there exists a constant $\delta > 0$ such that, for every $\tau \in (0,\delta)$,*

$$C(K,\tau) > (S-K)^+,$$

then

$$\lim_{\tau\to 0^+}\Sigma(K,\tau) = \begin{cases} \lim_{\tau\to 0^+}\dfrac{\sqrt{2\pi}C(K,\tau)}{S\sqrt{\tau}}, & \text{if } S=K,\\ \lim_{\tau\to 0^+}\dfrac{|\ln(K/S)|}{\sqrt{-2\tau\ln(C(K,\tau)-(S-K)^+)}}, & \text{if } S\neq K,\end{cases}$$

in the sense that the left-hand side limit exists (is infinite, respectively) if and only if the right-hand side limit exists (is infinite, respectively) and then they are equal.

Proof. The proof is a consequence of the following fact. If $u,v : (0,\infty) \to (0,\infty)$ and $u(\tau) \sim v(\tau)$ as $\tau \to 0^+$ then $u(\tau) = v(\tau)(1+\mathrm{o}(1))$ as $\tau \to 0^+$. $\square$

This result leads us to ask the question as to whether or not the no arbitrage property is consistent with non-convergent implied volatility. We now turn to answering this question.

7.3.4 Markets with Non-convergent Implied Volatility

It is not at all clear whether absence of arbitrage alone is sufficient for implied volatility to converge as time to expiry goes to zero. In this section, we will show that this may fail to hold even in the simple case of the Black-Scholes model with time-dependent volatility.

Lemma 7.3.7. *There exists a function $H : (0,\infty) \to [0,\infty)$ satisfying the following conditions:*

(1) H is continuously differentiable;

(2) *H is strictly increasing;*

(3) *there exists* $a, b > 0$ *such that* $a \leq H'(\tau) \leq b$ *for every* $\tau \in (0, \infty)$*;*

(4) $H(\tau) \to 0$ *as* $\tau \to 0^+$*; and*

(5) $\tau \mapsto (H(\tau)/\tau)^{1/2}$ *does not converge, to a finite or infinite limit, as* $\tau \to 0^+$.

Proof. The simplest function we are aware of is a slight modification of one reported in (Kli91). Let

$$H(\tau) = \int_0^\tau (3 + \sin(\ln(u)))\, \mathrm{d}u = 3\tau + \tau/2\left[\sin(\ln(\tau)) - \cos(\ln(\tau))\right].$$

Clearly,

$$H(\tau)/\tau = 3 + 1/2[\sin(\ln(\tau)) - \cos(\ln(\tau))]$$

is not convergent so that $(H(\tau)/\tau)^{1/2}$ does not converge as $\tau \to 0^+$. The other required properties are easy to check. □

We now show the existence of a model with non-convergent implied volatility.

Proposition 7.3.8. *There exist arbitrage-free markets in which implied volatility has no limit, finite or infinite, as time to expiry goes to zero.*

Proof. The simplest example is the Black-Scholes model with time-dependent, but deterministic volatility. Now, in the case that the volatility function σ is bounded above and below by strictly positive constants, it is well known (see (MR05)), that the corresponding implied volatility is the root-mean-square volatility, i.e.,

$$\Sigma(\tau) = \Sigma(K, \tau) = \left(\frac{1}{\tau}\int_0^\tau \sigma^2(s)\, \mathrm{d}s\right)^{1/2}.$$

Let H be defined as in the proof of Lemma 7.3.7. It is then sufficient to take σ to be defined by $\sigma^2(t) = H'(t)$ for $t > 0$, while the value at $t = 0$ is inconsequential. □

Remark 7.3.9. The failure of the presented counter-example to have a right-continuous volatility is not sufficient for implied volatility to fail to converge. Indeed, if we define

$$H(\tau) = \int_0^\tau (2 + \sin(1/x))\, \mathrm{d}x,$$

and take volatility to be given by $\sigma^2(t) = H'(t)$ for $t > 0$, then implied volatility converges as $\tau \to 0^+$. See Proposition A.7.1 starting on page 147 of the Appendix.

Remark 7.3.10. The presented example sheds light on a result of Berestycki *et al.* (see (BBF02)), where they studied the small time to expiry asymptotics of the implied volatility in local volatility models. Their analysis is applicable to the case of time-dependent volatility functions, as we have considered above. In this context, their key assumption is that the volatility function is bounded and continuous on $[0, T]$, $T > 0$. In this case, they showed that the limit of implied volatility as expiry goes to zero must exist and, in fact, calculated it explicitly. The example we have presented has a volatility function that is both bounded and continuous on $(0, 1]$ but fails to be continuous on $[0, 1]$. Our example therefore illustrates the importance of the extra regularity condition postulated in (BBF02).

Remark 7.3.11. We now comment on the implications of our results for the development of stochastic implied volatility (SIV) models (see, for example, (BFG07), (BGKW01), (BGvdHW02), (LSC98), (Sch99), and (SW08)). One of the basic questions is to determine which of the different possible parameterisations of the implied volatility surface should be used in the model equations. Various possibilities are considered in (BGKW01). One possibility is simply to model the implied volatility itself. As Proposition 7.3.8 shows, this quantity can behave very badly under the no arbitrage assumption. Any attempt to rule out such behaviour by the imposition of other restrictions will necessarily restrict the market dynamics that such models can describe. This appears undesirable. An alternative is to express the model equations in terms of the time-scaled implied volatility. Proposition 7.3.2 informs us that this quantity is very well-behaved and that this good behaviour in no way restricts the admissible dynamics of such models; the implied volatility can always be recovered. It seems reasonable to conclude that SIV models should therefore be parameterised in terms of the time-scaled implied volatility, or (the derivative of) its square, as in (SW08).

7.4 Summary of Results and Conclusion

In this chapter, we have obtained

1. Exact small time to expiry asymptotics at fixed strike of implied volatility in terms of the call price, strike price, and current stock price.

2. A number of convenient properties of time-scaled implied volatility

3. Answered an open problem in SIV modelling: conditions under which the "feedback condition" (i.e. $\sqrt{\tau}\Sigma(K,\tau) \to 0$ as $\tau \to 0^+$ for all $K > 0$), see (SIV.4) on p. 270 of (MR05).

4. Small time to expiry asymptotics of the Black-Scholes call price.

5. Implied volatility may not converge to any finite or infinite value.

6. SIV models may best be set up in terms of time-scaled implied volatility rather than implied volatility itself.

Chapter 8

Small Time to Expiry Asymptotics: Exponential Lévy Models

In this chapter, we examine the small time to expiry behaviour of implied volatility in models of exponential Lévy type. We do this by investigating the small time to expiry behaviour of European call options in such models. We then combine this with the results of Chapter 7 that precisely relate the small time to expiry implied volatility to the small time to expiry behaviour of call options.

In the at-the-money case, it turns out that the implied volatility converges to the square root of the Gaussian member of the driving Lévy process' characteristic triplet. In particular, the limit is zero if the Lévy process has no Gaussian part. In the not at-the-money case, there are a number of possible behaviours. In most cases of interest, however, it turns out that the implied volatility converges to infinity as time to expiry goes to zero.

Levendorskiĭ (see (Lev08)) calculates small time to expiry asymptotics for the European put and call in exponential Lévy models. Attention is restricted to the not at-the-money case. His results require some regularity conditions on the driving Lévy process that we are able to do without. He has presented similar results in (Lev04). Our approach to the small time to expiry asymptotics of European calls and puts is completely different to that of Levendorskiĭ.

Let S be a non-negative martingale, not necessarily of exponential Lévy type. Under some weak assumptions on S, Carr and Wu (see (CW03)) claim that

$$\mathbb{E}\left((S_{t+\tau}-K)^{+}|\mathscr{F}_{t}\right)-(S_{t}-K)^{+}=\mathrm{O}\left(\tau\right), \quad \text{as } \tau\to 0^{+}, \tag{8.1}$$

in the case that $S_t \neq K$. The left-hand side of Equation (8.1) is termed the time value of a European call.

In the case of S being an exponential Lévy process, Carr and Wu ((CW03)) conjecture exact asymptotics for the time value, for example for the out of the money call, in terms of $\tau\int(S_0e^x-K)^+\nu(dx)$, where ν is the Lévy measure of the driving Lévy process. We establish rigorously this conjecture. They also claim that the time value of at-the-money European calls in the pure jump models that they consider decay at the rate $O(\tau^p)$ as $\tau\to 0^+$ for some p in $(0,1]$. For stochastic volatility plus jumps models (i.e. noise with non-vanishing Brownian part as well as a jump process part) they claim that the decay rate is $O(\tau^p)$ as $\tau\to 0^+$ for some p in $(0,1/2]$. The decay rate for out-of-the money calls is claimed to be $O(\tau)$ in the pure jump models and stochastic volatility plus jumps cases. In the purely continuous case, it is claimed that the decay rate is $O(e^{-c/\tau})$ as $\tau \to 0^+$ for some $c > 0$.

The behaviour of at-the-money implied volatility in stochastic volatility models with bounded (spot) volatility and a finite variation jump component has been considered by Durrleman (see (Dur08)). The limiting value of the implied volatility turns out to be the "instantaneous spot volatility" of the model, see Chapter 9. This agrees with the results we obtain in this chapter.

Carr and Wu (see (CW03)) claim that in the not at-the-money case implied volatility explodes as time to expiry goes to zero once jumps are included in their stochastic volatility with jumps model. We show that this is not necessarily the case even in the simpler exponential Lévy models that we consider here. To the best of our knowledge there has been no general, rigorous study of the behaviour of implied volatility in exponential Lévy models as expiry goes to zero.

We proceed as follows. Section 8.1 gives the background necessary for the formulation and proof of our main results and supporting lemmas. Supporting lemmas are given in Section 8.2 and the main results are given in Section 8.3. Section 8.4 gives examples of our main results. Section 8.5 concludes our study of implied volatility close to expiry in exponential Lévy models.

8.1 Background

8.1.1 Setup

We first recall the definition of a Lévy process. We are assuming that we are working on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions.

Definition 8.1.1 (Lévy process)**.** Let X be a real-valued process with $X_0 = 0$ $\mathbb{P}$-a.s. and

(1) X has increments independent of the past; that is $X_t - X_s$ is independent of $\mathscr{F}_s$, $0 \leq s < t < \infty$; and

(2) X has stationary increments; that is $X_t - X_s$ has the same distribution as X_{t-s}, $0 \leq s < t < \infty$; and

(3) X is continuous in probability; that is, $\lim_{t \to s} X_t = X_s$, where the limit is taken in probability.

See (Pro04), p. 20 for this definition.

Remark 8.1.2. Note Theorem 30 on p.20 of (Pro04). It gives that if X is a Lèvy process, then there exists a unique modification Y of X which is càdlàg and also a Lévy process. We will work throughout with this modification. To avoid the introduction of further notation, we will abuse the notation and use X to denote the càdlàg modification.

We recall that a Lévy process is described by its characteristic triplet (b, σ^2, ν) where $\sigma, b \in \mathbb{R}$, $\sigma \geq 0$, and ν is a non-negative Radon measure satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \int (1 \wedge y^2) \nu(\,\mathrm{d}y) < \infty,$$

see Bertoin (Ber96), p. 3. We model the stock as the exponential of a Lévy process, that is

$$S_\tau = S_0 e^{X_\tau}, \quad \forall \tau \geq 0, \tag{8.2}$$

where X is a Lévy process and $S_0 > 0$ is some finite constant. So that our stock price process is a martingale we also demand that

$$\int_{|y| \geq 1} e^y \,\nu(\mathrm{d}y) < \infty \text{ and } b = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^y - 1 - y \mathbb{1}_{|y| \leq 1}) \,\nu(\mathrm{d}y) \tag{8.3}$$

(see p. 354 of Cont and Tankov (CT04)).

For simplicity, we assume the interest rate and dividend yield are both zero. The stock is a martingale under $\mathbb{P}$ and the driving Lévy process satisfies the constraints set out in Equation (8.3). The model is presented under the pricing measure, $\mathbb{P}$, chosen by the market so that we price options as expectations under $\mathbb{P}$ of their payoff. Since S is a time-homogeneous Markov process, there is no loss of generality in assuming that we are at time zero.

8.1.2 Auxiliary Facts

We need some definitions and results from the literature to prove our claims.

Definition 8.1.3. A non-negative locally bounded function $k : \mathbb{R} \to \mathbb{R}$ is submultiplicative if there exists a constant $\alpha > 0$ such that $k(x+y) \leq \alpha k(x)k(y)$ for all $x, y \in \mathbb{R}$ (see (FL08)).

Definition 8.1.4. A non-negative locally bounded function $q : \mathbb{R} \to \mathbb{R}$ is subadditive provided that there exists a constant $\beta > 0$ such $q(x+y) \leq \beta(q(x) + q(y))$ for all $x, y \in \mathbb{R}$ (see (FL08)).

We will use the following class of "dominating functions".

Definition 8.1.5 (The class $\mathscr{S}(\nu)$)**.** Suppose that ν is a Lévy measure. A function $u : \mathbb{R} \to \mathbb{R}$ is of class $\mathscr{S}(\nu)$ if

(1) $u(x) = q(x)k(x)$ for some functions q and k, where q is subadditive and k is submultiplicative; and

(2) $\int_{|x|>1} |u(x)| \, \nu(\mathrm{d}x) < \infty$

(see (FL08)).

Remark 8.1.6. Note that membership of a function, say g, in $\mathscr{S}(\nu)$ depends both on properties of the function g and also of the reference Lévy measure ν.

We use the following result a number of times in our proofs.

Theorem 8.1.7 (Figueroa-López, Theorem 1.1 in (FL08), abbreviated)**.** *Let X be a Lévy process with characteristic triplet (b, σ^2, ν). Let $w : \mathbb{R} \to \mathbb{R}$ satisfy*

(1) $w(x) = o\left(x^2\right)$ as $x \to 0$;

(2) w is locally bounded;

(3) w is ν-a.e. continuous; and

(4) there exists a function $u \in \mathscr{S}(\nu)$ for which

$$\limsup_{|x| \to \infty} \frac{|w(x)|}{u(x)} < \infty.$$

Then

$$\tau^{-1} \mathbb{E}\left(w(X_\tau)\right) \to \int w(x) \, \nu(\mathrm{d}x), \quad \text{as } \tau \to 0^+.$$

If conditions (2)-(4) hold, but (1) is replaced with

(1') $w(x) \sim x^2$ as $x \to 0$,

then

$$\tau^{-1}\mathbb{E}\left(w(X_\tau)\right) \to \sigma^2 + \int w(x)\,\nu(\mathrm{d}x) \quad \text{as } \tau \to 0^+.$$

We now recall Sato's classification of Lévy processes.

Definition 8.1.8 (Sato, p. 65 of (Sat99))**.** Let X be a Lévy process on $\mathbb{R}$ with characteristic triplet (b, σ^2, ν). Then

1. if $\sigma = 0$ and $\nu(\mathbb{R}) < \infty$, then X is of type A;
2. if $\sigma = 0$, $\nu(\mathbb{R}) = \infty$, and $\int_{|x|\le 1} |x|\ \nu(\mathrm{d}x) < \infty$, then X is of type B;
3. if $\sigma \neq 0$ or $\int_{|x|\le 1} |x|\,\nu(\mathrm{d}x) = \infty$, then X is of type C.

In order to obtain Lévy processes with zero implied volatility we need the following result.

Theorem 8.1.9 (Sato, Theorem 24.10 in (Sat99), abbreviated)**.** *Suppose that X is a Lévy process with characteristic triplet $(b, 0, \nu)$. Suppose that 0 is in the support of ν. Further, assume that X is either of Type A or Type B, see Definition 8.1.8, then, if the support of ν is a subset of $[0, \infty)$, we have $\mathbb{P}\left(X_\tau \in [b\tau, \infty)\right) = 1$. If the support of ν is a subset of $(-\infty, 0]$, then $\mathbb{P}\left(X_\tau \in (-\infty, b\tau]\right) = 1$.*

So as to handle at-the-money implied volatilities we need the following result of Jacod (see (Jac07)).

Lemma 8.1.10 (Jacod, Lemma 4.1, p. 181 of (Jac07); abbreviated)**.** *Let $\widetilde{X}$ be a Lévy process with no Gaussian part, then $\tau^{-1/2}\widetilde{X}_\tau \xrightarrow{\mathbb{P}} 0$ as $\tau \to 0^+$.*

Proof. A simple and original proof of this result is given in the Appendix starting from page 149. $\square$

8.2 Lemmas

In this section, we present some definitions and lemmas that are used to prove the main results of this chapter. The proofs of the lemmas are relegated to the Appendix.

We will frequently use the following call and put functions.

Definition 8.2.1. For a fixed $K > 0$ and $S_0 > 0$, let

$$\begin{aligned} C : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto (S_0 e^x - K)^+, \end{aligned}$$

which is the call function.

Definition 8.2.2. For a fixed $K > 0$ and $S_0 > 0$, let

$$\begin{aligned} P : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto (K - S_0 e^x)^+, \end{aligned}$$

which is the put function.

We also single out a trivial example of our model: the constant process.

Definition 8.2.3. A Lévy process with characteristic triplet $(b, 0, 0)$, with $b \in \mathbb{R}$, is a trivial process.

Obviously, for the exponential of a trivial process to be a martingale it must have zero drift. So henceforth we take a "trivial (Lévy) process" to be one with characteristic triplet $(0, 0, 0)$.

In order to use the results of Chapter 7, we need some results about the conditional expectation of the call payoff.

Lemma 8.2.4. *Let*

$$S_\tau = S_0 e^{X_\tau}, \quad \forall \tau \geq 0,$$

where X is a Lévy process satisfying the constraints in Equation (8.3) *and $S_0 > 0$ is some finite constant. Then, for each fixed $K > 0$,*

1. $(S_0 - K)^+ \leq \mathbb{E}\left((S_0 e^{X_\tau} - K)^+\right) \leq S_0$, *$\mathbb{P}$-a.s.;*

2. $\tau \mapsto \mathbb{E}\left((S_0 e^{X_\tau} - K)^+\right)$ *is $\mathbb{P}$-a.s. right-continuous on $[0, \infty)$; and*

3. $\tau \mapsto \mathbb{E}\left((S_0 e^{X_\tau} - K)^+\right)$ *is $\mathbb{P}$-a.s. non-decreasing.*

Proof. See the Appendix starting from page 150. $\square$

For ease of referencing we include part of the content of this last lemma with the following.

Lemma 8.2.5. *Let S be defined by Equation* (8.2) *and assume that S satisfies the constraints set out in Equation* (8.3). *For every $\tau > 0$ and $K > 0$,*

(Ai) $\mathbb{E}\left((S_\tau - K)^+\right) < S_0$.

(Aii) $\mathbb{E}\left((S_\tau - K)^+\right) \geq (S_0 - K)^+$.

(Aiii) If $S_0 > K$, then $\mathbb{E}\left((S_\tau - K)^+\right) = (S_0 - K)^+$ if and only if
$\mathbb{P}\left(S_\tau < K\right) = \mathbb{P}\left(X_\tau < \ln(K/S_0)\right) = 0.$

Proof. Given in the Appendix starting from page 150. $\square$

Lemma 8.2.6. *Let S be defined by Equation* (8.2) *and assume that the driving Lévy process satisfies the constraints set out in Equation* (8.3). *Consider the functions*

(1) P, given by $P(x) = (K - S_0 e^x)^+$, with the additional restriction that $0 < K < S_0$; and

(2) C, given by $C(x) = (S_0 e^x - K)^+$, with the additional restriction that $K > S_0 > 0$.

Then conditions (1)-(4) of Theorem 8.1.7 are satisfied by P and C under the respective stated conditions on S_0 and K.

Proof. See the Appendix starting from page 151. $\square$

Lemma 8.2.7. *Suppose that U is a non-negative process with representation*

$$U_\tau = U_0 e^{b\tau + \sigma W_\tau + Y_\tau}, \quad \tau \geq 0,$$

where $b \in \mathbb{R}, \sigma \geq 0$, and $U_0 > 0$ are finite constants, W is a standard Brownian motion, and Y is a compound Poisson process with constant, finite intensity $\lambda > 0$. We denote the sequence of summand random variables comprising Y as $\left(\widehat{Y}_i\right)_{i \geq 1}$; they are i.i.d. random variables with an exponential mean. We assume that the compound Poisson part has a finite exponential moment, i.e.

$$\mathbb{E}\left(e^{Y_\tau}\right) < \infty, \quad \forall \tau \geq 0.$$

Then

$$\lim_{\tau \to 0^+} \tau^{-1/2} \mathbb{E}\left(\left(U_0 - U_0 e^{b\tau + \sigma W_\tau + Y_\tau}\right)^+\right) = \frac{\sigma U_0}{\sqrt{2\pi}}.$$

Proof. See the Appendix starting from page 152. $\square$

Lemma 8.2.8. *Let*

$$\widetilde{P} : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto (1 - \exp(x))^+.$$

Then $\widetilde{P}$ is globally Lipschitz continuous with Lipschitz constant 1.

Proof. See the Appendix starting from page 156. □

8.3 Main Results

In this section, we present the main results of this chapter. We begin by establishing small time to expiry asymptotics of the call/put option. We then invoke the results of Chapter 7 to derive the small time to expiry asymptotics of implied volatility in exponential Lévy models.

8.3.1 Call Option Asymptotics

We begin by examining the call option asymptotics in the at-the-money case.

Theorem 8.3.1. *Let S be defined by Equation* (8.2) *and assume that the driving Lévy process, X, satisfies the constraints set out in Equation* (8.3). *Then*

$$\lim_{\tau \to 0^+} \frac{\mathbb{E}\left((S_\tau - S_0)^+\right)}{\tau^{1/2}} = \lim_{\tau \to 0^+} \frac{\mathbb{E}\left((S_0 - S_\tau)^+\right)}{\tau^{1/2}} = \frac{\sigma S_0}{\sqrt{2\pi}}, \tag{8.4}$$

and, in particular, if $\sigma = 0$, then

$$\lim_{\tau \to 0^+} \frac{\mathbb{E}\left((S_\tau - S_0)^+\right)}{\tau^{1/2}} = \lim_{\tau \to 0^+} \frac{\mathbb{E}\left((S_0 - S_\tau)^+\right)}{\tau^{1/2}} = 0. \tag{8.5}$$

Proof. Since S is a martingale, the first equality in both Equations (8.4) and (8.5) follows from put-call parity. It remains to show the second equality of Equation (8.4) for which it is clearly enough to suppose that $S_0 = 1$. The final equality of Equation (8.5) will then be clear as we will nowhere in the proof use that $\sigma \neq 0$. X is a Lévy process, with characteristic triplet (b, σ^2, ν), satisfying the constraints set out in Equation (8.3). By the Lévy-Itô decomposition we have that there exists a probability space on which X is the sum of four independent Lévy processes

$$X_\tau = b\tau + \sigma W_\tau + Y_\tau + \widetilde{Y}_\tau, \quad \tau \geq 0,$$

where W is a Wiener process, Y is a compound Poisson process, and $\widetilde{Y}$ is a square-integrable pure jump martingale.

Since $\widetilde{P}$ is globally Lipschitz with Lipschitz constant 1, by Lemma 8.2.8, we have that

$$\begin{aligned}\mathbb{E}\left(\left|\widetilde{P}(X_\tau) - \widetilde{P}(b\tau + \sigma W_\tau + Y_\tau)\right|\right) &\leq \mathbb{E}\left(\left|X_\tau - (b\tau + \sigma W_\tau + Y_\tau)\right|\right) \\ &= \mathbb{E}\left(\left|\widetilde{Y}_\tau\right|\right). \end{aligned} \tag{8.6}$$

We now proceed to show uniform integrability of $\left(\tau^{-1/2}\widetilde{Y}_\tau\right)_{\tau\in(0,\epsilon)}$ (for some $\epsilon > 0$) using Theorem 8.1.7. Then we will be able to use Lemma 8.1.10 to get that

$$\lim_{\tau\to 0^+} \tau^{-1/2}\mathbb{E}\left(\left|\widetilde{Y}_\tau\right|\right) = 0. \tag{8.7}$$

The final step will be to approximate $\tau^{-1/2}\mathbb{E}\left(\widetilde{P}(X_\tau)\right)$ by $\tau^{-1/2}\mathbb{E}\left(\widetilde{P}(b\tau + \sigma W_\tau + Y_\tau)\right)$ for which we have an explicit limiting expression.

We now proceed to show that Equation (8.7) holds. We begin by showing that

$$\lim_{\tau\to 0^+} \tau^{-1}\mathbb{E}\left(\widetilde{Y}_\tau^2\right)$$

exists and is finite. This is done by an application of Theorem 8.1.7. Denote the Lévy measure of $\widetilde{Y}$ by $\widetilde{\nu}$; it may be null. We now check the conditions of Theorem 8.1.7. Write $g(y) = y^2$ for $y \in \mathbb{R}$. Trivially, $g(y) \sim y^2$ as $y \to 0$, which is condition (1′) of Theorem 8.1.7. Local boundedness and $\widetilde{\nu}$ continuity, which are conditions (2) and (3) of Theorem 8.1.7, are obvious. For condition (4) of Theorem 8.1.7 we take $u = g$. We have that $g \in \mathscr{S}(\widetilde{\nu})$, since $\limsup_{|x|\to\infty} |g(x)|/g(x) = 1 < \infty$, and $y \mapsto g(y)\cdot 1$ is the product of a subadditive and submultiplicative function for which

$$\int_{|y|>1} y^2 \cdot 1\, \widetilde{\nu}(\mathrm{d}y) < \infty. \tag{8.8}$$

We know that Equation (8.8) holds because of the square-integrability of $\widetilde{Y}$. Indeed, $\mathbb{E}\left(\widetilde{Y}_\tau^2\right) < \infty$ for each $\tau > 0$ if and only if

$$\int_{|y|>1} y^2\, \widetilde{\nu}(\mathrm{d}y) < \infty, \tag{8.9}$$

by Example 25.12, p. 163 of Sato ((Sat99)). We have shown that the conditions of Theorem 8.1.7 are satisfied and hence, by Theorem 8.1.7,

$$\lim_{t\to 0^+} \tau^{-1}\mathbb{E}\left(\widetilde{Y}_\tau^2\right) = \int y^2\, \widetilde{\nu}(\mathrm{d}y) < \infty, \tag{8.10}$$

where we used that $\widetilde{Y}$ has no Gaussian part. We know that this limit is finite since by Equation (8.9) and the definition of a Lévy measure we have that

$$\int y^2\,\widetilde{\nu}(\mathrm{d}y) = \int_{|y|>1} y^2\,\widetilde{\nu}(\mathrm{d}y) + \int_{|y|\le 1} y^2\,\widetilde{\nu}(\mathrm{d}y) < \infty.$$

It follows from Equation (8.10) that there exists an $\epsilon > 0$ such that $\left(\frac{\left|\widetilde{Y}_\tau\right|}{\sqrt{\tau}}\right)_{\tau\in(0,\epsilon)}$ is uniformly integrable. In addition,

$$\left|\tau^{-1/2}\widetilde{Y}_\tau\right| \xrightarrow{\mathbb{P}} 0, \text{ as } \tau \to 0^+.$$

from Lemma 8.1.10. It is therefore clearly the case that

$$\lim_{\tau\to 0^+} \mathbb{E}\left(\left|\tau^{-1/2}\widetilde{Y}_\tau\right|\right) = 0, \tag{8.11}$$

as was claimed as Equation (8.7).

From Equations (8.6) and (8.11) we therefore have that

$$\lim_{\tau\to 0^+} \tau^{-1/2}\mathbb{E}\left(\left|\widetilde{P}(X_\tau) - \widetilde{P}(b\tau + \sigma W_\tau + Y_\tau)\right|\right) = 0. \tag{8.12}$$

From Lemma 8.2.7 we have

$$\lim_{\tau\to 0^+} \tau^{-1/2}\mathbb{E}\left(\widetilde{P}(b\tau + \sigma W_\tau + Y_\tau)\right) = \frac{\sigma}{\sqrt{2\pi}}. \tag{8.13}$$

Recalling that $S_0 = 1$ by assumption, it follows from both Equations (8.12) and (8.13) that the last equality of Equation (8.4) holds. To see this, we first note that Equation (8.12) implies that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $\tau \in (0, \delta)$

$$\tau^{-1/2}\mathbb{E}\left(\left|\widetilde{P}(X_\tau) - \widetilde{P}(b\tau + \sigma W_\tau + Y_\tau)\right|\right) < \epsilon.$$

But, $\mathbb{E}\left(\widetilde{P}(X_\tau) - \widetilde{P}(b\tau + \sigma W_\tau + Y_\tau)\right)$ exists for every $\tau > 0$ since $\widetilde{P}$ is bounded above and below. Therefore $\forall\epsilon > 0\ \exists\delta > 0$ such that $\forall\tau \in (0, \delta)$

$$\tau^{-1/2}\left|\mathbb{E}\left(\widetilde{P}(X_\tau)\right) - \mathbb{E}\left(\widetilde{P}(b\tau + \sigma W_\tau + Y_\tau)\right)\right| < \epsilon$$

from which, $\forall\epsilon > 0\ \exists\delta > 0$ such that $\forall\tau \in (0, \delta)$,

$$\begin{aligned}\tau^{-1/2}\mathbb{E}\left(\widetilde{P}(b\tau + \sigma W_\tau + Y_\tau)\right) - \epsilon &< \tau^{-1/2}\mathbb{E}\left(\widetilde{P}(X_\tau)\right)\\ &< \tau^{-1/2}\mathbb{E}\left(\widetilde{P}(b\tau + \sigma W_\tau + Y_\tau)\right) + \epsilon,\end{aligned}$$

which implies that

$$\lim_{\tau \to 0^+} \tau^{-1/2} \mathbb{E}\left(\widetilde{P}(X_\tau)\right) = \frac{\sigma S_0}{\sqrt{2\pi}},$$

using Equation (8.13) and recalling the definition of $\widetilde{P}$ and that we assumed that $S_0 = 1$. ☐

We now turn to the not at-the-money case. This case is much simpler than the at-the-money case, although we are unable to reach the same level of generality. We begin with a result of general interest.

Recall that $C(x) = (S_0 e^x - K)^+$ and $P(x) = (K - S_0 e^x)^+$ for $x \in \mathbb{R}$.

Theorem 8.3.2 (Small-expiry asymptotics of not at-the-money European calls: Part I)**.**

Suppose that $S_\tau = S_0 e^{X_\tau}$, for all $\tau \geq 0$, where X is a Lévy process satisfying (8.3) *and $S_0 > 0$. Fix $K > 0$ with $K \neq S_0$.*

Then, for every $K > 0$ where $K \neq S_0$

$$\mathbb{E}\left((S_\tau - K)^+\right) - (S_0 - K)^+ = O(\tau), \quad \tau \to 0^+.$$

Proof. By Lemma 8.2.6, we may apply Theorem 8.1.7 to get that

$$\lim_{\tau \to 0^+} \frac{1}{\tau} \mathbb{E}\left((S_\tau - K)^+\right) = \int_{\mathbb{R}} C(x)\, \nu(\mathrm{d}x) < \infty, \quad (S_0 < K), \tag{8.14}$$

and

$$\lim_{\tau \to 0^+} \frac{1}{\tau} \mathbb{E}\left((K - S_\tau)^+\right) = \int_{\mathbb{R}} P(x)\, \nu(\mathrm{d}x) < \infty, \quad (S_0 > K). \tag{8.15}$$

Now use put-call parity in (8.15). The claim is now clear: the right hand sides of Equations (8.14) and (8.15) lie in $[0, \infty)$. For example, if X has only positive jumps, then the right hand side of (8.15) will be zero. ☐

We now sharpen the result of the previous theorem to make it applicable to implied volatility asymptotics.

Theorem 8.3.3 (Small-expiry asymptotics of not at-the-money European calls: Part II)**.** *Suppose that $S_\tau = S_0 e^{X_\tau}$, for all $\tau \geq 0$, where X is a Lévy process satisfying* (8.3) *and $S_0 > 0$. Fix $K > 0$. Recall that $C(x) = (S_0 e^x - K)^+$ and $P(x) = (K - S_0 e^x)^+$.*

Then, for all $S_0, K > 0$ with $K \neq S_0$, *(i) If X has characteristic triplet $(-\sigma^2/2, \sigma^2, 0)$ with $\sigma \neq 0$, then*

$$\mathbb{E}\left((S_\tau - K)^+\right) = S_0 \Phi\left(\frac{-\ln(K/S_0)}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}\right) - K\Phi\left(\frac{-\ln(K/S_0)}{\sigma\sqrt{\tau}} - \frac{\sigma\sqrt{\tau}}{2}\right).$$

(ii) If $S_0 < K$, and $\int C(x)\nu(\mathrm{d}x) > 0$, then

$$\mathbb{E}\left((S_\tau - K)^+\right) - (S_0 - K)^+ \sim \tau \int C(x)\nu(\mathrm{d}x), \quad \tau \to 0^+.$$

(iii) If $S_0 > K$, and $\int_{\mathbb{R}} P(x)\nu(\mathrm{d}x) > 0$, then

$$\mathbb{E}\left((S_\tau - K)^+\right) - (S_0 - K)^+ \sim \tau \int P(x)\nu(\mathrm{d}x), \quad \tau \to 0^+.$$

(iv) Otherwise

$$\mathbb{E}\left((S_\tau - K)^+\right) - (S_0 - K)^+ = o(\tau), \quad \tau \to 0^+.$$

Proof.

(i) This is nothing but the Black-Scholes model (with zero interest rates and dividend yield). See, for example, Musiela and Rutkowski ((MR05)).

(ii) See the proof of Theorem 8.3.2.

(iii) See the proof of Theorem 8.3.2.

(iv) See the proof of Theorem 8.3.2.

□

8.3.2 Implied Volatility

We now apply the results of the previous subsection on call option asymptotics to our primary area of interest: small time to expiry asymptotics of implied volatility. This is made easy by the results of Chapter 7.

Theorem 8.3.4. *Suppose that $S_\tau = S_0 e^{X_\tau}$, for all $\tau \geq 0$, where X is a Lévy process satisfying* (8.3) *and $S_0 > 0$. Then, for every $K, \tau > 0$, the implied volatility of the European call for every expiry τ and strike K, i.e. $\Sigma(K,\tau)$, satisfies*

$$0 \leq \Sigma(K,\tau) < \infty. \tag{8.16}$$

Moreover, there exists a non-trivial Lévy process X such that the lower bound in Equation (8.16) *is obtained for some $K > 0$ and τ small enough.*

Proof. It is obvious that the lower bound is obtained for $X \equiv 0$. We will show that, there exists a non-trivial X such that the lower bound is obtained for some K.

From Lemma 8.2.5 (Ai), $\mathbb{E}\left((S_\tau - K)^+\right) < S_0$ for all τ and $K > 0$. By Proposition 6.2.16, we then have $\Sigma(K, \tau) < \infty$ for $\tau, K > 0$.

From Lemma 8.2.5 (Aii), $\mathbb{E}\left((S_\tau - K)^+\right) \geq (S_0 - K)^+$ for all τ and $K > 0$. Therefore, $\Sigma(K, \tau) \geq 0$ for all $K, \tau > 0$.

To show that the lower bound is attained, we construct a non-trivial Lévy process, X, for which e^X has $\Sigma(K, \tau) = 0$ for all τ small enough and some (unattainable) K. Of course, S must be a martingale.

From Proposition 6.2.16, $\Sigma(K, \tau) = 0$ only when $\mathbb{E}\left((S_\tau - K)^+\right) = (S_0 - K)^+$, with $K > 0$ and $\tau > 0$.

From Lemma 8.2.5 (Aiii), we have that if $S_0 > K$, then $\mathbb{E}\left((S_\tau - K)^+\right) = (S_0 - K)^+$ if and only if $\mathbb{P}\left(S_\tau < K\right) = \mathbb{P}\left(X_\tau < \ln(K/S_0)\right) = 0$. For simplicity take $S_0 = 1$.

Let X be a Lévy process of Type A or B with characteristic triplet $(b, 0, \nu)$. Suppose that ν is not null and $b \neq 0$. Moreover, e^X is a martingale. Suppose that 0 is in the support of ν and the support of ν is a subset of $[0, \infty)$. With $S_0 > K$, we have $\mathbb{E}\left((S_\tau - K)^+\right) = (S_0 - K)^+$ if and only if $\mathbb{P}\left(S_\tau < K\right) = \mathbb{P}\left(X_\tau < \ln(K/S_0)\right) = 0$, see Lemma 8.2.5 (Aiii). Of course, $\mathbb{E}\left((S_\tau - K)^+\right) = (S_0 - K)^+$ implies that $\Sigma(K, \tau) = 0$.

For simplicity, take $S_0 = 1$. Since X has only positive jumps, we have by Theorem 8.1.9 that $\mathbb{P}\left(X_\tau < b\tau\right) = 0$ where $b(< 0)$ is the drift of X. We need to choose a K^* such that $\mathbb{P}\left(X_\tau < \ln K^*\right) = 0$ for all τ smaller than some constant time, but with K^* fixed. Choose $\ln K^* = 2b$, so that $K^* = \exp(2b)$. Obviously, $K^* < S_0 = 1$ since $b < 0$. What is more $\mathbb{P}\left(X_\tau < \ln K^*\right) = \mathbb{P}\left(X_\tau < 2b\right) = 0$ for all $0 < \tau \leq 1$. But, from Lemma 8.2.5 (Aiii), $\mathbb{E}\left((S_\tau - K^*)^+\right) = (S_0 - K^*)^+$ for $\tau \in (0, 1)$, from which $\Sigma(K^*, \tau) = 0$ for all $\tau \in (0, 1)$. $\mathbb{E}\left((S_\tau - K^*)^+\right) = (S_0 - K^*)^+$ for $\tau \in (0, 1)$, from which $\Sigma(K^*, \tau) = 0$ for $\tau \in (0, 1)$. $\square$

Theorem 8.3.5 (Limiting implied volatility)**.**

Suppose that $S_\tau = S_0 e^{X_\tau}$*, for all* $\tau \geq 0$*, where* X *is a Lévy process satisfying Equation* (8.3) *and* $S_0 > 0$*. Fix* $K > 0$*. Assume that* X *is not the trivial process. Also,to avoid* *trivialities we assume that* $\mathbb{E}\left((S_\delta - K)^+\right) > (S_0 - K)^+$ *for every* $\tau \in (0, \delta)$ $(\exists \delta > 0)$

$$\mathbb{E}\left((S_\tau - K)^+\right) > (S_0 - K)^+.$$

With X having characteristic triplet (b, σ^2, ν)*, the at-the-money implied volatility satisfies*

$$\lim_{\tau \to 0^+} \Sigma(S_0, \tau) = \sigma,$$

and if, in particular, $\sigma = 0$*, then*

$$\lim_{\tau \to 0^+} \Sigma(S_0, \tau) = 0.$$

Then the not at-the-money implied volatility of a $K(> 0), K \neq S_0$ *strike European call satisfies the following. Assume that* $K \neq S_0$ *and both are strictly positive.*

(i) If X has characteristic triplet $(-\sigma^2/2, \sigma^2, 0)$ *with* $\sigma \in (0, \infty)$*, then, for every* $K, \tau > 0$, $\Sigma(K, \tau) = \sigma$ *so that*

$$\lim_{\tau \to 0^+} \Sigma(K, \tau) = \sigma$$

(ii) If $S_0 < K$*, and* $\int_{\mathbb{R}} (S_0 e^x - K)^+ \nu(\mathrm{d}x) > 0$*, then*

$$\lim_{\tau \to 0^+} \Sigma(K, \tau) = \infty.$$

(iii) If $S_0 > K$*, and* $\int_{\mathbb{R}} (K - S_0 e^x)^+ \nu(\mathrm{d}x) > 0$*, then*

$$\lim_{\tau \to 0^+} \Sigma(K, \tau) = \infty.$$

(iv) If it is not the case that for every $\tau \in (0, \delta)$ $(\exists \delta > 0)$ *such that*

$$\mathbb{E}\left((S_\tau - K)^+\right) > (S_0 - K)^+,$$

then there exist non-trivial Lévy processes, $\widetilde{X}$*, such that* $S = e^{\widetilde{X}}$ *is a martingale, and for some* $K \neq S_0$

$$\lim_{\tau \to 0^+} \Sigma(K, \tau) = 0.$$

Proof. We have $S_0 > 0$ by assumption. Also, we assumed that $\mathbb{E}\left((S_\delta - K)^+\right) > (S_0 - K)^+$ for every $\tau \in (0, \delta)$ $(\exists \delta > 0)$

$$\mathbb{E}\left((S_\tau - K)^+\right) > (S_0 - K)^+.$$

We can therefore apply Lemma 8.2.4 and use Corollary 7.3.6 to obtain the implied volatility limit from the call option limit obtained in Theorem 8.3.1.

We now consider the not at-the-money case.

Statements (i) is trivial. The statements (ii) and (iii) follow from Theorem 8.3.3 (ii) and (iii). and the fact that $\tau \ln(A\tau) \to 0$ as $\tau \to 0^+$ for $A > 0$. In the model constructed in Theorem 8.3.4, it is trivially the case that there exists a K^* such that $\lim_{\tau\to 0^+} \Sigma(K^*, \tau) = 0$. The considered Lévy process is not trivial. $\square$

8.4 Examples

Let X be either a Generalised Hyperbolic, Variance Gamma, Normal Inverse Gaussian, CGMY, or Meixner process. Then the Lévy measure of X has a density that is typically positive under most parameter specifications of interest in finance at each point of $\mathbb{R}$ except zero. Recall that X has no Brownian component. (See Cont and Tankov (CT04) and Schoutens (Sch03)). Defining a stock price model as $S = S_0 e^X$ (with $S_0 > 0$) for any of these processes such that the Lévy measure of X has a density that is positive at each point of $\mathbb{R}$ except zero, we find that

$$\lim_{\tau\to 0^+} \Sigma(K, \tau) = \begin{cases} \infty, & \text{if } K \neq S_0 \\ 0, & \text{if } K = S_0. \end{cases}$$

Let Y be Merton's model or Kou's model (see (CT04)). Then the Lévy measure has a density that is positive at each point of $\mathbb{R}$ outside of zero, and it contains a Brownian component. For the model $S = S_0 e^Y$, with $S_0 > 0$ and Y either Merton's or Kou's models we have that

$$\lim_{\tau\to 0^+} \Sigma(K, \tau) = \begin{cases} \infty, & \text{if } K \neq S_0 \\ \sigma, & \text{if } K = S_0. \end{cases}$$

where σ is the square root of the Brownian component of Y's characteristic triplet (see (CT04)).

8.5 Summary of Results and Conclusion

In this chapter, we presented a study of the small time to expiry asymptotics of implied volatility in models of exponential Lévy type that are of interest in mathematical finance. We found that:

(1) Implied volatility is restricted to $[0, \infty)$ where there is a non-trivial Lévy process X such that e^X attains the lower bound for some K and expiries small enough.

(2) In *all* exponential Lévy models, the at-the-money implied volatility converges to zero if the driving Lévy process has no Gaussian part and σ if it has Gaussian part σ^2.

(3) Not at-the-money implied volatility converges to infinity in most examples of interest in mathematical finance because they have a Lévy density which is positive for all $x \in \mathbb{R} \setminus \{0\}$. There exist non-trivial examples where the limiting implied volatility is zero, at least for some strikes.

(4) Small time to expiry asymptotics of the European call.

Chapter 9

Small Time To Expiry Asymptotics: Stochastic Volatility Models (At-the-money case)

In this chapter, we examine the convergence of at-the-money implied volatility to the instantaneous spot volatility as time to expiry goes to zero. Our results are applicable to various continuous models of the stock price. In particular, we present a generalisation of the known conditions under which this holds. By spot volatility, we mean the process σ that appears in the equation

$$S_t = S_0 + \int_0^t \sigma_u S_u \, \mathrm{d}W_u$$

which determines the stock price.

This convergence result plays an important role in the works of Medvedev and Scaillet (see (Med04) and (MS07)) who obtain asymptotics for call option prices and also in Brace *et al.* (see (BGKW01)), where it appears as a model assumption. It was first observed by Ledoit and Santa-Clara (see (LSC98)) that at-the-money implied volatility converges to the instantaneous spot volatility as time to expiry decreases to zero. They worked with continuous processes driven by Wiener noise. Daglish *et. al* (see (DHS07)) prove the same result with an argument somewhat similar to that of Ledoit and Santa-Clara. In both papers, the authors are not entirely precise about the conditions that need to be placed on the (spot) volatility process for the result to hold. However, the basic idea of the proofs in these two papers is correct: when $S_t = S_0 + \int_0^t \sigma_u S_u \, \mathrm{d}W_u$ for a suitable process σ, we have for small h that

$$\frac{S_{t+h} - S_t}{\sqrt{h}} \approx \frac{\sigma_t S_t (W_{t+h} - W_t)}{\sqrt{h}} \stackrel{d}{=} \sigma_t S_t Z,$$

where Z is a standard normal random variable. Clearly, this idea needs to be made precise, which is what we do here.

Let us also note that the convergence of the at-the-money implied volatility to the instantaneous spot volatility as time to expiry goes to zero is an easily proven consequence of Berestycki *et al.*'s (see (BBF02) and (BBF04)) general results on diffusion models of the stock.

Recently, Durrleman (see (Dur08)) has considered this convergence result in a stochastic volatility models with jumps in the stock price. The stochastic volatility model is not assumed to be of diffusion type.

It is assumed that the stochastic volatility process is right-continuous and globally bounded above and below by strictly positive constants and also that the jump process is of finite variation. Durrleman proved that the convergence of the implied volatility to the instantaneous spot volatility still holds in this setup. Our method of proof is completely different to that of Durrleman.

Finally, let us note the work of Carr and Wu (see (CW03)). They present convergence rates for the time value of a call option, i.e.

$$\mathbb{E}\left((S_{t+\tau}-K)^{+}|\mathscr{F}_{t}\right)-(S_{t}-K)^{+} \tag{9.1}$$

in continuous, pure jump and mixed models of the stock. Using the results of Chapter 7, our approach to implied volatility small time to expiry asymptotics essentially amounts to establishing these rates of convergence. We should therefore note that Carr and Wu claim that for at-the-money options the time value goes to zero at rate $\mathrm{O}\left(\sqrt{\tau}\right)$ in purely continuous models of the stock, at rate $\mathrm{O}\left(\tau^{p}\right)$ for $p\in(0,1]$ in pure jump models of the stock, and at rate $\mathrm{O}\left(\tau^{p}\right)$, $p\in(0,1/2]$ in mixed stochastic volatility with jump models. They claim that in mixed processes the rate is dominated by the component with the slowest convergence to zero. However, Carr and Wu's paper is somewhat imprecise as to the exact conditions that must be satisfied by the stock price process for these results to hold.

To the best of my knowledge, Durrleman's work ((Dur08)) is the only rigorous proof that appears in the literature for models containing jumps and also the first that handles non-diffusion stochastic volatility models. As mentioned previously, however, Durrleman's result is also a simple consequence of the rigorous results of Berestycki *et al.* (see (BBF02) and (BBF04)), when there is no jump part.

9.1 Background

9.1.1 Setup

In this chapter, we consider non-negative martingale models of the form

$$S_{t}=S_{0}+\int_{0}^{t}\sigma_{u}S_{u}\,\mathrm{d}W_{u},\quad S_{0}>0$$

on a filtered probability space $(\Omega,\mathscr{F},\mathbb{F}=(\mathscr{F}_{t})_{t\geq0},\mathbb{P})$ satisfying the usual conditions and where S_{0} is a constant.

9.2 Lemmas

We need the following lemma so that we can apply the results of Chapter 7 relating the small time to expiry asymptotics of implied volatility and the call option price.

Lemma 9.2.1. *Let*

$$S_t = S_0 + \int_0^t \sigma_u S_u \, \mathrm{d}W_u, \quad S_0 > 0$$

on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ *satisfying the usual conditions and where* S_0 *is a constant. Suppose also that* S *is a (non-negative) martingale. Then for every* $t \geq 0$ *and* $K \geq 0$

$$\lim_{\tau\to 0^+} \mathbb{E}\left((S_{t+\tau} - K)^+ | \mathscr{F}_t\right) = (S_t - K)^+,$$

$\mathbb{P}$*-a.s.*

Proof. S is obviously càdlàg and non-negative. Also the filtration satisfies the usual conditions. The result now follows from Proposition 4.3.8. □

Lemma 9.2.2. *Let* S *be a non-negative, càdlàg martingale on a filtered probability space* $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ *satisfying the usual conditions. Fix* $t \geq 0$*. Then if* $S_t \neq 0$ *and there exists a constant* $\delta > 0$ *such that, for every* $\tau \in (0, \delta)$*,* $\mathbb{E}\left((S_{t+\tau} - K)^+ | \mathscr{F}_t\right) > (S_t - K)^+$*,* $\mathbb{P}$*-a.s. then*

$$\Sigma_t(S_t, \tau) \sim \sqrt{2\pi} \frac{\mathbb{E}\left((S_{t+\tau} - S_t)^+ | \mathscr{F}_t\right)}{\sqrt{\tau} S_t}$$

$\mathbb{P}$*-a.s. and as* $\tau \to 0^+$*.*

Proof. Fix $t \geq 0$. Note that Corollary 7.3.6 applies with

$$C(S_t, \tau) = \mathbb{E}\left((S_{t+\tau} - S_t)^+ | \mathscr{F}_t\right)$$

because $S_t > 0$ by assumption and the three requisite properties on $C(\cdot, \cdot)$ hold by Lemma 9.2.1 and Proposition 4.3.4 (M3) and (M5). □

9.3 Main Results

We have two conditions under which implied volatility converges to the spot volatility.

Theorem 9.3.1. *Let S be a non-negative martingale on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ with $\mathbb{F}$ satisfying the usual conditions. Suppose that S has the representation*

$$S_t = S_0 + \int_0^t \sigma_u S_u \, \mathrm{d}W_u, \quad S_0 > 0, \quad t \geq 0, \tag{9.2}$$

for some Brownian motion W on $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ where σ is a non-negative $\mathbb{F}$ adapted process such that Equation (9.2) makes sense. Then, if $S_t = 0$ or there exists a $\delta > 0$ such that $\mathbb{E}\left((S_{t+\delta} - S_t)^+ \middle| \mathscr{F}_t\right) = 0$, then

$$\lim_{\tau \to 0^+} \Sigma_t(S_t, \tau) = 0,$$

$\mathbb{P}$*-a.s.*

Proof. Trivial. □

Theorem 9.3.2. *Let S be a non-negative martingale on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ with $\mathbb{F}$ satisfying the usual conditions. Suppose that S has the representation*

$$S_t = S_0 + \int_0^t \sigma_u S_u \, \mathrm{d}W_u, \quad S_0 > 0$$

for some Brownian motion W on the same filtered probability space where σ is a non-negative $\mathbb{F}$ adapted process. Fix $t \geq 0$ and suppose that $S_t > 0$; suppose also that there exists a $\delta > 0$ such that $\mathbb{E}\left((S_{t+\delta} - S_t)^+ | \mathscr{F}_t\right) > 0$, $\mathbb{P}$-a.s.
Suppose that at least one of the following holds:

(i) For every $t \geq 0$

$$\lim_{\tau \to 0^+} \mathbb{E}\left(\sup_{t \leq h \leq t+\tau} |\sigma_h S_h - \sigma_t S_t| \,\middle|\, \mathscr{F}_t \right) = 0, \quad \mathbb{P}\text{-a.s.} \tag{9.3}$$

(ii) S is conditionally square integrable, i.e. $\mathbb{E}\left(S_{t+\tau}^2 | \mathscr{F}_t\right) < \infty$ for every $t, \tau \geq 0$; and for every $t \geq 0$

$$\lim_{\tau \to 0^+} \mathbb{E}\left((\sigma_{t+\tau} S_{t+\tau} - \sigma_t S_t)^2 \,\middle|\, \mathscr{F}_t \right) = 0, \quad \mathbb{P}\text{-a.s.} \tag{9.4}$$

Then for every $t \geq 0$

$$\lim_{\tau \to 0^+} \Sigma_t(S_t, \tau) = \sigma_t, \quad \mathbb{P}\text{-a.s.}$$

Proof. By the conditions of the theorem, we may use Corollary 7.3.6 to find the limiting implied volatility by obtaining

$$\lim_{\tau \to 0^+} \frac{\sqrt{2\pi}\mathbb{E}\left((S_\tau - S_0)^+\right)}{S_0\sqrt{\tau}}.$$

Fix $t \geq 0$. Statements involving conditional expectations are to be interpreted in the almost sure sense. We will write

$$Y_h := \int_t^{t+h} (\sigma_u S_u - \sigma_t S_t)\, \mathrm{d}W_u, \quad h \geq 0.$$

Suppose that S_t and $\mathbb{E}\left((S_{t+\tau} - S_t)^+ \middle| \mathscr{F}_t\right)$ are both positive for all $\tau > 0$. By Lemma 9.2.2, it is enough to show that

$$\frac{\mathbb{E}\left((S_{t+\tau} - S_t)^+ \middle| \mathscr{F}_t\right)}{\sqrt{\tau}} \to \frac{\sigma_t S_t}{\sqrt{2\pi}}, \quad \text{as } \tau \to 0^+.$$

We have that

$$\begin{aligned}
&\frac{1}{\sqrt{\tau}}\mathbb{E}\left((S_{t+\tau} - S_t)^+ \middle| \mathscr{F}_t\right) \\
&= \frac{1}{\sqrt{\tau}}\mathbb{E}\left(\int_t^{t+\tau} \sigma_t S_t \, \mathrm{d}W_u \middle| \mathscr{F}_t\right) \\
&\quad + \frac{1}{\sqrt{\tau}}\mathbb{E}\left(\left(\int_t^{t+\tau} \sigma_u S_u \, \mathrm{d}W_u\right)^+ - \left(\int_t^{t+\tau} \sigma_t S_t \, \mathrm{d}W_u\right)^+ \middle| \mathscr{F}_t\right) \\
&= \frac{\sigma_t S_t}{\sqrt{2\pi}} + \frac{1}{\sqrt{\tau}}\mathbb{E}\left(\left(\int_t^{t+\tau} \sigma_u S_u \, \mathrm{d}W_u\right)^+ - \left(\int_t^{t+\tau} \sigma_t S_t \, \mathrm{d}W_u\right)^+ \middle| \mathscr{F}_t\right).
\end{aligned}$$

Therefore, since $x \mapsto x^+$ is a Lipschitz function, it is enough to show that

$$\lim_{\tau \to 0^+} \frac{1}{\sqrt{\tau}}\mathbb{E}\left(|Y_\tau| \middle| \mathscr{F}_t\right) = 0, \tag{9.5}$$

$\mathbb{P}$-a.s.

We now consider each of the conditions (i) and (ii) in turn.

(i) From the conditional Burkholder-Davis-Gundy Inequality we have that for a certain $c > 0$

$$\begin{aligned}
\mathbb{E}\left(|Y_\tau| \middle| \mathscr{F}_t\right) &\leq c\mathbb{E}\left(\sqrt{\langle Y \rangle_\tau} \middle| \mathscr{F}_t\right) \\
&\leq c\mathbb{E}\left(\sqrt{\tau \sup_{0 \leq h \leq \tau} (\sigma_u S_u - \sigma_t S_t)^2} \middle| \mathscr{F}_t\right).
\end{aligned}$$

Therefore, Equation (9.5) holds by Equation (9.3).

(ii) Fix $h > 0$. By Chebyshev's Inequality and Doob's Maximal Submartingale Inequality, we have that

$$\mathbb{P}\left(\sup_{0\leq\delta\leq h}|Y_\delta| \geq \sqrt{h}\epsilon\middle|\mathscr{F}_t\right) \leq \frac{\mathbb{E}\left((\sup_{0\leq\delta\leq h}|Y_\delta|)^2\middle|\mathscr{F}_t\right)}{h\epsilon^2} \leq 4\frac{\mathbb{E}\left(Y_h^2\middle|\mathscr{F}_t\right)}{h\epsilon^2}.$$

By Tonelli's Theorem we therefore have

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leq\delta\leq h}|Y_\delta| \geq \sqrt{h}\epsilon\middle|\mathscr{F}_t\right) &\leq 4\frac{\mathbb{E}\left(Y_h^2\middle|\mathscr{F}_t\right)}{h\epsilon^2} \\ &\leq 4\frac{\mathbb{E}\left(\int_t^{t+h}(\sigma_u S_u - \sigma_t S_t)^2\,\mathrm{d}u\middle|\mathscr{F}_t\right)}{h\epsilon^2} \\ &= 4\frac{\int_t^{t+h}\mathbb{E}\left((\sigma_u S_u - \sigma_t S_t)^2\middle|\mathscr{F}_t\right)\,\mathrm{d}u}{h\epsilon^2},\end{aligned}$$

so that it is certainly the case that

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leq\delta\leq h}|Y_\delta| \geq \sqrt{h}\epsilon\middle|\mathscr{F}_t\right) &\leq 4\frac{\int_t^{t+h}\sup_{t\leq u\leq t+h}\mathbb{E}\left((\sigma_u S_u - \sigma_t S_t)^2\middle|\mathscr{F}_t\right)\mathrm{d}u}{h\epsilon^2} \\ &\leq 4\frac{\sup_{t\leq u\leq t+h}\mathbb{E}\left((\sigma_u S_u - \sigma_t S_t)^2\middle|\mathscr{F}_t\right)}{\epsilon^2}.\end{aligned}$$

It follows from Equation (9.4) that this last quantity converges to zero as $h \to 0^+$. Therefore

$$\lim_{\tau\to 0^+}|Y_\tau|/\sqrt{\tau} = 0$$

in the almost sure sense. Since we have just established that

$$\lim_{\tau\to 0^+}\mathbb{E}\left(\frac{Y_\tau^2}{\tau}\middle|\mathscr{F}_t\right) = 0,$$

it is clear that $(|Y_\tau|/\sqrt{\tau})_{\tau\in(0,\epsilon')}$ is uniformly integrable for some $\epsilon' > 0$. The claim follows.

□

9.4 Examples

The CEV Process

We need the following results on the CEV process.

Lemma 9.4.1. *Let X be a CEV process, i.e. the solution to the following SDE*

$$dX_t = \lambda X_t^c dW_t, \quad c \in (0,1), \quad \lambda > 0, \quad X_0 > 0,$$

with an absorbing boundary at zero.

(i) if $p \geq 2$, then

$$\mathbb{E}\left(X_t^p\right) \leq \alpha(t,p) + \theta e^t \int_0^t \alpha(s,p) e^{-s} ds,$$

where $\theta = \lambda^2 p(p-1)/2$ and $\alpha(t,p) = X_0^p + \theta t$.

(ii) $(X_t^{2c})_{t\in[0,a]}$ and $(X_t^c)_{t\in[0,a]}$ are uniformly integrable for any $a > 0$.

(iii) For every $K > 0$ and $t > 0$, $\mathbb{E}\left((X_t - K)^+\right) > (X_0 - K)^+$.

Proof. (i) We use parts of the proof of Proposition 5.2 in Andersen and Piterbarg (AP07).

We have

$$dX_t^p = \frac{\lambda^2}{2} p(p-1) X_t^{p-2+2c} dt + p\lambda X_t^{p-1+c} dW_t$$

Therefore, by a localisation argument and the Fatou lemma,

$$\mathbb{E}\left(X_t^p\right) \leq X_0^p + \frac{\lambda^2 p(p-1)}{2} \int_0^t \mathbb{E}\left(X_s^{p-2+2c}\right) ds.$$

By Hölder's Inequality with

$$r = \frac{p}{p-2+2c}$$

and $1/q = 1 - 1/r$, we get that

$$\begin{aligned}
\mathbb{E}\left(X_t^{p-2+2c}\right) &\leq 1 \cdot \left(\mathbb{E}\left((X_t^{p-2+2c})^r\right)\right)^{1/r} \\
&= \left(\mathbb{E}\left((X_t^{p-2+2c})^{p/(p-2+2c)}\right)\right)^{(p-2+2c)/p} \\
&= \left(\mathbb{E}\left(X_t^p\right)\right)^{(p-2+2c)/p}
\end{aligned}$$

The application of Hölder's Inequality can be seen to be valid using that $p \geq 2$ and $c \in (0,1)$.

We then have that,

$$\begin{aligned}
\mathbb{E}\left(X_t^p\right) &\leq X_0^p + \frac{\lambda^2 p(p-1)}{2} \int_0^t \mathbb{E}\left(X_s^{p-2+2c}\right) ds \\
&\leq X_0^p + \frac{\lambda^2 p(p-1)}{2} \int_0^t \left(\mathbb{E}\left(X_s^p\right)\right)^{(p-2+2c)/p} ds.
\end{aligned}$$

Using that $0 < c < 1$ and $p \geq 2$, we deduce that

$$0 \leq \frac{p-2+2c}{p} < 1.$$

We now use that $x^d \leq 1 + x$ for all $x \geq 0$, provided that $0 \leq d \leq 1$.

From the preceding we have that

$$\left(\mathbb{E}\left(X_s^p\right)\right)^{(p-2+2c)/p} \leq 1 + \mathbb{E}\left(X_s^p\right).$$

We now write $F(t)$ for $\mathbb{E}\left(X_t^p\right)$. Using this notation we have

$$\begin{aligned}
F(t) &\leq X_0^p + \theta \int_0^t (F(s))^{(p-2+2c)/p}\, ds \\
&\leq X_0^p + \theta \int_0^t (1 + F(s))\, ds \\
&= X_0^p + \theta t + \theta \int_0^t F(s) ds \\
&= \alpha(t,p) + \theta \int_0^t F(s) ds.
\end{aligned}$$

By Gronwall's inequality, we therefore have that

$$\begin{aligned}
F(t) &\leq \alpha(t,p) + \int_0^t \alpha(s,p)\theta e^{\int_s^t \theta dr} ds \\
&\leq \alpha(t,p) + \int_0^t \alpha(s,p)\theta e^{\theta(t-s)} ds.
\end{aligned}$$

(ii) For the uniform integrability we use the first part of the claim. Fix $a > 0$. Obviously X^c and X^{2c} are non-negative processes. Observe that the assumption that $c \in (0,1)$ gives that both $1/c$ and $2/c > 1$, from which

$$\mathbb{E}\left((X_t^c)^{2/c}\right) = \mathbb{E}\left(X_t^2\right)$$

and

$$\mathbb{E}\left((X_t^{2c})^{1/c}\right) = \mathbb{E}\left(X_t^2\right).$$

The first part of this claim then gives that

$$\sup_{t\in[0,a]} \mathbb{E}\left(X_t^2\right) \leq \sup_{t\in[0,a]} \left[\alpha(t,2) + \theta e^t \int_0^t \alpha(s,2) e^{-s} ds\right] < \infty.$$

Hence, $(X_t^c)_{t\in[0,a]}$ and $(X_t^{2c})_{t\in[0,a]}$ are both uniformly integrable.

(iii) Fix $K > 0$ and $X_0 > 0$. It is easily seen that $\mathbb{E}\left((X_t - K)^+\right) > (X_0 - K)^+$, by using the representation

$$\mathbb{E}\left((X_t - K)^+\right) = (X_0 - K)^+ + \frac{\lambda^2 K^{2c}}{2} \int_0^t q(u, X_0, K) du,$$

where q is the continuous part of the law of the CEV process, as may be obtained from Proposition 10.3.1, and where q is strictly positive.

□

Proposition 9.4.2. *Let S be a CEV process starting from $S_0 > 0$. Then*

$$\lim_{\tau \to 0^+} \Sigma(S_0, \tau) = \lambda S_0^{c-1}.$$

Proof. We suppose that $S_0 > 0$ and $c \in (0, 1)$. We use Lemma 9.4.1 (iii). Using Lemma 9.4.1 (ii) on the uniform integrability of the CEV process as well as its continuity, we see that

$$\begin{aligned}
\lim_{\tau \to 0^+} \mathbb{E}\left((\sigma_\tau S_t - \sigma_0 S_0)^2\right) &= \lim_{\tau \to 0^+} \mathbb{E}\left((\lambda S_\tau^c - \lambda S_0^c)^2\right) \\
&= \lim_{\tau \to 0^+} \mathbb{E}\left(\lambda^2 S_\tau^{2c}\right) - 2\mathbb{E}\left(\lambda S_0^c \lambda S_\tau^c\right) + \lambda^2 S_0 S_0^{2c} \\
&= \lim_{\tau \to 0^+} \mathbb{E}\left(\lambda^2 S_\tau^{2c}\right) - 2\lambda^2 S_0^c \mathbb{E}\left(S_\tau^c\right) + \lambda^2 S_0^{2c} \\
&= \lambda^2 S_0^{2c} - 2\lambda^2 S_0^{2c} + \lambda^2 S_0^{2c}
\end{aligned}$$

It is straightforward that S satisfies:

1. $(S_0 - K) \leq \mathbb{E}\left((S_\tau - K)^+\right) \leq S_0$ for all $K, S_0 > 0$ $\mathbb{P}$-a.s.

2. $\tau \mapsto \mathbb{E}\left((S_\tau - K)^+\right)$ is a non-decreasing function of τ for each K,$\mathbb{P}$-a.s.

3. $\lim_{\tau \to 0^+} \mathbb{E}\left((S_\tau - K)^+\right) = S_0$.

The result is now an straightforward consequence of condition (ii) of Theorem 9.3.2, however, we need to establish that $\mathbb{E}\left((S_\tau - S_0)^+\right) > 0$ for all τ small enough. This is established in Lemma 9.4.1 (iii). □

Remark 9.4.3. The CEV process is not covered by the results of Durrleman (see (Dur08)) since $x \mapsto x^{\beta-1}$ is not bounded.

Bounded Volatility

Proposition 9.4.4. *Let $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F})_{t\geq 0}, \mathbb{P})$ be a filtered probability space under the usual conditions and let W be an $\mathbb{F}$-Brownian motion on this space.*
Define S by

$$S_t = S_0 + \int_0^t \sigma_u S_u \,\mathrm{d}W_u, S_0 = s > 0 \quad t \geq 0, \tag{9.6}$$

where σ is a strictly positive, right-continuous, $\mathbb{F}$-adapted process bounded above by a strictly positive constant $\overline{\sigma}$. To avoid trivialities assume that $\mathbb{P}$-a.s. $\mathbb{E}\left((S_\tau - S_0)^+\right) > 0$ for all τ small enough. Then

$$\lim_{\tau\to 0^+} \Sigma(S_0, \tau) = \sigma_0,$$

$\mathbb{P}$-*a.s.*

Proof. From the assumption of this Proposition, it is clear that we may use Corollary 7.3.6 to get implied volatilities from call option prices. Now, taking the n-th power, we have

$$\begin{aligned}
&\mathbb{E}\left(S_{t+\tau}^n|\mathscr{F}_t\right) \\
&= S_t\mathbb{E}\left(\exp\left(n\int_t^{t+\tau}\sigma_u \,\mathrm{d}Wu - \frac{n}{2}\int_t^{t+\tau}\sigma_u^2 \,\mathrm{d}u\right)\middle|\mathscr{F}_t\right) \\
&= S_t\mathbb{E}\left(\exp\left(n\int_t^{t+\tau}\sigma_u \,\mathrm{d}Wu - \frac{n^2}{2}\int_t^{t+\tau}\sigma_u^2 \,\mathrm{d}u + \frac{n^2}{2}\int_t^{t+\tau}\sigma_u^2 \,\mathrm{d}u - \frac{n}{2}\int_t^{t+\tau}\sigma_u^2 \,\mathrm{d}u\right)\middle|\mathscr{F}_t\right) \\
&= S_t\mathbb{E}\left(\exp\left(\int_t^{t+\tau}n\sigma_u \,\mathrm{d}W_u - \frac{1}{2}\int_t^{t+\tau}n^2\sigma_u^2 \,\mathrm{d}u\right)\exp\left(\frac{n^2}{2}\int_t^{t+\tau}\sigma_u^2 \,\mathrm{d}u - \frac{n}{2}\int_t^{t+\tau}\sigma_u^2 \,\mathrm{d}u\right)\middle|\mathscr{F}_t\right) \\
&\leq S_t\mathbb{E}\left(\exp\left(\int_t^{t+\tau}n\sigma_u \,\mathrm{d}W_u - \frac{1}{2}\int_t^{t+\tau}n^2\sigma_u^2 \,\mathrm{d}u\right)\exp\left(\frac{n^2}{2}\tau\overline{\sigma}^2\right)\middle|\mathscr{F}_t\right) \\
&\leq S_t \exp\left(\frac{n^2}{2}\tau\overline{\sigma}^2\right),
\end{aligned}$$

$\mathbb{P}$-a.s. Therefore,

$$\begin{aligned}
\mathbb{E}\left(\sigma_{t+\tau}^2 S_{t+\tau}^2|\mathscr{F}_t\right) &\leq \overline{\sigma}^2\mathbb{E}\left(S_{t+\tau}^2|\mathscr{F}_t\right) \\
&\leq \overline{\sigma}^2 S_t \exp\left(2\overline{\sigma}^2\tau\right),
\end{aligned} \tag{9.7}$$

$\mathbb{P}$-a.s. Similarly,

$$\begin{aligned}
\mathbb{E}\left(\sigma_{t+\tau}^4 S_{t+\tau}^4|\mathscr{F}_t\right) &\leq \overline{\sigma}^4\mathbb{E}\left(S_{t+\tau}^4|\mathscr{F}_t\right) \\
&\leq \overline{\sigma}^4 S_t \exp\left(8\overline{\sigma}^2\tau\right),
\end{aligned} \tag{9.8}$$

$\mathbb{P}$-a.s. We therefore have uniform integrability of $(\sigma_t^2 S_t^2)_{t\in(0,\epsilon)}$ and of $(\sigma_t S_t)_{t\in(0,\epsilon)}$, there exists an $\epsilon > 0$.

It follows that

$$\lim_{\tau\to 0^+} \mathbb{E}\left((\sigma_{t+\tau} S_{t+\tau} - \sigma_t S_t)^2 | \mathscr{F}_t \right) = 0,$$

$\mathbb{P}$-a.s.

The claim then follows as in the proof of Proposition (9.4.2). □

9.5 Summary of Results and Conclusion

We have examined the convergence of at-the-money implied volatilities to the instantaneous spot volatility in continuous models of the stock price. We presented two sets of sufficient conditions for the convergence of the at-the-money implied volatilities to the instantaneous spot volatility as time to expiry goes to zero. Examples were provided: the CEV model and a model with bounded volatility. This extends some results of Durrleman (see (Dur08)). We have also presented a new method of showing convergence of the at-the-money implied volatilities to the instantaneous (spot) volatility.

Chapter 10

Local times and European Call Options

In this chapter, we prove the formula

$$\mathbb{E}\left((S_T-K)^+\right) = (S_0-K)^+ + \frac{1}{2}\mathbb{E}\left(L_T^K\right) \tag{10.1}$$

$$= (S_0-K)^+ + \frac{1}{2}\int_0^T \mathbb{E}\left(\sigma_t^2 | S_t = K\right) f(t,K)\,\mathrm{d}t \tag{10.2}$$

where $f(t,\cdot)$ is the density of the marginal law of S_t at the strike K, L^K is the local time process of S at the level K, and σ is the absolute volatility of S, for example if $S_t = \int_0^t \sigma_u \,\mathrm{d}W_u$ then σ is the absolute volatility.

It is assumed that S is a diffusion process. We prove the formulae under weaker assumptions than have previously appeared in the literatue (see (Kle02)).

We present two proofs of Equation (10.2). In the one-dimensional case, we allow the volatility function to be piecewise continuous in K. We use a smoothing argument and work with the European put instead of the call. We approximate $x \mapsto (K-x)^+$ by a twice continuously differentable function and then take a limit. Somewhat surprisingly, we were unable to find such a proof in the literature. In the multi-dimensional case, it is difficult to see how to carry through this smoothing argument unless the volatility function is bounded. The popular stochastic volatility models in the literature do not satisfy this requirement, so we instead use the technique of Klebaner (Kle02). The results of this chapter are developed for the purpose of application in Chapter 11, which investigates not at-the-money implied volatility asymptotics in local volatility models. We consider the multi-dimensional case as well since it is of independent interest.

We note that Equation (10.2) is related to Dupire's formula (see (Dup94)) and Berestycki *et al.*'s notion of effective volatility (see (BBF04)). Related results may be found in (Sav02) and the references of (Kle03). It appears that the representation of the call option price using local time was first obtained by Carr and Jarrow (see (CJ90)).

Equations (10.1) and (10.2) appear along with a proof in (Kle02) under the assumption that the martingale S is in $\mathscr{H}^1$ (recall that a continuous martingale, M, is in $\mathscr{H}^1$ when $\mathbb{E}\left(\sqrt{\langle M\rangle_\infty}\right) < \infty$, see (Pro04)). Klebaner allows for continuous processes, although of a more general type than diffusions.

This chapter is organised as follows. In Section 10.1, we present the necessary mathematical background. Section 10.2 presents some lemmas. In 10.3, we investigate first the one-dimensional diffusion case and then the multi-dimensional diffusion case. 10.4 concludes.

10.1 Background

10.1.1 Setup

We deal with diffusion processes. The setup is slightly different for each of our main results and is given as part of the theorem statements.
In the one-dimensional case, we will use the following smoothing of the put payoff function $x \mapsto (K - x)^+$.

Definition 10.1.1. We define a compactly supported function

$$\begin{aligned} \Delta : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto (1 - |x|)\mathbb{1}_{\{|x| \leq 1\}}. \end{aligned}$$

Then, for each $a, K > 0$ we let

$$\begin{aligned} \xi_a^K : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \frac{1}{a}\Delta\left(\frac{x - K}{a}\right), \\ \Upsilon_a^K : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \int_x^\infty \xi_a^K(y)\,\mathrm{d}y, \end{aligned}$$

and

$$\begin{aligned} \Psi_a^K : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \int_x^\infty \Upsilon_a^K(z)\,\mathrm{d}z. \end{aligned}$$

Remark 10.1.2. It is clear that for each fixed $K > 0$, that $\Psi_a^K(\cdot)$ converges pointwise to $(K - \cdot)^+$ as $a \to 0^+$.

10.1.2 Auxiliary Facts

In preparation for later sections in this chapter, we develop the relationship between the local time of the stock price process and the expectation of the call option payoff. Under the assumption that S is a non-negative, continuous martingale it follows from the Itô-Tanaka formula (see Theorem 1.2, p. 222 of (RY99)) that for each $K > 0$

$$(S_T - K)^+ = (S_0 - K)^+ + \int_0^T \mathbb{1}_{\{S_t > K\}} \, \mathrm{d}S_t + \frac{1}{2} L_T^K, \quad T \geq 0,$$

where L^K is the local time of S at the level K, and then

$$\mathbb{E}\left((S_T - K)^+\right) = (S_0 - K)^+ + \mathbb{E}\left(\int_0^T \mathbb{1}_{\{S_t > K\}} \, \mathrm{d}S_t\right) + \frac{1}{2}\mathbb{E}\left(L_T^K\right), \quad T \geq 0.$$

What is not *a priori* clear and requires a proof is that

$$\mathbb{E}\left(\int_0^T \mathbb{1}_{\{S_t > K\}} \, \mathrm{d}S_t\right) = 0. \tag{10.3}$$

In (Kle02), Klebaner imposes an additional restriction on S to ensure that Equation (10.3) holds. We do not do this and instead use a result of Madan and Yor (see (MY06)) to get that Equation (10.3) holds when S is a non-negative, continuous martingale.

Theorem 10.1.3 (Madan and Yor, Theorem 1 in (MY06))**.** *We let S be a continuous, non-negative local martingale on a filtered probability space with a filtration that satisfies the usual conditions. We suppose that S is started at S_0 where S_0 is a finite, positive constant. Fix $K > 0$ and $T \geq 0$. Then*

$$\mathbb{E}\left((S_T - K)^+\right) = (S_0 - K)^+ + \frac{1}{2}\mathbb{E}\left(L_T^K\right) + \mathbb{E}\left(S_T\right) - S_0,$$

where L_T^K is the local time of S at K up to time T. In particular, when S is a (continuous, non-negative) martingale we have that

$$\mathbb{E}\left((S_T - K)^+\right) = (S_0 - K)^+ + \frac{1}{2}\mathbb{E}\left(L_T^K\right).$$

10.2 Lemmas

To prove the martingale property of a certain term in the expression for $\Psi_a^K(S)$, we require the following result.

Lemma 10.2.1. *For every $K > 0$, $x \geq 0$, and $a \in (0, K]$, it holds that*

$$\Psi_a^K(x) \leq K,$$

where Ψ_a^K is defined in Definition 10.1.1.

Proof. See the Appendix starting from page 158. ☐

Lemma 10.2.2. *Let S be the unique in law weak solution of a stochastic differential equation of the form*

$$\mathrm{d}S_t = \sigma(S_t, t)\,\mathrm{d}W_t, \quad S_0 > 0,$$

for a function σ such that the equation makes sense. Assume that S is a non-negative (continuous) martingale. Then, for $0 < a \leq K$, $\int_0^t \Upsilon_a^K(S_u)\,\mathrm{d}S_u$ is a martingale, where Υ_a^K is defined in Definition 10.1.1.

Proof. The proof uses a number of ideas of Madan and Yor's proof of Theorem 10.1.3. Fix $K > 0$ and $0 < a < K$. By Itô's Lemma, we have

$$\Psi_a^K(S_\tau) = \Psi_a^K(S_0) - \int_0^\tau \Upsilon_a^K(S_u)\,\mathrm{d}S_u + \frac{1}{2}\int_0^\tau \sigma^2(S_u, u)\xi_a^K(S_u)\,\mathrm{d}u.$$

To ease the notation we write

$$I_\tau^K = \int_0^\tau \Upsilon_a^K(S_u)\,\mathrm{d}S_u, \quad \tau \geq 0$$

and

$$U_\tau^K = \frac{1}{2}\int_0^\tau \sigma^2(S_u, u)\xi_a^K(S_u)\,\mathrm{d}u, \quad \tau \geq 0.$$

Note first that I^K is a local martingale. To show that it is a martingale we show that there exists an integrable random variable, say H, with the property that $\left|I_\tau^K\right| \leq H$ for all $\tau \geq 0$.

Now,

$$I_\tau^K = -\Psi_a^K(S_\tau) + \Psi_a^K(S_0) + U_\tau^K, \tag{10.4}$$

and by Lemma 10.2.1 we can bound the first two terms on the right hand side of this equation. It remains therefore to establish a bound on U^K.

Let $(\sigma_n)_{n\geq 1}$ be a localising sequence of stopping times for I^K. Rearranging Equation (10.4) we get

$$\begin{aligned}
\mathbb{E}\left(U_{\tau\wedge\sigma_n}^K\right) &= \mathbb{E}\left(\Psi_a^K(S_{\tau\wedge\sigma_n}) - \Psi_a^K(S_0) + I_{\tau\wedge\sigma_n}^K\right) \\
&\leq \mathbb{E}\left(\Psi_a^K(S_{\tau\wedge\sigma_n})\right) \\
&\leq K,
\end{aligned}$$

by Lemma 10.2.1 and the claim follows easily.

By the non-negativity of the process U^K, we may apply Fatou's Lemma to get, using the last display, that

$$\mathbb{E}\left(U_\tau^K\right) = \mathbb{E}\left(\liminf_{n\to\infty} U_{\tau\wedge\sigma_n}^K\right) \leq \liminf_{n\to\infty} \mathbb{E}\left(U_{\tau\wedge\sigma_n}^K\right) \leq K.$$

Because U^K is bounded below we may apply the Fatou Lemma again, to get that

$$\mathbb{E}\left(U_\infty^K\right) = \mathbb{E}\left(\liminf_{\tau\to\infty} U_\tau^K\right) \leq \liminf_{n\to\infty} \mathbb{E}\left(U_\tau^K\right) \leq K. \tag{10.5}$$

Using the non-negativity of U^K, it follows from Equation (10.5) that

$$U_\infty^K < \infty, \quad \mathbb{P}\text{-a.s.}$$

Recalling Equation (10.4) and Lemma 10.2.1 we get that

$$\begin{aligned}\left|I_\tau^K\right| &= \left|-\Psi_a^K(S_\tau) + \Psi_a^K(S_0) + U_\tau^K\right| \\ &\leq \left|-\Psi_a^K(S_\tau)\right| + \left|\Psi_a^K(S_0)\right| + \left|U_\tau^K\right| \\ &\leq 2K + U_\infty^K \\ &< \infty,\end{aligned}$$

$\mathbb{P}$-a.s. □

10.3 Main Results

10.3.1 One-dimensional Diffusions

Smooth Case

We begin by deriving Equation (10.1) in the one-dimensional case with
$\mathbb{E}\left(\sigma_t^2 | S_t = K\right) \equiv \sigma^2(K,t)$ and where σ is continuous in space and time.

Proposition 10.3.1. *Suppose that*

$$dS_t = \sigma(S_t,t)\,\mathrm{d}W_t, \quad S_0 = s \geq 0, \tag{10.6}$$

with, if it is possible to do so, an absorbing boundary condition imposed at zero. We suppose that Equation (10.6) *has a unique weak solution in the sense of probability law which is in addition a (non-negative) martingale. The law of S_t for some ts may have an* *atom at zero. We assume that $\sigma : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is jointly measurable and, in addition, for each $t \geq 0$*

$$x \mapsto \sigma(x,t)$$

is a continuous function and so, in particular, is everywhere finite. We make the following assumptions on the probability law of S:

(1) For each fixed $t \geq 0$ and $x, \tau > 0$ the law of $S_{t+\tau}$ given that $S_t = x$ is absolutely continuous on $(0,\infty)$, but may have an atom at zero. For the absolutely continuous part we write

$$\mathbb{P}\left(S_{t+\tau} \in \mathrm{d}z | S_t = x\right) = p(t,x,t+\tau,z)\,\mathrm{d}z.$$

(2) For each $t \geq 0$ and $\tau, x \in (0,\infty)$

$$z \mapsto p(t,x,t+\tau,z)$$

is a continuous function and, in particular, is everywhere finite.

Then for every $t,h,s \geq 0$ and $K > 0$

$$\mathbb{E}\left((S_{t+h} - K)^+ \middle| S_t = s\right) = (s-K)^+ + \frac{1}{2}\int_t^{t+h} \sigma^2(K,u) p(t,S,u,K)\,\mathrm{d}u.$$

Proof. Let $((S,W),(\Omega,\mathscr{F},\mathbb{P}),\mathbb{F})$ be the unique in law weak solution of Equation (10.6). In statements involving conditional expectations, we will omit the rider $\mathbb{P}$-a.s. We will first prove that for all $t,h,s \geq 0$ and $K > 0$

$$\mathbb{E}\left((K - S_{t+h})^+ \middle| S_t = s\right) = (K-s)^+ + \int_t^{t+h} \frac{\sigma^2(K,u)}{2} p(t,s,u,K)\,\mathrm{d}u. \tag{10.7}$$

We take the straightforward approach of approximating the function $(K-\cdot)^+$ by smooth functions and taking a limit. Recall the definitions of $\Psi_a^K, \Upsilon_a^K, \xi_a^K$, and Δ from Definition 10.1.1. Fix $t \geq 0$, $K > 0$, and, for the moment, $0 < a < K$. If $h = 0$ or $s = 0$, then the claim is trivial so we choose and fix $h > 0$ and $s > 0$.
We may apply Itô's Lemma and Lemma 10.2.2, to get

$$\begin{aligned}
&\mathbb{E}\left(\Psi_a^K(S_{t+h}) \middle| S_t = s\right) \\
&= \mathbb{E}\left(\Psi_a^K(S_t) - \int_t^{t+h} \Upsilon_a^K(S_u)\,\mathrm{d}S_u + \frac{1}{2}\int_t^{t+h} \xi_a^K(S_u)\sigma^2(S_u,u)\,\mathrm{d}u \middle| S_t = s\right) \\
&= \Psi_a^K(s) + \mathbb{E}\left(\frac{1}{2}\int_t^{t+h} \xi_a^K(S_u)\sigma^2(S_u,u)\,\mathrm{d}u \middle| S_t = s\right).
\end{aligned}$$

By Tonelli's Theorem and the fact that Ψ_a^K is bounded for all a smaller than some constant, say $A > 0$.

$$\begin{aligned}
&\mathbb{E}\left(\int_t^{t+h} \xi_a^K(S_u)\sigma^2(S_u,u)\,\mathrm{d}u \,\middle|\, S_t = s\right) \\
&= \int_t^{t+h} \mathbb{E}\left(\xi_a^K(S_u)\sigma^2(S_u,u)|S_t = s\right)\,\mathrm{d}u \\
&< \infty.
\end{aligned}$$

Without loss of generality, we suppose that $u \in [t, t+h]$ is fixed and such that

$$\mathbb{E}\left(\xi_a^K(S_u)\sigma^2(S_u,u)|S_t = s\right) < \infty.$$

We now show that there exists an $\epsilon > 0$ such that

$$\sup_{a\in(0,\epsilon)} \mathbb{E}\left(\xi_a^K(S_u)\sigma^2(S_u,u)\,\middle|\, S_t = s\right) < \infty. \tag{10.8}$$

By our assumption on σ, it is locally bounded, so that there exists an $\epsilon' > 0$ such that for all $0 < a < \epsilon'$

$$\begin{aligned}
\mathbb{E}\left(\xi_a^K(S_u)\sigma^2(S_u,u)|S_t = s\right) &= \int_{K-a}^{K+a} \xi_a^K(x)\sigma^2(x,u)p(t,s,u,x)\,\mathrm{d}x \\
&\leq M\mathbb{E}\left(\xi_a^K(S_u)|S_t = s\right)
\end{aligned}$$

where $M \geq 0$ is some constant independent of a. It is also assumed that a is small enough so that $K - a > 0$. Hence, the possibility of the law of S_u having an atom at zero is irrelevant. Therefore, since

$$\lim_{a\to 0^+} \int_{K-a}^{K+a} \xi_a^K(x)p(t,s,u,x)\,\mathrm{d}x = p(t,s,u,K),$$

by the assumed continuity of p in its second argument on $(0,\infty)$, it holds that for every $\epsilon'' > 0$, there exists a $\delta > 0$ such that for all $a \in (0, \delta \wedge \epsilon')$

$$\mathbb{E}\left(\xi_a^K(S_u)\sigma^2(S_u,u)\,\middle|\, S_t = s\right) < Mp(t,s,u,K)\epsilon'',$$

and Equation (10.8) follows immediately. Therefore,

$$\begin{aligned}
&\lim_{a\to 0^+} \mathbb{E}\left(\Psi_a^K(S_{t+h})\right) \\
&= \lim_{a\to 0^+}\left(\Psi_a^K(s) + \frac{1}{2}\int_t^{t+h}\int_0^\infty \xi_a^K(x)\sigma^2(x,u)p(t,s,u,x)\,\mathrm{d}x\,\mathrm{d}u\right) \\
&= (K-s)^+ + \frac{1}{2}\int_t^{t+h} \lim_{a\to 0^+}\int_0^\infty \xi_a^K(x)\sigma^2(x,u)p(t,s,u,x)\,\mathrm{d}x\,\mathrm{d}u \qquad (10.9) \\
&= (K-s)^+ + \frac{1}{2}\int_t^{t+h} \sigma^2(K,u)p(t,s,u,K)\,\mathrm{d}u,
\end{aligned}$$

where we used the assumed continuity of σ in the space variable and of p in its second argument. Now, $\Psi_a^K(x)$ is bounded, so Dominated Convergence yields

$$
\begin{aligned}
(K-s)^+ + \frac{1}{2}\int_t^{t+h} \sigma^2(K,u)p(t,s,u,K)\,\mathrm{d}u &= \lim_{a\to 0^+} \mathbb{E}\left(\Psi_a^K(S_{t+h})\middle|\, S_t = s\right) \\
&= \mathbb{E}\left(\lim_{a\to 0^+} \Psi_a^K(S_{t+h})\middle|\, S_t = s\right) \\
&= \mathbb{E}\left((K-S_{t+h})^+ | S_t = s\right).
\end{aligned}
$$

We have proved that Equation (10.7) holds. Since S is a martingale we may apply put-call parity to get that

$$
\mathbb{E}\left((S_{t+h}-K)^+\middle|\, S_t = s\right) = \mathbb{E}\left((K-S_{t+h})^+\middle|\, S_t = s\right) + (s-K)
$$

and the original claim follows after some simple algebra. □

Discontinuous Diffusion Co-efficient

There is some interest in the mathematical finance literature in diffusions with a discontinuous diffusion coefficient (see, for example, (DDSG04) and references therein). We therefore relax the conditions of Proposition 10.3.1 and allow for discontinuities in the volatility function.

Proposition 10.3.2. *Suppose that*

$$
dS_t = \sigma(S_t,t)\,\mathrm{d}W_t, \quad S_0 = s \geq 0, \tag{10.10}
$$

with, if it is possible to do so, an absorbing boundary condition imposed at zero. We suppose that Equation (10.10) *has a unique weak solution in the sense of probability law which is in addition a non-negative martingale. The law of S_t for some ts may have an atom at zero. We make the following assumption on $\sigma : [0,\infty)\times[0,\infty)\to\mathbb{R}$:*

(1) For each $t \geq 0$

$$
x \mapsto \sigma(x,t)
$$

is locally bounded.

(2) σ is jointly measurable.

(3) *What is more, for Lebesgue almost every $t \geq 0$ and every $x_0 \in [0, \infty)$*

$$\lim_{x \to x_0^+} \sigma(x, t) =: \sigma(x_0+, t) \in [0, \infty) \text{ and } \lim_{x \to x_0^-} \sigma(x, t) = \sigma(x_0-, t) \in [0, \infty)$$

where both limits exist and are finite.[1]

(4) *σ is jointly measurable.*

We make the following assumptions on the probability law of S:

(1) *For each fixed $t \geq 0$ and $x, \tau > 0$ the law of $S_{t+\tau}$ given that $S_t = x$ is absolutely continuous on $(0, \infty)$ and for the absolutely continuous part on $(0, \infty)$, we write*

$$\mathbb{P}\left(S_{t+\tau} \in \mathrm{d}z | S_t = x\right) = p(t, x, t+\tau, z)\, \mathrm{d}z.$$

There may be an atom at zero.

(2) *For each $t \geq 0$ and $\tau, x \in (0, \infty)$*

$$z \mapsto p(t, x, t+\tau, z)$$

is a continuous function and, in particular, is everywhere finite.

Then for every $t, h, s \geq 0$ and K strictly greater than 0

$$\mathbb{E}\left((S_{t+h} - K)^+ \middle| S_t = s\right) = (s - K)^+ + \frac{1}{2} \int_t^{t+h} p(t, s, u, K) \frac{\sigma^2(K+, u) + \sigma^2(K-, u)}{2}\, \mathrm{d}u.$$

Proof. The changes at Equation (10.9) of the proof of Proposition 10.3.1 are obvious. □

10.3.2 Representation of the Call Price of a Multi-dimensional Diffusion Process via Local Time

In order to simplify the notation and make things more definite, we will restrict attention to a generic two-dimensional stochastic volatility model. Extension to more dimensions is straightforward.

[1]Obviously when $x_0 = 0$ we are only concerned with the first limit.

Definition 10.3.3 (Model A)**.** We consider the system

$$\begin{aligned}
S_\tau &= S_0 + \int_0^\tau \sigma(S_t, Y_t, t)t \, \mathrm{d}W_t \\
Y_\tau &= Y_0 + \int_0^t b(S_t, Y_t, t) \, \mathrm{d}t + \int_0^\tau v(S_t, Y_t, t) \, \mathrm{d}\overline{W}_t \\
\mathrm{d}\langle W, \overline{W} \rangle_t &= \rho \, \mathrm{d}t
\end{aligned}$$

where $\rho \in [-1, 1]$. We suppose that S is a strictly positive martingale. The functions $\sigma, b, v : \mathbb{R} \times \mathbb{R} \times [0, \infty) \to \mathbb{R}$ are assumed sufficiently regular that a (unique) weak solution (in law) exists. In addition, we suppose that

1. the stochastic differential equation possesses a (unique) transition density, i.e. the law of S_t has a density for each $t \geq 0$; and

2. the transition density is strictly positive.

We denote the transition density by

$$\mathbb{P}\left((S_t, Y_t) \in (\mathrm{d}s, \mathrm{d}y) | (S_0, Y_0) = (s_0, y_0)\right) = p(0, (s_0, y_0), t, (s, y)) \, \mathrm{d}s \, \mathrm{d}y$$

In the next theorem, we extend the main result of (Kle02) (see Theorem (Kle02)) and using essentially his proof.

Theorem 10.3.4. *In Model A, for every $K > 0$ and $\tau \geq 0$*

$$\begin{aligned}
\mathbb{E}\left((S_\tau - K)^+\right) &= (S_0 - K)^+ + \frac{1}{2}\mathbb{E}\left(L_\tau^K\right) \\
&= (S_0 - K)^+ + \frac{1}{2}\int_0^\tau \mathbb{E}\left(\sigma^2(S_t, Y_t, t) | S_t = K\right) f(t, K) \, \mathrm{d}t \\
&= (S_0 - K)^+ + \frac{1}{2}\int_0^\tau \left(\int_0^\infty \sigma^2(K, y, t) p(0, (S_0, Y_0), t, (K, y)) \, \mathrm{d}y\right) \mathrm{d}t
\end{aligned}$$

where $f(t, K)$ is the density of S_t at time t.

Proof. The first equality comes from Theorem 10.1.3. Fix $\tau \geq 0$ and $K > 0$. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous, positive, bounded function with compact support. It is convenient to let

$$\sigma_t = \sigma(S_t, Y_t, t), \quad \forall t \geq 0.$$

By the definition of local time

$$\int L_\tau^K g(K) \, \mathrm{d}K = \int_0^\tau g(S_t) \mathrm{d}\langle S \rangle_t = \int_0^\tau g(S_t) \sigma_t^2 \, \mathrm{d}t, \tag{10.11}$$

$\mathbb{P}$-a.s., by the Occupation Times formula (see Corollary 1.6, p. 224 of (RY99)). Taking expectations and using Tonelli's Theorem, we get

$$\int \mathbb{E}\left(L_\tau^K\right) g(K)\,\mathrm{d}K = \int_0^\tau \mathbb{E}\left(g(S_t)\sigma_t^2\right)\,\mathrm{d}t < \infty. \tag{10.12}$$

The finiteness of the integral uses the bound (valid for each T) that

$$\mathbb{E}\left(L_T^K\right) \leq 2S_0,$$

which is obtained in the proof of Theorem 10.1.3 in (MY06). In addition, we use that g is bounded and continuous with compact support. From Equation (10.12), we therefore get from Equations (10.12) and (10.11).

The left-hand side, and hence the right-hand side, is finite. This is because

$$\int g(K)\mathbb{E}\left(L_T^K\right)\,\mathrm{d}K \leq 2S_0 \int g(K)\,\mathrm{d}K < \infty,$$

using the proof of Theorem 10.1.3 (see (MY06)) which gives the bound $\mathbb{E}\left(L_T^K\right) \leq 2S_0$, valid for all $T, K > 0$. We also used that g is bounded with compact support.

Now note that

$$\int_0^\tau \mathbb{E}\left(g(S_t)\sigma_t^2\right)\,\mathrm{d}t < \infty$$

implies that $\mathbb{E}\left(g(S_t)\sigma_t^2\right) < \infty$ and so $\mathbb{E}\left(\sigma_t^2\right) < \infty$ for Lebesgue almost every $t \in [0, \tau]$.

Now,

$$\begin{aligned}
\int \mathbb{E}\left(L_\tau^K\right) g(K)\,\mathrm{d}K &= \int_0^\tau \mathbb{E}\left(g(S_t)\sigma_t^2\right)\,\mathrm{d}t \\
&= \int_0^\tau \mathbb{E}\left(\mathbb{E}\left(g(S_t)\sigma_t^2 | S_t\right)\right)\,\mathrm{d}t \\
&= \int_0^\tau \int_0^\infty \mathbb{E}\left(\sigma_t^2 | S_t = K\right) g(K) f(t,K)\,\mathrm{d}K\,\mathrm{d}t \\
&= \int_0^\tau \int_0^\infty g(K)\mathbb{E}\left(\sigma_t^2 | S_t = K\right) f(t,K)\,\mathrm{d}K\,\mathrm{d}t \\
&= \int_0^\infty g(K) \int_0^\tau \mathbb{E}\left(\sigma_t^2 | S_t = K\right) f(t,K)\,\mathrm{d}t\,\mathrm{d}K,
\end{aligned}$$

where we used Tonelli's Theorem.

Since g is a continuous, positive, bounded function with compact support, we get that

$$\mathbb{E}\left(L_\tau^K\right) = \int_0^\tau \mathbb{E}\left(\sigma_t^2 | S_t = K\right) f(t,K)\,\mathrm{d}t. \tag{10.13}$$

Therefore,

$$\mathbb{E}\left((S_\tau - K)^+\right) = (S_0 - K)^+ + \frac{1}{2}\mathbb{E}\left(L_\tau^K\right)$$
$$= (S_0 - K)^+ + \frac{1}{2}\int_0^\tau \mathbb{E}\left(\sigma_t^2 | S_t = K\right) f(t,K)\,\mathrm{d}t$$

by Theorem 10.1.3. As Model A describes a diffusion, we have that

$$\mathbb{E}\left(\sigma_t^2 | S_t = K\right) = \frac{\int_0^\infty \sigma^2(K,y,t)p(0,(s_0,y_0),t,(K,y))\,\mathrm{d}y}{f(t,K)}.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\left((S_\tau - K)^+ | (S_0, Y_0) = (s_0, y_0)\right) \\
&= (s_0 - K)^+ + \frac{1}{2}\int_0^\tau \mathbb{E}\left(\sigma_t^2 | S_t = K\right) f(t,K)\,\mathrm{d}t \\
&= (s_0 - K)^+ + \frac{1}{2}\int_0^\tau \frac{\int_0^\infty \sigma^2(K,y,t)p(0,(s_0,y_0),t,(K,y))\,\mathrm{d}y}{f(t,K)} f(t,K)\,\mathrm{d}t \\
&= (s_0 - K)^+ + \frac{1}{2}\int_0^\tau \int_0^\infty \sigma^2(K,y,t)p(0,(s_0,y_0),t,(K,y))\,\mathrm{d}y\,\mathrm{d}t.
\end{aligned}$$

□

10.4 Summary of Results and Conclusions

We obtained an expression for the expected value of the call option payoff. The representations, which integrate over time to expiry, are ideally suited for examining the behaviour of call option prices in the small time to expiry limit. We studied both one- and multi- dimensional diffusions, using a different argument in each case. The one-dimensional case easily allows for the law of the stock process to have an atom at zero and also for a discontinuous volatility function. The argument in this case is new. We also proved an extension of a result of Klebaner on the representation of call option prices via local time, by using a result of Madan and Yor (see (MY06)).

Chapter 11

Small Time to Expiry Asymptotics: Local Volatility Models (Not At-the-money case)

In this chapter, we examine the small time to expiry behaviour of implied volatility in local volatility models. We restrict our attention to the not at-the-money case, since the at-the-money case has already been addressed in Chapter 9.

The main contribution of this chapter to the literature is an elementary proof of the small time to expiry implied volatility asymptotics in local volatility models previously presented by Berestycki *et al.* in (BBF02). Note, however, that the models we consider are less general than those considered by Berestycki *et al.* in (BBF02). We build on the work of previous chapters and obtain the limiting not at-the-money implied volatility by calculating

$$\lim_{\tau\to 0^+} \frac{|\ln(K/S_0)|}{\sqrt{-2\tau \ln(\mathbb{E}\left((S_\tau - K)^+\right) - (S_0-K)^+)}},$$

this being the same thing, by Corollary 7.3.6. Obviously, the challenge is to determine

$$\lim_{\tau\to 0^+} -2\tau \ln(\mathbb{E}\left((S_\tau - K)^+\right) - (S_0-K)^+),$$

or to show that the limit does not exist.

We will now briefly review the literature on the small time to expiry behaviour of implied volatility. The literature on small time asymptotics of the transition density of diffusions is very large. We do not attempt to review the literature on this subject in this thesis. Our review will include both local and stochastic volatility models although we only consider local volatility models in the remainder of the chapter.

The papers by Hagan and Woodward (see (HW99)) and Hagan *et al.* (see (HKLW02)) appear to have been particularly influential. In (HW99), Hagan and Woodward study models of the form

$$\mathrm{d}S_t = \alpha(t)A(S_t)\,\mathrm{d}W_t, \quad S_0 = s > 0, \tag{11.1}$$

for suitably defined functions α and A. A detailed analysis of the CEV (Constant Elasticity of Variance) model, i.e.

$$\mathrm{d}S_t = \alpha S_t^\beta\,\mathrm{d}W_t, \quad S_0 = s > 0, 0 < \beta < 1, \tag{11.2}$$

is included. In (HKLW02), Hagan *et al.* analyse a particular stochastic volatility model which they term the "SABR" model. It takes the form

$$\begin{aligned} \mathrm{d}S_t &= \sigma_t S_t^B \,\mathrm{d}W_t^1, \quad S_0 > 0 \\ \mathrm{d}\sigma_t &= \nu\sigma_t \,\mathrm{d}W_t^2, \quad \sigma_0 = \overline{\sigma} > 0, \end{aligned}$$

and $d\langle W^1, W^2\rangle_t = \rho\,\mathrm{d}t$, where $\rho \in [0,1]$. The relationship to the CEV model is clear. In the works of Hagan *et. al.*, singular perturbtation techniques are employed (see (KC81)). Following from the works of Hagan, Woodward and their co-authors, Bourgade and Croissant (see (BC05)) and Labordere (see (HL05)) provide asymptotic formulae for implied volatility in quite general classes of stochastic volatility models. These authors rely on results from differential geometry and the small time asymptotics for the transition densities of diffusion process. The results of Hagan *et al.* are used at crucial stages in the work of (BC05). We should also note the works of Medvedev and Scaillet (see (Med04) and (MS07)). We note that the works cited above are practical in nature; it was not the primary concern of the authors to determine precise conditions under which their results hold, nor to provide rigorous proofs.

Berestycki *et al.* (BBF02) rigorously examine local volatility models where the volatility function is globally bounded above and below, uniformly continuous, and time-inhomogeneous. Amongst other results they prove that

$$\lim_{\tau\to 0^+} \frac{1}{\widetilde{\Sigma}(x,\tau)} = \int_0^1 \frac{\mathrm{d}v}{\sigma(vx,0)}, \quad \text{uniformly in } x \in \mathbb{R},$$

where $\widetilde{\Sigma}(x,\tau)$ is the Black-Scholes implied volatility of an option with time to expiry $\tau = T - t$ and log-moneyness $x = \ln(s/K)$. We used s to denote the stock price value at time zero. In (BBF04), Berestycki *et al.* generalise their results to stochastic volatility models (of diffusion type). They assume Hölder continuity and certain growth and boundedness conditions on the drift and diffusion coefficient functions.

Obłój (Obł07) compares the results of Berestycki *et al.* (BBF04) and the results of Hagan *et al.* ((HW99) and (HKLW02)). He notes that the implied volatility limit in the CEV model reported in (HW99) is different to that obtained from (HKLW02). He computes the implied volatility limit using the results of (BBF04) and finds that it agrees with (HKLW02) in the case that the SABR model collapses to the CEV model and also when $\beta = 1$, but that there is a difference when $\beta < 1$. We note that it appears that when $\beta \in (0, 1)$ the results of (BBF04) are not directly applicable to the SABR model. In addition, when $\beta = 1$ we have the log-normal stochastic volatility model which is known to be a strictly local martingale when $\rho > 0$, see Chapter 5 and Musiela and Lions (LM06).

11.1 Background

11.1.1 Auxiliary results

We will use the following classical result of Varadhan. We have written it in terms of stochastic differential equations, even though Varadhan used partial differential equations and Markov theory.

Theorem 11.1.1 (Varadhan, Theorem 6.4 in (Var67a))**.** *Let*

$$\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + g(X_t)\,\mathrm{d}W_t.$$

1. *the drift coefficient satisfies a uniform Hölder condition:*

$$|f(x) - f(y)| \leq M_1 |x - y|^{\alpha_1}, \quad \exists M_1, \alpha_1 > 0,$$

   for all $x, y \in \mathbb{R}$;

2. *the drift coefficient is bounded:*

$$|f(x)| \leq M_2 < \infty, \quad \exists M_2 > 0,$$

   for all $x \in \mathbb{R}$;

3. *The diffusion coefficient satisfies a uniform Hölder condition:*

$$\left|g^2(x) - g^2(y)\right| \leq M_3 |x - y|^{\alpha_2}, \quad \exists M_3, \alpha_2 > 0,$$

   for all $x, y \in \mathbb{R}$;

4. *The diffusion coefficient satisfies a uniform ellipticity condition:*

$$0 < m_1 \leq g^2(x) \leq M_4 < \infty, \quad \exists m_1, M_4 > 0,$$

   for all $x \in \mathbb{R}$.

Then

$$\lim_{t\to 0^+} -2t\ln(p(t,x,y)) = d_X^2(x,y) = \left(\int_y^x \frac{\mathrm{d}z}{g(z)}\right)^2 \tag{11.3}$$

uniformly in x and y over compact sets.

Remark 11.1.2. The function d_X in a setup such as that in the previous theorem is a metric on $\mathbb{R}^2$ and has deep connections in differential geometry. See (dC92).

Remark 11.1.3. It is standard that under these conditions the diffusion X possesses a unique, strictly positive, continuous transition density (see, for example, (Dyn65), (Var67b), or (Var67a)). We denote the density p^X.

11.2 Lemmas

Lemma 11.2.1. *Let $f, g : (0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ with $n\in\mathbb{N}$. Suppose that*

(A.1) $\lim_{\tau\to 0^+} \tau \ln g(\tau,\mathbf{x}) = 0$*; and*

(A.2) There exists a non-negative function $D : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{\tau\to 0^+} \tau \ln f(\tau,\mathbf{x}) = -D(\mathbf{x});$$

and the limits in both (A.1) and (A.2) are uniform in $\mathbf{x}$ ranging over any fixed compact subset , denote it B, of $\mathbb{R}^n$. It then holds that

$$\lim_{\tau\to 0^+} \tau \ln \int_0^\tau f(u,\mathbf{x})g(u,\mathbf{x})\,\mathrm{d}u = -D(\mathbf{x}),$$

uniformly in $\mathbf{x}$ ranging over B.

Proof. See the Appendix starting from page 158. □

Lemma 11.2.2. *Let S be a CEV (Constant Elasticity of Variance) process. Let q denote the continuous part of the law of S_t. Then, for $s, K > 0$,*

$$\lim_{t\to 0^+} -2t\ln(q(t,s,K)) = \frac{\left(S_0^{1-\beta} - K^{1-\beta}\right)^2}{\sigma^2(1-\beta)^2}.$$

Proof. See page 160 of the Appendix for the proof. □

11.3 Examples

11.3.1 CEV Model

Proposition 11.3.1. *Let S be a CEV (Constant Elasticity of Variance) process, i.e. the (strong) solution to the stochastic differential equation*

$$\mathrm{d}S_t = \sigma S_t^{\beta}\,\mathrm{d}W_t, \quad S_0 > 0,$$

with $\beta \in (0,1)$, $\sigma > 0$, $S_0 \neq K$, and an absorbing boundary condition at zero. Then

$$\lim_{\tau \to 0^+} \Sigma_0(K,\tau) = \frac{\sigma(1-\beta)\ln(S_0/K)}{S_0^{1-\beta} - K^{1-\beta}},$$

provided that $S_0 > 0$.

Proof. One of the reasons that the CEV model is interesting is that for time $t > 0$ its law has a continuous density on $(0,\infty)$ and an atom at zero. Using our results from the previous chapter, this atom has no part in the expression of the call price as a time-varying integral.

We first establish the applicability of Corollary 7.3.6. First of all note that S_0 is strictly positive. Now note that S is a continuous, non-negative martingale on a filtered probability space with filtration satisfying the usual conditions. Therefore, for all $K > 0$

(1) $(S_0 - K)^+ \leq \mathbb{E}\left((S_\tau - K)^+\right) \leq S_0$;

(2) $\lim_{\tau \to 0^+} \mathbb{E}\left((S_\tau - K)^+\right) = (S_t - K)^+$;

(3) $\tau \mapsto \mathbb{E}\left((S_\tau - K)^+\right)$ is a non-decreasing function.

Finally, we note that by Proposition 10.3.1 that

$$\begin{aligned}\mathbb{E}\left((S_\tau - K)^+\right) &= \mathbb{E}\left((S_\tau - K)^+ | S_0 = s\right)\\ &= (S_0 - K)^+ + \frac{\sigma^2 K^{2\beta}}{2}\int_0^{\tau} q(u, S_0, K)\,\mathrm{d}u\\ &> (S_0 - K)^+, \quad \text{provided that } \tau > 0.\end{aligned}$$

Since q is strictly positive, we may therefore use Corollary 7.3.6 to get the small time to expiry limit of implied volatility in the CEV model.

From Lemma 11.2.2, we have that

$$\lim_{t\to 0+} -2t\ln(q(t,S_0,K)) = \frac{\left(S_0^{1-\beta}-K^{1-\beta}\right)^2}{\sigma^2(1-\beta)^2},$$

where $q(t,s,K)$ is the continuous part of the density of the law of S_t having started at $S_0 = s > 0$. It is now easy to obtain the not at-the-money implied volatility asymptotics. We have

$$\lim_{\tau\to 0^+} -2\tau\ln\left(\mathbb{E}\left((S_\tau - K)^+ - (S_0-K)^+\right)\right)$$

which, by Proposition 10.3.1, is

$$= \lim_{\tau\to 0^+} -2\tau\ln\left(\sigma^2\frac{K^{2\beta}}{2}\int_0^\tau q(t,S_0,K)\,\mathrm{d}t\right)$$
$$= \lim_{\tau\to 0^+} -2\tau\ln\left(\int_0^\tau q(t,S_0,K)\,\mathrm{d}t\right)$$

which, by Lemma 11.2.1, is

$$= \lim_{\tau\to 0^+} -2\tau\ln\left(q(\tau,S_0,K)\right)$$

which, by Lemma 11.2.2, is

$$= \frac{\left(S_0^{1-\beta}-K^{1-\beta}\right)^2}{\sigma^2(1-\beta)^2}.$$

We may then apply Corollary 7.3.6 to get that

$$\lim_{\tau\to 0^+} \Sigma_0(K,\tau) = \frac{\sigma(1-\beta)\ln(S_0/K)}{S_0^{1-\beta}-K^{1-\beta}},$$

which agrees with the formal result of Obłój (see (Obł07)). □

Remark 11.3.2. The result in Proposition 11.3.1 is the same as a formal application of the results of Berestycki *et al.* (see (BBF02) and (BBF04)) give. However, it should also be noted that it is not an immediate corollary of the results of Berestycki since the volatility is not bounded.

11.3.2 Model V

If we start with an equation for S, e.g.

$$\mathrm{d}S_t = \sigma(S_t)S_t\,\mathrm{d}W_t,$$

with transition density p^S we run into problems applying Varadhan's Theorem 11.1.1 because of, for example, the possible unboundedness of $x \mapsto \sigma(x)x$. In

addition, Varadhan's Theorem is written for processes which have state space $\mathbb{R}$. We therefore use a standard trick and consider the model for $\ln(S) =: Y$.
We now consider the model of $Y(= \ln S)$, handled by Varadhan's Theorem (see Theorem 11.1.1) and term it Model V.

Proposition 11.3.3. *Let*

$$\mathrm{d}S_t = \sigma(S_t)S_t \, \mathrm{d}W_t,$$

then consider

$$\mathrm{d}Y_t = -\frac{1}{2}\widetilde{\sigma}^2(Y_t)\,\mathrm{d}t + \widetilde{\sigma}(Y_t)\,\mathrm{d}W_t,$$

where $\widetilde{\sigma} = \sigma \circ \exp$ *and* $Y = \ln S$. *We assume that*

(1) *There are constants* $m, M > 0$ *such that* $0 < m \leq \widetilde{\sigma}(x) \leq M$ *for all* $x \in \mathbb{R}$;

(2) $\widetilde{\sigma}^2$ *satisfies a uniform Hölder condition:*

$$\left|\widetilde{\sigma}^2(x) - \widetilde{\sigma}^2(y)\right| \leq M_1 \left|x - y\right|^{\alpha_1},$$

for some M_1, α_1 *and all* $x, y \in \mathbb{R}$.

We call the above setup Model V.
Then, assuming that $S_0, K > 0$ *with* $S_0 \neq K$

$$\lim_{\tau \to 0^+} \Sigma(K,\tau) = \left|\frac{\ln(S_0/K)}{d_Y(\ln S_0, \ln K)}\right|,$$

where d_Y *is defined by*

$$\lim_{\tau \to 0^+} -2\tau \ln(p^Y(\tau, x, y)) = d_Y^2(x,y),$$

and $d_Y(x,y) = \int_y^x \frac{\mathrm{d}z}{\widetilde{\sigma}(z)}$.

Proof. As in Proposition 11.3.1, we begin by establishing the applicability of Corollary 7.3.6. First of all note that S_0 is strictly positive. Now note that S is a a continuous, non-negative martingale on a filtered probability space with filtration satisfying the usual conditions. Therefore, for all $K > 0$

(1) $(S_0 - K)^+ \leq \mathbb{E}\left((S_\tau - K)^+\right) \leq S_0$;

(2) $\lim_{\tau \to 0^+} \mathbb{E}\left((S_\tau - K)^+\right) = (S_t - K)^+$;

(3) $\tau \mapsto \mathbb{E}\left((S_\tau - K)^+\right)$ is a non-decreasing function.

Finally, we note that by Proposition 10.3.1,

$$\begin{aligned}
\mathbb{E}\left((S_\tau - K)^+\right) &= \mathbb{E}\left((S_\tau - K)^+ | S_0 = s\right) \\
&= (S_0 - K)^+ + \frac{\sigma^2(K)K^2}{2}\int_0^\tau p^S(u, S_0, K)\,\mathrm{d}u \qquad (11.4)\\
&> (S_0 - K)^+ \text{ if } \tau > 0,
\end{aligned}$$

where p^S is the transition density for the process defined by

$$dS_t = \sigma(S_t)S_t\,\mathrm{d}W_t, \quad S_0 = s > 0.$$

The conditions on the function σ come via the restrictions we have placed on $\widetilde{\sigma} = \sigma \circ \exp$. Now, from Equation 11.4, and the positivity of p^S – see Remark 11.1.3 – it is clear that there exists no $\delta > 0$ such that

$$\mathbb{E}\left((S_\delta - K)^+\right) = (S_0 - K)^+. \qquad (11.5)$$

We may therefore use Corollary 7.3.6 for which we will need the small time behaviour of the density p^S. This is somewhat difficult to do directly so we instead find the limiting behaviour of p^S via the limiting behaviour of p^Y. Henceforth, let Y be defined as in the statement of this Proposition.
It is elementary that

$$p^Y(t, x, y) = \exp(y)p^S(t, \exp(x), \exp(y)).$$

Then

$$\begin{aligned}
d_Y^2(x, y) &= \lim_{\tau \to 0^+} -2\tau \ln p^Y(t, x, y) \\
&= \lim_{\tau \to 0^+} -2\tau \ln\left(\exp(y)p^S(t, \exp(x), \exp(y))\right) \\
&= \lim_{\tau \to 0^+} -2\tau \ln\left(p^S(t, \exp(x), \exp(y))\right).
\end{aligned}$$

Therefore,

$$\lim_{\tau \to 0^+} -2\tau \ln\left(p^S(t, x, y)\right) = d_Y^2(\ln x, \ln y).$$

We now proceed much as in Proposition 11.3.1. We have

$$\lim_{\tau \to 0^+} -2\tau \ln\left(\mathbb{E}\left((S_\tau - K)^+ - (S_0 - K)^+\right)\right)$$

which, by Proposition 10.3.1, is

$$
\begin{aligned}
&= \lim_{\tau\to 0^+} -2\tau \ln\left(\frac{\sigma^2(K)K^2}{2}\int_0^\tau p(t,S_0,K)\,\mathrm{d}t\right)\\
&= \lim_{\tau\to 0^+} -2\tau \ln\left(\int_0^\tau p^S(t,S_0,K)\,\mathrm{d}t\right)
\end{aligned}
$$

which, by Lemma 11.2.1, is

$$
\begin{aligned}
&= \lim_{\tau\to 0^+} -2\tau \ln\left(p^S(\tau,S_0,K)\right)\\
&= d_Y^2(\ln S_0, \ln K)
\end{aligned}
$$

We may then apply Corollary 7.3.6 to get that

$$\lim_{\tau\to 0^+} \Sigma_0(K,\tau) = \left|\frac{\ln(S_0/K)}{d_Y(\ln S_0,\ln K)}\right|,$$

where

$$d_Y(x,y) = \int_y^x \frac{\mathrm{d}z}{\widetilde{\sigma}(z)}.$$

□

11.4 Summary of Results and Conclusions

We obtained the small time to expiry limit of the implied volatility in the CEV model and also a general local volatility model. The general model can be handled easily by Varadhan's results on small time asymptotics of transition densities for diffusion processes (see (Var67a)). The small time to expiry limit in case of the CEV model has not been rigorously examined in the literature; we filled that gap. Our method of solving the more general model demonstrates the very close relationship between the large deviations theory and the small time to expiry asymptotics of implied volatility in diffusion models.

Appendix

A.5 Asymptotics of Some Special Functions

Lemma A.5.1. *Recall that the error function is defined as*

$$\begin{aligned} \mathrm{erf} : \mathbb{R} &\to \mathbb{R} \\ \theta &\mapsto \frac{2}{\sqrt{\pi}} \int_0^\theta \exp(-t^2)\,\mathrm{d}t. \end{aligned}$$

Then

$$\mathrm{erf}(\theta) \sim \frac{2}{\sqrt{\pi}}\theta, \quad \theta \to 0.$$

Proof. Clearly, $\exp(-t^2) \sim 1$ as $t \to 0$. Hence, for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |t| < \delta$, then

$$\left|\exp(-t^2) - 1\right| < \epsilon.$$

Therefore, for $\forall \epsilon > 0\ \exists \delta > 0$ such that if $0 < |\theta| < \delta$ then

$$(1-\epsilon)\frac{2}{\sqrt{\pi}}\theta < \frac{2}{\sqrt{\pi}} \int_0^\theta \exp(-t^2)\,\mathrm{d}t < (1+\epsilon)\frac{2}{\sqrt{\pi}}\theta.$$

□

Lemma A.5.2. *Fix $\alpha \geq 0$. Let $\Lambda_\alpha(z)$ be the unique positive solution of*

$$y^\alpha e^y = z, \tag{A.6}$$

for $z > 0$. Then

$$\Lambda_\alpha(z) \sim \ln(z), \quad z \to \infty.$$

Proof. This result may be obtained from (Com70), or (JCHK95). We give a direct proof. Since $\alpha \geq 0$, we have $y \mapsto y^\alpha e^y$ is strictly increasing on $[0, \infty)$. Taking the logarithm of Equation (A.6), we get $\alpha \ln(y) + y = \ln(z)$. Observe that $\Lambda_\alpha(z) \to \infty$ as $z \to \infty$. Observe also that $y \sim \alpha \ln(y) + y$ as $y \to \infty$. Equivalently, $y \sim \ln(z)$ as $z \to \infty$. But $y = \Lambda_\alpha(z)$. The claim follows. □

Lemma A.5.3. *For each fixed $a \in \mathbb{R}$,*

$$\Gamma(a, z) \sim z^{a-1}e^{-z}, \quad z \to \infty,$$

where

$$\begin{aligned} \Gamma : \mathbb{R} \times (0, \infty) &\to \mathbb{R} \\ (a, z) &\mapsto \int_z^\infty t^{a-1}e^{-t}\,\mathrm{d}t \end{aligned}$$

is termed the Upper Incomplete Gamma function.

Proof. We note that authors often restrict a to be positive. It is therefore prudent to first show that the Upper Incomplete Gamma Function is a well-defined function on the domain that we have specified.

Fix $a \in \mathbb{R}$ and $z' \in (0, \infty)$. Note first that there exists $M_1, M_2 \geq 0$ such that

$$0 < t^{a-1} \leq M_1 + M_2 e^{t/2} < \infty, \quad \forall t \in [z', \infty).$$

Therefore,

$$0 < t^{a-1} e^{-t} \leq (M_1 + M_2 e^{t/2}) e^{-t} = M_1 e^{-t} + M_2 e^{-t/2}, \quad \forall t \in [z', \infty).$$

Hence

$$\begin{aligned} 0 < \Gamma(a, z') &= \int_{z'}^{\infty} t^{a-1} \exp(-t) \, \mathrm{d}t \\ &\leq M_1 \int_{z'}^{\infty} \exp(-t) \, \mathrm{d}t + M_2 \int_{z'}^{\infty} e^{-t/2} \, \mathrm{d}t \\ &= M_1 e^{-z'} + 2 M_2 e^{-z'/2} \\ &< \infty. \end{aligned}$$

Therefore, Γ is a well-defined, real valued function at each point in $\mathbb{R} \times (0, \infty)$.

We now establish the asymptotics of the Upper Incomplete Gamma Function. We do so using L'Hôpital's rule. Fix $a \in \mathbb{R}$. Observe that

$$\begin{aligned} \frac{\partial_z (z^{a-1} e^{-z})}{\partial_z \Gamma(a, z)} &= \frac{z^{a-2} e^{-z} ((a-1) - z)}{-e^{-z} z^{a-1}} \\ &= 1 + \frac{1-a}{z} \\ &\to 1 \end{aligned}$$

as $z \to \infty$. $\square$

A.6 Appendix to Chapter 4

Lemma 4.2.1

Let $f : (0, \infty) \to \mathbb{R}$ be a function that admits a continuous extension, call it g, to $[0, \infty)$. We consider the following conditions on f.

(i) f is convex;

(ii) f is non-increasing;

(iii) $f(x) \to 0$ as $x \to \infty$; and

(iv) for some $a \in [0, \infty)$, $(a - x)^+ \leq f(x) \leq a$ for every $x > 0$.

Then

(i′) if f satisfies (i), then g is convex (on $[0, \infty)$);

(ii′) if f satisfies (ii), then g is non-increasing (on $[0, \infty)$);

(iii′) if f satisfies (iii), then $g(x) \to 0$ as $x \to \infty$;

(iv′) if f satisfies (iv), then $(g(0) - x)^+ = (a - x)^+ \leq g(x) \leq a = g(0)$ for every $x \geq 0$;

(v′) if f satisfies (iv), then $a = g(0) = \lim_{x \to 0^+} g(x)$;

(vi′) if f satisfies (i), (ii), and (iv), then the right-hand derivative of g, write it g'_+, exists, and is non-decreasing and right-continuous on $x \geq 0$;

(vii′) if f satisfies (i), (ii), and (iv), then for every $x \geq 0$, $-1 \leq g'_+(x) \leq 0$; and

(viii′) if f satisfies (i), (ii), and (iv), then for every $x > 0$, $-1 \leq \dfrac{g(x) - g(0)}{x} \leq g'_+(x) \leq 0$.

Proof. (i′) Clearly, g is convex on $(0, \infty)$, because f is convex there. It is therefore enough to show that

$$g(\alpha x + (1 - \alpha)y) \leq \alpha g(x) + (1 - \alpha)g(y)$$

for all $\alpha \in [0,1], x = 0$ and $y > 0$. But this is obvious

$$\begin{aligned} g(\alpha \cdot 0 + (1-\alpha)y) &= g(\lim_{x \to 0^+} (\alpha x + (1-\alpha)y)) \\ &= \lim_{x \to 0^+} g(\alpha x + (1-\alpha)y) \\ &\leq \lim_{x \to 0^+} \alpha g(x) + (1-\alpha)g(y) \\ &= \alpha g(0) + (1-\alpha)g(y). \end{aligned}$$

We used the right-continuity of g at zero in the last line.

(ii′) Since f is non-increasing, it is enough to show that $g(0) \geq g(x)$ for every $x > 0$. But $g(x_1) \geq g(x_2)$ for $0 < x_1 \leq x_2$, so that

$$\lim_{x_1 \to 0^+} g(x_1) \geq g(x_2).$$

So, since $\lim_{x_1 \to 0^+} g(x_1) = g(0)$, by the right-continuity of g at zero, we are done.

(iii′) Trivial.

(iv′) We have $a = a^+ = \lim_{x \to 0^+} (a-x)^+ \leq \lim_{x \to 0^+} g(x) = g(0) \leq a$, using the right-continuity of g at zero and the non-negativity of a.

(v′) Follows from (iv).

It is convenient to prove properties (vi′), (vii′), and (viii′) together.

The fact that zero is not an interior point of the domain of g leads to some complications. This is because some of the useful properties of convex functions that we would like to use generally only hold at interior points of their domain. However, we can get around this by introducing a function h that coincides with g on $[0,\infty)$ and is convex on $\mathbb{R}$ such that, in particular, zero is an interior point of the domain of h. We now introduce such a function.

$$h(x) = \begin{cases} g(x), & \text{if } x \geq 0 \\ g(0) - x, & \text{if } x < 0. \end{cases}$$

It is clear that h has support line at every $x \in \mathbb{R}$. Therefore, the right-hand derivative of g, write it g'_+, exists, and is non-decreasing and right-continuous on $x \geq 0$.

To get that for every $x \geq 0$, $-1 \leq g'_+(x) \leq 0$, we argue as follows. First note that for $0 < b < x$.

$$g'_+(x) \geq \frac{g(x) - g(b)}{x - b}.$$

Let $x > 0$. By (ii'), g is non-increasing on $[0, \infty)$, so for $0 < b < x$

$$g'_+(x) \geq \frac{g(x) - g(0)}{x - b}$$

so

$$\lim_{b \to 0^+} g'_+(x) \geq \frac{g(x) - g(0)}{\lim_{b \to 0^+}(x - b)},$$

so by (v')

$$g'_+(x) \geq \frac{g(x) - a}{x},$$

and by (iv')

$$\begin{aligned} g'_+(x) &\geq \frac{(a - x)^+ - a}{x} \\ &= \begin{cases} -a/x, & \text{if } a \leq x \\ -x/x, & \text{if } a > x \end{cases} \\ &\geq -1. \end{aligned}$$

Therefore,

$$g'_+(0) = \lim_{x \to 0^+} g'_+(x) \geq -1$$

The upper bound on g'_+ follows from the fact that g is non-increasing on $[0, \infty)$ by (ii'):

$$g'_+(x_0) = \lim_{h \to 0^+} \frac{g(x_0 + h) - g(x_0)}{h} \leq \lim_{h \to 0^+} \frac{0}{h} = 0,$$

for $x_0 \geq 0$. $\square$

A.7 Appendix to Chapter 7

Lemma 7.2.1

The Black-Scholes call pricing function, C^{BS} (cf. Definition 3.3.1 on page 15), admits the representation

$$C^{BS}(K,\tau,\sigma) = (S-K)^+ + S\int_0^\theta \phi\left(-\frac{x}{v}+\frac{v}{2}\right)\mathrm{d}v, \tag{A.7}$$

where $x = \ln(K/S)$, $\theta = \sigma\sqrt{\tau}$ and ϕ is the standard normal density.

Proof. To ease the computations, we introduce the map

$$f : (x,\theta) \mapsto \Phi\left(-\frac{x}{\theta}+\frac{\theta}{2}\right) - \exp(x)\Phi\left(-\frac{x}{\theta}-\frac{\theta}{2}\right)$$

defined for $\theta > 0$ and $x \in \mathbb{R}$. It is simply C^{BS}/S expressed in terms of the reduced variables x and θ defined in the statement of the lemma. For brevity, let

$$D_1 = -x/\theta + \theta/2 \quad \text{and} \quad D_2 = -x/\theta - \theta/2.$$

For $\theta \in (0,\infty)$ and $x \in \mathbb{R}$, we have

$$\frac{\partial f(x,\theta)}{\partial \theta} = \phi(D_1)\left(\frac{x}{\theta^2}+\frac{1}{2}\right) - \exp(x)\phi(D_2)\left(\frac{x}{\theta^2}-\frac{1}{2}\right).$$

Since $\phi(z) = \exp(-z^2/2)/\sqrt{2\pi}$ and

$$-\frac{1}{2}\left(-\frac{x}{\theta}+\frac{\theta}{2}\right)^2 = x - \frac{1}{2}\left(-\frac{x}{\theta}-\frac{\theta}{2}\right)^2,$$

we have that $\phi(D_1) = \exp(x)\phi(D_2)$. It follows that

$$\frac{\partial f(x,\theta)}{\partial \theta} = \phi(D_1)\left(\left(\frac{x}{\theta^2}+\frac{1}{2}\right) - \left(\frac{x}{\theta^2}-\frac{1}{2}\right)\right) = \phi(D_1) = \phi\left(-\frac{x}{\theta}+\frac{\theta}{2}\right),$$

on $\mathbb{R}\times(0,\infty)$ and this function admits a continuous extension to $\mathbb{R}\times[0,\infty)$. It is easy to see that $f(x,0+) = (1-\exp(x))^+$. It is therefore clear that

$$f(x,\theta) = (1-\exp(x))^+ + \int_0^\theta \phi\left(-\frac{x}{v}+\frac{v}{2}\right)\mathrm{d}v. \tag{A.8}$$

Multiply both sides of Equation (A.8) by S and recall that $S > 0$. The result for $0 < \sigma < \infty$ then follows after recalling the relation between f and C^{BS}.
When $\sigma = 0$, the right-hand side of Equation (A.7) is easily seen to be $(S-K)^+$ as required. It is possible to show directly that

$$\int_0^\infty \phi\left(-\frac{x}{v}+\frac{v}{2}\right)\mathrm{d}v = \begin{cases} 1, & \text{if } x > 0, \\ K/S, & \text{if } x \le 0, \end{cases} \tag{A.9}$$

so that the limit of the right-hand side of Equation (A.7) is S, which, of course, agrees with $C^{BS}(K,\tau,\infty) = S$. $\square$

Lemma 7.2.4

Recall that $F(x,\theta) = \int_0^\theta \phi(\frac{-x}{v} + \frac{v}{2})\,\mathrm{d}z$, *with* ϕ *the Gaussian density. For every* $x \in \mathbb{R}$, $F(x,\cdot)$ *is continuous and strictly increasing. For every fixed non-negative* x *(non-positive* x*, respectively)* $F(x,\cdot)$ $(\exp(-x)F(x,\cdot)$, *respectively) is a strictly increasing, continuous cumulative distribution function.*

Proof. Fix $x \in \mathbb{R}$. Since the integrand is strictly positive, $F(x,\cdot)$ and $\exp(x)F(x,\cdot)$ are strictly increasing. It is clearly the case that $F(x,\cdot)$ and $\exp(-x)F(x,\cdot)$ are both continuous and that $F(x,0) = \exp(-x)F(x,0) = 0$. Recalling Equation (A.9) and the convention $x = \ln(K/S)$, we have that $F(x,\infty) = 1$ when $x \geq 0$ and $F(x,\infty) = \exp(x)$ when $x \leq 0$. The result follows easily. $\square$

Lemma 7.2.5

For $x = 0$,

$$F(x,\theta) = \operatorname{erf}\left(\frac{\theta}{2\sqrt{2}}\right) \sim \frac{\theta}{\sqrt{2\pi}}, \quad \theta \to 0^+.$$

Proof. When $x = 0$,

$$\begin{aligned}
F(x,\theta) &= \int_0^\theta \phi(v/2)\,\mathrm{d}v \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\theta \exp\left(-v^2/8\right)\,\mathrm{d}v \\
&= \frac{2}{\sqrt{\pi}} \int_0^{\theta/(2\sqrt{2})} \exp(-u^2)\,\mathrm{d}u \\
&= \operatorname{erf}\left(\frac{\theta}{2\sqrt{2}}\right).
\end{aligned}$$

Using Lemma A.5.1, we conclude that

$$\operatorname{erf}\left(\frac{\theta}{2\sqrt{2}}\right) \sim \frac{\theta}{\sqrt{2\pi}}, \quad \theta \to 0^+.$$

$\square$

Lemma 7.2.6

For $x \neq 0$,

$$F(x,\theta) \sim \frac{|x|\exp(x/2)}{4\sqrt{\pi}}\Gamma\left(-\frac{1}{2},\frac{x^2}{2\theta^2}\right) \tag{A.10}$$

$$\sim \frac{\theta^3}{\sqrt{2\pi}x^2}\exp\left(-\frac{x^2-\theta^2 x}{2\theta^2}\right), \tag{A.11}$$

both as $\theta \to 0^+$.

Proof. Fix $x \neq 0$. We have that

$$F(x,\theta) = \frac{\exp(x/2)}{\sqrt{2\pi}}\int_0^\theta \exp\left(-\frac{1}{2}\left(\frac{x^2}{v^2}+\frac{v^2}{4}\right)\right)\,\mathrm{d}v \tag{A.12}$$

and it is enough to consider the asymptotics of the simpler integral appearing in Equation (A.12). It is elementary that

$$\lim_{\theta\to 0^+}\frac{\int_0^\theta \exp\left(-\frac{1}{2}\left(\frac{x^2}{v^2}+\frac{v^2}{4}\right)\right)\,\mathrm{d}v}{\int_0^\theta \exp\left(-\frac{x^2}{2v^2}\right)\,\mathrm{d}v} = \lim_{\theta\to 0^+}\frac{\exp\left(-\frac{1}{2}\left(\frac{x^2}{\theta^2}+\frac{\theta^2}{4}\right)\right)}{\exp\left(-\frac{x^2}{2\theta^2}\right)} = 1.$$

Now observe that the simple change of variables $u = x^2/2v^2$ yields

$$\int_0^\theta \exp\left(-\frac{x^2}{2v^2}\right)\,\mathrm{d}v = \frac{|x|}{2\sqrt{2}}\int_{\frac{x^2}{2\theta^2}}^\infty u^{-3/2}\exp(-u)\,\mathrm{d}u = \frac{|x|}{2\sqrt{2}}\Gamma\left(-\frac{1}{2},\frac{x^2}{2\theta^2}\right),$$

by definition of the Upper Incomplete Gamma function. Recalling Equation (A.12), the validity of Equation (A.10) follows. As for Equation (A.11), it is shown in Lemma 7.1.3 that

$$\Gamma(a,z) \sim z^{a-1}\exp(-z), \quad z\to\infty,$$

for a in $\mathbb{R}$, so that

$$\frac{|x|}{2\sqrt{2}}\Gamma\left(-\frac{1}{2},\frac{x^2}{2\theta^2}\right) \sim \frac{|x|}{2\sqrt{2}}\left(\frac{x^2}{2\theta^2}\right)^{-3/2}\exp\left(-\frac{x^2}{2\theta^2}\right) = \frac{\theta^3}{x^2}\exp\left(-\frac{x^2}{2\theta^2}\right),$$

as $\theta \to 0^+$. Now use Equation (A.12) to get Equation (A.11). ☐

Proposition A.7.1. *Let*

$$f : (0,1) \to \mathbb{R}$$
$$x \mapsto 2+\sin(1/x).$$

Then $\lim_{x\to 0^+} f(x)$ *does not exist, but*

$$\lim_{h\to 0^+}\frac{\int_0^h f(x)\,\mathrm{d}x}{h} = 2. \tag{A.13}$$

Proof. It is elementary that f does not converge as $x \to 0^+$. We may not use L'Hôpital's Rule to evaluate the limit in Equation (A.13).

It will be enough to deal with $f(x) - 2 = \sin(1/x)$. Integration by parts gives

$$\int_0^h \sin(1/x)\,\mathrm{d}x = h\sin(1/h) + \int_0^h \frac{\cos(1/x)}{x}\,\mathrm{d}x,$$

which, after making the substitution $1/x = y$, is

$$\begin{aligned} &= h\sin(1/h) + \int_{1/h}^{\infty} \frac{\cos(y)}{y}\,\mathrm{d}y, \\ &= h\sin(1/h) - \mathrm{Ci}(1/h), \end{aligned}$$

where Ci is the cosine integral, see (AS84), for example. Now, as $z \to \infty$ we have

$$\mathrm{Ci}(z) = \frac{\sin(z)}{z}\left(1 + \mathrm{O}\left(\frac{1}{z^2}\right)\right) - \frac{\cos(z)}{z^2}\left(1 + \mathrm{O}\left(\frac{1}{z^2}\right)\right),$$

see (Wol). So, as $h \to 0^+$,

$$\begin{aligned} \frac{h\sin(1/h) - \mathrm{Ci}(1/h)}{h} &= \frac{-\sin(1/h)\mathrm{O}\left(h^3\right) + \cos(1/h)h^2 + \cos(1/h)\mathrm{O}\left(h^4\right)}{h} \\ &\to 0, \end{aligned}$$

and the claim follows. $\square$

A.8 Appendix to Chapter 8

Lemma 8.1.10

Let $\widetilde{X}$ be a Lévy process with no Gaussian part, then

$$\tau^{-1/2}\widetilde{X}_\tau \xrightarrow{\mathbb{P}} 0 \text{ as } \tau \to 0^+. \tag{A.14}$$

Proof. The result is due to Jacod ((Jac07), Lemma 4.1, p. 181). However, this proof is original.

We prove the convergence in distribution to zero, i.e.

$$\widetilde{X}_\tau/\sqrt{\tau} \xrightarrow{d} 0, \quad \text{as } \tau \to 0^+,$$

which of course implies Equation (A.14).

We write $\widetilde{\Psi}$ for the characteristic exponent of $\widetilde{X}$. The characteristic function of $\widetilde{X}_\tau/\sqrt{\tau}$, is given by

$$\phi_{\widetilde{X}_\tau/\sqrt{\tau}}(\lambda) = \phi_{\widetilde{X}_\tau}(\lambda/\sqrt{\tau}) = \exp\left(-\tau\widetilde{\Psi}\left(\frac{\lambda}{\sqrt{\tau}}\right)\right).$$

To prove the claim we must show that

$$\phi_{\widetilde{X}_\tau/\sqrt{\tau}}(\lambda) \to 1 \text{ as } \tau \to 0^+ \text{ for each } \lambda \in \mathbb{R}.$$

But Bertoin ((Ber96), Proposition 2 (i), p. 16), gives that

$$\lim_{|\lambda|\to\infty} \lambda^{-2}\widetilde{\Psi}(\lambda) = 0, \tag{A.15}$$

since $\widetilde{X}$ has no Gaussian part.

We prove the claim by showing that

$$\lim_{\tau\to 0^+} \tau\widetilde{\Psi}\left(\frac{\lambda}{\sqrt{\tau}}\right) = 0,$$

for each $\lambda \in \mathbb{R}$. Fix $\lambda \in \mathbb{R}$. If $\lambda = 0$, then

$$\phi_{\widetilde{X}_\tau/\sqrt{\tau}}(\lambda) = \phi_{\widetilde{X}_\tau}\left(\lambda/\sqrt{\tau}\right) = \exp\left(-\tau\widetilde{\Psi}\left(0\right)\right) = \exp(-\tau \cdot 0) = 1, \quad \forall \tau > 0.$$

For non-zero λ we perform a change of variables: $\widetilde{\lambda} = \widetilde{\lambda}(\tau) = \lambda/\sqrt{\tau}$. We only consider strictly positive τ. From Equation (A.15), we have

$$\begin{aligned}
0 &= \lim_{|\widetilde{\lambda}|\to\infty} \widetilde{\lambda}^{-2}\widetilde{\Psi}\left(\widetilde{\lambda}\right) \\
&= \lim_{\tau\to 0^+} \left(\frac{\lambda}{\sqrt{\tau}}\right)^{-2} \widetilde{\Psi}\left(\frac{\lambda}{\sqrt{\tau}}\right) \\
&= \frac{1}{\lambda^2} \lim_{\tau\to 0^+} \tau\widetilde{\Psi}\left(\frac{\lambda}{\sqrt{\tau}}\right).
\end{aligned}$$

□

Lemma 8.2.4 *Let*

$$S_\tau = S_0 e^{X_\tau}, \quad \forall \tau \geq 0,$$

where X is a Lévy process satisfying the constraints in Equation (8.3) *and $S_0 > 0$ is the initial value of the process S. Then, for each fixed $K > 0$,*

1. $(S_0 - K)^+ \leq \mathbb{E}\left((S_0 e^{X_\tau} - K)^+\right) \leq S_0$, *for each K;*

2. $\tau \mapsto \mathbb{E}\left((S_0 e^{X_\tau} - K)^+\right)$ *is right-continuous on $[0, \infty)$ for each $K > 0$; and*

3. $\tau \mapsto \mathbb{E}\left((S_0 e^{X_\tau} - K)^+\right)$ *is non-decreasing for each $K > 0$.*

Proof. The process S is càdlàg, since X is. S is a non-negative martingale because X satisfies the constraints set out in Equation (8.3) (also see the statement of this Lemma). Finally, the filtration is assumed to satisfy the usual conditions. Applying Proposition 4.3.8, we are done. □

Lemma 8.2.5 *Introduce*

$$S_\tau = S_0 e^{X_\tau}, \quad \forall \tau \geq 0,$$

where $S_0 > 0$ is a constant and X is a Lévy process satisfying (8.3)*, i.e.*

$$\int_{|y|\geq 1} e^y \,\nu(\mathrm{d}y) < \infty \text{ and } b = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^y - 1 - y\mathbb{1}_{|y|\leq 1})\,\nu(\mathrm{d}y).$$

Then for every $\tau > 0$ and $K > 0$,

(Ai) $\mathbb{E}\left((S_\tau - K)^+\right) < S_0$.

(Aii) $\mathbb{E}\left((S_\tau - K)^+\right) \geq (S_0 - K)^+$.

(Aiii) If $S_0 > K$, then $\mathbb{E}\left((S_\tau - K)^+\right) = (S_0 - K)^+$ if and only if $\mathbb{P}\left(S_\tau < K\right) = \mathbb{P}\left(X_\tau < \ln(K/S_0)\right) = 0$.

Proof. Fix $\tau, K > 0$.

(Ai) Clearly, $\mathbb{E}\left((S_\tau - K)^+\right) \leq S_0$. Suppose that $\mathbb{E}\left((S_\tau - K)^+\right) = S_0$. By Proposition 4.3.9 (Aiii), it then holds that $\mathbb{P}\left(S_\tau = 0\right) = 1$, equivalently $\mathbb{P}\left(X_\tau = -\infty\right) = 1$, but this is impossible.

(Aii) Well known. Use Jensen's Inequality and then the martingale assumption on S.

(Aiii) See Proposition 4.3.9 (Aii).

□

Lemma 8.2.6 *Introduce*

$$S_\tau = S_0 e^{X_\tau}, \quad \forall \tau \geq 0,$$

where $S_0 > 0$ is a constant and X is a Lévy process satisfying (8.3), i.e.

$$\int_{|y| \geq 1} e^y \, \nu(\mathrm{d}y) < \infty \text{ and } b = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^y - 1 - y \mathbb{1}_{|y| \leq 1}) \, \nu(\mathrm{d}y). \tag{A.16}$$

Then consider the functions

(1) $P(\cdot)$, defined in Notation 8.2.2 as $P(x) = (K - S_0 e^x)^+$, with the additional restriction that $0 < K < S_0$; and

(2) $C(\cdot)$, defined in Notation 8.2.1 as $C(x) = (S_0 e^x - K)^+$, with the additional restriction that $K > S_0 > 0$.

Then conditions (1)-(4) of Theorem 8.1.7 are satisfied by P and C under the respective stated conditions on S_0 and K.

Proof. Clearly P and C are locally bounded and ν continuous; hence conditions (2)-(3) of Theorem 8.1.7 are satisfied.
With $K > S_0$, C vanishes in a neighbourhood of the origin so that certainly $C(x) = \mathrm{o}\left(x^2\right)$ as $x \to 0$ and condition (1) of Theorem 8.1.7 is satisfied.
For $K < S_0$, P vanishes in a neighbourhood of the origin so that $P(x) = \mathrm{o}\left(x^2\right)$ as $x \to 0$ and condition (1) of Theorem 8.1.7 is again satisfied.

It remains to check condition (4) for C and P. Without further comment, we note that $z \mapsto 1$ is both subadditive and submultiplicative. Now, for C, consider $x \mapsto 1 \cdot e^x$. By Equation (A.16), the function $x \mapsto 1 \cdot e^x$ is in $\mathcal{S}(\nu)$: it satisfies

$$\limsup_{|x|\to\infty} \frac{(S_0 e^x - K)^+}{1 \cdot e^x} < \infty.$$

For P, consider $x \mapsto K$. Clearly $x \mapsto K \cdot 1$ is in $\mathcal{S}(\nu)$: it satisfies

$$\limsup_{|x|\to\infty} \frac{(K - S_0 e^x)^+}{1 \cdot K} < \infty$$

and

$$\int_{|x|>1} K \, \nu(\mathrm{d}x) < \infty,$$

since ν is a Lévy measure. □

Lemma 8.2.7 *Suppose that U is a non-negative process with representation*

$$U_\tau = U_0 e^{b\tau + \sigma W_\tau + Y_\tau}, \quad \tau \geq 0,$$

where $b \in \mathbb{R}, \sigma \geq 0$, and $U_0 > 0$ are finite constants, W is a standard Brownian motion, and Y is a compound Poisson process with constant, finite intensity $\lambda > 0$. We denote the sequence of summand random variables comprising Y as $\left(\widehat{Y}_i\right)_{i\geq 1}$; they are i.i.d. random variables with an exponential mean. We assume that the compound Poisson part has a finite exponential moment, i.e.

$$\mathbb{E}\left(e^{Y_\tau}\right) < \infty, \quad \forall \tau \geq 0.$$

Then

$$\lim_{\tau\to 0^+} \tau^{-1/2} \mathbb{E}\left(\left(U_0 - U_0 e^{b\tau + \sigma W_\tau + Y_\tau}\right)^+\right) = \frac{\sigma U_0}{\sqrt{2\pi}}.$$

Proof.

It is clearly enough to prove the claim for $U_0 = 1$.

Case 1: No compound Poisson part

First suppose that U has representation

$$U_\tau = b\tau + \sigma W_\tau, \quad \tau \geq 0$$

where $b, \sigma \in \mathbb{R}$, $\sigma \geq 0$, and W is a standard Wiener process.

We claim that

$$\lim_{t \to 0^+} \tau^{-1/2} \mathbb{E}\left(\left(1 - e^{b\tau + \sigma W_\tau}\right)^+\right) = \frac{\sigma}{\sqrt{2\pi}}.$$

If $\sigma = 0$, then it is trivially the case that $\lim_{t \to 0^+} \tau^{-1/2} \mathbb{E}\left((1 - e^{b\tau})^+\right) = 0$. Observe that b can be any real number and this same limit holds.

Suppose now that $\sigma > 0$ and continue to let $b \in \mathbb{R}$.

We will use that for $\theta \in \mathbb{R}$,

$$\operatorname{erf}(\theta\sqrt{\tau}) \sim \frac{2\theta\sqrt{\tau}}{\sqrt{\pi}}, \quad \tau \to 0^+,$$

see Lemma A.5.1, also $\operatorname{erf}(-x) = -\operatorname{erf}(x)$ for $x \in \mathbb{R}$, and

$$\Phi(x) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right).$$

See (AS84) or (Olv97) for the definition and facts about the error function.

It is well known that

$$dU_\tau = rU_\tau \, d\tau + \sigma U_\tau \, dW_\tau, \quad U_0 = 1$$

has solution

$$U_\tau = \exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma W_\tau\right), \quad \tau \geq 0.$$

Indeed, this is just the Black-Scholes model with r the risk-neutral interest rate. Using the well-known formula for the price of an at-the-money put option under this model (see, for example, (FPS00)) we obtain that

$$\begin{aligned} & e^{-r\tau}\mathbb{E}\left(\left(1 - \exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma W_\tau\right)\right)^+\right) \\ = \quad & e^{-r\tau}\Phi\left(-\frac{(r - \sigma^2/2)\sqrt{\tau}}{\sigma}\right) - \Phi\left(-\frac{(r + \sigma^2/2)\sqrt{\tau}}{\sigma}\right), \end{aligned}$$

from which

$$
\begin{aligned}
&\mathbb{E}\left(\left(1-\exp\left(\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma W_\tau\right)\right)^+\right)\\
&=\Phi\left(-\frac{(r-\sigma^2/2)\sqrt{\tau}}{\sigma}\right)-e^{r\tau}\Phi\left(-\frac{(r+\sigma^2/2)\sqrt{\tau}}{\sigma}\right)\\
&=\frac{1}{2}-\frac{1}{2}\operatorname{erf}\left(\frac{(r-\sigma^2/2)\sqrt{\tau}}{\sigma\sqrt{2}}\right)-\frac{e^{r\tau}}{2}+\frac{e^{r\tau}}{2}\operatorname{erf}\left(\frac{(r+\sigma^2/2)\sqrt{\tau}}{\sigma\sqrt{2}}\right)\\
&=\frac{1}{2}(1-e^{r\tau})-\frac{1}{2}\left(\operatorname{erf}\left(\frac{(r-\sigma^2/2)\sqrt{\tau}}{\sigma\sqrt{2}}\right)-e^{r\tau}\operatorname{erf}\left(\frac{(r+\sigma^2/2)\sqrt{\tau}}{\sigma\sqrt{2}}\right)\right)\\
&=\mathrm{O}(\tau)-\frac{1}{2}\left(\left(\frac{(r-\sigma^2/2)\sqrt{2\tau}}{\sqrt{\pi}\sigma}+\mathrm{O}(\tau)\right)-(1+\mathrm{O}(\tau))\left(\frac{(r+\sigma^2/2)\sqrt{2\tau}}{\sqrt{\pi}\sigma}+\mathrm{O}(\tau)\right)\right)\\
&=\mathrm{O}(\tau)-\frac{1}{2}\frac{(r-\sigma^2/2)\sqrt{2}}{\sigma\sqrt{\pi}}\sqrt{\tau}-\frac{1}{2}\mathrm{O}(\tau)+\frac{1}{2}\frac{(r+\sigma^2/2)\sqrt{2}}{\sigma\sqrt{\pi}}\sqrt{\tau}\\
&\qquad+\frac{1}{2}\mathrm{O}(\tau)+\frac{1}{2}\mathrm{O}\left(\tau^{3/2}\right)+\frac{1}{2}\mathrm{O}\left(\tau^2\right)\\
&=-\frac{1}{2}\frac{(r-\sigma^2/2)\sqrt{2}}{\sigma\sqrt{\pi}}\sqrt{\tau}+\frac{1}{2}\frac{(r+\sigma^2/2)\sqrt{2}}{\sigma\sqrt{\pi}}\sqrt{\tau}+\mathrm{O}(\tau)\\
&=\frac{\sqrt{2}}{2\sigma\sqrt{\pi}}\left(-(r-\sigma^2/2)+(r+\sigma^2/2)\right)\sqrt{\tau}+\mathrm{O}(\tau)\\
&=\frac{\sigma}{\sqrt{2\pi}}\sqrt{\tau}+\mathrm{O}(\tau),
\end{aligned}
$$

all as $\tau \to 0^+$.

Observe that we could have chosen r in such a way that $b = r - \sigma^2/2$ and there would be no difference in the final result. We therefore have that

$$\lim_{\tau\to 0^+}\tau^{-1/2}\mathbb{E}\left((1-\exp(b\tau+\sigma W_\tau))^+\right)=\frac{\sigma}{\sqrt{2\pi}},$$

for all $b \in \mathbb{R}$ and $\sigma \geq 0$.

Case 2: Compound Poisson part included

Suppose now that U has representation

$$U_\tau = b\tau + \sigma W_\tau + Y_\tau, \quad \tau \geq 0$$

where W is a standard Wiener process and Y is a compound Poisson process which is such that each of the random variables comprising Y have an exponential moment.

Now,

$$\begin{aligned}
\mathbb{E}\left((1-U_\tau)^+\right) &= \sum_{n=0}^{\infty} \mathbb{E}\left((1-U_\tau)^+ \middle| N_\tau = n\right) \mathbb{P}\left(N_\tau = n\right) \\
&= \mathbb{E}\left((1-U_\tau)^+ \middle| N_\tau = 0\right) \mathbb{P}\left(N_\tau = 0\right) \\
&\quad + \sum_{n=1}^{\infty} \mathbb{E}\left((1-U_\tau)^+ \middle| N_\tau = n\right) \mathbb{P}\left(N_\tau = n\right) \\
&=: A_\tau^1 + A_\tau^2.
\end{aligned}$$

For the first term we can just apply the first part (Case 1) of this proof to get

$$\begin{aligned}
\lim_{\tau\to 0^+} \tau^{-1/2}\mathbb{E}\left((1-e^{b\tau+\sigma W_\tau})^+\right) \mathbb{P}\left(N_\tau = 0\right) &= \lim_{\tau\to 0^+} e^{-\lambda\tau}\tau^{-1/2}\mathbb{E}\left((1-e^{b\tau+\sigma W_\tau})^+\right) \\
&= \frac{\sigma}{\sqrt{2\pi}},
\end{aligned}$$

so

$$\lim_{\tau\to 0^+} \tau^{-1/2} A_\tau^1 = \frac{\sigma}{\sqrt{2\pi}}.$$

For the second,

$$\begin{aligned}
A_\tau^2 &= \sum_{n=1}^{\infty} \mathbb{E}\left((1-U_\tau)^+ \middle| N_\tau = n\right) \mathbb{P}\left(N_\tau = n\right) \\
&\leq \sum_{n=1}^{\infty} \mathbb{E}\left(|1-U_\tau| \middle| N_\tau = n\right) \mathbb{P}\left(N_\tau = n\right) \\
&\leq \sum_{n=1}^{\infty} \mathbb{E}\left(1+|U_\tau| \middle| N_\tau = n\right) \mathbb{P}\left(N_\tau = n\right) \\
&= \sum_{n=1}^{\infty} \mathbb{P}\left(N_\tau = n\right) + \sum_{n=1}^{\infty} \mathbb{E}\left(U_\tau \middle| N_\tau = n\right) \mathbb{P}\left(N_\tau = n\right) \\
&= 1 - e^{-\lambda\tau} + \sum_{n=1}^{\infty} \mathbb{E}\left(U_\tau \middle| N_\tau = n\right) \mathbb{P}\left(N_\tau = n\right) \\
&= \mathrm{O}\left(\tau\right) + \sum_{n=1}^{\infty} \mathbb{E}\left(U_\tau \middle| N_\tau = n\right) \mathbb{P}\left(N_\tau = n\right) \\
&= \mathrm{O}\left(\tau\right) + e^{b\tau+\sigma^2\tau/2} \sum_{n=1}^{\infty} \mathbb{E}\left(e^{\sum_{i=1}^{n} \widehat{Y}_i} \middle| N_\tau = n\right) \mathbb{P}\left(N_\tau = n\right) \\
&= \mathrm{O}\left(\tau\right) + e^{b\tau+\sigma^2\tau/2} \sum_{n=1}^{\infty} \left(\mathbb{E}\left(\widehat{Y}_i\right)\right)^n \mathbb{P}\left(N_\tau = n\right).
\end{aligned}$$

as $\tau \to 0^+$. Therefore, by introducing $m := \mathbb{E}\left(\exp(\widehat{Y}_1)\right)$, we find that

$$\begin{aligned}
A_\tau^2 &\leq \mathrm{O}(\tau) + e^{b\tau+\sigma^2\tau/2-\lambda\tau}\left(e^{m\lambda\tau}-1\right) \\
&= \mathrm{O}(\tau) + (1+\mathrm{O}(\tau))\left(m\lambda\tau + \mathrm{O}\left(\tau^2\right)\right) \\
&= \mathrm{O}(\tau) + m\lambda\tau + \mathrm{O}\left(\tau^2\right) + \mathrm{O}\left(\tau^3\right) \\
&= \mathrm{O}(\tau),
\end{aligned}$$

all as $\tau \to 0^+$. Therefore, since $A_\tau^2 = \mathrm{O}(\tau)$, we have

$$\lim_{t\to 0^+} \tau^{-1/2} A_\tau^2 = 0.$$

That is

$$\begin{aligned}
\tau^{-1/2}\mathbb{E}\left((1-U_\tau)^+\right) &= \mathbb{E}\left((1-U_\tau)^+ \middle| N_\tau = 0\right)\mathbb{P}\left(N_\tau = 0\right) \\
&\quad + \sum_{n=1}^{\infty}\mathbb{E}\left((1-U_\tau)^+ \middle| N_\tau = n\right)\mathbb{P}\left(N_\tau = n\right) \\
&= \tau^{-1/2}(A_\tau^1 + A_\tau^2) \\
&\to \frac{\sigma}{\sqrt{2\pi}} \text{ as } \tau \to 0^+.
\end{aligned}$$

□

Lemma 8.2.8 *Let*

$$\begin{aligned}
\widetilde{P} : \mathbb{R} &\to \mathbb{R} \\
x &\mapsto (1-\exp(x))^+.
\end{aligned}$$

Then $\widetilde{P}$ is globally Lipschitz continuous with Lipschitz constant 1.

Proof. If $x = y$, the claim is trivial. We therefore suppose throughout that $x \neq y$. Suppose first that $x, y \geq 0$. Then

$$\left|\widetilde{P}(x) - \widetilde{P}(y)\right| = |0-0| \leq |x-y|.$$

Suppose now that $x, y \leq 0$. Without loss of generality suppose that $x < y \leq 0$. We may consider $\widetilde{P}$ restricted to $(-\infty, 0]$. By the mean value theorem, there exists $x_0 \in (x, y)$ such that

$$\left|\widetilde{P}(x) - \widetilde{P}(y)\right| = \left|\widetilde{P}'(x_0)(x-y)\right| \leq \sup_{x_0<0}\left|\widetilde{P}'(x_0)\right||x-y|,$$

where, because $x, y \leq 0$, $\sup_{x_0<0} \left|\widetilde{P}'(x_0)\right| = 1$. Finally, we may have either $x \leq 0$ and $y \geq 0$ or $y \leq 0$ and $x \geq 0$. Without loss of generality, we consider the case $x \leq 0$ and $y \geq 0$. Then

$$\left|\widetilde{P}(x) - \widetilde{P}(y)\right| = \left|\widetilde{P}(x)\right| = \left|(1 - \exp(x))^+\right| \leq |x - y|,$$

for all $x \leq 0$ and $y \geq 0$. □

A.9 Appendix to Chapter 10

Lemma 10.2.1 *For every $K > 0$, $x \geq 0$, and $a \in (0, K)$, it holds that*

$$\Psi_a^K(x) \leq K,$$

where Ψ_a^K is defined in Definition 10.1.1 on page 117.

Proof. Fix $K > 0$. We first obtain that $\Psi_K^K(0) = K$. From a simple diagram and some elementary geometry, one finds that

$$\Upsilon_K^K(z) = \begin{cases} 1 - \frac{1}{2}\frac{z^2}{K^2}, & \text{if } 0 \leq z \leq K \\ 2 - \frac{2z}{K} + \frac{1}{2}\frac{z^2}{K^2}, & \text{if } K \leq z \leq 2K \\ 0, & \text{if } z \geq 2K \end{cases}$$

Integrating, we find that

$$\Psi_K^K(0) = K.$$

We now obtain that $\partial_a \Psi_a^K(0) = 0$ for all $0 < a < K$. Suppose that $0 < a < K$. Again, from a simple diagram and some elementary geometry, we get that

$$\Upsilon_a^K(z) = \begin{cases} 1, & \text{if } 0 \leq z \leq K - a \\ -\dfrac{(K-z)^2 + a(2z - 2K - a)}{2a^2}, & \text{if } K - a \leq z \leq K \\ \dfrac{(a+K)^2 + z(z - 2K - 2a)}{2a^2}, & \text{if } K \leq z \leq K + a \\ 0, & \text{if } z \geq K + a. \end{cases}$$

Simple integration then shows that

$$\Psi_a^K(0) = K.$$

Therefore, $\partial_a \Psi_a^K(0) = 0$ for all $0 < a < K$.

Since $x \mapsto \Psi_a^K(x)$, is non-increasing on $[0, \infty)$, we are done. $\square$

A.10 Appendix to Chapter 11

Lemma 11.2.1

Let $f, g : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ with $n \in \mathbb{N}$. Suppose that

(A.1) $\lim_{\tau\to 0^+} \tau \ln g(\tau, \mathbf{x}) = 0$*; and*

(A.2) There exists a non-negative function $D : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$\lim_{\tau\to 0^+} \tau \ln f(\tau, \mathbf{x}) = -D(\mathbf{x});$$

and the limits in both (A.1) and (A.2) are uniform in $\mathbf{x}$ *ranging over any fixed compact subset , denote it* B*, of* $\mathbb{R}^n$*. It then holds that*

$$\lim_{\tau\to 0^+} \tau \ln \int_0^\tau f(u, \mathbf{x}) g(u, \mathbf{x})\, \mathrm{d}u = -D(\mathbf{x}), \tag{A.17}$$

uniformly in $\mathbf{x}$ *ranging over* B*.*

Proof. All limits are taken as $\tau \to 0^+$. Let us fix a non-empty compact set of positive measure, say B, in $\mathbb{R}^n$.

We prove two facts which we subsequently use to prove the claim.

Fact 1

Clearly,

$$\lim \tau \ln(f(\tau, \mathbf{x}) g(\tau, \mathbf{x})) = -D(\mathbf{x}),$$

uniformly for $\mathbf{x} \in B$ using that $\tau \ln g(\tau, \mathbf{x}) \to 0$ and also the uniform convergence in both (A.1) and (A.2).

Fact 2

Let $h : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ and $\alpha, \beta > 0$ be given. Then if there exists a $\delta_1 > 0$, that may be chosen independently of $\mathbf{x} \in B$, such that for all $\tau \in (0, \delta_1)$

$$\exp(-\alpha/\tau) \leq h(\tau, \mathbf{x}) \leq \exp(-\beta/\tau),$$

then it holds

$$(\forall \epsilon_2 > 0)(\exists \delta_2 > 0)(\forall \tau \in (0, \delta_2)) \quad -\alpha - \epsilon_2 < \tau \ln \int_0^\tau h(u, \mathbf{x})\, \mathrm{d}u < -\beta + \epsilon_2.$$

where δ_2 is independent of the choice of $\mathbf{x} \in B$.

We now prove Fact 2. We are given $\epsilon_2 > 0$. From the supposition of Fact 2 we have that

$$\int_0^\tau h(u, \mathbf{x})\, \mathrm{d}u \leq \int_0^\tau \exp(-\beta/u)\, \mathrm{d}u, \quad \forall \tau \in (0, \delta_1),$$

where $\delta_1 > 0$ may be chosen independently of $\mathbf{x} \in B$. Making the change of variable $v = \beta/u$ in the right hand side integral we find that

$$\int_0^\tau h(u,\mathbf{x})\,\mathrm{d}u \le \beta \int_{\beta/\tau}^\infty v^{-2} \exp(-v)\,\mathrm{d}v = \beta\,\Gamma(-1,\beta/\tau), \quad \tau \in (0,\delta_1),$$

and again δ_1 is independent of $\mathbf{x} \in B$. By Lemma A.5.3, we have

$$\Gamma(-1,\beta/\tau) \sim \left(\frac{\tau}{\beta}\right)^2 \exp(-\beta/\tau), \quad \tau \to 0^+.$$

Therefore, there exists a $\delta_1' > 0$, that may be chosen independently of $\mathbf{x} \in B$, such that

$$\tau \ln \int_0^\tau h(u,\mathbf{x})\ \mathrm{d}u < -\beta + \epsilon_2$$

for all $\tau \in (0,\delta_1')$. We used that $\tau \ln(\tau A)$ converges to zero as $\tau \to 0^+$ for every $A > 0$.

A similar argument shows that there exists a $\delta_1'' > 0$ such that

$$\tau \ln \int_0^\tau h(u,\mathbf{x})\,\mathrm{d}u > -\alpha - \epsilon_2,$$

for every $\tau \in (0,\delta_1'')$ and δ_1'' may be chosen independently of $\mathbf{x} \in B$. Taking $\delta_2 = \min\{\delta_1', \delta_1''\}$ we see that Fact 2 holds.

Proof of the Lemma

Let $h = fg$. We have that Fact 1 holds. Let $\epsilon_3 > 0$ be given. It follows from Fact 1 that there exists a $\delta_3 > 0$ that may be chosen independently of $\mathbf{x} \in B$ such that

$$\exp\left(\frac{-D(\mathbf{x}) - \epsilon_3/2}{\tau}\right) < h(\tau,\mathbf{x}) < \exp\left(\frac{-D(\mathbf{x}) + (\epsilon_3 \wedge D(\mathbf{x}))/2}{\tau}\right),$$

for all $\tau \in (0,\delta_3)$. Observe that $-D(\mathbf{x}) + (\epsilon_3 \wedge D(\mathbf{x}))/2 < 0$, so that Fact 2 is applicable. Applying Fact 2 and taking, in particular, $\epsilon_2 = \epsilon_3/2$, we have that there exists a $\delta_3' > 0$ such that

$$-D(\mathbf{x}) - \epsilon_3 < \tau \ln \int_0^\tau h(u,\mathbf{x})\,\mathrm{d}u < -D(\mathbf{x}) + (\epsilon_3 \wedge D(\mathbf{x}))/2 + \epsilon_3/2 \le -D(\mathbf{x}) + \epsilon_3$$

for all $\tau \in (0,\delta_3')$ and δ_3' is independent of $\mathbf{x} \in B$. $\square$

Lemma 11.2.2

Let S be a CEV (Constant Elasticity of Variance) solution to the stochastic differential equation

$$\mathrm{d}S_t = \sigma S_t^\beta\,\mathrm{d}W_t, \quad S_0 = s > 0,$$

with $\beta \in (0,1)$ with an absorbing boundary condition at zero if necessary. Let q denote the continuous part of the law of S_t. Then, for $s, K > 0$,

$$\lim_{t \to 0+} -2t \ln(q(t,s,K)) = \frac{\left(s^{1-\beta} - K^{1-\beta}\right)^2}{\sigma^2(1-\beta)^2}.$$

Proof. It is well known, see (DL01), that the continuous part of the law of S_t, where S is a CEV process, is given by

$$q(t,s,K) = \frac{K^{-2\beta+1/2}s^{1/2}}{\sigma^2\,|\beta-1|\,t} \exp\left(-\frac{s^{-2(\beta-1)} + K^{-2(\beta-1)}}{2\sigma^2(\beta-1)^2 t}\right) I_{\frac{1}{2|\beta-1|}}\left(\frac{s^{-\beta+1}K^{-\beta+1}}{\sigma^2(\beta-1)^2 t}\right),$$

for $s, K, t > 0$ and where I_v is the modified Bessel function of the first kind (see (Olv97)). We have

$$I_v(z) \sim \frac{\exp(z)}{\sqrt{2\pi z}} \quad \text{as } z \to \infty, \tag{A.18}$$

(see (Olv97)). Fix $s, K, \sigma > 0$ and $\beta \in (0,1)$. Let

$$\begin{aligned} f : (0,\infty) &\to \mathbb{R} \\ t &\mapsto \frac{s^{-\beta+1}K^{-\beta+1}}{\sigma^2(\beta-1)^2 t}. \end{aligned}$$

We have

$$f(t) = \gamma t^{-1}.$$

for

$$\gamma := \gamma(\sigma,\beta,s,K) := \frac{s^{-\beta+1}K^{-\beta+1}}{\sigma^2(\beta-1)^2}.$$

Now, from Equation (A.18), we have

$$\begin{aligned} t\ln\left(I_{\frac{1}{2|\beta-1|}}(f(t))\right) &= t\ln\left(\frac{\exp(f(t))}{\sqrt{2\pi f(t)}}\,(1+\mathrm{o}(1))\right), && \text{as } t \to 0^+, \\ &= \gamma - \frac{t}{2}\ln\left(2\pi\gamma t^{-1}\right) + t\ln(1+\mathrm{o}(1)), && \text{as } t \to 0^+, \end{aligned}$$

so that it is clear that

$$\lim_{t\to 0^+} t\ln\left(I_{\frac{1}{2|\beta-1|}}(f(t))\right) = \gamma. \tag{A.19}$$

It is obvious that

$$\begin{aligned} &\lim_{t\to 0^+} t\ln\left(\frac{K^{-2\beta+1/2}s^{1/2}}{\sigma^2\,|\beta-1|\,t}\exp\left(-\frac{s^{-2(\beta-1)}+K^{-2(\beta-1)}}{2\sigma^2(\beta-1)^2 t}\right)\right) \\ &= -\frac{s^{-2(\beta-1)}+K^{-2(\beta-1)}}{2\sigma^2(\beta-1)^2}. \end{aligned} \tag{A.20}$$

Combining Equations (A.19) and (A.20) we find that

$$\begin{aligned}
&\lim_{t \to 0^+} t \ln q(t, s, K) \\
&= \frac{1}{\sigma^2(\beta-1)^2} \left(-\frac{s^{-2(\beta-1)} + K^{-2(\beta-1)}}{2} + s^{-\beta+1} K^{-\beta+1} \right) \\
&= -\frac{(s^{1-\beta} - K^{1-\beta})^2}{2\sigma^2(\beta-1)^2},
\end{aligned}$$

and the claim follows. □

Bibliography

[AFV01] E. Amerio, G. Fusai, and A. Vulcano, *Pricing of implied volatility derivatives: a risk neutral model for market implied volatility*, Tech. report, Università del Piemonte, 2001.

[ALV06] E. Alòs, J. A. León, and J. Vives, *On the short-time behaviour of the implied volatility for jump-diffusion models with stochastic volatility*, June 2006, Available at SSRN: `http://ssrn.com/abstract=1002308`.

[AP07] L. B. G. Andersen and V. V. Piterbarg, *Moment explosions in stochastic volatility models*, Finance and Stochastics **11** (2007), no. 1, 29–50.

[AS84] M. Abramowitz and I. A. Stegun (eds.), *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, John Wiley & Sons Inc., New York, 1984.

[Ave05] M. Avellaneda, *From SABR to geodesics*, November 2005, Conference presentation slides. Available at `http://www.math.nyu.edu/faculty/avellane/AvellanedaQCongress2005.pdf`.

[BBF02] H. Berestycki, J. Busca, and I. Florent, *Asymptotics and calibration of local volatility models*, Quantitative Finance **2** (2002), no. 1, 61–69.

[BBF04] ———, *Computing the implied volatility in stochastic volatility models*, Communications on Pure and Applied Mathematics **57** (2004), no. 10, 1352–1373.

[BC05] P. Bourgade and O. Croissant, *Heat kernel expansion for a family of stochastic volatility models: δ-geometry*, Tech. report, Ixis-CIB, 2005, arxiv:cs.CE/0511024.

[BCS96] M. A. Bharadia, N. Christofides, and G. R. Salkin, *Computing the Black Scholes implied volatility*, Advances in Futures and Options Research **8** (1996), 15–29.

[Ber96] J. Bertoin, *Lévy processes*, Cambridge University Press, Cambridge, 1996.

[BFG07] A. Brace, G. Fabbri, and B. Goldys, *An Hilbert space approach for a class of arbitrage free implied volatilities models*, Tech. report, UNSW, September 2007, arXiv:0712.1343v2.

[BGKW01] A. Brace, B. Goldys, F. Klebaner, and R. Wommersley, *Market model of stochastic implied volatility with application to the BGM model*, 2001, Working paper S01-1.

[BGM97] A. Brace, D. Gatarek, and M. Musiela, *The market model of interest rate dynamics*, Mathematical Finance **7** (1997), 127–155.

[BGvdHW02] A. Brace, B. Goldys, F. van der Hoek, and R. Wommersley, *Markovian models in the stochastic implied volatility framework*, September 2002, University of New South Wales: Statistics preprint.

[BL78] D. Breeden and R. Litzenberger, *Prices of state contingent claims implicit in option prices*, Journal of Business **51** (1978), 621–651.

[BS88] M. Brenner and M. G. Subrahmanyam, *A simple formula to compute the implied standard deviation*, Financial Analysts Journal **44** (1988), no. 5, 80–83.

[Bue06] H. Buehler, *Expensive martingales*, Quantitative Finance **6** (2006), no. 3, 207–218.

[CCIV06] J. Chargoy-Corona and C. Ibarra-Valdez, *A note on Black-Scholes implied volatility*, Physica A **370** (2006), no. 2, 681–688.

[CGH+96] R. Corless, G. Gonnet, D. Hare, D. Jeffrey, and D. Knuth, *On the Lambert W Function*, Advances in Computational Mathematics **5** (1996), no. 1, 329–359.

[CH05] A. M. G. Cox and D. G. Hobson, *Local martingales, bubbles and option prices*, Finance and Stochastics **9** (2005), no. 4, 477–492.

[Cha96] D. M. Chance, *A generalized simple formula to compute the implied volatility*, The Financial Review **31** (1996), no. 4, 859–867.

[CJ90] P. Carr and R. Jarrow, *The Stop-Loss Start-Gain Paradox and Option Valuation: A New Decomposition into Intrinsic and Time Value*, Review of Financial Studies **3** (1990), no. 3, 469–492.

[CM96] C. J. Corrado and T. W. Miller, Jr., *A note on a simple, accurate formula to compute implied standard deviations*, Journal of Banking and Finance **20** (1996), 595–603.

[CM05] P. Carr and D. Madan, *A note on sufficient conditions for no arbitrage*, Finance Research Letters **2** (2005), no. 3, 125–130.

[CN01] D. R. Chambers and S. K. Nawalkha, *An improved approach to computing implied volatility*, The Financial Review **38** (2001), no. 3, 89–100.

[Com70] L. Comtet, *Inversion de $y^\alpha e^\gamma$ et $y \log^\alpha y$ au moyen des nombres de Stirling*, C. R. Acad. Sci. Paris Sér. A-B **270** (1970), A1085–A1088.

[CT04] R. Cont and P. Tankov, *Financial modelling with jump processes*, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[CW03] P. Carr and L. Wu, *What type of process underlies options? A simple robust test*, The Journal of Finance **58** (2003), no. 6, 2581–2610.

[dC92] M. do Carmo, *Riemannian geometry*, Birkhäuser Boston Inc., Boston, MA, 1992, Translated from the second Portuguese edition by Francis Flaherty.

[DDSG04] Marc Decamps, Ann De Schepper, and Marc Goovaerts, *Applications of δ-function perturbation to the pricing of derivative securities*, Phys. A **342** (2004), no. 3-4, 677–692.

[DH07] M. H. A. Davis and D. G. Hobson, *The range of traded option prices*, Mathematical Finance **17** (2007), no. 1, 1–14.

[DHS07] T. Daglish, J. Hull, and W. Suo, *Volatility surfaces: theory, rules of thumb, and empirical evidence*, Quantitative Finance **7** (2007), no. 5, 507–524.

[DL01] D. Davydov and V. Linetsky, *Pricing and hedging path-dependent options under the CEV process*, Management Science **47** (2001), no. 7, 949–965.

[Dup94] B. Dupire, *Pricing with a smile*, Risk **7** (1994), no. 1, 18–20.

[Dur03] V. Durrleman, *A note on initial volatility surface*, Unpublished manuscript, February 2003.

[Dur05] ______, *From Implied to Spot Volatilities*, Available at `http://math.stanford.edu/~valdo/papers/FmImplied2SpotVols.pdf`, 2005.

[Dur08] ______, *Convergence of at-the-money implied volatilities to the spot volatility*, Journal of Applied Probability **45** (2008), no. 2, 542–550.

[Dyn65] E. Dynkin, *Markov processes*, vol. II, Academic Press, 1965.

[FL08] J. Figueroa-Lopéz, *Small-time moment asymptotics for Lévy processes*, Statistics and Probability Letters **78** (2008), no. 18, 3355–3365.

[FPS00] J.-P. Fouque, G. Papanicolaou, and K. Sircar, *Derivatives in financial markets with stochastic volatility*, Cambridge University Press, 2000.

[FS04] H. Föllmer and A. Schied, *Stochastic finance: an introduction in discrete time*, second revised and extended ed., Walter de Gruyter & Co., Berlin, 2004.

[Gat99] J. Gatheral, *The volatility skew: Arbitrage constraints and asymptotic behaviour*, 1999, Merrill Lynch.

[Gat04] ———, *A parsimonious arbitrage-free implied volatility parameterization with application to the valuation of volatility derivatives*, http://www.math.nyu.edu/fellows_fin_math/gatheral/madrid2004.pdf, May 2004, Conference presentation slides.

[Gat06] ———, *The volatility surface: A practitioner's guide*, Wiley, 2006.

[HHK07] V. Henderson, D. Hobson, and T. Kluge, *Is there an informationally passive benchmark for option pricing incorporating maturity?*, February 2007, pp. 75–86.

[HKLW02] P. S. Hagan, D. Kumar, A. S. Lesniewski, and D. E. Woodward, *Managing smile risk*, Wilmott Magazine **1** (2002), 84–108.

[HL05] P. Henry-Labordere, *A general asymptotic implied volatility for stochastic volatility models*, Tech. report, Barclays Capital, April 2005, Available at SSRN: http://ssrn.com/abstract=698601.

[HW99] P. S. Hagan and D. E. Woodward, *Equivalent Black volatilities*, Applied Mathematical Finance **6** (1999), no. 3, 147–157.

[Jac07] J. Jacod, *Asymptotic properties of power variations of Lévy processes*, ESAIM: Probability and Statistics **11** (2007), 173–196.

[JCHK95] D. J. Jeffrey, R. M. Corless, D. E. G. Hare, and D. E. Knuth, *Sur l'inversion de $y^\alpha e^y$ au moyen des nombres de Stirling associés*, C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), no. 12, 1449–1452.

[KC81] J. Kevorkian and J. D. Cole, *Perturbation methods in applied mathematics*, Springer-Verlag, 1981.

[Kel72] H. G. Kellerer, *Markov-Komposition und eine Anwendung auf Martingale*, Mathematische Annalen **198** (1972), 99–122.

[Kle02] F. Klebaner, *Option price when the stock is a semimartingale*, Electronic Communications in Probability **7** (2002), 79–83 (electronic).

[Kle03] ———, *Correction to: Option price when the stock is a semimartingale*, Electronic Communications in Probability **8** (2003).

[Kli91] J. Klippert, *On the right-hand derivative of a certain integral function*, The American Mathematical Monthly **98** (1991), no. 8, 751–752.

[KS91] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*, second ed., Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 1991.

[Lee04] R. Lee, *The moment formula for implied volatility at extreme strikes*, Mathematical Finance **14** (2004), no. 3, 469–480.

[Lee05] ———, *Implied volatility: statics, dynamics, and probabilistic interpretation*, Recent advances in applied probability, Springer, New York, 2005, pp. 241–268.

[Lev04] S. Levendorskiĭ, *American and European options near expiry, under Lévy processes*, Tech. report, The University of Texas, February 2004, Available at SSRN: `http://ssrn.com/abstract=520062`.

[Lev08] ———, *American and European options in multi-factor jump-diffusion models, near expiry*, Finance and Stochastics **12** (2008), no. 4, 541–560.

[Li05] S. Li, *A new formula for computing implied volatility*, Applied Mathematics and Computation **170** (2005), no. 1, 611–625.

[LM06] P.-L. Lions and M. Musiela, *Some properties of diffusion processes with singular coefficients*, Communications in Applied Analysis **10** (2006), no. 1, 109–126.

[LSC98] O. Ledoit and P. Santa-Clara, *Relative pricing of options with stochastic volatility*, March 1998, University of California-Los Angeles Finance Working Paper 9-98. Available at SSRN: `http://ssrn.com/abstract=121257`.

[Med04] A. Medvedev, *Asymptotic methods for computing implied volatilities under stochastic volatility*, October 2004, NCCR-Finrisk Working Paper No. 204. Available at SSRN: `http://ssrn.com/abstract=667281`.

[Mer73] R. Merton, *Theory of rational option pricing*, Bell Journal of Economics and Management Science **4** (1973), 141–183.

[MR05] M. Musiela and M. Rutkowski, *Martingale methods in financial modelling*, 2nd ed., vol. 36, Springer-Verlag, Berlin, 2005.

[MS07] A. Medvedev and O. Scaillet, *Approximation and calibration of short-term implied volatilities under jump-diffusion stochastic volatility*, Review of Financial Studies **20** (2007), no. 2, 427–459.

[MY02] D. B. Madan and M. Yor, *Making Markov martingales meet marginals: with explicit constructions*, Bernoulli **8** (2002), no. 4, 509–536.

[MY06] D. Madan and M. Yor, *Itô's integrated formula for strict local martingales*, Séminaire de Probabilités XXXIX, Lecture Notes in Mathematics, vol. 1874, Springer, 2006, pp. 157–170.

[Obł07] J. Obłój, *Fine-tune your smile: Correction to Hagan et al*, August 2007, arXiv:0708.0998v1.

[Olv97] F. W. J. Olver, *Asymptotics and special functions*, A K Peters Ltd., Wellesley, MA, 1997.

[Pro04] P. Protter, *Stochastic Integration and Differential Equations*, 2nd ed., Springer, New York, 2004.

[RR] M. Roper and M. Rutkowski, *On the Relationship Between the Call Price Surface and the Implied Volatility Surface Close to Expiry*, International Journal of Theoretical and Applied Finance, forthcoming.

[RY99] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, 3rd ed., Springer, 1999.

[Sat99] K. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge, U.K., 1999.

[Sav02] A. Savine, *A theory of volatility*, Available at `http://www.reech.com/company/research.papers/downloads/theory_of_volatility.pdf`, 2002.

[Sch99] P. Schönbucher, *A market model for stochastic implied volatility*, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci. **357** (1999), no. 1758, 2071–2092.

[Sch03] W. Schoutens, *Lévy processes in finance: pricing financial derivatives*, John Wiley and Sons, 2003.

[SW08] M. Schweizer and J. Wissel, *Term structures of implied volatilities: absence of arbitrage and existence results*, Mathematical Finance **18** (2008), no. 1, 77–114.

[Var67a] S. R. S. Varadhan, *Diffusion processes in a small time interval*, Communications on Pure and Applied Mathematics **20** (1967), 659–685.

[Var67b] ______, *On the behavior of the fundamental solution of the heat equation with variable coefficients*, Communications on Pure and Applied Mathematics **20** (1967), 431–455.

[Wis08] J. Wissel, *Arbitrage-free market models for liquid options*, Ph.D. thesis, ETH Zurich, 2008, Diss. ETH No. 17538.

[Wol] *Wolfram functions site*, `http://functions.wolfram.com/06.38.06.0006.01`.

[Yeh95] J. Yeh, *Martingales and stochastic analysis*, World Scientific, 1995.