Escape and metastability in deterministic and random dynamical systems

Author:
Stancevic, Ognjen

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ESCAPE AND METASTABILITY IN DETERMINISTIC AND RANDOM DYNAMICAL SYSTEMS

by

Ognjen Stančević

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Dynamical systems that are close to non-ergodic are characterised by the existence of subdomains or regions whose trajectories remain confined for long periods of time. A well-known technique for detecting such metastable subdomains is by considering eigenfunctions corresponding to large real eigenvalues of the Perron-Frobenius transfer operator. The focus of this thesis is to investigate the asymptotic behaviour of trajectories exiting regions obtained using such techniques. We regard the complement of the metastable region to be a hole , and show in Chapter 2 that an upper bound on the escape rate into the hole is determined by the corresponding eigenvalue of the Perron-Frobenius operator. The results are illustrated via examples by showing applications to uniformly expanding maps of the unit interval. In Chapter 3 we investigate a non-uniformly expanding map of the interval to show the existence of a conditionally invariant measure, and determine asymptotic behaviour of the corresponding escape rate. Furthermore, perturbing the map slightly in the slowly expanding region creates a spectral gap. This is often observed numerically when approximating the operator with schemes such as Ulam s method. We investigate the asymptotic scaling of the spectral gap as the perturbation vanishes. In Chapter 4 we consider escape rate from random sets under the action of random dynamics and prove a result analogous to that of Chapter 2. We also show, under fairly weak assumptions, that in Oseledets subspaces Lyapunov exponents with respect to different norms are equal. The results are applied to Rychlik random dynamical systems. Finally, Chapter 5 deals with the main themes of the earlier chapters in the settings of deterministic and random shifts of finite type. There, we demonstrate methods to decompose shifts into complementary subshifts of large entropy. Much of the material in this thesis has either appeared in a scientific journal or has been submitted to one for publication.

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Abstract

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Contents

Frequently used symbols

X	Phase space
$\mathcal{B}$	σ-algebra/Borel σ-algebra on X
m	Reference measure on $(X, \mathcal{B})$
T	map on X governing the dynamics
μ	Invariant measure
ℓ	Lebesgue measure
$\mathcal{P}$	Perron-Frobenius operator of T
P, P_N	Ulam approximation of $\mathcal{P}$
H, H_n	Hole in X
$A = X \setminus H$	Complement of the hole
$T_A = T\|_A$	Map with a hole
$\mathcal{P}_A$	Conditional Perron-Frobenius operator of T_A
λ	Largest eigenvalue of $\mathcal{P}_A$
$E(A) = -\log \lambda$	Escape rate from A
μ_A, μ_n	Conditionally invariant measure
$(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$	Dynamical system modelling randomness
λ, λ_i	Lyapunov exponent
R	Return time
Δ	Young Tower
ν	Reference measure on Δ
F	Tower map
$\mathcal{A}$	Alphabet of symbols
$\Sigma, \Sigma_M, \Sigma_{\mathbb{F}}$	Shift space
$\mathbb{F}$	Forbidden sequence
M	Transition matrix
σ	Left shift map on Σ
$\mathfrak{B}_k(\Sigma)$	Number of allowed k-blocks in Σ
$h_{top}(\Sigma)$	Topological entropy of Σ

Introduction

In the ergodic theory of dynamical systems it is common to seek a decomposition into parts which may be studied on their own, with the obvious advantage that this enables us to understand systems otherwise too complex to study. A system or a component that cannot be decomposed any further is said to be *ergodic*. Nevertheless, many ergodic systems possess regions that remain close to invariant for long periods of time. In such cases it is still fruitful to study these "close-to-invariant" regions separately.

In deterministic settings, sets which confine typical trajectories for longer than usual periods of time are said to be *almost-invariant* or *metastable*. We may think of the dynamics on such sets as "close to non-ergodic".

The same idea also translates to non-autonomous (time-dependent) or random dynamical systems. Here, the randomness or time-dependence provides different "rules" at each application of the dynamics. In return, it makes sense for the metastable sets to also vary with time. To encompass the idea that they may possibly be non-static, such sets have been given the name *coherent structures* or *random metastable sets*.

Applications of the theory of metastability are numerous, including areas of molecular dynamics [102], where the metastable sets are regions in the phase space that ensure stable molecular conformations; astrodynamics [40], where the metastable sets are regions from which asteroid escape is rare; physical oceanography [37, 63], where metastable regions are stable structures such as gyres and eddies; and atmospheric science [64, 100], where vortices in the stratosphere form time-dependent metastable regions.

While metastability is usually quantified by measuring the amount of mass that is exchanged in *finite* time (in fact, in a single application of the dynamics), our approach in this thesis will primarily be to investigate the long-term or *asymptotic* behaviour of the dynamics in the presence of metastability. The central theme of our work shall be dealing with *rate of escape* — a quantity that describes the asymptotic speed at which typical trajectories exit a given region, never to return. We study escape rates through the theory of *open dynamical systems* and their *conditionally invariant measures* — measures that remain invariant under the condition of non-escape. Provided that the escape rate of an open system is sufficiently low, naturally, one may regard the corresponding subdomain as metastable.

One needs to be careful here in order to distinguish between the two types of metastability we have just described because, while in practice they often are one and the same, we will show in this thesis that there do exist simple but counterintuitive counterexamples.

Throughout we shall pursue the idea of regarding a closed dynamical system, with two or more metastable sets, as (two or more) open dynamical systems where the domain of each is metastable.

The primary tools that we shall use in both the deterministic and random settings are *Perron-Frobenius* (transfer) operators. Their isolated non-unit eigenvalues, or respectively isolated nonzero Lyapunov exponents, indicate presence of eigenfunctions whose decay rates are slower than the exponential separation of nearby trajectories. These eigenfunctions have been used to heuristically decompose the domain into two metastable regions, linking the slow exponential decay of eigenfunctions with slow exchange of trajectories; see e.g. [39]. One may then ask *how the rate of escape from such metastable regions is related to the corresponding isolated spectral values?* We will answer this question for deterministic systems in Chapter 2 and, later in Chapter 4, extend it to the random setting. Roughly speaking, the isolated spectral values determine upper bounds on the escape rate from either of the metastable domains in the decomposition.

In the third chapter we shall investigate the anomalous case of a non-uniformly expanding interval map with an indifferent fixed point at the origin. The spectrum of the corresponding Perron-Frobenius operator does not contain any isolated eigenvalues and this presents a difficulty in determining a good metastable decomposition, as it is unclear which non-unit eigenvalue should be chosen to obtain optimal metastability. We present two different but related solutions that provide us with more insight into the problem.

Firstly, by excising a small hole in the problematic region, we will prove the existence of a unique conditionally invariant measure, and show the limiting behaviour of escape rate as the hole closes. Secondly, by introducing a small random perturbation in the same region we will show the existence of a spectral gap and determine the asymptotic behaviour of the isolated second eigenvalue as the noise vanishes. This analysis explains commonly observed behaviour when one tries to apply Ulam's method [106] to this class of intermittent maps. The majority of the material in Chapter 3 has appeared in [61] as joint work with Gary Froyland and Rua Murray.

The random perturbation exercise of Chapter 3, in conjunction with the results of Chapter 2 on deterministic dynamical systems motivates for investigation of similar phenomena in a completely random setting. To this we dedicate Chapter 4, first by introducing the concept of escape rate to random dynamical systems and then translating our results of Chapter 2 accordingly. Perron-Frobenius operators become *Perron-Frobenius operator cocycles* and the spectrum of eigenvalues becomes the spectrum of *Lyapunov exponents*. We will show that coherent structures (random metastable sets) obtained from eigenfunctions corresponding to large L^1-Lyapunov exponents possess escape rates whose upper bounds are given by the absolute value of the corresponding Lyapunov exponent. We will then further extend these results to other types of Lyapunov exponents (not just L^1), in particular those calculated using the variation norm, provided the setting is such that an Oseledets splitting [92] holds.

While Chapter 5 slightly diverges from the material preceding it, we use similar techniques to approximate lower bounds of topological entropies of some *symbolic dynamical systems*. In *symbolic dynamics* or *shift spaces*, roughly speaking, a cylindrical hole may be thought of as a "forbidden sequence" and the corresponding escape rate may be interpreted as the loss in topological entropy. Thus in this setting, detecting holes with low escape is, in a sense, equivalent to detecting subshifts with high topological entropy. We shall study the spectral properties of *adjacency matrices* rather than Perron-Frobenius operators. More precisely, by considering eigenvectors of adjacency matrices that correspond to large real subdominant eigenvalues, we will decompose a shift of finite type into two complementary subshifts in such way that each subshift retains a high topological entropy. In the spirit of Chapter 4, we generalise our results to random shifts of finite type, where we consider the Lyapunov spectrum of cocycles of adjacency matrices. All our results will be illustrated through simple examples.

Chapter 1

Background and Literature Review

This first chapter is a brief introduction to closed dynamical systems and the somewhat lesser-known open dynamical systems, together with some motivation and useful results that shall be needed for later parts of the thesis. Basic knowledge of measure theory and L^1 spaces is assumed. For an elementary introduction to dynamical systems from the measure-theoretic point of view we recommend the book of Lasota and Mackey [79]. Some other books the reader may find useful are by Bogachev [12], for a comprehensive treatment of measure theory; Walters [109], for an introduction to ergodic theory; Boyarsky and Góra [19], for a focus, within this theme, on the study of expanding interval maps; Lind and Marcus [85] for a beginner's introduction to symbolic dynamics; and Arnold [1] for a comprehensive introduction to random dynamical systems.

We also attempt to provide an up-to-date literature survey in the areas related to open dynamics. This is by no means a complete or self-contained introduction, and the reader is encouraged to consult the references given throughout the chapter. We note that a recent survey paper of Demers and Young [44] provides a good starting point to a reader interested in venturing into the area of open dynamical systems.

1.1 Closed Dynamical Systems

Measurable transformations are the central objects of study in the ergodic theory of dynamical systems.

Definition 1.1 (Measurable transformation)**.** Let $(X, \mathcal{B})$ be a measurable space. A transformation $T\colon (X, \mathcal{B}) \circlearrowleft$ is *measurable* if $T^{-1}\mathcal{B} \subseteq \mathcal{B}$; that is, $T^{-1}B \in \mathcal{B}$ for any $B \in \mathcal{B}$.

Let m be a natural finite reference measure on $(X, \mathcal{B})$. For example, when X is a Eucledian space, m is naturally the Lebesgue measure in which case we denote it ℓ.

Definition 1.2 (Non-singular transofmation)**.** A measurable transformation $T\colon (X, \mathcal{B}) \circlearrowleft$ is said to be *non-singular with respect to m* if $m(B) = 0$ implies that $m(T^{-1}B) = 0$ for all $B \in \mathcal{B}$.

In this thesis we regard a *(closed) dynamical system*, denoted by the tuple $(X, \mathcal{B}, m, T)$, to be the action of a measurable non-singular transformation T on the finite measure space $(X, \mathcal{B}, m)$. One studies *orbits* or *trajectories* of points $x \in X$ under the iterates of T, given by $\{x, T(x), T^2(x), \dots\}$.

Definition 1.3 (Invariant measure)**.** A measure μ on $(X, \mathcal{B})$ is said to be *invariant* under T if $\mu(B) = \mu(T^{-1}B)$ for all $B \in \mathcal{B}$. The transformation T is said to *preserve μ* while the dynamical system $(X, \mathcal{B}, \mu, T)$ is said to be *measure-preserving*.

Invariant measures are important objects in the ergodic theory of dynamical systems as they convey useful statistical information about long-term behaviour of trajectories (cf. Birkhoff's Individual Ergodic Theorem [10]).

Definition 1.4 (Ergodicity)**.** Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving dynamical system. A set $B \subseteq X$ is *invariant* if $T^{-1}B = B$. If every invariant $B \in \mathcal{B}$ is trivial, that is either $\mu(B) = 0$ or $\mu(X \setminus B) = 0$, then (T, μ) is said to be *ergodic*.

The ergodic property is commonly assumed in the study of dynamical systems, and we will often do so in this thesis. It is, however, not an overly restrictive assumption as in the absence of ergodicity one is free to partition the domain into *ergodic components* (non-trivial invariant sets) and study the dynamics on each of these separately. Concepts describing higher levels of complexity of dynamical systems, such as *mixing* and *exactness*, are sometimes also useful. For definitions and further information we refer the reader to e.g. [79] or [46].

Definition 1.5. A dynamical system $(X^*, \mathcal{B}^*, m^*, T^*)$ is a (metric) *factor* of $(X, \mathcal{B}, m, T)$ if there exists a measurable map $\pi : (X, \mathcal{B}) \to (X^*, \mathcal{B}^*)$ (called the *factor map* or *semi-conjugacy*) such that π is measure-preserving ($m \circ \pi^{-1} = m^*$) and $T^* \circ \pi = \pi \circ T$, that is the diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \pi\big\downarrow & & \pi\big\downarrow \\ X^* & \xrightarrow{T^*} & X^* \end{array}$$

commutes. If, in addition, π has a measurable inverse then the two systems are (metrically) *isomorphic*.

Sometimes, besides just measurability, the phase space X enjoys the additional structure of a smooth Riemannian manifold equipped with a metric d. In such cases we shall require $\mathcal{B}$ to be the Borel σ-algebra and T to possess a Jacobian derivative JT (or T') almost everywhere.

1.1.1 Perron-Frobenius Operator

Rather than studying trajectories of individual points under the action of a dynamical system $(X, \mathcal{B}, m, T)$, often a more effective approach is to study "trajectories" of whole distributions of points. We shall do this via the corresponding *Perron-Frobenius (transfer) operator* $\mathcal{P}$, acting on real-valued integrable functions of $(X, \mathcal{B}, m)$, defined below.

Definition 1.6 (Perron-Frobenius operator)**.** Let $T\colon (X, \mathcal{B}, m) \circlearrowleft$ be a non-singular measure-preserving transformation. The *Perron-Frobenius operator* associated with T is the unique operator $\mathcal{P}\colon L^1(X, \mathcal{B}, m) \circlearrowleft$ satisfying the following integral equation:

$$\int_B \mathcal{P}f \,\mathrm{d}m = \int_{T^{-1}B} f \,\mathrm{d}m \quad \forall B \in \mathcal{B}, \quad \forall f \in L^1(m). \tag{1.1}$$

When X is a smooth Riemannian manifold and T is C^1, equation (1.1) may also be written in the more explicit form:

$$\mathcal{P}f(x) = \sum_{y \in T^{-1}x} \frac{f(y)}{|JT(y)|}, \quad \forall f \in L^1(m),\ \forall x \in X, \tag{1.2}$$

where JT stands for the Jacobian determinant of T, while $|\cdot|$ is a modulus sign.

Recall that a measure ν on X is *absolutely continuous* with respect to m, denoted $\nu \ll m$, if $\nu(B) = 0$ whenever $m(B) = 0$ for all $B \in \mathcal{B}$. If both $\nu \ll m$ and $m \ll \nu$ then the two measures are *equivalent*, denoted $\nu \sim m$. Radon-Nikodym Theorem [91] (see e.g. [12, Theorem 3.2.2]) asserts that there exists a derivative $f \in L^1(m)$ such that $f = \mathrm{d}\nu/\mathrm{d}m \geq 0$. If f is normalised, it is called a *density*[1].

It is a well-known fact that (normalised) stationary points of the Perron-Frobenius operator are densities of absolutely continuous invariant (probability) measures (ACI(P)M), $\mu \ll m$. That is, if $\mathcal{P}f = f$ and $f \geq 0$, then the measure $\mu := f \cdot m$ given by

$$\mu(B) = \int_B f \,\mathrm{d}m, \quad \forall B \in \mathcal{B}$$

is invariant. In fact the converse is also true: any density of an absolutely continuous invariant measure is a stationary point of the Perron-Frobenius operator.

If the Perron-Frobenius operator admits a *unique* stationary density (that is, its eigenvalue is of multiplicity one), then the corresponding measure is ergodic; see e.g. [79, Theorem 4.2.2]. The distance to the next-largest eigenvalue generally provides some information on how close to non-ergodic the measure is.

Spectral Gap and Quasi-compactness

Let $\mathrm{sp}(\mathcal{L})$ denote the spectrum of a bounded linear operator $\mathcal{L}$ on a Banach space $(Y, \|\cdot\|_Y)$ (see e.g. [78] for the definition of a spectrum of a linear operator). The *spectral radius* of $\mathcal{L}$ is then

$$\mathcal{R}(\mathcal{L}) := \{\sup |z| \;:\; z \in \mathrm{sp}(\mathcal{L})\}$$

and the *essential spectral radius* $\mathcal{R}_{ess}(\mathcal{L})$ is the smallest number such that any $z \in \mathrm{sp}(\mathcal{L})$ with $|z| > \mathcal{R}_{ess}(\mathcal{L})$ is an isolated eigenvalue (therefore there are at most countably many eigenvalues outside the essential spectrum and accumulations are only possible on the essential radius). Operators whose spectral radius is strictly greater than the essential spectral radius are said to be *quasi-compact*. This property is desirable for Perron-

[1] For signed measures f need not be non-negative.

Frobenius operators as it guarantees the existence of a *spectral gap* — the difference between 1 and the modulus of the "second" eigenvalue. More information and discussion on the spectrum of transfer operators may be found in, for example, the book of Baladi [7].

1.1.2 Functions of Bounded Variation

Let $\mathrm{I} = [0, 1]$ and let ℓ be the Lebesgue measure on I. For maps $T : (\mathrm{I}, \ell) \circlearrowleft$ it is often useful to work with Perron-Frobenius operators acting not on L^1, but on the space of *functions of bounded variation*.

Definition 1.7. The *variation* (or *total variation*) of a function $f \in L^1(\mathrm{I}, \ell)$, denoted $\mathrm{var}(f)$ is defined to be

$$\mathrm{var}(f) = \inf_g \sup \left\{ \sum_{i=0}^{n-1} |g(x_{i+1}) - g(x_i)| \ : \ 0 = x_0 < \cdots < x_n < 1 \right\}$$

where the infimum is taken over all *versions* of f, that is all $g \in L^1(\mathrm{I}, \ell)$ satisfying $\ell(f - g) = 0$.

Let BV be the set of all $f \in L^1(\mathrm{I}, \ell)$ such that $\mathrm{var}(f) < \infty$ and define the BV-*norm*

$$\|f\|_{\mathrm{BV}} := \max\{\mathrm{var}(f), \|f\|_{L^1}\}.$$

The Banach space $(\mathrm{BV}, \|\cdot\|_{\mathrm{BV}})$ is called the *space of functions of bounded variation*.

Perron-Frobenius operators of many expanding maps of the interval (e.g. some Lasota-Yorke [80] or Rychlik maps [99]) do not possess a spectral gap in L^1 but do so in BV. This setting has historically been a standard testbed for spectral analysis of chaotic dynamical systems. Lasota and Yorke [80] showed that for piecewise C^2 uniformly expanding[2] maps with finitely many branches, the inequality (now known as the *Lasota-Yorke inequality*)

$$\|\mathcal{P}^n f\|_{\mathrm{BV}} \leq \theta^n \|f\|_{\mathrm{BV}} + C\|f\|_{L^1} \tag{1.3}$$

holds for some $\theta \in (0, 1)$, $C > 0$ and all $f \in \mathrm{BV}$. Hofbauer and Keller [72], and Rychlik

[2] A map T is *expanding* if $|JT| > 1$ almost everywhere. It is *uniformly expanding* if $\operatorname{ess\,inf} |JT| > 1$.

[99] showed under the relaxation to allowing a countable number of branches, with requirements that $1/|T'|$ is of bounded variation and

$$1/\tau := \lim_{n\to\infty} \left(\|1/(T^n)'\|_\infty\right)^{1/n} > 1, \tag{1.4}$$

that $\mathcal{P}\colon \mathrm{BV} \circlearrowleft$ is quasi-compact. Soon after, Keller [74] proved that $1/\tau$ is the essential spectral radius of $\mathcal{P}$.

1.1.3 Shifts of Finite Type

Shifts of finite type (also known as *topological Markov chains*) are special types of dynamical systems acting on sequence spaces. Below we will introduce the main concepts. For proofs of our claims and any additional details, the reader may wish to read the relevant parts in the books of Lind and Marcus [85], and Kitchens [77].

Definition 1.8 (Shift space)**.** Let $\mathcal{A}$ be an *alphabet* — a finite collection of K symbols. Define a one-sided (resp. two-sided) *full N-shift* to be a collection of all infinite (bi-infinite) sequences of elements of $\mathcal{A}$, respectively

$$\mathcal{A}^{\mathbb{Z}^+} := \{x = (x_i)_{i\in\mathbb{Z}^+} \ : \ x_i \in \mathcal{A}, \quad \forall i \in \mathbb{Z}^+\}, \tag{1.5}$$

$$\mathcal{A}^{\mathbb{Z}} := \{x = (x_i)_{i\in\mathbb{Z}} \ : \ x_i \in \mathcal{A}, \quad \forall i \in \mathbb{Z}\}. \tag{1.6}$$

We write $x = x_0x_1x_2\ldots$ or $x = \ldots x_{-2}x_{-1}x_0x_1x_2\ldots$ for their respective elements. In this subsection we concentrate on one-sided shifts and most definitions and properties translate naturally to their two-sided counterparts. The *left shift map* $\sigma : \mathcal{A}^{\mathbb{Z}^+} \circlearrowleft$ acts according to

$$(\sigma x)_i = x_{i+1},$$

that is, it shifts all elements of x to the left by one. This map is invertible on $\mathcal{A}^{\mathbb{Z}}$ but not on $\mathcal{A}^{\mathbb{Z}^+}$. A *shift* consists of any σ-invariant set $\Sigma \subseteq \mathcal{A}^{\mathbb{Z}^+}$, together with σ itself, denoted (Σ, σ). A *subshift* of (Σ, σ) is any shift (Σ', σ) such that $\Sigma' \subseteq \Sigma$. In particular, any shift coded on $\mathcal{A}$ is a subshift of the full shift $(\mathcal{A}^{\mathbb{Z}^+}, \sigma)$.

A *block* (or *word*) of length k is any finite sequence of k symbols from the alphabet, written as $b = [b_0 b_1 \ldots b_{k-1}]$, where $b_i \in \mathcal{A}$. For a point $x \in \mathcal{A}^{\mathbb{Z}^+}$, denote by $x_{[k,k+n]}$ the block $[x_k \ldots x_{k+n}]$. Given a collection of *forbidden blocks* $\mathbb{F}$, the set of points that do not contain any of the forbidden blocks is invariant under σ, and therefore defines a shift. When $\mathbb{F}$ is finite, this set is called a *shift of finite type*, denoted $(\Sigma_{\mathbb{F}}, \sigma)$ (or (Σ, σ) if there is no ambiguity in regard to what $\mathbb{F}$ is). The *memory* of a shift of finite type is always one less than the length of the longest forbidden block.

Definition 1.9. Let $\mathfrak{B}_k(\Sigma_{\mathbb{F}})$ be the set of all allowed[3] blocks of length k in a shift of finite type $\Sigma_{\mathbb{F}}$. The *topological entropy* of $\Sigma_{\mathbb{F}}$ is defined to be the exponential growth rate of the number of elements of $\mathfrak{B}_k$:

$$h_{top}(\Sigma_{\mathbb{F}}) = \lim_{k \to \infty} \frac{1}{k} \log |\mathfrak{B}_k(\Sigma_{\mathbb{F}})|.$$

Topological entropy (or just entropy) may be thought of as a measure of the dynamic complexity of a shift. A subshift's entropy is always lower than or equal to the entropy of its parent shift. Observe that the topological entropy of a full N-shift is $\log N$, so this is the maximum entropy of any subshift encoded with N symbols.

By appropriately changing the alphabet $\mathcal{A}$ and the forbidden blocks $\mathbb{F}$, one can always *recode* any shift of finite type into a conjugate shift of memory 1, so that every forbidden block of the recoded shift is of length two. Such shifts $(\Sigma_{\mathbb{F}}, \sigma)$ may be represented by their $0-1$ *transition matrices* $M \in \mathcal{M}_{N \times N}(\{0,1\})$ where $M_{ij} = 0$ if and only if $[ij] \in \mathbb{F}$ (or equivalently $M_{ij} = 1$ if and only if $[ij] \in \mathfrak{B}_2(\Sigma_{\mathbb{F}})$). Since M determines $\mathbb{F}$ we shall often write Σ_M instead of $\Sigma_{\mathbb{F}}$. We may also represent any memory-1 shift of finite type Σ_M by a directed graph, determined by M serving as the adjacency matrix in the obvious way [4].

By the Perron-Frobenius Theorem for non-negative matrices [52], M has a real positive eigenvalue, equal to its spectral radius $\mathcal{R}(M)$. The topological entropy of a shift of finite type (Σ_M, σ) is then simply

$$h_{top}(\Sigma_M) = \log \mathcal{R}(M). \tag{1.7}$$

[3]Those that are not forbidden.

[4]Let $\mathcal{V} = \mathcal{A}$ be the vertices, and define the edges $\mathcal{E}$ according to $(v, w) \in \mathcal{E}$ if and only if $M_{vw} = 1$. The set of all infinite random walks on the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ represents the shift space (Σ_M, σ).

Now we describe the procedure to impose a measure on a shift of finite type. For a block $b = [b_0 b_1 \dots b_{k-1}]$ and position j, a *cylinder* $\mathcal{C}_j(b) \subseteq \Sigma$ is

$$\mathcal{C}_j(b) = [b_0 b_1 \dots b_{k-1}]_j := \{x \in \Sigma \; : \; x_{[j,j+k]} = b\}.$$

Regarding cylinders as open sets, the collection of all cylinders in Σ generates a topology (equal to the infinite product of discrete topologies on $\mathcal{A}$). We may then define a Borel σ-algebra $\mathcal{B}$, creating a measurable space $(\Sigma, \mathcal{B})$. Observe that in this setting σ is continuous and measurable. A common way to obtain a measure on a shift of finite type Σ_M is to take a row-stochastic matrix P *compatible* with M (that is $P_{ij} > 0$ implies that $M_{ij} = 1$), and its left stationary vector p (satisfying $pP = p$). The corresponding *Markov measure* $\mu_{(p,P)}$ is defined by its value on cylinders:

$$\mu_{(p,P)}([b_0 b_1 \dots b_{k-1}]) := p_{b_0} P_{b_0 b_1} \cdots P_{b_{k-2} b_{k-1}}.$$

It can be checked that the resulting dynamical system $(\Sigma_M, \mathcal{B}, \mu_{(p,P)}, \sigma)$ is measure-preserving. A Markov measure whose Kolmogorov-Sinai entropy equals the topological entropy $h_{top}(\Sigma_M)$ is called *maximal measure* or *measure of maximal entropy*. An advantage of studying shifts of finite type (equipped with a Markov measure) over other types of dynamical systems is that every point contains information on its orbit under the iteration of σ, that is the left shift map is trivial, while all of the dynamical complexity is contained in the space Σ and the measure $\mu_{(p,P)}$.

Also, recall that a non-negative square matrix P is *irreducible* if for all indices i, j there exists an integer n such that $(P^n)_{ij} > 0$. If the compatible stochastic matrix P of a shift Σ_M is irreducible, it follows that P has a unique left stationary probability vector p and the corresponding Markov measure is ergodic.

One may also find it useful to impose a metric d on Σ (compatible with the topology generated by cylinders) as follows:

$$d(x,y) := \begin{cases} 2^{-k}, & x \neq y \text{ and } k \text{ is maximal so that } x_{[-k,k]} = y_{[-k,k]} \\ 0, & x = y. \end{cases}$$

1.1.4 Markov Partition

Often in the study of dynamical systems it is convenient, if possible, to utilise the tools and machinery of shifts of finite type. In order to do so, one needs the concept of a *Markov partition*. Below we recall the definition of a Markov partition for expanding maps. A similar concept exists more generally for hyperbolic (Axiom A) maps.

Definition 1.10 (Markov partition [15])**.** Let $T\colon X \circlearrowleft$ be an expanding map. A partition (modulo sets of zero measure) $\eta = \{B_0, \ldots, B_{N-1}\}$ of X is said to be *Markov* if for every $i = 0, \ldots, N-1$, $T(B_i)$ is (exactly) a union of sets in η.

Dynamical systems that possess a finite Markov partition may be studied via their symbolic dynamics. We do this by assigning to every point $x \in X$ a sequence $y = (y_i)_{i=0}^{\infty} \in \{0, \ldots, N-1\}^{\mathbb{Z}^+}$ according to $y_i = k$ if and only if $T^i(x) \in B_k$; that is, y contains the itinerary of x, with respect to the elements of the Markov partition, under the iteration of T. The set of all such y, together with the left shift map, defines a shift of finite type (Σ_M, σ), where the transition (adjacency) matrix $M \in \mathcal{M}_{N\times N}(\{0,1\})$ is given by

$$M_{ij} = \begin{cases} 1, & B_j \subset \overline{T(B_i)}, \text{ the closure of } T(B_i) \\ 0, & \text{otherwise.} \end{cases}$$

The map $\pi : X \to \Sigma_M$, given by $\pi(x) = y$, is continuous (see e.g. [85, Proposition 6.5.8]) and the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ \Sigma_M & \xrightarrow{\sigma} & \Sigma_M \end{array}$$

Thus (Σ_M, σ) is a topological factor of (X, T) where π is the corresponding topological semi-conjugacy. Since π is one-to-one almost everywhere, with an appropriate choice of measure on Σ_M one may be able to obtain a dynamical system on Σ_M that is metrically isomorphic to the original one on X.

Example 1.11. Let $T\colon \mathrm{I} \circlearrowleft$ be the doubling map defined by

$$T(x) := 2x \pmod 1 = \begin{cases} 2x, & 0 \le x \le 1/2 \\ 2x-1, & 1/2 < x \le 1. \end{cases}$$

It is easy to check that $\eta = \{[0,1/2],[1/2,1]\}$ is a Markov partition for T. We encode the elements of the partition with $\{0,1\}$. The corresponding adjacency matrix is

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

thus T is modelled by the one-sided full 2-shift $\Sigma_M = \{0,1\}^{\mathbb{Z}^+}$. For every $x \in \mathrm{I}$ the corresponding $y = \pi(x) \in \Sigma_M$ is given by the fractional part of the binary representation of x (which is unique Lebesgue-almost everywhere). The stochastic matrix $P = (1/2)M$ is compatible with M, the row-vector $p = (1/2,1/2)$ uniquely satisfies $pP = p$ and the corresponding ergodic Markov measure is given by

$$\mu_{(p,P)}([b_0 b_1 \dots b_{k-1}]_j) = 2^{-k}.$$

Observe that $\pi^{-1}[b_0 b_1 \dots b_{k-1}]_j$ is a union of dyadic intervals and that $\mu_{(p,P)} \circ \pi^{-1} = \ell$. This is enough to show that (I, ℓ, T) is metrically isomorphic to $(\Sigma_M, \mu_{(p,P)}, \sigma)$.

1.1.5 Ulam's Method and Numerics

Ulam's method [82, 106] is a well-known scheme used to discretise the Perron-Frobenius operator.

Given a non-singular measurable transformation $T : (X, \mathcal{B}, m) \circlearrowleft$ and its Perron-Frobenius operator $\mathcal{P}$, take any finite partition $\eta = \{B_1, \dots, B_N\}$ of X. Let $\{\chi_B\}_{B\in\eta}$ be the set of *characteristic functions* on elements of η, defined by

$$\chi_{B_i}(x) = \begin{cases} 1, & x \in B_i \\ 0, & \text{otherwise.} \end{cases}$$

Define a projection Π_η from $L^1(m)$ to the finite N-dimensional space spanned by $\{\chi_B\}_{B\in\eta}$ by

$$\Pi_\eta f := \sum_{B\in\eta} \frac{\int_B f \,\mathrm{d}m}{m(B)} \chi_B, \quad f \in L^1(m). \tag{1.8}$$

The operator $\mathcal{P}_\eta := \Pi_\eta \circ \mathcal{P} \circ \Pi_\eta$ is the *Ulam approximation* of $\mathcal{P}$. Its matrix representation P_η, with respect to the normalised basis $\{(1/m(B))\chi_B\}_{B\in\eta}$ in $L^1(m)$ (and standard basis in $\mathbb{R}^N$ with left multiplication) is called the *Ulam matrix* and its entries are given by

$$(P_\eta)_{ij} = \frac{m(B_i \cap T^{-1}B_j)}{m(B_i)}, \quad 1 \le i,j \le N.$$

The Ulam matrix P_η (also denoted P_N) is row-stochastic and defines a Markov chain, which is a finite state model of the original dynamical system T. In Chapter 3 we shall utilise the observation that Ulam's approximation of $\mathcal{P}$ may be thought of as the Perron-Frobenius operator of a small random perturbation of T [53, 68].

Left eigenvectors of P_η are often good numerical approximations of the eigenfunctions of $\mathcal{P}$, and similarly for the corresponding eigenvalues.

If η is a Markov partition, observe that that the Ulam matrix is compatible with the adjacency matrix of the corresponding shift of finite type.

1.2 Open Dynamical Systems

The main theme of this thesis is the study of *open dynamical systems*, also known as *dynamical systems with holes*. Here we introduce the basic concepts. The book of Dorfman [47] also contains some introductory material to the area.

While the idea behind open dynamical systems is simple and there are many similarities to closed systems, there are also important distinctions. For example, the position and the size (measure) of the hole are factors that play important roles in the dynamical behaviour [21], but do not possess an analogue in closed dynamics.

An open dynamical system consists of a map T on a measurable domain A in a measure space $(X, \mathcal{B}, m)$ such that $A \subset T(A)$. Trajectories may eventually leave A to fall into the *hole*, $H = X \setminus A$, at which stage the dynamics are terminated. Although often only measurability is necessary, if X is a topological space, A is sometimes assumed to be open [95] (or closed [44]).

A common (but not the only) way to obtain an open system is to consider a closed dynamical system $(X, \mathcal{B}, m, T)$ and introduce a measurable hole $H \in \mathcal{B}$ so that $A = X \setminus H$ is the domain. Then the *restriction*

$$T_A := T|_A : A \to X,$$

together with $\mathcal{B}_A = \mathcal{B} \cap A$ and $m_A = m|_A$ define an open dynamical system, sometimes denoted $(A, \mathcal{B}_A, m_A, T_A)$. As T_A is only defined on A, *pre-images* of sets under the open map T_A are given by $T_A^{-1}B = T^{-1}B \cap A$ for any $B \in \mathcal{B}$. While we may start with T defined on all of X, for the purposes of studying open dynamics it is irrelevant what values T takes on H.

Although the example below is not obtained in this way, it illustrates well the concept of an open dynamical system.

Example 1.12. Perhaps the most famous open dynamical system is the horseshoe map, first studied by Stephen Smale in 1967 [103]. The domain A is a square, which under the action of T is "compressed" vertically and "stretched" horizontally in an area-preserving fashion, and then "folded back", as shown in Figure 1.1. The points that are found outside the original square are discarded before the next iteration of the map when the stretching and folding process repeats. After any finite number of iterations, a set of points of positive Lebesgue measure remains. Nonetheless, almost every point exits the square in finite time. The set of points that forever remain in the square is invariant, and the *closed* dynamics on this set is what is often studied in the horseshoe map (see e.g. [73]).

1.2.1 Escape Rates and Survivor Sets

Let the *time of escape* of a point $x \in A$ be the smallest positive integer $\xi(x)$ such that $T^{\xi(x)}(x) \in H$. Define A^n to be the set of all points that stay in A up to the n^{th} iterate of T;

Figure 1.1: One iteration of the horseshoe map of Example 1.12.

that is, A^n consists of all points that have not yet escaped by time n:

$$\begin{aligned} A^n &= \{x \in A \;:\; \xi(x) > n\} \\ &= \bigcap_{i=0}^{n} T^{-i}A \\ &= T_A^{-n}A. \end{aligned} \tag{1.9}$$

Because the points in A^n may be seen as the points that "survive" up to n iterates we refer to this set as the *n-step survivor*. The set of points that never escape is given by

$$\bigcap_{n=1}^{\infty} T^{-i}A =: A^{\infty}, \tag{1.10}$$

often called just *the survivor* [86] or *the repeller* [7]. Since A^{∞} is invariant, the action of T on A^{∞} determines a closed dynamical system, often studied in its own right, as is the case with the horseshoe map.

Escape rate is the rate of asymptotic decay of the measure of n-step survivors, defined more precisely below.

Definition 1.13 (Escape rate)**.** *Upper* and *lower escape rates* of a measure m from a measurable set $A \subset X$ under the action of $T\colon X \circlearrowleft$ are respectively

$$\overline{E}(A;m) := -\liminf_{n\to\infty} \frac{1}{n} \log m(A^n),$$

$$\underline{E}(A;m) := -\limsup_{n\to\infty} \frac{1}{n} \log m(A^n),$$

where A^n is as in (1.9). If $\overline{E}(A;m) = \underline{E}(A;m)$, then we say that the *escape rate* from A (with respect to m) exists and is given by

$$E(A;m) := -\lim_{n\to\infty} \frac{1}{n} \log m(A^n) \in [0,\infty]. \tag{1.11}$$

While the notion of *escape rate* as we have described above applies to the domain A of an open dynamical system, the definition works just as well for any other measurable set of an open or closed dynamical system. When it is clear what the underlying measure is, we may omit writing it: $E(A;m) = E(A)$. On the other hand, if we are talking about multiple transformations, say T and S, we shall denote escape rate from A under T and S respectively as $E(A;m,T)$ and $E(A;m,S)$.

Proposition 1.14. *Let $(X,\mathcal{B},m,T)$ be a closed dynamical system and let $A,B \subseteq X$ be measurable sets. Below are some useful (and well-known) properties of escape rates:*

(i) if $A \subseteq B$ then $E(A) \geq E(B)$;

(ii) $E(A^N) = E(A)$ for all $N \in \mathbb{N}$;

(iii) if $m(X \setminus A) = 0$ then $E(A) = 0$;

(iv) if $m(A) = 0$ then $E(A) = \infty$.

Proof.

(i) Since $A \subseteq B$ then we have for any integer n, $A^n \subseteq B^n$, hence $m(A^n) \leq m(B^n)$. Taking logarithms, dividing by n and taking the limit gives the required result.

(ii) First we will show that for all $n, N \geq 0$, $(A^N)^n = A^{N+n}$; that is, the points in A^N that survive for n steps under T are exactly those points in A that survive for $N+n$ steps. This may or may not be immediately obvious, but nevertheless it is a simple exercise in set theory:

$$(A^N)^n = \bigcap_{i=0}^{n} T^{-i}(A^N)$$

$$
\begin{aligned}
&= \bigcap_{i=0}^{n} T^{-i}\left(\bigcap_{j=0}^{N} T^{-j}(A)\right) \\
&= \bigcap_{i=0}^{n}\bigcap_{j=0}^{N} T^{-(i+j)}(A) \\
&= \bigcap_{k=0}^{n+N} T^{-k}(A) = A^{N+n}.
\end{aligned}
$$

The result then follows:

$$
\begin{aligned}
E(A^N) &= -\lim_{n\to\infty}\frac{1}{n}\log m((A^N)^n) \\
&= -\lim_{n\to\infty}\frac{1}{n}\log m(A^{N+n}) \\
&= -\lim_{n\to\infty}\frac{1}{n-N}\log m(A^n) = E(A).
\end{aligned}
$$

Points (iii) and (iv) are a consequence of the non-singularity of T. We will leave the proofs as an exercise. □

Proposition 1.15. *Let $(X, \mathcal{B}, m, T)$ be a dynamical system and let $A \subset X$ be measurable. Then for any $\gamma \in (0, 1]$*

(i) if $m(A^{n+1}) \geq \gamma m(A^n)$ for all $n \geq 0$, then $\overline{E}(A) \leq -\log\gamma$;

(ii) if $m(A^{n+1}) \leq \gamma m(A^n)$ for all $n \geq 0$, then $\underline{E}(A) \geq -\log\gamma$.

Proof.
(i) Inductively, $m(A^n) \geq \gamma^n m(A)$ thus

$$
\begin{aligned}
\overline{E}(A) &= -\liminf_{n\to\infty}\frac{1}{n}\log m(A^n) \\
&\leq -\liminf_{n\to\infty}\frac{1}{n}\log(\gamma^n m(A)) \\
&= -\log\gamma.
\end{aligned}
$$

(ii) is analogous to *(i)* with inequalities reversed. □

Corollary 1.16. *If the following limiting ratio exists*

$$\lim_{n\to\infty} \frac{m(A^{n+1})}{m(A^n)} =: \lambda, \tag{1.12}$$

then the escape rate from A exists and equals $-\log\lambda$.

Proof. For any $\epsilon > 0$ there exists a sufficiently large $N \in \mathbb{N}$ such that for all $i \geq 0$

$$\lambda - \epsilon \leq \frac{m(A^{N+i+1})}{m(A^{N+i})} \leq \lambda + \epsilon.$$

Hence by Proposition 1.15

$$-\log(\lambda+\epsilon) \leq E(A^N) \leq -\log(\lambda-\epsilon),$$

and by Proposition 1.14 (ii) the same inequality holds for $E(A)$, but since ϵ is arbitrary we must have $E(A) = -\log\lambda$.

□

Proposition 1.17. *The converse of the statement of Corollary 1.16 is generally not true.*

Proof. We provide a counterexample to the converse. Let $T : \mathrm{I} \circlearrowleft$ be an expanding one-branch map of the interval that is piecewise affine on a countable number of subintervals, and whose endpoints $(x_n)_{n\geq 0}$ satisfy $x_{2k} = 2^{-k}3^{-k}$ and $x_{2k+1} = 2^{-k}3^{-k-1}$, $k \geq 0$. Then T is constructed so that $T(x_n) = x_{n-1}$, from which we see that $A^n = [0, x_n]$ and $\ell(A^n) = x_n$. The ratio $m(A^{n+1})/m(A^n)$ oscillates between $1/2$ and $1/3$ hence the limiting ratio in (1.12) does not exist. However, escape rate exists and equals $\log\sqrt{6}$. □

Definition 1.18. An integrable function $f \in L^1$ is said to be *bounded away from zero and infinity* if $\inf f > 0$ and $\sup f < \infty$. If an L^1 version of f satisfies this (that is $\operatorname{ess\,inf} f > 0$ and $\operatorname{ess\,sup} f < \infty$), then we shall say that f is *essentially bounded away from zero and infinity*.

It is useful to keep in mind the following well-known result regarding escape with respect to "equivalent" measures (in the stronger sense of equivalence given by Definition 1.18).

Proposition 1.19. *If ν and m are equivalent measures and the Radon-Nikodym derivative* $\mathrm{d}\nu/\mathrm{d}m$ *is essentially bounded away from zero and infinity, then one has $E(A;\nu) = E(A;m)$ for any measurable set A.*

Proof. As $\mathrm{d}\nu/\mathrm{d}m$ is essentially bounded away from zero and infinity, there exists a positive constant C such that almost everywhere

$$C^{-1} \leq \frac{\mathrm{d}\nu}{\mathrm{d}m} \leq C. \tag{1.13}$$

Hence for any positive integer n we have

$$\begin{aligned} \nu(A^n) &= \int_{A^n} \frac{\mathrm{d}\nu}{\mathrm{d}m}\,\mathrm{d}m \\ &\leq \int_{A^n} C\,\mathrm{d}m \\ &= Cm(A^n). \end{aligned}$$

Thus

$$\begin{aligned} E(A;\nu) &= -\lim_{n\to\infty} \frac{1}{n} \log \nu(A^n) \\ &\geq -\lim_{n\to\infty} \frac{1}{n} \log(Cm(A^n)) \\ &= -\lim_{n\to\infty} \frac{1}{n} \log C - \lim_{n\to\infty} \frac{1}{n} \log m(A^n) \\ &= 0 + E(A;m). \end{aligned}$$

Similarly, by considering the lower bound of $\mathrm{d}\nu/\mathrm{d}m$ in (1.13) we obtain the reverse inequality $E(A;\nu) \leq E(A;m)$, and the result follows. □

Another useful fact, stated below, is that escape rate is an invariant of a metric isomorphism or a metric factor (see Bunimovich and Yurchenko [21, Lemma 2.3.5] for a proof).

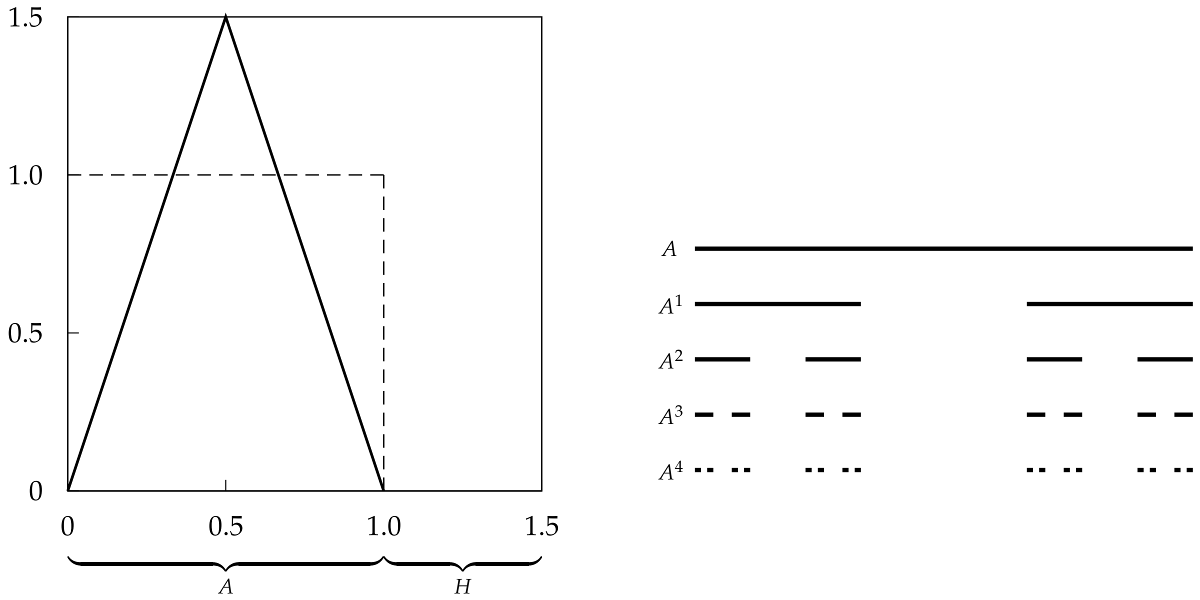
Proposition 1.20. *Let $(X^*, \mathcal{B}^*, m^*, T^*)$ be a factor of $(X, \mathcal{B}, m, T)$ with some factor map π, and suppose that for $A^* \in \mathcal{B}^*$, the escape rate exists. Then $E(A;m,T)$, the escape rate from $A = \pi^{-1}A^*$, exists and equals $E(A^*;m^*,T^*)$.*

1.2.2 Some Simple Examples

Example 1.21 (Tent map with a hole [47]). For $\epsilon > 0$ consider the following map $T : [0,1] \to [0,(1-\epsilon)^{-1}]$ defined by

$$T(x) = \begin{cases} \frac{2}{1-\epsilon}x, & 0 \le x \le \frac{1}{2} \\ \frac{2}{1-\epsilon}(1-x), & \frac{1}{2} \le x \le 1. \end{cases}$$

The domain is $A := [0,1]$ and the hole is $H := (1,(1-\epsilon)^{-1}]$ as shown in Figure 1.2(a) for $\epsilon = 1/3$. Let $m = \ell$ be the Lebesgue measure. The diagram in Figure 1.2(b) shows the n-step survivors for $n \in \{0,1,2,3,4\}$, and $\epsilon = 1/3$. Observe that the sequence of A^n describes the familiar construction of the middle-thirds Cantor set, where A^∞ is the Cantor set itself.



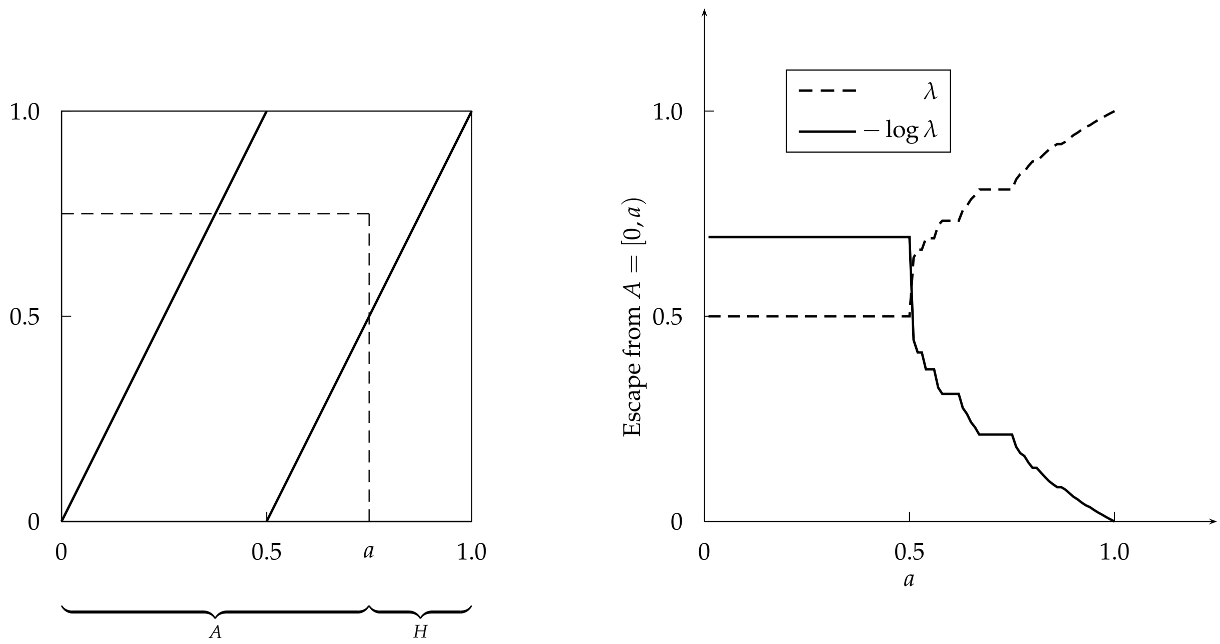
(a) Graph of T (b) Survivor sets

Figure 1.2: Tent map with a hole.

Let us calculate the escape rate of the Lebesgue measure from A. Since at each step a proportion ϵ falls into the hole, it is easy to see that $\ell(A^n) = (1-\epsilon)^n$. Hence

$$\begin{aligned} E(A;\ell) &= -\lim_{n\to\infty} \frac{1}{n} \log \ell(A^n) \\ &= -\lim_{n\to\infty} \frac{1}{n} \log(1-\epsilon)^n \\ &= -\log(1-\epsilon). \end{aligned}$$

Example 1.22 (Doubling map with a varying hole)**.** Let $T : \mathrm{I} \circlearrowleft$ be the usual doubling map, given by $T(x) := 2x \pmod 1$. For $a \in (0,1]$ let $A := [0,a)$ and $H := [a,1]$, as shown in Figure 1.3(a).



(a) Graph of T. (b) Escape from A as a function of a.

Figure 1.3: Doubling map with a hole.

First, we will consider the case of $a = 3/4$. The first three n-step survivors are

$$\begin{aligned} A &= [0,3/4) \\ A^1 &= T^{-1}[0,3/4) \cap [0,3/4) = [0,3/8) \cup [4/8,6/8) \end{aligned}$$

$$A^2 = T^{-1}([0,3/8) \cup [4/8,6/8)) \cap [0,3/4) = [0,3/16) \cup [4/16,6/16) \cup [8/16,12/16).$$

Since it is difficult to see any pattern here, we resort to a coding approach. In Example 1.11 we stated that the one-sided full two-shift (Σ_M, σ) with its Markov measure $\mu_{(p,P)}$ is metrically isomorphic to (T, ℓ) with isomorphism $\pi : \mathrm{I} \to \Sigma_M$. The hole $H = [3/4, 1]$ corresponds to the cylinder $H_* := [11]_0$ as $H = \pi^{-1} H_*$. Let A_* be the complement of the hole H_*. The first few n-step survivors under σ are:

$$\begin{aligned}
A_*^0 &= \{[00]_0 \cup [01]_0 \cup [10]_0\}, \\
A_*^1 &= \{[000]_0 \cup [001]_0 \cup [010]_0 \cup [100]_0 \cup [101]_0\} \\
A_*^2 &= \{[0000]_0 \cup [0001]_0 \cup [0010]_0 \cup [0100]_0 \cup [0101]_0 \cup [1000]_0 \cup [1001]_0 \cup [1010]_0\}.
\end{aligned}$$

The pattern is now much more obvious: A_*^n is the union of all cylinders of length $(n+2)$ at position 0 that do not contain the block $[11]$. The recursive formula for the number of $(n+2)$-cylinders in each A_*^n is

$$\#A_*^n = \#A_*^{n-1} + \#A_*^{n-2},$$

hence $\#A_*^n$ are the Fibonacci numbers $\{3, 5, 8, \dots\}$ and $\#A_*^n$ is approximated[5] by ϕ^n for large n, where $\phi = \frac{\sqrt{5}+1}{2}$ is the golden ratio. We can then calculate the escape rate from A_* under the action of the left shift:

$$\begin{aligned}
E(A_*; \mu_{(p,P)}, \sigma) &= -\lim_{n\to\infty} \frac{1}{n} \log \mu_{(p,P)}(A_*^n) \\
&= -\lim_{n\to\infty} \frac{1}{n} \log \left(2^{-(n+2)} \#A_*^n\right) \\
&= -\lim_{n\to\infty} \frac{1}{n} \log \left(2^{-(n+2)} \phi^n\right) \\
&= -\log(\phi/2) \\
&= \log 2 - \log \phi, \qquad (1.14)
\end{aligned}$$

but by Proposition 1.20 this equals $E(A; \ell, T)$. We should mention that this result is

[5] The ratio $\#A_*^n/\phi^n$ approaches a constant as $n \to \infty$ and in particular $\lim(1/n)\log(\#A_*^n/\phi^n) = 0$.

hardly new or surprising. The hole corresponds to $[11]$, which is the forbidden block of the golden mean shift $\mathcal{A}^\infty = \Sigma_{\{[11]\}}$. The topological entropies of the full 2-shift and the golden-mean shift are $\log 2$ and $\log\phi$ respectively. It is well-known that for shifts of finite type, escape rate corresponds to the difference in topological entropies hence (1.14).

A similar[6] method may be applied to $a = 2/3$ to show that the corresponding escape rate also equals $-\log(\phi/2)$, thus, as escape is monotone (Proposition 1.14 (i)) it must be constant for all $a \in [2/3, 3/4]$. More methods and results regarding escape from dyadic intervals of the doubling map can be found in [21].

For a whole range of values of a we approximate escape from $[0, a)$ by using Ulam's method (for open systems outlined in Section 1.2.7) and plot the results in Figure 1.3(b). Note the "devil's staircase" structure, and that the plot verifies our claim that escape is constant for $a \in [2/3, 3/4]$. It is also an interesting fact that the mapping $a \mapsto E([0, a))$ is continuous on all of (0,1] and differentiable at $a = 1$, with derivative $-1/2$ (cf. [21, 75]).

1.2.3 Conditionally Invariant Measures

As we mentioned previously, *invariant measures* are important objects in the study of closed dynamical systems. Their open-system-analogues are *conditionally invariant measures* — measures that are invariant under the condition of non-escape. We describe these formally below.

Definition 1.23. Consider a dynamical system $(X, \mathcal{B}, m, T)$ and a measurable domain $A \subseteq X$. A measure μ_A on X is said to be *conditionally invariant* with respect to A if

$$\mu_A(B)\mu_A(T^{-1}A \cap A) = \mu_A(T^{-1}B \cap A), \quad \text{for all } B \in \mathcal{B}, \tag{1.15}$$

or, perhaps more intuitively[7]

$$\mu_A(T_A^{-1}B) = \mu_A(T_A^{-1}A)\mu_A(B), \quad \text{for all } B \in \mathcal{B}. \tag{1.16}$$

[6] But not identical; since $H = [2/3, 1]$ is not a Markov hole the calculation is a little more involved.
[7] Compare (1.16) with definition of *conditional probability*.

Substituting $B = A$ into either equation we see that $\mu_A(A) = 1$ so μ_A is necessarily a probability on A. Since we do not care what the value of μ_A is on H, to avoid ambiguity, we shall generally assume that μ_A is supported on A. Note that if $A = X$ then (1.15) becomes

$$\mu_A(B) = \mu_A(T^{-1}(B)), \quad \text{for all } B \in \mathcal{B}$$

hence $\mu_A = \mu$ is invariant. Also any invariant measure on an invariant subset of the survivor set is trivially conditionally invariant. Both of these facts suggest that conditionally invariant measures in open systems are a natural generalisation of invariant measures in closed systems.

Observe that $\mu_A(A^n) = (\mu_A(A^1))^n$, therefore escape rate with respect to a conditionally invariant measure is trivial to compute:

$$E(A;\mu_A) = -\lim \frac{1}{n} \log(\mu_A(A^1))^n = -\log \mu_A(A^1).$$

Equation (1.16) may also be written more concisely as $\mu_A \circ T_A^{-1} = \lambda \mu_A$ where $\lambda := \mu_A(A^1)$. For this reason[8], λ is sometimes called the *eigenvalue*[9] of the conditionally invariant measure μ_A. Since it is often more convenient, we may work with the eigenvalue $\lambda = \lambda(A) \in [0,1]$, rather than with the escape rate $E(A) = -\log \lambda(A) \in [0,\infty]$.

Example 1.24. Let us return to the tent map with a hole from Example 1.21. It is easy to see that setting μ_A to be the Lebesgue measure on $[0,1]$ will satisfy (1.15).

1.2.4 Conditional Perron-Frobenius Operator

Definition 1.25. Let $(X, \mathcal{B}, m, T)$ be a closed dynamical system with Perron-Frobenius operator $\mathcal{P}$, and let $A \subseteq X$ be measurable. The *conditional Perron-Frobenius operator* with respect to A, denoted by $\mathcal{P}_A$, is defined for any $f \in L^1(X, \mathcal{B}, m)$ by

$$\mathcal{P}_A(f) = \chi_A \mathcal{P}(\chi_A \cdot f).$$

[8]And its connection to the Perron-Frobenius operator, which will be described later.

[9]Other proposals have been to call λ the *geometric rate of escape* or *retention ratio*.

We have already noted that in a closed system, stationary points of the Perron-Frobenius operator are densities corresponding to absolutely continuous (with respect to m) invariant measures. Analogously, it is a well-known fact that, in an open system, positive eigenfunctions of the conditional Perron-Frobenius operator are densities of the absolutely continuous conditionally invariant measures (ACCIM) [95]. For completeness we formalise and prove this below.

Proposition 1.26. *Let A be a measurable subdomain of a dynamical system $(X, \mathcal{B}, m, T)$ and let $\mathcal{P}_A : L^1(X, \mathcal{B}, m) \circlearrowleft$ be the corresponding conditional Perron-Frobenius operator. Suppose that a non-negative function $f \in L^1(X, \mathcal{B}, m)$ satisfies $\int_A f \, \mathrm{d}m = 1$ and $\mathcal{P}_A f = \lambda f$ for some $\lambda \in (0, 1]$. Then the measure with density f, $\mu_A := f \cdot m$, is conditionally invariant with eigenvalue λ.*

Proof. Since $\mathcal{P}_A f = \lambda f$ we have for any set $B \in \mathcal{B}$

$$\begin{aligned}
\lambda \mu_A(B) &= \lambda \int_B f \, \mathrm{d}m \\
&= \int_B \mathcal{P}_A f \, \mathrm{d}m \\
&= \int_B \mathcal{P}(\chi_A \cdot f) \, \mathrm{d}m \\
&= \int_{T^{-1}B \cap A} f \, \mathrm{d}m \\
&= \mu(T^{-1}B \cap A).
\end{aligned}$$

Substituting $B = A$ into the expression above, we see that $\lambda = \mu(A^1)$. Hence μ satisfies $\mu(A^1)\mu(B) = \mu(T^{-1}B \cap A)$ and is therefore conditionally invariant. □

Definition 1.27. Often it is useful to define the *normalised conditional operator*, $\hat{\mathcal{P}}_A$, by

$$\hat{\mathcal{P}}_A f = \frac{\mathcal{P}_A f}{\|\mathcal{P}_A f\|_{L^1}}, \quad f \in L^1(X, \mathcal{B}, m). \tag{1.17}$$

Nonnegative fixed points of $\hat{\mathcal{P}}_A$ are densities of absolutely continuous conditionally invariant measures.

Example 1.28. Consider the doubling map with a hole from Example 1.22. The conditional Perron-Frobenius operator $\mathcal{P}_A$ is given by

$$\mathcal{P}_A f(x) = \begin{cases} \frac{1}{2} f\left(\frac{x}{2}\right) + \frac{1}{2} f\left(\frac{x+1}{2}\right), & 0 \leq x < 2a-1 \\ \frac{1}{2} f\left(\frac{x}{2}\right), & 2a-1 \leq x < a \\ 0, & \text{otherwise.} \end{cases}$$

When $a = 3/4$ we find that $\mathcal{P}_A f = \lambda f$ is satisfied by the following density

$$f(x) = \begin{cases} \frac{1}{\phi^2}, & 0 \leq x \leq 1/2 \\ \frac{1}{\phi^3}, & 1/2 < x \leq 3/4 \\ 0, & \text{otherwise,} \end{cases}$$

with $\lambda = \phi/2$. Therefore, by Proposition 1.26, the measure $\mu_A = f \cdot \ell$ is conditionally invariant.

1.2.5 Existence and Uniqueness of Conditionally Invariant Measures

An important question in the study of open dynamical systems is whether there exists a conditionally invariant measure. Sufficient conditions for an open dynamical system were given by Collet *et al.* [28, 29].

As is the case of invariant measures in closed systems, conditionally invariant measures in open systems are rarely physically relevant and most do not provide useful information about the underlying dynamics. The paper of Demers and Young [44] gives insightful discussions on when a conditionally invariant measure is *natural* or *physically relevant*. Absolute continuity (usually with respect to Lebesgue or SRB) is not enough, and the authors demonstrate this by constructing uncountably many absolutely continuous conditionally invariant measures (ACCIM) in a fairly general setting. They conclude that a *natural conditionally invariant measure would be one that is absolutely continuous and whose density f is a limit point of* $\mathcal{P}_A^n 1$.

Even though ACCIMs are rarely unique, it is often the case that when restricted to densities in a set of sufficiently regular functions essentially bounded away from zero and infinity, a unique ACCIM exists. This is a desirable case as, by Proposition 1.19, the corresponding eigenvalue determines the Lebesgue escape rate. The standard approach here is to look for fixed points of the normalised conditional operator and often an open-systems-equivalent of a Lasota-Yorke inequality needs to be satisfied[10]. First results concerning ACCIMs were proved in Pianigiani and Yorke [95] for open expanding maps on $\mathbb{R}^n$ that possess a finite Markov partition[11], followed by results of Collet *et al.* [30, 31]. More recently, many authors have studied ACCIMs of open interval maps with non-Markov holes, often with the condition that the hole is sufficiently small; see for example [20, 42, 43, 86, 107]. Other similar work has been done in the settings of Anosov diffeomorphisms, [23–26], open billiards [88], Markov chains [50, 108] and topological Markov chains [32].

1.2.6 More Examples and Useful Results

Absorbing-state Markov Chains

Recall that a state i of a Markov chain is said to be *absorbing* if its transitional probability $p_{ii} = 1$. That is, once the state i is entered the process remains in this state forever. This closely resembles the behaviour of an open system, where once an orbit enters the hole, it is terminated. It is thus not surprising that the idea of conditionally invariant measures was originally borrowed from the area of absorbing-state Markov chains first studied by Vere-Jones [108].

Repellers and Thermodynamic Formalism

As we mentioned earlier, survivor sets are also called repellers. A *repeller* is defined to be a compact, T-invariant set K that has an open neighbourhood U so that

$$K = \{x \in U \ : \ T^n(x) \in U, \, \forall n \geq 0\}. \tag{1.18}$$

[10] Unlike closed dynamical systems, proving a Lasota-Yorke inequality for the conditional Perron-Frobenius operator does not automatically imply the existence of an ACCIM for the underlying open system.

[11] If the corresponding system without a hole possesses a Markov partition, and the hole is *Markov* (a union of elements of the partition), then the system with hole will satisfy this property.

The *escape rate* from a repeller is assumed to be the escape rate from any neighbourhood U satisfying (1.18).

In particular, any hyperbolic fixed point of a dynamical systems is a repeller, but more interestingly, many dynamical systems possess so called *strange repellers* which are (generalised) Cantor sets. In his book, Falconer [48] refers to repellers of expanding interval maps T as "cookie-cutter sets" and notes that these sets arise as attractors of related iterated function schemes, which may be viewed as inverse branches of T.

Recall that the *topological pressure* of a potential φ is defined as

$$P(T,\varphi) := \sup\left\{h_\nu(T) + \int \varphi \, \mathrm{d}\nu \;:\; \nu \text{ is a } T\text{-invariant probability}\right\}, \tag{1.19}$$

where $h_\nu(T)$ is the Kolmogorov-Sinai entropy of T (see e.g. [109] for the definitions). Any measure that achieves this supremum is called an *equilibrium state*. Note that an equilibrium state for $\varphi = 0$ is a measure of maximal entropy and the pressure coincides with the topological entropy of the system.

For T that is uniformly expanding and C^2 on a repeller K in a smooth Riemannian manifold, we have the *escape rate formula*:

$$E(K;\ell,T) = -P(T|_K, -\log|JT|) \tag{1.20}$$

$$= -\sup\left\{h_\mu(T) - \int \log|JT| \, \mathrm{d}\mu\right\}, \tag{1.21}$$

where the supremum is taken over all T-invariant Borel probability measures μ on K [16, 18, 30]. The result also holds for uniformly hyperbolic repellers in which case the potential $-\log|JT|$ is additionally restricted to unstable manifolds[12] [22, 24, 25, 88, 110]. Non-uniformly hyperbolic counterexamples of Young [110] and Baladi *et al.* [8], however, show that more generally (1.20) is not true and only an upper bound on escape rate remains.

There also exists an interesting relation between topological pressure and dimension, known as the *Bowen-Ruelle formula* [17]. More precisely, the Hausdorff dimension of an

[12] The formula in (1.21) may be viewed as a generalisation to Pesin's Entropy Formula [93], which states that in certain closed systems Kolmogorov-Sinai entropy equals the sum of positive Lyapunov exponents.

expanding repeller is the unique $s \geq 0$ such that

$$P(T|_K, -s \log |JT|) = 0. \tag{1.22}$$

Let us return to Example 1.21 with $\epsilon = 1/3$. Here K is the middle-thirds Cantor set and $JT = 3$ everywhere. Hence (1.22) becomes

$$\sup \{h_\nu(T_K) - s \log 3\} = 0.$$

Now on K, T is topologically conjugate to the full 2-shift and has topological entropy equal to $\log 2$. Hence the Hausdorff dimension of the Cantor set is $s = \log 2 / \log 3$.

Shifts of Finite Type

In our final chapter we will investigate topological entropy of shifts of finite type. As suggested by Example 1.22, when we introduce a cylindrical hole $C_j(b)$ to a shift of finite type $\Sigma_{\mathbb{F}}$, the resulting survivor set is a subshift of $\Sigma_{\mathbb{F}}$ with collection of forbidden blocks $\mathbb{F} \cup \{b\}$. The escape rate into the hole (of the measure of maximal entropy) may the seen as the loss in topological entropy, that is

$$E(X_{\mathbb{F}} \setminus C_j(b)) = h_{top}(X_{\mathbb{F}}) - h_{top}(X_{\mathbb{F} \cup \{b\}}). \tag{1.23}$$

Similarly, this formula holds when the hole is a union of cylinders. This is well-known and (1.23) may be regarded as a special case of (1.21). Relevant studies include the paper of Lind [84] where loss in topological entropy is investigated for small perturbations, work of Collet *et al.* [32] regarding escape into cylindrical holes and more recently work of Ferguson and Pollicott [49] where the authors generalise some of the results of [84] and [21]. In this light, the problem of maximising topological entropy of subshifts of finite type is equivalent to the problem of minimizing escape rate into holes that are unions of cylinders. We will discuss this idea further in Chapter 5.

A Sequence of Shrinking Holes

Consider a closed dynamical system with a sequence of holes H_n, where $m(H_n) \to 0$. Suppose that for every H_n a unique natural ACCIM μ_n exists. Do these converge to the natural ACIM of the closed system as $n \to \infty$? This is an important question concerning whether an open system with a small hole may be viewed as a perturbation of the corresponding closed system. Demers [42, 43], using Young towers [111], shows that the answer is affirmative in the setting of uniformly expanding interval maps and certain logistic maps with holes. In Chapter 3 will use similar techniques to prove such convergence for Pomeau-Manneville maps [97] with holes.

Another related problem is in regard to the behaviour of escape rate as the hole closes. For maps that are uniformly expanding, it is easy to show that[13] $E(X \setminus H_n) \sim m(H_n)$ as $m(H_n) \to 0$. Bunimovich and Yurchenko [21] study the doubling map with the Lebesgue measure and consider Markov holes, H_n of length 2^{-n}. By studying the isomorphic full 2-shift with the Bernoulli measure, they compute the asymptotics of $E(A \setminus H_n)$ as $n \to \infty$ to first order and relate them to the period of the "infinitesimal hole" — the unique point contained in every H_n. Subsequently these results have been generalised to escape from shifts other than the full shift in [49]. Keller and Liverani [75] also study the escape rates of systems with small holes as an application of an abstract perturbation result. They consider Lasota-Yorke maps with possibly countably many branches and a family of compact interval holes shrinking to a point z as $\epsilon \to 0$. All three papers provide formulae for the limiting ratio of escape rate to the size of the hole, dependent on the periodicity of the infinitesimal hole. In Chapter 3 we will consider a similar problem of determining the asymptotics of escape rate with size of the hole for non-uniformly expanding interval maps.

[13] Throughout the thesis we will use the following (standard) Big-O notation: $f(x) = \mathcal{O}(g(x))$ as $x \to a$ if and only if there exist positive real numbers M and δ such that $|f(x)| \leq M|g(x)|$ whenever $|x - a| < \delta$ (if $a = \infty$ then we replace "whenever $|x - a| < \delta$" by "for all sufficiently large x"). If both $f(x) = \mathcal{O}(g(x))$ and $g(x) = \mathcal{O}(f(x))$ as $x \to a$, then we write $f(x) \sim g(x)$ as $x \to a$ (not to be confused with equivalence of measures, where we use the same symbol).

Escape Rate and the Position of the Hole

It is shown in [21] that for the doubling map with a hole the escape rate is related to the first return time of a positive measure subset of the hole: longer return time to the hole implies faster escape rate into the hole. More precisely, for times longer than the return time, longer return time to the hole implies smaller survivor sets. Unfortunately, the proofs rely heavily on combinatorial arguments based upon the full 2-shift (or N-shift) structure, and thus are specific to the doubling map and systems metrically conjugate to the doubling map. Even for reasonably simple systems such as piecewise affine expanding Markov maps, similar results are not known.

In Chapter 2 we will consider a related problem of minimising escape rate while keeping constraints on the measure of the hole.

Using Open Systems to Model Closed Systems

In this thesis, one of the main ideas is that a metastable system may be viewed as a combination of two or more open systems with low escape rate. We describe two recent papers which explore similar concepts.

Tokman *et al.* [105] study piecewise smooth maps of the interval that possess two invariant subintervals of positive Lebesgue measure and exactly two ergodic ACIPMs. The pre-image of the boundary is called the *infinitesimal hole* and both ACIPMs are required to be positive on it. They perturb such maps slightly to destroy the two invariant subsets and show that the (now unique) ACIPM may be approximated by a convex combination of the two initial ergodic ACIPMs. The perturbation needs to be such that, regarding the dynamics on each previously invariant subinterval as open, no holes are created near the boundary point. The authors show that the unique ACIPM may be approximated as a convex combination of the two ACIPMs of the unperturbed system, where the weights are determined by the limiting ratio of escape rates of the corresponding open systems in presence of the perturbation.

Góra *et al.* [69] generalise results of [105] to higher dimensions. Their approach is to start with two maps: one that preserves two or more disjoint invariant sets in $\mathbb{R}^n$ and the other which does not. Based on these, they define a collection of random maps which model the perturbation.

1.2.7 Ulam's Method in Open Dynamical Systems

Accurate numerical approximation of escape rate by "brute force" attempts to evaluate the limit in (1.11) or (1.12) is generally extremely difficult. A reason for this is because the limiting set A^∞ is often fractal and errors propagate in calculations of the measure of the sets A^n. A more effective approach is to approximate a conditionally invariant measure whose eigenvalue determines the escape rate. If we know (or assume) that the density of this ACCIM is bounded away from zero and infinity, the approximation of its corresponding eigenvalue gives us the Lebesgue escape rate.

Most work in the topic of rigorous numerical approximation of escape rates has been by Bahsoun *et al.* [4–6], who provide algorithms that use a modified version of Ulam's method [106] to perform necessary computations.

Given an open dynamical system $T: A \to X$, for a finite partition η of X we will require that either $H \in \eta$ or H is a union of sets in η. Ulam's approximation of the conditional Perron-Frobenius operator $\mathcal{P}_A$ is given by

$$\mathcal{P}_{A,\eta} = \Pi_\eta \circ \mathcal{P}_A \circ \Pi_\eta,$$

where as in (1.8), Π_η is the projection onto the space spanned by characteristic functions of sets in η. The corresponding matrix representation with respect to the normalised basis of characteristic functions is $P_{A,\eta}$ where

$$(P_{A,\eta})_{ij} = \begin{cases} (P_\eta)_{ij}, & B_i, B_j \nsubseteq H \\ 0, & \text{otherwise.} \end{cases}$$

Any non-negative left eigenvector of $P_{A,\eta}$ and its corresponding eigenvalue will approximate a conditionally invariant density and the escape rate (w.r.t. m), respectively.

Chapter 2

Relating Open and Closed Dynamical Systems

In this chapter we present some results motivated by the problem of identifying regions of slow mixing in closed dynamical systems. A well-known heuristic approach is first to detect eigenfunctions f of the Perron-Frobenius operator that correspond to large real non-unit eigenvalues $\rho \lesssim 1$. This is often indicative of *almost-invariant sets* [36, 39, 89]. One then partitions the domain into two regions A_+ and A_- according to the positive and negative supports of the acquired eigenfunction f.

In our approach we shall use the same algorithm, but consider the two elements A_+ and A_- of the partition as the domains of two disjoint *open* dynamical systems. We will show in Theorem 2.5 that the escape from each set is bounded above by $-\log \rho$.

We will also show that in order to obtain meaningful results, one requires the Perron-Frobenius operator to possess a spectral gap. We apply our results in the setting of Lasota-Yorke maps, where the spectral gap is achieved in the space of functions of bounded variation.

The material in this chapter, apart from the final section on flows, has appeared in [65].

2.1 Almost-invariant Sets

Definition 2.1 (Invariance ratio [36, 39])**.** Let $T : (X, \mathcal{B}, m) \circlearrowleft$ be a measurable non-singular transformation. For a measurable set $A \in \mathcal{B}$ the *invariance ratio* is defined to be

$$\varrho(A; m) := \frac{m(T^{-1}A \cap A)}{m(A)}. \tag{2.1}$$

Almost-invariant or *metastable sets* [39, 54, 56] are sets A for which $\varrho(A)$ is close to 1. That is to say that the probability for a point in A to remain in A after one application of T is close to one. Dynamical systems that are close to non-ergodic typically have a decomposition into non-trivial sets, each of which has a high invariance ratio. The identification of such *almost-invariant* or *metastable* sets is often very difficult; see [62] for a recent computational study. Application areas include molecular dynamics [102] and ocean dynamics [37, 63].

Definition 2.2. Let $(X, \mathcal{B}, m)$ be a measure space. For a function $f \in L^1(m)$ we denote by $\mathrm{supp}(f) := \{x \in X : f(x) \neq 0\}$ the *support* of f.

By $f^+ := \max(f, 0)$ and $f^- := \max(-f, 0)$, we define the positive and negative parts of $f \in L^1(m)$. It has been known for a while that in the presence of a large real second eigenvalue of the Perron-Frobenius operator, the supports of the positive and negative parts of the corresponding eigenfunction are usually almost-invariant. That is, if $\mathcal{P}f = \rho f$ with $\rho \lesssim 1$, then the sets

$$A_+ = \mathrm{supp}(f^+), \quad \text{and} \quad A_- := \mathrm{supp}(f^-) \tag{2.2}$$

may be seen as m-almost invariant. Dellnitz and Junge [39] formalised this claim for a measure $|\nu|$, obtained from a signed measure ν with density f (with respect to m).

In the Lasota-Yorke [83] (or Rychlik [99]) map setting, with $\mathcal{P} : BV \circlearrowleft$, the value τ in (1.4) is intimately connected with the average rate of expansion experienced along orbits. Thus BV spectral points of $\mathcal{P}$ larger than $1/\tau$ in magnitude cannot be explained by local expansion of T and must be due to the influence of global structures such as almost-invariant and metastable sets, producing decay rates *slower than the average local expansion rate*.

Almost-invariant sets have formally been associated with isolated spectral points of $\mathcal{P}$ [38]. In such a setting, if the map possesses a Markov partition and one restricts oneself to searching for almost-invariant sets that are unions of Markov sets, then lower and upper bounds for the largest possible almost-invariance ratio are given by the second largest eigenvalue of an associated Markov chain [54].

In our main result of this chapter (Theorem 2.5) we will demonstrate that sets A_+ and A_- constructed in (2.2) also possess a low escape rate (of the measure m). Thus, there is a strong connection between almost-invariant sets and the construction we have used to define our slow escape sets A_+ and A_-. One might therefore naively expect that sets with low escape rate should have a high invariance ratio and vice-versa. However, escape rate is an asymptotic quantity, while almost-invariance measures exchange under just one iteration of a map. We give examples below to demonstrate that a set may simultaneously have (i) high almost-invariance and high escape rate and (ii) low almost-invariance and low escape rate.

Example 2.3 (High almost-invariance, infinite escape rate)**.** Let $T\colon S^1 \circlearrowleft$ be the irrational rotation of the circle, $T(x) := x + 2\pi\alpha$ where $\alpha \neq 0$ is small. Let $A = [0, \pi/2]$. The pre-image of A is given by $T^{-1}A = [2\pi\alpha, \pi/2 + 2\pi\alpha]$. Thus the invariance ratio of A with respect to Haar measure on the circle is $(\pi/2 - 2\pi\alpha)/(\pi/2) \approx 1$. However, for $1 < 4n\alpha < 3$ we have $T^{-n}A \cap A = \emptyset$, therefore escape rate from A with respect to any measure is infinite; see Figure 2.1.

Example 2.4 (Low almost-invariance, arbitrarily low escape rate)**.** Let $T\colon \mathrm{I} \circlearrowleft$ be defined as follows:

$$T(x) = \begin{cases} (1+\epsilon)x, & 0 \leq x \leq 1/4; \\ 2x \text{ (mod 1)}, & 1/4 < x \leq 1. \end{cases}$$

Let $A = [0, 1/2]$. The invariance ratio of A with respect to the Lebesgue measure equals to $1/2$. However its escape rate is $\log(1+\epsilon) \approx 0$; see Figure 2.2.

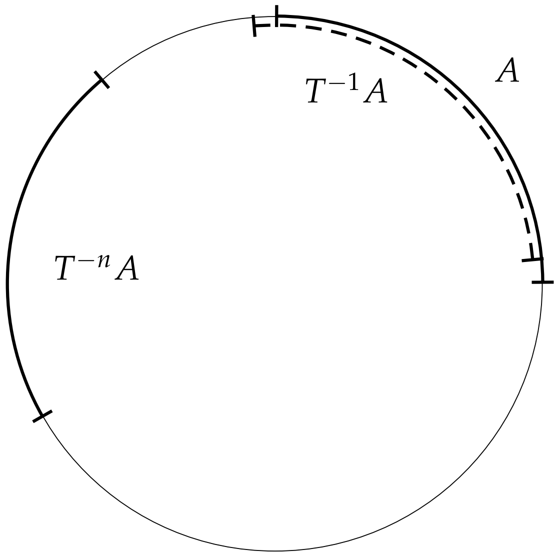


Figure 2.1: Illustration of Example 2.3 with $\alpha = 1/72$ and $n = 26$.

2.2 Escape Rates and the Perron-Frobenius Spectrum

Here we state and prove the main result of this chapter, which provides a bound on escape rate from sets in (2.2) by relating the eigenvalue ρ of the Perron-Frobenius operator $\mathcal{P}$ to the largest eigenvalues of the conditional operators $\mathcal{P}_{A_+}$ and $\mathcal{P}_{A_-}$.

Theorem 2.5. *Let $T : X \circlearrowleft$ be a non-singular transformation on the finite measure space $(X, \mathcal{B}, m)$ and let $\mathcal{P} \colon L^1(X, \mathcal{B}, m) \circlearrowleft$ be the corresponding Perron-Frobenius operator. Suppose that $\mathcal{P}$ has a real positive eigenvalue $0 < \rho < 1$, with corresponding bounded eigenfunction $-\infty < f < \infty$. Define the measurable sets $A_+, A_- \subset X$ by*

$$A_+ := \operatorname{supp}(f^+) \quad \textit{and} \quad A_- := \operatorname{supp}(f^-).$$

Then one has $\overline{E}(A_+; m) \leq -\log \rho$ and $\overline{E}(A_-; m) \leq -\log \rho$.

Proof. Define a finite measure ν on X by

$$\nu(B) := \int_B |f| \, \mathrm{d}m, \quad B \in \mathcal{B}.$$

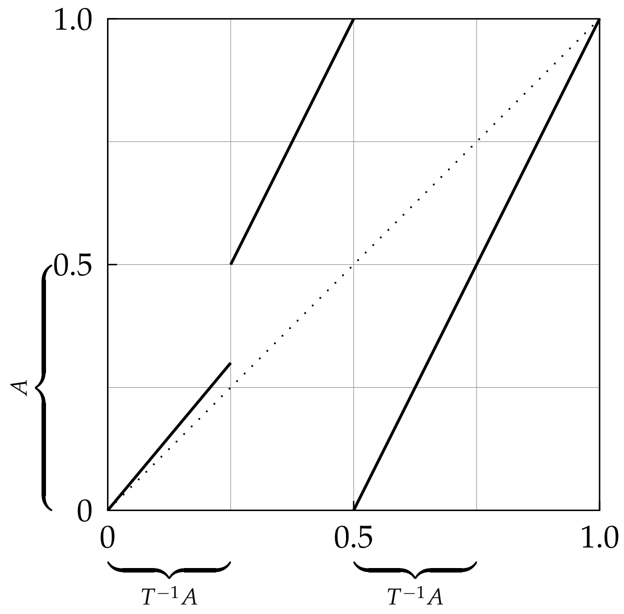


Figure 2.2: Graph of T in Example 2.4, with $\epsilon = 0.2$.

Now note that for all $n \geq 0$ we have $f > 0$ on A_+^n. Also $A_+^{n+1} = T^{-1}A_+^n \cap A_+$, therefore

$$\begin{aligned}
\rho\nu(A_+^n) &= \rho \int_{A_+^n} f \,\mathrm{d}m \\
&= \int_{A_+^n} \mathcal{P}f \,\mathrm{d}m \\
&= \int_{T^{-1}A_+^n} f \,\mathrm{d}m \\
&= \int_{T^{-1}A_+^n \cap A_+} f \,\mathrm{d}m + \int_{T^{-1}A_+^n \cap (X \setminus A_+)} f \,\mathrm{d}m \\
&\leq \int_{T^{-1}A_+^n \cap A_+} f \,\mathrm{d}m \\
&= \nu(A_+^{n+1}),
\end{aligned}$$

where the inequality above is due to $f \leq 0$ on $X \setminus A_+$. Since $\nu(A_+^{n+1}) \geq \rho\nu(A_+^n)$, by (i) of Proposition 1.15 we have $\overline{E}(A_+;\nu) \leq -\log\rho$. It remains to show that $\overline{E}(A_+;m) \leq \overline{E}(A_+;\nu)$. Since $f \leq C$ for some constant $C > 0$, we have $\nu(A_+^n) \leq Cm(A_+^n)$ for all $n \geq 0$.

This gives

$$\begin{aligned}\overline{E}(A_+;\nu) &= -\liminf_{n\to\infty}\frac{1}{n}\log(\nu(A_+^n))\\ &\geq -\liminf_{n\to\infty}\frac{1}{n}\log(Cm(A_+^n))\\ &= \overline{E}(A_+;m).\end{aligned}$$

Thus

$$\overline{E}(A_+;m)\leq\overline{E}(A_+;\nu)\leq-\log\rho$$

The inequality for A_- is obtained by considering $-f$ in place of f and following the same procedure. □

Remark 2.6. At the time of writing this thesis, it was pointed out to me that Lawler and Sokal used a similar approach in [81, Lemma 3.4]. The setting of [81] is reversible Markov processes with killing and the authors relate the spectrum of a self-adjoint L^2 operator describing a Markov process to the spectral radii of two operators associated with the processes with killing. Our results do not assume reversibility and apply in a Banach space, thus are more general.

Corollary 2.7. *Let $T:(X,\mathcal{B},m)\circlearrowleft$ be non-singular with Perron-Frobenius operator $\mathcal{P}: L^1(X,\mathcal{B},m)\circlearrowleft$ that admits a positive real eigenvalue $0<\rho<1$. Then*

$$\inf_{A\in\mathcal{B}}\max\{E(A;m),E(X\setminus A;m)\}\leq-\log\rho. \tag{2.3}$$

Remark 2.8. If one wishes to create a 2-partition of X such that each element of the partition has upper escape rate lower than $-\log\rho$, then the set $\{f=0\}$ may be absorbed into either A_+ or A_-. Enlarging A_+ does not increase $\overline{E}(A_+)$ so Theorem 2.5 also holds for $A_\oplus := X\setminus A_-$ and $A_\ominus := X\setminus A_+$. The desired 2-partition is then $\{A_+,A_\ominus\}$ or $\{A_\oplus,A_-\}$ (or any other redistribution of $\{f=0\}$ among the two sets).

Remark 2.9. By Proposition 1.14 (ii), we may replace T_A with T_{A^1} and obtain an open system with an identical escape rate. We may think of T_{A^1} as an open system on A with hole $A\setminus T^{-1}A$. Consider now our partition $\{A_+,A_\ominus\}$ of X formed from the positive and non-positive parts of some $f \in L^1$ satisfying $\mathcal{P}f = \rho f$, $0 < \rho < 1$. By the above remarks, the open system T_{A_+} has the same escape rate as the open system $T_{A^1_+}$, where the hole for the latter system is $A_+ \setminus T^{-1}A_+ = A_+ \cap T^{-1}A_\ominus \subset A_+$. Thus, while the hole $H = A_\ominus$ for the open system T_{A_+} is very large in measure, we may easily construct another system $T_{A^1_+}$ with the same escape rate, but a hole $H = A_+ \cap T^{-1}A_\ominus$ that is likely to be much smaller in terms of m. Similarly, we may define an open system $T_{A^1_\ominus}$, with hole $A_\ominus \setminus T^{-1}A_\ominus = A_\ominus \cap T^{-1}A_+ \subset A_\ominus$; this open system has the same escape rate as $T_{A_\ominus}$.

2.3 Spectrum of $\mathcal{P}$ in L^1

As a motivation for this section we begin with a result of Ding *et al.* [45].

Theorem 2.10 (Corollary 3.2 [45])**.** *Let $(X, \mathcal{B}, m)$ be a σ-finite measure space and $T\colon X \circlearrowleft$ be a non-singular transformation, whose Perron-Frobenius operator $\mathcal{P}$ has a positive fixed density. If $0 \in \mathrm{sp}(\mathcal{P})$, then $\mathrm{sp}(\mathcal{P}) = \{z \in \mathbb{C} \,:\, |z| \le 1\}$.*

Consider now a smooth Riemannian manifold X with measure m and $T : X \circlearrowleft$ differentiable almost everywhere so that its Perron-Frobenius operator $\mathcal{P}$, given by (1.2), has a fixed ACIM. The following lemma, more specifically than Theorem 2.10, states that if 0 is an eigenvalue of $\mathcal{P}$, then every point in the open unit disk is also an eigenvalue.

Lemma 2.11. *Let $\mathcal{P}$ be as above and suppose that there exists $h \in L^\infty(m)$ such that $h > 0$ and $\mathcal{P}h = h$. Suppose also there is a nonzero $\hat{f} \in L^\infty(m)$ satisfying $\mathcal{P}\hat{f} = 0$. Every $\rho \in \mathbb{C}$ such that $|\rho| < 1$ is an eigenvalue of $\mathcal{P}$ with corresponding eigenfunction $f \in L^\infty(m)$.*

Proof. This proof appears in a slightly different context in the proof of [7, Theorem 1.5 (7)]. If $\rho = 0$ we are done. Let $\rho \neq 0$. Then $f := \sum_{n=0}^\infty \rho^n (\hat{f}/h) \circ T^n \cdot h$ is an eigenfunction with eigenvalue ρ. To see this, we note that $f \in L^\infty$ and compute

$$
\begin{aligned}
\mathcal{P}f(x) &= \sum_{y \in T^{-1}x} \sum_{n=0}^{\infty} \rho^n (\hat{f}/h) \circ T^n(y) \cdot h(y)/|JT(y)| \\
&= \sum_{y \in T^{-1}x} \hat{f}(y)/|JT(y)| + \sum_{y \in T^{-1}x} \sum_{n=1}^{\infty} \rho^n (\hat{f}/h) \circ T^n(y) \cdot h(y)/|JT(y)|
\end{aligned}
$$

$$= 0 + \rho \sum_{y \in T^{-1}x} \sum_{n=0}^{\infty} \rho^n (\hat{f}/h) \circ T^n(x) \cdot h(y)/|JT(y)|$$

$$= \rho \sum_{n=0}^{\infty} \rho^n (\hat{f}/h) \circ T^n(x) \sum_{y \in T^{-1}x} h(y)/|JT(y)|$$

$$= \rho f(x).$$

□

Remark 2.12. A related result of Collet and Isola [27] shows that if T is a piecewise C^∞ expanding Markov map with bounded first and second derivatives, then the spectrum of $\mathcal{P}$, acting on C^0 functions, is the entire unit disk and every spectral point is an eigenvalue of infinite multiplicity.

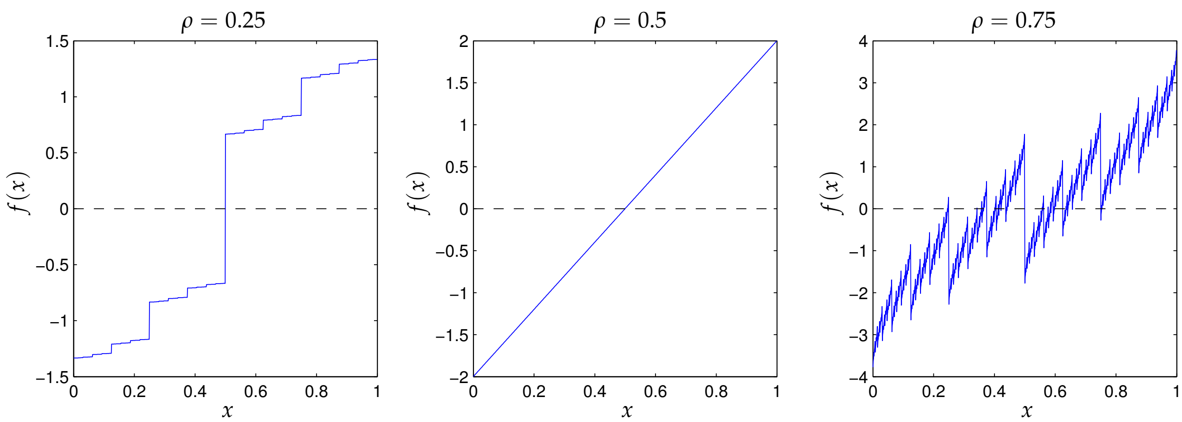


Figure 2.3: Graphs of L^1 eigenfunctions for the doubling map $x \mapsto 2x$ for $\rho = 0.25, 0.5$, and 0.75.

Example 2.13. Figure 2.3 shows three eigenfunctions for the doubling map on $[0, 1]$. We may apply Theorem 2.5 to any one of these eigenfunctions to obtain two open systems, both of which have escape rates slower than $-\log \rho$. In order to be able to apply Lemma 2.11 we note that $\hat{f} = \chi_{[0,1/2]} - \chi_{(1/2,1]}$ satisfies $\mathcal{P}\hat{f} = 0$ and $h \equiv 1$ satisfies $\mathcal{P}h = h$. Each eigenfunction produces a very large hole (of Lebesgue measure $1/2$), and Lemma 2.11 says that one may set ρ as close to unity as one wishes, to obtain very slow escape rates. The penalty that one pays for producing escape rates less than $\log 2$ are sets A_+ and A_- that may be very complicated. We discuss this further in the next section.

2.4 Application to Lasota-Yorke Maps

Let $\mathrm{I} = [0, 1]$ and let ℓ denote the Lebesgue measure on I. Firstly, we shall formally define Lasota-Yorke maps [80].

Definition 2.14 (Lasota-Yorke map [80])**.** A transformation $T\colon \mathrm{I} \circlearrowleft$ is said to be a *Lasota-Yorke map* if the following conditions are satisfied:

(LY1) There exists a finite partition $\{a_0, a_1, \ldots a_n\}$ with $a_0 = 0$ and $a_n = 1$ so that T is monotone and C^2 on the interior of each interval (a_{i-1}, a_i), $i = 1, \ldots, n$.

(LY2) T is uniformly expanding; that is $\tau := \inf |T'| > 1$ where the infimum is taken over all points in $[0, 1]$ for which the derivative exists.

Lasota-Yorke maps were shown [80] to possess absolutely continuous conditionally invariant measures with density of bounded variation. The following lemma states that for interesting Lasota-Yorke maps, zero is in the L^1-spectrum of the Perron-Frobenius operator, thus by Lemma 2.11 we can expect the L^1-spectrum of $\mathcal{P}$ to be the entire unit disk.

Lemma 2.15. *Let T be a Lasota-Yorke map and suppose that there are two monotone branches $T_i := T|_{(a_{i-1},a_i)}$ and $T_j := T|_{(a_{j-1},a_j)}$, $i \neq j$, for which*

$$T_i(a_{i-1}, a_i) \cap T_j(a_{j-1}, a_j) \neq \emptyset.$$

Furthermore, suppose that the distortion estimate

$$\operatorname*{ess\,sup}_{x,y\in \mathrm{I}} \left| \frac{T'(x)}{T'(y)} \right| < C \tag{2.4}$$

holds for some constant $C \in \mathbb{R}$. Then there exists a nonzero $\hat{f} \in L^\infty(\ell)$ such that $\mathcal{P}\hat{f} = 0$.

Proof. We construct a nonzero $\hat{f} \in L^1(\ell)$ with $\mathcal{P}\hat{f} = 0$. As T_i and T_j are monotone and expanding, the set $T_i(a_{i-1}, a_i) \cap T_j(a_{j-1}, a_j)$ is an interval, which we denote (x_1, x_2).

Define $\hat{f}$ by

$$\hat{f}(x) = \begin{cases} 0, & x \in [0,1] \setminus (T_i^{-1}(x_1, x_2) \cup T_j^{-1}(x_1, x_2)) \\ 1, & x \in T_i^{-1}(x_1, x_2) \\ \zeta(x), & x \in T_j^{-1}(x_1, x_2) \end{cases}$$

We now determine the value of $\zeta(x)$ so that $\mathcal{P}\hat{f} = 0$. For $x \in (x_1, x_2)$ we have

$$\begin{aligned} \mathcal{P}\hat{f}(x) &= \sum_{y \in T^{-1}x} \frac{\hat{f}(y)}{|T'(y)|} \\ &= \frac{1}{|T_i'(T_i^{-1}(x))|} + \frac{\zeta(T_j^{-1}(x))}{|T_j'(T_j^{-1}(x))|}. \end{aligned} \tag{2.5}$$

Equating (2.5) with zero and rearranging, we obtain

$$\zeta(T_j^{-1}(x)) = -\frac{|T_j'(T_j^{-1}(x))|}{|T_i'(T_i^{-1}(x))|},$$

therefore $\zeta = -|T_j'/T_i'|$ which, by (2.4) is essentially bounded so $\hat{f} \in L^\infty$. For $x \notin (x_1, x_2)$ clearly $\mathcal{P}\hat{f}(x)$ is also zero by the definition of $\hat{f}$. $\square$

Thus we have shown that the L^1 Perron-Frobenius spectrum for a large class of Lasota-Yorke maps is the whole unit disk and the scenario of Example 2.13 holds.

2.5 Spectrum of $\mathcal{P}$ in BV

Here, we investigate the spectrum of $\mathcal{P}$ in the space of function of bounded variation, and the corresponding implication on the escape rate.

By replacing the Banach space $(L^1(\ell), \|\cdot\|_{L^1})$ with $(\mathrm{BV}, \|\cdot\|_{\mathrm{BV}})$, the space of functions of bounded variation, as per the discussion in Section 1.1.2 the operator $\mathcal{P} : (\mathrm{BV}, \|\cdot\|_{\mathrm{BV}}) \circlearrowleft$ becomes *quasi-compact*. Eigenfunctions of $\mathcal{P}$ that lie in BV give rise to sets $A_\pm$ with a relatively simple structure.

Definition 2.16. Let $\mathcal{I}$ be the family of sets $A \subset \mathrm{I}$ where each $A \in \mathcal{I}$ may be written as a countable union of intervals, including possibly singleton sets of the form $\{x\} = [x, x]$.

Proposition 2.17 (Li and Yorke [83])**.** *If* $f \in \mathrm{BV}$ *then* $\mathrm{supp}(f) \in \mathcal{I}$.

Corollary 2.18. *If* $f \in \mathrm{BV}$ *then* $f^+, f^- \in \mathrm{BV}$*; thus the sets* A_+ *and* A_- *of Theorem 2.5 belong to* $\mathcal{I}$.

Example 2.19. Returning to the doubling map $x \mapsto 2x \pmod 1$, it is well known that the spectrum of $\mathcal{P} : \mathrm{BV} \circlearrowleft$ is contained in $\{|z| \leq 1/2\} \cup \{1\}$. Thus, all BV eigenfunctions corresponding to eigenvalues $0 < \rho < 1$ must in fact have $\rho \leq 1/2 = 1/\tau$. In particular, this excludes the third, more irregular L^1 eigenfunction in Figure 2.3.

Thus, for the doubling map in the BV setting, Theorem 2.5 guarantees the existence of open subsystems defined on reasonably regular domains (in the sense of Definition 2.16) with escape rates less than $\log C$ where $C \geq \tau$; the theorem does not, however, guarantee the existence of open systems on regular domains with escape rates *less than* $\log 2 = \log \tau$. In the following section we shall investigate a map for which Theorem 2.5 does predict open systems on regular domains with escape rates slower than $\log \tau$.

2.6 A Map with Escape Rate Slower than $\log \tau$

In this section we exhibit a map for which we identify two disjoint open subdomains, both of which have an escape rate slower than $\log \tau$. The sets A_+ and A_- constructed in Theorem 2.5 are one good way to define such open systems. Via numerical exploration, we investigate whether there are other decompositions into open systems with even slower escape rates than the decomposition identified by Theorem 2.5.

As an exponential version of (2.3), given a closed system, we propose to maximise the following quantity

$$\psi(A) := \min\{\lambda(A), \lambda(\mathrm{I} \setminus A)\}, \quad A \in \mathcal{I}.$$

Example 2.20. Consider the following piecewise affine map $T\colon \mathrm{I} \circlearrowleft$ [55].

$$T(x) = \begin{cases} 4x, & x \in [0, 1/8); \\ 4x - 1/2, & x \in [1/8, 2/8); \\ 4x - 1, & x \in [2/8, 4/8); \\ 4x - 2, & x \in [4/8, 6/8); \\ 4x - 5/2, & x \in [6/8, 7/8); \\ 4x - 3, & x \in [7/8, 1]. \end{cases}$$

The graph of T is shown in Figure 2.4. The Perron-Frobenius operator of T has an isolated second largest eigenvalue $\rho_2 = 1/2$ with the corresponding eigenfunction $f_2 \in \mathrm{BV}$, shown in Figure 2.5.

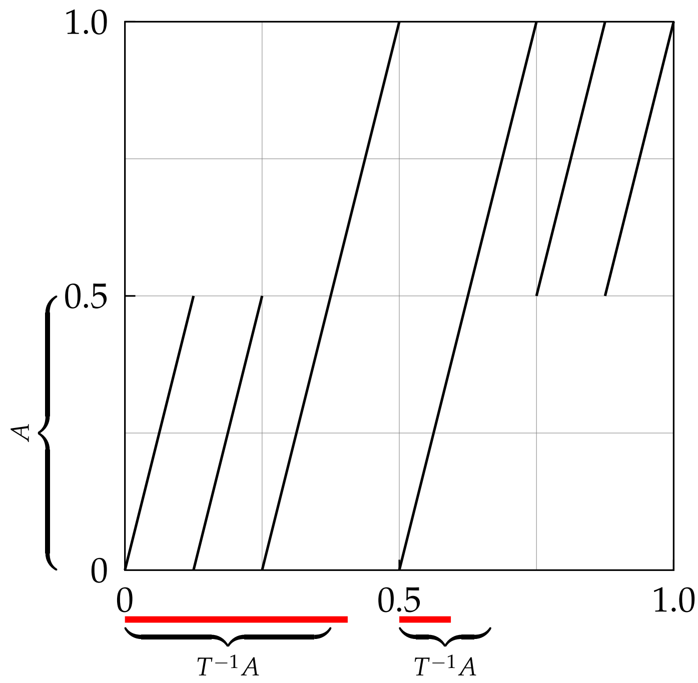


Figure 2.4: Graph of T in Example 2.20. The braces indicate the set $A = [0, 1/2]$ and the two components of its pre-image. The two red lines indicate the components of a set that maximises ψ in the σ-algebra generated by 32 dyadic intervals.

By considering where f_2 is positive and where it is negative, we can partition the

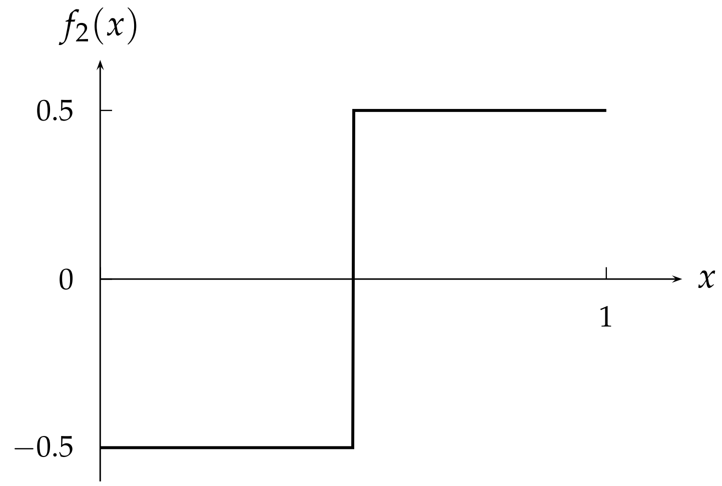


Figure 2.5: Graph of second eigenfunction f_2 of $\mathcal{P}$.

domain of T into two sets, $A_- = [0, 1/2)$ and $A_+ = [1/2, 1]$. The escape rate of both of these sets is much lower than $\log \tau$:

$$E(A_-) = E(A_+) = -\log 3/4 = \log 4/3,$$

compared to $\log \tau = \log 4$, and both satisfy the inequality of Theorem 2.5. Sets with even lower escape rates do exist (for example, if we take $A = [0, 1 - \epsilon]$ for small enough ϵ, then we can make $E(A)$ as close as we like to zero). However it is not immediately obvious that there exists a set $A \in \mathcal{I}$ with $\psi(A) > 3/4$; that is, the escape rate from both A and the complement of A is lower than $-\log 3/4$ (note that the escape rate of $\mathrm{I} \setminus A = \mathrm{I} \setminus [0, 1-\epsilon] = (1-\epsilon, 1]$ is $-\log 1/4$).

Intervals of length $1/2$. First, we will maximise $\psi(A)$ over the class of all intervals of length $1/2$. Let $\mathrm{I}_{\alpha,1/2}$ be an interval of length $1/2$ centered at $x = \alpha \in [1/4, 3/4]$. Figure 2.6 suggests that $\psi(\mathrm{I}_{\alpha,1/2})$ is maximised when $\alpha = 1/4$, that is $\mathrm{I}_{\alpha,1/2} = [0, 1/2]$, coinciding with the set A_- identified by Theorem 2.5.

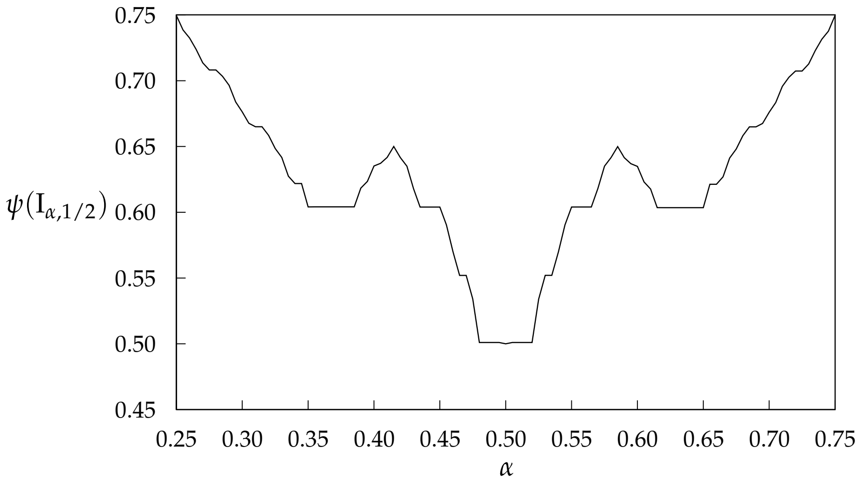


Figure 2.6: Graph of $\psi(\mathrm{I}_{\alpha,1/2})$ where $\mathrm{I}_{\alpha,1/2}$ is an interval of length $1/2$ with varying center point α.

Intervals of varying lengths. We also consider intervals $\mathrm{I}_{\alpha,l}$ with centres and lengths $\alpha, l \in \{i/512\}_{i=0,\ldots,255}$. Again, we found that $\psi(\mathrm{I}_{\alpha,l}) \leq 3/4$ for all α, l considered, with the maximum achieved by $\mathrm{I}_{1/4,1/2}$.

Finite unions of intervals. We may also consider A to be a finite union of elements from an interval partition of I. We maximise $\psi(A)$ over all unions of intervals in the partition $\mathcal{I}_{16} := \{[i/16,(i+1)/16) : i = 0,\ldots,15\}$ and find $\psi(A) \approx 0.799$ for $A = [0,7/16) \cup [1/2,9/16)$. If we repeat on the finer partition $\mathcal{I}_{32} := \{[i/32,(i+1)/32) : i = 0,\ldots,31\}$ we obtain maximal $\psi(A) \approx 0.8198$ for $A = [0,13/32) \cup [16/32,19/32)$. This set is coloured in red in Figure 2.4.

If we allow more complicated sets than those in $\mathcal{I}$, then combining Theorem 2.5, Lemma 2.15, and Lemma 2.11 we see that $\sup\{\psi(A) : A \subset I\} = 1$ as per the discussion in Section 2.3 for the doubling map.

2.7 Related Work

Bunimovich and Yurchenko [21] demonstrate for the doubling map that keeping measure of a hole constant, escape rate is dependent on the position of the hole. More generally, numerical investigations such as Figure 2.6 clearly display the dependence of escape rate on the position of the hole, and support our observation that the holes identified by Theorem 2.5 are positioned so as to form open systems with very low escape rates.

The holes considered in Tokman *et al.* [105] are the holes $A_+ \cap T^{-1}A_\ominus \subset A_+$ and $A_\ominus \cap T^{-1}A_+ \subset A_\ominus$ discussed in Remark 2.9. Our results of Theorem 2.5 may be viewed as generalised converses to [105], who study the particular setting of Lasota-Yorke maps and require very precise knowledge on the initial closed dynamical system. In contrast, we begin with a closed system about which we know very little, apart from the existence of eigenvalues for its Perron-Frobenius operator. From the eigenvalue and eigenfunction information, we are able to *determine* two holes and form two open systems the rate of escape from which is guaranteed to be slower than the rate suggested by the eigenvalue. In general, the identification of such open systems is far from obvious. Our approach may handle very general settings (only non-singularity is required to define the Perron-Frobenius operator), and provides useful information even for macroscopic holes when the closed system may be far from non-ergodic.

2.8 Open Flows

As an aside, in this section we will describe a method to apply Theorem 2.5 to *flows*. Flows are dynamical systems in continuous time. For a reference to relevant definitions and discussions on flows see for example [79, Chapter 7].

Let X be a Hausdorff space with a Borel σ-algebra and a finite measure m. A *flow* $\{\phi_t\}_{t\in\mathbb{R}}$ on $(X, \mathcal{B}, m)$ is a family of mappings $\phi : X \times \mathbb{R} \to X$ satisfying the following properties:

(F1) the map $(x, t) \mapsto \phi(x, t)$ is continuous;

(F2) $\phi(x, 0) = x$ for all $x \in X$;

(F3) $\phi(\phi(x, t), t') = \phi(x, t' + t)$ for all $x \in$ X and $t', t \in \mathbb{R}$.

For notational convenience and to emphasise the similarity to discrete dynamical systems one usually abbreviates $\phi(x, t) = \phi^t(x)$.

The concept of *open flows*, apart from related work on continuous Markov processes with killing (e.g. [81]) and recent work in Froyland *et al.* [57, Section 3.3], is largely an unresearched area of dynamical systems. In contrast to the related concept to entropy, which cannot be completely translated to the setting of continuous time, defining escape rate for flows is both natural and intuitive. As in Definition 1.13, for a measurable set $A \subseteq X$ one defines upper and lower escape rates to be respectively

$$\overline{E}(A; m, \phi) := -\liminf_{t\to\infty} \frac{1}{t} \log m(A^{t,\phi});$$

and

$$\underline{E}(A; m, \phi) := -\limsup_{t\to\infty} \frac{1}{t} \log m(A^{t,\phi}),$$

where

$$A^{t,\phi} := \{x \in X : \phi^s(x) \in A, \forall s \in [0,t]\}.$$

If the upper and lower escape rates coincide, then *escape rate* from A, $E(A)$ exists and equals to either of these.

It is often useful to model a flow as a discrete-time dynamical system. Let $\tau > 0$ and consider $T_\tau := \phi^\tau$ as a time-τ map on X. Observe that for any $0 < \tau \le t \in \mathbb{R}$

$$A^{t,\phi} = \bigcap_{s\in[0,t]} \phi^{-s} A \subseteq \bigcap_{i=0}^{n} T_\tau^{-i} A =: A^{n,T_\tau},$$

where $n := \lfloor t/\tau \rfloor$. Hence $m(A^{t,\phi}) \le m(A^{n,T_\tau})$ and, provided all the limits below exist, we have

$$\begin{aligned} E(A; m, \phi) &= -\lim_{t\to\infty} \frac{1}{t} \log m(A^{t,\phi}) \\ &\ge -\lim_{t\to\infty} \frac{1}{t} \log m(A^{n,T_\tau}) \\ &= -\frac{1}{\tau} \lim_{t\to\infty} \frac{\tau}{t} \log m(A^{n,T_\tau}) \\ &= -\frac{1}{\tau} \lim_{n\to\infty} \frac{1}{n} \log m(A^{n,T_\tau}) \\ &= \frac{1}{\tau} E(A; m, T_\tau). \end{aligned}$$

The difference between the sets $A^{t,\phi}$ and A^{n,T_τ} is precisely the set of points in A which make faster than τ-long "excursions" out of A; that is $B_{t,\tau} = A^{n,T_\tau} \setminus A^{t,\phi}$ where

$$B_{t,\tau} := \left\{ x \in A \ : \ (\phi^\tau)^i(x) \in A, i = 0, \ldots, n \text{ and } \exists s \in [0,t] \ \text{ so that } \phi^s(x) \notin A \right\}.$$

Lemma 2.21. *Let ϕ be a flow on $(X, \mathcal{B}, m)$ and let $A \in \mathcal{B}$. Suppose there is a $C \geq 1$ such that for all sufficiently large t and sufficiently small τ one has*

$$m(B_{t,\tau}) \leq C m(A^{t,\phi}). \tag{2.6}$$

Then

$$\lim_{\tau \to 0} \frac{1}{\tau} E(A; m, T_\tau) = E(A; m, \phi).$$

Proof. Using the assumption in (2.6) we have

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{t} \log m(A^{t,\phi}) &\leq \lim_{t\to\infty} \frac{1}{t} \log \left[m(A^{t,\phi}) + m(B_{t,\tau}) \right] \\
&\leq \lim_{t\to\infty} \frac{1}{t} \log \left[(1 + C) m(A^{t,\phi}) \right] \\
&= \lim_{t\to\infty} \frac{1}{t} \log(1 + C) + \lim_{t\to\infty} \frac{1}{t} m(A^{t,\phi}) \\
&= \lim_{t\to\infty} \frac{1}{t} \log m(A^{t,\phi}).
\end{aligned}$$

Therefore $\lim_t (1/t) \log \left[m(A^{t,\phi}) + m(B_{t,\tau}) \right] = \lim_t (1/t) \log m(A^{t,\phi})$ which yields the result:

$$\begin{aligned}
\lim_{\tau\to 0} \frac{1}{\tau} E(A; m, T_\tau) &= - \lim_{\tau\to 0} \frac{1}{\tau} \lim_{n\to\infty} \frac{1}{n} \log m(A^{n,T_\tau}) \\
&= - \lim_{\tau\to 0} \frac{1}{\tau} \lim_{t\to\infty} \frac{1}{\lfloor t/\tau \rfloor} \log \left[m(A^{t,\phi}) + m(B_{t,\tau}) \right] \\
&= - \lim_{t\to\infty} \frac{1}{t} \log \left[m(A^{t,\phi}) + m(B_{t,\tau}) \right] \\
&= - \lim_{t\to\infty} \frac{1}{t} \log m(A^{t,\phi}) \\
&= E(A; m, \phi).
\end{aligned} \tag{2.7}$$

□

Recall that for a flow $\phi : X \times \mathbb{R} \to X$, the *Perron-Frobenius* operator is defined to be the unique operator $\mathcal{P} : L^1(m) \times \mathbb{R} \to L^1(m)$ that satisfies

$$\int_B \mathcal{P}_t f \, \mathrm{d}m = \int_{\phi^{-t}B} f \, \mathrm{d}m \quad \forall B \in \mathcal{B},\ f \in L^1(m),\ t \in \mathbb{R}.$$

The corresponding *infinitesimal generator*, acting on all $f \in L^1(m)$ for which the following limit exists, is given by

$$\mathcal{A}f = \lim_{t \to 0} \frac{\mathcal{P}_t f - f}{t}.$$

Froyland *et al.* [57, Theorem 3.5] extend our Theorem 2.5 to continuous time to apply to infinitesimal generators $\mathcal{A}_\epsilon$ with ϵ-diffusion. Their proof, however, also holds in the non-diffusive case when $\epsilon = 0$. We summarise this result and our discussion above in the following.

Theorem 2.22 (Froyland *et al.* [57])**.** *Let $\mathcal{A}$ be the infinitesimal generator of a flow ϕ. Suppose that $\mathcal{A}f = \rho f$ for some $\rho < 0$ and $f \in L^\infty(m)$, and define $A_\pm := \{\pm f > 0\}$. Then for all $\tau > 0$*

$$\frac{1}{\tau} E(A_\pm; m, \phi^\tau) \leq -\rho. \tag{2.8}$$

Theorem 2.23. *In the setting of and Lemma 2.21 and Theorem 2.22 we have*

$$E(A_\pm; m, \phi) \leq -\rho.$$

Proof. The result follows from taking the limit as $\tau \to 0$ in (2.8) and applying Lemma 2.21.

□

Chapter 3

Escape from an Intermittent Map with a Hole

In this chapter we study Pomeau-Manneville (PM) maps, which are simple examples of non-uniformly expanding interval maps. Unlike Lasota-Yorke maps from the previous chapter, the corresponding Perron-Frobenius operators do not exhibit a spectral gap in BV. This would present a challenge in applying Theorem 2.5, as it is unclear which non-unit eigenvalue should be chosen to partition the interval into two metastable sets.

Through either creating a small hole in the non-uniformly expanding region, or introducing a small random perturbation in this region, one can make the dynamics uniformly expanding. Both procedures may be seen as a result of numerical approximation (*coarse graining*) of the Perron-Frobenius operator.

We will present results on existence and convergence of conditionally invariant measures of PM map with a hole and we will also describe the spectral behaviour of the Perron-Frobenius operator under coarse-graining.

The material in this chapter is joint work with Rua Murray and Gary Froyland and has appeared in [61].

We shall begin this chapter by describing the Pomeau-Manneville map in Section 3.1, state some well-known results, introduce the hole and give some preliminary results on asymptotics. In Section 3.2 we provide some motivation for the material to follow, by demonstrating our toy model — a two-state metastable Markov chain. Section 3.3

deals with the Young tower construction and existence of an ACCIM on the domain of uniform expansion. In Section 3.5 we state the results on the existence of a second eigenfunction of the closed system with bounds on the position of the second eigenvalue. Finally, Section 3.6 describes Ulam's method or coarse-graining and presents numerical results illustrating the various scalings with hole sizes.

3.1 Pomeau-Manneville Map with a Hole

Let ℓ be the Lebesgue measure on $\mathrm{I} := [0,1]$ and let $T : \mathrm{I} \circlearrowleft$ be a Pomeau-Manneville map [87, 97] which near 0 has the form

$$T(x) = x + c_\alpha \, x^{1+\alpha} + g(x) \tag{T}$$

where g is C^2 and the derivative $g'(x) = o(x^\alpha)$ (in conventional little-o notation[1]). Suppose also that T has two branches with breakpoint x_0 such that T is one-to-one and onto $(0,1)$ on both $(0, x_0)$ and $(x_0, 1)$. We suppose also that T is C^2 on both $(x_0, 1)$ and (ϵ, x_0) for every $\epsilon > 0$ and that $T' > 1$ on both $(0, x_0)$ and $(x_0, 1)$. Note that $T'(0) = 1$ so $x = 0$ is an indifferent fixed point; see Figure 3.1.

We assume that $\alpha \in (0,1)$, which ensures that these maps support a unique absolutely continuous invariant (probability) measure[2] μ^*, the dynamics of (T, μ^*) is exact, and T exhibits polynomial decay of correlations with rate $\mathcal{O}(k^{1-1/\alpha})$ (see e.g. [94, 112]). The slow decay of correlations occurs because typical orbits of T require anomalously long times to escape from the neighbourhood of 0.

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[1] That is $\lim_{x\to 0} \frac{g'(x)}{x^\alpha} = 0$.

[2] We reserve unstarred μ here for the corresponding ACIM on the Young Tower (to be introduced later on).

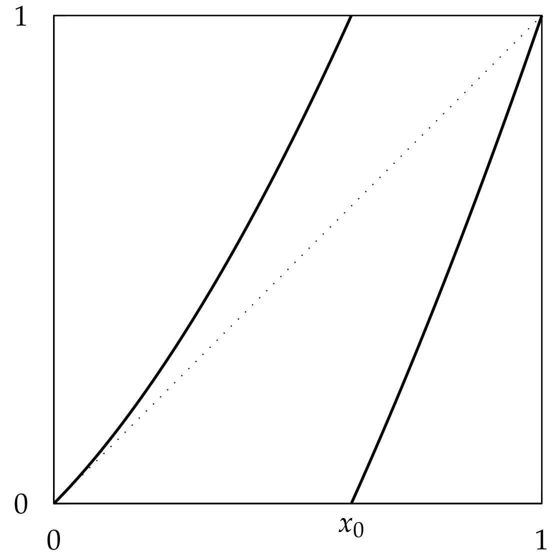


Figure 3.1: An example of a Pomeau-Manneville map with $g(x) = 0$, $c_\alpha = 1$ and $\alpha = 0.9$.

3.1.1 Asymptotic Behaviour in the Neighbourhood of the Fixed Point

Define the sequence (x_n) of pre-images of x_0 in $(0, x_0)$ recursively by $T(x_n) = x_{n-1}$. Computations of Young [112, Section 6] show that

$$x_n \sim n^{-1/\alpha}, \tag{3.1}$$

$$\text{and} \quad \ell(x_{n+1}, x_n) \sim n^{-1-1/\alpha}. \tag{3.2}$$

Let (γ_n) be the sequence of the corresponding points in the right branch, that satisfy $T(\gamma_n) = x_{n-1}$.

The density $h = d\mu^*/d\ell$ of the ACIM arises as a unique fixed point of the Perron-Frobenius operator for T. It has been shown [112] that h is bounded away from zero, and that it admits a singularity at $x = 0$ with $h(x) \sim x^{-\alpha}$ as $x \to 0$.

To this end, we fix $\epsilon_0 \in (0, x_0)$ and partition $[0,1] = \mathrm{I}_{\epsilon_0,1} \cup \mathrm{I}_{\epsilon_0,2}$ where $\mathrm{I}_{\epsilon_0,1} = [0, \epsilon_0]$ and $\mathrm{I}_{\epsilon_0,2} = (\epsilon_0, 1]$. For all formal results we assume that ϵ_0 is a preimage of x_0, that is $\epsilon_0 = x_{n-1}$ for some integer n, so that the hole $[0, \epsilon_0]$ is Markov. A simple integration then shows that

$$
\begin{aligned}
\mu^*([0,\epsilon_0]) &= \int_{[0,\epsilon_0]} h(x)\, \mathrm{d}\ell(x) \\
&\sim \int_{[0,\epsilon_0]} x^{-\alpha}\, \mathrm{d}\ell(x) \\
&\sim \epsilon_0^{1-\alpha}.
\end{aligned}
\tag{3.3}
$$

For small ϵ_0 the set $[0,\epsilon_0]$ is almost-invariant. As mentioned in Section 2.1, almost-invariant sets are often associated with isolated eigenvalues outside the essential spectrum of the Perron-Frobenius operator. However, for Pomeau-Manneville-type maps[3], the eigenvalue 1 corresponding to the invariant density is *not* isolated from the essential spectrum on any reasonable subspace of L^1, leaving no room for isolated "second" eigenvalues. Nevertheless, small random perturbations, or certain numerical approximations of $\mathcal{P}$ (such as Ulam's method) do possess a spectral gap.

Some of the main goals of this chapter will be to obtain the asymptotic scaling of escape rate from $\mathrm{I}_{\epsilon_0,2}$ and to explain the scaling of the spectral gap created from the Ulam approximation. Our preliminary attempt involves a two-state toy model of the dynamics.

3.2 A Two-state Metastable Model

Our first approximate version of the Perron-Frobenius operator $\mathcal{P}$ is a crude two-state Markov chain approximation, which nevertheless turns out to be an accurate descriptor of the important dynamics. For any probability measure m one can construct a 2-state Markov chain with transition matrix

$$
P_{\epsilon_0,m} = \begin{pmatrix} 1-a_{\epsilon_0} & a_{\epsilon_0} \\ b_{\epsilon_0} & 1-b_{\epsilon_0} \end{pmatrix},
$$

[3]Indeed any expanding maps with indifferent periodic points.

where

$$(P_{\epsilon_0,m})_{ij} = \frac{m(\mathrm{I}_{\epsilon_0,i} \cap T^{-1}\mathrm{I}_{\epsilon_0,j})}{m(\mathrm{I}_{\epsilon_0,i})}.$$

This Markov chain describes the movement between a small neighbourhood of 0 and the rest of the interval. Note that the numbers $1 - a_{\epsilon_0}$ and $1 - b_{\epsilon_0}$ are the invariance ratios $\varrho(\mathrm{I}_{\epsilon_0,1})$ and $\varrho(\mathrm{I}_{\epsilon_0,2})$ respectively, as given by (2.1). The normalised left stationary vector of $P_{\epsilon_0,m}$ is $p = (\frac{b_{\epsilon_0}}{a_{\epsilon_0}+b_{\epsilon_0}}, \frac{a_{\epsilon_0}}{a_{\epsilon_0}+b_{\epsilon_0}})$.

We can view this two-state model in two ways:

- As two one-state open systems where the *geometric escape rate*[4] from the two states equals the invariance ratios, $1 - a_{\epsilon_0}$ and $1 - b_{\epsilon_0}$.

- As a coarse-grained closed system; the rate of mixing is determined by the second eigenvalue of $P_{\epsilon_0,m}$ which equals $1 - a_{\epsilon_0} - b_{\epsilon_0}$ and the scaling of the rate of mixing as $\epsilon_0 \to 0$ is determined by whichever $a_{\epsilon_0}, b_{\epsilon_0}$ approaches zero most slowly.

We will now attempt to explain the statistical behaviour of T by this crude two-state model. The numbers $a_{\epsilon_0}, b_{\epsilon_0}$ are determined by the choice of measure m and there are a couple of natural choices:

(i) $m = \ell$ — one has as $\epsilon_0 \to 0$

$$\begin{aligned} a_{\epsilon_0} &= \frac{\ell([0,\epsilon_0] \cap T^{-1}[\epsilon_0,1])}{\ell([0,\epsilon_0])} \\ &= \frac{\ell([x_{n+1}, x_n])}{\ell([0,x_n])} \\ &\sim \frac{n^{-1-1/\alpha}}{n^{-1/\alpha}} \\ &= n^{-1} \sim \epsilon_0^{\alpha}, \end{aligned}$$

and similarly

$$\lim_{\epsilon_0 \to 0} \frac{b_{\epsilon_0}}{\epsilon_0} = \lim_{\epsilon_0 \to 0} \frac{\ell([\epsilon_0,1] \cap T^{-1}[0,\epsilon_0])}{\epsilon_0 \ell([\epsilon_0,1])}$$

[4] λ where $-\log \lambda$ is the escape rate.

$$
\begin{aligned}
&= \lim_{\epsilon_0 \to 0} \frac{y_{n+1} - x_0}{\epsilon_0} \\
&= \lim_{x \to x_0^+} \frac{1}{T'(x)}.
\end{aligned}
$$

Hence $b_{\epsilon_0} \sim \epsilon_0$ as $\epsilon_0 \to 0$.

(ii) $m = \mu$ — although no closed formula is available, it is well-known that the invariant density $h(x) \sim x^{-\alpha}$ as $x \to 0$ and $\inf h > 0$. Thus we obtain

$$
\begin{aligned}
a_{\epsilon_0} &\sim \frac{\int_{x_{n+1}}^{x_n} x^{-\alpha} \, \mathrm{d}\ell(x)}{\int_0^{x_n} x^{-\alpha} \, \mathrm{d}\ell(x)} \\
&\sim \frac{x_n^{-\alpha}(x_n - x_{n+1})}{x_n^{1-\alpha}} \\
&\sim \frac{(n^{-1/\alpha})^{-\alpha} n^{-1-1/\alpha}}{n^{1-1/\alpha}} \\
&= n^{-1} \sim \epsilon_0^{\alpha}
\end{aligned}
$$

and since h is bounded in the neighbourhood of x_0 we also have $b_{\epsilon_0} \sim \epsilon_0$ thus the behaviour is the same as when $m = \ell$.

Below we note some rates of scaling for this two-state model.

- The stationary measure given by the two-state model gives the interval $\mathrm{I}_{\epsilon_0,1}$ a mass of $\frac{b_{\epsilon_0}}{a_{\epsilon_0} + b_{\epsilon_0}} \sim \epsilon_0^{1-\alpha}$ which matches the previously calculated ACIM scaling of $\mu^*([0, \epsilon_0])$ in (3.3).

- The rate of escape from the second state is $-\log(1 - b_{\epsilon_0}) \approx b_{\epsilon_0} \sim \epsilon_0$. We will show in the next two sections (Theorem 3.9) that this matches the escape from $\mathrm{I}_{\epsilon_0,2}$ with respect to the ACCIM μ^*.

- The rate of escape from the first state is $-\log(1 - a_{\epsilon_0}) \approx a_{\epsilon_0} \sim \epsilon_0^{\alpha}$. We know that escape from $\mathrm{I}_{\epsilon_0,1}$ is subexponential, giving escape rate of 0. The escape predicted by the two-state model, however, is effectively the rate experienced when the map T is perturbed slightly to become uniformly expanding on $\mathrm{I}_{\epsilon_0,1}$.

- The second eigenvalue is $1 - a_{\epsilon_0} - b_{\epsilon_0}$. Since $a_{\epsilon_0} + b_{\epsilon_0} \sim \epsilon_0^{\alpha}$, the spectral gap scales like ϵ_0^{α}. We will see later (Theorem 3.10) that this matches the scaling of the second eigenvalue of the Ulam matrix.

Thus, despite its simplicity, this two-state Markov model

(i) captures well the relative mass of $\mathrm{I}_{\epsilon_0,1}$ and $\mathrm{I}_{\epsilon_0,2}$;

(ii) provides escape rates from the two states consistent with true or perturbed escape rates for T;

(iii) and captures well the mixing rate of a perturbed version of T.

These properties, expressed very clearly with only two states, will carry across to matrices arising from Ulam approximations of $\mathcal{P}$.

3.3 Young Tower Construction for PM Map with and without a Hole

We study the open and closed dynamics of T via a Young tower of returns to an interval away from the indifferent fixed point at 0. For the ACIM, the construction is standard and can be found in Young [111, 112]. For the ACCIM, we puncture the tower for the closed dynamics, and look for a fixed point of the normalised conditional Perron-Frobenius operator on the open tower. Because of the normalisation, the fixed point must have growing mass concentration as height increases up the tower; some effort is needed to control this growth.

Recall that $T\colon \mathrm{I} \circlearrowleft$ satisfies (T) and has two, onto one-to-one branches, with a discontinuity at x_0. Set $\Delta_0 = [x_0, 1]$ to be the *base*. The Young tower Δ will be constructed as the tower of first returns to Δ_0. For $x \in \Delta_0$ let

$$R(x) := \min\{n > 0 \ : \ T^n(x) \in (x_0, 1)\}$$

be the *return time* to Δ_0. We partition Δ_0, according to the return times, into $\{\Delta_{0,i}\}_{i=1}^{\infty}$ where $\Delta_{0,i} := \{x \in \Delta_0 \ : \ R(x) = i\}$. The tower is defined to be

$$\Delta := \{(x,n) \in \Delta_0 \times \mathbb{Z}^+ \ : \ n < R(x)\}$$

so that above any $x \in \Delta_0$ the *height* is $R(x) - 1$. For $l \geq 1$ the upper *levels* of the tower are $\Delta_l = \Delta \cap \{n = l\}$ partitioned as $\{\Delta_{l,i}\}_{i=l+1}^{\infty}$ where $\Delta_{l,i} = \{(x,l) \in \Delta \ : \ x \in \Delta_{0,i}\}$. A natural measure ν on Δ is Lebesgue on Δ_0 lifted by upwards translation[5].

For $\alpha \in (0,1)$, R is integrable with respect to ℓ [112], that is $\int_{\Delta_0} R \, \mathrm{d}\ell < \infty$. The *tower map* $F\colon \Delta \circlearrowleft$ is defined by

$$F(x,l) = \begin{cases} (x, l+1), & l < R(x) - 1 \\ (T^R(x), 0), & l = R(x) - 1. \end{cases}$$

The projection map $\pi : \Delta \to \mathrm{I}$ given by $\pi(x,l) = T^l(x)$ defines a semi-conjugacy $T \circ \pi = \pi \circ F$.

Note that F is non-singular and because $\{[0,x_0],[x_0,1]\}$ is a Markov partition for T, the tower map F maps each top level $\Delta_{l,l+1}$ injectively onto Δ_0. These facts are used in our arguments below.

For $x, y \in \Delta$ define the *separation time* $s(x,y)$ to be the smallest number of returns n to Δ_0 such that $(F^R)^n(x)$ and $(F^R)^n(y)$ are in different elements $\Delta_{0,i}$ of the partition of Δ_0.

Proposition 3.1. *Let $T\colon \mathrm{I} \circlearrowleft$ be a Pomeau-Manneville map satisfying* (T) *and let $F\colon \Delta \circlearrowleft$ be the tower map as described above. There exist constants $\beta \in (0,1)$ and $c < \infty$ such for any x, y in the same level of the tower Δ_l the Jacobian of F satisfies the regularity condition*

$$\left| \frac{JF(x)}{JF(y)} \right| \leq \exp(c\, \beta^{s \circ F(x,y)}). \tag{JF}$$

Proof. Let $\tau_0(x) := \min\{n > 0 \ : \ T^n(x) \in [x_0, 1]\}$ denote the first passage/return time to Δ_0. In order to choose β such that (JF) is satisfied, note that standard estimates (see for

[5] More formally, ν is the product of the Lebesgue measure and counting measure on the levels of the tower.

example [112]) give a constant c_0 such that

$$\log \left| \frac{JT^{\tau_0}(x)}{JT^{\tau_0}(y)} \right| \leq c_0 \left| T^{\tau_0}(x) - T^{\tau_0}(y) \right|$$

when x, y are in the same one-to-one branch of T^{τ_0}. Choosing $\beta < 1$ large enough so that $|JT(x)|\, \beta > 1$ for all $x \in [x_1, 1]$ ensures that $|J(T^{\tau_0})(x)| > \beta^{-1}$ for almost every $x \in (0, 1]$. Hence, distances between points are expanded by at least β^{-1} on every visit to Δ_0. If $s(x, y) = n$ then x, y lie in the same one-to-one branch of $(T^{\tau_0})^n$ so

$$\begin{aligned} \left| T^{\tau_0}(x) - T^{\tau_0}(y) \right| &\leq \beta^{n-1} \left| (T^{\tau_0})^n(x) - (T^{\tau_0})^n(y) \right| \\ &\leq \beta^{s \circ F(x,y)}\, \nu(\Delta_0) \end{aligned}$$

and (JF) follows. □

Truncation and Escape from the Tower

Next, for each n we impose a hole in the tower:

$$H_n := \cup_{l \geq 1, i \geq n+1} \Delta_{l,i};$$

that is, H_n consists of all elements directly above $H_n^1 := \cup_{i \geq n+1} \Delta_{0,i}$. Note that the highest level of $\Delta \setminus H_n$ is $n - 1$ and

$$H_n^1 = (\Delta \setminus H_n) \cap F^{-1} H_n; \tag{3.4}$$

that is H_n^1 consists of all the points that fall into the hole in exactly one iteration of the map F (see Figure 3.2).

If $\mathcal{P} \colon L^1(\Delta, \nu) \circlearrowleft$ is the Perron-Frobenius operator on the tower then

$$\mathcal{P}\varphi(x) = \sum_{y \in F^{-1}x} \frac{\varphi(y)}{|JF(y)|}, \quad \text{for } \varphi \in L^1(\Delta)$$

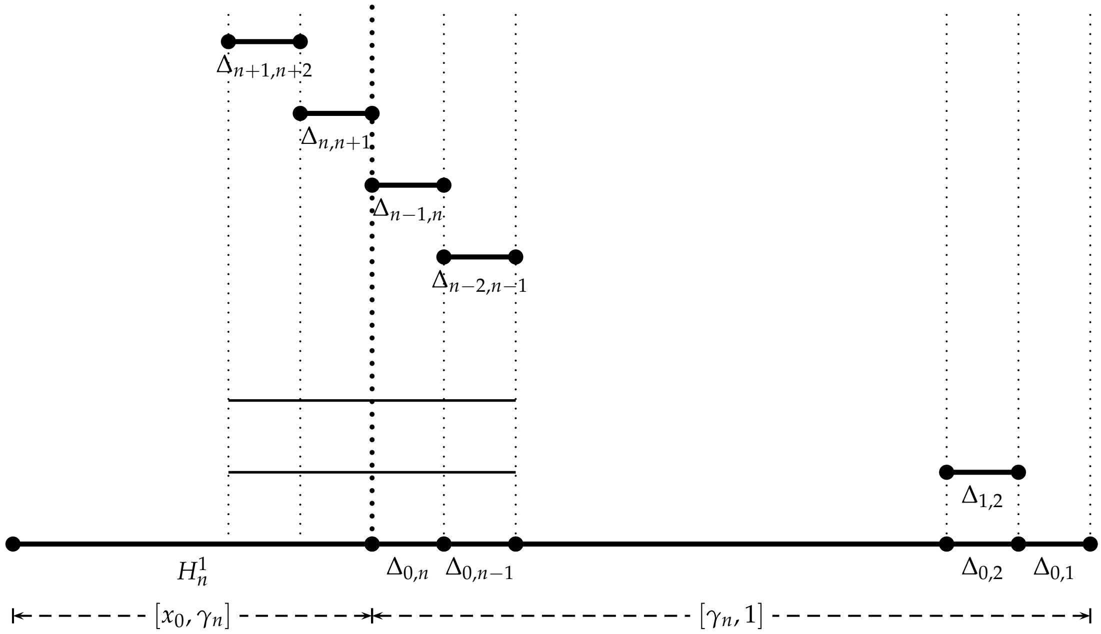


Figure 3.2: Young Tower for Pomeau-Manneville map with base identified with $[x_0, 1]$.

and for each hole H_n the conditional operator $\mathcal{P}_n : L^1(\Delta, \nu) \circlearrowleft$ is given by

$$\begin{aligned}\mathcal{P}_n\varphi(x) &= \chi_{\Delta\setminus H_n}(x)\, \mathcal{P}(\varphi \cdot \chi_{\Delta\setminus H_n})(x) \\ &= \chi_{\Delta\setminus H_n}(x) \sum_{y\in F^{-1}x\setminus H_n} \frac{\varphi(y)}{|JF(y)|}.\end{aligned}$$

The normalised conditional Perron-Frobenius operator is $\hat{\mathcal{P}}_n : L^1(\Delta, \nu) \circlearrowleft$ which acts according to

$$\hat{\mathcal{P}}_n\varphi = \frac{\mathcal{P}_n\varphi}{\|\mathcal{P}_n\varphi\|_{L^1}}. \tag{3.5}$$

Distribution of R on the Base

The distribution of R on Δ_0 is determined by the exponent α. Note that $R(x) = i$ precisely when $T(x) \in (x_{i-1}, x_{i-2})$, hence $R(x) \geq i$ when $T(x) \in (0, x_{i-1})$. Using (3.1) we have

$$\ell(\{x \ : \ R(x) \geq i\}) \sim i^{-1/\alpha}$$

and the tail of the return time distribution is

$$\sum_{i\geq n+1} \ell(\{R\geq i\}) \sim \sum_{i\geq n+1} i^{-1/\alpha} \sim n^{1-1/\alpha}$$

(giving polynomial decay of correlations with rate $\mathcal{O}(n^{1-1/\alpha})$ [112, Theorem 5]) and

$$\nu(H_n^1) = \ell(\{R \geq n\}) \sim n^{-1/\alpha}. \tag{3.6}$$

3.3.1 Existence and Uniqueness of ACCIM

In this section we prove the existence and uniqueness of an absolutely continuous conditionally invariant probability measure of $F : \Delta \circlearrowleft$ with hole H_n. It should be noted that Pianigiani and Yorke [95, Theorems 1 & 2], may be directly applied in our setting to show these claims. Nonetheless, we shall duplicate the corresponding results on a suitable tower. Obtaining explicit, uniform bounds on the density of the ACCIMs (independently of hole size) is necessary for our estimates of escape rates and for showing convergence of conditionally invariant measures to the invariant measure as the hole closes[6].

For a fixed constant $C > 0$ let $\mathcal{C}_*$ be a set of *regular* functions in $L^1(\Delta, \nu)$ defined as

$$\mathcal{C}_* := \left\{\varphi \in L^1(\Delta,\nu) \ : \ \varphi \geq 0,\ \varphi(x) \leq \varphi(y) e^{C\beta^{s(x,y)}} \text{ for a.e. comparable } x, y\right\}.$$

Above, $x, y \in \Delta$ are considered *comparable* if either they are both in the same cell of the partition $\Delta_{l,i}$, or if they are both in Δ_0.

Furthermore for every $n \in \mathbb{Z}^+$ let $\mathcal{C}_n \subseteq \mathcal{C}_*$ be a family of regular densities on $\Delta \setminus H_n$, that is

$$\mathcal{C}_n := \left\{\varphi \in \mathcal{C}_* \ : \ \int_{\Delta\setminus H_n} \varphi \, d\nu = 1, \quad \varphi|_{H_n} = 0\right\}.$$

Lemma 3.2. *For each $B > 0$ the set $\mathcal{C}_*^B := \{\varphi \in \mathcal{C}_* \ : \ \|\varphi\|_{L^\infty} \leq B\}$ is compact in $L^1(\Delta, \nu)$. In addition, for each n there is a $B = B(n)$ such that $\mathcal{C}_n \subseteq \mathcal{C}_*^B$ so that each $\mathcal{C}_n$ is also compact.*

[6]The ACCIMs we construct have uniformly bounded densities on Δ (Corollary 3.4), but not when projected back to the interval $[0, 1]$.

Before we start the proof, recall that a family of real-valued continuous functions $\{\varphi_s\}_{s\in S}$ on a compact metric space X are said to be *equicontinuous* if, for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $s \in S$, $|\varphi_s(x) - \varphi_s(y)| < \epsilon$ whenever $d(x,y) < \delta$. The Arzelà-Ascoli Theorem [2, 3, 51] states that $\{\varphi_s\}_{s\in S}$ is relatively compact (in the space of continuous functions with the uniform norm) if an only if it is equicontinuous and uniformly bounded.

Proof of Lemma 3.2. Straight-forward arguments show that $\mathcal{C}_*^B$ and $\mathcal{C}_n$ are closed up to equivalence a.e. We concentrate on showing that both sets are relatively compact. For any positive integer M let $X_M = \cup_{0\le l<i\le M}\Delta_{l,i}$. For another positive integer j let

$$X_M^j = \{x \in X_M \ : \ (F^R)^i(x) \in X_M,\ i = 1,2,\ldots,j\}$$

and define $X_M^\infty = \cap_{j=1}^\infty X_M^j$.

Define a metric d_β on Δ by $d_\beta(x,y) = \beta^{s(x,y)}$ (for non-comparable x and y set $s(x,y) = 0$). We claim that X_M^∞ is compact in d_β. Note that since F maps each $\Delta_{l,l+1}$ onto Δ_0, for any $x' \in \{1,2,\ldots,M\}^{\mathbb{N}}$ and l,i such that $0 \le l < i \le M$ there exists an $x \in X_M^\infty \cap \Delta_{l,i}$ such that $(F^R)^n(x) \in \Delta_{0,(x')_n}$ for all $n \in \mathbb{N}$. Suppose that $\{x_j\} \subset X_M^\infty$ is a sequence converging to some $x \in \Delta$. Then for any $n \in \mathbb{Z}^+$ there is an x_j such that $s(x_j,x) \ge n$. In particular $(F^R)^n(x) \in X_M$ for all $n \in \mathbb{Z}^+$. Therefore $x \in X_M^\infty$, hence X_M^∞ is closed. Now for any $\epsilon > 0$ let N be the smallest integer such that $\beta^N \le \epsilon$. Define the set

$$\eta_\epsilon := \bigcup_{0\le l<i\le M} \left\{x \in X_M^\infty \cap \Delta_{l,i} \ : \ (F^R)^n(x) \in \Delta_{0,1}\ \forall n > N\right\}.$$

Then it is easy to see that η_ϵ is a finite ϵ-net for X_M^∞ therefore X_M^∞ is totally bounded. As X_M^∞ is a closed and totally bounded subset of a complete metric space it follows that it is compact.

Now we will show that $\mathcal{C}_*^B$ is equicontinuous. For a given $\varphi \in \mathcal{C}_*^B$ and any $\epsilon > 0$ choose $\delta = \min(\beta, C^{-1}\log(1+\epsilon/B))$. Take any $x,y \in \Delta$ such that $d_\beta(x,y) < \delta$. As $\delta \le \beta$ we have $s(x,y) \ge 1$ so x and y are in the same cell of the partition of Δ. Then

$$|\varphi(x) - \varphi(y)| \le \varphi(x)|1 - \exp(C\beta^{s(x,y)})|$$

$$
\begin{aligned}
&\leq B(\exp(C\beta^{s(x,y)}) - 1) \\
&\leq B(e^{C\delta} - 1) \\
&\leq \epsilon.
\end{aligned}
$$

Thus $\mathcal{C}_*^B$ is equicontinuous and in particular it is equicontinuous when restricted to the compact set X_M^∞. In addition $\mathcal{C}_*^B$ is uniformly bounded (by B) so by Arzelà-Ascoli Theorem the restriction to a compact set $\mathcal{C}_*^B|_{X_M^\infty}$ is relatively compact in the uniform norm.

To show that $\mathcal{C}_*^B$ is relatively compact in $L^1(\Delta, \nu)$ suppose (φ_n) is a sequence in $\mathcal{C}_*^B$. Given $\epsilon > 0$ fix an integer K such that $B(\exp(C\beta^K) - 1) < \epsilon$ and then choose M large enough so that $\nu(\Delta \setminus X_M^K) < \epsilon$. As $\mathcal{C}_*^B|_{X_M^\infty}$ is relatively compact, there exists a Cauchy subsequence (φ_{n_j}) and an integer J so that for all $j, k > J$

$$
|\varphi_{n_j}(y) - \varphi_{n_k}(y)| < \epsilon, \quad \forall y \in X_M^\infty. \tag{3.7}
$$

We proceed to show that (φ_{n_j}) is a Cauchy sequence in $L^1(\Delta, \nu)$. For any $x \in X_M^K$, choose $y \in X_M^\infty$ so that $s(x,y) \geq K$. This is always possible, as for $x \in \Delta_{l,i}$ we can choose $y \in \Delta_{l,i} \cap X_M^\infty$ so that x and y are in the same $\Delta_{0,j}$ after each of the first K returns to Δ_0. Then

$$
|\varphi_{n_j}(x) - \varphi_{n_j}(y)| \leq B(e^{C\beta^K} - 1) < \epsilon \tag{3.8}
$$

and similarly

$$
|\varphi_{n_k}(x) - \varphi_{n_k}(y)| < \epsilon. \tag{3.9}
$$

Using (3.7), (3.8) and (3.9), we obtain that for all $x \in X_M^K$ and $j, k > J$

$$
\begin{aligned}
|\varphi_{n_j}(x) - \varphi_{n_k}(x)| &\leq |\varphi_{n_j}(x) - \varphi_{n_j}(y)| + |\varphi_{n_j}(y) - \varphi_{n_k}(y)| + |\varphi_{n_k}(y) - \varphi_{n_k}(x)| \\
&< 3\epsilon.
\end{aligned}
$$

Finally, for any $j, k > J$ we have

$$
\begin{aligned}
\|\varphi_{n_j} - \varphi_{n_k}\|_{L^1} &= \int_{\Delta \setminus X_M^K} |\varphi_{n_j} - \varphi_{n_k}| \, \mathrm{d}\nu + \int_{X_M^K} |\varphi_{n_j} - \varphi_{n_k}| \, \mathrm{d}\nu \\
&< 2B\nu(\Delta \setminus X_M^K) + 3\epsilon\nu(X_M^K)
\end{aligned}
$$

$$\leq \epsilon(2B + 3\nu(\Delta)).$$

Thus (φ_{n_j}) is Cauchy, therefore (φ_n) has a limit point. As $\mathcal{C}_*^B$ is also closed, we conclude that it is compact in the Banach space $L^1(\Delta, \nu)$.

Now consider $\varphi \in \mathcal{C}_n$. For any $\Delta_{l,i}$, $1 \leq l < i \leq n$ by the Integral Mean Value Theorem there exists $x^* \in \Delta_{l,i}$ such that

$$\varphi(x^*) = \frac{1}{\nu(\Delta_{l,i})} \int_{\Delta_{l,i}} \varphi \, \mathrm{d}\nu,$$

so

$$\begin{aligned}
\operatorname*{ess\,sup}_{\Delta_{l,i}} \varphi &\leq e^{C\beta} \varphi(x^*) \\
&= \frac{e^{C\beta}}{\nu(\Delta_{l,i})} \int_{\Delta_{l,i}} \varphi \, \mathrm{d}\nu \\
&\leq \frac{e^{C\beta}}{\nu(\Delta_{l,i})}.
\end{aligned}$$

Similarly on the base of the tower we obtain $\operatorname{ess\,sup}_{\Delta_0} \varphi \leq \frac{e^C}{\nu(\Delta_0)}$. If we choose

$$B = B(n) := e^C \max\left(\frac{1}{\nu(\Delta_0)}, \max_{1 \leq l < i \leq n} \frac{1}{\nu(\Delta_{l,i})}\right),$$

then $\mathcal{C}_n \subseteq \mathcal{C}_*^B$ hence $\mathcal{C}_n$ is also compact. □

Theorem 3.3. *Let $C \geq c/(1-\beta)$. For each $n \in \mathbb{Z}^+$ the normalised conditional operator $\hat{\mathcal{P}}_n$ admits a fixed point in $\mathcal{C}_n$.*

Proof. Note that $C\beta + c \leq C$. First, we will show that $\hat{\mathcal{P}}_n \mathcal{C}_n \subseteq \mathcal{C}_n$ from which a standard fixed point argument will follow. Let $\varphi \in \mathcal{C}_n$; it suffices to show that $\mathcal{P}_n \varphi \in \mathcal{C}_*$. Now let $(z, l), (w, l) \in \Delta_{l,i}$ where $1 \leq l < i \leq n$ so that both (z, l) and (w, l) have only one pre-image of F, namely $(z, l-1)$ and $(w, l-1)$. Here, separation time is invariant under F so $s((z, l), (w, l)) = s((z, l-1), (w, l-1))$. Moreover, $JF = 1$ on these levels, since the translation is straight upwards. Hence

$$\begin{aligned}(\mathcal{P}_n\varphi)(z,l) &= \frac{\varphi(z,l-1)}{JF(z,l-1)}\\ &\le \frac{\varphi(w,l-1)\exp(C\beta^{s((z,l-1),(w,l-1))})}{1}\\ &= (\mathcal{P}_n\varphi)(w,l)\exp(C\beta^{s((z,l),(w,l))}).\end{aligned}$$

On the base, the story is different as for each $(z,0),(w,0)\in\Delta_0$ there are n pre-images on the top levels of the tower. For $l=0,\dots,n-1$ let $(z_l,l)\in\Delta_{l,l+1}$ be such that $F(z_l,l)=(z,0)$ and similarly for (w_l,l). Now we have $s((z_l,l),(w_l,l))=s((z,0),(w,0))+1$ for every $l=0,\cdots,n-1$. Then for any $\varphi\in\mathcal{C}_n$

$$\begin{aligned}(\mathcal{P}_n\varphi)(z,0) &= \sum_{l=0}^{n-1}\frac{\varphi(z_l,l)}{|JF(z_l,l)|}\\ &\le \sum_{l=0}^{n-1}\frac{\varphi(w_l,l)}{|JF(w_l,l)|}\exp(C\beta^{s((z_l,l),(w_l,l))})\exp(c\beta^{s\circ F((z_l,l),(w_l,l))})\\ &\le (\mathcal{P}_n\varphi)(w,0)\max_l\exp(C\beta^{s((z_l,l),(w_l,l))})\exp(c\beta^{s\circ F((z_l,l),(w_l,l))})\\ &= (\mathcal{P}_n\varphi)(w,0)\max_l\exp(C\beta^{s((z,0),(w,0))+1})\exp(c\beta^{s\circ F((z_l,l),(w_l,l))})\\ &= (\mathcal{P}_n\varphi)(w,0)\exp((C\beta+c)\beta^{s((z,0),(w,0))}).\end{aligned}$$

Since $C\beta+c\le C$ we have $\hat{\mathcal{P}}_n\varphi\in\mathcal{C}_n$ for all $\varphi\in\mathcal{C}_n$ and therefore $\hat{\mathcal{P}}_n\mathcal{C}_n\subseteq\mathcal{C}_n$.

It is easy to see that $\mathcal{C}_n$ is convex as for any $\varphi,\phi\in\mathcal{C}_n$ and $x,y\in\Delta\setminus H_n$, we have

$$t\varphi(x)+(1-t)\phi(x)\le(t\varphi(y)+(1-t)\phi(y))\exp(C\beta^{s(x,y)}).$$

Moreover, the operator $\mathcal{P}_n$ is continuous as $\mathcal{P}$ is contractive:

$$\begin{aligned}\|\mathcal{P}_n\varphi-\mathcal{P}_n\phi\|_{L^1} &= \|\mathcal{P}((\varphi-\phi)\cdot\chi_{\Delta\setminus H_n})\|_{L^1}\\ &\le \|(\varphi-\phi)\cdot\chi_{\Delta\setminus H_n}\|_{L^1}\\ &\le \|\varphi-\phi\|_{L^1}.\end{aligned}$$

By the Integral Mean Value Theorem and the conditions on $\varphi \in \mathcal{C}_*^B$,

$$\frac{1}{\nu(H_n^1)} \int_{H_n^1} \varphi \, \mathrm{d}\nu \leq e^C \frac{1}{\nu(\Delta_0 \setminus H_n^1)} \int_{\Delta_0 \setminus H_n^1} \varphi \, \mathrm{d}\nu.$$

Hence

$$\begin{aligned}
1 = \int_\Delta \varphi \, \mathrm{d}\nu &= \int_{H_n^1} \varphi \, \mathrm{d}\nu + \int_{\Delta_0 \setminus H_n^1} \varphi \, \mathrm{d}\nu + \int_{\Delta \setminus \Delta_0} \varphi \, \mathrm{d}\nu \\
&\leq \left(e^C \frac{\nu(H_n^1)}{\nu(\Delta_0 \setminus H_n^1)} + \frac{\nu(\Delta_0 \setminus H_n^1)}{\nu(\Delta_0 \setminus H_n^1)} \right) \int_{\Delta_0 \setminus H_n^1} \varphi \, \mathrm{d}\nu + \int_{\Delta \setminus \Delta_0} \varphi \, \mathrm{d}\nu \\
&< e^C \frac{\nu(\Delta_0)}{\nu(\Delta_0 \setminus H_n^1)} \left(\int_{\Delta_0 \setminus H_n^1} \varphi \, \mathrm{d}\nu + \int_{\Delta \setminus \Delta_0} \varphi \, \mathrm{d}\nu \right) \\
&= e^C \frac{\nu(\Delta_0)}{\nu(\Delta_0 \setminus H_n^1)} \int_{\Delta \setminus H_n^1} \mathcal{P}_n \varphi \, \mathrm{d}\nu \\
&=: \alpha \int_\Delta \mathcal{P}_n \varphi \, \mathrm{d}\nu.
\end{aligned}$$

Since $1/\|\mathcal{P}_n\varphi\|_{L^1} < \alpha$, the normalisation map $\mathcal{P}_n\varphi \mapsto \mathcal{P}_n\varphi/\|\mathcal{P}_n\varphi\|_{L^1}$ is continuous. Combining the above results with Lemma 3.2 we see that $\mathcal{C}_n$ is a compact, convex set, invariant under the continuous map $\hat{\mathcal{P}}_n$. The Schauder Fixed Point Theorem [101] asserts that $\hat{\mathcal{P}}_n$ has a fixed point $\varphi_n \in \mathcal{C}_n$. □

We prove uniqueness of φ_n in $\mathcal{C}_n$ below.

Corollary 3.4. *Let C satisfy the hypothesis of Theorem 3.3. For each $n \in \mathbb{Z}^+$ there is a unique $\varphi_n \in \mathcal{C}_n$ such that $\mathcal{P}_n\varphi_n = \lambda_n\varphi_n$ where $\lambda_n = \|\mathcal{P}_n\varphi_n\|_{L^1}$. In addition, φ_n is essentially bounded above and below by positive constants.*

Proof. Let $\varphi_n \in \mathcal{C}_n$ be a fixed point of $\hat{\mathcal{P}}_n$. Then φ_n satisfies $\mathcal{P}_n\varphi_n = \lambda_n\varphi_n$ where $\lambda_n = \|\mathcal{P}_n\varphi_n\|_{L^1}$. Now if $\operatorname{ess\,inf} \varphi_n = 0$ then the regularity of φ_n ensures that $\varphi_n|_{\Delta_{l,i}} \equiv 0$ a.e. on some $\Delta_{l,i}$. Take any $x \in \Delta_{0,i}$. Then $0 = \lambda_n^l \varphi_n(F^l(x)) = (\mathcal{P}_n^l\varphi_n)(F^l(x)) = \varphi_n(x)$, hence $\varphi_n|_{\Delta_{0,i}} \equiv 0$. All of Δ_0 is comparable so this forces φ_n to vanish on Δ_0 and hence on all of Δ. Clearly this is not possible as φ_n is a density so necessarily $\operatorname{ess\,inf} \varphi_n > 0$.

Now, since φ_n is essentially bounded above[7] and below by positive constants, $-\log \lambda_n$ is the Lebesgue escape rate into the hole H_n so λ_n is unique (cf. Proposition 1.19).

[7]Note that $\varphi_n \in \mathcal{C}_n$ and see Lemma 3.2.

In the proof of uniqueness of φ_n we borrow a technique from [95]. Suppose that there is another eigenfunction ϕ_n with the same eigenvalue λ_n. For any $s \in \mathbb{R}$ we are able to construct another eigenfunction ξ_s of $\mathcal{P}_n$, namely

$$\xi_s := s\varphi_n + (1-s)\phi_n.$$

Let $\sigma > 1$ be the largest real number so that $\operatorname{ess\,inf} \xi_s \geq 0$ for all $s \in (1, \sigma]$. Then necessarily $\operatorname{ess\,inf} \xi_\sigma = 0$ and $\xi_\sigma = \lim_{s\to\sigma} \xi_s \in \mathcal{C}_n$. We have already seen in the first part of the proof that this cannot be, hence φ_n is unique in $\mathcal{C}_n$. □

Corollary 3.5. *Let C be such that Theorem 3.3 holds and let φ_n and λ_n be as in Corollary 3.4. For each $n \in \mathbb{Z}^+$ let μ_n be the measure on Δ with density $\varphi_n = d\mu_n/d\nu$. Then μ_n is an absolutely continuous conditionally invariant probability measure for the open system with hole H_n. In particular $\lambda_n = 1 - \mu_n(H_n^1)$.*

Proof. In Proposition 1.26 we stated that nonnegative normalised eigenfunctions of the Perron-Frobenius operator are densities of absolutely continuous conditionally invariant probability measures. As μ_n is conditionally invariant with eigenvalue λ_n we have

$$\begin{aligned}
\lambda_n &= \mu_n(F^{-1}(\Delta \setminus H_n) \setminus H_n) \\
&= \mu_n(F^{-1}(\Delta) \setminus H_n) - \mu_n(F^{-1}(H_n) \setminus H_n) \\
&= 1 - \mu_n(H_n^1).
\end{aligned}$$

□

3.3.2 Convergence of ACCIM to the ACIM of the Closed System

Lemma 3.6. *Let $\varphi_n \in \mathcal{C}_n$ be as in Theorem 3.3. There exist positive constants a and b (independent of n) such that*

$$\operatorname*{ess\,inf}_{\Delta\setminus H_n} \varphi_n \geq a \quad \textit{and} \quad \operatorname*{ess\,sup}_{\Delta\setminus H_n} \varphi_n \leq b$$

for all $n \in \mathbb{Z}^+$.

Proof. Fix $n \in \mathbb{Z}^+$ and let $\varphi_{n,i} = \varphi_n|_{\Delta_{0,i}}$ for $1 \le i \le n$ and $\varphi_{n,n+1} = \varphi_n|_{H_n^1}$. First we approximate a lower bound of λ_n and then obtain a uniform upper bound for λ_n^{-n}. We begin with the result of Corollary 3.5 to obtain

$$\begin{aligned}\lambda_n &= 1 - \int_{H_n^1} \varphi_n \,\mathrm{d}\nu \\ &\ge 1 - \nu(H_n^1) e^C \frac{\int_{\Delta_0} \varphi_n \,\mathrm{d}\nu}{\nu(\Delta_0)} \\ &\ge 1 - \frac{\nu(H_n^1)}{\nu(\Delta_0)} e^C,\end{aligned}$$

where the first inequality above is a consequence of the Integral Mean Value Theorem and the property that $\varphi(x) \le e^C \varphi(y)$ for all $x, y \in \Delta_0$. Now using the fact that $\nu(H_n^1) \cdot n < \nu(\Delta)$ we obtain

$$\begin{aligned}\lambda_n &\ge 1 - \frac{e^C \nu(\Delta)}{n \cdot \nu(\Delta_0)} \\ &= 1 - \frac{C'}{n}\end{aligned}$$

for the constant $C' := e^C \nu(\Delta)/\nu(\Delta_0)$ independent of n. Next choose $n^* \in \mathbb{Z}^+$ so that $C'/n < 1/2$ for all $n \ge n^*$. By the Mean Value Theorem there is a constant C'' such that $\log(1 - C'/n) \ge -C''/n$ for all $n \ge n^*$ and hence

$$\begin{aligned}\lambda_n^{-n} &\le \left(1 - \frac{C'}{n}\right)^{-n} \\ &= e^{-n \log(1 - C'/n)} \\ &\le e^{C''}.\end{aligned} \tag{3.10}$$

Using the bound on λ_n^{-n} and the fact that φ_n is an eigenvector of norm 1 we obtain

$$\begin{aligned}1 = \|\varphi_n\|_{L^1} &= \|\varphi_{n,n+1}\|_{L^1} + \sum_{i=1}^{n} \sum_{j=1}^{i} \lambda_n^{-(j-1)} \|\varphi_{n,i}\|_{L^1} \\ &\le (\operatorname*{ess\,inf}_{\Delta_0} \varphi_n) e^C \left(\nu(H_n^1) + \sum_{i=1}^{n} \sum_{j=1}^{i} \lambda_n^{-(j-1)} \nu(\Delta_{0,i}) \right)\end{aligned}$$

$$
\begin{aligned}
&\leq (\operatorname*{ess\,inf}_{\Delta_0}\ \varphi_n) e^{C} \left(\nu(H_n^1) + \sum_{i=1}^{n} i \cdot e^{C''} \nu(\Delta_{0,i}) \right) \\
&\leq (\operatorname*{ess\,inf}_{\Delta_0}\ \varphi_n) e^{C+C''} \left(\sum_{i=1}^{\infty} i \cdot \nu(\Delta_{0,i}) \right) \\
&= (\operatorname*{ess\,inf}_{\Delta_0}\ \varphi_n) e^{C+C''} \nu(\Delta).
\end{aligned}
$$

Hence we have a uniform lower bound on φ_n on Δ_0 and therefore on all of $\Delta \setminus H_n$ for all $n \geq n^*$:

$$
\operatorname*{ess\,inf}_{\Delta \setminus H_n}\ \varphi_n = \operatorname*{ess\,inf}_{\Delta_0}\ \varphi_n \geq \left(e^{C+C''} \nu(\Delta) \right)^{-1} > 0.
$$

With (3.10) we are also able estimate an upper bound of φ_n, for $n \geq n^*$:

$$
\begin{aligned}
\operatorname*{ess\,sup}_{\Delta \setminus H_n}\ \varphi_n &\leq \lambda_n^{-n} \operatorname*{ess\,sup}_{\Delta_0}\ \varphi_n \\
&\leq e^{C''} \frac{e^{C}}{\nu(\Delta_0)} \int_{\Delta_0} \varphi_n \, \mathrm{d}\nu \\
&\leq \frac{e^{C+C''}}{\nu(\Delta_0)}.
\end{aligned}
$$

Since n^* is finite, we conclude that there exist constants $a > 0$ and $b > 0$ such that $\operatorname{ess\,inf}\ \varphi_n \geq a$ and $\operatorname{ess\,sup}\ \varphi_n \leq b$ for all $n \geq 1$. □

Corollary 3.7. *Let the hypotheses of Theorem 3.3 hold. There are constants a and b (independent of n) such that*

$$
a \leq \lim_{n \to \infty} \frac{-\log \lambda_n}{\nu(H_n^1)} \leq b.
$$

Proof. Using $\lim_{x \to 1} \frac{\log x}{1-x} = -1$ and $1 - \lambda_n = \int_{H_n^1} \varphi_n \, d\nu$ in conjunction with the result of Lemma 3.6 proves the claim. □

Theorem 3.8. *For every positive integer n let $\varphi_n \in \mathcal{C}_n$ and $\lambda_n < 1$ be as in Corollary 3.4. Then $\varphi_n \xrightarrow{L^1} \varphi$, where φ is the density of the unique absolutely continuous invariant probability measure μ of the closed system $F \colon \Delta \circlearrowleft$.*

Proof. The result of Lemma 3.6 ensures that all φ_n are elements of $\mathcal{C}_*^b$, which, as seen in Lemma 3.2, is compact. Hence a subsequence of (φ_n), say (φ_{n_i}), converges to some density φ'. Let (μ_{n_i}) and μ' be the corresponding measures. Then for any measurable $A \subseteq \Delta$ we have

$$\begin{aligned}
\mu'(F^{-1}A) &= \lim_{i\to\infty} \mu_{n_i}(F^{-1}A) \\
&\leq \lim_{i\to\infty} \mu_{n_i}(F^{-1}(A \setminus H_{n_i})) \\
&= \lim_{i\to\infty} \lambda_{n_i}\mu_{n_i}(A \setminus H_{n_i}) \\
&= \lim_{i\to\infty} \lambda_{n_i}\mu_{n_i}(A) = \mu'(A).
\end{aligned}$$

But $\mu' \circ F^{-1} \leq \mu'$ is possible only if μ' is invariant, therefore $\mu' = \mu$ and $\varphi' = \varphi$ almost everywhere. Hence $\varphi_n \to \varphi$ in $L^1(\Delta, \nu)$ as required. □

3.4 Realisation of ACCIM for T

We have mentioned earlier that there is a semi-conjugacy π between T and F. This enables us to translate all of the results for the tower down to the interval $[0,1]$. We summarize these below.

Theorem 3.9. *Let T be a Pomeau-Manneville map that satisfies* (T) *with $\alpha \in (0,1)$. Let $H_n^* = [0, x_n]$, be a nested sequence of Markov holes with $T^n(x_n) = x_0$ and $x_n < x_{n-1}$ for each $n \geq 1$. The following are true:*

(i) T admits a finite unique ACIM μ^* whose density h is bounded away from zero.

(ii) The map $T|_{(x_n,1]}$ with hole H_n^* admits an ACCIM μ_n^* with eigenvalue λ_n which is unique in the set of probability measures with densities bounded away from zero and infinity. Moreover the corresponding density $h_n := \mathrm{d}\mu_n^*/\mathrm{d}\ell$ is uniformly (independently of n) bounded away from zero.

(iii) The density of the ACCIM converges in L^1 to the density of ACIM; that is $h_n \xrightarrow{L^1} h$ as $n \to \infty$.

(iv) The escape rate $E((x_n, 1]; \ell)$ exists and equals $-\log \lambda_n \sim x_n$ as $n \to \infty$.

(v) *For an arbitrary hole $[0, \epsilon]$, the (upper and lower) escape rates into the hole asymptotically scale like ϵ.*

Proof.

(i) Existence of a unique ACIM is well-known. Proof and an argument on a lower bound on the density h can be found in [112, Theorem 5 and Section 6.3].

(ii) By Corollary 3.4 and Corollary 3.5 there is a unique ACCIM μ_n on the tower with density φ_n bounded above and below. Let $\pi : \Delta \to \mathrm{I}$ be the factor map for F and T and define $\mu_n^* := \mu_n \circ \pi^{-1}$. It is easy to check that μ_n^* is conditionally invariant with eigenvalue λ_n. Now let $\mathcal{P}_\pi : L^1(\Delta, \nu) \to L^1(\mathrm{I}, \ell)$ be the Perron-Frobenius operator of π (given by either (1.1) or (1.2)). Then the density $h_n = \mathrm{d}\mu_n^*/\mathrm{d}\ell$ satisfies $h_n = \mathcal{P}_\pi \varphi_n$. Obtaining a uniform lower bound on h_n is straightforward: if $x \in [x_0, 1]$, then $h_n(x) = \varphi_n(x, 0) \geq a$ (where a is as in Lemma 3.6); if $x \in [x_n, x_0)$, let $z \in (x_0, \gamma_1)$ be such that $T(z) = x$. Then

$$\begin{aligned} h_n(x) &= \mathcal{P}_\pi \varphi_n(x) \\ &= \sum_{(y,l) \in \pi^{-1}x} \frac{\varphi_n(y,l)}{J(T^l)(y)} \\ &\geq \frac{\varphi_n(z,1)}{JT(z)} \\ &\geq \frac{a}{JT(\gamma_1)}. \end{aligned}$$

Similarly, using the fact that π has a finite number of pre-images in each $\Delta \setminus H_n$, and by the uniform upper bound on φ_n we may obtain a (non-uniform) upper bound on h_n, dependent on n. Repeating the uniqueness argument of Corollary 3.4 we see that each μ_n^* is unique in a set of measures supported on $\Delta \setminus H_n$ with density bounded away from zero and infinity.

(iii) Clearly the Perron-Frobenius operator $\mathcal{P}_\pi$ is continuous. By Theorem we have 3.8 $\varphi_n \to \varphi$ in $L^1(\Delta, \nu)$ so by continuity $\mathcal{P}_\pi \varphi_n \to \mathcal{P}_\pi \varphi$ in $L^1(\mathrm{I}, \ell)$, that is $h_n \xrightarrow{L^1} h$.

(iv) Clearly $-\log \lambda_n$ is the escape rate of μ_n^* into H_n^*. To show that this is also the escape rate of the Lebesgue measure, we need to show that h_n is bounded away from zero and infinity. For every $x \in [x_n, 1]$ there are at most n points $(z, l) \in \Delta \setminus H_n$ such that $x = T^l(z)$, thus

$$\begin{aligned} h_n(x) &= \sum_{(y,l)\in\pi^{-1}x} \frac{\varphi_n(y,l)}{J(T^l)(y)} \\ &< nb, \end{aligned}$$

where b is as in Lemma 3.6. We have already shown in (ii) that f_n is bounded below by a positive constant, thus by Proposition 1.19, $E([x_n, 1]; \ell)$ exists and equals $-\log\lambda_n$. Finally, from Corollary 3.7

$$-\log\lambda_n \sim \nu(H_n^1) = \ell([x_0, \gamma_n]) \approx \frac{x_n}{JT(x_0)} \sim x_n.$$

(v) For any $\epsilon \in (0, x_0)$ we can find $n \in \mathbb{N}$ such that $x_n \leq \epsilon < x_{n-1}$. Hence

$$\frac{E([x_n, 1])}{x_n} \leq \frac{\underline{E}([\epsilon, 1])}{\epsilon} \leq \frac{\overline{E}([\epsilon, 1])}{\epsilon} \leq \frac{E([x_{n-1}, 1])}{x_{n-1}}.$$

Taking the limit as the denominators approach zero, and using the result of (iv), we get the required result $\underline{E}([\epsilon, 1]) \sim \overline{E}([\epsilon, 1]) \sim \epsilon$ as $\epsilon \to 0$.

□

3.5 On Second Eigenfunctions

We have previously mentioned that although the Perron-Frobenius operator of the Pomeau-Manneville map does not possess a spectral gap, certain approximations of it do. Here we will formalise this and give bounds on the second eigenvalue with the size of the perturbation. The results in this section are not a part of the original contribution of this thesis and are included for completeness. We refer the reader to [61, Section 5] for details of the proofs.

The proof showing existence of a second eigenfunction with bounds on its eigenvalue uses a novel setup of a signed Young tower, consisting of two "subtowers", $\Delta^{\epsilon,+}$ and $\Delta^{\epsilon,-}$ joined through the hole H_n. The first subtower $\Delta^{\epsilon,+} = \Delta \setminus H_n$ is as in the previous section. Points from $\Delta^{\epsilon,+}$ that enter H_n are no longer lost to the system but enter the other subtower $\Delta^{\epsilon,-}$ where the dynamics continue (with uniform expansion) before the orbit returns back to $\Delta^{\epsilon,+}$. The second eigenfunction of the corresponding Perron-Frobenius operator has the property that it is positive on $\Delta^{\epsilon,+}$ and negative on $\Delta^{\epsilon,-}$. In this thesis we will not deal with this double tower and will only state the results concerning the factor map T. For more information on the tower construction and the proofs, we refer the reader to our original paper [61].

As before let (x_n) be a sequence in $[0, x_0]$ defined recursively by $T(x_n) = x_{n-1}$ and let (γ_n) be a corresponding sequence in $(x_0, 1)$ defined by $T(\gamma_n) = x_{n-1}$. For $\epsilon > 0$ let $n = n(\epsilon)$ be the smallest integer such that $\gamma_n - x_0 < \epsilon$ and assign the following values to constants ϵ_0, ϵ_1 and ϵ_2:

$$\epsilon_0 := x_{n-1}, \tag{3.11}$$

$$\epsilon_1 := \gamma_n - x_0, \tag{3.12}$$

$$\epsilon_2 := \gamma_n - \gamma_{n+1}. \tag{3.13}$$

Now, we wish to perturb T so that when an orbit enters $[0, \epsilon_0]$, its new location is determined not by T, but by an appropriately distributed random variable. We can formalise this in the following way. Let z_k represent the points in an orbit of the perturbed system and let ξ_k be i.i.d. random variables on $[0, \epsilon_0]$ with density function

$$\rho^\epsilon(z) = \frac{1}{\epsilon_1 T'(T^{-1}_{right}(z))},$$

where $T_{right} : [x_0, x_0 + \epsilon_1] \to [0, \epsilon_0]$ is the right branch of T in the neighbourhood of the break point. The density ρ^ϵ is chosen to be the push-forward of the uniform density on $[x_0, x_0 + \epsilon_1]$ by the Perron-Frobenius operator $\mathcal{P}$ of T, that is for $f = \epsilon_1^{-1}$ on $[x_0, x_0 + \epsilon_1)$

and $f = 0$ otherwise we have

$$\mathcal{P}f(z) = \sum_{y \in T^{-1}z} \frac{f(y)}{|T'(y)|} = \frac{\epsilon_1^{-1}}{T'(T_{right}^{-1}(z))} = \rho^\epsilon(z).$$

When ϵ is small, ρ^ϵ is close to constant (since T is C^2 on the right-hand branch). The perturbed dynamics act according to

$$z_{k+1} = \begin{cases} \xi_{k+1}, & T(z_k) \in [0, \epsilon_0) \\ T(z_k), & \text{otherwise.} \end{cases} \tag{3.14}$$

Following the ideas of [79, Chapter 10], we will now derive the transfer operator of the process in (3.14). Let f_k be the density of z_k and let $B \in \mathcal{B}(\mathrm{I})$. Then

$$\begin{aligned} \text{Prob}\{z_{k+1} \in A\} &= \text{Prob}\{z_{k+1} \in A \text{ and } T(z_k) \in [0, \epsilon_0)\} \\ &\quad + \text{Prob}\{z_{k+1} \in A \text{ and } T(z_k) \in (\epsilon_0, 1]\} \\ &= \text{Prob}\{\xi_{k+1} \in A \text{ and } z_k \in T^{-1}[0, \epsilon_0]\} \\ &\quad + \text{Prob}\{z_k \in T^{-1}(A \cap (\epsilon_0, 1])\} \\ &= \int_A \rho^\epsilon \, \mathrm{d}\ell \int_{T^{-1}[0,\epsilon_0]} f_k \, \mathrm{d}\ell + \int_{T^{-1}(A \cap (\epsilon_0, 1])} f_k \, \mathrm{d}\ell \\ &= \int_A \rho^\epsilon \, \mathrm{d}\ell \int_0^{\epsilon_0} \mathcal{P} f_k \, \mathrm{d}\ell + \int_A \chi_{(\epsilon_0,1]} \mathcal{P} f_k \, \mathrm{d}\ell, \end{aligned}$$

so the distribution of z_{k+1} is given by

$$f_{k+1}(z) = \rho^\epsilon(z) \int_0^{\epsilon_0} \mathcal{P} f_k \, \mathrm{d}\ell + \chi_{(\epsilon_0,1]}(z) \mathcal{P} f_k(z),$$

and the transfer operator of the perturbed system $\mathcal{P}^\epsilon$ is then given by

$$\mathcal{P}^\epsilon f(z) = \begin{cases} \rho^\epsilon(z) \int_0^{\epsilon_0} f \, \mathrm{d}\ell, & z \in [0, \epsilon_0] \\ \mathcal{P} f(z), & \text{otherwise.} \end{cases}$$

Below we summarise results of [61, Theorem 5.6] concerning $\mathcal{P}^\epsilon$ and its spectrum.

Theorem 3.10 ([61]).

(i) *For any* $f \in L^1([0,1],\ell)$, $\mathcal{P}^\epsilon f \xrightarrow{L^1} \mathcal{P} f$ *as* $\epsilon \to 0$,

(ii) $\mathcal{P}^\epsilon$ *has an eigenvector* f^ϵ *satisfying* $\mathcal{P}^\epsilon f^\epsilon = f^\epsilon$, *and* $f^\epsilon \xrightarrow{L^1} f^*$, *where* f^* *is the density of* μ^*, *the unique ACIM for* T.

(iii) $\mathcal{P}^\epsilon$ *has an eigenvector* h^ϵ *satisfying* $\mathcal{P}^\epsilon h^\epsilon = \lambda^\epsilon h^\epsilon$ *where* $1 - \lambda^\epsilon \in \left(\frac{\epsilon_2}{\epsilon_1}, \frac{2\epsilon_2}{\epsilon_1}\right)$ *and* $[h^\epsilon]^+ \xrightarrow{L^1} \frac{1}{2} f^*$ *as* $\epsilon \to 0$.

3.6 Numerics

Ulam's method is an effective method for studying T numerically via its Perron-Frobenius operator. We create a partition of size $N \in \mathbb{N}$ by dividing $[0,1]$ uniformly into subintervals of length $1/N$, and construct the corresponding Ulam matrix P_N. The leading eigenvalue of P_N is 1, and the corresponding stationary eigenvector is a numerical approximation of a fixed point of $\mathcal{P}$. Surprisingly, given the absence of a spectral gap for $\mathcal{P}$, these fixed points converge to the density of the unique ACIM of T as $N \to \infty$ [11, 90].

Each Ulam matrix P_N is extremely sparse (having $\mathcal{O}(N)$ nonzero entries), and their eigenvalues can be found quickly by iterative methods. Because the dynamics of T are transitive, each P_N is irreducible, so the eigenvalue 1 has strictly larger modulus than the other eigenvalues. Interestingly, we observe that the *spectral gap* of P_N scales as $N^{-\alpha}$.

3.6.1 Eigenvalue Scaling

The two-state model of Section 3.2 showed that when the escape rate from the set $(0,\epsilon_0]$ approached zero more slowly than the escape rate from the set $[\epsilon_0, 1]$, the gap from 1 of the second eigenvalue of the two-state Markov chain scaled like the slower escape rate from $[0,\epsilon_0]$; namely ϵ_0^α.

We now replace the two-state model with the "N-state model" P_N arising from Ulam's method. The matrix P_N is row-stochastic, representing the transitions of a finite state

Markov chain whose i^{th} state is identified with the subinterval

$$J_i := [(i-1)/N, i/N).$$

The indifferent fixed point at 0 can be associated naturally with the subinterval $J_1 = [0, 1/N) \approx [0, \epsilon_0)$. The only nonzero conditional transition probabilities out of state 1 are $(P_N)_{11}$ and $(P_N)_{12}$ given by

$$\begin{aligned}(P_N)_{11} &= \frac{\ell(J_1 \cap T^{-1}J_1)}{\ell(J_1)} = 1 - N\,T^{-1}(1/N)\\(P_N)_{12} &= 1 - (P_N)_{11}.\end{aligned}$$

Thus the rate of escape from J_1 is

$$-\log(P_N)_{11} \approx 1 - (P_N)_{11} \sim N^{-\alpha} \sim \epsilon_0^{\alpha},$$

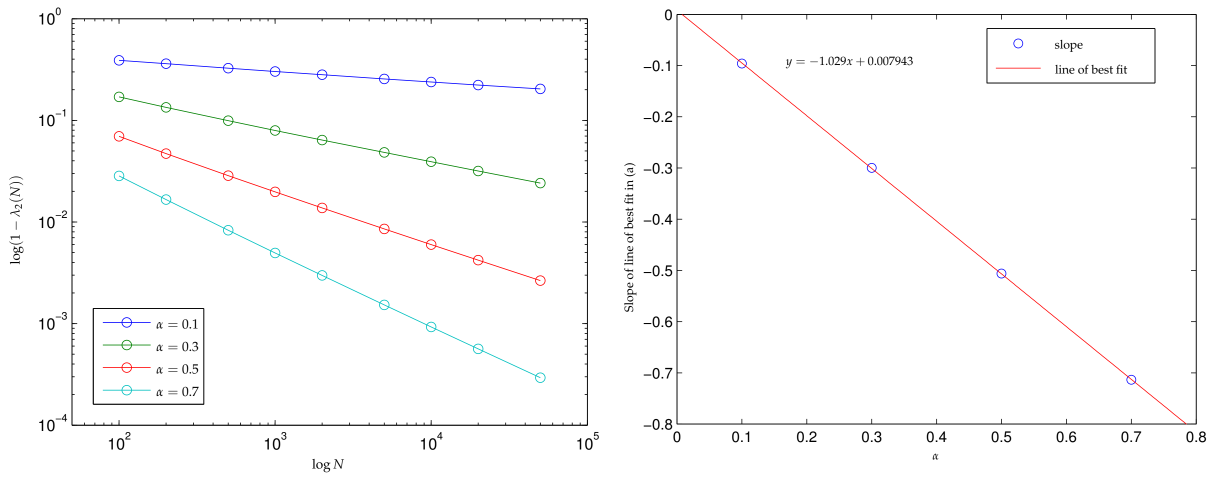
and this is of the same order as previously computed for the two-state model of Section 3.2.

We find numerically that despite increasing the number of states from two to N, the second eigenvalue of our N-state Ulam matrix retains the scaling predicted by the two-state model when $\epsilon_0 = 1/N$, namely $1 - \lambda_2(N) \sim N^{-\alpha}$; see Figure 3.3.

Connection with the Perron-Frobenius operator $\mathcal{P}^{\epsilon}$. Theorem 3.10 claims the existence of a second eigenvalue[8] λ^{ϵ}. The matrix P_N successfully reproduces the dynamics responsible for this eigenvalue, and we now explicitly describe the connection. Set $\epsilon = 1/(N\,JT(x_0^+))$ and choose $n, \epsilon_1, \epsilon_2$ as in (3.12) and (3.13). Then

$$T[x_0, x_0 + \epsilon_1] =: [0, \epsilon_0] \approx [0, 1/N) = J_1.$$

[8]More precisely, it is shown that there is another real eigenvalue very close to 1; based on numerical computations we conjecture that the eigenvalue λ^{ϵ} is indeed the second-largest real eigenvalue.



(a) (b)

Figure 3.3: (a) Variation of the second eigenvalue of P_N with N and α. (b) Slope of line of best fit for each α. Note: Computed by `eigs` in MATLAB, with Ulam matrices for the PM map [90, Example 3].

Theorem 3.10 predicts an eigenvalue of $\mathcal{P}^\epsilon$,

$$\lambda^\epsilon \in \left(1 - \frac{2\epsilon_2}{\epsilon_1}, 1 - \frac{\epsilon_2}{\epsilon_1}\right).$$

Numerical computations with P_N for a range of N produce second eigenvalues within this range. In fact, the upper limit is a very good estimate; see Table 3.1.

3.6.2 Ulam's Method and the Escape Rate from $[1/N, 1]$

We conclude this chapter with some simple remarks on how to observe the ACCIMs and their escape rates, numerically. The measure $\mu^* = \mu \circ \pi^{-1}$ is an ACIM for T, and $\mu_n^* = \mu_n \circ \pi^{-1}$ is an ACCIM for $T|_{(x_n,1]}$ with escape rate $-\log \lambda_n \sim x_n$. Hence, if $x_n \approx 1/N$, then one expects the escape rate from $(x_n, 1]$ to scale like $1/N$. Now partition the $N \times N$ Ulam matrix P_N as

$$P_N = \left[\begin{array}{c|c} (P_N)_{11} & \mathbf{a}^T \\ \hline \mathbf{b} & P_N^o \end{array}\right]$$

N	$1-\lambda_2(N)$	ϵ_2/ϵ_1
100	0.069494728128226	0.060750416292176
200	0.047118990434159	0.042626262679704
500	0.028582682402957	0.026696029895732
1000	0.019751285772241	0.018706181316717
2000	0.013727390048589	0.013165183357731
5000	0.008542396305559	0.008301674655368
10000	0.005988977377968	0.005866565930472
20000	0.004208535921532	0.004150111773511
50000	0.002646628586393	0.002621525600809

Table 3.1: Comparison of $1-\lambda_2(N)$ computed numerically as a second eigenvalue of the $N \times N$ Ulam matrix and the corresponding lower bound ϵ_2/ϵ_1 obtained from Theorem 3.10 ($\alpha = 0.5$).

where $\mathbf{a}, \mathbf{b}$ are $(N-1)$-vectors and P_N^o is an $(N-1) \times (N-1)$ matrix. In fact, P_N^o is the Ulam approximation to the conditional Perron-Frobenius operator $\chi_{[1/N,1]}\mathcal{P}(\cdot\,\chi_{[1/N,1]})$. In Figure 3.4 we present numerical evidence that the leading eigenvalue $\lambda_1^o(N)$ of P_N^o has the scaling $1-\lambda_1^o(N) \sim 1/N$, *independently of* α.

Finally, Theorem 3.9 predicts the convergence of the ACCIMs μ_n^* to the ACCIM as the size of the hole H_n^* shrinks to 0. We illustrate this convergence numerically as follows. For a large N_* (we have used $N_* = 10^5$), form P_{N_*} and calculate the leading eigenvector. This is a good approximation to the density of the ACIM for T [90], and we use it as a reference measure. Next, for a sequence of smaller N_k (we used the values from the first column of Table 3.1), calculate the leading eigenvector of $P_{N_k}^o$. Comparing the probability measure induced by these eigenvectors with the reference measure from the Ulam approximation P_{N_*} we see good convergence in Figure 3.5.

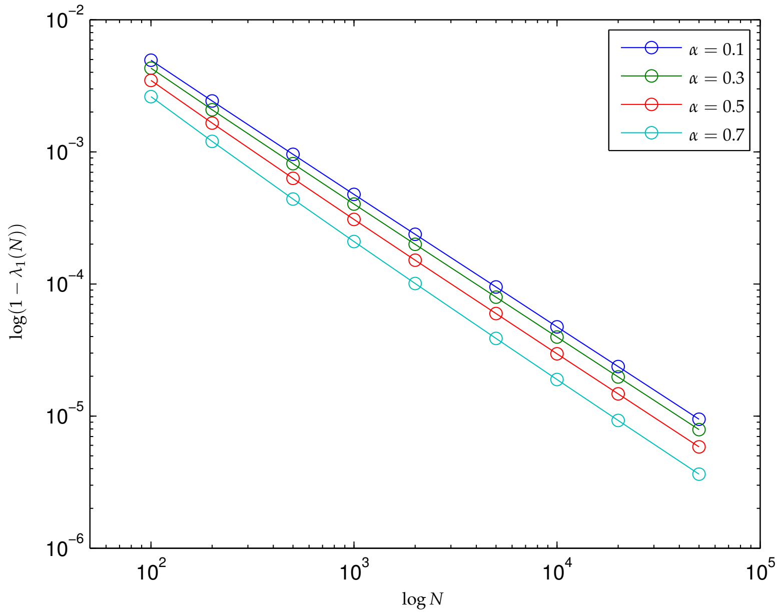


Figure 3.4: Variation of escape rate from $[1/N, 1]$ with N and α.

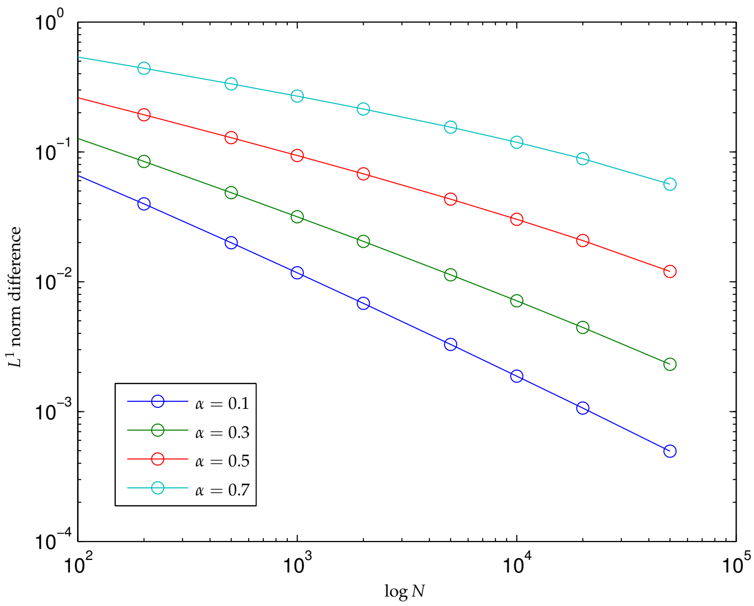


Figure 3.5: Total variation norm error between Ulam approximate ACCIMs with a hole $[0, 1/N]$ and the Ulam approximate ACIM with a bin size of 10^{-5}. A range of different PM maps are used, (α is order of tangency of the indifferent fixed point).

Chapter 4

Open Random Dynamical Systems

Random dynamical systems (RDS) provide a setting to model non-autonomous or time-dependent phenomena. They may also serve as a model for noise or uncertainty in otherwise deterministic dynamical systems. We will provide the details later, but for now we note that the randomness is modelled on an abstract (probability) space Ω and the rules/behaviour of the system are dependent on $\omega \in \Omega$.

The concepts of metastability and almost-invariant sets were extended to random dynamical systems in Froyland *et al.* [58, 59], where these sets are referred to as *coherent structures*. To move from deterministic to random (or time-dependent) concepts of metastability, the authors introduced the *Lyapunov spectrum* for cocycles of random Perron-Frobenius operators $\mathcal{P}_\omega$, replacing the spectrum of a single deterministic Perron-Frobenius operator $\mathcal{P}$. The associated random *Oseledets subspaces* now play the role of eigenfunctions when determining the random metastable sets. Numerical algorithms and experiments based on this theory were detailed in [60]. Other work on metastability of random or perturbed dynamical systems includes papers of Colonius *et al.* [34, 35].

Our goal here is to link the slow decay of random (ω-dependent) functions induced by the Perron-Frobenius cocycle with escape rates from random metastable sets. Studies of escape rates for random dynamical systems have, to our knowledge, only been concerned with escape from *fixed* (ω-invariant) sets under random or randomly-perturbed maps (see for example [33, 70, 71]; and for more recent work [41, 98]). In this chapter we shall, however, deal with the more general concept of escape from a *random* set under a *random* map. We extend the results of Chapter 2 to random dynamical systems by showing a relationship between Lyapunov spectrum and the corresponding random escape rates from metastable sets. The material of this chapter has appeared in [66].

A summary of this chapter is as follows. In Section 4.2, we will show in a rather general setting (X measurable, non-singular dynamics) that given a Lyapunov exponent $\lambda(\omega, f) \lesssim 0$ of $\omega \in \Omega$ and $f \in L^\infty$, one can define metastable random sets $A_\pm$ along the orbit of ω as $A_\pm(\vartheta^n\omega) = \{\pm\mathcal{P}_\omega^{(n)} f > 0\}$. Our first main result (Theorem 4.7) analogously to Theorem 2.5 states that the escape rates from $A_\pm$ (with respect to ω) are slower than $-\lambda(\omega, f)$. In Section 4.4 we will extend these results to quasi-compact random dynamical systems that admit an Oseledets splitting and, in particular, to Rychlik random dynamical systems where the dynamics are given by random expanding piecewise C^2 interval maps and $\mathcal{P}$ acts on BV. In this setting of random Rychlik maps, Froyland *et al.* [58] proved a result parallel to the result of Keller [74], relating the average expansion of trajectories to the essential spectral radius (cf. Section 1.1.2). Our main result in Section 4.4 will show a relation between the escape rate from random almost-invariant sets and isolated values in the Lyapunov spectrum of $\mathcal{P}$.

4.1 A Brief Introduction to Random Dynamical Systems

We will follow the definitions and notation of Arnold [1], whose book we recommend to the reader as a thorough introduction to random dynamical systems.

There are two key ingredients that constitute a random dynamical system. The first is an invertible *deterministic* measure-preserving dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ (where $\mathbb{P}$ is a probability), which serves as a model of noise or randomness. For technical reasons we shall additionally assume that singletons of Ω are $\mathcal{F}$-measurable. This system is called the *base*. The second ingredient is a space Z and a collection of endomorphisms or transformations on Z indexed by Ω, $\tilde{\Phi} : \Omega \to \mathrm{End}(Z)$. We will assume that for each $\omega \in \Omega$, $\tilde{\Phi}(\omega)$ preserves whatever structure Z may have (such as linearity or measurability).

We refer to the tuple $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta, Z, \tilde{\Phi})$ as a *random dynamical system*. The dynamics on points in Z are determined by the mapping $\Phi : \mathbb{N} \times \Omega \to \mathrm{End}(Z)$, given by

$$\Phi(n,\omega) = \Phi^{(n)}(\omega) = \Phi^{(n)}_{\omega} := \tilde{\Phi}(\vartheta^{n-1}\omega) \circ \cdots \circ \tilde{\Phi}(\vartheta\omega) \circ \tilde{\Phi}(\omega) \tag{4.1}$$

satisfying the *cocycle property*:

(C1) $\Phi(0,\omega) = id$;

(C2) $\forall m, n \in \mathbb{N}, \Phi(m+n,\omega) = \Phi(m,\vartheta^n\omega) \circ \Phi(n,\omega)$.

In (4.1) for notational convenience we have adopted the convention of showing dependence on ω by subscripting and dependence on n by superscripting. We will refer to Φ as the *cocycle* while $\tilde{\Phi}$ will be its corresponding *generator*. Two (related) types of cocycles that we will study in this chapter are *measurable map cocycles* and their *Perron-Frobenius operator cocycles*. In the next chapter we shall deal with *adjacency matrix cocycles*.

Measurable Cocycles

We assume at first that Z is a measure space $(X, \mathcal{B}, m)$ where $\mathcal{B}$ is its σ-algebra m finite measure. A *measurable (map) cocycle* is a mapping $T\colon \mathbb{N} \times \Omega \to \mathrm{End}(X)$ satisfying (C1) and (C2) and such that $(\omega, x) \mapsto T^{(n)}_{\omega}(x)$ is $(\mathcal{F} \otimes \mathcal{B}, \mathcal{B})$-measurable for each $n \in \mathbb{N}$, while every $T_{\omega} : X \circlearrowleft$ is non-singular with respect to m.

Define a *random set*[1] to be any set-valued function $A : \Omega \to \mathcal{B}$ such that the graph $\{(\omega, A(\omega)) \ : \ \omega \in \Omega\}$ is measurable in the product σ-algebra $\mathcal{F} \otimes \mathcal{B}$. We are now in a position to define *rate of escape* from a given random set under the random dynamics of the cocycle.

Definition 4.1. Let $T : \mathbb{N} \times \Omega \to \mathrm{End}(X, \mathcal{B}, m)$ be a measurable cocycle over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ and let $A : \Omega \to \mathcal{B}$ be a measurable random set. The *random escape rate* with respect to m is the non-negative valued function $E(A, \cdot) : \Omega \to \mathbb{R}$ given by

$$E(A,\omega) := -\limsup_{n\to\infty} \frac{1}{n} \log m(A^{(n)}(\omega)), \quad \omega \in \Omega, \tag{4.2}$$

[1] Our definition of a random set is slightly weaker than Arnold's [1] definition of a closed random set, where X is additionally Polish (with metric d) and for every $x \in X$ the mapping $\omega \mapsto d(x, A(\omega))$ is measurable.

where

$$A^{(n)}(\omega) := \bigcap_{i=0}^{n-1} T(i,\omega)^{-1} A(\vartheta^i \omega). \tag{4.3}$$

Limit supremum is used above in order to have a well-defined rate of escape even when the limit does not exist. Strictly speaking, what we refer to as 'escape rate' in this chapter corresponds to what we considered to be 'lower escape rate' in the previous chapters.

In Definition 4.1 we defined escape rate from a random set. It is, however, often of interest in dynamical systems to study properties of single orbits. In order to study the escape rate along a sample orbit we can restrict the domain of a random set just to this particular orbit.

Definition 4.2. Let $A : \Omega \to \mathcal{B}$ be a random set and let $\omega^* \in \Omega$. We shall refer to the restriction of A to the orbit $\{\vartheta^n \omega^*\}_{n\in\mathbb{Z}^+}$ as an *orbit set*.

The following proposition shows that any mapping $A : \{\vartheta^n \omega^*\}_{n\in\mathbb{Z}^+}$ is an orbit set, as it may be trivially extended to a random set.

Proposition 4.3. *For a fixed $\omega^* \in \Omega$, any mapping $A : \{\vartheta^n \omega^*\}_{n\in\mathbb{Z}^+} \to \mathcal{B}$ may be extended to a random set by defining $A(\omega) = X$ for all $\omega \in \Omega \setminus \{\vartheta^n \omega^*\}_{n\in\mathbb{Z}^+}$.*

Proof. To see that this extension indeed produces a set $\{(\omega, A(\omega)\} \in \mathcal{F} \otimes \mathcal{B}$ note that we may write the graph of A as the union of $(\Omega \setminus \{\vartheta^n \omega^*\}_{n\in\mathbb{Z}^+}) \times X$ and $\bigcup_{n\in\mathbb{Z}^+}(\vartheta^n\omega^*, A(\vartheta^n\omega^*))$. The former set is a rectangle in $\mathcal{F} \times \mathcal{B}$ and the latter set is a countable union of measurable rectangles, as all singletons are $\mathcal{F}$-measurable. Thus the graph of A is $(\mathcal{F} \otimes \mathcal{B})$-measurable. $\square$

Below we show that, provided the base system is ergodic, escape rate is constant almost everywhere.

Proposition 4.4. *Assume that the base system $(\vartheta, \mathbb{P})$ is ergodic and that for almost every $\omega \in \Omega$ the Radon-Nikodym derivative $\frac{\mathrm{d}(m \circ T_\omega^{-1})}{\mathrm{d}m}$ is bounded. For any fixed random set $A : \Omega \to \mathcal{B}$, $E(A, \omega)$ is constant almost everywhere.*

Proof. We begin with the observation that $A^{(n)}(\omega) = T_\omega^{-1} A^{(n-1)}(\vartheta\omega) \cap A(\omega)$ (see (4.3)) so that $m(A^{(n)}(\omega)) \leq m(T_\omega^{-1} A^{(n-1)}(\vartheta\omega))$. Using the boundedness of the Radon-Nikodym derivative one sees that $E(A,\omega) \leq E(A,\vartheta\omega)$. We will now show that $E(A,\omega) = E(A)$ for almost any $\omega \in \Omega$. Assume otherwise. As A is a random set, $E(A,\cdot)$ is measurable and there exists $c \in \mathbb{R}$ such that the set $S := \{\omega \ : \ E(A,\omega) \geq c\}$ has $\mathbb{P}(S) \in (0,1)$. Since $\vartheta^{-1}(S) \supseteq S$ and ϑ preserves $\mathbb{P}$ we must have $\vartheta^{-1}S = S$ almost everywhere, but this cannot be since $\mathbb{P}$ is ergodic. □

Perron-Frobenius Operator Cocycles

On the other hand, if we let Z be the space of integrable functions $L^1(X,\mathcal{B},m)$ (or a subspace of $L^1(X,\mathcal{B},m)$), we may define a *Perron-Frobenius operator cocycle* as follows.

Definition 4.5. Let $T : \mathbb{N} \times \Omega \to \mathrm{End}(X,\mathcal{B},m)$ be a measurable map cocycle over $(\Omega,\mathcal{F},\mathbb{P},\vartheta)$. The corresponding *Perron-Frobenius operator cocycle* is a linear cocycle $\mathcal{P} : \mathbb{N} \times \Omega \to \mathrm{End}(L^1(X,\mathcal{B},m))$ whose generator $\tilde{\mathcal{P}}$ is given by

$$\int_B \tilde{\mathcal{P}}(\omega) f \,\mathrm{d}m = \int_{T_\omega^{-1}B} f \,\mathrm{d}m, \quad \forall \omega \in \Omega,\ \forall B \in \mathcal{B},\ \forall f \in L^1(X,\mathcal{B},m). \tag{4.4}$$

Definition 4.6 (Lyapunov exponent)**.** Let $\mathcal{P}\colon \mathbb{N} \times \Omega \to \mathrm{End}(L^1(X,\mathcal{B},m))$ be the Perron-Frobenius operator cocycle corresponding to a measurable map cocycle $T\colon \mathbb{N} \times \Omega \to \mathrm{End}(X)$. For any $f \in L^1(X,\mathcal{B},m)$, and $\omega \in \Omega$ the *Lyapunov exponent* is defined to be

$$\lambda(\omega,f) := \limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{P}_\omega^{(n)} f\|_{L^1}.$$

We also define the *Lyapunov spectrum* to be the set of all Lyapunov exponents:

$$\Lambda(\omega) := \{\lambda(\omega,f) \ : \ f \in L^1(X,\mathcal{B},m)\},$$

and the quantity $\lambda(\omega) \in \mathbb{R}$ by

$$\lambda(\omega) := \limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{P}_\omega^{(n)}\|_{op}.$$

As each $\mathcal{P}_\omega : L^1(m) \circlearrowleft$ is a Markov operator we have $\|\mathcal{P}_\omega f\|_{L^1} \leq \|f\|_{L^1}$ and therefore $\Lambda(\omega) \subseteq [-\infty, 0]$. With the definition of the Perron-Frobenius cocycle in (4.4), it is natural to use the L^1-norm for calculating the Lyapunov spectrum. However we will see later in Section 4.4 that when working with subspaces of L^1 other norms are sometimes more informative.

4.2 A Result on Escape Rate for a General Random Dynamical System

Our main theorem of this chapter relates the Lyapunov exponents of a Perron-Frobenius operator cocycle $\mathcal{P}$ to the rates of escape from particular orbit sets under the corresponding measurable map cocycle T.

Theorem 4.7. *Let $T : \mathbb{N} \times \Omega \to \mathrm{End}(X, \mathcal{B}, m)$ be a measurable map cocycle over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ and let $\mathcal{P}! : \mathbb{N} \times \Omega \to \mathrm{End}(L^1(X, \mathcal{B}, m))$ be the corresponding Perron-Frobenius cocycle as defined in* (1.1). *Fix an aperiodic $\omega^* \in \Omega$ and suppose that there exists an $f \in L^\infty$ such that $\lambda(\omega^*, f) < 0$. Let $A_+, A_- : \{\vartheta^n \omega^*\}_{n \in \mathbb{Z}^+} \to \mathcal{B}$ be defined by*

$$A_\pm(\vartheta^n \omega^*) := \{x \in X \ : \ \pm \mathcal{P}^{(n)}_{\omega^*} f(x) > 0\}, \quad n \in \mathbb{Z}^+. \tag{4.5}$$

Then $A_\pm$ are orbit sets and one has $E(A_\pm, \omega^*) \leq -\lambda(\omega^*, f)$.

The fact that the sets defined in (4.5) are orbit sets was shown in Proposition 4.3. The proof of the rest of Theorem 4.7 follows after a preliminary lemma.

Lemma 4.8. *In the notation of Theorem 4.7 we have for every $n \in \mathbb{Z}^+$*

$$\int_{A_+(\vartheta^n \omega^*)} \mathcal{P}^{(n)}_{\omega^*} f \, \mathrm{d}m = \frac{1}{2} \|\mathcal{P}^{(n)}_{\omega^*} f\|_{L^1}.$$

Proof. Firstly we will show that $\int_X \mathcal{P}^{(n)}_{\omega^*} f \, \mathrm{d}m = 0$ for all $n \geq 0$. From (4.4) one can see that $\mathcal{P}_{\omega^*}$ preserves integrals over all of X therefore $\int_X \mathcal{P}^{(n)}_{\omega^*} f \, \mathrm{d}m = M$, a constant for all

$n \geq 0$. This implies that $\|\mathcal{P}_{\omega^*}^{(n)} f\|_{L^1} \geq |M|$ for all $n \geq 0$. Suppose that $M \neq 0$. Then

$$\lambda(\omega^*, f) = \limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{P}_{\omega^*}^{(n)} f\|_{L^1} \geq \limsup_{n\to\infty} \frac{1}{n} \log |M| = 0.$$

This is a contradiction as $\lambda(\omega^*, f) < 0$, therefore $M = 0 = \int_X \mathcal{P}_{\omega^*}^{(n)} f \; \mathrm{d}m$. Now we have

$$0 = \int_{A_+(\vartheta^n\omega^*)} \mathcal{P}_{\omega^*}^{(n)} f \; \mathrm{d}m + \int_{X\setminus A_+(\vartheta^n\omega^*)} \mathcal{P}_{\omega^*}^{(n)} f \; \mathrm{d}m \tag{4.6}$$

and

$$\|\mathcal{P}_{\omega^*}^{(n)} f\|_{L^1} = \int_{A_+(\vartheta^n\omega^*)} \mathcal{P}_{\omega^*}^{(n)} f \; \mathrm{d}m - \int_{X\setminus A_+(\vartheta^n\omega^*)} \mathcal{P}_{\omega^*}^{(n)} f \; \mathrm{d}m. \tag{4.7}$$

Adding equations (4.6) and (4.7) takes us to the required result. □

Proof of Theorem 4.7. The proof is a random version of the proof of Theorem 2.5. Let j, n be integers such that $0 \leq j \leq n$ and let $B \in \mathcal{B}$. Using (4.4) we derive the following:

$$\begin{aligned}
\int_B \mathcal{P}_{\omega^*}^{(j+1)} f \; \mathrm{d}m &= \int_{T_{\vartheta^j\omega^*}^{-1} B} \mathcal{P}_{\omega^*}^{(j)} f \; \mathrm{d}m \\
&= \int_{T_{\vartheta^j\omega^*}^{-1} B} (\mathcal{P}_{\omega^*}^{(j)} f)\chi_{A_+(\vartheta^j\omega^*)} \; \mathrm{d}m + \int_{T_{\vartheta^j\omega^*}^{-1} B} (\mathcal{P}_{\omega^*}^{(j)} f)\chi_{X\setminus A_+(\vartheta^j\omega^*)} \; \mathrm{d}m \\
&\leq \int_{T_{\vartheta^j\omega^*}^{-1} B} (\mathcal{P}_{\omega^*}^{(j)} f)\chi_{A_+(\vartheta^j\omega^*)} \; \mathrm{d}m \\
&= \int_{T_{\vartheta^j\omega^*}^{-1} B\cap A_+(\vartheta^j\omega^*)} \mathcal{P}_{\omega^*}^{(j)} f \; \mathrm{d}m.
\end{aligned}$$

Hence

$$\int_B \mathcal{P}_{\omega^*}^{(j+1)} f \; \mathrm{d}m \leq \int_{T_{\vartheta^j\omega^*}^{-1} B\cap A_+(\vartheta^j\omega^*)} \mathcal{P}_{\omega^*}^{(j)} f \; \mathrm{d}m.$$

Now letting $B = A_+^{(n-j-1)}(\vartheta^{j+1}\omega^*)$ (defined as in (4.3)) we have for all $j \geq 0$

$$\int_{A_+^{(n-j-1)}(\vartheta^{j+1}\omega^*)} \mathcal{P}_{\omega^*}^{(j+1)} f \; \mathrm{d}m \leq \int_{A_+^{(n-j)}(\vartheta^j\omega^*)} \mathcal{P}_{\omega^*}^{(j)} f \; \mathrm{d}m,$$

where we have used the relation

$$A_+^{(n-j)}(\vartheta^j\omega^*) = A_+(\vartheta^j\omega^*) \cap T_{\vartheta^j\omega^*}^{-1}(A_+^{(n-j-1)}(\vartheta^{j+1}\omega^*)),$$

easily obtainable from (4.3). By considering all $j = 0, 1, \ldots, n-1$ we arrive at the following series of inequalities:

$$\int_{A_+^{(0)}(\vartheta^n\omega^*)} \mathcal{P}_{\omega^*}^{(n)} f \, \mathrm{d}m \leq \int_{A_+^{(1)}(\vartheta^{n-1}\omega^*)} \mathcal{P}_{\omega^*}^{(n-1)} f \, \mathrm{d}m \leq \cdots \leq \int_{A_+^{(n)}(\omega^*)} f \, \mathrm{d}m.$$

Hence

$$\begin{aligned} \frac{1}{2}\|\mathcal{P}_{\omega^*}^{(n)} f\|_{L^1} &= \int_{A_+(\vartheta^n\omega^*)} \mathcal{P}_{\omega^*}^{(n)} f \, \mathrm{d}m \\ &\leq \int_{A_+^{(n)}(\omega^*)} f \, \mathrm{d}m \\ &\leq \|f\|_{L^\infty} m(A_+^{(n)}(\omega^*)), \end{aligned}$$

where the equality above is due to Lemma 4.8, and the second inequality holds because $f \in L^\infty(m)$. By taking logarithms, dividing by n and taking limit supremum as $n \to \infty$ we arrive at the required inequality $E(A_+, \omega^*) \leq -\lambda(\omega^*, f)$. The inequality for $E(A_-, \omega^*)$ is obtained by repeating the procedure, while considering $-f$ in place of f. □

Remark 4.9. We note that, if for a random set $A : \Omega \to \mathcal{B}$ (or an orbit set $A : \{\vartheta^n\omega^*\}_{n\in\mathbb{Z}^+} \to \mathcal{B}$) one defines a *conditional operator cocycle* $\mathcal{P}_A$ by $\tilde{\mathcal{P}}_A(\omega) f := \tilde{\mathcal{P}}(\omega)(f\chi_{A(\omega)})$ for all $\omega \in \Omega$ (or $\omega \in \{\vartheta^n\omega^*\}_{n\in\mathbb{Z}^+}$) and $f \in L^1$, then the Lebesgue escape rate is given by

$$E(A, \omega) = -\limsup \frac{1}{n} \log \|\mathcal{P}_A(n, \omega)\mathbb{1}\|_{L^1},$$

which equals in absolute value to the Lyapunov exponent of a constant function with respect to this conditional cocycle.

Remark 4.10. Note that the sets $A_\pm$ defined by (4.5) are only guaranteed to be orbit sets when ω^* is aperiodic. If ω is periodic, one would further require f to be an eigenfunction of $\mathcal{P}_{\omega^*}^{(p)}$ (where p is the period), in which case the equivalent result would be given by

Theorem 2.5 of Chapter 2.

4.2.1 Choosing a Metastable Partition

Theorem 4.7 presents a method of finding pairs of orbit sets whose ω-fibres form 2-partitions of X. Both of these orbit sets have low escape rates. Theorem 4.7 applies to a large class of random dynamical systems. For the remainder of this section we will investigate some of the consequences of this result. Similarly to its deterministic counterpart (Lemma 2.11), Lemma 4.12 will show that in a very general setting one may choose any $\rho \in [-\infty, 0)$, find an appropriate $f \in L^1(m)$ with Lyapunov exponent $\lambda(\omega, f) = \rho$ and obtain two random sets with escape lower than $-\rho$. In particular there is no spectral gap and ρ may be arbitrarily close to 0, however, as will be suggested in Example 4.13, such ρ often results in highly irregular random metastable sets.

Definition 4.11. A mapping $h : \Omega \to L^1(X, \mathcal{B}, m)$ is said to be a *random L^1-function* if $(\omega, x) \mapsto h(\omega, x)$ is $(\mathcal{F} \otimes \mathcal{B}, \mathcal{B}(\mathbb{R}))$-measurable. If each h_ω is a density in $L^1(X, \mathcal{B}, m)$, it is called a *random density*. Such a density is said to be *preserved* by a Perron-Frobenius operator cocycle $\mathcal{P}$ if $\mathcal{P}_\omega h_\omega = h_{\vartheta\omega}$ for almost every $\omega \in \Omega$.

Lemma 4.12. *Let $\mathcal{P} : \mathbb{N} \times \Omega \to \mathrm{End}(L^1(X, \mathcal{B}, m))$ be a Perron-Frobenius operator cocycle (of a measurable map cocycle T) over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ that preserves a positive random density $h : \Omega \to L^\infty(X, \mathcal{B}, m)$. Suppose that there exists a random function $g : \Omega \to L^\infty(X, \mathcal{B}, m)$ so that $\mathcal{P}_\omega g_\omega = 0$ for almost all $\omega \in \Omega$. Then for every $\rho \in [-\infty, 0]$ there exists a random function $f : \Omega \to L^\infty(X, \mathcal{B}, m)$ such that $\lambda(\omega, f_\omega) = \rho$ for almost every $\omega \in \Omega$.*

Proof. We adapt the argument of proof of Lemma 2.11 to the random setting. Define f so that $f_\omega := \sum_{n=0}^{\infty} e^{\rho n} (g_{\vartheta^n\omega} / h_{\vartheta^n\omega}) \circ T_\omega^{(n)} \cdot h_\omega$ for every $\omega \in \Omega$. The facts that f is measurable and each $f_\omega \in L^\infty(m)$ are inherited from the corresponding properties of g and h. For any $B \in \mathcal{B}$ we have

$$\begin{aligned}\int_B \mathcal{P}_\omega f_\omega \, \mathrm{d}m &= \int_{T_\omega^{-1}B} f_\omega \, \mathrm{d}m \\ &= \int_{T_\omega^{-1}B} \sum_{n=0}^{\infty} e^{\rho n} (g_{\vartheta^n\omega} / h_{\vartheta^n\omega}) \circ T_\omega^{(n)} \cdot h_\omega \, \mathrm{d}m\end{aligned}$$

$$
\begin{aligned}
&= \int_{T_\omega^{-1}B} g_\omega \,\mathrm{d}m + \int_{T_\omega^{-1}B} \sum_{n=1}^{\infty} e^{\rho n} (g_{\vartheta^n\omega}/h_{\vartheta^n\omega}) \circ T_\omega^{(n)} \cdot h_\omega \,\mathrm{d}m \\
&= 0 + e^{\rho} \int_{T_\omega^{-1}B} \sum_{n=0}^{\infty} e^{\rho n} (g_{\vartheta^{n+1}\omega}/h_{\vartheta^{n+1}\omega}) \circ T_\omega^{(n+1)} \cdot h_\omega \,\mathrm{d}m \\
&= e^{\rho} \int_{B} \sum_{n=0}^{\infty} e^{\rho n} (g_{\vartheta^{n+1}\omega}/h_{\vartheta^{n+1}\omega}) \circ T_{\vartheta\omega}^{(n)} \cdot h_{\vartheta\omega} \,\mathrm{d}m \\
&= e^{\rho} \int_B f_{\vartheta\omega}.
\end{aligned}
$$

Thus $\mathcal{P}_\omega f_\omega = e^\rho f_{\vartheta\omega}$ almost everywhere. Now for $\epsilon > 0$ let

$$\Omega_\epsilon := \{\omega \in \Omega \;:\; \|f_\omega\|_{L^1} \geq \epsilon\}.$$

Since $\omega \mapsto \|f_\omega\|_{L^1}$ is measurable, the set Ω_ϵ is also measurable. Fix ϵ sufficiently small so that $\mathbb{P}(\Omega_\epsilon) > 0$. The Poincaré Recurrence Theorem [96, Chapter 26] (see [9] for an in-depth historical account) asserts that $\mathbb{P}$-almost surely there is a sequence $m_k \uparrow \infty$ such that $\vartheta^{m_k}\omega \in \Omega_\epsilon$. Hence

$$0 \geq \limsup_{n\to\infty} \frac{1}{n} \log \|f_{\vartheta^n\omega}\|_{L^1} \geq \limsup_{k\to\infty} \frac{1}{m_k} \log \|f_{\vartheta^{m_k}\omega}\|_{L^1} \geq 0,$$

from which we obtain

$$
\begin{aligned}
\lambda(\omega, f) &= \limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{P}_\omega^{(n)} f_\omega\|_{L^1} \\
&= \limsup_{n\to\infty} \frac{1}{n} \log e^{\rho n} \|f_{\vartheta^n\omega}\|_{L^1} \\
&= \rho + \limsup_{n\to\infty} \frac{1}{n} \log \|f_{\vartheta^n\omega}\|_{L^1} = \rho.
\end{aligned}
$$

□

It is clear that the set-valued mappings $A_\pm : \Omega \to \mathcal{B}$ defined by $A_\pm(\omega) := \{\pm f_\omega\}$ obtained from a random function f are indeed random sets. Thus an application of Theorem 4.7 to f in Lemma 4.12 implies that for any negative ρ, arbitrarily close to zero, there exist complementary random sets whose random rate of escape is slower than $-\rho$.

Example 4.13. Let $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ be the full two-sided 2-shift on $\{0,1\}$ equipped with the σ-algebra $\mathcal{F}$ generated by cylinders, and with Bernoulli probability measure $\mathbb{P}$ ($\mu_{(p,P)}$ of Example 1.11). Let $\tilde{T}\colon \Omega \to \mathrm{End}([0,1])$ be the generator of a cocycle T, constant on cylinders, given by $\tilde{T}(\omega) := T_{\omega_0}$ where $T_i(x) := 2x + \alpha_i \pmod 1$ for $\alpha_i \in \mathbb{R}$, $i = 0,1$. It is easy to check that the corresponding Perron-Frobenius operator cocycle $\mathcal{P}$ satisfies Lemma 4.12 with $h_\omega \equiv 1$ for all ω and $g_\omega = g_{\omega_0}$ where

$$g_i(x) = \begin{cases} -1/2, & 0 \le x - \alpha_i \pmod 1 \le 1/2 \\ 1/2, & 1/2 < x - \alpha_i \pmod 1 \le 1. \end{cases}, \quad i \in \{0,1\}.$$

After applying Lemma 4.12 we conclude that any $\rho \in [-\infty, 0]$ is a Lyapunov exponent with an essentially bounded Lyapunov function, hence by Theorem 4.7, there exist complementary random sets with arbitrarily low escape rates.

For a numerical demonstration we set $\alpha_0 = 0$ and $\alpha_1 = 0.6$. We choose $\omega^* \in \Omega$ such that $\omega^*_i = 0$ for all $i < 0$ and ω^*_i equals the $(i+1)^{\text{th}}$ digit in the fractional part of the binary expansion of π for $i \ge 0$. The first few central elements of ω^*, with the zeroth element underlined, are:

$$\omega^* = (\dots, 0, 0, \underline{0}, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, \dots).$$

Numerical approximations of f_{ω^*} for some values of ρ are shown in Figure 4.1. For $\rho = -1$ applying the construction in Theorem 4.7 we see from the graph of f_{ω^*} that $A_-(\omega^*) = [0, 1/2)$ and $A_+(\omega^*) = [1/2, 1]$. As ρ becomes closer to 0 we can see more oscillations in the f_{ω^*} and, subsequently, higher disconnectedness of the corresponding sets $A_\pm(\omega^*)$.

4.3 Grassmannians

Before we start the discussion on Oseledets splitting, we first give a brief introduction to Grassmannians and the topology that we use. We follow the setup of [58, Section 2].

Let $(Y, \|\cdot\|_Y)$ be a Banach space. A subspace E of Y is said to be *closed complemented* if

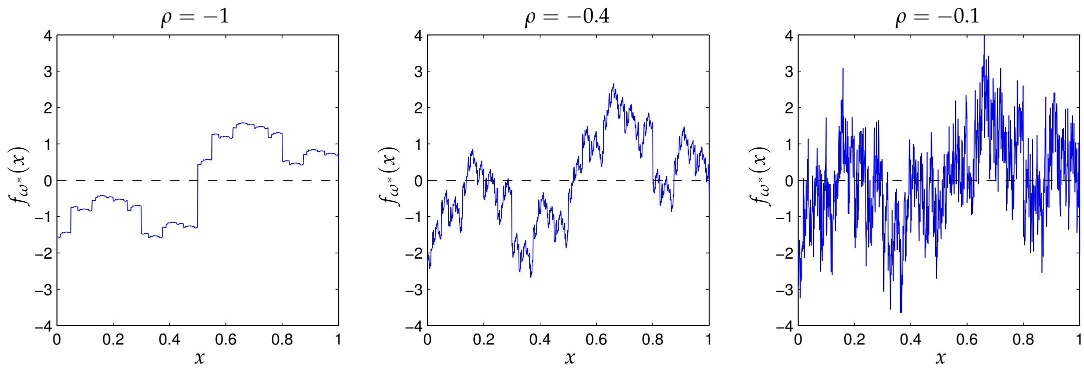


Figure 4.1: Graphs of f_{ω^*} corresponding to different Lyapunov exponents in Example 4.13. Note the increased irregularity as ρ approaches zero.

it is closed and there exists a closed subspace F of Y such that $E \cap F = \{0\}$ and $E + F = Y$, where '$+$' denotes the direct sum; that is, any nonzero element of Y can be uniquely written as $e + f$ with $e \in E$ and $f \in F$. A natural linear map on Y is the *projection* onto F along E, defined by $\mathrm{Pr}_{F\|E}(e + f) = f$.

The *Grassmannian* $\mathcal{G}(Y)$ of the space Y is the set of all closed complemented subspaces of Y. For any $E_0 \in \mathcal{G}(Y)$ there exists at least one $F_0 \in \mathcal{G}(Y)$ such that $E_0 \oplus F_0 = Y$, where '$\oplus$' now denotes topological direct sum. Every such F_0 defines a *neighbourhood* $U_{F_0}(E_0)$ of E_0 by

$$U_{F_0}(E_0) := \{E \in \mathcal{G}(Y) \ : \ E \oplus F_0 = Y\}.$$

Furthermore, on every such neighbourhood we can define the (E_0, F_0)-*local norm* by

$$\|E\|_{(E_0,F_0)} := \| \mathrm{Pr}_{F_0\|E} |_{E_0} \|_{op}.$$

This induces a topological structure of a Banach manifold on $\mathcal{G}(Y)$. In particular, given a suitable topology on Ω, the continuity of maps $\Omega \to \mathcal{G}(Y)$ is well-defined. In a similar fashion, by taking the corresponding Borel σ-algebra $\mathcal{B}(\mathcal{G}(Y))$ and $\mathcal{B}(\Omega)$ (or $\mathcal{F}$), we may also define measurability of such maps.

By $\mathcal{G}_d(Y)$ and $\mathcal{G}^c(Y)$ we will denote the subspaces of the Grassmannian $\mathcal{G}(Y)$ of Y consisting only of subspaces of dimension d and codimension c respectively.

Recall that a function $f : Y \to \mathbb{R}$ is said to be *upper semi-continuous* at $x_0 \in Y$ if for every $\epsilon > 0$ there is an open neighbourhood U_{x_0} such that $f(x) \leq f(x_0) + \epsilon$ for all $x \in U_{x_0}$. We shall use the fact that upper semi-continuity implies measurability.

Lemma 4.14. *Let d be a fixed integer and let $(Y, \|\cdot\|_*)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces with $\|\cdot\|_* \leq \|\cdot\|_Y$. For a fixed finite d, the function $\psi : \mathcal{G}_d(Y) \to \mathbb{R}$ defined as*

$$\psi(E) = \sup_{\xi \in E} \frac{\|\xi\|_Y}{\|\xi\|_*}$$

is upper semi-continuous and therefore measurable.

Proof. Each $E \in \mathcal{G}_d(Y)$ is finite-dimensional. Since all norms on finite dimensional spaces are equivalent, the function ψ is well-defined and $1 \leq \psi(E) < \infty$ for all $E \in \mathcal{G}_d(Y)$. For any $E_0 \in \mathcal{G}_d(Y)$, let $F_0 \in \mathcal{G}^d(Y)$ be such that $E_0 \oplus F_0 = Y$. For any $\epsilon \in (0, \psi(E_0)^{-1})$ let $\mathcal{N}_\epsilon \subset U_{F_0}(E_0)$ be a neighbourhood of E_0 such that for all $E \in \mathcal{N}_\epsilon$,

$$\|E\|_{(E_0,F_0)} = \| \mathrm{Pr}_{F_0\|E} |_{E_0}\|_{op} < \epsilon.$$

Take any $E \in \mathcal{N}_\epsilon$. For any $x \in E$ write $x = y - z$ where $y \in E_0$ and $z \in F_0$. Then $z = \mathrm{Pr}_{F_0\|E}(y)$ and $\|z\|_*/\|y\|_Y \leq \|z\|_Y/\|y\|_Y < \epsilon$. Now

$$\begin{aligned} \frac{\|x\|_Y}{\|x\|_*} &= \frac{\|y + x - y\|_Y}{\|y + x - y\|_*} \\ &\leq \frac{\|y\|_Y + \|z\|_Y}{\|y\|_* - \|z\|_*} \\ &< \frac{\|y\|_Y + \epsilon\|y\|_Y}{\|y\|_* - \epsilon\|y\|_Y} \\ &\leq \frac{1+\epsilon}{\psi(E_0)^{-1} - \epsilon}. \end{aligned}$$

The right hand side converges to $\psi(E_0)$ as $\epsilon \to 0$. As E_0 and ϵ are arbitrary, this establishes upper semi-continuity of ψ on all of $\mathcal{G}_d(Y)$. □

4.4 Oseledets Splitting and Applications

In this section we extend Theorem 4.7 to apply in a Banach space $(Y, \|\cdot\|_Y)$, with $Y \subset L^1(m)$, in which the Perron-Frobenius cocycle admits an *Oseledets splitting*. We then apply these new results to expanding maps of the unit interval, where Y is taken to be the Banach space of functions of bounded variation BV.

Definition 4.15 ([58, 104])**.** A linear operator cocycle[2] $\mathcal{P} : \mathbb{N} \times \Omega \to \mathrm{End}(Y, \|\cdot\|_Y)$ over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is said to be *quasi-compact* if for almost every $\omega \in \Omega$ there exists an $\alpha < \lambda(\omega)$ such that the set $\mathcal{V}_\alpha := \{y \in Y \ : \ \lambda(\omega, y) < \alpha\}$ is finite co-dimensional. We will denote the infimal such α by $\alpha(\omega)$.

Quasi-compact cocycles have the property that Lyapunov exponents larger than $\alpha(\omega)$ are isolated. For an isolated Lyapunov exponent $r > \alpha(\omega)$, let $\epsilon > 0$ be small enough so that $\Lambda(\omega) \cap (r - \epsilon, r) = \emptyset$. If the co-dimension of $\mathcal{V}_{r-\epsilon}(\omega)$ in $\mathcal{V}_r(\omega)$ is d then we call r a Lyapunov exponent of *multiplicity* d. There are at most countably many of these and we refer to them as *exceptional Lyapunov exponents*. The *exceptional Lyapunov spectrum* is the set of pairs of exceptional Lyapunov exponents and their multiplicities, $\{(\lambda_i(\omega), d_i(\omega))\}_{i=1}^{p(\omega)}$. For the rest of this chapter we retain the assumption that the base system $(\vartheta, \mathbb{P})$ is ergodic, which ensures that λ_i, d_i and p are all constant almost everywhere; see [58] for more details.

Definition 4.16 (Oseledets splitting [104])**.** A quasi-compact linear operator cocycle $\mathcal{P} \colon \mathbb{N} \times \Omega \to \mathrm{End}(Y, \|\cdot\|_Y)$ over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ with exceptional spectrum $\{(\lambda_i, d_i)\}_{i=1}^{p}$, $p \leq \infty$, admits a *Lyapunov filtration* over a ϑ-invariant set $\tilde{\Omega} \subseteq \Omega$ of full measure, if there exists a collection of maps $\{V_i : \Omega \to \mathcal{G}^{c_i}(Y)\}_{i=1}^{p}$, such that for all $\omega \in \tilde{\Omega}$ and all $i = 1, \ldots, p$

(i) $Y = V_1(\omega) \supset \cdots \supset V_i(\omega) \supset V_{i+1}(\omega)$

(ii) $\mathcal{V}_{\alpha(\omega)} \subseteq \cap_{i=1}^{p} V_i(\omega)$, with equality if and only if p is infinite;

(iii) $\mathcal{P}_\omega V_i(\omega) = V_i(\vartheta\omega)$;

[2] Not necessarily Perron-Frobenius.

(iv) $\lambda(\omega, v) = \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{P}_\omega^{(n)} v\|_Y = \lambda_i$ if and only if $v \in V_i(\omega) \setminus V_{i+1}(\omega)$. If p is finite, we take $V_{p+1}(\omega) := \mathcal{V}_{\alpha(\omega)}(\omega)$.

An *Oseledets splitting* for $\mathcal{P}$ is a Lyapunov filtration with an additional family of maps $\{E_i : \Omega \to \mathcal{G}_{d_i}(Y)\}_{i=1}^p$ such that for all $\omega \in \tilde{\Omega}$ and $i = 1, \ldots, p$

(v) $V_i(\omega) = E_i(\omega) \oplus V_{i+1}(\omega)$ (with $V_{p+1}(\omega) := \mathcal{V}_{\alpha(\omega)}(\omega)$ for $p < \infty$);

(vi) $\mathcal{P}_\omega E_i(\omega) = E_i(\vartheta\omega)$;

(vii) $\lambda(\omega, v) = \lambda_i$ if $v \in E_i(\omega) \setminus \{0\}$.

A Lyapunov filtration is measurable if each $V_i : \Omega \to \mathcal{G}^{c_i}(Y)$ is $(\mathcal{F}, \mathcal{B}(\mathcal{G}^{c_i}(Y)))$-measurable. An Oseledets splitting is measurable if its Lyapunov filtration is measurable and each of the maps $E_i : \Omega \to \mathcal{G}_{d_i}(Y)$ is measurable.

In order to connect the Y-Lyapunov spectrum to escape rate, we first need to relate the Y-Lyapunov exponents to the L^1-Lyapunov exponents used in Theorem 4.7. For this we shall require a certain relation between the two norms.

Theorem 4.17. *Let $\mathcal{P} : \mathbb{N} \times \Omega \to \mathrm{End}(Y, \|\cdot\|_Y)$ be a quasi-compact linear operator cocycle over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ with exceptional spectrum $\{(\lambda_i, d_i)\}_{i=1}^p$ and a measurable Oseledets splitting $\{E_i\}_{i=1}^p$ on $\tilde{\Omega}$. Let $\|\cdot\|_*$ be a second norm on Y such that $\|\cdot\|_* \le C\|\cdot\|_Y$ for some $C > 0$. Then for almost any $\omega \in \tilde{\Omega}$, $i \in \{1, \ldots, p\}$ and any $f \in E_i(\omega)$, we have $\lambda_{\|\cdot\|_*}(\omega, f) = \lambda_{\|\cdot\|_Y}(\omega, f) = \lambda_i$; that is, the Lyapunov exponents with respect to the two norms are equal almost everywhere.*

Proof. Firstly note that scaling a norm by a constant does not change the Lyapunov exponent, hence without loss of generality we may assume that $C = 1$. Fix $i \in \{1, \ldots, p\}$. Since $\|\cdot\|_* \le \|\cdot\|_Y$ the inequality $\lambda_{\|\cdot\|_*}(\omega, f) \le \lambda_{\|\cdot\|_Y}(\omega, f)$ for all $\omega \in \tilde{\Omega}$ follows trivially. Now for the reverse inequality. Define a function $c : \tilde{\Omega} \to \mathbb{R}$ by

$$c(\omega) = \sup_{\xi \in E_i(\omega)} \frac{\|\xi\|_Y}{\|\xi\|_*} = \psi \circ E_i(\omega),$$

where $\psi : \mathcal{G}_{d_i} \to \mathbb{R}$ is as in Lemma 4.14. Since E_i is $(\mathcal{F}, \mathcal{B}(\mathcal{G}_{d_i}(X)))$-measurable and ψ is $(\mathcal{B}(\mathcal{G}_{d_i}(X)), \mathcal{B}(\mathbb{R}))$-measurable, it follows that c is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable.

For a positive integer N let $\chi_{\{c<N\}}$ be the characteristic function of the (measurable) set $\{\omega \ : \ c(\omega) < N\}$. Given any $\omega \in \tilde{\Omega}$, the function $c(\omega)$ is finite so $\chi_{\{c<N\}}(\omega) = 1$ for all $N > c(\omega)$. Thus $\chi_{\{c<N\}} \to 1$ pointwise. By Lebesgue's Dominated Convergence Theorem, we see that $\mathbb{P}(\{c < N\}) \to 1$ as $N \to \infty$. Thus we may choose an N large enough so that $\mathbb{P}(\{c < N\}) > 0$. By the Poincaré Recurrence Theorem there almost surely exists a sequence $m_k \uparrow \infty$ such that $\vartheta^{m_k}\omega \in \{c < N\}$. Then

$$\begin{aligned}\lambda_{\|\cdot\|_*}(\omega, f) &\geq \limsup_{k\to\infty} \frac{1}{m_k} \log \|\mathcal{P}_\omega^{(m_k)} f\|_* \\ &\geq \lim_{k\to\infty} \frac{1}{m_k} \log N^{-1} \|\mathcal{P}_\omega^{(m_k)} f\|_Y \\ &= \lambda_i(\omega),\end{aligned}$$

which completes the proof. □

Remark 4.18. By reversing the appropriate inequalities in the proof of Theorem 4.17 and a similar modification of Lemma 4.14 one can see that the same result holds when the two norms satisfy the relation $C\|\cdot\|_* \geq \|\cdot\|_Y$ for some $C > 0$. In particular Theorem 4.17 is satisfied when the two norms are equivalent.

Now we relate the results of this section back to Perron-Frobenius operator cocycles. A direct consequence of Theorem 4.7 and Theorem 4.17 is the following corollary.

Corollary 4.19. *Let $T : \mathbb{N} \times \Omega \to \mathrm{End}(X, \mathcal{B}, m)$ be a measurable map cocycle over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ and let its Perron-Frobenius cocycle be $\mathcal{P} : \mathbb{N} \times \Omega \to \mathrm{End}(Y, \|\cdot\|_Y)$, where $Y \subseteq L^1(X)$ and $\|\cdot\|_{L^1} \leq \|\cdot\|_Y$. Suppose that $\mathcal{P}$ is quasi-compact, with exceptional spectrum $\{(\lambda_i, d_i)\}_{i=1}^p$, admitting a measurable Oseledets splitting $E_i : \Omega \to \mathcal{G}(X)$. For $\mathbb{P}$-almost all $\omega^* \in \tilde{\Omega}$ and any $f \in E_i(\omega^*)$ if $A_\pm$ the orbit sets given by $A_\pm(\vartheta^n\omega^*) = \{\pm\mathcal{P}_{\omega^*}^{(n)} f > 0\}$, then $E(A_\pm, \omega^*) \leq -\lambda_i$, $i = 2, \ldots, p$.*

This result extends the applicability of Theorem 4.7 to Perron-Frobenius cocycles on Banach spaces for which the cocycle is quasi-compact and the Banach space norm dominates the L^1-norm. Note that our result also applies to periodic ω^* as, in this case, the corresponding Oseledets subspaces $E_i(\omega^*)$ would indeed be eigenspaces.

4.4.1 Application to Cocycles of Expanding Interval Maps

We now focus on the unit interval, $\mathrm{I} = [0, 1]$, one-dimensional map cocycles $T : \mathbb{N} \times \Omega \to \mathrm{End}(\mathrm{I})$, and their Perron-Frobenius operators. In [58] it is shown that the Perron-Frobenius cocycle is quasi-compact if the *index of compactness* (a quantity corresponding to the essential spectral radius in the deterministic setting)

$$\kappa := \lim_{n\to\infty} \frac{1}{n} \log(1/\operatorname{ess\,inf}((T^n_\omega)'(x)))$$

is less than zero. Such systems are said to be *expanding-on-average*. This formula for κ suggests that any Lyapunov spectral points lying between κ and 0 (the latter corresponding to the random invariant density) are associated with large-scale coherent structures responsible for rates of mixing slower than the local expansion of trajectories can account for. We apply the results of Corollary 4.19 to show that these sets also posses a slow rate of escape, bounded by the corresponding exponent in the Lyapunov spectrum. Firstly, we outline a generalisation to Lasota-Yorke maps — Rychlik maps and their cocycles.

Definition 4.20 (Rychlik map [99])**.** A map $T : \mathrm{I} \circlearrowleft$ is *Rychlik* if

(R1) T is C^2 on an open subset $U_T \subseteq \mathrm{I}$ of full measure

(R2) $T|_B$ extends to a homeomorphism from $\overline{B}$ (closure of B) to a subinterval of I, for each connected component $B \subseteq U_T$

(R3) the function g_T, where g_T equals $1/|JT|$ on U_T and 0 otherwise, has bounded variation.

Let $\Omega \subseteq \{1, \ldots, k\}^{\mathbb{Z}}$ be a shift space on k symbols with the left shift map $\vartheta : \Omega \circlearrowleft$ given by $(\vartheta\omega)_j = \omega_{j+1}$. Furthermore, suppose $\mathcal{F}$ is the Borel σ-algebra generated by cylinders in Ω and suppose that $\mathbb{P}$ is an ergodic shift-invariant probability measure on Ω.

A *Rychlik map cocycle* is a cocycle $T : \mathbb{N} \times \Omega \to \mathrm{End}(\mathrm{I})$ obtained from a collection of *k Rychlik maps* $\{T_i\}_{i=1}^k$ where the generator $\tilde{T}$ is given by $\tilde{T}_\omega = T_{\omega_0}$. We will denote the corresponding Perron-Frobenius operator cocycle $\mathcal{P} : \mathbb{N} \times \Omega \to \mathrm{End}(\mathrm{BV})$. For more details we refer the reader to [58].

In [58, Corollary 28] it is shown that the Perron-Frobenius cocycle of any Rychlik map cocycle that is *expanding-on-average* (i.e. $\kappa < 0$) admits a $\mathbb{P}$-continuous (and therefore measurable) Oseledets splitting in BV. We combine this result with Corollary 4.19 to obtain the following.

Corollary 4.21. *Let $T : \mathbb{N} \times \Omega \to \mathrm{End}(I)$ be a Rychlik map cocycle which is expanding on average and let $\mathcal{P} : \mathbb{N} \times \Omega \to \mathrm{End}(BV)$ be its Perron-Frobenius operator cocycle, which admits a measurable Oseledets splitting on a set of full $\mathbb{P}$-measure $\tilde{\Omega} \subseteq \Omega$. For any isolated Lyapunov exponent $\lambda_i < 0$ and $\mathbb{P}$-almost any $\omega^* \in \Omega$ there exist orbit sets $A_\pm$ such that ω-fibres of $A_\pm$ partition I and $E(A_\pm, \omega^*) \leq -\lambda_i$.*

Proof. Since $\|\cdot\|_{L^1} \leq \|\cdot\|_{\mathrm{BV}}$, a direct application of Corollary 4.19 shows that any pair of orbit sets $A_\pm$ satisfying $A_\pm(\vartheta^n\omega^*) = \{\pm\mathcal{P}^{(n)}_{\omega^*} f > 0\}$ have escape rates lower than $-\lambda_i$. □

Moreover, by an application of Corollary 2.18 to BV functions we see that each $A_\pm(\omega)$, $\omega \in \{\vartheta^n\omega^*\}_{n\in\mathbb{Z}^+}$, may be written as a *countable union of closed sets* (including possibly singleton sets). Thus, as we saw in the deterministic setting, the orbit sets $A_\pm$, from which we are bounding the rate of escape, have a relatively simple topological form.

We will use the rest of this chapter to illustrate our techniques via a numerical example. Firstly, though, we outline the algorithm of [59, Section 6], which we use to numerically compute Oseledets subspaces.

Algorithm 4.22 ([59])**.** Let $\mathcal{P} : \mathbb{N} \times \Omega \to \mathrm{End}(L^1(X, \mathcal{B}, m))$ be a quasi-compact linear Perron-Frobenius operator cocycle over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ with a measurable Oseledets splitting $\{(E_i, \lambda_i)\}$. For a test point $\omega^* \in \Omega$ we apply the following steps to numerically approximate the corresponding Oseledets subspaces and their Lyapunov exponents:

(A1) Choose integers $I, J > 0$ and for all n such that $-I \leq n \leq J - I$ compute the Ulam approximations $P(\vartheta^n\omega^*)$ of $\mathcal{P}_{\vartheta^n\omega^*}$, with respect to an appropriate partition. The corresponding Ulam matrix cocycle is defined in the usual way, denoted $P^{(n)}(\omega)$.

(A2) Form the matrix

$$\Psi^{(J)}(\vartheta^{-I}\omega^*) := (P^{(J)}(\vartheta^{-I}\omega^*)^T P^{(J)}(\vartheta^{-I}\omega^*))^{1/2J}.$$

(A3) Calculate the orthonormal eigenspace decomposition of $\Psi^{(J)}(\vartheta^{-I}\omega^*)$, denoted

$$U_i^{(J)}(\vartheta^{-I}\omega^*), \quad i = 1, \ldots, k.$$

(A4) Define

$$E_i^{(I,J)}(\omega^*) := P^{(J)}(\vartheta^{-I}\omega^*)U_i^{(J)}(\vartheta^{-I}\omega^*).$$

(A5) The finite dimensional space $E_i^{(I,J)}(\omega^*)$ is a numerical approximation to $E_i(\omega^*)$.

Example 4.23. This example is borrowed from [59, p. 746] and we refer the reader to the original article for additional details. It is easy to check that the cocycle T described below is Rychlik and expanding-on-average. The base dynamical system is given by a shift ϑ on sequence space $\Omega = \Sigma_E \subset \{1, \ldots, 6\}^{\mathbb{Z}}$ with adjacency matrix

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix},$$

equipped with the σ-algebra generated by 1-cylinders and the Markov probability measure $\mathbb{P}$ determined by the stochastic matrix $\frac{1}{2}E$.

The map cocycle T is generated by maps $\tilde{T} \colon \Omega \to \mathrm{End(I)}$ given by $\tilde{T}(\omega) = T_{\omega_0}$ where $\{T_i\}_{i=1}^6$ is a collection of six Lebesgue-preserving, piecewise affine, Markov expanding maps of the interval, which share a common Markov partition, graphs of which are shown in Figure 4.2. The map cocycle T has been designed so that at each step, a particular (random) interval of length $1/3$ (selected from $[0, 1/3]$, $[1/3, 2/3]$ and $[2/3, 1]$) is approximately shuffled (with some escape) to another of these three intervals. For example, the map T_1 approximately shuffles $[0, 1/3]$ to $[1/3, 2/3]$. These particular random intervals are the metastable sets or coherent sets for this random system from which we show the escape rate is slow.

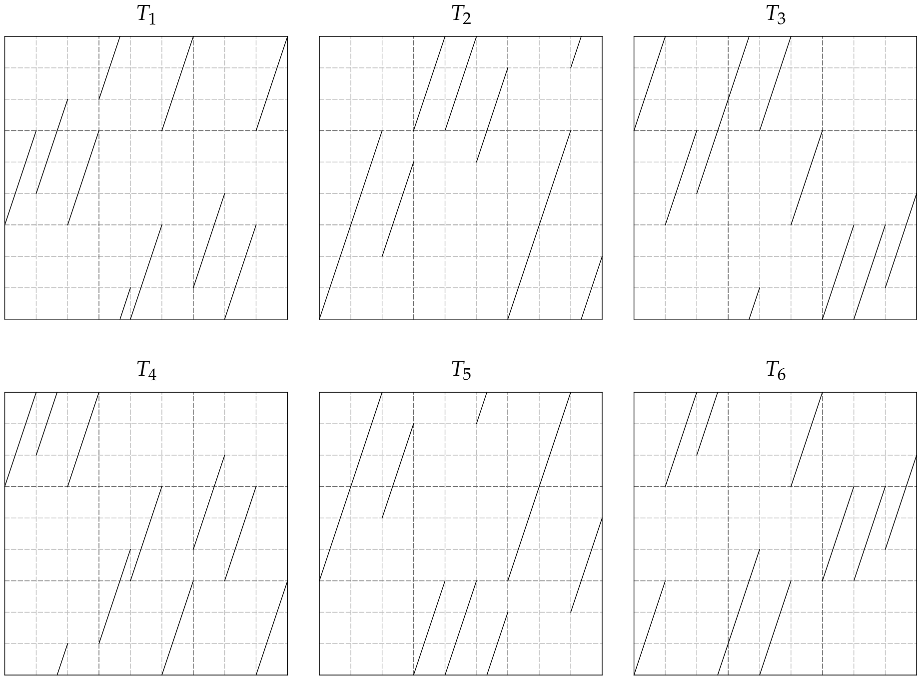


Figure 4.2: Graphs of maps $T_1, \ldots, T_6$, reproduced from [59, Figures 1 & 5].

A test sequence $\omega^* \in \Omega$ is obtained in the following way. Let $\alpha \in \{0,1\}^{\mathbb{Z}}$ be such that $\alpha_0 = 0$, and for $i \geq 1$, α_i is the $(2i)^{\text{th}}$ digit in the binary expansion of the fractional part of π while α_{-i} is the $(2i-1)^{\text{th}}$ digit of the same expansion. Let $h : \Omega \to \{0,1\}^{\mathbb{Z}}$ be such that

$$h(\omega)_i = \begin{cases} 0, & \omega_i \in \{1,2,3\} \\ 1, & \omega_i \in \{4,5,6\}. \end{cases}$$

Observe that h is three-to-one and that we may uniquely choose $\omega^* \in h^{-1}\{\alpha\}$ that satisfies $\omega_0^* = 1$. Shown below are some of the central elements of ω^*, with the zeroth element underlined:

$$\omega^* = (\ldots, 3, 4, 6, 5, 4, 3, 4, 6, 5, 1, 2, 3, 4, 3, \underline{1}, 2, 3, 4, 3, 1, 5, 4, 6, 5, 1, 2, 3, 1, 2, \ldots).$$

It is shown in [59] that $\Lambda(\omega^*) \subset [-\infty, \log 1/3] \cup \{\lambda_2(\omega^*)\} \cup \{0\}$ where $\lambda_2(\omega^*) \approx$

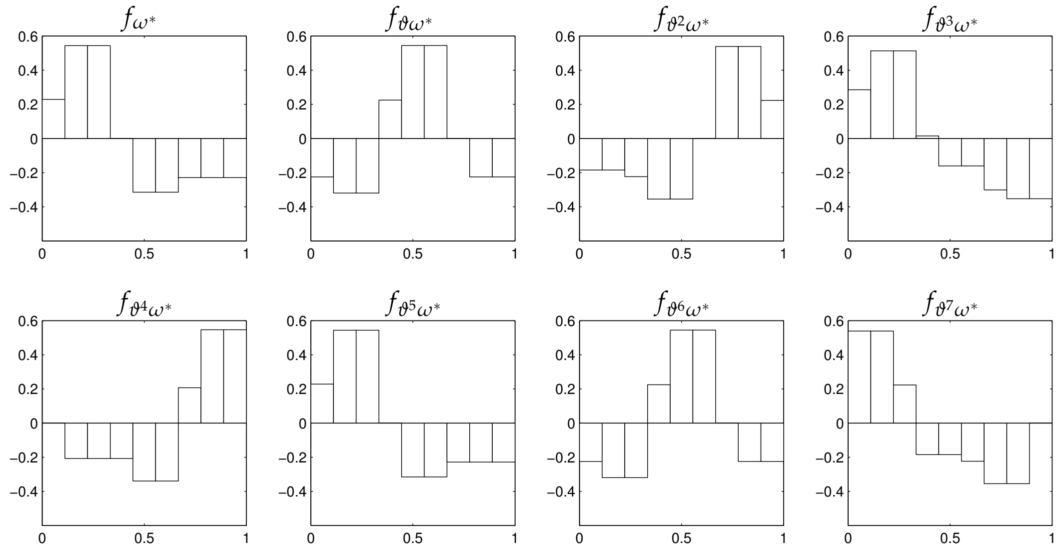


Figure 4.3: Functions $f_{\vartheta^i\omega^*}$, spanning second Oseledets subspaces $E_2(\vartheta^i\omega^*)$ for $i = 0, \ldots, 7$.

$\log 0.81$, approximated using Algorithm 4.22. The functions $f_{\vartheta^n\omega^*} = \mathcal{P}^{(n)}_{\omega^*} f_{\omega^*}$ spanning the corresponding Oseledets subspaces $E_2(\vartheta^n\omega^*)$ are shown in Figure 4.3. One can see that, when compared to those in Figure 4.1, these functions are more regular (i.e. lower variation). We also determine the random metastable sets or coherent sets $A_\pm(\vartheta^n\omega^*) = \{\pm f_{\vartheta^n\omega^*} > 0\}$ for the first eight values on the forward orbit of ω^*:

$$\begin{aligned}
A_+(\omega^*) &= [0, 3/9]\,, & A_-(\omega^*) &= [3/9, 1)\,,\\
A_+(\vartheta\omega^*) &= [3/9, 6/9]\,, & A_-(\vartheta\omega^*) &= [0, 3/9) \cup (6/9, 1]\,,\\
A_+(\vartheta^2\omega^*) &= [6/9, 1]\,, & A_-(\vartheta^2\omega^*) &= [0, 6/9)\,,\\
A_+(\vartheta^3\omega^*) &= [0, 4/9]\,, & A_-(\vartheta^3\omega^*) &= (4/9, 1]\,,\\
A_+(\vartheta^4\omega^*) &= [6/9, 1]\,, & A_-(\vartheta^4\omega^*) &= [0, 6/9)\,,\\
A_+(\vartheta^5\omega^*) &= [0, 3/9]\,, & A_-(\vartheta^5\omega^*) &= [3/9, 1)\,,\\
A_+(\vartheta^6\omega^*) &= [3/9, 6/9]\,, & A_-(\vartheta^6\omega^*) &= [0, 3/9) \cup (6/9, 1]\,,\\
A_+(\vartheta^7\omega^*) &= [0, 3/9]\,, & A_-(\vartheta^7\omega^*) &= [3/9, 1)\,.
\end{aligned}$$

As per the discussion in Remark 4.9 we can approximate the rates of escape from A_+ and A_- by computing the largest Lyapunov exponent of the matrix approximations of the corresponding conditional cocycle (we use $I = 0$ and $J = 20$ for parameters I and J in Algorithm 4.22). We then find that $E(A_+, \omega^*) \approx -\log 0.83$ and $E(A_-, \omega^*) \approx -\log 0.89$. This is in agreement with Corollary 4.21 as both escape rates are less than the previously computed $-\lambda_2(\omega^*) \approx -\log 0.81$.

By inspecting $T_{\vartheta^k \omega^*}$ we see that $A_+(\vartheta^k \omega^*)$ is mostly mapped onto $A_+(\vartheta^{k+1}\omega^*)$, $k = 0, \ldots, 6$. This phenomenon is the cause of the slow escape from the random set A^+. By Corollary 4.21, the presence of a Lyapunov spectral value close to 0 forces the existence of a random set with escape rate slower than that spectral value.

Chapter 5

Bounds on Topological Entropy in Symbolic Dynamics

In this final chapter we turn our focus to symbolic dynamics of shifts of finite type, both deterministic and random. We will use techniques similar in theme to those in the preceding chapters, but this time applied to transition matrices rather than Perron-Frobenius operators. As per the discussion in Chapter 1, many dynamical systems (more precisely, those that possess a Markov partition) are semi-conjugate to shifts of finite type and the results here, rather than being an extension, are somewhat of a simplification to the results of Chapter 2 and 4. Nevertheless, interesting applications to topological entropy arise.

This chapter consists of two parts. In Section 5.1 we investigate deterministic shifts of finite type. This material has appeared in the final section of [65]. In Section 5.2 we deal with the analogous results in the setting of random shifts of finite type, which has appeared as the final section of [66].

5.1 Entropy Bound for Shifts of Finite Type

As earlier mentioned in Chapter 1, in the study of shifts of finite type, topological entropy quantifies the exponential growth rate of the number of allowed blocks with block length. Equivalently, if one considers a random walk on the corresponding graph, entropy is the growth rate on the number of distinct k-paths, and in a sense measures the connectedness of the graph. We discussed in Section 1.2.6 that the escape rate formula of thermodynamic formalism translates in this setting to formula (1.23), equating escape rate to the difference in topological entropies of a shift of finite type and the corresponding subshift on the survivor set.

In the same spirit of detecting metastable sets that we have exercised in Chapter 2, our approach to metastability within a shift of finite type is to detect two disjoint subshifts, both of which possess high topological entropies relative to the original subshift. In the corresponding graph analogue, this may be seen as an attempt at determining highly connected subgraphs.

Theorem 5.1. *Let (Σ_M, σ) be a memory-1 shift of finite type, with corresponding $N \times N$ adjacency matrix M. Let $0 < \rho < \mathcal{R}(M)$ be a real eigenvalue of M with eigenvector $v \in \mathbb{R}^N$. Define $\mathcal{A}_+$ and $\mathcal{A}_-$ to be the two sets of indices (sub-alphabets) for which v is positive and negative, respectively:*

$$\mathcal{A}_+ := \{i \in \mathcal{A} \;:\; v_i > 0\}\,, \quad \mathcal{A}_- := \{i \in \mathcal{A} \;:\; v_i < 0\}\,.$$

Let M_+ and M_- be the restrictions of M to indices in $\mathcal{A}_+$ and $\mathcal{A}_-$ respectively. These adjacency matrices define two disjoint memory-1 subshifts of Σ_M on disjoint symbol sets ($\mathcal{A}_+$ and $\mathcal{A}_-$), denoted by Σ_{M_+} and Σ_{M_-}. One then has $h_{top}(\Sigma_{M_+}) \geq \log \rho$ and $h_{top}(\Sigma_{M_-}) \geq \log \rho$.

Proof. It is sufficient to show that $\mathcal{R}(M_+) \geq \rho$, where $\mathcal{R}(M_+)$ is the spectral radius of M_+. For every $i \in A_+$ we have

$$\begin{aligned}
\rho v_i &= \sum_{j \in \mathcal{A}} M_{ij} v_j \\
&= \sum_{j \in \mathcal{A}_+} M_{ij} v_j + \sum_{j \notin \mathcal{A}_+} M_{ij} v_j \\
&\leq \sum_{j \in \mathcal{A}_+} (M_+)_{ij} v_j.
\end{aligned}$$

It follows that $(\rho^n v^+)_i \leq (M_+^n v^+)_i$ for all $n \geq 1$ and $i \in \mathcal{A}_+$, where v^+ is the restriction of v to $\mathcal{A}_+$; thus $\rho^n \|v^+\| \leq \|M_+^n v^+\|$. By Gelfand's Spectral Radius Formula [67] (see e.g.

[14, Theorem 12.9]) we obtain $\rho \leq \mathcal{R}(M_+)$, therefore $h_{top}(\Sigma_{M_+}) = \log r(M_{A_+}) \geq \log \rho$. By considering $-v$ in place of v we also obtain $h_{top}(\Sigma_{M_-}) \geq \log \rho$. $\square$

Example 5.2. Let $\mathcal{A} = \{0, 1, \ldots, 8\}$ and $\Sigma_M \subset \mathcal{A}^{\mathbb{Z}^+}$ be the one-sided memory-1 shift whose allowed transition graph is shown in Figure 5.1.

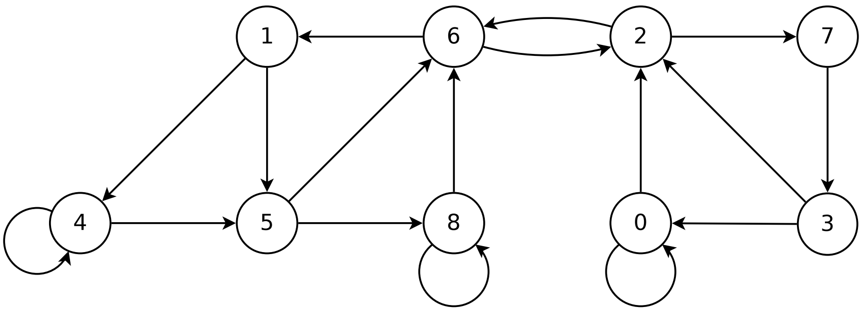


Figure 5.1: Transition graph of $X_{\mathbb{F}}$.

The adjacency matrix M of Σ_M is given by

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

We have deliberately constructed the shift Σ_M so that its graph of allowed transitions consists of two weakly-linked subgraphs, each of which is highly internally linked. The dynamics restricted to each of the two subgraphs generates almost as much entropy as the dynamics on the whole graph. We expect that the adjacency matrix M has a real positive eigenvalue ρ close to $\mathcal{R}(M)$. If so, we may use Theorem 5.1 to identify two disjoint subshifts of Σ_M, namely Σ_{M_+} and Σ_{M_-} with entropy of each larger than $\log \rho$.

We find that M has largest eigenvalue $\mathcal{R}(M) \approx 1.92$, and second largest eigenvalue $\rho \approx 1.42$. The eigenvector v corresponding to ρ is shown in Figure 5.2; thus we define $\mathcal{A}_+ = \{0,2,3,7\}$ and $\mathcal{A}_- = \{1,4,5,6,8\}$. This corresponds to breaking both connections

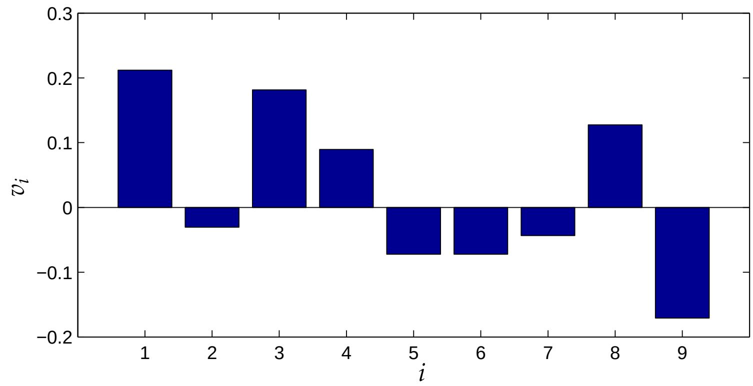


Figure 5.2: Second eigenvector of M.

between vertices '6' and '2' in Figure 5.1, and taking each of the two connected components to be the transition graphs of Σ_{M_+} and Σ_{M_-}. We calculate the topological entropy of each of the newly obtained subshifts: $h_{top}(\Sigma_{M_+}) \approx \log 1.47$ and $h_{top}(\Sigma_{M_-}) \approx \log 1.76$. Both entropies are greater than $\log \rho$, as guaranteed by Theorem 5.1. In this example, we guessed a good partitioning of the set of states and this guess coincided with the conclusion of Theorem 5.1. For larger, more complicated examples, Theorem 5.1 can be used to discover good partitions that may not be as immediately obvious as in this example.

Example 5.3. As we remarked in the introductory chapter, one can always recode a shift of higher memory into a conjugate memory-1 shift. Thus, using our technique of Theorem 5.1, we may also partition memory-2 or higher shifts by first conducting the appropriate recoding. In Example 5.2, $\Sigma_M = \Sigma_{\mathbb{F}}$ is a recoding of a memory-2 shift $\Sigma_{\mathbb{F}_*} \subset \mathcal{A}_*^{\mathbb{Z}^+}$ where $\mathcal{A}_* = \{0,1,2\}$ and

$$\mathbb{F}_* = \{001, 010, 022, 101, 110, 121, 200, 211, 212, 221\}.$$

The *sliding block code* $\pi : \Sigma_{\mathbb{F}_*} \to \Sigma_{\mathbb{F}}$ given by $\pi(y) = x$ for $y \in \Sigma_{\mathbb{F}_*}$ where $x_i = 3y_{i-1} + y_i$, for all $i \in \mathbb{Z}^+$, conjugates the two shifts. As in Example 5.2 we partition $\Sigma_{\mathbb{F}} = \Sigma_M$ into two disjoint subshifts $\Sigma_{\mathbb{G}} = \Sigma_{M_+}$ and $\Sigma_{\mathbb{H}} = \Sigma_{M_-}$ of high topological entropy, relative to the entropy of $\Sigma_{\mathbb{F}}$. By applying π^{-1} and using the fact that π is a conjugacy, we create a partition of $\Sigma_{\mathbb{F}_*}$ into two disjoint subshifts of high entropy: $\Sigma_{\mathbb{G}_*} = \pi^{-1}\Sigma_{\mathbb{G}}$ and $\Sigma_{\mathbb{H}_*} = \pi^{-1}\Sigma_{\mathbb{H}}$. From the calculations in Example 5.2 we have $h_{top}(\Sigma_{\mathbb{F}_*}) \approx \log 1.92$, $h_{top}(\Sigma_{\mathbb{G}_*}) \approx \log 1.47$ and $h_{top}(\Sigma_{\mathbb{H}_*}) \approx \log 1.76$. Moreover,

$$\mathbb{G}_* = \mathbb{F}_* \cup \{011, 012, 111, 112, 120, 122, 201, 220, 222\}$$

and

$$\mathbb{H}_* = \mathbb{F}_* \cup \{000, 002, 020, 021, 100, 102, 202, 210\}.$$

thus $\mathbb{G}_* \cap \mathbb{H}_* = \mathbb{F}_*$ and $\mathbb{G}_* \cup \mathbb{H}_*$ contains all words of length three on $\mathcal{A}_*$.

5.2 Entropy Bound for Random Shifts of Finite Type

In this section we use our machinery to obtain results on partitioning random shifts of finite type into disjoint subshifts of high entropy. We begin by defining random transition matrices, the corresponding random shifts of finite type and some important properties such as aperiodicity. We alter some of our notation to match the notation usually applied to shifts. For a more detailed description of random shifts of finite type see for example the paper of Bogenschütz and Gundlach [13] and the references therein.

As in Chapter 4, we shall assume that $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is an abstract ergodic base dynamical system.

Definition 5.4. For any integer $N \geq 2$, a *random transition matrix* is defined to be a measurable $N \times N$ transition-matrix-valued function $M : \Omega \to \mathcal{M}_{N\times N}(\{0, 1\})$. For $\omega \in \Omega$ and $n \in \mathbb{N}$ write the *matrix cocycle* as

$$M^{(n)}(\omega) := M(\omega)M(\vartheta\omega) \cdots M(\vartheta^{n-1}\omega).$$

Note that the map $(n, \omega) \mapsto M^n(\omega)$ satisfies the cocycle properties (C1) and (C2) of

Chapter 4.

Definition 5.5. Let $\mathcal{A} = \{1, \ldots, N\}$ be an alphabet and $\mathcal{A}^{\mathbb{Z}^+}$ the space of all one-sided $\mathcal{A}$-valued sequences. A random matrix $M : \Omega \to \mathcal{M}_{N\times N}$ defines a subset of $\mathcal{A}^{\mathbb{Z}^+}$ for every $\omega \in \Omega$ by

$$\Sigma_M(\omega) := \{x \in \mathcal{A}^{\mathbb{Z}^+} \ : \ M_{x_i x_{i+1}}(\vartheta^i\omega) = 1 \text{ for all } i \in \mathbb{Z}^+\}.$$

Let σ be the left shift map on each $\Sigma_M(\omega)$. Then σ does not preserve each $\Sigma_M(\omega)$, hence $\Sigma_M(\omega)$ is not a shift space in the deterministic sense. However, we may study the bundle random dynamical system determined by the family of maps

$$\{\sigma : \Sigma_M(\omega) \to \Sigma_M(\vartheta\omega),\ \omega \in \Omega\},$$

and we refer to it as a *random shift of finite type*. The set $\Sigma_M := \{(\omega, \Sigma_M(\omega)),\ \omega \in \Omega\}$ is called a *random shift space*.

Definition 5.6. A random transition matrix $M : \Omega \to \mathcal{M}_{k\times k}(\{0,1\})$ is *aperiodic* (or irreducible) if for almost every $\omega \in \Omega$ there exists a positive integer $K = K(\omega)$ such that $M^{(K)}(\omega) > 0$. If K is independent of ω then M is said to be *uniformly aperiodic*. We will also use the terms "aperiodic" and "uniformly aperiodic" to describe the corresponding random shift space Σ_M.

Define

$$\mathcal{C}_n(\omega) := \{[x_0x_1 \ldots x_{n-1}] \ : \ M_{x_i x_{i+1}}(\vartheta^i\omega) = 1 \text{ for all } 0 \le i < n-2\}$$

to be the set of all n-cylinders of $\Sigma_M(\omega)$ beginning at position 0.

Proposition 5.7. *The following limit exists and is constant $\mathbb{P}$-almost everywhere:*

$$h_{top}(\Sigma_M(\omega)) := \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{C}_n(\omega)|.$$

Proof. Observe that $|\mathcal{C}_{n+m}(\omega)| \le |\mathcal{C}_n(\omega)| \cdot |\mathcal{C}_m(\vartheta^n\omega)|$, thus the sequence $\{\log|\mathcal{C}_n(\omega)|\}_{n\in\mathbb{Z}^+}$ is subadditive. By Kingman's Subadditive Ergodic Theorem [76] (see e.g. [1, Theorem

3.3.2]) there exists a measurable function $f : \Omega \to \mathbb{R} \cup \{-\infty\}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathcal{C}_n(\omega)| = f(\omega)$$

and $f \circ \vartheta = f$ almost everywhere. As $(\vartheta, \mathbb{P})$ is ergodic, f is constant almost everywhere (see e.g. [79, Theorem 4.2.1]). □

The quantity $h_{top}(\Sigma_M(\omega))$ is called the *topological entropy* of $\Sigma_M(\omega)$. Denote by $h_{top}(\Sigma_M)$ the constant where $h_{top}(\Sigma_M) = h_{top}(\Sigma_M(\omega))$ almost everywhere.

Proposition 5.8. $|\mathcal{C}_n(\omega)| = \sum_{i,j} M_{ij}^{(n-1)}(\omega)$ *for every* $n \geq 2$.

Proof. The proof of this result is largely identical to the proof of its deterministic analogue (see for example [85, Proposition 2.2.12]). □

Definition 5.9. Let $\{\sigma : \Sigma_M(\omega) \to \Sigma_M(\vartheta\omega)\}$ and $\{\sigma : \Sigma_Q(\omega) \to \Sigma_Q(\vartheta\omega)\}$ be two random shifts of finite type with common base dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$. The random shift Σ_Q is a *subshift* of Σ_M if

$$(Q_{ij}(\omega) = 1) \implies (M_{ij}(\omega) = 1) \text{ for all } i, j \in \mathcal{A}, \omega \in \Omega.$$

A subshift may not utilise all the symbols of its parent shift for different values of ω. We may think of this as either a subshift whose alphabet, while finite, changes with ω or as a subshift on all of the alphabet of its parent shift, but possibly containing isolated vertices in the associated adjacency graph. We now introduce the notion of a *complementary subshift*. Roughly speaking for each $\omega \in \Omega$ two complementary subshifts of Σ_M utilise disjoint subsets of $\mathcal{A}$ in a maximal way.

Definition 5.10. Let Σ_M be a random shift of finite type and let Σ_Q be a subshift of Σ_M. The *complementary* subshift of Σ_M to Σ_Q is the subshift $\Sigma_{Q'}$ whose elements $Q'_{ij} = 1$ if and only if $M_{ij} = 1$ and $Q_{ik} = Q_{kj} = 0$ for all $k \in \mathcal{A}$.

We state a recent extended version of the classical Oseledets Multiplicative Ergodic Theorem (MET) [92] which guarantees the existence of an Oseledets splitting of $\mathbb{R}^N$ even when the adjacency matrices $M(\omega)$ are not invertible. This is the case in many interesting examples, including some random shifts of finite type. The MET is a central piece of machinery which we use to determine complementary subshifts with large topological entropies. Later we will see that the leading Lyapunov exponent λ_1 yields the topological entropy of the shift, while the second Lyapunov exponent λ_2, if close to λ_1, indicates the presence of metastability and the ability to form complementary subshifts with large topological entropies relative to that of the original shift. The following is [59, Theorem 4.1] specialised to adjacency matrices.

Theorem 5.11 (Froyland *et al.* [59]). *Suppose $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is an invertible ergodic base dynamical system and consider a random transition matrix $M : \Omega \to \mathcal{M}_{N\times N}(\{0,1\})$. There exists a forward ϑ-invariant full $\mathbb{P}$-measure subset $\tilde{\Omega} \subset \Omega$, numbers $\lambda_r < \cdots < \lambda_1$ and dimensions $d_1, \ldots, d_r \in \mathbb{N}$ satisfying $\sum_l d_l = N$ such that for all $\omega \in \tilde{\Omega}$:*

(i) There exist subspaces $W_l(\omega) \subset \mathbb{R}^N$, $l = 1, \ldots, r$, $\dim(W_l(\omega)) = d_l$;

(ii) $\mathbb{R}^N = W_1(\omega) \oplus \cdots \oplus W_r(\omega)$ for $\omega \in \tilde{\Omega}$;

(iii) $M(\omega)W_l(\omega) \subseteq W_l(\vartheta\omega)$ with equality if $\lambda_l > -\infty$;

(iv) For $v \in W_l(\omega) \setminus \{0\}$ the Lyapunov exponent

$$\lambda(\omega, v) := lim_{n\to\infty} \frac{1}{n} \log \|vM^{(n)}(\omega)\|_1$$

exists and equals λ_l.

The subspaces W_l are called *Oseledets subspaces* and the splitting in (ii) is an *Oseledets splitting*.

Remark 5.12. The result of [59, Theorem 4.1] applies to all measurable random matrices whose logarithm of the norm is integrable. We have stated the theorem above in the specific setting of transition matrices where the integrability condition always holds.

The following lemma states that $W_1(\omega)$ always contains the first quadrant of $\mathbb{R}^N$.

Lemma 5.13. *Under the hypothesis of Theorem 5.11, for all $\omega \in \tilde{\Omega}$ and for all vectors $v > 0$ one has $\lambda(\omega, v) = \lambda_1$. If, in addition, M is uniformly aperiodic, then for all $\omega \in \tilde{\Omega}$ and for all vectors $v \geq 0$ one has $\lambda(\omega, v) = \lambda_1$.*

Proof. Let $v_1 \in \mathbb{R}^N$ satisfy $\lambda(\omega, v_1) = \lambda_1$. Since $|v_1|M^{(n)}(\omega) \geq |v_1 M^{(n)}(\omega)|$, we must also have $\lambda(\omega, |v_1|) = \lambda_1$, where the absolute values and the inequalities are taken element-wise. Thus, the leading exponent λ_1 is achieved by a nonnegative vector, namely, $v_1' = |v_1| \geq 0$.

Suppose first that in fact $v_1' > 0$. For any $v > 0$, there exist positive constants c and C such that $cv_1' \leq v \leq Cv_1'$ and therefore for any $n \in \mathbb{N}$ we have $c\|v_1' M^{(n)}(\omega)\|_1 \leq \|vM^{(n)}(\omega)\|_1 \leq C\|v_1' M^{(n)}(\omega)\|_1$. We conclude that $\lambda(\omega, v) = \lambda(\omega, v_1')$ for all positive v.

Secondly, we consider the case where v_1' is merely non-negative and nonzero. Since M is uniformly aperiodic, for every ω there exists an integer K such that $M^{(K)}(\omega)$ is positive and therefore $v_1' M^{(K)}(\omega)$ is also positive. Using the argument above for positive vectors and the fact that $\lambda(\omega, v) = \lambda(\vartheta^K \omega, vM^{(K)}(\omega))$ we obtain $\lambda(\omega, v) = \lambda(\omega, v_1') = \lambda_1$ for all $v \geq 0$. □

Corollary 5.14. *For all $\omega \in \tilde{\Omega}$ one has $h_{top}(\Sigma_M(\omega)) = h_{top}(\Sigma_M) = \lambda_1$.*

Proof. Let $\mathbb{1}$ denote the vector in $\mathbb{R}^N$ with all entries 1. From Proposition 5.8, clearly $|\mathcal{C}_n(\omega)| = \|\mathbb{1}M^{(n-1)}(\omega)\|_1$, thus $h_{top}(\Sigma_M(\omega)) = \lambda(\omega, \mathbb{1})$. By Lemma 5.13, this equals λ_1 for all $\omega \in \tilde{\Omega}$. □

Now we state our main result of this section in the following theorem.

Theorem 5.15. *Let Σ_M be a uniformly aperiodic random shift of finite type with corresponding random adjacency matrix $M : \Omega \to \mathcal{M}_{k\times k}(\{0,1\})$. Fix $\omega^* \in \tilde{\Omega}$. Let $v^* \in W_\ell(\omega^*)$ with $\ell > 1$. Define the sequence of vectors $v : \{\vartheta^n \omega^*\}_{n\in\mathbb{Z}^+} \to \mathbb{R}^k$ on the orbit of ω^* by*

$$v(\vartheta^n \omega^*) := \frac{v^* M^{(n)}(\omega^*)}{\|v^* M^{(n)}(\omega^*)\|_1} \in W_\ell(\vartheta^n \omega^*)$$

and a sequence of sub-alphabets $\mathcal{A}_+$ by $\mathcal{A}_+(\vartheta^n \omega^*) := \{i \in \mathcal{A} \,:\, v_i(\vartheta^n \omega^*) > 0\}$. Suppose Σ_Q is a subshift of Σ_M such that on the orbit of ω^* the random matrix Q takes the following values:

$$Q_{ij}(\vartheta^n \omega^*) = \begin{cases} M_{ij}(\vartheta^n \omega^*) & \text{if } i \in \mathcal{A}_+(\vartheta^n \omega^*) \text{ and } j \in \mathcal{A}_+(\vartheta^{n+1} \omega^*) \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Then the topological entropy of $\Sigma_Q(\omega^*)$ is less than or equal to λ_ℓ, that is $h(\Sigma_Q(\omega^*)) \geq \lambda_\ell$. If $\Sigma_{Q'}$ is the complementary subshift to Σ_B then also $h(\Sigma_{Q'}(\omega^*)) \geq \lambda_\ell$.

Proof. Firstly, we will show by induction that for all $i = 1, \ldots, k$

$$(v(\omega^*)M^{(n)}(\omega^*))_i \leq (v(\omega^*)Q^{(n)}(\omega^*))_i. \tag{5.2}$$

Let $v = v^+ + v^-$ denote the decomposition of the vector v into nonnegative and nonpositive parts. Then we have $(v(\omega^*)M(\omega^*))_i \leq (v(\omega^*)^+M(\omega^*))_i = (v(\omega^*)Q(\omega^*))_i$ so (5.2) holds for $n = 1$. Assuming that (5.2) is true for some $n \geq 1$, we proceed with the inductive step

$$\begin{aligned}
\frac{(v(\omega^*)M^{(n+1)}(\omega^*))_i}{\|v(\omega^*)M^{(n)}(\omega^*)\|_1} &= (v(\vartheta^n\omega^*)M(\vartheta^n\omega^*))_i \\
&= \sum_j v_j(\vartheta^n\omega^*)M_{ji}(\vartheta^n\omega^*) \\
&= \sum_{j\in\mathcal{A}_+(\vartheta^n\omega^*)} v_j(\vartheta^n\omega^*)M_{ji}(\vartheta^n\omega^*) + \sum_{j\notin\mathcal{A}_+(\vartheta^n\omega^*)} v_j(\vartheta^n\omega^*)M_{ji}(\vartheta^n\omega^*) \\
&\leq \sum_{j\in\mathcal{A}_+(\vartheta^n\omega^*)} v_j(\vartheta^n\omega^*)M_{ji}(\vartheta^n\omega^*) \\
&= \frac{1}{\|v(\omega^*)M^{(n)}(\omega^*)\|_1} \sum_{j\in\mathcal{A}_+(\vartheta^n\omega^*)} (v(\omega^*)M^{(n)}(\omega^*))_j M_{ji}(\vartheta^n\omega^*) \\
&\leq \frac{1}{\|v(\omega^*)M^{(n)}(\omega^*)\|_1} \sum_{j\in\mathcal{A}_+(\vartheta^n\omega^*)} (v(\omega^*)Q^{(n)}(\omega^*))_j M_{ji}(\vartheta^n\omega^*) \quad (5.3) \\
&= \frac{(v(\omega^*)Q^{(n+1)}(\omega^*))_i}{\|v(\omega^*)M^{(n)}(\omega^*)\|_1}, \quad (5.4)
\end{aligned}$$

where we have used the inductive hypothesis to obtain inequality (5.3). Thus (5.2) holds for all $n \geq 1$ and all $i \in \mathcal{A}$. Noting that for $i \in \mathcal{A}_+(\vartheta^n\omega^*)$ both sides of (5.2) are positive, we have

$$\|(v(\omega^*)M^{(n)}(\omega^*))^+\|_1 \leq \|v(\omega^*)Q^{(n)}(\omega^*)\|_1.$$

Thus,

$$\lim_{n\to\infty} \frac{1}{n} \log \|v(\omega^*)M^{(n)}(\omega^*))^+\|_1 \leq \lim_{n\to\infty} \frac{1}{n} \log \|v(\omega^*)Q^{(n)}(\omega^*)\|_1$$

$$\leq \lim \frac{1}{n} \log \|\mathbb{1} Q^{(n)}(\omega^*)\|_1 = h(\Sigma_Q(\omega^*)).$$

Next we will show that $\lim_n (1/n) \log \|v(\omega^*)M^{(n)}(\omega^*)\|_1 \leq \lim_n (1/n) \log \|(v(\omega^*)M^{(n)}(\omega^*))^+\|_1$ to finally obtain that $\lambda_\ell \leq h(\Sigma_Q(\omega^*))$. We need to have some control over the relative size of the positive and negative parts of v along the orbit of ω^*. To continue the proof of Theorem 5.15 we first state and prove the following claim.

Claim: Let $\omega \in \{\vartheta^n \omega^*\}_{n \in \mathbb{Z}^+}$ and let $N = N(\omega)$ be smallest integer such that $M^{(N)}(\omega) > 0$. Then

$$\frac{1}{k^N} \leq \frac{\|v(\omega)^+\|_1}{\|v(\omega)^-\|_1} \leq k^N. \tag{5.5}$$

Proof of claim: As M is a $0-1$ matrix, then $\max_{i,j} M_{ij}^{(N)}(\omega) \leq k^N$. From the definition of N we also have $\min_{i,j} M_{ij}^{(N)}(\omega) \geq 1$. The proof of (5.5) is by contradiction. Suppose that $\|v(\omega)^+\|_1 > k^N \|v(\omega)^-\|_1$. Then for every $i = 1, \ldots, k$ we have

$$\begin{aligned}
v_i(\vartheta^N \omega)\|v(\omega)M^{(N)}(\omega)\|_1 &= (v(\omega)M^{(N)}(\omega))_i \\
&= (v(\omega)^+ M^{(N)}(\omega) + v(\omega)^- M^{(N)}(\omega))_i \\
&= \sum_j (v(\omega)^+)_i M_{ij}^{(N)}(\omega) + \sum_j (v(\omega)^-)_i M_{ij}^{(N)}(\omega) \\
&\geq \|v(\omega)^+\|_1 - k^N \|v(\omega)^-\|_1 \\
&> k^N \|v(\omega)^-\|_1 - k^N \|v(\omega)^-\|_1 = 0.
\end{aligned}$$

Therefore $v(\vartheta^N \omega) \in W_\ell(\vartheta^N \omega)$ is a positive vector, but this is a contradiction because the Lyapunov exponent of any positive vector equals $\lambda_1 \neq \lambda_\ell$. The inequality $1/k^N \leq \|v(\omega)^+\|_1 / \|v(\omega)^-\|_1$ is proven similarly. $\square$

We continue the proof of the theorem as follows

$$\begin{aligned}
\lambda_\ell &= \lim_{n\to\infty} \frac{1}{n} \log \|(v(\omega^*)M^{(n)}(\omega^*))\|_1 \\
&= \lim_{n\to\infty} \frac{1}{n} \log \left(\|(v(\omega^*)M^{(n)}(\omega^*))^+\|_1 + \|(v(\omega^*)M^{(n)}(\omega^*))^-\|_1 \right) \\
&\leq \lim_{n\to\infty} \frac{1}{n} \log((1+k^N)\|(v(\omega^*)M^{(n)}(\omega^*))^+\|_1)
\end{aligned}$$

$$= \lim_{n\to\infty} \frac{1}{n} \log \|(v(\omega^*)M^{(n)}(\omega^*))^+\|_1.$$

Hence we have shown (5.5), therefore $\lambda_l \leq h_{top}(\Sigma_Q)$.

By considering $-v$ in place of v we obtain a subshift $\Sigma_{Q'}$ with sub-alphabet $\mathcal{A}_- := \mathcal{A} \setminus \mathcal{A}_+$ that is complementary to Σ_Q and the inequality $\lambda_l \leq h_{top}(\Sigma_{Q'})$ also holds. □

Theorem 5.15 may be used to decompose a metastable random shift space into two complementary random subshifts, with each possessing a large topological entropy. One chooses $v(\omega) \in W_2(\omega)$ corresponding to the second largest Lyapunov exponent λ_2 and partitions according to the positive and negative parts of the push-forwards of v by the matrix cocycle of M. We illustrate this with the following example.

Example 5.16. Let $\Omega = \{0,1\}^{\mathbb{Z}}$ and let $\vartheta : \Omega \circlearrowleft$ be the full two-sided shift on two symbols. Consider the random matrix $M : \Omega \to \mathcal{M}_{4\times 4}$ given by $M(\omega) = M_{\omega_0}$ where

$$M_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \text{and } M_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

We consider a generic point $\omega^* \in \tilde{\Omega}$ where ω_i^* is the $(20+i)^{\text{th}}$ digit of the fractional part of the binary expansion of π for $i > -20$ (and $\omega_i^* = 0$ for $i \leq -20$). The first few elements of ω^*, with the zeroth element underlined, are given below:

$$\omega^* = (\dots, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, \underline{0}, 1, 1, 0, 1, 0, 0, 0, 1, 1, 0, \dots).$$

Using Algorithm 4.22 (excluding (A1) and with $J = 2I = 40$) we approximate the largest Lyapunov exponent $\lambda_1(\omega^*) \approx \log 2.20$, which, by Corollary 5.14, equals the topological entropy of the random shift, that is $h_{top}(\Sigma_M(\omega^*)) \approx \log 2.20$. The second Lyapunov exponent of this system is $\lambda_2 \approx \log 1.21$. Thus, by Theorem 5.15 we can decompose the shift Σ_M into two complementary subshifts Σ_Q and $\Sigma_{Q'}$, each with topological entropy larger than $\log 1.21$. Moreover, the decomposition is given by the Oseledets subspaces for λ_2. These Oseledets subspaces W_l are spans of the vectors $\{v(\omega^*), v(\vartheta\omega^*), v(\vartheta^2\omega^*), \dots\}$, whose graphs are shown in Figure 5.3. The sub-alphabets $\mathcal{A}_+$ and $\mathcal{A}_-$ have the following values on the first four points in the forward orbit of ω^*:

$$\begin{aligned}
\mathcal{A}_+(\omega^*) &= \{1,2\}, & \mathcal{A}_-(\omega^*) &= \{3,4\},\\
\mathcal{A}_+(\vartheta\omega^*) &= \{2,4\}, & \mathcal{A}_-(\vartheta\omega^*) &= \{1,3\},\\
\mathcal{A}_+(\vartheta^2\omega^*) &= \{1,3\}, & \mathcal{A}_-(\vartheta^2\omega^*) &= \{2,4\},\\
\mathcal{A}_+(\vartheta^3\omega^*) &= \{1,2\}, & \mathcal{A}_-(\vartheta^3\omega^*) &= \{3,4\}.
\end{aligned}$$

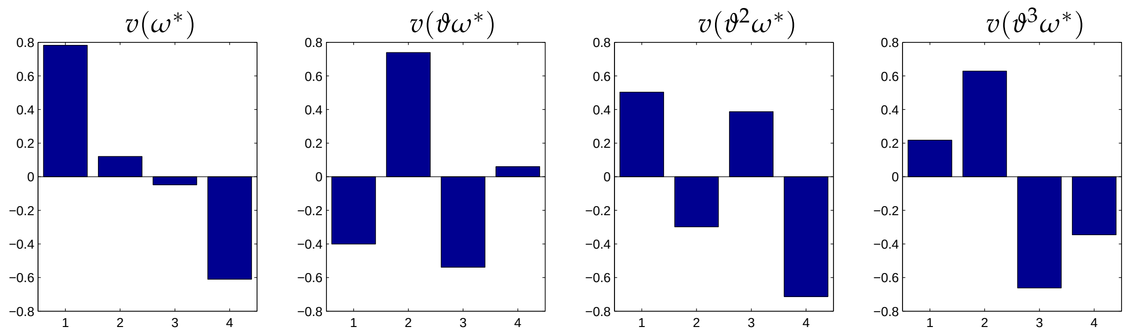


Figure 5.3: Vectors spanning Oseledets subspaces corresponding to the second Lyapunov exponent.

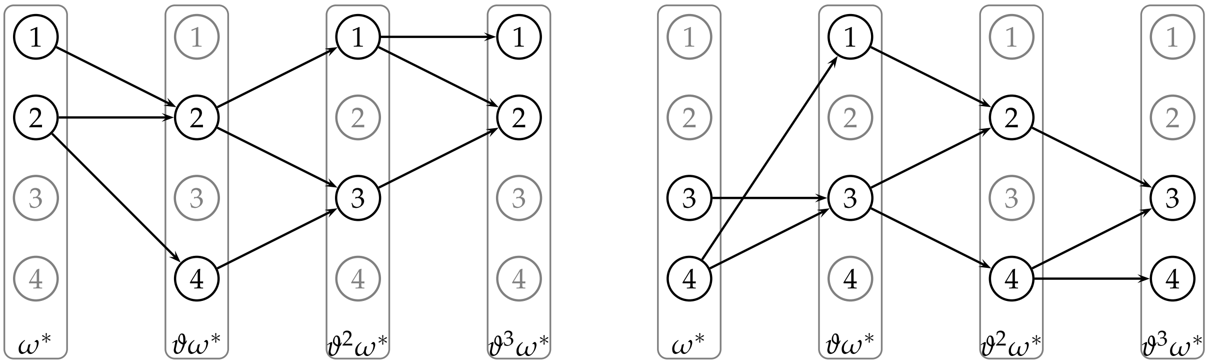
We construct the matrix Q of the random subshift according to (5.1) in Theorem 5.15. On the first three elements of the forward orbit of ω^*, Q takes the following values:

$$Q(\omega^*) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Q(\vartheta\omega^*) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad Q(\vartheta^2\omega^*) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly the adjacency matrices of the complementary subshift $\Sigma_{Q'}$ begin with

$$Q'(\omega^*) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \ Q'(\vartheta\omega^*) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ Q'(\vartheta^2\omega^*) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The graphs of Σ_Q and $\Sigma_{Q'}$ for the first four elements of the forward orbit of ω^* are shown in Figure 5.4.



(a) Graph of Σ_Q. (b) Graph of $\Sigma_{Q'}$.

Figure 5.4: Graphs of Σ_Q and $\Sigma_{Q'}$ for first four transitions along the orbit of ω^*. The grayed-out nodes in each belong to the corresponding complementary subshift.

Using Algorithm 4.22 (with $J = 20$ and $I = 0$) we estimate the largest Lyapunov exponents, and therefore the topological entropies, of these two subshifts to be $h_{top}(\Sigma_Q(\omega^*)) \approx \log 1.62$ and $h_{top}(\Sigma_{Q'}(\omega^*)) \approx \log 1.58$. Both are larger than $\lambda_2 \approx \log 1.21$, as predicted by Theorem 5.15.

Summary

We started with the aim of investigating escape rates in dynamical systems, with the premise that metastability is closely linked to low escape.

In the first chapter, we defined the main notions and provided a few useful and well-known results, together with a literature survey of the area of open dynamical systems.

We began Chapter 2 by providing two examples emphasising the distinctions between almost invariance and low escape rate. We then proved our first main result (Theorem 2.5), which showed that the metastable regions constructed from spectral analysis of the Perron-Frobenius operator do possess low escape rates. More precisely, if $\mathcal{P}f = \rho f$ for real ρ close to 1 then the metastable sets $A_{\pm} := \{\pm f > 0\}$ possess low escape rate, bounded above by $-\log \rho$. Corollary 2.7 then asserts that

$$\inf_{A} \max\{E(A), E(X \setminus A)\} \leq -\log \rho.$$

We demonstrated numerically that in the absence of a spectral gap one loses control of the regularity of the sets A. In order to ensure the existence of a spectral gap for Lasota-Yorke maps we considered $\mathcal{P}$ acting on BV, the space of functions of bounded variation. In this setting one has some control on the regularity of the partition (Corollary 2.18).

We continued further developing our ideas on escape and spectral gap in Chapter 3, focussing on the class of Pomeau-Manneville maps with two full branches and an indifferent fixed point at the origin. We recalled that these maps do not exhibit a spectral gap on any reasonable Banach space of functions on the interval. After creating a small Markov hole $H_n = [0, x_n]$ in the problematic neighbourhood at the origin we demonstrated (Theorem 3.9) that in a set of regular measures with densities bounded away from zero and infinity, a unique absolutely continuous conditionally invariant measure μ_n^* exists. Moreover, we showed that μ_n^* converges in L^1 to the ACIM of the closed system and, for an arbitrary hole $[0, \epsilon]$, the Lebesgue escape rate scales linearly as $\epsilon \to 0$ (Theorem 3.9). Thereafter, we provided numerical evidence for the scaling of second eigenvalue of the coarse-grained Ulam approximation of the operator, which accurately represents the Perron-Frobenius operator of the map slightly perturbed in the critical region. The asymptotic behaviour, somewhat surprisingly, agreed with our simple two-state Markov chain model of the dynamics.

Motivated by the work of Chapter 3 on random perturbations, in Chapter 4 we engaged into defining and investigating escape rates from fully random dynamical systems. In the presence of randomness, Perron-Frobenius operators became cocycles and their Lyapunov spectrum took the place of the deterministic eigenvalue spectrum. We succeeded in translating our deterministic results to this setting and showed in Theorem 4.7 that random sets $A_\pm$ that satisfy $A_\pm(\vartheta^n\omega) = \{\pm\mathcal{P}_\omega^n f > 0\}$ for $f \in L^\infty$ possess escape rates that are bounded above by the absolute value of the corresponding Lyapunov exponent $|\lambda(\omega, f)|$. We proved in Theorem 4.17 that, provided an Oseledets splitting holds in a Banach space $(Y, \|\cdot\|_Y)$ with $Y \in L^1(X)$ and $C\|\cdot\|_Y \geq \|\cdot\|_{L^1}$, we have in the isolated Lyapunov spectrum $\lambda_{\|\cdot\|_{L^1}} = \lambda_{\|\cdot\|_Y}$. We then applied this result to demonstrate the validity of Theorem 4.7 in the setting of Perron-Frobenius cocycles of Rychlik random dynamical systems in BV.

Finally, in the fifth chapter we adapted our methods to deterministic and random shifts of finite type, where Perron-Frobenius operators and their cocycles were replaced by adjacency matrices and their cocycles, respectively. Rather than reducing escape rate, we considered the equivalent problem of partitioning a shift of finite type into two complementary subshifts in a way that ensures a large topological entropy is retained in each element of the partition (Theorem 5.1 and Theorem 5.15).

In conclusion, we successfully demonstrated, in both the deterministic and random settings, that effective methods for detecting almost-invariant sets are also useful in the detection of sets with low escape rates. We also showed that our techniques are applicable in the area of symbolic dynamics when one searches for complementary subshifts of high topological entropy.

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