School of Mathematics and Statistics
Faculty of Science

# Derivations with values into ideals of a semifinite von Neumann algebra

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One of the classic problems in operator algebra theory is the question whether every derivation from an algebra N into an N-bimodule J is automatically inner. In the present thesis, we study derivations with values into ideals of a semifinite von Neumann algebras M. Precisely, we characterise the symmetric ideals J of M such that every derivation from an arbitrary C*-subalgebra (resp. von Neumann subalgebra) of M into J is automatically inner.

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# Abstract

One of the classic problems in operator algebra theory is the question whether every derivation from an algebra $\mathcal{N}$ into an $\mathcal{N}$-bimodule $J$ is automatically inner [56, 63]. In the present thesis, we study derivations with values into ideals of a semifinite von Neumann algebra $\mathcal{M}$. Precisely, we characterise the symmetric ideals $\mathcal{J}$ of $\mathcal{M}$ such that every derivation from an arbitrary $C^*$-subalgebra (resp. von Neumann subalgebra) of $\mathcal{M}$ into $\mathcal{J}$ is automatically inner.

# Acknowledge

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# Introduction

A corner stone of classical analysis is the theory of differentiability. During the twentieth century, a great effort was made in attempt to establish the theory of differential operators in various classes of spaces. In abstract algebra, a *derivation* is a mapping on an algebra $\mathcal{A}$ over $\mathbb{C}$ which generalises certain features of a differential operator. Precisely, if $\mathcal{J}$ is an $\mathcal{A}$-bimodule, a linear map $\delta : \mathcal{A} \to \mathcal{J}$ that satisfies the Leibniz law is called a derivation, that is,

$$\delta(ab) = \delta(a)b + a\delta(b), \ \forall a, b \in \mathcal{A}.$$

In particular, if $k \in \mathcal{J}$, then $\delta_k(x) := kx - xk$, $x \in \mathcal{A}$, is a derivation. Such derivations implemented by elements in $\mathcal{J}$ are called *inner* [7, 125].

Derivations appeared for the first time at a fairly early stage in the field of $C^*$-algebras and were initiated in the 1950s, by Kaplansky [83], by Singer and Wermer [126], and later by Sakai etc [118, 119]. The study of derivations in operator algebras continues to be one of the central branches in the field [121]. During the late 1960s and early 1970s, a great deal of work was done (by Kadison, Johnson and Ringrose etc.) in developing the theory of derivations, which also led to the study of Hochschild cohomology of $C^*$-algebras (see e.g. [63, 64, 66, 72–74, 74–76]). The most famous result in this field is the so-called Kadison–Sakai theorem [70, 119], which shows that every derivation from a von Neumann algebra into itself is inner. We provide a short survery of the theory of derivations in operator algebras in Chapter 3.

The present thesis concentrates on derivations having values into ideals of a von Neumann algebra $\mathcal{M}$ and identifies those ideals $\mathcal{J}$ of $\mathcal{M}$ such that every derivation $\delta : \mathcal{A} \to \mathcal{J}$ is necessarily inner for any $C^*$-/von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$.

A beautiful extension of the Kadison–Sakai theorem was obtained recently by Ber and Sukochev [16, 17], who showed that for a von Neumann algebra $\mathcal{M}$ and an arbitrary ideal $J$ of $\mathcal{M}$, every derivation from $\mathcal{M}$ into $J$ is automatically inner. However, the case for general subalgebras of $\mathcal{M}$ is more complicated and requires new techniques/ideas. In 1972, Johnson and Parrott [67] showed that derivations from an abelian/properly infinite von Neumann subalgebra of $B(\mathcal{H})$ into the algebra $K(\mathcal{H})$ of all compact operators on $\mathcal{H}$ are inner. However, they failed to resolve the case when $\mathcal{A}$ is a type $II_1$ von Neumann algebra, which remained open until resolved by Popa [110] in 1987. This result is now known as the so-called the Johnson–Parrott–Popa theorem.

**Theorem** (Johnson–Parrott–Popa)**.** *Every derivation from an arbitrary von Neumann algebra $\mathcal{A}$ of $B(\mathcal{H})$ into the algebra $K(\mathcal{H})$ is inner.*

A natural development of the Johnson–Parrott–Popa theorem is to establish a suitable semifinite version of the result. In 1985, Kaftal and Weiss [81] considered Johnson and Parrott's derivation problem in a more general setting where $B(\mathcal{H})$ is replaced with a semifinite von Neumann algebra $\mathcal{M}$ and $K(\mathcal{H})$ is replaced with the uniform norm closed ideal $\mathcal{J}(\mathcal{M})$ generated by all finite projections in $\mathcal{M}$. It is shown in [81] that if $\mathcal{A}$ is an abelian (or properly infinite) von Neumann subalgebra of $\mathcal{M}$ containing the center $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$, then any derivation

$$\delta : \mathcal{A} \to \mathcal{J}(\mathcal{M})$$

is inner. This result was latter extended by Popa and Rădulescu [112] in 1988. In particular, they showed that the result by Kaftal and Weiss holds if $\mathcal{A}$ is an arbitrary type $II_1$ (or properly infinite) subalgebra of $\mathcal{M}$. However, for an arbitrary von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$, any derivation $\delta : \mathcal{A} \to \mathcal{J}(\mathcal{M})$ is inner only under additional condition on the center of $\mathcal{A}$ and $\mathcal{M}$ (see precise condition in Theorem 3.2.2). In particular, Popa and Rădulescu established the existence of non-inner derivations

$$\delta : \mathcal{A} \to \mathcal{J}(\mathcal{M})$$

for a specific semifinite von Neumann algebra $\mathcal{M}$ and an abelian von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$, which is the first example of non-inner derivations in von Neumann algebras.

In 1987, Christensen [24] introduced the notion of generalized compacts associated with a von Neumann algebra and showed that derivations from a properly infinite von Neumann algebra into the generalized compacts associated with this von Neumann algebra are inner. However, the question whether derivations from a type $II_1$ von Neumann algebra into the generalized compacts associated with this von Neumann algebra are inner was left open, which was recently answered in affirmative by Galatan and Popa [49].

One of the main results of the present thesis is the Johnson–Parrott–Popa theorem for another type of semifinite version of the ideal $K(\mathcal{H})$, namely the ideal $\mathcal{C}_0(\mathcal{M}, \tau)$ of $\tau$-compact operators, which is the uniform norm closure of the linear span of all $\tau$-finite projections in a semifinite von Neumann algebra $\mathcal{M}$ equipped with a semifinite faithful normal trace $\tau$.

**Theorem** (Theorem 5.6.1). *Let $\mathcal{A}$ be a von Neumann subalgebra of a semifinite von Neumann algebra $(\mathcal{M}, \tau)$. Then every derivation $\delta : \mathcal{A} \to \mathcal{C}_0(\mathcal{M}, \tau)$ is necessarily inner.*

Even though $\mathcal{C}_0(\mathcal{M}, \tau)$ and $\mathcal{J}(\mathcal{M})$ are similar in many respects (see Theorem 2.5.7), our result is in strong contrast with the result by Popa and Rădulescu [112], since we do not impose any additional condition on the von Neumann subalgebra $\mathcal{A}$.

Some attempts have been made to extend the Johnson–Parrott–Popa theorem in another direction, i.e., replacing $K(\mathcal{H})$ with some other ideals in $B(\mathcal{H})$. The Schatten $p$-classes $C_p(\mathcal{H})$ introduced in [122] are important examples of ideals in $B(\mathcal{H})$, which are the noncommutative counterpart of $l_p$-sequence spaces in the sense of Calkin [52,98]. In 1977, Hoover [56] used the Ryll-Nardzewski fixed point theorem (as suggested by Johnson [63,67]) and the reflexivity of the ideals $C_p(\mathcal{H})$, $1 < p < \infty$, to show that every derivation from a $C^*$-subalgebra of $B(\mathcal{H})$ into $C_p(\mathcal{H})$ is inner. Hoover [56] also resolved the special case when $p = 1$ by a completely different approach (see also [4] for a new proof).

However, when $B(\mathcal{H})$ is replaced by a general semifinite von Neumann algebra $\mathcal{M}$, the corresponding ideal $\mathcal{C}_p(\mathcal{M}, \tau)$ is not necessarily reflexive even for $1 < p < \infty$ and therefore the Ryll-Nardzewski fixed point theorem can not be applied directly (the method used in [4] is not applicable, either). In 1985, using Johnson and Parrott's trick [67], Kaftal and Weiss [81] showed that every derivation from an abelian (or properly infinite) von Neumann subalgebra of $\mathcal{M}$ into $\mathcal{C}_p(\mathcal{M}, \tau)$ is inner when $1 \leq p < \infty$. However, the case for general von Neumann subalgebra of $\mathcal{M}$ was left unanswered. The second main result of the thesis provides sharp conditions (we demonstrate the sharpness of our result in Theorem 4.2.1) on a symmetric ideal $\mathcal{E}$ of $\mathcal{M}$ such that any derivation from an arbitrary $C^*$-subalgebra of $\mathcal{M}$ into $\mathcal{E}$ is inner. Namely, we prove the following result (see Chapter 2 for definitions), which, in particular, fully resolves the untreated cases for derivations with values in $\mathcal{C}_p(\mathcal{M},\tau)$ in the paper [81] by Kaftal and Weiss.

**Theorem** (Theorem 4.1.4)**.** *Let $\mathcal{A}$ be a $C^*$-subalgebra of $\mathcal{M}$ and let $E(0,\infty)$ be a fully symmetric space on $(0,\infty)$ having the Fatou property and order continuous norm, i.e, $E(0,\infty)$ is a KB-space. Then every derivation $\delta$ from $\mathcal{A}$ into the corresponding symmetric ideal $\mathcal{E}(\mathcal{M},\tau)$ of $\mathcal{M}$ is inner.*

The thesis is structured such that the first two chapters consist of the necessary background material of noncommutative analysis. The classical results of derivations are surveyed in Chapter 3. In Chapter 4, we study derivation $\delta$ from a $C^*$-subalgebra $\mathcal{A}$ of a semifinite von Neumann algebra $\mathcal{M}$ into the symmetric ideals of $\mathcal{M}$. The main tools are the Ryll-Nardzewski fixed point theorem [96] and the properties of the so-called $p$-convexifications of a noncommutative symmetric space developed by P. Dodds, T. Dodds and B. de Pagter [38]. In Chapter 5, we show that derivations from an arbitrary von Neumann subalgebra of $\mathcal{M}$ into $\mathcal{C}_0(\mathcal{M},\tau)$ are necessarily inner. As an application of this result, we show that derivations from an arbitrary von Neumann subalgebra of $\mathcal{M}$ into $\mathcal{E}$ are necessarily inner for a wide class of symmetric ideals $\mathcal{E}$ of $\mathcal{M}$, which unifies the Johnson–Parrott–Popa theorem [67,110] and results by Kaftal and Weiss [81] with a substantial extension.

The main results in this thesis all stem from the articles *Derivations with values in ideals of semifinite von Neumann algebras* [11] and *Derivations with values in the ideal of $\tau$-compact operators affiliated with a semifinite von Neumann algebra* [12]. These results have been presented in:

1. The International Workshop on Operator Theory and Applications (IWOTA), Shanghai, 23-27 July 2018.

2. Mini Workshop on Noncommutative Analysis, Central South University, Changsha, 18 July 2018.

3. The fifth Annual Postgraduate Conference (Session chair), UNSW, Sydney, 8 June 2018.

4. The 61st annual meeting of the Australian Mathematical Society, Macquarie University, Sydney, 12-15 December 2017.

5. The fourth Annual Postgraduate Conference (Plenary Speaker), UNSW, Sydney, 7 June 2017.

# Contents

# Chapter 1

# von Neumann algebras

Von Neumann algebras were originally introduced by John von Neumann [97] in 1929, motivated by his study of single operators, group representations, ergodic theory and quantum mechanics. He and Francis Murray developed the basic theory, under the original name of rings of operators, in a series of papers written in the 1930s and 1940s [93–95,99–102]. In this section, we recall some notions of the theory of von Neumann algebras. For details on von Neumann algebra theory, the reader is referred to e.g. [34], [77, 78] or [132]. The book by Connes [29] discusses more advanced topics.

## 1.1 Algebras with an involution

Let $\mathcal{A}$ be an algebra over the complex numbers (the set of all complex numbers is denoted by $\mathbb{C}$). The mapping $x \mapsto x^*$ from $\mathcal{A}$ into itself is said to be an *involution* if

1. $(x+y)^* = x^* + y^*$;
2. $(\lambda x)^* = \bar{\lambda} x^*$;
3. $(xy)^* = y^* x^*$;
4. $(x^*)^* = x$,

whenever $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. An algebra equipped with an involution is called a $*$-algebra. An element $x \in \mathcal{A}$ is called *self-adjoint* (or *hermitian*) if $x^* = x$. The set of all self-adjoint elements in $\mathcal{A}$ is denoted by $\mathcal{A}_h$, which is clearly a real linear subspace of $\mathcal{A}$. If $x, y \in \mathcal{A}_h$, then $xy \in \mathcal{A}_h$ if and only if $xy = yx$. Note furthermore that $x^*x$ and $xx^*$ belong to $\mathcal{A}_h$ for every $x \in \mathcal{A}$. For $x \in \mathcal{A}$, we set

$$\mathrm{Re}(x) = \frac{1}{2}(x + x^*), \;\; \mathrm{Im}(x) = \frac{1}{2i}(x - x^*).$$

Clearly, $\mathrm{Re}(x), \mathrm{Im}(x) \in \mathcal{A}_h$ and $x = \mathrm{Re}(x) + i\,\mathrm{Im}(x)$ for all $x \in \mathcal{A}$. Conversely, if for a given $x \in \mathcal{A}$, we have $x = x_1 + ix_2$ with $x_1, x_2 \in \mathcal{A}_h$, then necessarily $x_1 = \mathrm{Re}(x)$ and $x_2 = \mathrm{Im}(x)$.

The $*$-algebra is called *unital* if it possesses a multiplicative identity, a *unit element*, denoted by $\mathbf{1} = \mathbf{1}_{\mathcal{A}}$. Note that $\mathbf{1}^* = \mathbf{1}$. An element $x$ in the unital algebra

$\mathcal{A}$ is said to be *invertible* if there exists $y \in \mathcal{A}$ such that $xy = yx = \mathbf{1}$; in this case, the element $y$ is unique and denoted by $x^{-1}$, the *inverse* of $x$. It is easy to see that $x \in \mathcal{A}$ is invertible if and only if $x^*$ is invertible and, in this case, $(x^{-1})^* = (x^*)^{-1}$.

An element $x \in \mathcal{A}$ is called *normal* if $x^*x = xx^*$. Furthermore, $u \in \mathcal{A}$ is said to be *unitary* if $u^*u = uu^* = \mathbf{1}$ (equivalently, $u$ is invertible and $u^* = u^{-1}$). All unitary elements in $\mathcal{A}$ form a (multiplicative) group, which we shall denote by $\mathcal{U}(\mathcal{A})$. An element $p \in \mathcal{A}$ is said to be a *projection* if $p^2 = p$ and $p^* = p$. The set of all projections in $\mathcal{A}$ is denoted by $\mathcal{P}(\mathcal{A})$.

A subset $\mathcal{S}$ of a $*$-algebra $\mathcal{A}$ is called self-adjoint if $x^* \in \mathcal{S}$ whenever $x \in \mathcal{S}$. A self-adjoint subalgebra $\mathcal{S}$ of $\mathcal{A}$ is said to be a *$*$-subalgebra* of $\mathcal{A}$ and, in this case, $\mathcal{S}$ itself is a $*$-algebra with respect to the algebraic operations and involution inherited from $\mathcal{A}$.

## 1.2 $C^*$-algebras

An algebra $\mathcal{A}$ equipped with a norm $\|\cdot\|_{\mathcal{A}}$ such that $\mathcal{A}$ is a Banach space and

(i) $\|xy\|_{\mathcal{A}} \leq \|x\|_{\mathcal{A}} \|y\|_{\mathcal{A}}$ for all $x, y \in \mathcal{A}$,

is called a *Banach algebra*. If $\mathcal{A}$ has a unit element $\mathbf{1}$, then we assume that $\|\mathbf{1}\|_{\mathcal{A}} = 1$. If $\mathcal{A}$ is a $*$-algebra and the norm also satisfies

(ii) $\|x^*\|_{\mathcal{A}} = \|x\|_{\mathcal{A}}$ for all $x, y \in \mathcal{A}$,

then $\mathcal{A}$ is called a *Banach $*$-algebra*.

A *$C^*$-algebra* is a $*$-algebra $\mathcal{A}$ equipped with a norm $\|\cdot\|_{\mathcal{A}}$, such that $\mathcal{A}$ is a Banach algebra and

(iii) $\|x^*x\|_{\mathcal{A}} = \|x\|^2_{\mathcal{A}}$ for all $x, y \in \mathcal{A}$.

If $\mathcal{A}$ is a $C^*$-algebra, then it is easy to see that the norm also satisfies condition (i). So, any $C^*$-algebra is a Banach $*$-algebra. Moreover, if the $C^*$-algebra has a unit element $\mathbf{1}$, then the equality $\|\mathbf{1}\|_{\mathcal{A}} = 1$ is automatically satisfied.

**Proposition 1.2.1** (see e.g. [18, II,3.2.12])**.** *Every element of a unital $C^*$-algebra $\mathcal{A}$ is a linear combination of four unitary elements of $\mathcal{A}$. In fact, if $x = x^* \in \mathcal{A}$ and $\|x\|_{\mathcal{A}} \leq 2$, then $x$ is a sum of two unitary elements of $\mathcal{U}(\mathcal{A})$.*

If $\mathcal{A}$ is a $C^*$-algebra and $\mathcal{A}_1$ is a closed $*$-subalgebra of $\mathcal{A}$, then $\mathcal{A}_1$ is $C^*$-algebra with respect to the structure inherited from $\mathcal{A}$ and we say that $\mathcal{A}_1$ is a *$C^*$-subalgebra* of $\mathcal{A}$.

We recall also the classical unitalization result for $C^*$-algebras [113].

**Theorem 1.2.2.** *Let $\mathcal{A}$ be a $C^*$-algebra. There exists a $C^*$-algebra $\mathcal{A}_1$ which is unital, contain $\mathcal{A}$ as a closed two-sided ideal and $\mathcal{A}_1/\mathcal{A} \cong \mathbb{C}$. Moreover, this $C^*$-algebra is unique.*

## 1.3 Topologies on $B(\mathcal{H})$

In what follows, $\mathcal{H}$ is a Hilbert space and $B(\mathcal{H})$ is the $*$-algebra of all bounded linear operators on $\mathcal{H}$ equipped with the uniform norm $\|\cdot\|_\infty$, and $\mathbf{1}$ is the identity operator on $\mathcal{H}$.

In addition to the uniform norm topology, there are a number of other important topologies on $B(\mathcal{H})$.

For every $\eta \in \mathcal{H}$, we define the semi-norm $\rho_\eta$ on $B(\mathcal{H})$ by $\rho_\eta(x) = \|T\eta\|_{\mathcal{H}}$, $x \in B(\mathcal{H})$. The locally convex Hausdorff topology on $B(\mathcal{H})$ generated by the family of semi-norms $\{\rho_\eta : \eta \in \mathcal{H}\}$ is called the *strong operator topology* (shortly, *so-topology*). A net $\{T_\alpha\}$ in $B(\mathcal{H})$ *so-converges* to an operator $T \in B(\mathcal{H})$, denoted by $T_\alpha \to_{so} T$, if and only if

$$\|T_\alpha \eta - T\eta\|_{\mathcal{H}} \to 0$$

for all $\eta \in \mathcal{H}$. Multiplication in $B(\mathcal{H})$ is continuous with respect to the so-topology in each factor separately, but in general not jointly so-continuous (however, multiplication is jointly so-continuous when restricted to norm bounded sets). The mapping $T \mapsto T^*$ is not so-continuous (unless $\mathcal{H}$ is finite dimensional).

For $\eta, \xi \in \mathcal{H}$, we define the semi-norm $\rho_{\eta,\xi}$ by

$$\rho_{\eta,\xi}(T) = |\langle T\eta, \xi\rangle|, \ T \in B(\mathcal{H}).$$

The locally convex Hausdorff topology on $B(\mathcal{H})$ generated by the family of semi-norms $\{\rho_{\eta,\xi} : \eta, \xi \in \mathcal{H}\}$ is called the *weak operator topology* (shortly, *wo-topology*). A net $\{T_\alpha\}$ in $B(\mathcal{H})$ wo-converges to an operator $T \in B(\mathcal{H})$, denoted by $T_\alpha \to_{wo} T$, if and only if

$$\langle T_\alpha \eta, \xi\rangle \to \langle T\eta, \xi\rangle, \ \forall \eta, \xi \in \mathcal{H}.$$

It is clear that the wo-topology is weaker than the so-topology and coincides with the latter only if $\mathcal{H}$ is finite dimensional. However, for a convex subset of $B(\mathcal{H})$, its wo-closure coincides with its so-closure.

**Proposition 1.3.1** (see e.g. [31, Chapter IX, Corollary 5.2])**.** *If $\mathcal{S}$ is a convex subset of $B(\mathcal{H})$, then the wo-closure of $\mathcal{S}$ equals the so-closure of $\mathcal{S}$.*

Multiplication is wo-continuous in each factor separately, but is not jointly wo-continuous (unless $\mathcal{H}$ is finite dimensional). The mapping $x \mapsto x^*$ is clearly wo-continuous.

A useful property of the wo-topology is given in the next theorem.

**Theorem 1.3.2** (see e.g. [31, Chapter IX, Proposition 5.5])**.** *The closed unit ball $B(\mathcal{H})_1$ of $B(\mathcal{H})$ is compact in the weak operator topology.*

Next, we consider the locally convex Hausdorff topology on $B(\mathcal{H})$ generated by the family of semi-norms defined by $\rho_{\{\eta_i\},\{\xi_i\}}(T) = |\sum_{i=1}^\infty \langle T\eta_i, \xi_i\rangle|$, where $\{\eta_i\}_{i=1}^\infty$ and $\{\xi_i\}_{i=1}^\infty$ are sequences in $\mathcal{H}$ satisfying $\sum_{i=1}^\infty \|\eta_i\|_{\mathcal{H}}^2 < \infty$ and $\sum_{i=1}^\infty \|\xi_i\|_{\mathcal{H}}^2 < \infty$. This topology is called the *ultra-weak operator topology* (shortly, *uwo-topology*) or

*$\sigma$-weak topology* or *weak* operator topology*. The ultra-weak operator topology is stronger than the wo-topology. On norm bounded subsets of $B(\mathcal{H})$, the uwo- and wo- topology coincide [132, Chapter II]. In particularly, $B(\mathcal{H})_1$ is uwo-compact.

Given a sequence $\{\eta_i\}$ in $\mathcal{H}$ satisfying $\sum_{i=1}^{\infty} \|\eta_i\|_{\mathcal{H}}^2 < \infty$, the semi-norm $\rho_{\{\eta_i\}}$ on $B(\mathcal{H})$ is defined by

$$\rho_{\{\eta_i\}}(T) = \left( \sum_{i=1}^{\infty} \|T\eta_i\|_{\mathcal{H}}^2 \right)^{\frac{1}{2}}.$$

The locally convex Hausdorff topology on $B(\mathcal{H})$ generated by the family of semi-norms given by $\rho_{\{\eta_i\}}$ is called the *ultra-strong operator topology* (briefly, *uso-topology*). The ultra-strong operator topology is stronger than the so- and the uwo-topologies and is weaker than the norm topology. On norm bounded subsets of $B(\mathcal{H})$, the uso- and so- topology coincide [132, Chapter II].

## 1.4 von Neumann algebras

Given a non-empty subset $\mathcal{S}$ of $B(\mathcal{H})$, the commutant $\mathcal{S}'$ of $\mathcal{S}$ is defined by

$$\mathcal{S}' = \{X \in B(\mathcal{H}) : XY = YX \ \forall Y \in \mathcal{S}\},$$

which is a (wo)-closed unital subalgebra of $B(\mathcal{H})$. If $\mathcal{S}$ is self-adjoint, then $\mathcal{S}'$ is a (wo)-closed unital $C^*$-subalgebra of $B(\mathcal{H})$. Defining the bi-commutant $\mathcal{S}''$ of $\mathcal{S}$ by $\mathcal{S}'' = (\mathcal{S}')'$, it is clear that $\mathcal{S} \subset \mathcal{S}''$ and $\mathcal{S}' = \mathcal{S}'''$.

**Definition 1.4.1.** *A $*$-subalgebra $\mathcal{M}$ of $B(\mathcal{H})$ is said to be a von Neumann algebra if $\mathcal{M} = \mathcal{M}''$.*

If $\mathcal{M}$ is a von Neumann algebra, then $\mathcal{M}$ is a wo-closed unital $C^*$-subalgebra of $B(\mathcal{H})$. The simplest examples of von Neumann algebras are given by the algebra $B(\mathcal{H})$ itself and the subalgebra $\mathbb{C}\mathbf{1} = \{\lambda\mathbf{1} : \lambda \in \mathbb{C}\}$. For any non-empty subset $\mathcal{S}$ of $B(\mathcal{H})$, the commutant $\mathcal{S}'$ is a von Neumann algebra. Similarly, $\mathcal{S}''$ is a von Neumann algebra. Actually, $\mathcal{S}''$ is the von Neuamnna algebra generated by $\mathcal{S}$, that is, the smallest von Neumann algebra containing $\mathcal{S}$.

The *center* $\mathcal{Z}(\mathcal{M})$ of a von Neumann algebra $\mathcal{M}$ is defined by

$$\mathcal{Z}(\mathcal{M}) = \{X \in \mathcal{M} : XY = YX, \ \forall Y \in \mathcal{M}\}.$$

Since $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$, it follows that $\mathcal{Z}(\mathcal{M})$ is a von Neumann algebra, which is clearly commutative. Note that $\mathcal{Z}(\mathcal{M}') = \mathcal{Z}(\mathcal{M})$. If $\mathcal{Z}(\mathcal{M}) = \mathbb{C}$, then the von Neumann algebra $\mathcal{M}$ said to be a *factor*. Since $B(\mathcal{H})' = \mathbb{C}_H$, it follows that $B(\mathcal{H})$ is a factor.

The following is the famous Double Commutant Theorem of J. von Neumann.

**Theorem 1.4.2.** *Let $\mathcal{M}$ be a unital $*$-subalgebra of $B(\mathcal{H})$ and let $\mathcal{M}_1$ be its unit ball (with respect to the operator norm). The following statements are equivalent.*

*(1). $\mathcal{M}$ is a von Neumann algebra, that is, $\mathcal{M} = \mathcal{M}''$.*

*(2). $\mathcal{M}$ is wo-closed (or, equivalently, so-, uwo-, uso-closed).*

*(3). $\mathcal{M}_1$ is wo-closed (or, equivalently, so-, uwo-, uso-closed).*

If $\mathcal{M}$ is a von Neumann algebra, then $\mathcal{M}_1 = \mathcal{M} \cap B(\mathcal{H})_1$ is wo-compact and hence, $\mathcal{M}_1$ is uwo-compact, as the wo- and uwo-topology coincide on norm bounded subsets of $B(\mathcal{H})$.

Let $P \in \mathcal{M}$ and $Q \in \mathcal{M}'$. The algebra $\mathcal{M}_P := P\mathcal{M}P = \{PXP : X \in \mathcal{M}\}$ is called the *reduced von Neumann algebra* of $\mathcal{M}$ with respect to $P \in \mathcal{P}(\mathcal{M})$. The algebra $\mathcal{M}_Q := Q\mathcal{M}Q = \{QXQ : X \in \mathcal{M}\}$ is called the *induced von Neumann algebra* by $\mathcal{M}$ on $Q(\mathcal{H})$.

In what follows, we shall frequently use the following notation concerning partial ordering. Let $(X, \leq)$ be a partially ordered set. If $D$ is a non-empty subset of $X$ for which the least upper bounded (or, supremum) exists, then this least upper bound is denoted by $\sup D$ or $\vee D$. Similarly, $\inf D$ or $\wedge D$ denotes the greatest lower bounded (or infimum) or $D$ whenever it exists. In the case when $D = \{x, y\}$, we also write $\sup D = x \vee y$ and $\inf D = x \wedge y$. A net $\{x_\alpha\}_{\alpha\in\Gamma}$ is called increasing (or, upwards directed) if $x_\alpha \leq x_\beta$ whenever $\alpha \leq \beta$ in $\Lambda$ (notation as $x_\alpha \uparrow$). If $\{x_\alpha\}_{\alpha\in\Lambda}$ is increasing and $x = \sup_\alpha x_\alpha$ exists, then we write $x_\alpha \uparrow x$. Decreasing nets are defined similarly and $x_\alpha \downarrow x$ means that the decreasing net $\{x_\alpha\}$ has infimum $x$.

A self-adjoint operator $A \in B(\mathcal{H})$ is called positive if $\langle A\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{H}$. The collection of all positive elements of $B(\mathcal{H})$ is denoted by $B(\mathcal{H})_+$. This set is a proper closed cone in $B(\mathcal{H})$ and it induces a partial ordering in the set of all self-adjoint operators from $B(\mathcal{H})$ by setting $A \leq B$ if and only if $B - A \in B(\mathcal{H})_+$. For a given von Neumann algebra $\mathcal{M}$, we set $\mathcal{M}_+ := \mathcal{M} \cap B(\mathcal{H})_+$, which is called the positive part (or positive cone) of $\mathcal{M}$.

Vigier's theorem states that von Neumann algebras have the least upper bound property.

**Theorem 1.4.3.** *If $\{A_i\}_{i\in I}$ is an increasing net in $\mathcal{M}_+$, bounded from above by $B$ in $B(\mathcal{H})$, then there exists $A \in \mathcal{M}_+$ such that $A_i \uparrow A$ in strong operator topology and $A \leq B$.*

## 1.5 Supports of projections

In this section, $\mathcal{M}$ is a von Neumann algebra on the Hilbert space $\mathcal{H}$. We denote by $\mathcal{P}(\mathcal{M})$ the collection of all (orthogonal) projections belonging to $\mathcal{M}$, that is,

$$\mathcal{P}(\mathcal{M}) = \{P \in \mathcal{M} : P^2 = P,\ P^* = P\}.$$

Evidently, $\mathcal{P}(\mathcal{M}) \subset \mathcal{P}(B(\mathcal{H}))$. For every $P \in \mathcal{P}(\mathcal{M})$, we denote by $P^\perp := \mathbf{1} - P$ the *complement* of $P$.

For any $X \in \mathcal{M}$, the *range* and *kernel* of a linear operator $X$ are denoted by $\mathrm{Ran}(X)$ and $\mathrm{Ker}(X)$, respectively.

**Definition 1.5.1.** *Let $X \in B(\mathcal{H})$. We define*

- *the projection onto* $\mathrm{Ker}(X)$ *is called the* null projection *of* $X$*, denote by* $n(X)$*;*
- *the projection onto* $\overline{\mathrm{Ran}(X)}$ *is called the* range projection *of* $X$*, denote by* $r(X)$*;*
- *the projection* $\mathbf{1} - n(X)$*, which is the projection onto* $\overline{\mathrm{Ran}(X^*)}$ *, is called the* support projection *of* $X$*, denote by* $s(X)$*.*

For any $X \in \mathcal{M}$, the support projection $s(X)$ and the range projection $r(X)$ both belong to $\mathcal{P}(\mathcal{M})$. Therefore, $s(X)$ (respectively, $r(X)$) is the smallest of all projections $P \in \mathcal{P}(\mathcal{M})$ satisfying $X = XP$ (respectively, $X = PX$).

Projections belonging to the center $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$ are called *central projections* in $\mathcal{M}$.

**Definition 1.5.2.** *For* $X \in \mathcal{M}$*, the central support* $z(X) \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ *is defined by*

$$z(X) = \inf\{P \in \mathcal{P}(\mathcal{Z}(\mathcal{M})) : X = XP\}.$$

Note that $X = Xz(X)$, so the above infimum is actually a minimal projection. For $P \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$, the conditions $X = XP$ and $X^* = X^*P$ are equivalent. Hence, $z(X) = z(X^*)$ for all $x \in \mathcal{M}$. Note furthermore that, for any $Q \in \mathcal{P}(\mathcal{M})$, we have

$$z(Q) = \inf\{P \in \mathcal{P}(\mathcal{Z}(\mathcal{M})) : P \leq Q\}.$$

In particular, $z(X) = z(s(X)) = z(r(X))$.

## 1.6 Comparison of projections

Recall that an operator $V \in B(\mathcal{H})$ is called a *partial isometry* if $\|V\xi\|_{\mathcal{H}} = \|\xi\|_{\mathcal{H}}$ for all $\xi \in \mathrm{Ker}(V)^{\perp}$.

**Definition 1.6.1.** *Let* $P, Q \in \mathcal{P}(\mathcal{M})$ *be given.*

- *The projections* $P$ *and* $Q$ *are said to be* equivalent *(relative to the von Neumann algebra* $\mathcal{M}$*) if there exists a partial isometry* $V \in \mathcal{M}$ *with initial projection* $P$ *and final projection* $Q$ *(that is,* $P = V^*V$ *and* $Q = VV^*$*). This is denoted by* $P \sim Q$ *(or by* $P \overset{\mathcal{M}}{\sim} Q$*, if it is necessary to emphasize the von Neumann algebra relative to which the projections are equivalent).*
- *The projection* $P$ *is said to be* majorized *by* $Q$ *(relative to* $\mathcal{M}$*) if there exists a projection* $P_1 \in \mathcal{P}(\mathcal{M})$ *such that* $P_1 \leq P$ *and* $P \sim Q$*. This is denoted by* $Q \precsim P$ *(or* $P \precsim_{\mathcal{M}} Q$*).*

If $X \in \mathcal{M}$ with polar decomposition $X = V|X|$, then $V \in \mathcal{M}$ and $V^*V = s(X)$ and $vv^* = r(X)$. Evidently, if $\mathcal{M}$ is an abelian von Neumann algebra and $P, Q \in \mathcal{P}(\mathcal{M})$, then $P \sim Q$ if and only if $P = Q$, and $P \precsim Q$ if and only if $P \leq Q$. In the next proposition, we list some of the properties of the relation $\sim$.

**Proposition 1.6.2.** *(i). If* $X \in \mathcal{M}$*, then* $s(X) \sim r(X)$*.*

*(ii).* *If $P, Q \in \mathcal{P}(\mathcal{M})$, then $P \vee Q - Q \sim P - P \wedge Q$. In particular, if $P \wedge Q = 0$, then $P \precsim Q^{\perp}$.*

*(iii).* *If $P, Q \in \mathcal{P}(\mathcal{M})$ and $P \sim Q$, then $z(P) = z(Q)$.*

*(iv).* *Given $P, Q \in \mathcal{P}(\mathcal{M})$, there exist $P_1, Q_1 \in \mathcal{P}(\mathcal{M})$ such that $P_1 \leq P$, $Q_1 \leq Q$ and $P \sim Q$ if and only if $z(P)z(Q) \neq 0$ (equivalently, there exists $X \in \mathcal{M}$ such that $PXQ \neq 0$).*

*(v).* *If $P, Q \in \mathcal{P}(\mathcal{M})$ such that $P \sim Q$, then $PZ \sim QZ$ for all $Z \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$.*

*(vi).* *Suppose that $\{P_i\}_{i \in I}$ and $\{Q_i\}_{i \in I}$ are two families of pairwise orthogonal projections in $\mathcal{P}(\mathcal{M})$. If $P_i \sim Q_i$ for all $i \in I$, then $\sum_{i \in I} P_i \sim \sum_{i \in I} Q_i$.*

Some properties of the relation $\precsim$ are collected in the following proposition.

**Proposition 1.6.3.** *(i).* *If $P, Q, R \in \mathcal{P}(\mathcal{M})$ are such that $P \precsim Q$ and $Q \precsim R$, then $P \precsim R$.*

*(ii).* *If $P, Q \in \mathcal{P}(\mathcal{M})$ are such that $P \precsim Q$ and $Q \precsim P$, then $P \sim Q$.*

*(iii).* *Suppose that $\{P_i\}_{i \in I}$ and $\{Q_i\}_{i \in I}$ are two families of pairwise orthogonal projections in $\mathcal{P}(\mathcal{M})$. If $P_i \precsim Q_i$ for all $i \in I$, then $\sum_{i \in I} P_i \precsim \sum_{i \in I} Q_i$.*

*(iv).* *If $P, Q \in \mathcal{P}(\mathcal{M})$, then there exists a central projection $Z \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ such that $PZ \precsim QZ$ and $P^{\perp}Z \precsim Q^{\perp}Z$.*

*(v).* *Suppose that $\mathcal{M}$ is a factor. For $P, Q \in \mathcal{P}(\mathcal{M})$, we have either $P \precsim Q$ or $Q \precsim P$.*

## 1.7 Type decomposition of von Neumann algebras

**Definition 1.7.1.** *Let $\mathcal{M}$ be a von Neumann algebra on the Hilbert space $\mathcal{H}$.*

1. *A projection $P \in \mathcal{P}(\mathcal{M})$ is said to be* finite *(relative to $\mathcal{M}$) if it follows from $Q \in \mathcal{P}(\mathcal{M})$, $P \sim Q$ and $Q \leq P$ that $P = Q$. If $P$ is not finite, then we say that $P$ is* infinite.

2. *A projection $P \in \mathcal{P}(\mathcal{M})$ is said to be* properly infinite *(relative to $\mathcal{M}$) if $P \neq 0$ and for every $Q \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$, either $PQ = 0$ or $PQ$ is infinite.*

We recall that a projection $P \in \mathcal{P}(\mathcal{M})$ is said to be *countably decomposable* (also called *$\sigma$-finite* or *of countable type*) if every system of $\{P_\alpha\}$ of non-zero pairwise orthogonal projections in $\mathcal{P}(\mathcal{M})$, satisfying $P_\alpha \leq P$ for all $\alpha$, is at most countable. On a separable Hilbert space, every $P \in \mathcal{P}(\mathcal{M})$ is clearly countably decomposable.

**Definition 1.7.2.** *A projection $P \in \mathcal{P}(\mathcal{M})$ is said to be abelian if the reduced von Neumann algebra $\mathcal{M}_P$ is abelian.*

Now, we discuss the type decomposition of von Neumann algebras.

**Definition 1.7.3.** *Let $\mathcal{M}$ be a von Neumann algebra on the Hilbert space $\mathcal{H}$.*

*(i). $\mathcal{M}$ is of type I if there exists an abelian projection $P \in \mathcal{P}(\mathcal{M})$ with $z(P) = \mathbf{1}$.*

*(ii). $\mathcal{M}$ is of type II if $\mathcal{M}$ does not contain any non-zero abelian projections and there exists a finite projection $P \in \mathcal{P}(\mathcal{M})$ such that $z(P) = \mathbf{1}$.*

*(iii). $\mathcal{M}$ is of type III if $\mathcal{M}$ does not contain any non-zero finite projection.*

*(iv). $\mathcal{M}$ is of type $I_n$, where $n$ is a cardinal number satisfying $1 \leq n \leq \dim\mathcal{H}$, if $\mathbf{1}$ is the sum of $n$ mutually equivalent abelian projections in $\mathcal{P}(\mathcal{M})$.*

*(v). If $\mathcal{M}$ is of type II, then $\mathcal{M}$ is said to be of type $II_1$ (respectively, type $II_\infty$), if $\mathbf{1}$ is a finite projection (respectively, $\mathbf{1}$ is a properly infinite projection).*

Type $I$ von Neumann algebras are also called *discrete* and type $III$ von Neumann algebras are also known as *purely infinite* von Neumann algebras. Observe that any von Neumann algebra $\mathcal{M}$ of type $I_n$, for some $n$, is also of type $I$.

**Theorem 1.7.4** (Type decomposition)**.** *Suppose that $\mathcal{M}$ is a von Neumann algebra on a Hilbert space $\mathcal{H}$.*

*(1). There exist unique, pairwise orthogonal, central projections $P_I, P_{II}, P_{III} \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$, satisfying $P_I + P_{II} + P_{III} = \mathbf{1}$, such that $\mathcal{M}_{P_I}$ is of type $I$ or $P_i = 0$, $\mathcal{M}_{P_{II}}$ is of type $II$ or $P_{II} = 0$, and $\mathcal{M}_{P_{III}}$ is of type III or $P_{III} = 0$.*

*(2). Suppose that $\mathcal{M}$ is of type $I$. There exists a unique system $\{P_n : 1 \leq n \leq \dim H\}$ of pairwise orthogonal projections in $\mathcal{P}(\mathcal{Z}(\mathcal{M}))$, satisfying $\sum_n P_n = \mathbf{1}$, such that $\mathcal{M}_{P_n}$ is of type $I_n$ or $P_n = 0$, for each $n$.*

*(3). Suppose that $\mathcal{M}$ is of type II. There exist unique, mutually orthogonal projections $P_1, P_\infty \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$, with $P_1 + P_\infty = \mathbf{1}$, such that $\mathcal{M}_{P_1}$ is of type $II_1$ or $P_1 = 0$, and $\mathcal{M}_{P_\infty}$ is of type $II_\infty$ or $P_\infty = 0$.*

**Corollary 1.7.5.** *A factor is either of type $I_n$ (for a unique cardinal $n$ satisfying $1 \leq n \leq \dim\mathcal{H}$), or type $II_1$, or type $II_\infty$, or type III.*

We introduce some further terminology.

**Definition 1.7.6.** *We use the notation introduced in Theorem 1.7.4.*

*(i). If $P_{III} = 0$, then $\mathcal{M}$ is said to be a* semifinite von Neumann algebra.

*(ii). If $P_I = 0$, then $\mathcal{M}$ is called a* continuous von Neumann algebra.

*(iii). If $\mathbf{1}$ is a finite projection, then $\mathcal{M}$ is called a* finite von Neumann algebra.

*(iv). If $\mathbf{1}$ is a properly infinite projection, then $\mathcal{M}$ is called a* properly infinite von Neumann algebra.

*(v). $\mathcal{M}$ is said to be* of type $\mathrm{I}_{fin}$ *if $\mathcal{M}$ is of type I and $\mathcal{M}$ is finite.*

*(vi).* $\mathcal{M}$ *is said to be* of type $\mathrm{I}_\infty$ *if* $\mathcal{M}$ *is of type I and* $\mathcal{M}$ *is properly infinite.*

*(vii).* $\mathbf{1}$ *is a countably decomposable, then* $\mathcal{M}$ *is said to be* countably decomposable *(or,* $\sigma$-finite*).*

## 1.8 Traces

Let $\mathcal{M}$ be a von Neumann algebra with the positive cone $\mathcal{M}_+$.

**Definition 1.8.1.** *A* weight *on a von Neumann algebra* $\mathcal{M}$ *is a function* $\tau$ *on the positive cone* $\mathcal{M}_+$ *with values in the extended positive reals* $[0, \infty]$ *satisfying*

*(i).* $\tau(A + B) = \tau(A) + \tau(B)$ *for all* $A, B \in \mathcal{M}_+$*;*

*(ii).* $\tau(\lambda A) = \lambda\tau(A)$ *for all* $A \in \mathcal{M}_+$ *and* $0 \leq \lambda \in \mathbb{R}$ *(with the convention that* $0 \cdot \infty = 0$*).*

*If* $\tau$ *has the additional property that*

*(iii).* $\tau(U^*AU) = \tau(A)$ *whenever* $A \in \mathcal{M}_+$ *and* $U \in \mathcal{U}(\mathcal{M})$,

*then* $\tau$ *is called a* trace *(or,* tracial weight*) on* $\mathcal{M}_+$.

If $\tau : \mathcal{M}_+ \to [0, \infty]$ is a weight, then it follows immediately from (i) in the above definition that $\tau(A) \leq \tau(B)$ whenever $A \leq B$ in $\mathcal{M}_+$. Furthermore, observe that a weight $\tau$ is a trace if and only if

$$\tau(X^*X) = \tau(XX^*)$$

for all $X \in \mathcal{M}$.

**Definition 1.8.2.** *A weight* $\tau : \mathcal{M}_+ \to [0, \infty]$ *is called*

*(i). finite if* $\tau(\mathbf{1}) < \infty$*;*

*(ii). semifinite if*

$$\tau(A) = \sup\{\tau(B) : B \in \mathcal{M}_+, B \leq A, \tau(B) < \infty\}$$

*for all* $A \in \mathcal{M}_+$*;*

*(iii). faithful if* $A \in \mathcal{M}_+$ *and* $\tau(A) = 0$ *imply that* $A = 0$*;*

*(iv). normal if* $A_\beta \uparrow A$ *in* $\mathcal{M}_+$ *implies that* $\tau(A_\beta) \uparrow \tau(A)$.

The following theorem characterizes finite and semifinite von Neumann algebras in terms of traces.

**Theorem 1.8.3.** *Let* $\mathcal{M}$ *be a von Neumann algebra.*

1. $\mathcal{M}$ *is finite if and only if for every non-zero* $X \in \mathcal{M}_+$*, there exists a finite trace* $\tau$ *on* $\mathcal{M}_+$ *such that* $\tau(X) > 0$.

*2. $\mathcal{M}$ is semifinite if and only if there exists a faithful normal semifinite trace $\tau$ on $\mathcal{M}_+$.*

In this thesis, we exclusively deal with semifinite von Neumann algebras. The following lemma provides a useful tool.

**Lemma 1.8.4.** *Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a semifinite faithful normal trace $\tau$. If $\{T_i\} \subset \mathcal{M}$ is a uniformly bounded net of self-adjoint operators converging to $T \in \mathcal{M}$ in the strong operator topology, then*

$$\liminf_i \tau(E^{T_i}(\varepsilon, \infty)) \geq \tau(E^T(\varepsilon, \infty))$$

*for any $\varepsilon \in \mathbb{R}$.*

*Proof.* Consider the characteristic function $\chi_{(\varepsilon,\infty)}$. There exists a sequence of positive continuous functions $f_k$ with compact support such that $f_k \uparrow \chi_{(\varepsilon,\infty)}$ pointwise. By [132, Lemma II 4.3], we have $f_k(T_i) \to_{so} f_k(T)$ for all $k \in \mathbb{N}$. Since $\tau$ is lower semicontinuous in the weak operator topology on a uniformly bounded set (see e.g. [132, Lemma II 2.5] and [133, Theorem VII 1.11]), it follows that

$$\tau(f_k(T)) \leq \liminf_i \tau(f_k(T_i)) \leq \liminf_i \tau(\chi_{(\varepsilon,\infty)}(T_i)) = \liminf_i \tau(E^{T_i}(\varepsilon, \infty)).$$

Note that $f_k \uparrow \chi_{(\varepsilon,\infty)}$ implies $\sup_k f_k(T) = \chi_{(\varepsilon,\infty)}(T) = E^T(\varepsilon, \infty)$. Hence, using the normality of the trace $\tau$, we conclude that

$$\tau(E^T(\varepsilon, \infty)) = \sup_{k \in \mathbb{N}} \tau(f_k(T)) \leq \liminf_i \tau(E^{T_i}(\varepsilon, \infty)).$$

□

The following proposition gives a necessary and sufficient condition for a von Neumann algebra being countably decomposable [103, Proposition 1.3.5 or Theorem 1.3.6].

**Proposition 1.8.5.** *For a von Neumann algebra $\mathcal{M}$ on a Hilbert space, the following are equivalent:*

*(i). $\mathcal{M}$ is countably decomposable;*

*(ii). $\mathcal{M}$ admits a finite normal faithful weight.*

# Chapter 2

# Symmetric spaces

In this chapter, we introduce the theory of noncommutative Banach function spaces. General facts concerning measurable operators may be found in [92], [124] (see also [133, Chapter IX] and the forthcoming book [45]).

In this theory, the notion of a measure space is replaced by the lattice of projections of a semifinite von Neumann algebra, the integral by a faithful normal semifinite trace, and measurable function by a (so-called) measurable operator, which is an operator (in general unbounded) affiliated with the underlying von Neumann algebra. The special case of noncommutative $L_p$-spaces was initiated by Dixmier [33] and Segal [124]. In particular, if the underlying von Neumann algebra is $B(\mathcal{H})$ for some Hilbert space, then these noncommutative $L_p$-spaces are special cases of the so-called trace ideals investigated by Schatten [122]. In this case, the seminal ideas may be traced back to a paper on $n \times n$-matrices due to von Neumann [98] and the principal features of this theory may be found in the book of Gohberg and Krein [52].

Initial contributions to the study of general symmetric spaces of measurable operators include those of Ovčinnikov [104] and Yeadon [139,140], based on methods from classical real analysis related to rearrangements and these methods continue to play a significant role in the present theory of noncommutative symmetric spaces. By special choice of the underlying von Neumann algebra, this study unifies important aspects of the classical theory of rearrangement-invariant normed Köthe spaces (as given in [89, 141]) with the theory of trace ideals given in [52] as well as the general features of the classical theory of the non-commutative $L_p$-spaces of Dixmier [33] and Segal [124].

## 2.1 Closed linear operators

Many of the linear operators we encounter are not bounded and are only defined on a (dense) subspace of the Hilbert space $\mathcal{H}$. Here we introduce the necessary notions to deal with such operators. For details on unbounded operators, the reader is referred to [123] and [45, Chapter I]. A linear operator $X$ in $\mathcal{H}$ is a linear mapping from its *domain* $\mathfrak{D}(X)$, which is a linear subspace of $\mathcal{H}$, into the space $\mathcal{H}$. Given two such linear operator $X$ and $Y$ in $\mathcal{H}$, the operator $Y$ is said to be an *extension* of $X$ (or, $X$ is a *restriction* of $Y$), if $\mathfrak{D}(X) \subset \mathfrak{D}(Y)$ and $X\xi = Y\xi$ for all $\xi \in \mathfrak{D}(X)$. This is denoted as $X \subset Y$. If $X \subset Y$ as well as $Y \subset X$, then $X = Y$. The *range* and *kernel* of a linear operator $X$ are denoted by $\mathrm{Ran}(X)$ and $\mathrm{Ker}(X)$ respectively.

We introduce the algebraic operations of scalar multiplication, addition and multiplication for linear operators as follows. Given linear operators $X, Y$ in $\mathcal{H}$ and $\lambda \in \mathbb{C}$, we define

- $\lambda X$ by setting $\mathfrak{D}(\lambda X) = \mathfrak{D}(X)$ and $(\lambda X)\xi = \lambda(X\xi)$ for all $\xi \in \mathfrak{D}(\lambda X)$;
- $X + Y$ by setting $\mathfrak{D}(X+Y) = \mathfrak{D}(X) \cap \mathfrak{D}(Y)$ and $(X+Y)\xi = X\xi + Y\xi$ for all $\xi \in \mathfrak{D}(X+Y)$;
- $XY$ by setting $\mathfrak{D}(XY) = \{\xi \in \mathfrak{D}(Y) : Y\xi \in \mathfrak{D}(X)\}$ and $(XY)\xi = X(Y\xi)$ for all $\xi \in \mathfrak{D}(XY)$;
- the inverse operator $X^{-1}$ whenever $X$ is injective, by setting $\mathfrak{D}(X^{-1}) = \mathrm{Ran}(X)$ and $X^{-1}\xi = \eta$ whenever $\xi = X\eta$ for some $\eta \in \mathfrak{D}(X)$.

We note that in general it may happen that $\mathfrak{D}(X+Y) = \{0\}$ or $\mathfrak{D}(XY) = \{0\}$.

For a linear operator $X$ in $\mathcal{H}$ the *graph* $\Gamma(X)$ is defined to be the linear subspace of $\mathcal{H} \times \mathcal{H}$ given by $\Gamma(X) := \{(\xi, X\xi) : \xi \in \mathfrak{D}(X)\}$. Note that $X \subset Y$ is equivalent to $\Gamma(X) \subset \Gamma(Y)$. A linear operator $X$ is called *closed* if $\Gamma(X)$ is a closed subspace of $\mathcal{H} \times \mathcal{H}$ (equipped with the natural product topology). In other words, $X$ is closed if and only if it follows from $\{\xi_n\}_{n=1}^{\infty} \subset \mathfrak{D}(X)$, $\xi, \eta \in \mathcal{H}$, $\xi_n \to \xi$ and $X\xi_n \to \eta$ as $n \to \infty$, that $\xi \in \mathfrak{D}(X)$ and $X\xi = \eta$. If $X$ is closed, then $\mathrm{Ker}(X)$ is a closed subspace of $\mathcal{H}$. It is clear that any bounded linear operator in $\mathcal{H}$ is closed. Conversely, if $X$ is a closed linear operator and if the domain $\mathfrak{D}(X)$ is a closed subspace of $\mathcal{H}$, then it follows from the Closed Graph Theorem that $X$ is bounded on its domain $\mathfrak{D}(X)$. This applies in particular if $\mathfrak{D}(X) = \mathcal{H}$. Furthermore, if $X$ is a closed injective linear operator in $\mathcal{H}$, then its inverse $X^{-1}$ is also closed. Consequently, if $X$ is in addition surjective, then $\mathfrak{D}(X^{-1}) = \mathrm{Ran}(X) = \mathcal{H}$ and so, $X^{-1} \in B(\mathcal{H})$.

A linear operator $X$ in $\mathcal{H}$ is called *densely defined* if $\mathfrak{D}(X)$ is a dense subspace of $\mathcal{H}$. Note that if $X$ is a closed and densely defined operator, then $X$ is bounded if and only if $\mathfrak{D}(X) = \mathcal{H}$. Now suppose that $X$ is a densely defined operator in $\mathcal{H}$ and consider the linear subspace $\mathfrak{D}$ of $\mathcal{H}$ given by

$$\mathfrak{D} := \{\eta \in \mathcal{H} : \exists \zeta \in \mathcal{H} \text{ such that } \langle X\xi, \eta\rangle = \langle \xi, \zeta\rangle,\ \forall \xi \in \mathfrak{D}(X)\}.$$

If $\eta \in \mathfrak{D}$, then the element $\zeta \in \mathcal{H}$ satisfying $\langle X\xi, \eta\rangle = \langle \xi, \zeta\rangle$ is uniquely determined by $\eta$, as $\mathfrak{D}(X)$ is dense in $\mathcal{H}$. Therefore, we may define the mapping $X^* : \eta \mapsto \zeta$ from $\mathfrak{D}$ into $\mathcal{H}$. It is readily verified that $X^*$ is a linear operator in $\mathcal{H}$ with domain $\mathfrak{D}(X^*) = \mathfrak{D}$. The operator $X^*$ is called the adjoint of $X$. Note that, be definition, we have

$$\langle X\xi, \eta\rangle = \langle \xi, X^*\eta\rangle,\ \xi \in \mathfrak{D}(X),\ \eta \in \mathfrak{D}(X^*).$$

It is evident that $X^*$ is closed.

A densely defined linear operator $A$ in $\mathcal{H}$ is called *self-adjoint* if $A = A^*$. A self-adjoint operator $A$ in $\mathcal{H}$ is called *positive* if $\langle A\xi, \xi \rangle \geq 0$ for all $\xi \in \mathfrak{D}(A)$. This is denoted by $A \geq 0$. Furthermore, as in the case of bounded linear operators on $\mathcal{H}$, a closed and densely defined linear operator $X$ in $\mathcal{H}$ is called *normal* when $X^*X = X^*X$. The $\sigma$-algebra of all Borel subsets of $\mathbb{C}$ is denoted by $\mathcal{B}(\mathbb{C})$. Suppose that $\Omega$ is a non-empty set and that $\mathcal{A}$ is a $\sigma$-algebra of subsets of $\Omega$, so $(\Omega, \mathcal{A})$ is a measurable space.

**Definition 2.1.1.** *A* spectral measure *on* $(\Omega, \mathcal{A})$ *is a mapping* $E : \mathcal{A} \to B(\mathcal{H})$ *such that*

*(1).* $E(\delta)$ *is a projection in* $\mathcal{H}$ *for each* $\delta \in \mathcal{A}$*;*

*(2).* $E(\emptyset) = 0$ *and* $E(\Omega) = \mathbf{1}$*;*

*(3).* $E(\delta_1 \cap \delta_2) = e(\delta_1)e(\delta_2)$ *for all* $\delta_1, \delta \in \mathcal{A}$*;*

*(4). if* $\delta_j \in \mathcal{A}$ *(*$j = 1, 2, \cdots$*) are pairwise disjoint, then* $E(\cup_{j=1}^{\infty} \delta_j) = \sum_{j=1}^{\infty} E(\delta_j)$*, where the series converges in the strong operator topology.*

**Theorem 2.1.2.** *If* $X : \mathfrak{D}(X) \to \mathcal{H}$ *is a normal operator, then there exists a uniquely determined spectral measure* $E^X : \mathcal{B}(\mathbb{C}) \to B(\mathcal{H})$ *such that*

$$X = \int_{\mathbb{C}} \lambda dE^X(\lambda).$$

**Definition 2.1.3.** *Given a normal operator* $X : \mathfrak{D}(X) \to \mathcal{H}$ *with spectral measure* $E^X : \mathcal{B}(\mathbb{C}) \to B(\mathcal{H})$*, we define*

$$f(X) = \int_{\mathbb{C}} f(\lambda) dE^X(\lambda).$$

## 2.2 The $*$-algebra of $\tau$-measurable operators

From now on, we always assume that $\mathcal{M}$ is a semifinite von Neumann algebra equipped with a fixed semifinite faithful normal trace $\tau$.

A closed, densely defined operator $X : \mathfrak{D}(X) \to \mathcal{H}$ with the domain $\mathfrak{D}(X)$ is said to be *affiliated* with $\mathcal{M}$ if $YX \subseteq XY$ for all $Y \in \mathcal{M}'$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$. A closed, densely defined operator $X : \mathfrak{D}(X) \to \mathcal{H}$ affiliated with $\mathcal{M}$ is said to be *measurable* if there exists a sequence $\{P_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathcal{M})$, such that $P_n \uparrow \mathbf{1}$, $P_n(\mathcal{H}) \subseteq \mathfrak{D}(X)$ and $\mathbf{1} - P_n$ is a finite projection (with respect to $\mathcal{M}$) for all $n$. The collection of all measurable operators with respect to $\mathcal{M}$ is denoted by $S(\mathcal{M})$, which is a unital $*$-algebra with respect to strong sums and products (denoted simply by $X + Y$ and $XY$ for all $X, Y \in S(\mathcal{M})$) [43, 45, 123].

Let $X$ be a self-adjoint operator affiliated with $\mathcal{M}$. We denote its spectral measure by $\{E^X\}$. It is well known that if $X$ is an operator affiliated with $\mathcal{M}$ with the polar decomposition $X = U|X|$, then $U \in \mathcal{M}$ and $E \in \mathcal{M}$ for all projections $E \in \{E^{|X|}\}$. Moreover, $X \in S(\mathcal{M})$ if and only if $E^{|X|}(\lambda, \infty)$ is a finite projection for some $\lambda > 0$. It follows immediately that in the case when $\mathcal{M}$ is a von Neumann algebra of type $III$ or a type $I$ factor, we have $S(\mathcal{M}) = \mathcal{M}$. For type $II$ von Neumann algebras, this is no longer true. From now on, let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace $\tau$.

An operator $X \in S(\mathcal{M})$ is called *$\tau$-measurable* if there exists a sequence $\{P_n\}_{n=1}^{\infty}$ in $P(\mathcal{M})$ such that $P_n \uparrow \mathbf{1}$, $P_n(\mathcal{H}) \subseteq \mathfrak{D}(X)$ and $\tau(\mathbf{1} - P_n) < \infty$ for all $n$. The collection $S(\mathcal{M}, \tau)$ of all $\tau$-measurable operators is a unital $*$-subalgebra of $S(\mathcal{M})$. It is well known that a linear operator $X$ belongs to $S(\mathcal{M}, \tau)$ if and only if $X \in S(\mathcal{M})$ and there exists $\lambda > 0$ such that $\tau(E^{|X|}(\lambda, \infty)) < \infty$. Alternatively, an unbounded operator $X$ affiliated with $\mathcal{M}$ is $\tau$-measurable (see [48]) if and only if

$$\tau\left(E^{|X|}(n, \infty)\right) \to 0, \quad n \to \infty.$$

**Definition 2.2.1.** *Let a semifinite von Neumann algebra $\mathcal{M}$ be equipped with a faithful normal semi-finite trace $\tau$ and let $X \in S(\mathcal{M}, \tau)$. The generalized singular value function $\mu(X) : t \to \mu(t; X)$ of the operator $X$ is defined by setting*

$$\mu(s; X) = \inf\{\|XP\| : \ P \in \mathcal{P}(\mathcal{M}) \text{ with } \tau(\mathbf{1} - P) \le s\}.$$

An equivalent definition in terms of the *distribution function* of the operator $X$ is the following. For every self-adjoint operator $X \in S(\mathcal{M}, \tau)$, setting

$$d_X(t) = \tau(E^X(t, \infty)), \quad t > 0,$$

we have (see e.g. [48])

$$\mu(t; X) = \inf\{s \ge 0 : \ d_{|X|}(s) \le t\}. \tag{2.1}$$

It is well-known that $d_|X|(\cdot)$ and $\mu(\cdot; X)$ are right-continuous.

### 2.2.1 Measure topology and local measure topology

For convenience of the reader, we also recall the definition of the *measure topology* $t_\tau$ on the algebra $S(\mathcal{M}, \tau)$. For every $\varepsilon, \delta > 0$, we define the set

$$\begin{aligned} V(\varepsilon, \delta) &= \{X \in S(\mathcal{M}, \tau) : \ \exists P \in P(\mathcal{M}) \text{ such that } \|X(\mathbf{1} - P)\|_\infty \le \varepsilon, \ \tau(P) \le \delta\} \\ &= \{X \in S(\mathcal{M}, \tau) : \ \tau(E^{|X|}(\varepsilon, \infty)) \le \delta\} \\ &= \{X \in S(\mathcal{M}, \tau) : \ \mu(\delta; X) \le \delta\}. \end{aligned}$$

The topology generated by the sets $V(\varepsilon, \delta)$, $\varepsilon, \delta > 0$, is called the measure topology $t_\tau$ on $S(\mathcal{M}, \tau)$ [35, 45, 48, 92]. It is well known that the algebra $S(\mathcal{M}, \tau)$ equipped with the measure topology is a complete metrizable topological algebra [60, 92]. A net $\{X_\alpha\}_{n=1}^{\infty} \subset S(\mathcal{M}, \tau)$ converges to zero with respect to measure topology $t_\tau$ if and only if $\tau\big(E^{|X_\alpha|}(\varepsilon, \infty)\big) \to_\alpha 0$ for all $\varepsilon > 0$ [45], or equivalently, $\mu(t; X_\alpha) \to_\alpha 0$ for all $t > 0$.

Another important vector topology on $S(\mathcal{M}, \tau)$ is the *local measure topology.* For convenience we denote by $\mathcal{P}_{fin}(\mathcal{M})$ the collection of all $\tau$-finite projections in

$\mathcal{M}$, that is the set of all $P \in \mathcal{P}(\mathcal{M})$ satisfying $\tau(P) < \infty$. A neighbourhood base for this topology is given by the sets $V(\varepsilon, \delta; P)$, $\varepsilon, \delta > 0$, $P \in \mathcal{P}_{fin}(\mathcal{M})$, where

$$V(\varepsilon, \delta; P) = \{X \in S(\mathcal{M}, \tau) \ : \ PXP \in V(\varepsilon, \delta)\}.$$

Obviously, local measure topology is weaker than measure topology [43]. It is clear that, if $\tau(\mathbf{1}) < \infty$, then the measure topology coincides with the local measure topology. However, if $\tau(\mathbf{1}) = \infty$, then $S(\mathcal{M}, \tau)$ need not be complete for the local measure topology, even in the case when $\mathcal{M}$ is commutative. We note here, that the local measure topology used in the present thesis differs from the local measure topology defined in e.g. [9, 10].

In general, local measure topology is not metrisable and multiplication in $S(\mathcal{M}, \tau)$ is not jointly continuous with respect to local measure topology. However, if $\{X_\alpha\} \subset S(\mathcal{M}, \tau)$ is a net and if $X_\alpha \to X \in S(\mathcal{M}, \tau)$ in local measure topology, then $YX_\alpha \to YX$ and $X_\alpha Y \to XY$ in local measure topology for all $Y \in S(\mathcal{M}, \tau)$.

If $\mathcal{M} = B(\mathcal{H})$ equipped with the canonical trace Tr, then the measure topology coincides with the operator norm topology while the local measure topology coincides with the weak operator topology.

### 2.2.2 Properties of generalized singular value functions

We collect some properties of generalized singular value functions below [43, 87].

**Proposition 2.2.2.** *If $A \in S(\mathcal{M}, \tau)$, then*

*(1). The function $t \mapsto \mu(t; A)$, $t > 0$, is decreasing and right-continuous.*

*(2). $\mu(t; A) \to \|A\|_\infty$ when $t \to 0$ for bounded $A$ and $\mu(t; A) \to \infty$ when $t \to 0$ for unbounded $A$.*

*(3). $\mu(t; A) = \mu(t; |A|)$ for all $t > 0$.*

*(4). If $\alpha \in \mathbb{C}$, then $\mu(t; \alpha A) = |\alpha|\mu(t; A)$ for all $t > 0$.*

*(5). If $0 \le B \le A$, then $\mu(t; B) \le \mu(t; A)$ for all $t > 0$.*

*(6). If $\tau(\mathbf{1}) = 1$, then $\mu(t; A) = 0$ for all $t > 1$.*

*(7). If $B, C \in \mathcal{M}$, then*

$$\mu(t; BAC) \le \|B\|_\infty \|C\|_\infty \mu(t; A), \ \ \mu(t; A^*) = \mu(t; A) \tag{2.2}$$

*(8). Let $B \in S(\mathcal{M}, \tau)$ and $t, s > 0$. Then,*

$$\mu(t + s; A + B) \le \mu(t; A) + \mu(s; B),$$

*and*

$$\mu(t + s; AB) \le \mu(t; A)\mu(s; B).$$

*(9). If $A \geq 0$ and if $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and increasing function, then $\mu(f(A)) = f(\mu(A))$.*

*(10). The operation $A \to \mu(t; A)$ is continuous in the uniform norm on $\mathcal{M}$. More precisely,*

$$|\mu(s; A) - \mu(s; B)| \leq \|A - B\|_\infty, \ \forall A, B \in \mathcal{M}, \ \forall s > 0.$$

**Lemma 2.2.3.** *For every $X \in S(\mathcal{M}, \tau)$ and $t > 0$, $\tau(E^{|X|}(a, \infty)) > t$ if and only if $\mu(s; X) > a$ for all $s \in [0, t]$.*

*Proof.* Necessity. By (2.1), we have $\mu(t; X) = \inf\{s \geq 0 : \tau(E^{|X|}(s, \infty)) \leq t\}$. Assume by contradiction that $\mu(t; X) \leq a$, then $\inf\{s \geq 0 : \tau(E^{|X|}(s, \infty)) \leq t\} \leq a$ and therefore $\tau(E^{|X|}(a + \varepsilon, \infty)) \leq t$ for any $\varepsilon > 0$. Since the distribution function $d_{|X|}(\cdot)$ is right-continuous (see e.g. [48]), it follows that $\tau(E^{|X|}(a, \infty)) \leq t$, which is a contradiction.

Sufficiency. By assumption, we have that $\mu(t; X) > a$. Using again (2.1), we obtain that $\inf\{s \geq 0 : \tau(E^{|X|}(s, \infty)) \leq t\} > a$, and therefore $\tau(E^{|X|}(a, \infty)) > t$. □

Suppose that $X \in S(\mathcal{M}, \tau)$. If $0 < \alpha \in \mathbb{R}$ and $E = E^{|X|}(\alpha, \infty)$, then

$$\mu(|X|E) = \mu(X)\chi_{[0,\tau(E))} \tag{2.3}$$

and

$$\mu(t; |X|E^\perp) = \mu(t + \tau(E); X) \tag{2.4}$$

for all $t \geq 0$ whenever $\tau(E) < \infty$.

Consider the algebra $\mathcal{M} = L^\infty(0, \infty)$ of all Lebesgue measurable essentially bounded functions on $(0, \infty)$. The algebra $\mathcal{M}$ can be seen as an abelian von Neumann algebra acting via multiplication on the Hilbert space $\mathcal{H} = L^2(0, \infty)$, with the trace given by integration with respect to Lebesgue measure $m$. It is easy to see that the algebra of all $\tau$-measurable operators affiliated with $\mathcal{M}$ can be identified with the subalgebra $S(0, \infty)$ of the algebra of Lebesgue measurable functions $L_0(0, \infty)$ which consists of all functions $x$ such that $m(\{|x| > s\})$ is finite for some $s > 0$. It should also be pointed out that the generalized singular value function $\mu(x)$ is precisely the decreasing rearrangement $\mu(x)$ of the function $|x|$ (see e.g. [86]) defined by

$$\mu(t; x) = \inf\{s \geq 0 : \ m(\{|x| \geq s\}) \leq t\}.$$

If $\mathcal{M} = B(\mathcal{H})$ (respectively, $l_\infty$) and $\tau$ is the standard trace Tr (respectively, the counting measure on $\mathbb{N}$), then it is not difficult to see that $S(\mathcal{M}) = S(\mathcal{M}, \tau) = \mathcal{M}$. In this case, for $X \in S(\mathcal{M}, \tau)$ we have

$$\mu(n; X) = \mu(t; X), \quad t \in [n, n + 1), \quad n \geq 0.$$

The sequence $\{\mu(n; X)\}_{n \geq 0}$ is just the sequence of singular values of the operator $X$.

## 2.3 Properties of submajorisations

We collect some properties of submajorisation below. For detailed on the theory of submajorisation, the reader is referred to [43,45,48,91]. For the sake of convenience, we denote

$$(L_1 + L_\infty)(\mathcal{M},\tau) := \left\{X \in S(\mathcal{M},\tau) :\ \mu(X)\chi_{(0,1)} \in L_1(0,1)\right\}.$$

**Proposition 2.3.1.** *Let $A, B \in (L_1 + L_\infty)(\mathcal{M},\tau)$. We have*

*(i). $\mu(A + B) \prec\prec \mu(A) + \mu(B)$.*

*(ii). If $A, B \geq 0$, then $\mu(A) + \mu(B) \prec\prec 2\sigma_{1/2}\mu(A + B)$.*

*(iii). $\mu(A) - \mu(B) \prec\prec \mu(A + B)$.*

**Proposition 2.3.2.** *(i). [61, 131] Assume that $0 \leq A \in (L_1 + L_\infty)(\mathcal{M},\tau)$ and $B$ is self-adjoint in $\in (L_1 + L_\infty)(\mathcal{M},\tau)$. If $-A \leq B \leq A$, then $B \prec\prec A$.*

*(ii). [44, Lemma 6.1] Assume that $X \in S(\mathcal{M},\tau)$ and $P_1, P_2, \cdots, P_n \in \mathcal{M}$ are projections with $P_iP_j = 0$, $i \neq j$. We have*

$$P_1XP_1 + P_2XP_2 + \cdots + P_nXP_n \prec\prec X. \tag{2.5}$$

*(iii). [87, Lemma 3.3.5] Let $0 \leq A_k \in (L_1+L_\infty)(\mathcal{M},\tau)$, $k \in \mathbb{N}$ Let $\alpha_k \in \mathbb{R}_+$, $k \in \mathbb{N}$, be such that $\sum_{k=1}^\infty \alpha_k \leq 1$. We have*

$$\sum_{k=1}^{\infty} \int_0^{\alpha_k a} \mu(s; A_k)ds \leq \int_0^a \mu(s; \sum_{k=1}^{\infty} A_k)ds,\ \forall a > 0. \tag{2.6}$$

*Here, we assume that the series $\sum_{k=1}^\infty A_k$ converges in $(L_1 + L_\infty)(\mathcal{M},\tau)$.*

The following proposition is an easy consequence of (2.5) and (2.6).

**Proposition 2.3.3.** *Assume that $P_1, P_2, \cdots, P_n \in \mathcal{M}$ are projections with $P_iP_j = 0$, $i \neq j$. Let $\alpha_i > 0$, $i \in \mathbb{N}$, be such that $\sum_{i=1}^n \alpha_i \leq 1$. For every $X \in (L_1+L_\infty)(\mathcal{M},\tau)$, we have*

$$\int_0^a \mu(t; X)dt \geq \sum_{i=1}^{n} \int_0^{\alpha_i a} \mu(t; |P_iXP_i|)dt = \sum_{i=1}^{n} \int_0^{\alpha_i a} \mu(t; P_iXP_i)dt,\ \forall a > 0. \tag{2.7}$$

*Proof.* Since $P_i$ are pairwise disjoint, it follows that

$$\begin{aligned}\mu(P_1XP_1 + P_2XP_2 + \cdots + P_nXP_n) &= \mu(|P_1XP_1 + P_2XP_2 + \cdots + P_nXP_n|)\\ &= \mu(|P_1XP_1| + |P_2XP_2| + \cdots + |P_nXP_n|).\end{aligned}$$

Therefore, by (2.5), we obtain that

$$\int_0^a \mu(t; |P_1XP_1| + |P_2XP_2| + \cdots + |P_nXP_n|)dt \leq \int_0^a \mu(t; X)dt.$$

The validity of (2.7) follows from (2.6). □

## 2.4 Symmetric spaces

**Definition 2.4.1.** *A linear subspace $E(\mathcal{M},\tau)$ of $S(\mathcal{M},\tau)$ equipped with a complete norm $\|\cdot\|_E$, is called symmetric space (of $\tau$-measurable operators) if $X \in S(\mathcal{M},\tau)$, $Y \in E(\mathcal{M},\tau)$ and $\mu(X) \le \mu(Y)$ imply that $X \in E$ and $\|X\|_E \le \|Y\|_E$.*

It is well-known that any symmetric space $E$ is a normed $\mathcal{M}$-bimodule, that is $AXB \in E(\mathcal{M},\tau)$ for any $X \in E(\mathcal{M},\tau)$, $A, B \in \mathcal{M}$ and $\|AXB\|_E \le \|A\|_\infty \|B\|_\infty \|X\|_E$ [43,45,128]. Further, $\|X\|_E = \|X^*\|_E = \||X|\|_E$. Moreover, the embedding of $E(\mathcal{M},\tau)$ in $S(\mathcal{M},\tau)$ is continuous with respect to the norm topology in $E(\mathcal{M},\tau)$ and the measure topology in $S(\mathcal{M},\tau)$ (see [43], see also [58,129]).

If $X, Y \in S(\mathcal{M},\tau)$, then $X$ is said to be *submajorised* (in the sense of Hardy–Littlewood–Polya) by $Y$, denoted by $X \prec\prec Y$, if

$$\int_0^t \mu(s;X)ds \le \int_0^t \mu(s;Y)ds$$

for all $t \ge 0$.

A symmetric space $E(\mathcal{M},\tau) \subset S(\mathcal{M},\tau)$ is called *strongly symmetric* if its norm $\|\cdot\|_E$ has the additional property that $\|X\|_E \le \|Y\|_E$ whenever $X, Y \in E(\mathcal{M},\tau)$ satisfy $X \prec\prec Y$. In addition, if $X \in S(\mathcal{M},\tau)$, $Y \in E(\mathcal{M},\tau)$ and $X \prec\prec Y$ imply that $X \in E(\mathcal{M},\tau)$ and $\|X\|_E \le \|Y\|_E$, then $E(\mathcal{M},\tau)$ is called *fully symmetric space* (of $\tau$-measurable operators).

A symmetric space $E(\mathcal{M},\tau)$ is said to have the *Fatou property* if for every upwards directed net $\{X_\beta\}$ in $E(\mathcal{M},\tau)^+$, satisfying $\sup_\beta \|X_\beta\|_E < \infty$, there exists an element $X \in E(\mathcal{M},\tau)^+$ such that $X_\beta \uparrow X$ in $E(\mathcal{M},\tau)$ and $\|X\|_E = \sup_\beta \|X_\beta\|_E$. Examples such as Schatten-von Neumann operator ideals, Lorentz operator ideals, Orlicz operator ideals, etc. all have symmetric norms which have the Fatou property. If $E \subset S(\mathcal{M},\tau)$ is a symmetric space, then the norm $\|\cdot\|_E$ is called *order continuous* if $\|X_\alpha\|_E \to 0$ whenever $\{X_\alpha\}$ is a downwards directed net in $E^+$ satisfying $X_\alpha \downarrow 0$.

The classical noncommutative $L_p$-space $L_p(\mathcal{M},\tau)$, $p \ge 1$, is the symmetric space corresponding to the classical $L_p$-space of functions $L_p(0,\infty)$, that is

$$L_p(\mathcal{M},\tau) = \{X \in S(\mathcal{M},\tau) : \mu(X) \in L_p(0,\infty)\}.$$

This space can be also described as the space of all $\tau$-measurable operator $X$, such that $\tau(|X|^p) < \infty$. It is well-known [43] that $L_\infty(\mathcal{M},\tau)$ has the Fatou property, and, for all $1 \le p < \infty$, the symmetric space $L_p(\mathcal{M},\tau)$ is fully symmetric, has Fatou property and order continuous norm. In addition, for $1 < p < \infty$, the space $L_p(\mathcal{M},\tau)$ is reflexive [109].

If $E(\mathcal{M},\tau)$ is a symmetric space, then the carrier projection $c_E \in \mathcal{P}(\mathcal{M})$ is defined by setting

$$c_E = \bigvee\{P : P \in P(\mathcal{M}),\ P \in E(\mathcal{M},\tau)\}.$$

If $E(\mathcal{M},\tau)$ is a symmetric space, then the Köthe dual $E(\mathcal{M},\tau)^\times$ of $E(\mathcal{M},\tau)$ is defined by

$$E(\mathcal{M},\tau)^\times = \{X \in S(\mathcal{M},\tau) : \sup_{\|Y\|_E \le 1, Y \in E} \tau(|XY|) < \infty\},$$

and for every $X \in E(\mathcal{M},\tau)^\times$, we set $\|X\|_{E^\times} = \sup\{\tau(|YX|) : Y \in E(\mathcal{M},\tau), \|Y\|_E \leq 1\}$ (see e.g. [43, Section 5.2], see also [39,87]). It is well-known that $\|\cdot\|_{E^\times}$ is a norm on $E(\mathcal{M},\tau)^\times$ if and only if the carrier projection $c_E$ of $E(\mathcal{M},\tau)$ is equal to $\mathbf{1}$. In this case, for a strongly symmetric space $E(\mathcal{M},\tau)$, the following statements are equivalent [41,43].

- $E(\mathcal{M},\tau)$ has the Fatou property.
- $E(\mathcal{M},\tau)^{\times\times} = E(\mathcal{M},\tau)$ and $\|X\|_E = \|X\|_{E^{\times\times}}$ for all $X \in E(\mathcal{M},\tau)$.
- The norm closed unit ball $B_E$ of $E(\mathcal{M},\tau)$ is closed in $S(\mathcal{M},\tau)$ with respect to the local measure topology .

Reflexivity of strongly symmetric spaces may be characterised as follows.

**Theorem 2.4.2.** *A strongly symmetric space $E(\mathcal{M},\tau)$ is reflexive if and only if*

*(i). The space $E$ has the Fatou property.*

*(ii). The norms on $E$ and $E^\times$ are order continuous.*

A fully symmetric space is called a *Kantorovich-Banach space* (or *$KB$-space*) if it has order continuous norm and the Fatou property. It is clear that the noncommutative $L_p$-spaces are $KB$-spaces for all $p \in [1,\infty)$.

**Definition 2.4.3.** *A linear subspace $E(\mathcal{M},\tau)$ of $S(\mathcal{M},\tau)$ is called a Calkin operator space if $B \in E(\mathcal{M},\tau)$ whenever $B \in S(\mathcal{M},\tau)$ and $\mu(B) \leq \mu(A)$ for some $A \in E(\mathcal{M},\tau)$.*

*A Calkin function (respectively, sequence) space is the term reserved for a Calkin operator space when $\mathcal{M} = L_\infty(0,1)$ or $\mathcal{M} = L_\infty(0,\infty)$ (respectively, $\mathcal{M} = l_\infty$).*

The following theorem extends the Calkin correspondence between two-sided ideals of $B(\mathcal{H})$ and their Calkin sequence spaces by showing that the singular value function maps bijectively Calkin operator spaces to Calkin function spaces.

**Theorem 2.4.4.** *Let $\mathcal{M}$ be an atomless (or atomic) von Neumann algebra equipped with a faithful normal semifinite trace $\tau$. If $E(\mathcal{M},\tau)$ is a Calkin operator space, then*

$$E := \{x \in S : \ \mu(x) = \mu(A), \ A \in E(\mathcal{M},\tau)\}$$

*is a Calkin function (or sequence) space, where $S = S(0,\tau(\mathbf{1}))$ (or $l_\infty$). If $E$ is a Calkin function (or sequence) space, then*

$$E(\mathcal{M},\tau) := \{A \in S(\mathcal{M},\tau) : \ \mu(A) \in E\}$$

*is a Calkin operator space.*

A wide class of symmetric operator spaces associated with the von Neumman algebra $\mathcal{M}$ can be constructed from concrete symmetric function spaces studied extensively in e.g. [86]. Let $(E(0,\infty), \|\cdot\|_{E(0,\infty)})$ be a symmetric function space on the semi-axis $(0,\infty)$. One can construct an operator space by defining

$$E(\mathcal{M},\tau) = \{X \in S(\mathcal{M},\tau) : \mu(X) \in E(0,\infty)\}, \quad \|X\|_{E(\mathcal{M},\tau)} := \|\mu(X)\|_{E(0,\infty)}.$$

If $E(0,\infty)$ is a strongly/fully symmetric function space on $(0,\infty)$, then it is clear that $E(\mathcal{M},\tau)$ is also a strongly/fully symmetric space [36, 37]. However, when $E(0,\infty)$ is symmetric rather than strongly/fully symmetric, the question as to whether $\|\cdot\|_E$ is a norm, and not merely a quasi-norm turns out to be highly non-trivial and was solved only recently by Kalton and Sukochev [82] (see also [87]). In the case when $(\mathcal{M},\tau) = (B(\mathcal{H}), \mathrm{Tr})$, we use notation $E(\mathcal{H})$.

**Theorem 2.4.5.** *If $E(0,\infty)$ is a symmetric function space, then the functional $\|\cdot\|_{E(\mathcal{M},\tau)}$ is a complete norm on $E(\mathcal{M},\tau)$. In particular, $c_E = \mathbf{1}$.*

Further, for any symmetric space $E(0,\infty)$, we have

$$L_1 \cap L_\infty(\mathcal{M},\tau) \subset E(\mathcal{M},\tau) \subset (L_1 + L_\infty)(\mathcal{M},\tau).$$

For convenience, we denote $\|\cdot\|_{E(\mathcal{M},\tau)}$ by $\|\cdot\|_E$. Many properties of symmetric spaces, such as reflexivity, Fatou property, order continuity of the norm as well as Köthe duality carry over from commutative symmetric function space $E(0,\infty)$ to its noncommutative counterpart $E(\mathcal{M},\tau)$ (see e.g. [41, 43, 46]).

## 2.5 $\tau$-compact operators

A projection $P \in \mathcal{P}(\mathcal{M})$ is called *$\tau$-finite* if $\tau(P) < \infty$. If $P \in \mathcal{P}(\mathcal{M})$ is $\tau$-finite, then $P$ is a finite projection. The two-sided ideal $\mathcal{F}(\mathcal{M},\tau)$ in $\mathcal{M}$ consisting of all elements of $\tau$-finite range is defined by setting

$$\mathcal{F}(\mathcal{M},\tau) = \{X \in \mathcal{M} \ : \ \tau(r(X)) < \infty\} = \{X \in \mathcal{M} \ : \ \tau(s(X)) < \infty\}.$$

**Definition 2.5.1** (see e.g. [87, Definition 2.6.8]). *The set $\mathcal{C}_0(\mathcal{M},\tau)$ of all* $\tau$-compact *bounded operators is the closure in the norm $\|\cdot\|_\infty$ of the linear span of all $\tau$-finite projections.*

This notion is a direct generalization of the ideal of compact operators on a Hilbert space $\mathcal{H}$. If $\tau$ is finite, then every projection is $\tau$-finite and, therefore, $\mathcal{C}_0(\mathcal{M},\tau) = \mathcal{M}$ (see e.g. [87, Page 64]). The next lemma shows that $\mathcal{C}_0(\mathcal{M},\tau)$ is the noncommutative counterpart of the algebra $\mathcal{C}_0(0,\infty)$ of bounded functions vanishing at infinity.

**Proposition 2.5.2** (see e.g. [87, Lemma 2.6.9]). *The space $\mathcal{C}_0(\mathcal{M},\tau)$ is associated to the ideal of essentially bounded functions vanishing at infinity (see [87, Lemma 2.6.9]), that is,*

$$\mathcal{C}_0(\mathcal{M},\tau) = \{A \in S(\mathcal{M},\tau) : \mu(A) \in L_\infty(0,\infty),\ \mu(\infty; A) := \lim_{t\to\infty} \mu(t; A) = 0\}.$$

*Proof.* Let $A \in \mathcal{M}$ and let $\mu(\infty; A) = 0$. Let $A = U|A|$ be the polar decomposition for $A$. For a given $n \in \mathbb{N}$, define $A_n \in \mathcal{M}$ by setting

$$A_n := U \sum_{k=1}^{n-1} \frac{k \|A\|_\infty}{n} E^{|A|}(\frac{k\|A\|_\infty}{n}, \frac{(k+1)\|A\|_\infty}{n}].$$

By the spectral theorem, $A_n \to A$ uniformly in $\mathcal{M}$. Since $A_n \in \mathcal{C}_0(\mathcal{M}, \tau)$, it follows that $A \in \mathcal{C}_0(\mathcal{M}, \tau)$.

Conversely, $A \in \mathcal{C}_0(\mathcal{M}, \tau)$ and let $A_n$, $n \in \mathbb{N}$, be a finite linear combination of $\tau$-finite projections such that $A_n \to A$ in $\mathcal{M}$. If $a_n = \tau(E^{A_n}(0, \infty))$, then $\mu(t; A_n) = 0$ for $t \geq a_n$. It follows from Proposition 2.2.2 that

$$\mu(\infty; A) \leq \mu(a_n; A) = |\mu(a_n; A) - \mu(a_n; A_n)| \leq \|A - A_n\|_\infty.$$

Since $A_n \to A$ uniformly in $\mathcal{M}$, it follows that $\mu(\infty; A) = 0$. $\square$

Equivalently, $\mathcal{C}_0(\mathcal{M}, \tau)$ is set of all elements $X \in \mathcal{M}$ such that $\tau(E^{|X|}(\lambda, \infty)) < \infty$ for every $\lambda > 0$ (see e.g. Lemma 2.2.3). By (2.2), $\mathcal{C}_0(\mathcal{M}, \tau)$ is a two-sided ideal of $\mathcal{M}$.

**Definition 2.5.3.** *The space $S_0(\mathcal{M}, \tau)$ of $\tau$-compact operators is the space associated to the algebra of functions from $S(0, \infty)$ vanishing at infinity [39, 43, 127], that is,*

$$S_0(\mathcal{M}, \tau) = \{A \in S(\mathcal{M}, \tau) : \ \mu(\infty; A) = 0\}.$$

$S_0(\mathcal{M}, \tau)$ is a two-sided ideal in $S(\mathcal{M}, \tau)$ [43, 45] and, clearly, $\mathcal{C}_0(\mathcal{M}, \tau) = S_0(\mathcal{M}, \tau) \cap \mathcal{M}$. It is known that every symmetric space having order continuous norm is a subspace of $S_0(\mathcal{M}, \tau)$ (see e.g. [45, Chapter IV, Lemma 8.5] or [61, Remark 2.9]). For the sake of completeness, we provide a short proof below.

**Proposition 2.5.4.** *If $E(\mathcal{M}, \tau)$ is a symmetric space having order continuous norm, then $E(\mathcal{M}, \tau) \subset S_0(\mathcal{M}, \tau)$.*

*Proof.* It is clear that $S_0(\mathcal{M}, \tau) = S(\mathcal{M}, \tau)$ if $\tau$ is finite. Without loss of generality, we may assume that $\tau$ is infinite.

Assume by contradiction that $E(\mathcal{M}, \tau) \not\subseteq S_0(\mathcal{M}, \tau)$. Then, there exists an operator $T \in E(\mathcal{M}, \tau)$ such that $\mu(\infty; T) = c > 0$. Since $\mu(c\mathbf{1}) = c \leq \mu(T)$, it follows from Definition 2.4.1 that $c\mathbf{1} \in E(\mathcal{M}, \tau)$, and, therefore, $\mathbf{1} \in E(\mathcal{M}, \tau)$. Since $\tau$ is a semifinite trace, it follows that there exists an increasing net $P_i$ with $\tau(P_i) < \infty$ and $\vee P_i = \mathbf{1}$. Since $\|\cdot\|_E$ is an order continuous norm, it follows that $\|\mathbf{1} - P_i\|_E \to 0$. However, since $\tau(P_i) < \infty$, it follows that $\mu(\mathbf{1}) = \mu(\mathbf{1} - P_i)$ for every $i$. Hence, $\|\mathbf{1} - P_i\|_E = \|\mathbf{1}\|_E$, which is a contradiction. $\square$

The following lemma provides a sufficient condition for an operator $X \in \mathcal{M}$ to be not $\tau$-compact. This condition plays a crucial role in the proof of Theorem 5.2.2.

**Lemma 2.5.5.** *Let $X \in (L_1 + L_\infty)(\mathcal{M}, \tau)$ and $\{\alpha_i > 0\}_i$ be an arbitrary sequence of real numbers increasing to infinity. If there exists a number $c > 0$ such that*

$$\int_0^{\alpha_i} \mu(t; X)dt \geq \alpha_i c$$

*for every $\alpha_i$, then $\mu(t; X) \geq c$ for all $t > 0$, that is, $X$ is not $\tau$-compact. In other words, if $\mu(X) \succ\succ c$, then $\mu(X) \geq c$.*

*Proof.* Assume by contradiction that $\mu(n_0; X) < c$ for some $n_0 > 0$ and therefore, $\mu(t; X) \leq \mu(n_0; X) < c$ for every $t \geq n_0$. By the assumption that $X \in (L_1 + L_\infty)(\mathcal{M}, \tau)$, we have

$$\begin{aligned} \infty > \int_0^{n_0} \mu(t; X)dt + (\alpha_i - n_0)\mu(n_0; X) &\geq \int_0^{n_0} \mu(t; X)dt + \int_{n_0}^{\alpha_i} \mu(t; X)dt \\ &= \int_0^{\alpha_i} \mu(t; X)dt \geq \alpha_i c \end{aligned}$$

for any $\alpha_i \geq n_0$. It follows that $\mu(n_0; X) \geq \frac{\alpha_i c - \int_0^{n_0} \mu(t;X)dt}{\alpha_i - n_0}$ for every $\alpha_i \geq n_0$. By assumption, we have that $\alpha_i \to_i \infty$ as $i \to \infty$, and therefore, $\mu(n_0; X) \geq c$, which is a contradiction. Thus, $\mu(t; X) \geq c$ for all $t > 0$, which implies that the operator $X$ is not $\tau$-compact. □

Recall that $\mathcal{J}(\mathcal{M})$ is the uniform norm closure of the linear span of all finite projections in $\mathcal{M}$, which was first studied by Kaftal [79, 80] (see also [111, 112]). Note that $\mathcal{C}_0(\mathcal{M}, \tau) \subset \mathcal{J}(\mathcal{M})$ for any semifinite algebra $\mathcal{M}$ because every $\tau$-finite projection is finite. It is known that $\mathcal{C}_0(\mathcal{M}, \tau) = \mathcal{J}(\mathcal{M})$ whenever $\mathcal{M}$ is a factor (see e.g. [112, 2.1.1.]).

**Remark 2.5.6.** *Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace $\tau$. It is easy to see that $\mathcal{C}_0(\mathcal{M}, \tau) \neq \mathcal{J}(\mathcal{M})$ if and only if there exists a finite projection $P \in \mathcal{M}$ such that $\tau(P) = \infty$ (see e.g. [79, Theorem 1.3]).*

We end this section with the following theorem, which gives a necessary and sufficient condition on the algebra $\mathcal{M}$ for the existence of a faithful normal semifinite trace $\tau$ on $\mathcal{M}$ with $\mathcal{C}_0(\mathcal{M}, \tau) \subsetneqq \mathcal{J}(\mathcal{M})$.

**Theorem 2.5.7.** *Let $\mathcal{M}$ be a semifinite von Neumann algebra. The following conditions are equivalent:*

*(i). There exists a faithful normal semifinite trace $\tau$ on $\mathcal{M}$ such that $\mathcal{C}_0(\mathcal{M}, \tau) \neq \mathcal{J}(\mathcal{M})$;*

*(ii).* $\dim(\mathcal{Z}(\mathcal{M})) = \infty$.

*Proof.* (i)⇒(ii). Assume by contradiction that $\dim(\mathcal{Z}(\mathcal{M})) < \infty$. We denote by $E_1, \ldots, E_n$, $n \in \mathbb{N}$, the finite family of atoms in $\mathcal{Z}(\mathcal{M})$. It is clear that $\mathcal{M}_{E_k}$ is a semifinite factor for all $k = 1, \ldots, n$. For every $k = 1, \ldots, n$, fix a trace $\tau_k$ on $\mathcal{M}_{E_k}$. It is clear that $\tau(X) = \sum_{k=1}^n \alpha_k \tau_k(X E_k)$ for some $\alpha_k > 0$.

By Remark 2.5.6, we can find a finite projection $P \in \mathcal{P}(\mathcal{M})$ such that $\tau(P) = \infty$. Therefore, $\tau_k(PE_k) = \infty$ for some $k$. However, this is impossible since $PE_k$ is a finite projection in the factor $\mathcal{M}_{E_k}$. This contradiction shows that $\dim(\mathcal{Z}(\mathcal{M})) = \infty$.

(ii)$\Rightarrow$(i). Let $\tau'$ be an arbitrary faithful normal semifinite trace on $\mathcal{M}$. By the assumption, there exists a sequence of pairwise disjoint non-zero projections $\{E_n\}_{n=1}^{\infty} \subset \mathcal{Z}(\mathcal{M})$ such that $\bigvee_{n=1}^{\infty} E_n = \mathbf{1}$. In every algebra $\mathcal{M}_{E_n}$, $n = 1, 2, 3, \cdots$, there exists a non-zero finite projection $P_n$. If for some $n$ we have that $\tau'(P_n) = \infty$, then the assertion follows from Remark 2.5.6.

Assume that $\tau'(P_n) < \infty$ for all $n$. Since $P_n \in \mathcal{M}_{E_n}$ and $E_n$ are pairwise disjoint, it follows that the central supports of $P_n$ are pairwise disjoint. Hence, $P := \bigvee_{n=1}^{\infty} P_n$ is also a finite projections. Set $\tau(X) := \sum_{n=1}^{\infty} n\tau'(XE_n)/\tau'(P_n)$. Clearly, $\tau$ is a faithful normal semifinite trace on $\mathcal{M}$ and $\tau(P) = \infty$. By Remark 2.5.6, we obtain the validity of (i). $\square$

**Example 2.5.8.** *Let $\mathcal{M}$ be the algebra $L_\infty(0,\infty)$ of all Lebesgue measurable essentially bounded functions on $(0,\infty)$. Since $L_\infty(0,\infty)$ is a commutative von Neumann algebra, it follows that it is a finite von Neumann algebra, and therefore, $\mathcal{M} = \mathcal{J}(\mathcal{M})$. However, when $\mathcal{M}$ is equipped with the standard trace, $\mathcal{C}_0(\mathcal{M},\tau)$ coincides with the set $\mathcal{C}_0(0,\infty)$ of bounded functions whose singular value functions vanish at infinity.*

# Chapter 3

# Derivations of operator algebras: review and general properties

Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. An additive abelian group $\mathcal{G}$ is called an *$\mathcal{A}$-bimodule* if there exist mappings $\mathcal{A} \times \mathcal{G} \to \mathcal{G}$ and $\mathcal{G} \times \mathcal{A} \to \mathcal{G}$, (written as $ag$ and $ga$, $a \in \mathcal{A}$, $g \in \mathcal{G}$ respectively), satisfies the following:

1. $a(g_1 + g_2) = ag_1 + ag_2$ and $(g_1 + g_2)a = g_1a + g_2a$;

2. $(a_1 + a_2)g = a_1g + a_2g$ and $g(a_1 + a_2) = ga_1 + ga_2$;

3. $a_1(a_2g) = (a_1a_2)g$ and $(ga_1)a_2 = g(a_1a_2)$.

In addition, if $\mathcal{G}$ is a Banach space and there exists a constant $M$ such that

$$\|agb\| \leq M \|a\|_{\mathcal{A}} \|b\|_{\mathcal{A}} \|g\|_{\mathcal{G}},$$

then $\mathcal{G}$ is called a *Banach $\mathcal{A}$-bimodule* [63, 125].

A *derivation* is a function on an algebra $\mathcal{A}$ over $\mathbb{C}$ which generalised certain features of the derivative operator. Namely, if $\mathcal{J}$ is an $\mathcal{A}$-bimodule, a linear map $\delta : \mathcal{A} \to \mathcal{J}$ that satisfies the Leibniz law is called a derivation, that is,

$$\delta(ab) = \delta(a)b + a\delta(b), \ \forall a, b \in \mathcal{A}.$$

In particular, if $k \in \mathcal{J}$, then $\delta_k(x) := kx - xk$ is a derivation. Such derivations implemented by elements in $\mathcal{J}$ are called *inner* [7, 125].

Recall that a Banach $\mathcal{A}$-bimodule $X$ is called a *dual $\mathcal{A}$-bimodule* if $X$ is isometrically isomorphic to the dual space of a Banach space $X_*$ [32, 125]. If $\mathcal{A}$ is a $C^*$-algebra acting on a Hilbert space $\mathcal{H}$ and, for each $x \in X$, the mappings

$$A \mapsto Ax, \ A \mapsto xA$$

are ultraweak-weak$^*$ continuous, we describe $X$ as a *dual normal $\mathcal{A}$-bimodule* (see e.g. [66, p. 75] and [63]. A Banach algebra $\mathcal{A}$ is called *amenable* if the derivations from $\mathcal{A}$ into an arbitrary dual $\mathcal{A}$-bimodules $X$ are all inner (see [63, Section 5], see also [21, 28, 53, 64, 66, 90]. A von Neuamnn algebra $\mathcal{A}$ is said to be *amenable* if, for every dual normal Banach $\mathcal{A}$-bimodule $X$, the derivations from $\mathcal{A}$ into $X$ are all inner (see [29, Chapter 5, Section 7], see also [27, 125]). The amenability is one of the most essential topics in the study of von Neumann algebras. In particular, Johnson, Kadison and Ringrose showed that all approximately finite dimensional von Neumann algebras are amenable (see [66], see also [125, Theorem 2.4.3] and [32, Theorem 10.8]). It is proved by Connes [27] that there are several equivalent conditions for a von Neumann algebra acting on a separable Hilbert space to be approximately finite. In particular, Connes [27] showed (see also [29, p. 505]) that a von Neumann algebra $\mathcal{A}$ on a separable Hilbert space $\mathcal{H}$ is approximately finite dimensional if and only if it is amenable. On the other hand, for an arbitrary given Banach algebra (or, more generally, a group [63, Section 10.11]) $\mathcal{A}$, it is desirable to identify those bimodules $X$ such that every derivation $\delta : \mathcal{A} \to X$ is automatically inner. During the past decades, a number of important special cases have been resolved (see e.g. [4, 63, 67, 88, 110, 117] for more details). For more details of the theory of derivations in operator algebras, we refer to [63, 84, 115, 116, 121] and the forthcoming book [7]. In this chapter, we concentrate on the following question (see e.g. [56, 63]):

> Let $\mathcal{A}$ be a $C^*$-algebra and $\mathcal{J}$ be an $\mathcal{A}$-bimodule. Is every derivation from $\mathcal{A}$ into $\mathcal{J}$ necessarily inner?

This chapter presents a survey of some results of the theory of derivations.

## 3.1 Derivations on algebras of bounded operators

In 1953, Kaplansky [83] showed that every derivation from a commutative $C^*$-algebra into itself is identically 0, which was later extended by Singer and Wermer [126], who showed that every derivation from a semi-simple commutative Banach algebra into itself is identically 0. Moreover, the authors of [126] also studied derivations from a Banach algebra into an larger algebra [126]. The early study of derivations by Kaplansky [83], Singer and Wermer [126] inspired the so-called *derivation problem*, which is one of the oldest unsettled problems in operator algebra theory. Let $\mathcal{A} \subset B(\mathcal{H})$ be a $C^*$-algebra. The so-called *derivation problem* is the following question.

**Question 3.1.1.** *Is every derivation $\delta : \mathcal{A} \to B(\mathcal{H})$ necessarily inner? That is, can we find a $T \in B(\mathcal{H})$ such that $\delta(\cdot) = \delta_T(\cdot) = [T, \cdot]$.*

For details of the study of the derivation problem, we refer to [22, 25, 108, 125]. Since any inner derivation is necessarily continuous (in the norm topology), the natural step in the study of the derivation problem is the question of automatic continuity of derivations. It was conjecture by Kaplansky [83], and proved by Sakai [118], that a derivation $\delta$ from a $C^*$-algebra into itself is automatically norm continuous. From this, Kadison [70, Lemma 3] deduced that $\delta$ is continuous also in the ultraweak topology, when the algebra is represented as an algebra of operators acting on a Hilbert space. Subsequently, Johnson and Sinclair [68] proved the automatic norm continuity of derivations of a semi-simple Banach algebra.

The results of Sakai [118] and Kadison [70] was further generalised by Ringrose [114] to a very general form (Ber, Chilin and Levitina [5] generalised Ringrose's theorem to the case of quasi-Banach $\mathcal{A}$-bimodules ).

**Theorem 3.1.2.** *Every derivation from a $C^*$-algebra $(\mathcal{A}, \|\cdot\|)$ into a (quasi-)Banach $\mathcal{A}$-bimodule $(\mathcal{J}, \|\cdot\|_{\mathcal{J}})$ is automatically continuous.*

It is known that every derivation from a $C^*$-algebra $\mathcal{A} \subset B(\mathcal{H})$ into $B(\mathcal{H})$ is also continuous in the ultraweak topology [66, 67]. Therefore, the derivation problem is equivalent to the following.

**Question 3.1.3.** *Is every derivation from a von Neumann subalgebra $\mathcal{A}$ of $B(\mathcal{H})$ into $B(\mathcal{H})$ necessarily inner?*

Even though the derivation problem is still open, there are results giving affirmative answers to it under some additional conditions on the subalgebra $\mathcal{A}$ [22, 25, 26, 32, 108, 125]. In all known cases, the derivation problem has affirmative answers. In particular, the case when $\mathcal{A}$ is of type $I_{fin}$ is an immediate consequence of the property of approximately finite von Neumann algebras (see e.g. [66, 125] or [32, Chapter 8]). The properly infinite case was proved by Christensen [22]. Indeed, the only remaining situation to consider is a type $II_1$ von Neumann subalgebra $\mathcal{A}$ with a type $II_\infty$ commutant [125].

Christensen [23] (see also [25]) established the characterisation of the inner derivations from $\mathcal{A}$ into $B(\mathcal{H})$ in terms of *complete boundedness* of derivations. Let $\mathcal{A}$ and $\mathcal{B}$ be two $C^*$-algebras. Let $\mathbb{M}_n$ be the space of all $n \times n$ matrices. The operator norms on $B(\mathcal{H} \oplus \cdots \oplus \mathcal{H})$ induce $C^*$-algebra norms on the matrix algebras $\mathbb{M}_n(\mathcal{A})$ over $\mathcal{A}$. Any linear map $\varphi : \mathcal{A} \to \mathcal{B}$ gives a family $\{\varphi_n : \mathbb{M}_n(\mathcal{A}) \to \mathbb{M}_n(\mathcal{B})\}$ defined by $\varphi_n(a_{ij}) = (\varphi(a_{ij}))$ for each $n \times n$ matrix $(a_{ij}) \in \mathbb{M}_n(\mathcal{A})$. We say that $\varphi : \mathcal{A} \to \mathcal{B}$ is *completely bounded*, if the sequence $\{\|\varphi_n\|\}_{n=1}^{\infty}$ is uniformly bounded [125, Chapter 1.2].

**Theorem 3.1.4.** *Let $\mathcal{A}$ be a $C^*$-subalgebra of $B(\mathcal{H})$. A derivation $\delta : \mathcal{A} \to B(\mathcal{H})$ is inner if and only if it is completely bounded.*

The derivation problem has several equivalent formulations. One of them is the so-called *similarity problem* raised by Kadison [69].

**Question 3.1.5.** *Let $\mathcal{A} \subset B(\mathcal{H})$ be a $C^*$-algebra and $\pi : \mathcal{A} \to B(\mathcal{H})$ be a unital homomorphism with $\|\pi\| < \infty$. Does there exist an invertible element $S$ in $B(\mathcal{H})$ such that $S\pi(\cdot)S^{-1}$ is a $*$-homomorphism.*

Kirchberg [85] proved the following equivalence.

**Theorem 3.1.6.** *Let $\mathcal{A}$ be an arbitrary $C^*$-algebra. The similarity problem for $\mathcal{A}$ and the derivation problem for $\mathcal{A}$ are equivalent.*

A natural generalisation of the derivation problem is the following.

**Question 3.1.7.** *Let $\mathcal{A}$ be a $C^*$-subalgebra of a von Neumann algebra $\mathcal{M}$. Is every derivation $\delta : \mathcal{A} \to \mathcal{M}$ necessarily inner? That is, does there exist a $T \in \mathcal{M}$ such that $\delta = \delta_T$?*

The classical result in this field is the following theorem due to Kadison [70] and Sakai [119] (see also [120, 121]), which should be considered as the first attack to Question 3.1.7.

**Theorem 3.1.8.** *Every derivation from a von Neumann algebra $\mathcal{M}$ into itself is automatically inner.*

However, when one considers more general algebras (for examples, $C^*$-algebras), there are examples of non-inner derivations from this algebra into itself.

**Example 3.1.9** (see e.g. [120, Example 4.1.8])**.** *Let $K(\mathcal{H})$ be the $C^*$-algebra of all compact operators on $\mathcal{H}$. Let $A \in B(\mathcal{H})$ with $A \notin K(\mathcal{H}) + \mathbb{C}\,\mathbf{1}$. Then, $\delta_A$ defined by $\delta_A(X) = [A, X]$, $X \in K(\mathcal{H})$ is not an inner derivation from $K(\mathcal{H})$ into $K(\mathcal{H})$.*

For the general case when $\mathcal{A} \neq \mathcal{M}$, very little is known for Question 3.1.7. So far, the best result in this area is the following theorem, which is a corollary of a result by Johnson, Kadison and Ringrose [66] (see also [27, 29], [125, Theorem 2.4.3] or [22, Corollary 5.6]).

**Theorem 3.1.10.** *Every derivation from an approximately finite von Neumann subalgebra of $\mathcal{M}$ into $\mathcal{M}$ is automatically inner.*

Since every abelian von Neumann algebra is approximately finite [32, Lemma 8.4], the above theorem yields that every derivation from an abelian von Neumann subalgebra of $\mathcal{M}$ into $\mathcal{M}$ is inner.

Using the Ryll-Nardzewski fixed point theorem (see Theorem 3.3.2), Christensen [22] showed that Question 3.1.7 has an affirmative answer for finite von Neumann algebras $\mathcal{M}$.

**Theorem 3.1.11.** *Let $\mathcal{M}$ be a finite von Neumann algebra. Then, every derivation from a $C^*$-subalgebra of $\mathcal{M}$ into $\mathcal{M}$ is inner.*

## 3.2 The Johnson–Parrott–Popa theorem

A special type of Question 3.1.7 is the case when a derivation takes values in a proper ideal of the von Neumann algebra $\mathcal{M}$. The first result in this direction is due to Johnson and Parrott [67], who considered the special case of Question 3.1.3 when the range of $\delta$ is contained in $K(\mathcal{H})$, the ideal of all compact operators on $\mathcal{H}$. In that paper, the authors proved that if $\mathcal{A}$ is an abelian von Neumman subalgebra of $\mathcal{M}$, then every derivation $\delta$ from $\mathcal{A}$ to $K(\mathcal{H})$ is automatically inner. As an easy consequence, they were also able to treat the case when $\mathcal{A}$ has no certain type $II_1$

factors as direct summands. The remaining case, when $\mathcal{A}$ is a von Neumann algebra of type $II_1$ was later resolved by Popa [110] by transforming the noncommutative framework of the problem into a commutative one. The results in [67] and [110] can be therefore formulated as follows.

**Theorem 3.2.1** (Johnson–Parrott–Popa)**.** *Let $\mathcal{A}$ be a von Neumann subalgebra of $B(\mathcal{H})$. Then, every derivation*

$$\delta : \mathcal{A} \to K(\mathcal{H})$$

*is inner.*

Note that the Johnson–Parrott–Popa theorem can not be extended to the case of $C^*$-subalgebras of $B(\mathcal{H})$ (see Example 3.1.9).

The semifinite version of the result by Johnson and Parrott [67] was first studied by Kaftal and Weiss [81]. Precisely, they considered the case when $B(\mathcal{H})$ is replaced with a semifinite von Neumann algebra $\mathcal{M}$ and $K(\mathcal{H})$ is replaced with the uniform norm closed ideal $\mathcal{J}(\mathcal{M})$ generated by all finite projections in $\mathcal{M}$. It is shown in [81] that if $\mathcal{A}$ is an abelian (or properly infinite) von Neumann subalgebra of $\mathcal{M}$ containing the center $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$, then any derivation of $\mathcal{A}$ into $\mathcal{J}(\mathcal{M})$ is inner. This result was later extended to the case when the type $I_{fin}$ direct sum of $\mathcal{A}$ is *locally compatible* with the center of $\mathcal{M}$ [112]. Recall that a subalgebra $\mathcal{N}$ of $\mathcal{M}$ is locally compatible with the center $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$, if there exists a partition of the unity $\{P_i\}_{i\in I}$ in the center $\mathcal{Z}(\mathcal{N})$ of $\mathcal{N}$ such that for each $i$, we have either

$$\mathcal{Z}(\mathcal{N})P_i \subset \mathcal{Z}(\mathcal{M})P_i \text{ or } \mathcal{Z}(\mathcal{M})P_i \subset \mathcal{Z}(\mathcal{N})P_i.$$

**Theorem 3.2.2.** *Let $\mathcal{M}$ be a semifinite von Neumann algebra and $\mathcal{J}(\mathcal{M})$ be the uniform norm closed ideal generated by all finite projections in $\mathcal{M}$. Let $\mathcal{A} \subset \mathcal{M}$ be a weak operator closed $*$-subalgebra of $\mathcal{M}$ and suppose the finite type $I$ summand of $\mathcal{N}$ is locally compatible with $\mathcal{Z}(\mathcal{M})$. Then, for any derivation $\delta : \mathcal{A} \to \mathcal{J}(\mathcal{M})$, there exists $K \in \mathcal{J}(\mathcal{M})$, $\|K\|_\infty \le 2\,\|\delta\|_{\mathcal{A}\to\mathcal{J}(\mathcal{M})}$ with $\delta = \delta_K$.*

In particular, every derivation $\delta : \mathcal{A} \to \mathcal{J}(\mathcal{M})$ is inner if $\mathcal{A}$ is of type $II_1$ or properly infinite. Popa and Rădulescu also established the existence of non-inner derivations $\delta : \mathcal{A} \to \mathcal{J}(\mathcal{M})$ for some specific semifinite von Neumann algebra $\mathcal{M}$ and abelian von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$ (see [112, Theorem 1.2 and Section 8]), which is somehow unexpected and is the first example of non-inner derivations in von Neumann algebras.

**Example 3.2.3.** *let $\mathcal{M} = L_\infty([0,1],\lambda)\overline{\otimes}B(L_2(\mathbb{T},\mu))$, $\mathcal{A} = \mathbf{1}\otimes L_\infty(\mathbb{T},\mu) \subset \mathcal{M}$, where $\mu$ is the Lebesgue measure on the torus $\mathbb{T}$ and $\lambda$ is the Lebesgue measure on the unit interval $[0,1]$. There exists an operator $T \in \mathcal{M}$ such that $\delta_T$ is a derivation from $\mathcal{A}$ into $\mathcal{J}(\mathcal{M})$ which is not inner, i.e., there exists no elements $K \in \mathcal{J}(\mathcal{M})$ such that $\delta_T = \delta_K$.*

Another semifinite version of the Johnson–Parrott–Popa theorem was initiated by Christensen [24], who introduced the notion of *generalized compacts* (see also [137]).

Let $X \in B(\mathcal{H})$ and let $\mathcal{M}$ be a von Neumann algebra. Then $X$ is said to be a *weakly compact multiplier* of $\mathcal{M}$ if the operator $r_X$ and $l_X$ of $\mathcal{M}$ into $B(\mathcal{H})$, given by $r_X M = MX$ and $l_X M = XM$, are weakly compact. The so-called generalized compacts $C(\mathcal{M})$ associated with $\mathcal{M}$ is the set of weakly compact multipliers of $\mathcal{M}$. If $\mathcal{M} = B(\mathcal{H})$, then the set $C(\mathcal{M})$ coincides with $K(\mathcal{H})$.

Christensen [24] showed that every derivation from $\mathcal{M}$ into $C(\mathcal{M})$ is inner if $\mathcal{M}$ is a properly inifinite von Neumann algebra. However, the case when $\mathcal{M}$ is a type $II_1$ von Neumann algebra was left open in [24] and was resolved by Galatan and Popa [49] in 2017.

There are some other extensions of the result by Johnson and Parrott [67]. In 1977, Hoover [56] obtained the following result where the algebra $K(\mathcal{H})$ in the Johnson–Parrott–Popa theorem was replaced by the Schatten $p$-class.

**Theorem 3.2.4.** *Let $\mathcal{A}$ be a unital $C^*$-subalgebra of $B(\mathcal{H})$ and let $C_p(\mathcal{H})$ be the Schatten $p$-class, $1 \leq p < \infty$. If $\delta : \mathcal{A} \to C_p(\mathcal{H})$ is a derivation, then there exists a $T \in C_p(\mathcal{H})$ such that $\delta = \delta_T$.*

Since $C_p(\mathcal{H})$, $1 < p < \infty$, is reflexive, it is a straightforward application of the Ryll-Nardzewski fixed point theorem (see [31, 96], see also Section 3.3) or the celebrated result by Johnson [63, Proposition 3.7] (see also Theorem 4.1.1). Ideas used by Hoover in the proof for the special case when $p = 1$ (the ideal coincides with the predual of $B(\mathcal{H})$) are subtle and the proof relies on the fact that $C_1(\mathcal{H}) \subset C_p(\mathcal{H})$ for any $p \geq 1$. However, when $B(\mathcal{H})$ is replaced by a general semifinite von Neumann algebra $\mathcal{M}$, the corresponding ideal $\mathcal{C}_p(\mathcal{M}, \tau)$ is not necessarily a reflexive space and therefore the Ryll-Nardzewski fixed point theorem can not be applied directly. In 1985, adapting the proof in [67], Kaftal and Weiss [81] obtained the following result on derivations into an $L_p$-ideal $\mathcal{C}_p(\mathcal{M}, \tau)$ $(:= L_p(\mathcal{M}, \tau) \cap \mathcal{M})$ of $\mathcal{M}$, $1 \leq p < \infty$.

**Theorem 3.2.5.** *Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a semifinite faithful normal trace $\tau$ and let $p \in [1, \infty)$. Assume that $\mathcal{A}$ is an abelian/properly infinite von Neumann subalgebra of $\mathcal{M}$. Then, every derivation*

$$\delta : \mathcal{A} \to \mathcal{C}_p(\mathcal{M}, \tau)$$

*is necessarily inner. That is, there exists $T \in \mathcal{C}_p(\mathcal{M}, \tau)$ such that $\delta = \delta_T$.*

However, the cases of type $I$ and type $II_1$ von Neumann subalgebras were left unresolved in [81], which led to the following question.

**Question 3.2.6.** *Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a semifinite faithful normal trace $\tau$ and let $p \in [1, \infty)$. Assume that $\mathcal{A}$ is an arbitrary von Neumann subalgebra of $\mathcal{M}$. Is every derivation*

$$\delta : \mathcal{A} \to \mathcal{C}_p(\mathcal{M}, \tau)$$

*automatically inner?*

In the special case when von Neumann subalgebra $\mathcal{A}$ coincides with $\mathcal{M}$, then Question 3.2.6 has an affirmative answer. Indeed, Ber and Sukochev [16, 17] have proved the following highly non-trivial result, which generalised the Kadison-Sakai Theorem (see Theorem 3.1.8) significantly.

**Theorem 3.2.7.** *Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{J}$ be an arbitrary ideal of $\mathcal{M}$. Then, every derivation from $\mathcal{M}$ into $\mathcal{J}$ is inner.*

In Chapter 4, we consider a question more general than Question 3.2.6. In particular, we resolve Question 3.2.6 affirmatively. An alternative proof is provided in Chapter 5.

## 3.3 Derivations with values into symmetric spaces

In this section, we collect some results on derivations into more general bimodules.

The so-call *Ryll-Nardzewski Fixed Point Theorem* is very important in this field. Before introducing the Ryll-Nardzewski Fixed Point Theorem, we recall the definition of *noncontracting family of maps*.

**Definition 3.3.1.** *Let $\mathfrak{X}$ be a locally convex space and let $Q$ be a nonempty subset of $\mathfrak{X}$. If $\mathcal{G}$ is a family of maps (not necessarily linear) of $Q$ into $Q$, then $\mathcal{G}$ is said to be a* noncontracting *family of maps if for any two distinct points $x$ and $y$ in $Q$, $0$ is not in the closure of*

$$\{T(x) - T(y) : T \in \mathcal{G}\}.$$

**Theorem 3.3.2** (Ryll-Nardzewski Fixed Point Theorem)**.** *If $\mathfrak{X}$ is a locally convex space, $Q$ is a weakly compact convex subset of $\mathfrak{X}$, and $\mathcal{G}$ is a noncontracting semigroup of weakly continuous affine maps of $Q$ into $Q$, then there is a point $x_0$ in $Q$ such that $T(x_0) = x_0$ for every $T \in \mathcal{G}$.*

The following theorem, due to Johnson [63] (see also [11, 15, 56, 136]), is a straightforward application of the Ryll-Nardzewski Fixed Point Theorem. We note that the assumption of a unit element in the $C^*$-algebra in [63] is omitted. The full proof of the following theorem can be founded in Chapter 4 (see Theorem 4.1.1).

**Theorem 3.3.3.** *Let $\mathcal{A}$ be a $C^*$-algebra and let $\mathcal{J}$ be a reflexive $\mathcal{A}$-bimodule. Then, for every derivation $\delta : \mathcal{A} \to \mathcal{J}$, there exists a $T \in \mathcal{J}$ such that $\delta = \delta_T$ and $\|T\|_{\mathcal{J}} \leq \|\delta\|_{\mathcal{A}\to\mathcal{J}}$. Moreover, $T \in \overline{\mathrm{co}\{\delta(U)U^* : U \in \mathcal{U}(\mathcal{A})\}}^{\|\cdot\|_{\mathcal{J}}}$, where $\mathrm{co}(S)$ denotes the convex hull of a set $S$.*

Throughout this section, we assume that $\mathcal{M}$ is a semifinite von Neumann algebra equipped with a semifinite faithful normal trace $\tau$. As a corollary of Theorem 4.1.1, we immediately obtain the following result.

**Corollary 3.3.4.** *Let $\mathcal{A}$ be a $C^*$-subalgebra of $\mathcal{M}$ and let $E(\mathcal{M}, \tau)$ be a reflexive symmetric space. Then, for every derivation $\delta : \mathcal{A} \to E(\mathcal{M}, \tau)$, there exists a $T \in E(\mathcal{M}, \tau)$ such that $\delta = \delta_T$ and $\|T\|_E \leq \|\delta\|_{\mathcal{A}\to E}$. Moreover, $T \in \overline{\mathrm{co}\{\delta(U)U^* : U \in \mathcal{U}(\mathcal{A})\}}^{\|\cdot\|_E}$.*

Noting that $L_p(\mathcal{M},\tau)$ is reflexive when $p>1$, we obtain the following corollary.

**Corollary 3.3.5.** *Let $\mathcal{A}$ be a unital $C^*$-subalgebra of $\mathcal{M}$ and $\delta:\mathcal{A}\to L_p(\mathcal{M},\tau)$, $p\geq 1$, be a derivation. Then, there exists an element $T\in L_p(\mathcal{M},\tau)$ such that $\delta=\delta_T$ and $\|T\|_{L_p}\leq\|\delta\|_{\mathcal{A}\to L_p}$.*

One should note that Corollary 3.3.5 does not recover Theorem 3.2.5 since derivation $\delta$ takes the value in the symmetric space $L_p(\mathcal{M},\tau)$ (of possibly unbounded operators), while Kaftal and Weiss showed (under some additional assumptions on the von Neumann subalgebra $\mathcal{A}$) that every derivation into the symmetric ideal $\mathcal{C}_p(\mathcal{M},\tau):=L_p(\mathcal{M},\tau)\cap\mathcal{M}$ is inner, and so there exists a bounded $T\in L_p(\mathcal{M},\tau)$, such that $\delta=\delta_T$.

It is well-known that $L_1(\mathcal{M},\tau)$ is not reflexive unless $\mathcal{M}$ is finite-dimensional. It was a long-standing open question whether every derivation a $C^*$-subalgebra of $\mathcal{M}$ into $L_1(\mathcal{M},\tau)$ is inner. The special case when $\mathcal{M}=B(\mathcal{H})$ was proved by Hoover (see Theorem 3.2.4). The case when $\mathcal{M}$ is a finite von Neumann algebra was resolved by Bunce and Paschke [20]. Recently, the question was resolved completely by Bader, Gelander and Monod [4]. Since the unit ball of $L_1$ is not weakly compact, the Ryll-Nardzewski Fixed Point Theorem can not be applied directly. Bader, Gelander and Monod considered the so-called Chebyshev center (which is weakly compact) in $L_1(\mathcal{M},\tau)$ rather than the $\|\cdot\|_1$-closure of convex hull of $\delta(U)U^*$ as in Theorem 4.1.1. Consequently, the Ryll-Nardzewski Fixed Point Theorem can be applied to this set, and the long standing open question on derivations with values in $L_1(\mathcal{M},\tau)$ was resolved.

**Theorem 3.3.6.** *[4] Let $\mathcal{N}$ be a von Neumann algebra and $\mathcal{A}$ be a $C^*$-subalgebra of $\mathcal{N}$. Every derivation from $\mathcal{A}$ into the predual of $\mathcal{N}$ is inner.*

**Remark 3.3.7.** *Pfitzner (see [107, Theorem 8.2]) indicated that the above theorem can be proved by applying an earlier result by Japón [62]. Moreover, the element implementing the derivation can be found in the closure (in the sense of the so-called* abstract measure topology*) of the convex hull of $\delta(U)U^*$.*

The special case when $\mathcal{A}=\mathcal{M}$ has been studied by Ber, Chilin and Sukochev [9, 10, 15]. In particular, they proved the following result.

**Theorem 3.3.8.** *Let $\mathcal{M}$ be a semifinite von Neumann algebra with a faithful semifinite normal trace $\tau$ and $\mathcal{E}$ be a Banach $\mathcal{M}$-bimodule (in particular, symmetric spaces) of $\tau$-measurable operators. Every derivation from $\mathcal{M}$ into $\mathcal{E}$ is inner.*

**Remark 3.3.9.** *One may consider derivations from the $*$-algebra $S(\mathcal{M},\tau)$ into itself. However, it is shown in [8] that a derivation $\delta:S(\mathcal{M},\tau)\to S(\mathcal{M},\tau)$ is not necessarily inner even when $\mathcal{M}$ is commutative. When $\mathcal{M}$ is a type I von Neumann algebra, a complete description of all derivations $\delta:S(\mathcal{M},\tau)\to S(\mathcal{M},\tau)$ has been obtained in [1]. The case when $\mathcal{M}$ is a properly infinite von Neumann algebra was established in [10].*

*It is a long-standing open question whether every derivation* $\delta : S(\mathcal{M},\tau) \to S(\mathcal{M},\tau)$ *is inner when* $\mathcal{M}$ *is of type* $II_1$ *[72, 73]. The complete resolution of this question was recently obtained by Ber, Kudaybergenov and Sukochev [13]. Precisely, they proved that every derivation from* $S(\mathcal{M},\tau)$ *into* $S(\mathcal{M},\tau)$ *is inner if and only if the type* $I_{fin}$ *summand of* $\mathcal{M}$ *is atomic.*

## 3.4 General properties results of derivations

Let $\delta$ be a derivation from a $C^*$-subalgebra $\mathcal{A}$ of a semifinite von Neumann algebra $\mathcal{M}$ into $S(\mathcal{M},\tau)$, i.e., $\delta : \mathcal{A} \to S(\mathcal{M},\tau)$ is a linear mapping satisfying the Leibniz law. The derivation $\delta$ is said to be *skew-adjoint* if $\delta = -\delta^*$, where $\delta^*$ is a derivation defined by $\delta^*(X) = (\delta(X^*))^*,\ x \in \mathcal{A}$. Actually, we can assume that the derivation $\delta$ is skew-adjoint because every derivation $\delta : \mathcal{A} \to S(\mathcal{M},\tau)$ can be decomposed into skew-adjoint components $\delta = \delta_1 + i \cdot \delta_2$, where

$$\delta_1(X) := \frac{\delta(X) - \delta(X^*)^*}{2} \text{ and } \delta_2(X) := \frac{\delta(X) + \delta(X^*)^*}{2i}.$$

**Remark 3.4.1.** *Assume that there exists an operator* $T \in S(\mathcal{M},\tau)$ *such that the skew-adjoint derivation* $\delta = \delta_T = [\cdot, T]$*. For every* $X \in \mathcal{A}$*, we have*

$$[X, T - T^*] = [X,T] - [X,T^*] = [X,T] + [X^*,T]^* = \delta(X) + (\delta(X^*))^* = 0,$$

*which implies that* $\mathrm{Im}(T) = \frac{T-T^*}{2i} \in \mathcal{A}'$*. Thus, for every* $X \in \mathcal{A}$*, we have*

$$\delta(X) = [X,T] = [X, \mathrm{Re}(T) + i\,\mathrm{Im}(T)] = [X, \mathrm{Re}(T)].$$

*Hence, without loss of generality, we can always assume that the operator* $T$ *implementing a skew-adjoint derivation* $\delta$ *is self-adjoint.*

In the following, we consider several types of reductions of a given derivation $\delta$ from $\mathcal{A}$ into an $\mathcal{M}$-bimodule $\mathcal{J}$ of $\tau$-measurable operators. The first one is reduction of $\delta$ by a given central projection $Z$ in the algebra $\mathcal{M}$. Recall that $\mathcal{J}_Z := \{ZX \in S(\mathcal{M}_Z,\tau) : X \in \mathcal{J}\} = Z\mathcal{J}Z$ (see e.g. [43, p. 215]).

**Lemma 3.4.2.** *Let* $\delta : \mathcal{A} \to \mathcal{J}$ *be a derivation and let* $Z \in \mathcal{Z}(\mathcal{M})$ *be a projection. The mapping* $\delta^{(Z)} : \mathcal{A}_Z \to \mathcal{J}_Z$ *given by* $\delta^{(Z)}(XZ) = Z\delta(X)Z,\ X \in \mathcal{A}$*, is a well-defined derivation from the induced von Neumann algebra* $\mathcal{A}_Z$ *into* $\mathcal{J}_Z$*.*

*Proof.* If $A, B \in \mathcal{A}$ such that $AZ = BZ$, then

$$\begin{aligned}
\delta^{(Z)}(AZ) - \delta^{(Z)}(BZ) &= Z\delta(A)Z - Z\delta(B)Z \\
&= Z\delta((A-B)E^{|A-B|}(0,\infty))Z \\
&= Z\delta(A-B)\cdot E^{|A-B|}(0,\infty)Z + Z(A-B)\cdot\delta(E^{|A-B|}(0,\infty))Z \\
&= Z\delta(A-B)\cdot E^{|A-B|}(0,\infty)Z.
\end{aligned}$$

Since $Z \in \mathcal{Z}(\mathcal{M})$, it follows that $E^{|A-B|}(0,\infty)Z$ is a projection with $E^{|A-B|}(0,\infty)Z \le E^{|A-B|}(0,\infty)$. However, the assumption, $(A-B)E^{|A-B|}(0,\infty)Z = (A-B)Z = 0$

implies that $E^{|A-B|}(0,\infty)Z = 0$ and therefore $\delta^{(Z)}(AZ) = \delta^{(Z)}(BZ)$. For every $X, Y \in \mathcal{A}$, we write

$$\begin{aligned}\delta^{(Z)}(ZXZ \cdot ZYZ) &= \delta^{(Z)}(ZXYZ) = Z\delta(XY)Z \\ &= Z\delta(X)YZ + ZX\delta(Y)Z \\ &= Z\delta(X)Z \cdot ZYZ + ZXZ \cdot Z\delta(Y)Z \\ &= \delta^{(Z)}(ZXZ)ZYZ + ZXZ\delta^{(Z)}(ZYZ),\end{aligned}$$

which implies that $\delta^{(Z)}$ is a well-defined derivation. □

**Remark 3.4.3.** *It is clear that if $\delta$ is skew-adjoint, then $\delta^{(Z)}$ is also skew-adjoint.*

From now on, we always assume that $\mathcal{A}$ is a von Neumann subalgebra of $\mathcal{M}$.

**Remark 3.4.4.** *In the special case when $\mathcal{A} = \mathcal{M}$, derivations $\delta : \mathcal{M} \to \mathcal{J}$ always vanish on $\mathcal{P}(\mathcal{Z}(\mathcal{M}))$ (see e.g. [10, Lemma 3.1]). However, it is not true for the general case when $\mathcal{A} \neq \mathcal{M}$.*

The other reduction of $\delta : \mathcal{A} \to \mathcal{J}$ we intend to use depends on the type of the algebra $\mathcal{A}$ with an additional assumption that $\delta|_{\mathcal{P}(\mathcal{Z}(\mathcal{A}))}$ vanishes. For every $Z \in \mathcal{P}(\mathcal{Z}(\mathcal{A}))$, the mapping $Z\delta(\cdot)Z$ is a derivation from $\mathcal{A}$ into $\mathcal{J}$. Moreover, if $\delta$ vanishes on $\mathcal{P}(\mathcal{Z}(\mathcal{A}))$, then $Z\delta(\cdot)Z$ is a derivation from $\mathcal{A}_Z$ into $\mathcal{J}_Z$, which coincides with $\delta(\cdot)$ on $\mathcal{A}_Z$. Let $Z_1, Z_2$ be two projections in $\mathcal{M}$ such that $Z_1Z_2 = 0$. For elements $X_1 \in \mathcal{M}_{Z_1} = Z_1\mathcal{M}Z_1$ and $X_2 \in \mathcal{M}_{Z_2} = Z_2\mathcal{M}Z_2$, we frequently identify $X_1 + X_2$ with $X_1 \oplus X_2$.

**Lemma 3.4.5.** *Let $\delta : \mathcal{A} \to \mathcal{J}$ be a derivation such that $\delta|_{\mathcal{P}(\mathcal{Z}(\mathcal{A}))} = 0$. If for $Z_1, Z_2 \in \mathcal{P}(\mathcal{Z}(\mathcal{A}))$ with $Z_1Z_2 = 0$, $\delta|_{\mathcal{A}_{Z_1}}$ and $\delta|_{\mathcal{A}_{Z_2}}$ are inner derivations implemented by $T_1 \in \mathcal{J}_{Z_1}$ and $T_2 \in \mathcal{J}_{Z_2}$, then $\delta|_{\mathcal{A}_{Z_1+Z_2}}$ is implemented by $T_1 \oplus T_2$.*

*Proof.* For every $X \in \mathcal{A}_{Z_1+Z_2}$, we have

$$\begin{aligned}\delta(X) &= \delta(XZ_1 + XZ_2) = \delta(XZ_1) + \delta(XZ_2) = \delta_{T_1}(XZ_1) + \delta_{T_2}(XZ_2) \\ &= [XZ_1, T_1] + [XZ_2, T_2] = [X, T_1 + T_2] = \delta_{T_1+T_2}(X),\end{aligned}$$

which completes the proof. □

Lemma 3.4.5 allows us to make the following reduction of the problem considered in Chapter 5.

**Remark 3.4.6.** *Let $P_1, P_2, P_3 \in \mathcal{Z}(\mathcal{A})$ be the central partition of unity (some of $P_i$ can be zero), such that $\mathcal{A}_{P_1}$ is of type $I_{fin}$, $\mathcal{A}_{P_2}$ is of type $II_1$, $\mathcal{A}_{P_3}$ is properly infinite. Assume that $\delta : \mathcal{A} \to \mathcal{J}$ vanishes on $\mathcal{Z}(\mathcal{A})$. By reducing $\delta$ to the algebras $\mathcal{A}_{P_i}$, $i = 1, 2, 3$, to prove that $\delta$ is inner derivation, it is sufficient to consider separately the cases when $\mathcal{A}$ is type I, type $II_1$ or properly infinite.*

*As we show in Section 5.2 (see Remark 5.2.4), the assumption that $\delta : \mathcal{A} \to \mathcal{C}_0(\mathcal{M}, \tau)$ vanishes on $\mathcal{Z}(\mathcal{A})$ can be imposed without loss of generality.*

Next, we introduce a special subset $K_\delta$ of the algebra $\mathcal{M}$ generated by derivation $\delta : \mathcal{A} \to \mathcal{M}$. As we prove later in Chapter 5, for any derivation $\delta : \mathcal{A} \to \mathcal{C}_0(\mathcal{M}, \tau)$, $K_\delta$ contains the operator implementing $\delta$.

**Definition 3.4.7.** *For a skew-adjoint derivation $\delta : \mathcal{A} \to \mathcal{M}$, we define by $K_\delta$ the weak* (or ultraweak) operator closure of $co\{U\delta(U^*) \mid U \in \mathcal{U}(\mathcal{A})\}$, where $co(S)$ denotes the convex hull of a set $S$.*

**Remark 3.4.8.** *Recall that, in Theorem 4.1.1, the operator implementing the derivation is in the $\|\cdot\|_{\mathcal{J}}$-closure of $co\{U\delta(U^*) \mid U \in \mathcal{U}(\mathcal{A})\}$.*

**Remark 3.4.9.** *Recall that the strong operator closure, the weak operator closure and the weak* operator closure of the convex hull of a uniformly bounded set in $\mathcal{M}$ coincide (see e.g. [132, Chapter II, Lemma 2.5] and [31, Chapter IX, Corollary 5.2]). By Ringrose's theorem (see Theorem 3.1.2), derivation $\delta : \mathcal{A} \to \mathcal{M}$ is bounded and therefore, the set $\{\delta(U) \mid U \in \mathcal{U}(\mathcal{A})\}$ is uniformly bounded. Thus,*

$$K_\delta = \overline{co}^{wo}\{U\delta(U^*) \mid U \in \mathcal{U}(\mathcal{A})\} = \overline{co}^{so}\{U\delta(U^*) \mid U \in \mathcal{U}(\mathcal{A})\}, \tag{3.1}$$

*where $\overline{co}^{so}(S)$ (respectively, $\overline{co}^{wo}(S)$) denotes the strong operator closure (respectively, weak operator closure) of convex hull of a set $S$. In particular, $\|X\|_\infty \le \|\delta\|_{\mathcal{A}\to\mathcal{M}}$ for every $X \in K_\delta$. Furthermore, since $\delta$ is assumed to be skew-adjoint, using Leibniz rule, for any unitary $U \in \mathcal{A}$, we have*

$$(U\delta(U^*))^* = -\delta(U)U^* = U\delta(U^*) - \delta(\mathbf{1}) = U\delta(U^*),$$

*which implies that every element in $K_\delta$ is self-adjoint.*

**Remark 3.4.10.** *Let $Z_1, Z_2, \cdots, Z_n \in \mathcal{Z}(\mathcal{A})$ be mutually disjoint projections such that $\delta(Z_i) = 0$ for $i = 1, 2, \cdots, n$. For every $Z_i$, we have*

$$\begin{aligned} K_\delta Z_i &= \overline{co}^{wo}\{U\delta(U^*) \mid U \in \mathcal{U}(\mathcal{A})\}Z_i = \overline{co}^{wo}\{U\delta(U^*)Z_i \mid U \in \mathcal{U}(\mathcal{A})\} \\ &= \overline{co}^{wo}\{UZ_i\delta(U^*Z_i) \mid U \in \mathcal{U}(\mathcal{A})\} = \overline{co}^{wo}\{U\delta(U^*) \mid U \in \mathcal{U}(\mathcal{A}_{Z_i})\}. \end{aligned}$$

*Since $\delta(\mathbf{1}) = 0$, it follows that $\delta(\mathbf{1} - \sum_{i=1}^n Z_i) = 0$. Therefore, since $Z_i$ are mutually disjoint, for every $U_1, U_2, \cdots U_n \in \mathcal{U}(\mathcal{A})$, we have that*

$$\begin{aligned} \sum_{i=1}^n U_i Z_i \delta(U_i^* Z_i) &= (\sum_{i=1}^n U_i Z_i)\delta(\sum_{i=1}^n U_i Z_i) \\ &= (\sum_{i=1}^n U_i Z_i + \mathbf{1} - \sum_{i=1}^n Z_i)\delta(\sum_{i=1}^n U_i Z_i + \mathbf{1} - \sum_{i=1}^n Z_i). \end{aligned}$$

*Note that $\sum_{i=1}^n U_i Z_i + \mathbf{1} - \sum_{i=1}^n Z_i \in \mathcal{U}(\mathcal{A})$. Thus, $\sum_{i=1}^n U_i Z_i \delta(U_i^* Z_i) \in K_\delta$. For any $X_1 \in K_\delta Z_1$, there is a net in $co\{U_1 Z_1 \delta(U_1^* Z_1)\}$ converging to $X_1$ in the weak operator topology. Hence, $X_1 \oplus (\oplus_{i=2}^n U_i Z_i \delta(U_i^* Z_i)) \in K_\delta$. By mathematical induction, we obtain that $\oplus_{i=1}^n X_i \in K_\delta$ for any $X_i \in K_\delta Z_i$. That is, $\sum_{i=1}^n K_\delta Z_i \subset K_\delta$.*

In the following proposition, we provide an auxiliary result which allows us to use Lemma 2.5.5 in the proof of Theorem 5.2.2.

**Proposition 3.4.11.** *Let $T \in K_\delta$ and let $\varepsilon, s > 0$. If $0 < s \le \tau(E^{|T|}(\varepsilon,\infty))$, then there is a unitary element $U \in \mathcal{U}(\mathcal{A})$ such that*

$$\int_0^{s/2} \mu(t;\delta(U))dt > \frac{s}{2}\varepsilon.$$

*Proof.* Since $T \in K_\delta$, it follows from (3.1) that there is a net $\{B_\alpha\}_\alpha$ with

$$B_\alpha := \sum_{i=1}^{n_\alpha} \lambda_\alpha^{(i)} U_\alpha^{(i)} \delta((U_\alpha^{(i)})^*), 1 \le n_\alpha < \infty,\ U_\alpha^{(i)} \in \mathcal{U}(\mathcal{A}),\ \sum_{i=1}^{n_\alpha} \lambda_\alpha^{(i)} = 1,$$

converging to $T$ in the strong operator topology. Note that every $B_\alpha$ is self-adjoint. By [106, Proposition 2.3.2], we have $|B_\alpha| \to_{so} |T|$, and therefore, employing Lemma 1.8.4, we infer that there exists a $B_\alpha$ such that $\tau(E^{|B_\alpha|}(\varepsilon,\infty)) > \frac{s}{2}$. Hence, Lemma 2.2.3 implies that

$$\mu\Big(t; \sum_{i=1}^{n_\alpha} \lambda_\alpha^{(i)} U_\alpha^{(i)} \delta((U_\alpha^{(i)})^*)\Big) = \mu(t; B_\alpha) > \varepsilon, \quad t \in [0, \frac{s}{2}]. \tag{3.2}$$

Now, it follows from [87, Theorem 3.3.3] (see also Proposition 2.3.1) that

$$\sum_{i=1}^{n_\alpha} \lambda_\alpha^{(i)} \int_0^{\frac{s}{2}} \mu(t; U_\alpha^{(i)} \delta((U_\alpha^{(i)})^*))dt \ge \int_0^{\frac{s}{2}} \mu(t; \sum_{i=1}^{n_\alpha} \lambda_\alpha^{(i)} U_\alpha^{(i)} \delta((U_\alpha^{(i)})^*))dt \overset{(3.2)}{>} \frac{s}{2}\varepsilon.$$

Thus, there exists $U_\alpha^{(i)} \in \mathcal{U}(\mathcal{A})$ such that

$$\int_0^{\frac{s}{2}} \mu(t; \delta((U_\alpha^{(i)})^*))dt \overset{(2.2)}{\ge} \int_0^{\frac{s}{2}} \mu(t; U_\alpha^{(i)} \delta((U_\alpha^{(i)})^*))dt > \frac{s}{2}\varepsilon.$$

□

# Chapter 4

# Derivations on $C^*$-subalgebras of a semifinite von Neumann algebra

Throughout this chapter, $\mathcal{M}$ is a semifinite von Neumann algebra equipped with a semifinite faithful normal trace $\tau$. The main result of this chapter characterises the ideals of $\mathcal{M}$ such that every derivation from an arbitrary $C^*$-subalgebra into $\mathcal{J}$ is inner. In particular, we answer to Question 3.2.6 in affirmative.

Theorem 4.1.4 is the main result of the present chapter, which substantially extends [81, Theorem 14] (see Theorem 3.2.5). The prototype of the proof of the following theorem for the case of Schatten ideals $C_p$ when the Hilbert space $\mathcal{H}$ is separable can be found in [56]. The main result of this chapter generalises Theorem 3.2.5 in two directions. Firstly, instead of imposing additional condition on the von Neumann subalgebra $\mathcal{A}$, we consider the case when $\mathcal{A}$ is an arbitrary $C^*$-algebra. Secondly, we extend significantly the class of symmetric ideals associated with $\mathcal{M}$ for which the result is applicable. We also demonstrate the sharpness of our assumptions on the symmetric ideals.

We note that throughout this chapter we denote symmetric space (of possible unbounded operators) affiliated with $\mathcal{M}$ by $(E(\mathcal{M},\tau), \|\cdot\|_E)$, while the corresponding ideal in $\mathcal{M}$ by $\mathcal{E}(\mathcal{M},\tau) := E(\mathcal{M},\tau) \cap \mathcal{M}$. The latter ideal is equipped with the norm $\|\cdot\|_E$, however no assumption on completeness of $\mathcal{E}(\mathcal{M},\tau)$ with respect to $\|\cdot\|_E$ is imposed.

The main result (Theorem 4.1.4) of this chapter is taken from the joint paper *Derivations with values in ideals of semifinite von Neumann algebras* [11].

## 4.1 Derivations with values in symmetric ideals

Before proceeding to the proof of the main result of this chapter, we provide a complete proof for Theorem 3.3.3. We note that the assumption of a unit element in the $C^*$-algebra in [63] can be omitted. We present the full proof for completeness of exposition.

**Theorem 4.1.1.** *Let $\mathcal{A}$ be a $C^*$-algebra and let $\mathcal{J}$ be a reflexive $\mathcal{A}$-bimodule. Then, for every derivation $\delta : \mathcal{A} \to \mathcal{J}$, there exists a $T \in \mathcal{J}$ such that $\delta = \delta_T$ and $\|T\|_{\mathcal{J}} \le \|\delta\|_{\mathcal{A}\to\mathcal{J}}$. Moreover, $T \in \overline{\mathrm{co}\{\delta(U)U^* : U \in \mathcal{U}(\mathcal{A})\}}^{\|\cdot\|_{\mathcal{J}}}$, where $\mathrm{co}(S)$ denotes the convex hull of a set $S$.*

*Proof.* We firstly consider the case when $\mathcal{A}$ is unital.

By Ringrose's theorem (see Theorem 3.1.2), the derivation $\delta : (\mathcal{A}, \|\cdot\|_\infty) \to (\mathcal{J}, \|\cdot\|_{\mathcal{J}})$ is bounded. Let us define the sets $K_{00} := \{\delta(U)U^* : U \in \mathcal{U}(\mathcal{A})\} \subset \mathcal{J}$ and $K_0 := \mathrm{co}(K_{00})$. It is clear that $K_{00}$, and therefore $K_0$, lie in the ball of radius $\|\delta\|_{\mathcal{A}\to\mathcal{J}}$ in $\mathcal{J}$.

We set $K := \overline{K_0}^{\|\cdot\|_{\mathcal{J}}}$. Note that $K$ is weakly closed in $\mathcal{J}$ (see e.g. [31, Chapter V, Theorem 1.4]). Since $\mathcal{J}$ is reflexive, it follows that $K$ is a convex weakly compact subset of $\mathcal{J}$ (see [31, Chapter V, Theorem 4.2]), contained in the ball of radius $\|\delta\|_{\mathcal{A}\to\mathcal{J}}$.

For every $U \in \mathcal{U}(\mathcal{A})$, we have $\delta(U) \in \mathcal{J}$, and therefore we can define the mapping $\alpha_U : \mathcal{J} \longrightarrow \mathcal{J}$, by setting

$$\alpha_U(X) := UXU^* + \delta(U)U^*.$$

For every $U, V \in \mathcal{U}(\mathcal{A})$, we have

$$\begin{aligned}\alpha_U(\alpha_V(X)) &= UVXV^*U^* + U\delta(V)V^*U^* + \delta(U)U^* \\ &= (UV)X(UV)^* + U\delta(V)V^*U^* + \delta(U)VV^*U^* \\ &= (UV)X(UV)^* + \delta(UV)(UV)^* = \alpha_{UV}(X).\end{aligned}$$

In addition, the equality $\delta(\mathbf{1}) = \delta(\mathbf{1}^2) = 2\delta(\mathbf{1})$ implies that $\delta(\mathbf{1}) = 0$, and therefore $\alpha_{\mathbf{1}}(X) = X$, $X \in \mathcal{J}$. Thus, $\alpha$ is an action of the group $\mathcal{U}(\mathcal{A})$ on $\mathcal{J}$.

We claim that the set $K$ is invariant with respect to $\alpha$. Since $\delta(U)U^* = \alpha_U(0)$, it follows that $K_{00}$ is an orbit of 0 with respect to $\alpha$, and therefore, is an invariant subset with respect to $\alpha$. In addition, for any positive scalars $s$ and $t$ with $s+t = 1$, we have

$$\begin{aligned}\alpha_U(sX + tY) &= sUXU^* + tUYU^* + (s+t)\delta(U)U^* \\ &= s\alpha_U(X) + t\alpha_U(Y), \quad X, Y \in \mathcal{J}.\end{aligned}$$

Hence, for every $U \in \mathcal{U}(\mathcal{A})$ the mapping $\alpha_U$ is affine, which implies that $K_0 = \mathrm{co}(K_{00})$ is also an invariant subset with respect to $\alpha$. Now, the equality $\alpha_U(X) - \alpha_U(Y) = U(X-Y)U^*$, $X, Y \in \mathcal{J}$ implies that every $\alpha_U, U \in \mathcal{U}(\mathcal{A})$, is an isometry on $\mathcal{J}$. Hence, $K$ is an invariant subset with respect to $\alpha$.

Furthermore, the fact that $\alpha_U$ is an isometry implies that the family $\{\alpha_U : U \in \mathcal{U}(\mathcal{A})\}$ is a noncontracting family of affine mappings (see e.g. [31, Chapter V, Lemma 10.7]). Clearly, $\alpha_U$ is weakly continuous for every $U \in \mathcal{U}(\mathcal{A})$. Thus, the set $K$ and the family $\{\alpha_U : U \in \mathcal{U}(\mathcal{A})\}$ satisfy the assumptions of Theorem 3.3.2. Hence, there exists a point $T \in K$ fixed with respect to $\alpha$, that is, we have $T =$

$\alpha_U(T) = UTU^* + \delta(U)U^*$ for every $U \in \mathcal{U}(\mathcal{A})$ and $\|T\|_{\mathcal{J}} \le \|\delta\|_{\mathcal{A}\to\mathcal{J}}$. Therefore $TU = UT + \delta(U)$ for every $U \in \mathcal{U}(\mathcal{A})$. Thus, $\delta(U) = [T, U]$ for every $U \in \mathcal{U}(\mathcal{A})$. Since every element $X \in \mathcal{A}$ is a linear combination of four elements from $\mathcal{U}(\mathcal{A})$, we obtain that $\delta = \delta_T$ on $\mathcal{A}$ and complete the proof for the case when $\mathcal{A}$ is unital.

Next, if $\mathcal{A}$ is not unital, then we define $\mathcal{A}_1 := \mathbb{C} \oplus \mathcal{A}$, which is a unital $C^*$-algebra equipped with the norm

$$\|(\lambda, A)\|_1 = |\lambda| + \|A\|_{\mathcal{A}}, \ \lambda \in \mathbb{C}, A \in \mathcal{A}.$$

For every $X \in \mathcal{J}$ and $A_1 = (\alpha, A), B_1 = (\beta, B) \in \mathcal{A}$, we define

$$A_1X = (\alpha, A)X = \alpha X + AX,$$

and

$$XB_1 = (\beta, B)X = \beta X + XB.$$

Define $\delta_1 : \mathcal{A}_1 \to \mathcal{J}$ by $\delta_1((\alpha, A)) = \delta(A)$, $A \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. It is clear that $\delta_1$ is a derivation from a unital $C^*$-subalgebra $\mathcal{A}_1$ into $\mathcal{J}$. Hence, the assertion follows from the proved above. $\square$

Recall that, for every reflexive symmetric function space $E(0, \infty)$, the corresponding operator space $E(\mathcal{M}, \tau)$ is also reflexive [39, Corollary 5.16] (see also [43, Theorem 54]).

**Proposition 4.1.2.** *Let $\mathcal{A}$ be a $C^*$-subalgebra of $\mathcal{M}$ and let $E(\mathcal{M}, \tau)$ be a reflexive symmetric space affiliated with $\mathcal{M}$. Then, for every derivation $\delta : \mathcal{A} \to \mathcal{E}(\mathcal{M}, \tau) := E(\mathcal{M}, \tau) \cap \mathcal{M}$, there exists an element $T \in \mathcal{E}(\mathcal{M}, \tau)$ such that $\delta = \delta_T$ with $\|T\|_E \le \|\delta\|_{\mathcal{A}\to E}$ and $\|T\|_\infty \le \|\delta\|_{\mathcal{A}\to\mathcal{M}}$.*

*Proof.* Since $E(\mathcal{M}, \tau)$ is reflexive, Theorem 4.1.1 implies that there exists a $T \in E(\mathcal{M}, \tau)$ such that $\delta = \delta_T$ and $\|T\|_E \le \|\delta\|_{\mathcal{A}\to E}$. Therefore, it remains to show that $T \in \mathcal{M}$ and $\|T\|_\infty \le \|\delta\|_{\mathcal{A}\to\mathcal{M}}$.

By the Ringrose's theorem (see Theorem 3.1.2), we have that $\delta : (\mathcal{A}, \|\cdot\|_\infty) \to (\mathcal{M}, \|\cdot\|_\infty)$ is a bounded mapping. Hence, $K_0 := \mathrm{co}\{\delta(U)U^* : U \in \mathcal{U}(\mathcal{A})\}$ lies in the ball of radius $\|\delta\|_{\mathcal{A}\to\mathcal{M}}$ in $\mathcal{M}$. By Theorem 4.1.1, we have $T \in \overline{K_0}^{\|\cdot\|_E}$. Let $\{X_n\} \subset K_0$ be such that $\|T - X_n\|_E \to 0$. By [43, Proposition 11] (see also Section 2.4), we have $X_n \to T$ in local measure topology. Since $\mathcal{M}$ has Fatou property (see Section 2.4), it follows that the closed ball in $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ with radius $\|\delta\|_{\mathcal{A}\to\mathcal{M}}$ is closed with respect to the local measure topology (see e.g. Section 2.4 or [41, Theorem 4.1]). Noting that $\|X_n\|_{\mathcal{M}} \le \|\delta\|_{\mathcal{A}\to\mathcal{M}}$ and $X_n \to T$ in local measure topology, we conclude that $T \in \mathcal{M}$ with $\|T\|_\infty \le \|\delta\|_{\mathcal{A}\to\mathcal{M}}$ . $\square$

Let $E(0, \infty)$ be a symmetric function space on $(0, \infty)$ and let $(E(\mathcal{M}, \tau), \|\cdot\|_E)$ be the corresponding noncommutative operator space. Following the notation introduced in [138], for $1 < p < \infty$, we set

$$E(\mathcal{M}, \tau)^{(p)} = \{X \in S(\mathcal{M}, \tau) : |X|^p \in E\}, \quad \|X\|_{E^{(p)}} = \||X|^p\|_E^{1/p}.$$

It is well-known (see e.g. [40, Proposition 3.1]) that $E^{(p)}(\mathcal{M},\tau) = E(\mathcal{M},\tau)^{(p)}$, where $E^{(p)}(\mathcal{M},\tau)$ is the symmetric space corresponding to the $p$-convexification $E^{(p)}(0,\infty)$ of the symmetric function space $E(0,\infty)$.

**Theorem 4.1.3.** *[38, 40, 45] Let $E(0,\infty)$ be a strongly symmetric space. Then,*

*(i). $E^{(p)}(\mathcal{M},\tau)$ is a strongly symmetric space.*

*(ii). If $E(0,\infty)$ has the Fatou property, then $E^{(p)}(\mathcal{M},\tau)$ has the Fatou property.*

*(iii). If $E(0,\infty)$ has order continuous norm, then $E^{(p)}(\mathcal{M},\tau)$ has order continuous norm.*

*(iv). If $p > 1$ and $E(0,\infty)$ is a KB-space (a fully symmetric space has order continuous norm and the Fatou property), then $E^{(p)}(\mathcal{M},\tau)$ is reflexive.*

As mentioned before, $L_1(\mathcal{M},\tau)$ is (usually) not reflexive. Therefore, Proposition 4.1.2 does not cover the case for $\mathcal{C}_1(\mathcal{M},\tau)$. Theorem 4.1.4 is the main result of the present chapter, which covers the case for $\mathcal{C}_1(\mathcal{M},\tau)$. In fact, we substantially enlarge the class of symmetric ideals to which the result is applicable.

**Theorem 4.1.4.** *Let $\mathcal{A}$ be a $C^*$-subalgebra of $\mathcal{M}$ and let $E$ be a fully symmetric function space on $(0,\infty)$ having Fatou property and order continuous norm. Then every derivation $\delta : \mathcal{A} \to \mathcal{E}(\mathcal{M},\tau)$ is inner, that is there exists an element $T \in \mathcal{E}(\mathcal{M},\tau)$ such that $\delta = \delta_T$ with $\|T\|_\infty \le \|\delta\|_{\mathcal{A}\to\mathcal{M}}$ and $\|T\|_E \le \|\delta\|_{\mathcal{A}\to E}$.*

*Proof.* Without loss of generality, we may assume that $\|\delta\|_{\mathcal{A}\to\mathcal{M}} \le 1$.

Since $\mathcal{E}(\mathcal{M},\tau) \subset \mathcal{M}$, it follows that $|X|^q$ is a bounded operator for every $X \in \mathcal{E}(\mathcal{M},\tau)$ and $q \ge 0$. Therefore, for $p \ge p' \ge 1$ and every $X \in \mathcal{E}^{(p')}(\mathcal{M},\tau) = E^{(p')}(\mathcal{M},\tau) \cap \mathcal{M}$ we have that

$$|X|^p = |X|^{p'} \cdot |X|^{p-p'} \in \mathcal{E}(\mathcal{M},\tau),$$

that is $X \in \mathcal{E}^{(p)}(\mathcal{M},\tau) = E^{(p)}(\mathcal{M},\tau) \cap \mathcal{M}$. Thus,

$$\mathcal{E}^{(p')}(\mathcal{M},\tau) \subset \mathcal{E}^{(p)}(\mathcal{M},\tau), \quad p \ge p' \ge 1. \tag{4.1}$$

In particular, from inclusion (4.1) we have that $\mathcal{E}(\mathcal{M},\tau) \subset \mathcal{E}^{(p)}(\mathcal{M},\tau)$ for every $p > 1$. Hence the derivation $\delta$ can be considered as a derivation defined on $\mathcal{A}$ with values in the symmetric ideal $\mathcal{E}^{(p)}(\mathcal{M},\tau)$. By Theorem 4.1.3 (see also [38, Propostion 5.3]), every $E^{(p)}(\mathcal{M},\tau)$, $p > 1$, is reflexive and therefore, it follows from Proposition 4.1.2 that there exists a $T_p \in \mathcal{E}^{(p)}(\mathcal{M},\tau)$ such that $\delta = \delta_{T_p}$ on $\mathcal{A}$ with $\|T_p\|_\infty \le \|\delta\|_{\mathcal{A}\to\mathcal{M}}$ and $\|T_p\|_{E^{(p)}} \le \|\delta\|_{\mathcal{A}\to E^{(p)}}$.

We note that for $p, p' > 1$ with $p \ge p'$, inclusions $T_{p'} \in \mathcal{E}^{(p')}(\mathcal{M},\tau)$ and (4.1) imply that $T_{p'} \in \mathcal{E}^{(p)}(\mathcal{M}, \tau)$. Moreover, since $\|T_{p'}\|_\infty \leq \|\delta\|_{\mathcal{A}\to\mathcal{M}} \leq 1$, we have that

$$\begin{aligned}
\|T_{p'}\|_{E^{(p)}}^p &= \big\| \, |T_{p'}|^p \big\|_E \leq \big\| \, |T_{p'}|^{p'} \big\|_E \cdot \big\| \, |T_{p'}|^{p-p'} \big\|_\infty \\
&\leq \|T_{p'}\|_{E^{(p')}}^{p'} \leq \|\delta\|_{\mathcal{A}\to E^{(p')}}^{p'} = \sup_{X\in\mathcal{A}, \|X\|_\infty=1} \Big\| |\delta(X)|^{p'} \Big\|_E \\
&\leq \sup_{X\in\mathcal{A}, \|X\|_\infty=1} \|\delta(X)\|_E \cdot \|\delta(X)\|_\infty^{p'-1} \\
&\leq \sup_{X\in\mathcal{A}, \|X\|_\infty=1} \|\delta(X)\|_E \cdot \|\delta\|_{\mathcal{A}\to\mathcal{M}}^{p'-1} \leq \|\delta\|_{\mathcal{A}\to E}.
\end{aligned} \tag{4.2}$$

We define $M := \{T_{1+\frac{1}{m}}\}_{m\in\mathbb{N}}$. Since $1 + \frac{1}{m} \leq 2, m \in \mathbb{N}$, inclusion (4.1) implies that $M \subset \mathcal{E}^{(2)}(\mathcal{M}, \tau)$. Now, let us construct inductively a subsequence $\{T_{n,m}\}_m$ of $M$ for every $n \geq 1$ such that

(i) for every fixed $n \geq 1$, $T_{n,m} \subset E^{(1+\frac{1}{n})}(\mathcal{M}, \tau), m \in \mathbb{N}$ and $T_{n,m} \to S_n \in E^{(1+\frac{1}{n})}(\mathcal{M}, \tau)$ as $m \to \infty$ in the weak topology of $E^{(1+\frac{1}{n})}(\mathcal{M}, \tau)$ with $\|S_n\|_{E^{(1+\frac{1}{n})}}^{1+\frac{1}{n}} \leq \|\delta\|_{\mathcal{A}\to E}$.

(ii) $\{T_{n+1,m}\}_m \subset \{T_{n,m}\}_m$ for every $n \geq 1$.

Let $M_{1,0} := M \subset E^{(2)}(\mathcal{M}, \tau)$ and $M_1 := \overline{\mathrm{co}(M_{1,0})}^{\|\cdot\|_{E^{(2)}}}$. It follows from [31, Chapter V, Theorem 1.4 and Theorem 4.2] that $M_1$ is a convex weakly compact subset of $E^{(2)}(\mathcal{M}, \tau)$. Hence, by the Eberlein-Šmulian Theorem (see e.g. [31, Chapter V, Section 13]), there is a subsequence $\{T_{1,m}\}$ of $M_{1,0}$ converging to an element $S_1 \in M_1 \subset E^{(2)}(\mathcal{M}, \tau)$ in the weak topology of $E^{(2)}(\mathcal{M}, \tau)$. Since $S_1 \in M_1$ and $M_1$ is the $\|\cdot\|_{E^{(2)}}$-norm closure of $\mathrm{co}(M_{1,0})$, inequality (4.2) implies that $\|S_1\|_{E^{(2)}}^2 \leq \|\delta\|_{\mathcal{A}\to E}$.

Assume that the construction up to $n-1$, $n \geq 2$, is completed. We let $M_{n,0} = \{T_{n-1,m}\}_m \cap \{T_{1+\frac{1}{m}} : m \geq n, m \in \mathbb{N}\} \subset E^{(1+\frac{1}{n})}(\mathcal{M}, \tau)$. Note, that this intersection is non-empty (and infinite) as the elements of $\{T_{n-1,m}\}_m$ are chosen from the sequence $\{T_{1+\frac{1}{m}}\}_{m\in\mathbb{N}}$. We set $M_n := \overline{\mathrm{co}(M_{n,0})}^{\|\cdot\|_{E^{(1+\frac{1}{n})}}}$. It follows again from [31, Chapter V, Theorem 1.4 and Theorem 4.2] that $M_n$ is a convex weakly compact subset of $E^{(1+\frac{1}{n})}(\mathcal{M}, \tau)$. Then, by the Eberlein-Šmulian Theorem [31, Chapter V, Section 13], there is a subsequence $\{T_{n,m}\}_m$ of $M_{n,0}$ converging to an $S_n \in M_n \subset E^{(1+\frac{1}{n})}(\mathcal{M}, \tau)$ in the weak topology of $E^{(1+\frac{1}{n})}(\mathcal{M}, \tau)$, in particular, $\|S_n\|_{E^{(1+\frac{1}{n})}}^{1+\frac{1}{n}} \leq \|\delta\|_{\mathcal{A}\to E}$, which completes the induction.

Now, we show that every $S_n$ belongs to $\mathcal{M}$. For every $n \geq 1$, there is a sequence $\{X_{n,m}\} \subset \mathrm{co}(M_{n,0})$ such that $\|X_{n,m} - S_n\|_{E^{(1+\frac{1}{n})}} \to 0$ as $m \to \infty$. Hence, by [43, Proposition 11] (see also Section 2.4), we have $X_{n,m} \to S_n$ as $m \to \infty$ in local measure topology. It follows from [41, Theorem 4.1] (see also Section 2.4) that the closed ball of radius $\|\delta\|_{\mathcal{A}\to\mathcal{M}}$ of $(\mathcal{M}, \|\cdot\|_\infty)$ is closed with respect to the local measure topology. Since $\mathrm{co}(M_{n,0})$ lies in the closed ball of radius $\|\delta\|_{\mathcal{A}\to\mathcal{M}}$ of $(\mathcal{M}, \|\cdot\|_\infty)$, it follows that $S_n \in \mathcal{M}$ and $\|S_n\|_\infty \leq \|\delta\|_{\mathcal{A}\to\mathcal{M}}$.

We claim that all of the $S_n$ are the same. Since $S_n$ and $S_{n+1}$ are $\tau$-compact operators, the operator $S_n - S_{n+1}$ is also $\tau$-compact. Let $S_n - S_{n+1} = U|S_n - S_{n+1}|$ be the polar decomposition. Then, for any $\varepsilon > 0$, $E^{|S_n - S_{n+1}|}(\varepsilon, \infty)$ is a $\tau$-finite projection. Hence, by [43, Proposition 23 and Lemma 25 (ii)] we have that

$E^{|S_n-S_{n+1}|}(\varepsilon,\infty) \in E^{(1+\frac{1}{n})}(\mathcal{M},\tau)^\times$ for every $n \in \mathbb{N}$. Since the Köthe dual space $E^{(1+\frac{1}{n})}(\mathcal{M},\tau)^\times$ can be identified with the subspace of the Banach dual, conditions (i) and (ii) imply that

$$\begin{aligned}\tau(S_nE^{|S_n-S_{n+1}|}(\varepsilon,\infty)U^*) &\overset{(i)}{=} \lim_{m\to\infty} \tau(T_{n,m}E^{|S_n-S_{n+1}|}(\varepsilon,\infty)U^*)\\ &\overset{(ii)}{=} \lim_{m\to\infty} \tau(T_{n+1,m}E^{|S_n-S_{n+1}|}(\varepsilon,\infty)U^*)\\ &\overset{(i)}{=} \tau(S_{n+1}E^{|S_n-S_{n+1}|}(\varepsilon,\infty)U^*)\end{aligned}$$

and therefore

$$\tau(|S_n - S_{n+1}|E^{|S_n-S_{n+1}|}(\varepsilon,\infty)) = \tau(U^*(S_n - S_{n+1})E^{|S_n-S_{n+1}|}(\varepsilon,\infty)) = 0$$

for any $\varepsilon > 0$, which implies that $S_n = S_{n+1}$. In what follows we denote $S_n$ by $T$. In particular, we have $\|T\|_\infty \le \|\delta\|_{\mathcal{A}\to\mathcal{M}}$ and $\|T\|^p_{E^{(p)}} \le \|\delta\|_{\mathcal{A}\to E}$ for every $p \in (1,2]$.

Next, we claim that $\delta = \delta_T$. Consider $\mathcal{E}(\mathcal{M},\tau)$ as a subspace of $E^{(2)}(\mathcal{M},\tau)$. For every $X \in \mathcal{A}$, $\delta_{T_p}(X) = \delta(X)$ for every $p > 1$. By condition (i) above, we have $T_{1,m} \to T$ in weak topology of $E^{(2)}(\mathcal{M},\tau)$. Thus, for every $f \in (E^{(2)}(\mathcal{M},\tau))^*$ and $X \in \mathcal{A}$, we have $f(T_{1,m}X) \to f(TX)$ and $f(XT_{1,m}) \to f(XT)$ as $m \to \infty$, which implies that $f(\delta_{T_{1,m}}(X)) \to f(\delta_T(X))$ as $m \to \infty$. That is, $\delta_{T_{1,m}}(X) \to \delta_T(X)$ in the weak topology of $E^{(2)}(\mathcal{M},\tau)$ as $m \to \infty$. On the other hand, every $\delta_{T_{1,m}}(X)$, $m \in \mathbb{N}$, is equal to $\delta(X)$, and therefore, we conclude that $\delta(X) = \delta_{T_{1,m}}(X) = \delta_T(X)$ for every $m$, and therefore $\delta = \delta_T$ on $\mathcal{A}$.

By the construction of $T$, we have that $T \in \cap_{p>1}E^{(p)}(\mathcal{M},\tau)$ and $\||T|^p\|_E = \|T\|^p_{E^{(p)}} \le \|\delta\|_{\mathcal{A}\to E}$ for every $p \in (1,2]$ . Since $\|T\|_\infty \le \|\delta\|_{\mathcal{A}\to\mathcal{M}} \le 1$, we have $|T|^p \uparrow |T|$ as $p \downarrow 1$. Since $(E(\mathcal{M},\tau), \|\cdot\|_E)$ has Fatou property, we have $T \in E(\mathcal{M},\tau)$ with $\|T\|_E \le \|\delta\|_{\mathcal{A}\to E}$, which completes our proof. □

It is well-known that the space $L_p(0,\infty)$ is fully symmetric and has Fatou property and order continuous norm. Therefore, as an immediate corollary of Theorem 4.1.4, we obtain the following result extending the earlier results by Kaftal and Weiss [81]. In particular, we answer Question 3.2.6 in affirmative.

**Corollary 4.1.5.** *Let $\mathcal{A}$ be a $C^*$-subalgebra of $\mathcal{M}$ and let $\delta : \mathcal{A} \to \mathcal{C}_p(\mathcal{M},\tau)$, $p \ge 1$, be a derivation. Then, there exists an element $T \in \mathcal{C}_p(\mathcal{M},\tau)$ such that $\delta = \delta_T$ and $\|T\|_p \le \|\delta\|_{\mathcal{A}\to\mathcal{L}_p}$.*

**Remark 4.1.6.** *We note that the assumption that the derivation $\delta$ takes values in an ideal of $\mathcal{M}$ is crucial for the proof of Theorem 4.1.4. The technique applied there is not applicable for the extension of Theorem 4.1.1 and Corollary 3.3.5 to the case of more general symmetric operator spaces (of possibly unbounded) operators affiliated with $\mathcal{M}$.*

We provide an alternative proof by the following extension of Proposition 4.1.2.

**Theorem 4.1.7.** *Let $\mathcal{A}$ be a $C^*$-subalgebra of $\mathcal{M}$ and let $E(\mathcal{M},\tau)$ be a reflexive symmetric space affiliated with $\mathcal{M}$. Assume that $F(\mathcal{M},\tau)$ is a strongly symmetric* *space having the Fatou property. Then, for any derivation $\delta : \mathcal{A} \to F(\mathcal{M},\tau) \cap E(\mathcal{M},\tau)$, there exists an element $T \in F(\mathcal{M},\tau) \cap E(\mathcal{M},\tau)$ such that $\delta = \delta_T$ with $\|T\|_F \le \|\delta\|_{\mathcal{A}\to F}$ and $\|T\|_F \le \|\delta\|_{\mathcal{A}\to E}$.*

*Proof.* Note that $\delta : \mathcal{A} \to F(\mathcal{M},\tau)$ can be considered as a derivation defined on $\mathcal{A}$ with values in $E(\mathcal{M},\tau)$. Since $E(\mathcal{M},\tau)$ is reflexive, Theorem 4.1.1 implies that there exists a $T \in E(\mathcal{M},\tau)$ such that $\delta = \delta_T$ and $\|T\|_E \le \|\delta\|_{\mathcal{A}\to E}$. Therefore, it remains to show that $T \in F(\mathcal{M},\tau)$ and $\|T\|_\infty \le \|\delta\|_{\mathcal{A}\to F}$.

By the Ringrose's theorem (see Theorem 3.1.2), we have that $\delta : (\mathcal{A}, \|\cdot\|_\infty) \to (F(\mathcal{M},\tau), \|\cdot\|_F)$ is a bounded mapping. Hence, $K_0 := \mathrm{co}\{\delta(U)U^* : U \in \mathcal{U}(\mathcal{A})\}$ lies in the ball of radius $\|\delta\|_{\mathcal{A}\to F}$ in $\mathcal{M}$. By Theorem 4.1.1, we have $T \in \overline{K_0}^{\|\cdot\|_E}$. Let $\{X_n\} \subset K_0$ be such that $\|T - X_n\|_E \longrightarrow 0$. By [43, Proposition 11] (see also Section 2.4), we have $X_n \longrightarrow T$ in local measure topology. Since $F(\mathcal{M},\tau)$ has Fatou property (see Section 2.4), it follows that the closed ball in $(F(\mathcal{M},\tau), \|\cdot\|_F)$ with radius $\|\delta\|_{\mathcal{A}\to F}$ is closed with respect to the local measure topology (see e.g. Section 2.4 or [43, Theorem 32]). Noting that $\|X_n\|_F \le \|\delta\|_{\mathcal{A}\to F}$ and $X_n \longrightarrow T$ in local measure topology, we conclude that $T \in \mathcal{M}$ with $\|T\|_\infty \le \|\delta\|_{\mathcal{A}\to F}$ . □

Theorem 4.1.4 is an straightforward consequence of Theorem 4.1.7. Indeed, recall that if $E(0,\infty)$ is a fully symmetric $KB$-space, then the $p$-convexification $E^{(p)}(\mathcal{M},\tau)$ of $E(\mathcal{M},\tau)$ is reflexive. By (4.1), we have $\mathcal{E}(\mathcal{M},\tau) \subset E^{(p)}(\mathcal{M},\tau)$ for any $1 < p < \infty$. To apply Theorem 4.1.7, one only need to notice that $\mathcal{E}(\mathcal{M},\tau)$ equipped with the norm $\|X\|_{\mathcal{E}} := \max\{\|X\|_\infty, \|X\|_E\}$ is a strongly symmetric space having the Fatou property.

## 4.2 Non-inner derivations

In conclusion we consider special cases, where by omitting various assumptions on the space $E(\mathcal{M},\tau)$ and considering a smaller space, we construct examples of non-inner derivations, which extends Example 3.1.9.

Let $E(\mathcal{M},\tau)$ be a symmetric space. In the following theorem, we consider $E_0(\mathcal{M},\tau)$, which is the $\|\cdot\|_E$-closure of all operators of $\tau$-finite rank (the range/support projection is $\tau$-finite) in $E(\mathcal{M},\tau)$ (see e.g. [43]). We note that $E_0(\mathcal{M},\tau) = E(\mathcal{M},\tau)$ if $E(\mathcal{M},\tau)$ has order continuous norm, and $E_0(\mathcal{M},\tau)$ has no Fatou property whenever $E_0(\mathcal{M},\tau) \ne E(\mathcal{M},\tau)$. For examples of sequence spaces when $E_0(\mathcal{M},\tau) \ne E(\mathcal{M},\tau)$, we refer the reader to [52, Chapter III, Section 6].

**Theorem 4.2.1.** [1] *Let $\mathcal{M}$ be a semifinite non-finite factor. If $E_0(\mathcal{M},\tau) \ne E(\mathcal{M},\tau)$, then we can always find a non-inner derivation from a $C^*$-subalgebra of $\mathcal{M}$ into $E_0(\mathcal{M},\tau) \cap \mathcal{M}$.*

*Proof.* We claim that $E_0(\mathcal{M},\tau) \subset S_0(\mathcal{M},\tau)$. Assume by contradiction that $E_0(\mathcal{M},\tau) \not\subseteq S_0(\mathcal{M},\tau)$. Then, there exists an operator $T \in E_0(\mathcal{M},\tau)$ with

[1] The original statement in [11, Theorem 3.8] requires the assumption that the symmetric space $E(\mathcal{M},\tau)$ is $\tau$-compact. However, in Theorem 4.2.1, this assumption is omitted.

$T \notin S_0(\mathcal{M}, \tau)$. That is, $\mu(\infty; T) > 0$. By the definition of symmetric spaces, we obtain that $\mathcal{M} \subset E_0(\mathcal{M}, \tau)$. For every $X \in E(\mathcal{M}, \tau)$, there exists an $n > 0$ such that $XE^{|X|}(n, \infty)$ of $\tau$-finite rank. Hence, $X = XE^{|X|}(n, \infty) + XE^{|X|}(0, n] \subset E_0(\mathcal{M}, \tau)$. That is, $E_0(\mathcal{M}, \tau) = E(\mathcal{M}, \tau)$, which is a contradiction.

Let $T \in \mathcal{M} \cap (E(\mathcal{M}, \tau) \setminus (E_0(\mathcal{M}, \tau) + \mathbb{C}\,\mathbf{1}))$. We assert that such an element $T$ exists. Assume that $E(\mathcal{M}, \tau) \subset S_0(\mathcal{M}, \tau)$ and $X \in E(\mathcal{M}, \tau) \setminus E_0(\mathcal{M}, \tau)$. For every $n > 0$, $XE^{|X|}(n, \infty) \in E_0(\mathcal{M}, \tau)$ and therefore, $XE^{|X|}(0, n] \notin E_0(\mathcal{M}, \tau)$. In this case, we define $T := XE^{|X|}(0, n]$, which is a bounded $\tau$-compact operator and not in $E_0(\mathcal{M}, \tau) + \mathbb{C}\,\mathbf{1}$. If $E(\mathcal{M}, \tau) \supset \mathcal{M}$, then $T$ can be chosen as any operator in $\mathcal{M} \setminus (\mathcal{C}_0(\mathcal{M}, \tau) + \mathbb{C}\,\mathbf{1})$.

We claim that $\delta_T$ is a non-inner derivation from some $C^*$-subalgebra of $\mathcal{M}$ into $E_0(\mathcal{M}, \tau) \cap \mathcal{M}$. Consider $\delta_T$ acting on $\mathcal{C}_0(\mathcal{M}, \tau)$. For every $X \in \mathcal{C}_0(\mathcal{M}, \tau)$, $E^{|X|}(\varepsilon, \infty)$ is $\tau$-finite for every $\varepsilon > 0$. Thus,

$$\|TX - TXE^{|X|}(\varepsilon, \infty)\|_E \leq \|TXE^{|X|}(0, \varepsilon]\|_E \leq \varepsilon \|T\|_E$$

implies that $TX \in E_0(\mathcal{M}, \tau)$. Similarly, $XT \in E_0(\mathcal{M}, \tau)$ and therefore $\delta_T(\mathcal{C}_0(\mathcal{M}, \tau)) \subset E_0(\mathcal{M}, \tau)$. Moreover, $T \in \mathcal{M}$ and $\mathcal{C}_0(\mathcal{M}, \tau) \subset \mathcal{M}$ imply that $\delta_T(\mathcal{C}_0(\mathcal{M}, \tau)) \subset E_0(\mathcal{M}, \tau) \cap \mathcal{M}$. Finally, if there exists an operator $K \in E_0(\mathcal{M}, \tau) \cap \mathcal{M}$ such that $\delta_T = \delta_K$, then $T - K \in \mathcal{C}_0(\mathcal{M}, \tau)'$. For every $B \in \mathcal{C}_0(\mathcal{M}, \tau)$ and $A \in \mathcal{C}_0(\mathcal{M}, \tau)'$, we have $BA = AB$. Then, noticing that $\mathcal{M}$ is the weak operator closure of $\mathcal{C}_0(\mathcal{M}, \tau)$ (see e.g. [87, Definition 2.6.8]), we have $BA = AB$ for every $B \in \mathcal{M}$ and $A \in \mathcal{C}_0(\mathcal{M}, \tau)'$ and therefore $\mathcal{C}_0(\mathcal{M}, \tau)' \subset \mathcal{M}'$. Since $\mathcal{M}' \subset \mathcal{C}_0(\mathcal{M}, \tau)'$, we have $\mathcal{C}_0(\mathcal{M}, \tau)' = \mathcal{M}'$. However, $\mathcal{M}$ is a factor and therefore $T - K \in \mathbb{C}\mathbf{1}$, which is a contradiction with the choice of $T$. $\square$

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. Noticing that $K(\mathcal{H})$ has order continuous norm, one can see from Example 3.1.9 that the assumption that $E(0, \infty)$ has Fatou property in Theorem 4.1.4 can not be omitted.

# Chapter 5

# Derivations on von Neumann subalgebras of a semifinite von Neumann algebra

One of the main results in this chapter is a semifinite version of the Johnson–Parrott–Popa theorem (see Theorem 3.2.1). The first attempt of establishing a semifinite Johnson–Parrott–Popa theorem was due to Kaftal and Weiss [81]. They considered derivations with values in the compact ideal $\mathcal{J}(\mathcal{M})$ of a semifinite von Neumann algebra $\mathcal{M}$, which is generated by all finite projections of $\mathcal{M}$. This result was later extended by Popa and Rădulescu [112]. Moreover, Popa and Rădulescu showed that there exist non-inner derivations into $\mathcal{J}(\mathcal{M})$. However, for a general semifinite von Neumann algebra $\mathcal{M}$, there is another notion of compactness, so-called $\tau$-compactness, which comes from a semifinite faithful normal trace $\tau$ defined on $\mathcal{M}$ (see Section 2.5). In this chapter, we consider derivations into the $\tau$-compact ideal $\mathcal{C}_0(\mathcal{M},\tau)$ of $\mathcal{M}$. Although, the ideals $\mathcal{C}_0(\mathcal{M},\tau)$ and $\mathcal{J}(\mathcal{M})$ are quite similar in many respects, the main result (Theorem 5.6.1) of this chapter is in strong contrast with Example 3.2.3, which is somewhat unexpected. Namely, the additional assumption on the type $I$ summand of the von Neumann subalgebra which plays an important role in the proof of Theorem 3.2.2 could be dispensed with in our current setting.

For an arbitrary von Neumann algebra $\mathcal{M}$ and any (not necessarily closed) ideal $\mathcal{E}$ of $\mathcal{M}$, it is known that any derivation $\delta : \mathcal{M} \to \mathcal{E}$ is inner (see Theorem 3.2.7). For a semifinite von Neumann algebra $\mathcal{M}$ equipped with a faithful normal semifinite trace $\tau$, and a $C^*$-subalgebra $\mathcal{A}$ of $\mathcal{M}$, it is proved that a derivation $\delta$ defined on $\mathcal{A}$ is necessarily inner provided that the values $\delta$ belong to the ideal $E(\mathcal{M},\tau) \cap \mathcal{M}$ generated by a fully symmetric $KB$-space $E(0,\infty)$ (see Theorem 4.1.4). So, it is natural to ask whether every derivation acting on a von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$ into an arbitrary proper ideal of $\mathcal{M}$ is inner. Popa and Rădulescu (see Example 3.2.3) constructed an example showing that this question has a negative answer.

The main object of this chapter is to characterise the symmetric ideals of $\mathcal{M}$ such that the following question has an affirmative answer.

**Question 5.0.1.** *Assume that $\mathcal{M}$ is a semifinite von Neumann algebra equipped with a faithful normal semifinite trace $\tau$. Let $\mathcal{E}$ be a symmetric ideal of $\mathcal{M}$ and let $\mathcal{A}$ be a von Neumann subalgebra of $\mathcal{M}$. Is every derivation $\delta$ from $\mathcal{A}$ into $\mathcal{E}$ inner?*

As an application of Theorem 5.6.1, we gives an affirmative answer to Question 5.0.1 for 'almost' every **proper** symmetric ideal $\mathcal{E}$ in $\mathcal{M}$ (see Theorem 5.6.3), that is, for most proper symmetric ideals $\mathcal{E}$ in $\mathcal{M}$, derivations from an arbitrary von Neumann subalgebra of $\mathcal{M}$ into $\mathcal{E}$ are automatically inner. In particular, Theorem 5.6.3 is a unification of the Johnson–Parrott–Popa theorem (see Theorem 3.2.1) and the result by Kaftal and Weiss (Theorem 3.2.5) with significant extension, which also answer Question 3.2.6 in affirmative.

One should note that the class of symmetric ideals characterized in this chapter (see Theorem 5.6.3) covers almost every ideal $\mathcal{E}$ corresponding to a symmetric function space in the sense of Calkin (see [82,87]). In the meantime, the ideal $\mathcal{J}(\mathcal{M})$ of all compact operators in $\mathcal{M}$ does not correspond to any symmetric function space whenever $\mathcal{M} \neq \mathcal{J}(\mathcal{M}) \neq \mathcal{C}_0(\mathcal{M},\tau)$ (see e.g. [112, Section 8] or Example 3.2.3).

The main result of this chapter is taken from the joint paper *Derivations with values in the ideal of $\tau$-compact operators affiliated with a semifinite von Neumann algebra* [12].

## 5.1 Preliminaries

Throughout this chapter, we assume that $\mathcal{M}$ is a semifinite von Neumann algebra equipped with a semifinite faithful normal trace $\tau$. In this section, we consider derivations $\delta$ from an arbitrary von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$ into $\mathcal{C}_0(\mathcal{M},\tau)$. Without loss of generality, we always assume that $\delta$ is skew-adjoint (see Section 3.4.5).

Recall that our aim is to show that any derivation $\delta : \mathcal{A} \to \mathcal{C}_0(\mathcal{M},\tau)$ inner. Hence, if we have a central partition of unity $\{Z_i\}$ of $\mathcal{A}$ such that $\delta$ is inner on every $\mathcal{A}_{Z_i}$ and is implemented by $K_i \in Z_i\mathcal{C}_0(\mathcal{M},\tau)Z_i$, then a natural choice of element implementing $\delta$ on $\mathcal{A}$ is $\oplus_i K_i$ (see Lemma 3.4.5). However, it can happen that $K_i \in Z_i\mathcal{C}_0(\mathcal{M},\tau)Z_i$, but $\oplus_i K_i \notin \mathcal{C}_0(\mathcal{M},\tau)$ (as an example, consider the algebra $\mathcal{M} = L_\infty(0,\infty)$ and partition $\{Z_i\} = \{\chi_{(i,i+1]}\}$). The latter fact is in direct contrast with [112, 2.11.], since if $\{Z_i\}_i$ is a central partition of the identity of $\mathcal{M}$, then the direct sum of a family of uniformly bounded operators $K_i \in \mathcal{J}(\mathcal{M}_{Z_i})$ is also in $\mathcal{J}(\mathcal{M})$. We tackle this issue for $\tau$-compact operators, by showing that under additional assumption that every operator $K_i$ is chosen from $Z_iK_\delta$ (see Definition 3.4.7), the direct sum $K := \oplus_i K_i$ is also $\tau$-compact.

**Theorem 5.1.1.** *Let $\mathcal{A}$ be a von Neumann subalgebra of $\mathcal{M}$ and let $\{Z_i \in \mathcal{P}(\mathcal{A})\}_i$ be a central partition of the unity in $\mathcal{A}$. Assume that $\delta(Z) = 0$ for every $Z \in \mathcal{Z}(\mathcal{A})$. If there exists $K_i \in \mathcal{C}_0(\mathcal{M}_{Z_i},\tau) \cap Z_iK_\delta$ such that $\delta = \delta_{K_i}$ on $\mathcal{A}_{Z_i}$ for every $i$, then $K := \oplus_i K_i \in \mathcal{C}_0(\mathcal{M},\tau) \cap K_\delta$ with $\delta = \delta_K$ on $\mathcal{A}$.*

*Proof.* Note that the operators $K_i$ and $K$ are self-adjoint. Since $\delta(Z_i)=0$ for every $i$, it follows that $Z_i\delta(X)=\delta(Z_iXZ_i)=Z_i\delta(X)Z_i=\delta(X)Z_i$ for every $X\in\mathcal{A}$. Hence, the fact that $\{Z_i\}$ is a central partition of unity, together with the assumption that $\delta=\delta_{K_i}$ on $\mathcal{A}_{Z_i}$ implies that for every $X\in\mathcal{A}$, we have

$$\begin{aligned}\delta(X)&=\oplus_i(Z_i\delta(X))=\oplus_i(Z_i\delta(X)Z_i)=\oplus_i\delta(Z_iXZ_i)\\&=\oplus_i\delta_{K_i}(Z_iXZ_i)=\oplus_i\delta_{K_i}(X)=\delta_K(X).\end{aligned}$$

We assert that $K\in K_\delta$. Let $\tau_{\mathcal{Z}(\mathcal{A})}$ be a semifinite faithful normal trace $\mathcal{Z}(\mathcal{A})$. It follows that there is an increasing net $\{R_\lambda\}\subset\{\mathcal{Z}(\mathcal{A})\}$ of $\tau_{\mathcal{Z}(\mathcal{A})}$-finite projections such that $R_\lambda\uparrow\mathbf{1}$. Since $R_\lambda$ is $\tau_{\mathcal{Z}(\mathcal{A})}$-finite, it follows that the reduced von Neumann algebra $\mathcal{Z}(\mathcal{A})_{R_\lambda}$ is finite and countably decomposable for every $R_\lambda$ (see e.g. [103, Theorem 1.3.6] for a proof of this fact). Thus, for every fixed $\lambda$, there are only countably many $Z_i$ such that $Z_iR_\lambda\neq 0$. We denote the sequence consists of non-zero elements from $\{Z_iR_\lambda\}$ by $\{P_n\}_{n=1}^\infty$. Note that for every $k$, we have

$$P_kKP_k\in P_kK_\delta P_k.$$

By Remark 3.4.10, $\oplus_{k=1}^nP_kKP_k\in K_\delta R_\lambda$ for every $n$. Since $\sum_{k=1}^\infty P_k=R_\lambda$, it follows that

$$R_\lambda KR_\lambda=\oplus_{k=1}^\infty P_kKP_k\in K_\delta R_\lambda.$$

Since $R_\lambda\in\mathcal{Z}(\mathcal{A})$, by Remark 3.4.10 again, we have that

$$R_\lambda KR_\lambda\in K_\delta R_\lambda\subset K_\delta.$$

Since $R_\lambda KR_\lambda\to_{so}K$, we obtain that $K\in K_\delta$.

Now, we prove that $K$ is $\tau$-compact. If the net $\{Z_i\}$ consists of finitely many projections, then $K$ is clearly $\tau$-compact. We assume that $\{Z_i\}$ contains infinitely many projections and $K\notin\mathcal{C}_0(\mathcal{M},\tau)$. By the definition of $\mathcal{C}_0(\mathcal{M},\tau)$, there exists an $\varepsilon>0$ such that $\infty=\tau(E^{|K|}(\varepsilon,\infty))=\tau(E^{\oplus_i|K_i|}(\varepsilon,\infty))$. Noting that $\tau$ is completely additive (see e.g. [133, Chapter VII, Theorem 1.11]), we obtain that $\sum_i\tau(E^{|K_i|}(\varepsilon,\infty))=\infty$. Hence, we can choose countably many distinct $T_j:=K_{i(j)}$ from $\{K_i\}$ such that

$$\tau(E^{\oplus_{j=1}^\infty|T_j|}(\varepsilon,\infty))=\sum_{j=1}^\infty\tau(E^{|T_j|}(\varepsilon,\infty))=\sum_{j=1}^\infty t_j=\infty,$$

where $t_j:=\tau(E^{|T_j|}(\varepsilon,\infty))\in(0,\infty)$, $1\le j<\infty$. We denote $Z_{i(j)}$ by $Q_j$.

Note that for every $1\le j<\infty$, we have

$$T_j\in\overline{co}^{so}\{U\delta(U^*)|U\in\mathcal{U}(\mathcal{A}_{Q_j})\}.$$

For every $j$, by Proposition 3.4.11, we can choose a $U_j\in\mathcal{U}(\mathcal{A}_{Q_j})$ such that

$$\int_0^{\frac{t_j}{2}}\mu(t;\delta(U_j))dt>\frac{t_j}{2}\varepsilon. \tag{5.1}$$

Let $U := \oplus_{j=1}^{\infty} U_j \in \mathcal{A}$. Since $\delta$ vanishes on $\{Q_j\}$, it follows that $\delta(U_j) = \delta(Q_j U Q_j) = Q_j \delta(U) Q_j$. Thus, for every $n$, we have

$$\int_0^{\sum_{j=1}^n \frac{t_j}{2}} \mu(t;\delta(U))dt \overset{(2.7)}{\geq} \sum_{j=1}^n \int_0^{\frac{t_j}{2}} \mu(t;\delta(U_j))dt \overset{(5.1)}{\geq} \sum_{j=1}^n \frac{t_j}{2}\varepsilon.$$

Noticing that $\delta(U) \in \mathcal{M}$ and recalling that $\sum_{j=1}^{\infty} t_j = \infty$, by Lemma 2.5.5, we obtain that $\delta(U)$ is not $\tau$-compact, which is a contradiction and hence $K \in \mathcal{C}_0(\mathcal{M},\tau)$. $\square$

We end this section by showing a fine property of inner derivations $\delta : \mathcal{A} \to \mathcal{C}_0(\mathcal{M},\tau)$, which is related to the set $K_\delta$. As we show in Proposition 5.1.2 below, for any inner derivation $\delta : \mathcal{A} \to \mathcal{C}_0(\mathcal{M},\tau)$, there exists an operator $T' \in K_\delta$ implementing $\delta$. We note that analogous property for inner derivations from $\mathcal{A}$ into $\mathcal{J}(\mathcal{M})$ is established in [112], however, our approach is completely different from that used in the proof of [112, Lemma 4.6]. Furthermore, our result holds if the assumption on $\mathcal{A}$ is relaxed to a weaker assumption that $\mathcal{A}$ is a unital (that is, $\mathbf{1}_{\mathcal{A}} = \mathbf{1}_{\mathcal{M}}$) $C^*$-subalgebra of $\mathcal{M}$.

**Proposition 5.1.2.** *Let $\mathcal{N}$ be a unital $C^*$-subalgebra of $\mathcal{M}$ and let $\delta : \mathcal{N} \to \mathcal{C}_0(\mathcal{M},\tau)$ be a derivation. If there exists $T \in \mathcal{C}_0(\mathcal{M},\tau)$ such that $\delta = \delta_T$, then there exists an element $T' \in K_\delta = \overline{co}^{wo}\{U\delta(U^*) \mid U \in \mathcal{U}(\mathcal{N})\}$ such that $\delta = \delta_{T'}$.*

*Proof.* Let $P_n := E^{|T|}(\frac{1}{n},\infty)$ and let $T_n := TP_n$. For every $n \in \mathcal{N}$, the projection $P_n$ is $\tau$-finite and $\|T - T_n\|_\infty \leq \frac{1}{n}$. In particular, $T_n \in L_2(\mathcal{M},\tau)$, where $L_2(\mathcal{M},\tau)$ denotes the noncommutative $L_2$-space affiliated with $\mathcal{M}$. Hence, $\delta_{T_n}$ has range inside $L_2(\mathcal{M},\tau) \cap \mathcal{M}$ and therefore, by Theorem 4.1.1 (see also [11, Theorem 3.1 and Proposition 3.4]), there exists

$$T_n' \in \overline{co\{U\delta_{T_n}(U^*) \mid U \in \mathcal{U}(\mathcal{N})\}}^{\|\cdot\|_2} = \overline{co\{T_n - UT_nU^* \mid U \in \mathcal{U}(\mathcal{N})\}}^{\|\cdot\|_2}$$

such that $\|T_n'\|_\infty \leq \|\delta_{T_n}\|_\infty \leq 2\|T_n\|_\infty \leq 2\|T\|_\infty$ and $\delta_{T_n} = \delta_{T_n'}$. Hence, by [31, Chapter IX, Proposition 5.5], there is a (wo)-cluster point $T' \in \cap_n \overline{\{T_n', T_{n+1}', \cdots\}}^{wo}$ for the sequence $\{T_n'\}$ in the ball of radius $2\|T\|_\infty$ in $\mathcal{M}$.

Since $\|\cdot\|_2$ induces the strong operator topology, and the strong operator closure and the weak operator closure of the convex hull of a uniformly bounded set coincide, it follows that $T_n' \in \overline{co}^{wo}\{T_n - UT_nU^* \mid U \in \mathcal{U}(\mathcal{N})\}$. Since $\|T_n - UT_nU^* - (T - UTU^*)\|_\infty \leq \frac{2}{n}$, it follows from the Kaplansky density theorem (see e.g. [132, Chapter II, Theorem 4.8]) that there is an element $B_n \in \overline{co}^{wo}\{T - UTU^* \mid U \in \mathcal{U}(\mathcal{N})\} = K_\delta$ such that

$$\|T_n' - B_n\|_\infty \leq \frac{2}{n}.$$

Thus, $T'$ is a (wo)-cluster point of $\{B_n\}$ and therefore $T' \in K_\delta$.

For every $X \in \mathcal{N}$, $\eta, \xi \in \mathcal{H}$, we set $\omega(\cdot) = \langle \cdot X\eta, \xi\rangle$ and $\rho(\cdot) = \langle \cdot\eta, X^*\xi\rangle$ on $\mathcal{M}$. For every $\varepsilon > 0$, there exists $N > 2/\varepsilon$ such that

$$|\omega(T' - T_N')|,\ |\rho(T' - T_N')| < \varepsilon.$$

Recall that $\delta_{T_N} = \delta_{T'_N}$. We have

$$\begin{aligned}|\langle [T-T',X]\eta,\xi\rangle| &\le |\langle [T-T_N,X]\eta,\xi\rangle| + |\langle [T_N-T'_N,X]\eta,\xi\rangle| + |\langle [T'_N-T',X]\eta,\xi\rangle| \\ &\le |\langle [T-T_N,X]\eta,\xi\rangle| + |\langle [T'_N-T',X]\eta,\xi\rangle| \\ &\le |\langle (T-T_N)X\eta,\xi)\rangle| + |\langle X(T-T_N)\eta,\xi)\rangle| \\ &\quad + |\omega(T'_N-T')| + |\rho(T'_N-T')| \\ &\le \frac{2}{N}\|X\|_\infty\|\eta\|_{\mathcal{H}}\|\xi\|_{\mathcal{H}} + 2\varepsilon \le \varepsilon\|X\|_\infty\|\eta\|_{\mathcal{H}}\|\xi\|_{\mathcal{H}} + 2\varepsilon.\end{aligned}$$

Since $\varepsilon$ is arbitrary, we infer that $[T-T',X]=0$ for every $X\in\mathcal{N}$. Hence, $T-T'\in\mathcal{N}'$ and therefore $\delta=\delta_T=\delta_{T'}$. □

## 5.2 The abelian case

In this section, we consider derivations $\delta:\mathcal{A}\to\mathcal{C}_0(\mathcal{M},\tau)$, where $\mathcal{A}$ is an abelian von Neumann subalgebra of $\mathcal{M}$. We show that in this case, any derivation $\delta$ is inner. In particular, this result allows us to assume in the following sections that we work with derivations vanishing on the center of the subalgebra $\mathcal{A}$ of $\mathcal{M}$.

We note that even though $\mathcal{C}_0(\mathcal{M},\tau)\subset\mathcal{J}(\mathcal{M})$ and $\mathcal{C}_0(\mathcal{M},\tau)$ behaves somewhat like $\mathcal{J}(\mathcal{M})$, the additional restrictions to the abelian subalgebra $\mathcal{A}$ in [81, 112] are no longer required. Moreover, since $\mathcal{C}_0(\mathcal{M},\tau)$ is not necessarily the dual space of a Banach space, the techniques used in [81, Theorem 14] are not applicable in this case.

Throughout this section, we assume that $\mathcal{A}$ is an abelian von Neumann subalgebra of $\mathcal{M}$.

Let $\delta:\mathcal{A}\to\mathcal{M}$ be a derivation. The following result is well-known (see e.g. [81, Section 3] and [67, Theorem 2.1]).

**Proposition 5.2.1.** *If $\mathcal{A}$ is an abelian von Neumann subalgebra of $\mathcal{M}$, then every derivation $\delta$ from $\mathcal{A}$ into $\mathcal{M}$ is inner. That is, $\delta=\delta_T$ for some $T\in\mathcal{M}$.*

In what follows, we consider derivations $\delta:\mathcal{A}\to\mathcal{C}_0(\mathcal{M},\tau)$. Since $\mathcal{C}_0(\mathcal{M},\tau)\subset\mathcal{M}$, $\delta$ is a derivation from $\mathcal{A}$ into $\mathcal{M}$ and therefore there exists an operator $T\in\mathcal{M}$ such that $\delta=\delta_T$. Thus, our aim in this section is to show that $T$ can be chosen to be $\tau$-compact.

Recall that an expectation $\Phi$ is a norm one projection from $B(\mathcal{H})$ onto a von Neumann algebra (see [32, Section 8]). Motivated by the idea related to an expectation from $B(\mathcal{H})$ onto $\mathcal{A}'$ used in [32, Theorem 10.9], we prove the main theorem of this section by techniques different from those used in [81], extending the results in [81] to the case of $\mathcal{C}_0(\mathcal{M},\tau)$.

**Theorem 5.2.2.** *Assume that $\mathcal{A}$ is an abelian von Neumann subalgebra of $\mathcal{M}$. For every derivation $\delta:\mathcal{A}\to\mathcal{C}_0(\mathcal{M},\tau)$, there exists $K\in\mathcal{C}_0(\mathcal{M},\tau)\cap K_\delta$ such that $\delta=\delta_K$. In particular, $\delta$ is inner.*

*Proof.* Without loss of generality, we may assume that $\delta$ is a skew-adjoint derivation from $\mathcal{A}$ into $\mathcal{M}$ (see Section 5.1). Proposition 5.2.1 guarantees that there exists $T \in \mathcal{M}$ such that $\delta = \delta_T$. In particular, we may assume that $T$ is self-adjoint (see Remark 3.4.1). Let $\Phi$ be an expectation from $B(\mathcal{H})$ onto $\mathcal{A}'$ given by [32, Theorem 8.3]. By the construction of $\Phi$ (see [32, Theorem 8.3], see also [81]), we have that $\Phi(T)$ belongs to the weak* operator closed convex hull of $\{UTU^* : U \in \mathcal{U}(\mathcal{A})\}$. In particular, $\Phi(T) \in \mathcal{M}$. Set $K := T - \Phi(T) \in \mathcal{M}$. It is clear that $\delta = \delta_K$ and $K$ belongs to the weak* operator closure of the convex hull of $\{U\delta(U^*) \mid U \in \mathcal{U}(\mathcal{A})\}$. It suffices to prove that $K \in \mathcal{C}_0(\mathcal{M}, \tau)$. Since $T$ is self-adjoint, it follows that $K$ is also self-adjoint.

Assume by contradiction that $K \notin \mathcal{C}_0(\mathcal{M}, \tau)$, i.e., there is an $\varepsilon > 0$ such that $\mu(\infty; K) > \varepsilon$. We claim that there exists $A \in \mathcal{A}$ such that $\delta(A)$ is not $\tau$-compact. To this end, we intend to use Lemma 2.5.5. For convenience, we divide the proof into several steps.

**(a)** Let

$$\mathcal{P} := \{P \in \mathcal{P}(\mathcal{A}) \mid \mu(\infty; PKP) > \varepsilon\}.$$

We claim that there is a maximal downwards directed chain $\{P_\gamma\}$ of infinitely many elements in $\mathcal{P}$ which satisfies $P_0 := \inf\{P_\gamma\} \notin \mathcal{P}$ and $P_\gamma - P_0 \in \mathcal{P}$ for every $\gamma$.

It is clear that $\mathcal{P}$ is not empty as $\mathbf{1} \in \mathcal{P}$. We note, in addition, that $\tau(P) = \infty$ for any $P \in \mathcal{P}$. Take an arbitrary $P \in \mathcal{P}$. Assume that $P$ is minimal in $\mathcal{A}$. Note that $PTP = 0$ (see e.g. the argument in the proof of [81, Lemma 8]). Since $P \in \mathcal{A} \subset \mathcal{A}'$, it follows from [32, Theorem 8.1] that

$$PKP = PTP - P\Phi(T)P = PTP - \Phi(PTP) = 0,$$

which is a contradiction to $P \in \mathcal{P}$. Thus, $\mathcal{P}$ contains no minimal element in $\mathcal{A}$.

Now, let $Q \in \mathcal{P}(\mathcal{A})$ be such that $0 \neq Q \lneqq P$ and let $Q_1 = Q, Q_2 = P - Q$. We have

$$\begin{aligned} PKP &= Q_1KQ_1 + Q_2KQ_2 + Q_1KQ_2 + Q_2KQ_1 \\ &= Q_1KQ_1 + Q_2KQ_2 + \delta(Q_1)Q_2 + \delta(Q_2)Q_1, \end{aligned} \tag{5.2}$$

where we used the fact that $Q_1 \perp Q_2$ and $\delta = \delta_K$ for the second equality. Since $\mu(\infty; PKP) > \varepsilon$, Theorem 2.2.2 (see also [87, Corollary 2.3.16]) implies that

$$\begin{aligned} &\mu(t; Q_1KQ_1 + Q_2KQ_2) + \mu(s_1; \delta(Q_1)Q_2) + \mu(s_2; \delta(Q_2)Q_1) \\ &\geq \mu(t + s_1 + s_2; PKP) \geq \mu(\infty; PKP) > \varepsilon \end{aligned}$$

for all $t, s_1, s_2 > 0$. Let $\varepsilon_1$ be such that $\mu(\infty; PKP) > \varepsilon_1 > \varepsilon$. Since $\mu(s_1; \delta(Q_1)Q_2), \mu(s_2; \delta(Q_2)Q_1) \to 0$ as $s_1, s_2 \to \infty$, we have

$$\mu(t; Q_1KQ_1 + Q_2KQ_2) > \varepsilon_1 > \varepsilon,\ t \in [0, \infty). \tag{5.3}$$

Assume that both projections $E^{|Q_1KQ_1|}(\varepsilon_1, \infty)$ and $E^{|Q_2KQ_2|}(\varepsilon_1, \infty)$ are $\tau$-finite. Since $Q_1 \perp Q_2$, it follows that $E^{|Q_1KQ_1|}(\varepsilon_1, \infty) + E^{|Q_2KQ_2|}(\varepsilon_1, \infty) =$

$E^{|Q_1KQ_1+Q_2KQ_2|}(\varepsilon_1,\infty)$, and therefore

$$\begin{aligned}\tau(E^{|Q_1KQ_1+Q_2KQ_2|}(\varepsilon_1,\infty)) &= \tau(E^{|Q_1KQ_1|+|Q_2KQ_2|}(\varepsilon_1,\infty)) \\ &= \tau(E^{|Q_1KQ_1|}(\varepsilon_1,\infty)) + \tau(E^{|Q_2KQ_2|}(\varepsilon_1,\infty)) < \infty.\end{aligned}$$

By Lemma 2.2.3, this is a contradiction to (5.3). Hence, either $\tau(E^{|Q_1KQ_1|}(\varepsilon_1,\infty)) = \infty$ or $\tau(E^{|Q_2KQ_2|}(\varepsilon_1,\infty)) = \infty$. By Lemma 2.2.3, either $\mu(\infty;Q_1KQ_1) \geq \varepsilon_1 > \varepsilon$ or $\mu(\infty;Q_2KQ_2) \geq \varepsilon_1 > \varepsilon$, which implies that either $Q_1$ or $Q_2$ belongs to $\mathcal{P}$. This shows that $\mathcal{P}$ has no minimal elements, that is, for every element $P \in \mathcal{P}$, we can always find an element $Q \in \mathcal{P}$ such that $Q \leq P$. Moreover, if $P \in \mathcal{P}$ and $Q \in \mathcal{P}(\mathcal{A})$ such that $Q \leq P$, then either $Q$ or $P - Q$ belongs to $\mathcal{P}$.

Let $\{P_\gamma\}$ be a maximal downwards directed chain in $\mathcal{P}$ and let $P_0 = \inf\{P_\gamma\}$. Obviously, $P_0 \notin \mathcal{P}$. Otherwise, there exists a $P \lneqq P_0$ with $P \in \mathcal{P}$, which contradicts the maximality of $\{P_\gamma\}$. By the property stated in the above paragraph, either $P_\gamma - P_0$ or $P_0$ must belong to $\mathcal{P}$. However, $P_0 \notin \mathcal{P}$. Thus, $P_\gamma - P_0 \in \mathcal{P}$.

**(b)** Now, let us construct a sequence $\gamma_1 \succ \gamma_2 \succ \cdots$ such that the projection $Q_k := P_{\gamma_k} - P_{\gamma_{k+1}}$ satisfies

$$\mu(t;Q_kKQ_k) > \varepsilon,\ t \in [0,2]. \tag{5.4}$$

Take an arbitrary $\gamma$ and set $\gamma_1 = \gamma$. Assume that the sequence $\gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_n$ is constructed for some $n \in \mathbb{N}$. Let $A_n := (P_{\gamma_n} - P_0)K(P_{\gamma_n} - P_0)$. We have that $A_n^* = A_n$. Furthermore, since $P_\gamma - P_0 \in \mathcal{P}$, it follows that $\mu(\infty;A_n) > \varepsilon$, which guarantees that $\tau(E^{|A_n|}(\varepsilon,\infty)) = \infty$ (see Lemma 2.2.3). Since $\mathbf{1} - P_\gamma + P_0 \uparrow \mathbf{1}$, it follows from [106, Proposition 2.3.2] that

$$so - \lim_\gamma |(\mathbf{1} - P_\gamma + P_0)A_n(\mathbf{1} - P_\gamma + P_0)| = |A_n|.$$

Then, by Lemma 1.8.4, we have

$$\liminf_\gamma \tau(E^{|(\mathbf{1}-P_\gamma+P_0)A_n(\mathbf{1}-P_\gamma+P_0)|}(\varepsilon,\infty)) \geq \tau(E^{|A_n|}(\varepsilon,\infty)) = \infty.$$

Hence, we can find $\gamma_{n+1} \succ \gamma_n$ such that $\tau(E^{|(\mathbf{1}-P_{\gamma_{n+1}}+P_0)A_n(\mathbf{1}-P_{\gamma_{n+1}}+P_0)|}(\varepsilon,\infty)) > 2$, and therefore, by Lemma 2.2.3, we have

$$\mu(t;(\mathbf{1} - P_{\gamma_{n+1}} + P_0)A_n(\mathbf{1} - P_{\gamma_{n+1}} + P_0)) > \varepsilon,\ t \in [0,2]. \tag{5.5}$$

Since $P_\gamma \downarrow$, it follows that $(P_{\gamma_n} - P_0)(\mathbf{1} - P_{\gamma_{n+1}} + P_0) = (P_{\gamma_n} - P_{\gamma_{n+1}})$, which implies that, setting $Q_n := P_{\gamma_n} - P_{\gamma_{n+1}}$, we obtain that $\mu(t;Q_nKQ_n) > \varepsilon$ for all $t \in [0,2]$.

**(c)** We claim that for every $k \in \mathbb{N}$, there is a $U_k \in \mathcal{U}(\mathcal{A})$ such that

$$\int_0^1 \mu(t;Q_k\delta(U_k)Q_k)dt > \varepsilon. \tag{5.6}$$

Since $\Phi(K) = \Phi(T - \Phi(T)) = 0$, by [32, Theorem 8.3], the operator

$$Q_kKQ_k = Q_k(K - \Phi(K))Q_k$$

belongs to the weak* operator closure of

$$co\{Q_k(K - UKU^*)Q_k : U \in \mathcal{U}(\mathcal{A})\} = co\{Q_k U\delta(U^*)Q_k : U \in \mathcal{U}(\mathcal{A})\}.$$

Since $\mathcal{A}$ is abelian, it follows that $Q_k\delta(\cdot)Q_k$ is a derivation from $\mathcal{A}$ into $\mathcal{C}_0(\mathcal{M},\tau)$ (see Section 5.1). Thus, $Q_kKQ_k \in K_{Q_k\delta(\cdot)Q_k}$. By the construction of $Q_k$ and Proposition 3.4.11, we conclude that $U_k \in \mathcal{U}(\mathcal{A})$ satisfying (5.6) exists.

**(d)** Finally, since $Q_k, U_k \in \mathcal{A}$, $Q_i \perp Q_j$ $(i \neq j)$ and $\mathcal{A}$ is abelian, the series $\sum_{k=1}^{\infty} Q_kU_k$ converges in $\mathcal{A}$ in the strong operator topology. We define

$$A := \sum_{k=1}^{\infty} Q_kU_k.$$

Since $\mathcal{A}$ is abelian, it follows that

$$Q_k\delta(Q_k^{\perp}X)Q_k = Q_k\delta(Q_k^{\perp}XQ_k^{\perp})Q_k = Q_kQ_k^{\perp}X\delta(Q_k^{\perp})Q_k + Q_k\delta(Q_k^{\perp}X)Q_k^{\perp}Q_k = 0$$

for every $X \in \mathcal{A}$, and therefore,

$$\begin{aligned} Q_k\delta(A)Q_k &= Q_k\delta(Q_kA)Q_k + Q_k\delta(Q_k^{\perp}A)Q_k \\ &= Q_k\delta(Q_kU_k)Q_k \\ &= Q_k\delta(Q_kU_k)Q_k + Q_k\delta(Q_k^{\perp}U_k)Q_k \\ &= Q_k\delta(U_k)Q_k. \end{aligned} \tag{5.7}$$

Hence, we obtain

$$\int_0^1 \mu(t; Q_k\delta(A)Q_k)dt \overset{(5.7)}{=} \int_0^1 \mu(t; Q_k\delta(U_k)Q_k)dt \overset{(5.6)}{>} \varepsilon. \tag{5.8}$$

Take an arbitrary $n \geq 1$. Since $Q_i \perp Q_j$ for $i \neq j$, it follows that

$$\int_0^n \mu(t; \delta(A))dt \overset{(2.7)}{\geq} \sum_{i=1}^{n} \int_0^1 \mu(t; Q_i\delta(A)Q_i)dt \overset{(5.8)}{>} n \cdot \varepsilon.$$

Now, by Lemma 2.5.5, we obtain that $\delta(A)$ is not $\tau$-compact, which is a contradiction. Thus, $K \in \mathcal{C}_0(\mathcal{M},\tau)$ as requipred. □

**Remark 5.2.3.** *Note that the so-called locally compatible condition on the abelian von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$ is required in studying derivations from $\mathcal{A}$ into $\mathcal{J}(\mathcal{M})$ (see [112, Proposition 4.3], see also [81]). When this condition is not fulfilled, derivations from $\mathcal{A}$ into $\mathcal{J}(\mathcal{M})$ are not necessarily inner (see Example 3.2.3). However, the "locally compatible" condition is redundant in our present setting, that is, the result of Theorem 5.2.2 holds without any additional assumption on the abelian subalgebra $\mathcal{A}$ of $\mathcal{M}$.*

**Remark 5.2.4.** *Assume that $\mathcal{A}$ is a von Neumann subalgebra of $\mathcal{M}$. By Theorem 5.2.2, for every derivation $\delta : \mathcal{A} \to \mathcal{C}_0(\mathcal{M},\tau)$, $\delta|_{\mathcal{Z}(\mathcal{A})}$ is implemented by a $\tau$-compact operator $K$. Hence, in the study of derivations $\delta : \mathcal{A} \to \mathcal{C}_0(\mathcal{M},\tau)$, we can consider linear mapping $\delta - \delta_K$ which is a derivation from $\mathcal{A}$ into $\mathcal{C}_0(\mathcal{M},\tau)$ vanishing on $\mathcal{Z}(\mathcal{A})$. That is, without loss of generality, we may assume that derivation $\delta$ vanishes on $\mathcal{Z}(\mathcal{A})$.*

## 5.3 The properly infinite Case

In this section, we show that any derivation $\delta$ on a properly infinite von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$ with values in the ideal $\mathcal{C}_0(\mathcal{M},\tau)$ is necessarily inner. We note, that the result for derivations from an abelian subalgebra of $\mathcal{M}$ (see Theorem 5.2.2) allows us to use the same approach for properly infinite algebras as in [81] (see also [67] and [32]).

Recall that if $\mathcal{A}$ is properly infinite von Neumann subalgebra of a semifinite von Neumann algebra $\mathcal{M}$, then there is an infinite countable decomposition of the identity into mutually orthogonal projections of $\mathcal{A}$, all equivalent in $\mathcal{A}$ to $\mathbf{1}$, and thus a fortiori equivalent in $\mathcal{M}$ to $\mathbf{1}$ [34, Part III, Chapter 8, Section 6, Corollary 2] (see also [78]).

Let $H_0 = \ell^2(\mathbb{Z})$. By [34, Part I, Section 2.4, Proposition 5], there is a *spatial* isomorphism

$$\phi : \mathcal{M} \to \tilde{\mathcal{M}} = \mathcal{M} \otimes B(H_0) \tag{5.9}$$

with

$$\phi(\mathcal{A}) = \tilde{\mathcal{A}} = \mathcal{A} \otimes B(H_0).$$

It is well-known [34, Section 1.5, Proposition 8] that a spatial isomorphism is isometric and is normal, i.e., for every bounded increasing net $\{X_i \in \mathcal{M}_+\}_i$ satisfying $X_i \uparrow X$, we have $\phi(X_i) \uparrow \phi(X)$. Recall also that the elements $B \in \tilde{\mathcal{M}}$ (or $\tilde{\mathcal{A}}$) are represented by matrices $[B_{ij}]$, $i,j \in \mathbb{Z}$, with entries in $\mathcal{M}$(or $\mathcal{A}$) by the formula

$$(\mathbf{1} \otimes E_{ij})B(\mathbf{1} \otimes E_{kl}) = B_{jk} \otimes E_{il},$$

where $E_{ij}$ is the canonical matrix unit of $B(H_0)$. In particular, if $\mathcal{L}$ (respectively, $\mathcal{D}$) is the maximal abelian subalgebras of $B(H_0)$ of Laurent (respectively, diagonal) matrices, then $B \in \mathcal{M}\otimes\mathcal{L}$ (respectively, $B \in \mathcal{M}\otimes\mathcal{D}$) if and only if $[B_{ij}]$ is a Laurent (respectively, a diagonal) matrix with entries in $\mathcal{M}$, i.e., $B_{ij} = B_{i-j}$ (respectively, $B_{ij} = \delta_{ij}B_{ii}$, where $\delta_{ij}$ stands for the Kronecker Delta), $i,j \in \mathbb{Z}$, where $B_k$ denotes the entry along the $k$th diagonal for all $k \in \mathbb{Z}$.

Let $\tau_0$ be the standard trace on $B(H_0)$ and $\tilde{\tau} := \tau \otimes \tau_0$. For the properties of tensor products of von Neumann algebras, we refer the reader to [133, Chapter IV]. It is well-known that the isomorphism $\phi$ introduced in (5.9) is trace-preserving.

Before we proceed to the proof of the main result of this section (see Theorem 5.3.3 below), we establish several properties of the isomorphism $\phi$ introduced in (5.9) related to the generalised singular value functions and $\tau$-compact operators.

**Proposition 5.3.1.** *Let $\phi$ be the spatial isomorphism from $\mathcal{M}$ onto $\tilde{\mathcal{M}}$ introduced in* (5.9) *. Then, for any $X \in \mathcal{M}$, we have*

*(i).* $\mu(X) = \mu(\phi(X))$.

*(ii).* $\mu(X) = \mu(X \otimes E_{00})$.

*Proof.* (i). Since $\phi$ is an isometric, trace-preserving isomorphism from $\mathcal{M}$ onto $\tilde{\mathcal{M}}$, it follows from the definition of generalised singular value function (see Definition 2.2.1) that

$$\begin{aligned}\mu(t;X) &= \inf\{\|XP\|_\infty : P \in \mathcal{P}(\mathcal{M}), \tau(\mathbf{1}-P) \le t\}\\ &= \inf\{\|\phi(X)\phi(P)\|_\infty : P \in \mathcal{P}(\mathcal{M}), \tau(\mathbf{1}-P) \le t\}\\ &= \inf\{\|\phi(X)\tilde{P}\|_\infty : \tilde{P} \in \mathcal{P}(\tilde{\mathcal{M}}), \tilde{\tau}(\mathbf{1}-\tilde{P}) \le t\} = \mu(t;\phi(X)).\end{aligned}$$

(ii). For every $t > 0$, we have

$$\begin{aligned}d_{|X\otimes E_{00}|}(t) &= \tilde{\tau}(E^{|X\otimes E_{00}|}(t,\infty))\\ &= \tilde{\tau}(E^{|X|\otimes E_{00}}(t,\infty))\\ &= \tilde{\tau}(E^{|X|}(t,\infty)\otimes E_{00})\\ &= \tau(E^{|X|}(t,\infty)) = d_{|X|}(t).\end{aligned}$$

Thus, by (2.1), we have $\mu(t; X \otimes E_{00}) = \mu(t;X)$. $\square$

**Proposition 5.3.2.** *Let $\phi$ be the spatial isomorphism from $\mathcal{M}$ onto $\tilde{\mathcal{M}}$ as in (5.9). We have that*

*(i). $\mathcal{C}_0(\tilde{\mathcal{M}},\tilde{\tau}) = \phi(\mathcal{C}_0(\mathcal{M},\tau))$;*

*(ii). If $K \otimes E_{00} \in \mathcal{C}_0(\tilde{\mathcal{M}},\tilde{\tau})$, then $K \in \mathcal{C}_0(\mathcal{M},\tau)$;*

*(iii). $(\mathcal{M}\otimes\mathcal{L}) \cap \mathcal{C}_0(\tilde{\mathcal{M}},\tilde{\tau}) = \{0\}$.*

*Proof.* Part (i) immediately follows from Proposition 5.3.1.

(ii). Suppose that $K \otimes E_{00} \in \mathcal{C}_0(\tilde{\mathcal{M}},\tilde{\tau})$. On one hand, part (i) guarantees that $\phi^{-1}(K\otimes E_{00}) \in \mathcal{C}_0(\mathcal{M},\tau)$. On the other hand, by Proposition 5.3.1, we have that $\mu(\phi^{-1}(K\otimes E_{00})) = \mu(K)$. Hence, we conclude that $K \in \mathcal{C}_0(\mathcal{M},\tau)$.

(iii). Let $\mathcal{J}(\tilde{\mathcal{M}})$ be the norm closure of the linear space of all finite projections of $\tilde{\mathcal{M}}$. Since $\mathcal{C}_0(\tilde{\mathcal{M}},\tilde{\tau}) \subset \mathcal{J}(\tilde{\mathcal{M}})$ (see section 2), it follows from [81, Lemma 12 (b)] that $(\mathcal{M}\otimes\mathcal{L}) \cap \mathcal{C}_0(\tilde{\mathcal{M}},\tilde{\tau}) = \{0\}$. $\square$

The lifting technique used in [81] (see also [32,67]) and the already proven abelian case play crucial roles in proving Theorem 5.3.3. However, we can simplify the proof since the condition that $\mathcal{A}$ contains the center of $\mathcal{M}$ imposed in [81, Theorem 4] is not required in Theorem 5.2.2.

**Theorem 5.3.3.** *Let $\mathcal{A}$ be a properly infinite von Neumann subalgebra of $\mathcal{M}$. For every derivation $\delta : \mathcal{A} \to \mathcal{C}_0(\mathcal{M},\tau)$, there exists $T \in \mathcal{C}_0(\mathcal{M},\tau) \cap K_\delta$ such that $\delta = \delta_T$ on $\mathcal{A}$.*

*Proof.* Let $\tilde{\delta} = \phi\circ\delta\circ\phi^{-1}$, where $\phi$ is a spatial isomorphism as in (5.9). Clearly, $\tilde{\delta}$ is also a derivation, and, by Proposition 5.3.2, we have that

$$\tilde{\delta} : \tilde{\mathcal{A}} \to \phi(\mathcal{C}_0(\mathcal{M},\tau)) = \mathcal{C}_0(\tilde{\mathcal{M}},\tilde{\tau}).$$

Let us define the following von Neumann algebras:

$$\tilde{\mathcal{A}}_1 = \mathbf{1}\otimes\mathcal{L}, \quad \mathcal{A}_1 = \phi^{-1}(\tilde{\mathcal{A}}_1), \quad \tilde{\mathcal{A}}_2 = \mathcal{A}\otimes\mathcal{L} \quad \text{and} \quad \tilde{\mathcal{A}}_3 = \mathcal{A}_1\otimes\mathcal{D}.$$

By [132, Chapter IV, Theorem 5.9 and Corollary 5.10], we have

$$\begin{aligned}\tilde{\mathcal{A}}_1' \cap \mathcal{C}_0(\tilde{\mathcal{M}}, \tilde{\tau}) &= (\mathbf{1} \otimes \mathcal{L})' \cap ((\mathcal{M} \otimes B(H_0) \cap \mathcal{C}_0(\tilde{\mathcal{M}}, \tilde{\tau})) \\ &= (\mathcal{M} \otimes \mathcal{L}') \cap \mathcal{C}_0(\tilde{\mathcal{M}}, \tilde{\tau}) \\ &= (\mathcal{M} \otimes \mathcal{L}) \cap \mathcal{C}_0(\tilde{\mathcal{M}}, \tilde{\tau}) \\ &\overset{P.\ 5.3.2\ (iii)}{=} \{0\}. \end{aligned} \tag{5.10}$$

Since the isomorphism $\phi$ is spatial, we infer that

$$\mathcal{A}_1' \cap \mathcal{C}_0(\mathcal{M}, \tau) = \phi^{-1}(\tilde{\mathcal{A}}_1') \cap \mathcal{C}_0(\mathcal{M}, \tau) = \phi^{-1}\left(\tilde{\mathcal{A}}_1' \cap \mathcal{C}_0(\tilde{\mathcal{M}}, \tilde{\tau})\right) = \{0\}. \tag{5.11}$$

We now study derivation $\tilde{\delta}$ on each of the algebras $\tilde{\mathcal{A}}_j$, $j = 1, 2, 3$, separately.

Since $\tilde{\mathcal{A}}_1$ is abelian, Theorem 5.2.2 applied to the derivation $\tilde{\delta}|_{\tilde{\mathcal{A}}_1}$ guarantees the existence of $T_1 \in \mathcal{C}_0(\tilde{\mathcal{M}}, \tilde{\tau})$ such that

$$\tilde{\delta}_1 := \tilde{\delta} - \delta_{T_1}$$

vanishes on $\tilde{\mathcal{A}}_1$. Moreover, $T_1 \in \overline{co}^{wo}\{U\tilde{\delta}(U^*) \mid U \in \mathcal{U}(\tilde{\mathcal{A}}_1)\}$.

Note that $\tilde{\mathcal{A}}_2 = \mathcal{A} \otimes \mathcal{L} \subset \mathcal{M} \otimes \mathcal{L} \subset \mathbf{1}' \otimes \mathcal{L} = \tilde{\mathcal{A}}_1'$. For any $A_1 \in \tilde{A}_1$ and $A_2 \in \tilde{A}_2$, we have

$$A_1\tilde{\delta}_1(A_2) = \tilde{\delta}_1(A_1A_2) = \tilde{\delta}_1(A_2A_1) = \tilde{\delta}_1(A_2)A_1,$$

that is, $\tilde{\delta}_1(A_2) \in \tilde{\mathcal{A}}_1'$. Therefore, it follows from (5.10) that

$$\tilde{\delta}_1(\tilde{\mathcal{A}}_2) \subset \tilde{\mathcal{A}}_1' \cap \mathcal{C}_0(\tilde{\mathcal{M}}, \tilde{\tau}) = \{0\},$$

which implies that the derivation $\tilde{\delta}_1$ also vanishes on $\tilde{\mathcal{A}}_2$.

Next, we consider $\tilde{\delta}_1$ on the algebra $\tilde{\mathcal{A}}_3$. Since $\tilde{\mathcal{A}}_1$ is abelian, it follows that $\mathcal{A}_1$ is also abelian and therefore, $\tilde{\mathcal{A}}_3$ is also abelian. Thus, we can apply Theorem 5.2.2 to the derivation $\tilde{\delta}_1|_{\tilde{\mathcal{A}}_3}$ to infer that there is a $T_2 \in \mathcal{C}_0(\tilde{\mathcal{M}}, \tilde{\tau})$ such that $\tilde{\delta}_1 = \delta_{T_2}$ on $\tilde{\mathcal{A}}_3$. We claim that $T_2 = 0$, that is, $\tilde{\delta}_1$ vanishes on $\tilde{\mathcal{A}}_3$.

Since $\mathcal{A}_1 \otimes \mathbf{1} \subset \mathcal{A} \otimes \mathbf{1} \subset \mathcal{A} \otimes \mathcal{L} = \tilde{\mathcal{A}}_2$, $\mathcal{A}_1 \otimes \mathbf{1} \subset \tilde{\mathcal{A}}_3$ and $\tilde{\delta}_1$ vanishes on $\tilde{\mathcal{A}}_2$, we have $\delta_{T_2}$ vanishes on $\mathcal{A}_1 \otimes \mathbf{1}$, i.e.,

$$T_2 \in (\mathcal{A}_1 \otimes \mathbf{1})' \cap \mathcal{C}_0(\tilde{\mathcal{M}}, \tilde{\tau}) = (\mathcal{A}_1' \otimes B(H_0)) \cap \mathcal{C}_0(\tilde{\mathcal{M}}, \tilde{\tau}).$$

Hence, for all $i, j \in \mathbb{Z}$, we have that $(T_2)_{ij} \in \mathcal{A}_1'$ and

$$(T_2)_{ij} \otimes E_{00} = (\mathbf{1} \otimes E_{0i})T_2(\mathbf{1} \otimes E_{j0}) \in \mathcal{C}_0(\tilde{\mathcal{M}}, \tilde{\tau}).$$

By Proposition 5.3.2 (ii), the latter condition implies that $(T_2)_{ij} \in \mathcal{C}_0(\mathcal{M}, \tau)$ and therefore, $(T_2)_{ij} \in \mathcal{A}_1' \cap \mathcal{C}_0(\mathcal{M}, \tau)$ for all $i, j \in \mathbb{Z}$. Appealing to (5.11), we conclude that $(T_2)_{ij} = 0$ for all $i, j \in \mathbb{Z}$, so $T_2 = 0$. Thus, the derivation $\tilde{\delta}_1$ vanishes on $\tilde{\mathcal{A}}_3$. In particular, $\tilde{\delta}_1$ vanishes on $\mathbf{1} \otimes \mathcal{D}$.

Finally, we claim that $\tilde{\delta}_1$ vanishes on $\tilde{\mathcal{A}}$, which would imply that $\tilde{\delta} = \delta_{T_1}$. Since $\mathcal{L}$ and $\mathcal{D}$ generate $B(H_0)$ in weak* operator topology, we have $\tilde{\mathcal{A}}_2 = \mathcal{A} \otimes \mathcal{L}$ and $\mathbf{1} \otimes \mathcal{D}$ generate $\tilde{\mathcal{A}}$ in weak* operator topology. Since $\tilde{\delta}$ is weak* topology continuous (see [67, Lemma 1.3]), it follows that $\tilde{\delta}_1 = \tilde{\delta} - \delta_{T_1} = 0$, i.e., $\tilde{\delta} = \delta_{T_1}$ on $\tilde{\mathcal{A}}$. Then, for every $X \in \tilde{\mathcal{A}}$, we have

$$\phi(\delta(\phi^{-1}(X))) = \tilde{\delta}(X) = \delta_{T_1}(X) = XT_1 - T_1X$$

and therefore

$$\delta(\phi^{-1}(X)) = \phi^{-1}(XT_1 - T_1X) = \phi^{-1}(X)\phi^{-1}(T_1) - \phi^{-1}(T_1)\phi^{-1}(X). \tag{5.12}$$

Since $\phi$ is a isomorphism from $\mathcal{A}$ onto $\tilde{\mathcal{A}}$, (5.12) implies that $\delta(Y) = \delta_{\phi^{-1}(T_1)}(Y)$ for every $Y \in \mathcal{A}$. Since $T_1 \in \mathcal{C}_0(\tilde{\mathcal{M}}, \tilde{\tau}) \cap \overline{co}^{wo}\{U\tilde{\delta}(U^*) \mid U \in \mathcal{U}(\tilde{\mathcal{A}}_1)\}$, we have that $\phi^{-1}(T_1) \in \mathcal{C}_0(\mathcal{M}, \tau)$ and $\phi^{-1}(T_1) \in \overline{co}^{wo}\{U\delta(U^*) \mid U \in \mathcal{U}(\mathcal{A}_1)\} \subset K_\delta$, which completes the proof. □

## 5.4 The type $I$ case

In this section, we consider the case when $\mathcal{A}$ is an arbitrary type $I$ von Neumann subalgebra of $\mathcal{M}$. Before we proceed to the proof for the type $I$ case, we need the following proposition.

**Proposition 5.4.1.** *Let $\mathcal{A}$ be a type $I_n$ von Neumann subalgebra of $\mathcal{M}$, $n \in \mathbb{N}$. Then, every derivation $\delta$ from $\mathcal{A}$ into $\mathcal{C}_0(\mathcal{M}, \tau)$ is inner, i.e., $\delta = \delta_E$ for some $E \in \mathcal{C}_0(\mathcal{M}, \tau)$. Moreover, $E \in co\{U\delta(U^*) \mid U \in \mathcal{U}(\mathcal{A})\}$.*

*Proof.* By [120, Theorem 2.3.3], we have $\mathcal{A} = M_n \otimes \mathcal{Z}(\mathcal{A})$, where $M_n$ stands for the algebra of all $n \times n$ matrices. For the sake of convenience, we denote $\mathcal{A} = M_n \otimes \mathcal{Z}(\mathcal{A})$ by $M_n(\mathcal{Z}(\mathcal{A}))$, and $E_{ij} \otimes \mathbf{1}_{\mathcal{A}}$ by $B_{ij}$, where $E_{ij}$ is the standard matrix units of $M_n$. In particular, every $A \in \mathcal{A}$ is in the form of $\sum_{i,j=1}^n A_{ij}B_{ij}$, $A_{ij} \in \mathcal{Z}(\mathcal{A})$.

We define

$$D_1 = \sum_{i=1}^n B_{i1}\delta(B_{1i}).$$

Since every $\delta(B_{1i})$ is $\tau$-compact, it follows that $D_1$ is a $\tau$-compact operator.

Equality $\delta(\mathbf{1}_{\mathcal{A}}) = 0$ together with the Leibniz rule implies that

$$\begin{aligned} D_1 &= \sum_{i=1}^n \Big(\delta(B_{i1}B_{1i}) - \delta(B_{i1})B_{1i}\Big) = \sum_{i=1}^n \Big(\delta(B_{ii}) - \delta(B_{i1})B_{1i}\Big) \\ &= \delta(\mathbf{1}_{\mathcal{A}}) - \sum_{i=1}^n \delta(B_{i1})B_{1i} = -\sum_{i=1}^n \delta(B_{i1})B_{1i}. \end{aligned} \tag{5.13}$$

Then, for every $k, l = 1, \ldots, n$ we have

$$\begin{aligned} [B_{kl}, D_1] = B_{kl}D_1 - D_1B_{kl} &\overset{(5.13)}{=} B_{kl}\sum_{i=1}^n B_{i1}\delta(B_{1i}) + \Big(\sum_{i=1}^n \delta(B_{i1})B_{1i}\Big)B_{kl} \\ &= B_{k1}\delta(B_{1l}) + \delta(B_{k1})B_{1l} = \delta(B_{k1}B_{1l}) = \delta(B_{kl}). \end{aligned} \tag{5.14}$$

Now, consider $X = \sum_{i,j=1}^n X_{ij}B_{ij} \in \mathcal{A}$, $X_{ij} \in \mathcal{Z}(\mathcal{A})$. Since $\sum_{k=1}^n X_{ij}B_{kk} \in \mathcal{Z}(\mathcal{A})$, we have that $\delta\big(\sum_{k=1}^n X_{ij}B_{kk}\big) = 0$ (see Remark 5.2.4). Hence, using the Leibniz rule, we write

$$\begin{aligned}\delta(X) &= \sum_{i,j=1}^n \delta(X_{ij}B_{ij}) = \sum_{i,j=1}^n \delta\Big(\big(\sum_{k=1}^n X_{ij}B_{kk}\big)B_{ij}\Big)\\ &= \sum_{i,j=1}^n \delta\big(\sum_{k=1}^n X_{ij}B_{kk}\big)B_{ij} + \sum_{i,j=1}^n \big(\sum_{k=1}^n X_{ij}B_{kk}\big)\delta(B_{ij})\\ &= \sum_{i,j=1}^n \big(\sum_{k=1}^n X_{ij}B_{kk}\big)\delta(B_{ij}).\end{aligned}$$

Therefore, referring to (5.14), we obtain that

$$\delta(X) = \sum_{i,j=1}^n \big(\sum_{k=1}^n X_{ij}B_{kk}\big)[B_{ij}, D_1].$$

Since $\sum_{k=1}^n X_{ij}B_{kk} \in \mathcal{Z}(\mathcal{A})$ and $\delta(\mathcal{Z}(\mathcal{A})) = 0$, it follows from the definition of $D_1$ that $\sum_{k=1}^n X_{ij}B_{kk}$ commutes with $D_1$. Hence, we obtain that

$$\delta(X) = \sum_{i,j=1}^n \big(\sum_{k=1}^n X_{ij}B_{kk}\big)[B_{ij}, D_1] = \sum_{i,j=1}^n \big[X_{ij}B_{ij}, D_1] = [X, D_1].$$

Arguing similarly, one can show that $D_j := \sum_{i=1}^n B_{ij}\delta(B_{ji})$ such that $\delta = \delta_{D_j}$ for every $j$. Define

$$E := \frac{1}{n}\sum_{j=1}^n D_j = \frac{1}{n}\sum_{i,j} B_{ij}\delta(B_{ji}). \tag{5.15}$$

Then, $\delta = \frac{1}{n}\sum_{j=1}^n \delta_{D_j} = \delta_E$. To complete the proof, it suffices to show that $E \in co\{U\delta(U^*) \mid U \in \mathcal{U}(\mathcal{A})\}$.

We denote by $S$ the collection of all (possibly empty) subsets of $\{1, \cdots, n\}$. There are $2^n$ sets in $S$. For $i \in \{1, \cdots, n\}$ and $K \in S$, we set $e_K^i = 1$ if $i \in K$ and $e_K^i = -1$ if $i \notin K$. Let

$$a_{ij} := \sum_{K \in S} e_K^i e_K^j. \tag{5.16}$$

Clearly, $a_{ii} = \sum_K 1 = 2^n$.

Let $i \neq j$. We denote by $S_1$ the subset of $S$, such that every $K \in S_1$ satisfies $K \supset \{i, j\}$ and denote by $S_2$ the subset of $S$ such that every $K \in S_2$ satisfies that $K \cap \{i, j\} = \varnothing$. Clearly, there are $2^{n-2}$ sets in $S_1$ and $2^{n-2}$ sets in $S_2$. For every $K \in S_1 \cup S_2$, we have $e_K^i e_K^j = 1$. Note that there are $2^{n-1}$ sets in $S \setminus (S_1 \cup S_2)$ and $e_K^i e_K^j = -1$ for every $K \in S \setminus (S_1 \cup S_2)$. Hence, for $i \neq j$, we have

$$a_{ij} = \sum_{K \in S} e_K^i e_K^j = \sum_{K \in S_1 \cup S_2} e_K^i e_K^j + \sum_{K \in S \setminus (S_1 \cup S_2)} e_K^i e_K^j = 2^{n-1} - 2^{n-1} = 0. \tag{5.17}$$

For $\sigma \in S_{(n)}$, the set of all permutations of $\{1, 2, \cdots, n\}$, and $K \in S$, we define a unitary operator

$$U_\sigma^K := \sum_{i=1}^n e_K^i B_{i,\sigma(i)}.$$

Then, by (5.16) and (5.17), we have

$$\begin{aligned}
\sum_{\sigma \in S_{(n)}} \sum_{K \in S} U_\sigma^K \delta((U_\sigma^K)^*) &= \sum_{\sigma \in S_{(n)}} \sum_{K \in S} \sum_{i,j} e_K^i e_K^j B_{j,\sigma(j)} \delta(B_{\sigma(i),i}) \\
&= \sum_{\sigma \in S_{(n)}} \sum_{i,j} B_{j,\sigma(j)} \delta(B_{\sigma(i),i}) \sum_{K \in S} e_K^i e_K^j \\
&= \sum_{\sigma \in S_{(n)}} \sum_{i} B_{i,\sigma(i)} \delta(B_{\sigma(i),i}) a_{ii} \\
&= 2^n \sum_{i} \sum_{\sigma \in S_{(n)}} B_{i,\sigma(i)} \delta(B_{\sigma(i),i}).
\end{aligned}$$

For every $i, j$, there are $(n-1)!$ permutations taking $i$ to $j$. Then, we obtain that

$$\sum_{\sigma \in S_{(n)}} \sum_{K \in S} U_\sigma^K \delta((U_\sigma^K)^*) = 2^n (n-1)! \sum_{i,j} B_{ij} \delta(B_{ji}) = 2^n \frac{n!}{n} \sum_{i,j} B_{ij} \delta(B_{ji}) \overset{(5.15)}{=} 2^n n! E,$$

which implies that $E \in co\{U\delta(U^*) \mid U \in \mathcal{U}(\mathcal{A})\}$. □

The following theorem is the main result of this section, which is a semifinite version of the so-called Johnson–Parrott theorem [67] (see also [32, Chapter 10]). Another semifinite version of the Johnson–Parrott theorem (see [112]) shows that derivations from a type $I$ von Neumann subalgebra of $\mathcal{M}$ into $\mathcal{J}(\mathcal{M})$, the ideal of all compact operators in $\mathcal{M}$, are not necessarily inner. However, in the following theorem, we show that derivations from an arbitrary type $I$ von Neumann subalgebra of $\mathcal{M}$ into $\mathcal{C}_0(\mathcal{M}, \tau)$ are necessarily inner.

**Theorem 5.4.2.** *If $\mathcal{A}$ is a type $I$ von Neumann subalgebra of $\mathcal{M}$, then for every derivation $\delta : \mathcal{A} \to \mathcal{C}_0(\mathcal{M}, \tau)$, there exists $K \in \mathcal{C}_0(\mathcal{M}, \tau) \cap K_\delta$ such that $\delta = \delta_K$.*

*Proof.* Since $\mathcal{A}$ is a type $I$ von Neumann algebra, there exists a central partition of unity $\{Z_n : n \in \mathbb{N}\}$ such that $Z_n\mathcal{A}$ is of type $I_n$ and $Z_0\mathcal{A}$ is properly infinite. Recall that we may always assume that $\delta|_{\mathcal{Z}(\mathcal{A})} = 0$ (see Remark 5.2.4). We have $\delta(Z_n\mathcal{A}) \subset Z_n\mathcal{C}_0(\mathcal{M}, \tau)Z_n$ for all $n \geq 0$. Since for $n \geq 1$, the algebra $Z_n\mathcal{A}$ is of type $I_n$, it follows from Proposition 5.4.1 that $\delta|_{Z_n\mathcal{A}} = \delta_{K_n}$ for some $K_n \in Z_n\mathcal{C}_0(\mathcal{M}, \tau)Z_n \cap Z_nK_\delta$. In addition, by Theorem 5.3.3, there exists $K_0 \in Z_0\mathcal{C}_0(\mathcal{M}, \tau)Z_0 \cap Z_0K_\delta$ such that $\delta|_{Z_0\mathcal{A}} = \delta_{K_0}$. Set $K = \sum_{n=0}^\infty Z_nK_n$. Appealing to Theorem 5.1.1, we conclude that that $\delta = \delta_K$ and $K \in \mathcal{C}_0(\mathcal{M}, \tau) \cap K_\delta$. □

## 5.5 The type $II_1$ case

As before, we assume that $\mathcal{M}$ is a semifinite von Neumann algebra with a faithful normal semifinite trace $\tau$. Let $\mathcal{A}$ be a von Neumann subalgebra of $\mathcal{M}$ and let

$\delta : \mathcal{A} \to \mathcal{C}_0(\mathcal{M}, \tau)$ be a derivation. As we showed in Theorems 5.3.3 and 5.4.2, the derivation $\delta$ is inner provided that $\mathcal{A}$ is properly infinite or of type $I$. Hence, by Remark 3.4.6, to complete the proof of Theorem 5.6.1, it remains to consider the case when $\mathcal{A}$ is of type $II_1$. We cover this remaining case in the present section.

In the setting of the present section, we consider a derivation $\delta : \mathcal{A} \to \mathcal{C}_0(\mathcal{M}, \tau)$, where $\mathcal{A}$ is a type $II_1$ algebra. Since $\mathcal{C}_0(\mathcal{M}, \tau) \subset \mathcal{J}(\mathcal{M})$, the main result of [112] guarantees that there exists $T \in \mathcal{J}(\mathcal{M})$ such that $\delta = \delta_T$. Hence, to prove that $\delta : \mathcal{A} \to \mathcal{C}_0(\mathcal{M}, \tau)$ is inner, it is sufficient to show that there exists $T' \in \mathcal{A}'$ such that $T - T' \in \mathcal{C}_0(\mathcal{M}, \tau)$.

### 5.5.1 Some preliminaries

Recall that $\mathcal{C}_2(\mathcal{M}, \tau) := \{X \in \mathcal{M} \mid \tau(X^*X) < \infty\}$ is the Hilbert-Schmidt class ideal in $\mathcal{M}$ equipped with the norm $\|X\|_\tau = \tau(X^*X)^{\frac{1}{2}}$, $X \in \mathcal{C}_2(\mathcal{M}, \tau)$. Let $\mathcal{H}_\tau$ be the Hilbert space completion of $\mathcal{C}_2(\mathcal{M}, \tau)$ in the norm $\|\cdot\|_\tau$, that is, $\mathcal{H}_\tau = L_2(\mathcal{M}, \tau)$. $\mathcal{M}$ is always regarded in its standard representation, acting on $\mathcal{H}_\tau$ by left multiplication.

In what follows, we introduce norms $|||\cdot|||$ and $|||\cdot|||_{ess}$ on $\mathcal{M}$. The norms $|||\cdot|||$ and $|||\cdot|||_{ess}$ play similar roles in this chapter as the uniform and usual essential norms do in [67] and [110] (see also [32, Chapter 10]).

By the well-known Holmstedt formula (see e.g. [57, Theorem 4.1]), $\|\cdot\|_{L_2+L_\infty}$ defined by $\|f\|_{L_2+L_\infty} = (\int_0^1 \mu(t; f)^2 dt)^{1/2}$, $f \in L_2(0,\infty) + L_\infty(0,\infty)$, is a complete norm on $L_2(0,\infty) + L_\infty(0,\infty)$. It follows immediately from the definition of the norm $\|\cdot\|_{L_2+L_\infty}$ that $(L_2+L_\infty)(0,\infty)$ equipped with the norm $\|\cdot\|_{L_2+L_\infty}$ is a strongly symmetric space. Hence, $(L_2 + L_\infty)(\mathcal{M}, \tau)$ is a strongly symmetric operator space equipped with norm $\|\cdot\|_{L_2+L_\infty}$ defined by $\|T\|_{L_2+L_\infty} = (\int_0^1 \mu(t; T)^2 dt)^{1/2}$, $T \in (L_2 + L_\infty)(\mathcal{M}, \tau)$ (see Chapter 2.4, see also [43, 45, 82]).

**Definition 5.5.1.** *For every $T \in \mathcal{M}$, we define $|||T||| := \|T\|_{L_2+L_\infty}$. It is clear that $|||T||| \le \|T\|_\infty$.*

The following proposition follows immediately from the fact that $(L_2+L_\infty)(\mathcal{M}, \tau)$ is a symmetric space (see 2.4).

**Proposition 5.5.2.** *If $T_1, T_2, T \in \mathcal{M}$, then $|||T_1TT_2||| \le \|T_1\|_\infty |||T||| \|T_2\|_\infty$ and $|||T||| = |||T^*||| = ||||T||||$.*

**Lemma 5.5.3.** *Let $K \in \mathcal{C}_0(\mathcal{M}, \tau)$ and $\{E_n\}$ be a sequence of mutually orthogonal projections in $\mathcal{M}$. Then, we have $|||KE_n||| \to_n 0$ and $|||E_nK||| \to_n 0$.*

*Proof.* By Proposition 5.5.2, we have $|||E_kK||| = |||K^*E_k|||$. Therefore, it is sufficient to show that $|||KE_k||| \to_k 0$.

Since $K$ is a $\tau$-compact operator, the projection $E^{|K|}(\varepsilon, \infty)$ is $\tau$-finite for every $\varepsilon > 0$. In particular, $KE^{|K|}(\varepsilon, \infty) \in \mathcal{F}(\mathcal{M}, \tau)$. Since $\left|\left|\left|K - KE^{|K|}(\varepsilon, \infty)\right|\right|\right| \le (\int_0^1 \varepsilon^2 dt)^{1/2} = \varepsilon$, it follows that $K \in \overline{\mathcal{F}(\mathcal{M}, \tau)}^{|||\cdot|||}$. Then, [43, Proposition 56] together with [42, Theorem 6.13 (iii)] implies that $|||K(\vee_{n\ge k} E_n)||| \to_k 0$. By Proposition 5.5.2, we have $|||KE_k||| \le |||K(\vee_{n\ge k} E_n)||| \to_k 0$. $\square$

**Theorem 5.5.4.** $|||\cdot|||$ *is inferior semicontinuous with respect to the weak operator topology, that is, if a net* $\{T_i\}$ *converges* $T$ *in the weak operator topology, then* $|||T||| \le \limsup_i |||T_i|||$.

*Proof.* By [87, Lemma 2.3.18], we may assume without loss of generality that $\mathcal{M}$ is atomless. We have that $\mu(T)^2 = \mu(|T|)^2 = \mu(|T|^2)$ (see Proposition 2.2.2). Therefore, by [87, Lemma 3.3.2], we have that

$$\int_0^1 \mu(s;T)^2 ds = \int_0^1 \mu(s;|T|^2)ds = \sup\{\|TP\|_\tau^2 : P \in \mathcal{P}(\mathcal{M}),\ \tau(P) \le 1\}. \tag{5.18}$$

and, similarly,

$$\int_0^1 \mu(s;T_i)^2 ds = \int_0^1 \mu(s;|T_i|^2)ds = \sup\{\|T_iP\|_\tau^2 : P \in \mathcal{P}(\mathcal{M}),\ \tau(P) \le 1\}. \tag{5.19}$$

Let $P \in \mathcal{P}(\mathcal{M})$ be such that $\tau(P) \le 1$. Since $|\tau(PT^*T_iP)| \le \|T_iP\|_\tau \|TP\|_\tau$, it follows that $\|TP\|_\tau^2 = \tau(PT^*TP) = \lim_i |\tau(PT^*T_iP)| \le \limsup_i \|T_iP\|_\tau \|TP\|_\tau$. Hence, we have

$$\|TP\|_\tau \le \limsup_i \|T_iP\|_\tau \overset{(5.19)}{\le} \limsup_i |||T_i|||,$$

which together with (5.18) implies that $|||T||| \le \limsup_i |||T_i|||$. □

**Definition 5.5.5.** *For* $T \in \mathcal{M}$, *we define* $|||T|||_{ess} := \inf\{|||T-K||| \mid K \in \mathcal{C}_0(\mathcal{M},\tau)\}$.

The norm $|||\cdot|||_{ess}$ can be described in terms of the singular value function.

**Proposition 5.5.6.** $|||T|||_{ess} = \mu(\infty;T)$ *for every* $T \in \mathcal{M}$.

*Proof.* If $T \in \mathcal{C}_0(\mathcal{M},\tau)$, then, by definition, $|||T|||_{ess} = 0$ and $\mu(\infty;T) = 0$. Hence, we may assume that $\mu(\infty;T) = \varepsilon$ for some $\varepsilon > 0$. For any $K \in \mathcal{C}_0(\mathcal{M},\tau)$ and $\Delta > 0$, there exists $t_0 > 0$ such that $\mu(t;T) - \mu(t;K) \ge \varepsilon - \Delta$ for every $t > t_0$. By Proposition 2.3.1, we obtain that

$$\varepsilon - \Delta \prec\prec \mu(T) - \mu(K) \prec\prec \mu(T-K)$$

for any $K \in \mathcal{C}_0(\mathcal{M},\tau)$ and $\Delta > 0$. By Lemma 2.5.5, the latter condition guarantees that $\mu(T-K) \ge \varepsilon - \Delta$. Thus, $|||T-K||| = (\int_0^1 \mu(t;T-K)^2 dt)^{1/2} \ge \varepsilon - \Delta$. Since $K$ and $\Delta$ are arbitrary, it follows that $|||T|||_{ess} \ge \varepsilon$.

To prove the converse inequality, assume that $\Delta > 0$ and choose $t > 0$ such that $\mu(t;T) \le \varepsilon + \Delta$. By Lemma 2.2.3, $E^{|T|}(\varepsilon+\Delta,\infty)$ is $\tau$-finite. In particular, $TE^{|T|}(\varepsilon+\Delta,\infty) \in \mathcal{C}_0(\mathcal{M},\tau)$. It follows from the definition of $|||\cdot|||_{ess}$ that

$$\begin{aligned}
|||T|||_{ess} &\le \left|\left|\left| T - TE^{|T|}(\varepsilon+\Delta,\infty) \right|\right|\right| \\
&= \left|\left|\left| TE^{|T|}[0,\varepsilon+\Delta] \right|\right|\right| \\
&= \left( \int_0^1 \mu(t; TE^{|T|}[0,\varepsilon+\Delta])^2 dt \right)^{1/2} \\
&\le \varepsilon + \Delta.
\end{aligned}$$

Since $\Delta$ is arbitrary, we conclude that $|||T|||_{ess} = \varepsilon$. □

Let $T_1, T_2 \in \mathcal{M}$ be two operators which are disjoint from the left and the right, that is, $(s(T_1) \vee r(T_1)) \perp (s(T_2) \vee r(T_2))$. The essential norm of $T_1 + T_2$ with respect to $\mathcal{J}(\mathcal{M})$ (see [112, Definition 2.6]) does not necessarily equal the maximum of the essential norms of $T_1$ and $T_2$ with respect to $\mathcal{J}(\mathcal{M})$ (see [112, Section 2.7]). However, similar to the usual essential norm in $B(\mathcal{H})$ (see e.g. [67] and [32, Chapter 10]), the essential norm with respect to $\mathcal{C}_0(\mathcal{M}, \tau)$ has the following property for disjointly supported operators.

**Proposition 5.5.7.** *Let $T \in \mathcal{M}$ and let $P_1, P_2$ be mutually orthogonal projections in $\mathcal{M}$. We have*

$$\max_i |||P_i T P_i|||_{ess} = |||P_1 T P_1 + P_2 T P_2|||_{ess}. \tag{5.20}$$

*Proof.* Without loss of generality, we assume that both $P_1TP_1$ and $P_2TP_2$ are not $\tau$-compact with $\varepsilon_1 := \mu(\infty; P_1TP_1) \geq \mu(\infty; P_2TP_2) =: \varepsilon_2$. By Lemma 2.2.3, for every $\Delta > 0$, we have

$$\begin{aligned}\tau(E^{|P_1TP_1+P_2TP_2|}(\varepsilon_1+\Delta,\infty)) &= \tau(E^{|P_1TP_1|}(\varepsilon_1+\Delta,\infty)) + \tau(E^{|P_2TP_2|}(\varepsilon_1+\Delta,\infty)) \\ &= M < \infty\end{aligned}$$

for some $M > 0$. Hence, using again Lemma 2.2.3, we obtain that

$$\mu(\infty; P_1TP_1 + P_2TP_2) \leq \mu(M; P_1TP_1 + P_2TP_2) \leq \varepsilon_1 + \Delta.$$

Moreover, since $|P_1TP_1| \leq |P_1TP_1| + |P_2TP_2| = |P_1TP_1 + P_2TP_2|$, it follows that

$$\varepsilon_1 = \mu(\infty; P_1TP_1) \leq \mu(\infty; P_1TP_1 + P_2TP_2).$$

Since $\Delta$ is arbitrary, we conclude that $\mu(\infty; P_1TP_1 + P_2TP_2) = \varepsilon_1$. The assertion now follows from Proposition 5.5.6. $\square$

**Remark 5.5.8.** *Note that for any semifinite von Neumann algebra $\mathcal{A}$ there exist pairwise orthogonal central projections $P_i$ with $\sum_i P_i = 1$ such that each $\mathcal{Z}(\mathcal{A})_{P_i}$ is countably decomposable. Hence, combining Theorem 5.1.1 together with Theorem 5.2.2, we may assume, without loss of generality, that the center $\mathcal{Z}(\mathcal{A})$ of the von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$ is countably decomposable. In particular, since $\mathcal{A}$ is of type $II_1$, we can always assume that $\mathcal{A}$ is a countably decomposable type $II_1$ von Neumann algebra (see e.g. [78, Corollary 8.2.9]).*

### 5.5.2 Some continuity results

In this subsection, we study the continuity of derivations $\delta : \mathcal{A} \to \mathcal{C}_0(\mathcal{M}, \tau)$. Similar results with respect to the ideal $\mathcal{J}(\mathcal{M})$ can be found in [112, Section 4]. In this subsection, unless otherwise stated, we always assume that algebra $\mathcal{A}$ is a countably decomposable type $II_1$ von Neumann subalgebra of $\mathcal{M}$ and therefore $\mathcal{A}$ has a normal faithful finite trace $\tau_{\mathcal{A}}$. For every $X \in \mathcal{A}$, we denote

$$\|X\|_2 := \tau_{\mathcal{A}}(X^*X)^{1/2}.$$

**Proposition 5.5.9.** *Let $\delta : \mathcal{A} \to \mathcal{C}_0(\mathcal{M}, \tau)$ be a derivation. Then, $\delta$ is continuous from the unit ball of $\mathcal{A}$ with the strong operator topology into $\mathcal{C}_0(\mathcal{M}, \tau)$ equipped with the norm $|||\cdot|||$.*

*Proof.* By Ringrose's theorem (see Theorem 3.1.2), the mapping $\delta : (\mathcal{A}, \|\cdot\|_\infty) \to (\mathcal{C}_0(\mathcal{M}, \tau), |||\cdot|||)$ is continuous. Hence, denoting by $\|\delta\|$ the operator norm this mapping, we can assume that $\|\delta\| \le 1$.

We firstly prove that if $\{P_n\}_{n\in\mathbb{N}}$ is a sequence of projections in $\mathcal{A}$ with $\tau_{\mathcal{A}}(P_n) \to 0$, then $|||\delta(P_n)||| \to 0$. Suppose that $|||\delta(P_n)|||$ does not converge to 0. Passing to a subsequence, if necessary, we may assume that $|||\delta(P_n)||| \ge c$ for some $c > 0$ for all $n$ and that $\sum \tau_{\mathcal{A}}(P_n) < \infty$. Define $G_n := \vee_{k\ge n} P_k$. We have

$$\tau_{\mathcal{A}}(G_n) \le \sum_{k\ge n} \tau_{\mathcal{A}}(P_k) \to 0$$

as $n \to \infty$. Denote by $S_{n,m}$ the support of $P_m G_n P_m$. It is clear that $S_{n,m} \le P_m$. Moreover, since $S_{n,m} = l(P_m G_n P_m) = r(P_m G_n P_m)$, it follows that $l(P_m G_n P_m) \le l(P_m G_n) \sim r(P_m G_n) \le G_n$, i.e. $S_{n,m} \preceq G_n$. Therefore, $\tau_{\mathcal{A}}(S_{n,m}) \le \tau_{\mathcal{A}}(G_n) \to_n 0$ for each $m$. Since $\{G_n\}_n$ is decreasing, it follows that for every fixed $m$, the sequence $\{P_m G_n P_m\}_n$ is decreasing, and so, $\{S_{n,m}\}$ is decreasing, too. In particular, $S_{n,m} \downarrow 0$ as $n \to \infty$. Thus, $\{P_m - S_{n,m}\}_n$ increases to $P_m$. Since $\delta$ is continuous in weak$^*$ operator topology (see [67, Lemma 1.3]), we obtain that $\{\delta(P_m - S_{n,m})\}$ converges to $\delta(P_m)$ in the weak$^*$ operator topology. By the inferior semicontinuity of the norm $|||\cdot|||$ (see Theorem 5.5.4), it follows that for a fixed $m$, we can find a sufficiently large $n$ such that

$$|||\delta(P_m - S_{n,m})||| \ge \frac{c}{2}.$$

Thus, by induction, we can find an increasing sequence of integers $n_1, n_2, \ldots$ such that for every $k$, the projection $H_k := P_{n_k} - S_{n_{k+1},n_k}$ satisfies $|||\delta(H_k)||| \ge \frac{c}{2}$. These projections also satisfy $\tau_{\mathcal{A}}(H_k) \le \tau_{\mathcal{A}}(P_{n_k}) \to_k 0$. Moreover, since $H_k \le P_{n_k}$ and $S_{n_{k+1},n_k}$ is the support of $P_{n_k} G_{n_{k+1}} P_{n_k}$, by the definition of $H_k$ we get

$$H_k G_{n_{k+1}} H_k = H_k P_{n_k} G_{n_{k+1}} P_{n_k} H_k = H_k P_{n_k} G_{n_{k+1}} P_{n_k} S_{n_{k+1},n_k} H_k = 0,$$

which implies that $H_k G_{n_{k+1}} = 0$. Recalling that $G_n = \vee_{k\ge n} P_k$, we conclude that $H_k H_l = 0$ for every $l \ge k+1$, which means that $H_k$ are mutually orthogonal projections.

Denote by $\mathcal{B}$ the abelian von Neumann subalgebra of $\mathcal{A}$ generated by $\{H_k\}$. By considering $\delta$ as a derivation from $\mathcal{B}$ into $\mathcal{C}_0(\mathcal{M}, \tau)$, we can apply Theorem 5.2.2 to obtain the existence of $K \in \mathcal{C}_0(\mathcal{M}, \tau)$, such that $\delta(H_k) = \delta_K(H_k)$ for $k \in \mathbb{N}$. On one hand, $|||\delta(H_k)||| \ge \frac{c}{2}$. On the other hand, since $H_n$ are mutually orthogonal projections, Lemma 5.5.3 implies that $|||\delta(H_k)||| = |||\delta_K(H_k)||| \le |||KH_k||| + |||H_k K||| \to 0$, which is a contradiction.

Now, we turn to the general case. It is well-known that $\|\cdot\|_2$ induces the strong operator topology on the unit ball of $\mathcal{A}$. Hence, it suffices to prove that if $\{X_n\}_n$ is a bounded sequence in $\mathcal{A}$ with $\|X_n\|_2 \to_n 0$ , then $|||\delta(X_n)||| \to_n 0$. Since $\|X_n\|_2 \to_n 0$, it follows that the $\|(\mathrm{Re}(X_n))_+\|_2 \to_n 0$, $\|(\mathrm{Re}(X_n))_-\|_2 \to_n 0$, $\|(\mathrm{Im}(X_n))_+\|_2 \to_n 0$ and $\|(\mathrm{Im}(X_n))_-\|_2 \to_n 0$. Hence, without loss of generality, we may assume that every element in $\{X_n\}$ is positive and $\|X_n\|_\infty \le 1$.

Let $0 \le X \le \mathbf{1}$ be arbitrary. Let $A_m = \cup_{i=1}^{2^{m-1}}((2i-1)/2^m, 2i/2^m]$, $m \ge 1$. We define

$$k_m := \chi_{A_m},\ m \ge 1.$$

Note that for every $\lambda \in [0,1]$, we have $\lambda = \sum_{m\ge 1} 2^{-m} k_m(\lambda)$. By functional calculus, we have

$$\begin{aligned} X &= \int \lambda dE_\lambda^X = \int \sum_{m\ge 1} 2^{-m} k_m(\lambda) dE_\lambda^X \\ &= \sum_{m\ge 1} 2^{-m} \int k_m(\lambda) dE_\lambda^X = \sum_{m\ge 1} 2^{-m} E_{A_m}^X. \end{aligned}$$

Thus, for every $X_n$, we can write the dyadic decomposition

$$X_n := \sum_{m\ge 1} 2^{-m} e_m^n,$$

where $e_m^n := E_{A_m}^{X_n}$.

Since $\|X_n\|_2 \to_n 0$, it follows that $\tau_{\mathcal{A}}(e_m^n) \to_n 0$ for each $m \ge 1$. Let $\varepsilon > 0$ be fixed and choose $m_0 \ge 1$ such that $2^{-m_0} \le \frac{\varepsilon}{2}$. By the first part of the proof, there exists $n_0$ such that for every $n \ge n_0$, $|||\delta(e_m^n)||| < \frac{\varepsilon}{2}$ for any $m \le m_0$. Recall that $\|\delta\| \le 1$. For $n \ge n_0$, we infer that

$$\begin{aligned} |||\delta(X_n)||| &\le \sum_{m=1}^{m_0} 2^{-m} |||\delta(e_m^n)||| + \|\delta\| \cdot \Big\| \sum_{m>m_0} 2^{-m} e_m^n \Big\|_\infty \\ &\le \sum_{m=1}^{m_0} 2^{-m} |||\delta(e_m^n)||| + \|\delta\| \cdot \sum_{m>m_0} 2^{-m} \\ &\le \Big( \sum_{m=1}^{m_0} 2^{-m} \Big) \frac{\varepsilon}{2} + 1 \cdot \frac{\varepsilon}{2} \cdot \sum_{m=1}^{\infty} 2^{-m} \\ &\le \varepsilon, \end{aligned}$$

which completes the proof. □

Recall that $K_\delta = \overline{co}^{wo}\{U\delta(U^*) \mid U \in \mathcal{U}(\mathcal{A})\}$ (see Section 3.4). Proposition 5.5.9 immediately implies the following corollary.

**Corollary 5.5.10.** *Given $\beta > 0$, there exists $\alpha > 0$ such that*

$$|||TX||| \le \beta \text{ and } |||XT||| \le \beta$$

*for all $T \in K_\delta$ and $X \in \mathcal{A}$, $\|X\|_\infty \le 1$, $\|X\|_2 \le \alpha$.*

*Proof.* By Proposition 5.5.9, there exists $\alpha > 0$ such that $|||\delta(X)||| < \beta/3$ for every $X \in \mathcal{A}$ with $\|X\|_\infty \le 1$ and $\|X\|_2 < \alpha$. For a unitary element $U$ in $\mathcal{A}$, we have $U\delta(U^*)X = U\delta(U^*X) - \delta(X)$ and $\|U^*X\|_2 = \|X\|_2$, which implies that

$$|||U\delta(U^*)X||| \le |||U\delta(U^*X)||| + |||\delta(X)||| < \frac{2}{3}\beta.$$

By taking convex combinations of $U\delta(U^*)$ and using the inferior semi-continuity of norm $|||\cdot|||$ in the weak operator topology (see Theorem 5.5.4), we get $|||TX||| \le \beta$ for all $T \in K_\delta$. The symmetricity of the norm $|||\cdot|||$ (see Proposition 5.5.2) implies that $|||XT||| \le \beta$. □

The following proposition is the main result of the present subsection, which is the key in the proof of Theorem 5.5.13 below.

**Proposition 5.5.11.** *Let $\delta : \mathcal{A} \to \mathcal{C}_0(\mathcal{M}, \tau)$ be a derivation. If $T \in K_\delta$ is such that $\delta = \delta_T$ on $\mathcal{A}$, then $T \in \mathcal{C}_0(\mathcal{M}, \tau)$.*

*Proof.* Since $\mathcal{A}$ is of type $II_1$, there exists a decreasing sequence of projections $\{E_n\}_{n\ge 0}$ in $\mathcal{A}$ with $E_0 = \mathbf{1}$, $E_{n+1} \sim E_n - E_{n+1}$ for all $n \ge 0$ (see e.g. [78, Lemma 6.5.6]).

Using mathematical induction, we show that $|||E_nTE_n|||_{ess} = |||T|||_{ess}$ for all $n$. For $n = 0$, the assertion is trivial. Assume that $|||E_kTE_k|||_{ess} = |||T|||_{ess}$ for all $k \le n$ for some fixed $n \ge 0$. For every $n$, by [134, Chapter XIV, Lemma 2.1], there is a unitary element $U_n \in \mathcal{A}$ such that

$$U_n^* E_{n+1} U_n = E_n - E_{n+1}. \tag{5.21}$$

Since $\delta = \delta_T$, it follows that that $U_n^*TU_n - T = \delta(U^*)U \in \mathcal{C}_0(\mathcal{M}, \tau)$. Therefore, by Proposition 5.5.6 and Definition 5.5.5, we have

$$\begin{aligned} |||E_{n+1}TE_{n+1}|||_{ess} &= |||U_nE_{n+1}TE_{n+1}U_n^*|||_{ess} \\ &= |||U_nE_{n+1}U_n^*TU_nE_{n+1}U_n^*|||_{ess} \\ &\overset{(5.21)}{=} |||(E_n - E_{n+1})T(E_n - E_{n+1})|||_{ess}. \end{aligned} \tag{5.22}$$

Now, using now Proposition 5.5.7, we infer that

$$\begin{aligned} |||E_nTE_n|||_{ess} &= |||E_{n+1}TE_{n+1} + (E_n - E_{n+1})T(E_n - E_{n+1}) \\ &\quad + \delta(E_{n+1})(E_n - E_{n+1}) + \delta(E_n - E_{n+1})E_{n+1}|||_{ess} \\ &= |||E_{n+1}TE_{n+1} + (E_n - E_{n+1})T(E_n - E_{n+1})|||_{ess} \\ &\overset{(5.20)}{=} \max\{|||E_{n+1}TE_{n+1}|||_{ess}, |||(E_n - E_{n+1})T(E_n - E_{n+1})|||_{ess}\} \\ &\overset{(5.22)}{=} |||E_{n+1}TE_{n+1}|||_{ess}. \end{aligned}$$

Therefore, $|||T||| = |||E_nTE_n|||_{ess} = |||E_{n+1}TE_{n+1}|||_{ess}$, which concludes the induction argument.

Assume now that $T \notin \mathcal{C}_0(\mathcal{M}, \tau)$, that is,

$$|||E_nTE_n|||_{ess} = |||T|||_{ess} =: c > 0 \tag{5.23}$$

for all $n$.

Since $\vee_{k\ge n}E_k \downarrow_n 0$, we have $\tau_{\mathcal{A}}(E_n) \le \tau_{\mathcal{A}}(\vee_{k\ge n}E_k) \to_n 0$. Since $T \in K_\delta$ and $\|E_n\|_2 \to 0$, Proposition 5.5.2 and Corollary 5.5.10 imply that

$$|||E_nTE_n||| \le |||TE_n||| \to_n 0,$$

which is a contradiction to (5.23). Thus, $T \in \mathcal{C}_0(\mathcal{M}, \tau)$, as required. □

### 5.5.3 The proof for the main result: the type $II_1$ case

Before proceeding to the proof of Theorem 5.5.13, we prove the special case when $\mathcal{Z}(\mathcal{M})$ is of countable type by using the auxiliary results obtained in Section 5.5.2 and [112, Section 7.4]. To prove the case for $\mathcal{J}(\mathcal{M})$, several reductions are needed in [112, Section 7]. However, rather than repeating the proof in [112, Section 7], we use the main result of [112] in the proof of the following proposition, which makes our proof more efficient.

**Proposition 5.5.12.** *If the center $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$ is countably decomposable, then every derivation $\delta$ from a type $II_1$ von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$ into $\mathcal{C}_0(\mathcal{M},\tau)$ is inner. Moreover, the element $T \in \mathcal{C}_0(\mathcal{M},\tau)$ implementing $\delta$ lies in $K_\delta$.*

*Proof.* It is proved in [112] that every derivation from a type $II_1$ von Neumann subalgebra $\mathcal{A}$ into $\mathcal{J}(\mathcal{M})$ is implemented by some element in $\mathcal{J}(\mathcal{M})$. Noticing that $\mathcal{C}_0(\mathcal{M},\tau) \subset \mathcal{J}(\mathcal{M})$, we conclude that there exists an element $K \in \mathcal{J}(\mathcal{M})$ such that $\delta = \delta_K$. Since $\mathcal{Z}(\mathcal{M})$ is countably decomposable, by [112, Lemma 4.6] (note that this lemma requires the condition that $\mathcal{Z}(\mathcal{M})$ is countably decomposable), there is a $\overline{T} \in \overline{co}^{wo}\{\delta(U)U^* \mid U \in \mathcal{U}(\mathcal{A})\} = -K_\delta$ such that $\delta(\cdot) = [\overline{T}, \cdot] = \delta_{-\overline{T}}(\cdot)$. Now, let $T = -\overline{T}$. Then, $T \in K_\delta$ with $\delta = \delta_T$. It follows from Proposition 5.5.11 that $T \in \mathcal{C}_0(\mathcal{M},\tau)$. $\square$

In the following theorem, we remove the condition that $\mathcal{Z}(\mathcal{M})$ is countably decomposable imposed in Proposition 5.5.12, proving the main result of this section.

**Theorem 5.5.13.** *Every derivation $\delta$ from a type $II_1$ von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$ into $\mathcal{C}_0(\mathcal{M},\tau)$ is inner. Moreover, the element implementing $\delta$ lies in $K_\delta$.*

*Proof.* Let $\{Z_i \in \mathcal{Z}(\mathcal{M})\}$ be a net of projections increasing to $\mathbf{1}$ such that $\mathcal{Z}(\mathcal{M}_{Z_i}) = \mathcal{Z}(\mathcal{M})_{Z_i}$ is countably decomposable. Since $\mathcal{A}$ is assumed to be countably decomposable (see Remark 5.5.8), it follows that $\mathcal{A}_{Z_i}$ is also countably decomposable. Define $\delta_i : \mathcal{A}_{Z_i} \to Z_i\mathcal{C}_0(\mathcal{M},\tau)Z_i = \mathcal{C}_0(\mathcal{M}_{Z_i},\tau)$ by $\delta_i(XZ_i) = Z_i\delta(X)Z_i$ for every $X \in \mathcal{A}$. Since $Z_i \in \mathcal{Z}(\mathcal{M})$, it follows from Lemma 3.4.2 that $\delta_i$ are well-defined derivations.

By Proposition 5.5.12, there exists $K_i \in \mathcal{C}_0(\mathcal{M}_{Z_i},\tau)$ with $K_i \in K_{\delta_i}$ such that $\delta_i = \delta_{K_i}$ on $\mathcal{A}_{Z_i}$. Since $\mathcal{U}(\mathcal{A}_{Z_i}) = \mathcal{U}(\mathcal{A})Z_i$ (see e.g. [77, Proposition 5.5.5]), it follows that

$$\begin{aligned} K_i \in K_{\delta_i} &= \overline{co}^{wo}\{U\delta_i(U^*) \mid U \in \mathcal{U}(\mathcal{A}_{Z_i})\} = \overline{co}^{wo}\{UZ_i\delta_i(U^*Z_i) \mid U \in \mathcal{U}(\mathcal{A})\} \\ &= \overline{co}^{wo}\{UZ_i\delta(U^*)Z_i \mid U \in \mathcal{U}(\mathcal{A})\} = K_\delta Z_i. \end{aligned}$$

Hence, for every $i$, there exists $T_i \in K_\delta$ such that $K_i = Z_iT_iZ_i$.

Note that $K_\delta$ is compact in the weak operator topology (see [31, Chapter IX, Proposition 5.5]). Let $T \in K_\delta$ be a limit point of a subnet of $\{T_i\}_i$ in the weak operator topology. Without loss of generality, we assume that $T_i \to_{wo} T$. For every $X \in \mathcal{A}$, we have that

$$Z_i\delta(X)Z_i = \delta_{K_i}(X) = \delta_{Z_iT_iZ_i}(X).$$

On one hand, since $Z_i \uparrow \mathbf{1}$, it follows that $Z_i\delta(X)Z_i = Z_i\delta(X) \to_{so} \delta(X)$ and therefore a fortiori $Z_i\delta(X)Z_i \to_{wo} \delta(X)$. On the other hand, since $\|U\delta(U^*)\|_\infty \leq \|\delta\|_{(\mathcal{A},\|\cdot\|_\infty)\to(\mathcal{C}_0(\mathcal{M},\tau),\|\cdot\|_\infty)} < \infty$ (see Theorem 3.1.2) and $T_i \in K_\delta$, it follows that $\|T_i\|_\infty \leq \|\delta\|_{(\mathcal{A},\|\cdot\|_\infty)\to(\mathcal{C}_0(\mathcal{M},\tau),\|\cdot\|_\infty)} < \infty$ for every $i$. Hence, $Z_iT_iZ_i \to_{wo} T$. Combining these two convergences, we conclude that

$$\delta(X) = wo\text{-}\lim_i Z_i\delta(X)Z_i = wo\text{-}\lim_i \delta_{Z_iT_iZ_i}(X) = \delta_T(X),$$

that is, $\delta = \delta_T$ on $\mathcal{A}$. Since $T \in K_\delta$, it follows from Proposition 5.5.11 that $T \in \mathcal{C}_0(\mathcal{M},\tau)$. □

## 5.6 Conclusions and applications

The following theorem is the main result of the present chapter.

**Theorem 5.6.1.** *Every derivation $\delta$ from a von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$ into the ideal $\mathcal{C}_0(\mathcal{M},\tau)$ of all $\tau$-compact operators is inner.*

*Proof.* By Theorem 5.2.2, there exists $T \in \mathcal{C}_0(\mathcal{M},\tau)$ such that $\delta|_{\mathcal{Z}(\mathcal{A})} = \delta_T$. Replacing $\delta$ with $\delta - \delta_T$, we can assume that $\delta$ vanishes on $\mathcal{Z}(\mathcal{A})$. By Remark 3.4.6, it suffices to prove the assertion in the case when $\mathcal{A}$ is of type $I$, type $II_1$ and properly infinite, separately. Hence, appealing to Theorem 5.4.2, Theorem 5.5.13 and Theorem 5.3.3, we conclude the proof. □

In the particular case when $\mathcal{M} = B(\mathcal{H})$ and $\tau$ is the standard trace, our result recovers the Johnson–Parrott–Popa theorem (see Theorem 3.2.1, see also [67, 110]). Furthermore, in the case of an arbitrary von Neumann algebra $\mathcal{M}$ equipped with a faithful normal finite trace $\tau$, we have that $\mathcal{C}_0(\mathcal{M},\tau) = \mathcal{M}$ (see e.g. [87, Page 64]), and therefore, Theorem 5.6.1 guarantees that any derivation $\delta : \mathcal{A} \to \mathcal{M}$ is inner if $\mathcal{M}$ is equipped with a faithful normal finite trace. In the following corollary, we extend this result to a general finite von Neumann algebra $\mathcal{M}$, recovering the main result of [22, Section 5] using completely different approach.

**Corollary 5.6.2.** *Every derivation $\delta$ from a von Neumann subalgebra $\mathcal{A}$ of a finite von Neumann algebra $\mathcal{M}$ into $\mathcal{M}$ is inner. Moreover, the element $K \in \mathcal{M}$ implementing $\delta$ can be chosen from $K_\delta$.*

*Proof.* Since $\mathcal{M}$ is finite, it follows that there is a net $\{P_i\}$ of projections in $\mathcal{Z}(\mathcal{M})$ with $P_i \uparrow \mathbf{1}$ such that $\mathcal{M}_{P_i}$ is countably decomposable (see e.g. [78, Corollary 8.2.9] and Proposition 1.8.5). Hence, $\mathcal{M}_{P_i}$ has a faithful normal finite trace $\tau_i$ (see e.g. Proposition 1.8.5), that is, $\mathcal{M}_{P_i} = \mathcal{C}_0(\mathcal{M}_{P_i},\tau_i)$. Therefore, by Theorem 5.6.1, the derivation $\delta_i : \mathcal{A}_{P_i} \to \mathcal{M}_{P_i}$ defined by $\delta_i(XP_i) = \delta(X)P_i$ is inner, that is, there exists $T_i \in \mathcal{M}_{P_i}$ such that $\delta_i = \delta_{T_i}$ on $\mathcal{A}_{P_i}$. By Proposition 5.1.2, there exists

$$\begin{aligned} K_i \in K_{\delta_i} &= \overline{co}^{wo}\{U\delta_i(U^*) \mid U \in \mathcal{U}(\mathcal{A}_{P_i})\} = \overline{co}^{wo}\{UP_i\delta_i(U^*P_i) \mid U \in \mathcal{U}(\mathcal{A})\} \\ &= \overline{co}^{wo}\{U\delta(U)P_i \mid U \in \mathcal{U}(\mathcal{A})\} = K_\delta P_i \end{aligned}$$

such that $\delta_i = \delta_{K_i}$ on $\mathcal{A}_{P_i}$ and there is a $K_i' \in K_\delta$ such that $K_i = K_i' P_i$. Since $K_\delta$ is compact in the weak operator topology (see e.g. [31, Chapter IX, Proposition 5.5]), there is a limit point $K \in K_\delta$ of a subnet of $\{K_i'\}$ in the weak operator topology. Without loss of generality, we denote that $K_i' \to_{wo} K$. We have

$$\delta(X)P_i = \delta_i(X) = \delta_{K_i}(X) = \delta_{K_i' P_i}(X)$$

for every $X \in \mathcal{A}$. Since $\{P_i\}$ converges strongly to the identity, it follows that $\delta = \delta_K$ on $\mathcal{A}$. □

We conclude this chapter with an application of Theorem 5.6.1 to derivations with values in a class of ideals of $\mathcal{M}$. We characterize a wide class of ideals $\mathcal{E}$ of $\mathcal{M}$ such that derivations with values in these ideals are automatically inner.

We note that the Fatou property is an analogue of the so-called "dual normal" property of bimodules over von Neumann algebras. It is known that every derivation from a hyperfinite von Neumann algebra $\mathcal{A}$ into a dual normal $\mathcal{A}$-bimodule is inner (see e.g. [114, Theorem 2] and [125, Theorem 2.4.3]). However, no additional conditions on the von Neumann subalgebra are needed in our setting.

**Theorem 5.6.3.** *Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace $\tau$, let $E(\mathcal{M},\tau)$ be a strongly symmetric space with the Fatou property and let $\mathcal{A}$ be a von Neumann subalgebra of $\mathcal{M}$. Then every derivation $\delta$ from $\mathcal{A}$ into $E(\mathcal{M},\tau) \cap \mathcal{C}_0(\mathcal{M},\tau)$ is necessarily inner, that is, there exists $T \in E(\mathcal{M},\tau) \cap \mathcal{C}_0(\mathcal{M},\tau)$ such that $\delta = \delta_T$.*

*Proof.* For any symmetric space $E(\mathcal{M},\tau)$ such that $E(\mathcal{M},\tau) \not\subseteq S_0(\mathcal{M},\tau)$, there exists an element $X \in E(\mathcal{M},\tau)$ such that $\mu(X) \geq \alpha\chi_{(0,\infty)} \geq \alpha\mu(\mathbf{1})$ for some $\alpha > 0$ (see Section 2.5 and Definition 2.2.1). By Definition 2.4.1, we obtain that $\mathbf{1} \in E(\mathcal{M},\tau)$, which implies that $\mathcal{C}_0(\mathcal{M},\tau) \subset \mathcal{M} \subset E(\mathcal{M},\tau)$ (see Section 2.4). That is, $E(\mathcal{M},\tau) \cap \mathcal{C}_0(\mathcal{M},\tau) = \mathcal{C}_0(\mathcal{M},\tau)$. By Theorem 5.6.1, it is sufficient to prove the case when $E(\mathcal{M},\tau) \subset S_0(\mathcal{M},\tau)$. In particular, $E(\mathcal{M},\tau)\cap\mathcal{C}_0(\mathcal{M},\tau) = E(\mathcal{M},\tau)\cap\mathcal{M}$.

We first assume that the carrier projection of $E(\mathcal{M},\tau)$ is $\mathbf{1}$ (see also Section 2.4). By [43, Theorem 32], we have that $E(\mathcal{M},\tau) = E(\mathcal{M},\tau)^{\times\times}$, that is, $E(\mathcal{M},\tau)$ is the Köthe dual of $E(\mathcal{M},\tau)^{\times}$. Since $E(\mathcal{M},\tau) \cap \mathcal{M} \subset \mathcal{C}_0(\mathcal{M},\tau)$, it follows from Theorem 5.6.1 and Proposition 5.1.2 that there is a $T \in K_\delta$ such that $\delta = \delta_T$. Hence, there exists a net $\{T_i\} \subset co\{U\delta(U^*) \mid U \in \mathcal{U}(\mathcal{A})\} \subset E(\mathcal{M},\tau) \cap \mathcal{C}_0(\mathcal{M},\tau)$ such that $T_i \to_{so} T$ with

$$\sup_i \tau(|T_i X|) \leq \sup_i \|T_i\|_E \leq \|\delta\|_{\mathcal{A}\to E} < \infty \tag{5.24}$$

for every $X$ in the unit ball of $E(\mathcal{M},\tau)^{\times}$ (see Section 2.4 and [114, Theorem 2]).

Fix $X \in E(\mathcal{M},\tau)^{\times}$ with $\|X\|_{E^\times} \leq 1$. Let $Z$ be an arbitrary operator in $L_1(\mathcal{M},\tau) \cap \mathcal{M}$ such that $Z \prec\prec X$. Since $E(\mathcal{M},\tau)^{\times}$ is a fully symmetric space (see e.g. [43, Theorem 27] or [42, Proposition 3.7]), it follows that $Z \in E(\mathcal{M},\tau)^{\times}$ and $\|Z\|_{E^\times} \leq 1$. Hence, $\|T_i Z\|_1 = \tau(|T_i Z|) \overset{(5.24)}{\leq} \|\delta\|_{\mathcal{A}\to E} < \infty$. Since $\|T_i\|_\infty \leq$

$\|\delta\|_{\mathcal{A}\to\mathcal{M}} < \infty$ (see Theorem 3.1.2 or [114, Theorem 2]) and $Z \in L_1(\mathcal{M},\tau)$, it follows from [3, Lemma 2.5] that $TZ \in L_1(\mathcal{M},\tau)$ with $\|TZ\|_1 \le \|\delta\|_{\mathcal{A}\to E} < \infty$. Noting that $X, T \in L_1(\mathcal{M},\tau) + \mathcal{M}$ (see e.g. [43, Lemma 25]), it follows from [39, Theorems 3.10 and 4.12] that

$$\tau(|TX|) \le \sup\{\tau(|TZ|) : Z \in L_1(\mathcal{M},\tau) \cap \mathcal{M}, Z \prec\prec X\} \le \|\delta\|_{\mathcal{A}\to E}.$$

Since $X \in E^{\times}(\mathcal{M},\tau)$, $\|X\|_{E^{\times}} \le 1$, is arbitrary and $E(\mathcal{M},\tau) = (E(\mathcal{M},\tau)^{\times})^{\times}$, it follows that $T \in E(\mathcal{M},\tau)$, as required.

Now, consider the general case. Let $c_E$ be the carrier projection of $E(\mathcal{M},\tau)$. Then, $\mathcal{M}_{c_E}$ is a von Neumann algebra with identity $c_E$. By Corollary 5.6.2, there is a $T \in \mathcal{C}_0(\mathcal{M},\tau)$ such that $\delta = \delta_T$ on $\mathcal{A}$. Note that $c_E$ is a central projection in $\mathcal{M}$ (see [43, Corollary 6]). Hence, $E(\mathcal{M}_{c_E},\tau) := E(\mathcal{M},\tau) \subset S(\mathcal{M}_{c_E},\tau)$ is a strongly symmetric space having the Fatou property and $\delta_T : \mathcal{A}_{c_E} \to E(\mathcal{M}_{c_E},\tau) \cap \mathcal{M}_{c_E}$ is also a derivation. By the first part of the proof, there is a $K \in E(\mathcal{M}_{c_E},\tau) \cap \mathcal{M}_{c_E}$ such that $\delta_T = \delta_K$ on $\mathcal{A}_{c_E}$. For every $X \in \mathcal{A}$, we have $\delta_T(X) \in E(\mathcal{M},\tau)$ and therefore $c_E\delta_T(X) = \delta_T(X)$ (see [43, Corollary 6]). Hence, for every $X \in \mathcal{A}$, we have $\delta_T(c_E^{\perp}X) = c_E^{\perp}\delta_T(X) = c_E^{\perp}c_E\delta_T(X) = 0$ and therefore,

$$\delta(X) = \delta_T(X) = \delta_T((c_E + c_E^{\perp})X) = \delta_T(c_E X) + \delta_T(c_E^{\perp}X) = \delta_K(c_E X).$$

Since $c_E$ is a central projection in $\mathcal{M}$ and $c_E K = K = K c_E$, it follows that $\delta(X) = \delta_K(c_E X) = \delta_K(X)$. Noting that $K \in E(\mathcal{M}_{c_E},\tau) \cap \mathcal{M}_{c_E} = E(\mathcal{M},\tau) \cap \mathcal{M}$, we complete the proof. $\square$

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