Statistical stability for deterministic and random dynamical systems

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Publication Date:
2021

DOI:
https://doi.org/10.26190/unsworks/22415

Statistical stability for deterministic and random dynamical systems

A thesis submitted for the degree of
Doctor of Philosophy

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January 2021

Thesis Title

Statistical stability for deterministic and random dynamical systems

Thesis Abstract

It is well known that sufficiently smooth, hyperbolic dynamical systems admit strong statistical descriptions e.g. limit laws such as a central limit theorem or large deviation principle. Given the existence of these laws one is led to the question of their stability: do nearby systems have similar statistical descriptions, and to what extent can one numerically approximate the statistics of any particular system? In this thesis such questions are investigated by building on the so-called functional analytic approach, and in particular the spectral perturbation theory of Keller and Liverani. For deterministic systems it is shown that the Keller-Liverani perturbation theory is compatible with the naive Nagaev-Guivarc'h method – the method used to obtain the aforementioned statistical limit laws – yielding a general framework for deducing the statistical stability of deterministic dynamical systems under a variety of perturbations. This theory is then applied to piecewise expanding maps in one and many dimensions, in addition to Anosov maps on tori. Of particular note is the development of new, efficient and rigorous numerical methods for the approximation of the statistical properties of multidimensional piecewise expanding maps and Anosov maps. In the second part of this thesis this program is begun again for random systems. Here there is no analogue of the Keller-Liverani pertur- bation theory, and so an appropriate random version of the theory is developed. This theory is then applied to smooth random expanding maps on the circle, and the stability of some basic statistical properties is deduced with respect to fiber-wise deterministic perturbations and a Fourier-analytic numerical method.

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> The results of part 1 are based on two papers with Gary Froyland; these are "Stability and approximation of statistical limit laws for multidimensional piecewise expanding maps" and "Fourier approximation of the statistical properties of Anosov maps on tori". Part 2 is based on my paper "Stability of hyperbolic Oseledets splittings for quasi-compact operator cocycles". There is an acknowledgement of this in the introduction to the thesis.

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Acknowledgements

I am very thankful for the life I have lived these past few years: for the fascinating people who are now my friends, for being able to travel and learn and share my research, and for being able to spend my days labouring away at something I love. It's been great.

I owe most of what I have achieved to Gary Froyland. For almost five years he has patiently and tactfully nurtured, pushed, funded, and encouraged me in my academic endeavours, despite much naivety on my part. Gary, I don't think I can ever pay you back, but thank you – you have been an amazing supervisor, and I am proud to have been your student.

I am thankful to Davor Dragičević, Cecilia González-Tokman, and Jason Atnip, all of whom have been very generous with their time and their minds in the last few years. I am grateful to Philippe Thieullen for his collaboration, and for hosting me at the Institut de Mathématiques de Bordeaux in late 2018, where he spent every afternoon for nearly two months working closely with me on our problem. Moreover, I am thankful to Philippe for introducing me to the graph representation of the Grassmannian and the associated graph transform, which were instrumental tools in my third paper [27]. I am also thankful to Benoît Saussol for his collaboration, and for hosting me at the Universit de Bretagne Occidentale for a very pleasant week.

I would like to acknowledge and thank the various institutions that have placed their trust in me: the Australian Government for supplying me with a Research Training Program (RTP) Scholarship, the School of Mathematics and Statistics at UNSW for a top-up scholarship and many employment opportunities, UNSW for travel funding under the Postgraduate Research Student Support (PRSS) Scheme, and Campus France for the PHC FASIC PhD mobility program travel grant to visit Philippe in 2018.

I am very grateful to the people at the School of Mathematics and Statistics. It has been a consistent pleasure working alongside the staff and students at the school. In particular, I am very thankful for the friendship of Fadi Antown, Kam Hung Yau, Thomas Scheckter, Chris Rock (no relation), Michael Denes, Fiona Kim, Susannah Waters and Sebastian Blefari. There are many others, but to include them all would be to make a long thesis longer. You know who you are.

I would like to thank my immediate family: my mum, my brother, my sister, and Rumi (I think you fit in this category too). I am indebted to your constant support, and for putting up with me.

Lastly, my thanks go to Maja for being a constant joy in my life. I don't think I can fully put it into words, which may come as a surprise, but I think you get it.

- Harry

Abstract

It is well known that sufficiently smooth, hyperbolic dynamical systems admit strong statistical descriptions e.g. limit laws such as a central limit theorem or large deviation principle. Given the existence of these laws one is led to the question of their stability: do nearby systems have similar statistical descriptions, and to what extent can one numerically approximate the statistics of any particular system? In this thesis such questions are investigated by building on the so-called functional analytic approach, and in particular the spectral perturbation theory of Keller and Liverani. For deterministic systems it is shown that the Keller-Liverani perturbation theory is compatible with the naive Nagaev-Guivarc'h method – the method used to obtain the aforementioned statistical limit laws – yielding a general framework for deducing the statistical stability of deterministic dynamical systems under a variety of perturbations. This theory is then applied to piecewise expanding maps in one and many dimensions, in addition to Anosov maps on tori. Of particular note is the development of new, efficient and rigorous numerical methods for the approximation of the statistical properties of multidimensional piecewise expanding maps and Anosov maps. In the second part of this thesis this program is begun again for random systems. Here there is no analogue of the Keller-Liverani perturbation theory, and so an appropriate random version of the theory is developed. This theory is then applied to smooth random expanding maps on the circle, and the stability of some basic statistical properties is deduced with respect to fiber-wise deterministic perturbations and a Fourier-analytic numerical method.

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Introduction

While the presence of uniform hyperbolicity is a well-known driver of chaotic behaviour in smooth dynamical systems, it is this same hyperbolicity that leads to precise statistical characterisations for such systems. For instance, uniform hyperbolicity guarantees the existence of physically meaningful invariant measures [19, 74, 93, 98], and, under additional hypotheses, stronger properties such as exponentially decaying correlations for smooth observables [60, 77] or statistical limit laws such as a Central Limit Theorem (CLT) or Large Deviation Principle (LDP) [85, 88, 92]. The research in this thesis is concerned with questions of the stability of such statistical properties. Here 'stability' is to be understood in quite a general sense, for a number of practically important questions can be recast as stability problems. To name a critical few: When (and how) does a system's statistics change continuously with the system itself? Are the statistics of an isolated system robust to the presence of small amounts of external random noise? Do the statistical properties of sufficiently granular numerical discretisations approximate those of the original system? This last question, on the potential for numerical approximation of the statistics of a given system, will be a common thread throughout this thesis and has motivated much of the research contained herein.

One of the most effective techniques for attacking these kinds of problems is the *functional analytic approach* (see, for a modern overview, any of [10, 11, 45, 78, 79]). This approach builds on the observation that many of the statistical properties of sufficiently smooth, uniformly hyperbolic dynamical systems are accessible via the spectral properties of the associated Perron-Frobenius operator, provided that one considers the operator on a Banach space that is appropriately matched to the dynamics. In particular, one wishes to find a Banach space that is small enough so that the Perron-Frobenius operator is quasi-compact, but rich enough so that one may study the statistics of as many observables as possible. Many relevant perturbations manifest naturally as perturbations to the Perron-Frobenius operator, and so the question of stability of statistical properties then becomes one of spectral stability. Unfortunately, very few perturbations continuously perturb the Perron-Frobenius operator in the operator norm, effectively ruling out the application of classical operator perturbation theory[1] (as in Kato [65]). The spectral perturbation theory of Keller and Liverani [68, 71] (and, later, Gouëzel [54]) was developed to address this problem, and the theory consequently forms the technical backbone of this thesis.

We start in Part I by focusing on deterministic (i.e. non-random) uniformly hyperbolic maps and on the stability of the variance of the CLT and the rate function of the LDP for such systems. The stability of the variance has been previously studied in a number of specific cases: stability to deterministic perturbations was proven for Anosov maps and flows [54, 24], Lorenz flows [8] and Lasota-Yorke maps [70], while various rigorous numerical methods have been developed for estimating the variance of one-dimensional expanding maps [7, 63, 106]. The stability and estimation of the rate function has received much less attention: a positive result is noted in [24] for deterministic perturbations to Anosov flows, and some partial results exist for numerical approximations of one-dimensional expanding systems [102]. In the functional analytic approach the CLT and LDP (and a host of others: [5, 22, 51, 52, 58, 80, 89, 92, 36]) are often obtained via the Nagaev-Guivarc'h method [82, 56, 58, 53], in which one aims to code the moment or cumulant generating functions of a sequence of Birkhoff sums in terms of an analytically twisted Perron-Frobenius operator. As a result of this coding one obtains various limit laws, the parameters of which are determined by the spectral data of the twisted Perron-Frobenius operators, including the variance of the CLT and the rate function of the LDP. The first main contribution of this thesis is to show that the perturbation theory of Keller and Liverani is compatible with the Nagaev-Guivarc'h method, resulting in a general, abstract stability theorem for the variance and rate function (Theorem 2.2.1).

Throughout the remainder of Part I we reap the rewards of Theorem 2.2.1 for a cohort of systems and perturbations. In Chapter 3 we examine a classical example: one-dimensional Lasota-Yorke maps. For these maps we consider the Perron-Frobenius operator on the space of functions of bounded variation, and examine stability to stochastic and deterministic perturbations [68], in addition to numerical approximations via Ulam's method [50, 75]. While these systems are well studied

[1] Analytic systems are a notable exception; for a recent example of stability theory for analytic maps see [13].

[20, 74] and the stability of the variance has been proven here by other methods [7, 70], our results on the stability of the rate function are new. We also take the opportunity in this relatively simple setting to illustrate with a case study on Ulam's method how one might use our theory to adapt existing numerical methods to approximate the variance and rate function. In Chapter 4 we apply our theory to the class of multidimensional piecewise expanding maps considered by Saussol in [94]. For these maps we show that stochastic perturbations and Ulam's method verify the conditions of Keller-Liverani stability theory when the Perron-Frobenius operator acts on quasi-Hölder functions, and so we obtain stability of various statistical parameters to these perturbations. Despite there being some earlier results on the numerical approximation of the absolutely continuous invariant measure associated to similar maps via Ulam's method [32, 81], our results are the first to show that the rate function and variance may also be approximated. Lastly, in Chapter 5 we turn to Anosov diffeomorphisms on tori and construct two new, efficient, Fourier-analytic numerical methods for approximating the statistical properties of Anosov maps that are compatible with Keller-Liverani perturbation theory and the anisotropic Banach spaces of Gouëzel and Liverani [54]. The computational experiments that conclude Chapter 5 constitute the first rigorous computations of the variance and rate function for any Anosov maps, although we note the existence of other numerical methods for estimating the statistical data of Anosov maps [37, 31, 14, 100].

In Part II we shift to considering random dynamical systems (RDSs) and their quenched statistical properties i.e. properties that are exhibited by almost every realisation of the system. The functional analytic approach, as formulated for deterministic systems, does not generalise to the random setting and, in particular, it does not suffice to consider the spectral data of a single Perron-Frobenius operator to access the system's statistical properties. Rather, one is lead to consider a random cocycle of Perron-Frobenius operators and apply tools from multiplicative ergodic theory to obtain a Oseledets splitting for the cocycle [43, 47, 48]. It is this Oseledets splitting, and its associated Lyapunov exponents, that encodes the quenched statistical behaviour of the system. This approach, which was pioneered in [42, 43] by extending the techniques of [103], has developed into a rapidly expanding literature: quenched statistical limit laws may be obtained under a random Nagaev-Guivarc'h method [33, 34], and some results on the stability (and linear response) for random equivariant measures exist [12, 35, 39, 83, 96, 97].

However, the question of stability of the Oseledets splittings and Lyapunov exponents for Perron-Frobenius cocycles has only been considered on a few prior occasions and is yet of high practical importance, having applications to the detection and characterisation of oceanic and atmospheric flows (see [46, Section 6] for an overview, and e.g. [44, 41, 40] for applications). Stability for these objects to 'asymptotically small random perturbations' of cocycles satisfying certain hyperbolicity conditions was proven in [16], but unfortunately this result only has limited applicability to Perron-Frobenius operator cocycles. Secondly, in [49] the stability, and lack thereof, of Lyapunov exponents was studied for the Perron-Frobenius operator cocycle associated to a RDS consisting of expanding Blaschke products when subjected to a variety of perturbations. This last setting is quite special, since the dynamics are all analytic and each Perron-Frobenius operator is compact. In particular, the perturbations considered are small in the operator norm, which is a substantial simplification, and so the techniques used for this result are unlikely to generalise to non-analytic maps.

Hence, as it stands, there is no analogue of the Keller-Liverani perturbation theory for the stability of Oseledets splittings and Lyapunov exponents that is well-adapted to random Perron-Frobenius cocycles. As such, the goal of Part II is to propose, develop and apply an appropriate generalisation of the Keller-Liverani perturbation theory for the random case. In Chapter 7 a Keller-Liverani-esque stability theorem (Theorem 7.1.7) is proved for hyperbolic splittings associated to certain linear automorphisms over vector bundles. Then, in Chapter 8, Theorem 7.1.7 is applied to deduce the stability of Oseledets splittings and Lyapunov exponents for cocycles with hyperbolic Oseledets splittings. Chapter 9 contains an application to the stability of the quenched statistical properties of random $\mathcal{C}^k$ expanding maps on the circle: we deduce stability under perturbations arising from a Fejér kernel numerical method and to fiber-wise deterministic perturbations.

A notable difference between our approach in Part II and the existing statistical stability literature [71, 54, 39, 96, 97, 35] is the usage of *Saks spaces*, which, among other things, allows us to weaken the hypotheses of Keller-Liverani stability results in the deterministic case (see Remark 8.1.12). The theory of Saks spaces unifies many of the concepts from the functional analytic approach such as the relationship between weak and strong norms, Lasota-Yorke inequalities, and the 'triple norm' of Keller-Liverani perturbation theory. We posit that by studying these spaces one may better understand the potential for, and limitations of, the functional analytic approach. For example, using Saks space theory one can precisely characterise the set of norms $|\cdot|$ on a Banach space $(X, \|\cdot\|)$ such that the closed unit $\|\cdot\|$-ball is $|\cdot|$-compact (see Theorems 6.2.22 and 6.2.23), which has applications to the construction of anisotropic Banach spaces adapted to hyperbolic dynamical systems.

A note on the originality of content in this thesis: the research contained herein has appeared previously in a number of journal articles. Part I contains the publications [28, 29], which have been lightly edited for coherency. Part II is wholly the article [27], which has also been lightly edited.

Part I

Deterministic Systems

Chapter 1

Spectral stability for twisted quasi-compact operators

The goal of this chapter is to prove some abstract spectral stability results for analytic families of quasi-compact operators with respect to the kinds of perturbations considered by Keller and Liverani [71]. In Section 1.1 we discuss previous results on the stability of the spectrum of individual quasi-compact operators and then in Section 1.2 we state our main results for this chapter (Theorem 1.2.2): a spectral stability theorem for 'analytically twisted' families of quasi-compact operators. Lastly, in Section 1.3, we prove a useful technical result on the robustness of the hypotheses for the main result in Section 1.2 to operator norm perturbations.

1.1 Spectral stability of quasi-compact operators

The aim of this section is to review the spectral stability theory for quasi-compact operators from [71], but first we recall some basic facts about quasi-compact operators. Let $(E, \|\cdot\|)$ be a complex Banach space, and denote by $L(E)$ the Banach space of bounded linear operators on E, and by $\|\cdot\|$ the operator norm on $L(E)$. For $A \in L(E)$ we denote the spectrum of A by $\sigma(A)$, and recall that the essential spectrum of A is

$$\sigma_{\text{ess}}(A) = \{\omega \in \sigma(A) : \omega \text{ is not an eigenvalue of } A \text{ of finite algebraic multiplicity}\}.$$

Denote the spectral radius and essential spectral radius of A by $\rho(A)$ and $\rho_{\text{ess}}(A)$, respectively. We say that A is *quasi-compact* if $\rho_{\text{ess}}(A) < \rho(A)$. If A is quasi-compact and $\sigma(A) \cap \{\omega : |\omega| = \rho(A)\}$ consists of a single simple eigenvalue λ then we call A a *simple quasi-compact operator* and say that λ is the *leading eigenvalue* of A. In this case A has decomposition [65, III.6.4-5]

$$A = \lambda\Pi + N, \tag{1.1}$$

where Π is the rank-one eigenprojection corresponding to λ, $N \in L(E)$ is such that $\rho(N) < \rho(A)$, and $N\Pi = \Pi N = 0$. We call (1.1) the *quasi-compact decomposition* of A. Let $|\cdot|$ be a norm on E such that the closed, unit ball in $(E, \|\cdot\|)$ is $|\cdot|$-pre-compact. After possibly scaling $|\cdot|$, we may assume that $|\cdot| \leq \|\cdot\|$. Define the norm $|||\cdot|||$ on $L(E)$ by

$$|||A||| = \sup_{\|f\|=1} |Af| .$$

It is classical that if A is a simple quasi-compact operator and $\|A' - A\|$ is sufficiently small for some $A' \in L(E)$ then A' is also a simple quasi-compact operator with leading eigenvalue close to λ (see e.g. [65, IV.3.5]). However, this condition of closeness in $\|\cdot\|$ is seldom satisfied in applications to dynamical systems. In [71] it was showed that if A' is close to A in the weaker topology of $|||\cdot|||$, both operators obey a Lasota-Yorke inequality, and growth restrictions are placed on the (various) operator norms of iterates of A and A', then one can recover appropriately modified versions of the spectral stability results from operator norm based perturbation theory. We now detail the requirements for these results, referring to [71] for exact statements.

Definition 1.1.1. *We say that a family of operators $\{A_\epsilon\}_{\epsilon\geq 0} \subseteq L(E)$ satisfies the Keller-Liverani (KL) condition if each of the following conditions is verified:*

(KL1) *There exists a monotone upper-semicontinuous function $\tau : [0, \infty) \to [0, \infty)$ such that $|||A_\epsilon - A_0||| \leq \tau(\epsilon)$ and $\tau(\epsilon) > 0$ whenever $\epsilon > 0$, and $\tau(\epsilon) \to 0$ as $\epsilon \to 0$.*

(KL2) *There exists $C_1, K_1 > 0$ such that $|A_\epsilon^n| \leq C_1 K_1^n$ for every $\epsilon \geq 0$ and $n \in \mathbb{N}$.*

(KL3) *There exists $C_2, C_3, K_2 > 0$ and $\alpha \in (0, 1)$ such that*

$$\|A_\epsilon^n f\| \leq C_2\alpha^n \|f\| + C_3 K_2^n |f| \tag{1.2}$$

for every $\epsilon \geq 0$, $f \in E$ and $n \in \mathbb{N}$.

Remark 1.1.2. The inequality (1.2) is known as a Lasota-Yorke inequality after [74], and provides a wealth of information about the spectral data of any operator that may satisfy it. In particular, the Ionescu-Tulcea–Marinescu Theorem [62] (also known as Hennion's Theorem after a later strengthening by Hennion [57]), asserts that if the closed, unit ball in $(E, \|\cdot\|)$ is $|\cdot|$-pre-compact, and if $A \in L(E)$ satisfies (KL2) and (KL3) then $\rho_{\text{ess}}(A) \leq \alpha$. This result provides a commonly-trodden path to proving the quasi-compactness of any particular operator: if, in addition, one can show that $\rho(A) > \alpha$, then A is quasi-compact.

The following result summarises the conclusions of [71].

Theorem 1.1.3 ([71]). *Let $\{A_\epsilon\}_{\epsilon \geq 0} \subseteq L(E)$ satisfy (KL), where A_0 is a simple quasi-compact operator with decomposition $A_0 = \lambda_0 \Pi_0 + N_0$ and $\alpha < |\lambda_0|$. For sufficiently small $\delta > 0$ and each r such that $\max\{\alpha, \rho(N_0)\} < r < |\lambda_0|$ there exists $\epsilon_{\delta,r} > 0$ such that A_ϵ is a simple quasi-compact operator with decomposition $A_\epsilon = \lambda_\epsilon \Pi_\epsilon + N_\epsilon$ whenever $\epsilon \in [0, \epsilon_{\delta,r})$. Furthermore, for each $\epsilon \in [0, \epsilon_{\delta,r})$ the spectral data of A_ϵ satisfies $\lambda_\epsilon \in B(\lambda_0, \delta)$ and $\rho(N_\epsilon) < r$, in addition to the following Hölder estimate: there exists C such that for all ϵ sufficiently small one has*

$$\max\left\{|\lambda_\epsilon - \lambda_0|, |||\Pi_\epsilon - \Pi_0|||, |||N_\epsilon - N_0|||\right\} \leq C\tau(\epsilon)^\eta,$$

where $\eta := \frac{\ln(r/\alpha)}{\ln(\max\{K_1, K_2\}/\alpha)}$.

Remark 1.1.4. If rates of convergence are not required in Theorem 1.1.3 then it suffices to prove that $|||A_\epsilon - A_0||| \to 0$ instead of constructing the function τ in (KL1). After possibly passing to a sub-family $\{A_\epsilon\}_{\epsilon \in [0, \epsilon')}$ for some $\epsilon' > 0$, the two conditions are equivalent. We will use this fact frequently without further comment.

1.2 Spectral stability of twisted quasi-compact operators

In the setting of the previous section, rather than just considering a single, simple quasi-compact operator one sometimes considers an analytic, operator-valued map[1] $A : U \to L(E)$, where $U \subseteq \mathbb{C}$ is an open neighbourhood of 0 and $A(0)$ is a simple quasi-compact operator. This is the case, for instance, when trying to prove the stability of statistical laws for dynamical systems, which is the topic of Chapter 2 and motivates the work of the present chapter. When one considers such an analytic, operator-valued map, classical analytic perturbation theory for linear operators [65] posits the existence of a $\delta > 0$ such that $A(z)$ is a simple quasi-compact operator for each $z \in D_\delta = \{\omega \in \mathbb{C} : |\omega| < \delta\}$. Moreover, the quasi-compact decomposition of $A(z)$ depends analytically on z i.e. there are analytic maps $\lambda : D_\delta \to \mathbb{C}$, $\Pi : D_\delta \to L(E)$, and $N : D_\delta \to L(E)$ such that $A(z)$ has quasi-compact decomposition

[1] Recall that an operator-valued map $P : U \to L(E)$ is analytic if U is an open subset of $\mathbb{C}$ and P is locally representable by a $\|\cdot\|$-convergent power series.

$A(z) = \lambda(z)\Pi(z) + N(z)$. In this section we consider the question of spectral stability of such analytic, operator-valued maps under conditions similar to (KL) and when the analytic families are induced by a 'twist':

Definition 1.2.1. *If $M : U \to L(E)$ is analytic on an open neighbourhood $U \subseteq \mathbb{C}$ of 0 and $M(0)$ is the identity, then we call M a* twist. *If $A \in L(E)$ then the operators $A(z) := AM(z)$ are said to be* twisted *by M. We say that M is compactly $|\cdot|$-bounded if for every compact $V \subseteq U$ there exists $C_{M,V} > 0$ such that*

$$\sup_{z \in V} |M(z)| \leq C_{M,V}.$$

Our main result asserts that one can 'uniformly extend' the application of Theorem 1.1.3 to a family of operators satisfying (KL) to the corresponding twisted family of operators in some neighbourhood of 0, provided the twist is compatible with $|\cdot|$. We use the superscript (n) to denote the nth derivative.

Theorem 1.2.2. *Let $\{A_\epsilon\}_{\epsilon \geq 0}$ satisfy (KL), where A_0 is a simple quasi-compact operator with leading eigenvalue λ_0 satisfying $\alpha < |\lambda_0|$, and let $M : U \to \mathbb{C}$ be a compactly $|\cdot|$-bounded twist. Then there exists $\theta > 0$ such that for every compact $V \subseteq D_\theta$ there exists $\epsilon_V > 0$ and, for each $\epsilon \in [0, \epsilon_V)$, analytic functions[2] $\lambda_\epsilon : V \to \mathbb{C}$, $\Pi_\epsilon : V \to L(E)$, and $N_\epsilon : V \to L(E)$ such that $A_\epsilon(z)$ is a simple quasi-compact operator with decomposition $A_\epsilon(z) = \lambda_\epsilon(z)\Pi_\epsilon(z) + N_\epsilon(z)$ whenever $z \in V$. Additionally, the derivatives of all orders of the spectral data of $A_\epsilon(z)$ satisfy the following uniform Hölder estimate: there exists $\eta(V)$, and, for each $n \in \mathbb{N}$, a constant O_n such that for all $z \in V$ and sufficiently small ϵ one has*

$$\max \left\{ \begin{array}{c} \left|\lambda_\epsilon^{(n)}(z) - \lambda_\epsilon^{(n)}(0)\right|, \left|\left|\left|\Pi_\epsilon^{(n)}(z) - \Pi_\epsilon^{(n)}(0)\right|\right|\right|, \\ \left|\left|\left|N_\epsilon^{(n)}(z) - N_\epsilon^{(n)}(0)\right|\right|\right| \end{array} \right\} \leq O_n \tau(\epsilon)^{\eta(V)}.$$

Let us describe the strategy for proving Theorem 1.2.2. Firstly, using the fact that $\{A_\epsilon\}_{\epsilon \geq 0}$ satisfies (KL), we show that there exists $\psi > 0$ such that $\{A_\epsilon(z)\}_{\epsilon \geq 0}$ satisfies (KL) uniformly in z on compact subsets of D_ψ. In our setting standard arguments [92, 82, 65, 58] imply that $A_0(z)$ is a simple quasi-compact operator on some D_θ, where we may also assume that $\theta \in (0, \psi)$. Using a technical lemma concerning the boundedness of the resolvents of $A_0(z)$ on every compact $V \subseteq D_\theta$, we then apply the theory of [71] to obtain a uniform version of Theorem 1.1.3 for

[2] Recall that a map is said to be analytic on an arbitrary compact subset V of $\mathbb{C}$ if it may be extended to an analytic map on some larger open subset of $\mathbb{C}$.

the family of operators $\{A_\epsilon(z)\}_{\epsilon\geq 0}$ whenever $z \in V$. Theorem 1.2.2 immediately follows.

Lemma 1.2.3. *Suppose $\{A_\epsilon\}_{\epsilon\geq 0}$ satisfies (KL), $M : U \to L(E)$ is a twist and $V \subseteq U$ is compact. Let $\tau_V : [0,\infty) \to [0,\infty)$ be defined by*

$$\tau_V(\epsilon) = \left(\sup_{z\in V} \|M(z)\|\right)\tau(\epsilon).$$

Then τ_V is an upper-semicontinuous function, and (KL1) holds for $\{A_\epsilon(z)\}_{\epsilon\geq 0}$ for every $z \in V$ with τ_V in place of τ.

Proof. Note that τ_V is finite as V is compact and M is continuous on V. For each $\epsilon > 0$ and $z \in V$ the definition of $|||\cdot|||$ implies that $|||A_\epsilon(z) - A_0(z)||| \leq |||A_\epsilon - A_0|||\, \|M(z)\|$, and so using (KL1) we find that

$$\sup_{z\in V} |||A_\epsilon(z) - A_0(z)||| \leq |||A_\epsilon - A_0||| \left(\sup_{z\in V} \|M(z)\|\right) \leq \tau_V(\epsilon),$$

as required. □

Lemma 1.2.4. *If $\{A_\epsilon\}_{\epsilon\geq 0}$ satisfies (KL), $M : U \to L(E)$ is a compactly $|\cdot|$-bounded twist, and $V \subseteq U$ is compact, then there exists $K_{1,V} > 0$ such that for every $\epsilon \geq 0$ and $n \in \mathbb{N}$ we have*

$$\sup_{z\in V} |A_\epsilon(z)^n| \leq K_{1,V}^n.$$

In particular, (KL2) holds for $\{A_\epsilon(z)\}_{\epsilon\geq 0}$ for every $z \in V$.

Proof. As M is compactly $|\cdot|$-bounded there exists $C_{M,V} > 0$ such that $|M(z)| \leq C_{M,V}$ for every $x \in V$. Set $K_{1,V} = C_1 K_1 C_{M,V}$. Then for each $z \in V$, $n \in \mathbb{N}$ and $\epsilon \geq 0$ we have

$$|A_\epsilon(z)^n| \leq |A_\epsilon|^n \, |M(z)|^n \leq (C_1 K_1)^n C_{M,V}^n = K_{1,V}^n.$$

□

Lemma 1.2.5. *Under the hypotheses of Theorem 1.2.2 for every $\beta \in (\alpha, 1)$ there exists $\psi(\beta) > 0$ and $C_{2,\beta}, C_{3,\beta}, K_{2,\beta} > 0$ such that $D_{\psi(\beta)} \subseteq U$ (the domain of the twist M) and*

$$\|A_\epsilon(z)^n f\| \leq C_{2,\beta}\beta^n \|f\| + C_{3,\beta} K_{2,\beta}^n |f|$$

for every $z \in D_{\psi(\beta)}$, $f \in E$, $n \in \mathbb{N}$ and $\epsilon \geq 0$.

Proof. Fix m sufficiently large so that $2C_2\alpha^m < \beta^m$. Using (KL3) for $\{A_\epsilon\}_{\epsilon\geq 0}$ yields

$$\begin{aligned} \|A_\epsilon(z)^m f\| &\leq \|A_\epsilon^m f\| + \|A_\epsilon(z)^m - A_\epsilon^m\| \, \|f\| \\ &\leq \left(2^{-1}\beta^m + \|A_\epsilon(z)^m - A_\epsilon^m\|\right) \|f\| + C_3 K^m \, |f| \,. \end{aligned} \tag{1.3}$$

By telescoping and applying (KL3) again we obtain

$$\begin{aligned} \|A_\epsilon(z)^m - A_\epsilon^m\| &\leq \sum_{k=0}^{m-1} \|A_\epsilon\|^{k+1} \, \|M(z) - M(0)\| \, \|A_\epsilon(z)\|^{m-1-k} \\ &\leq \|M(z) - M(0)\| \sum_{k=0}^{m-1} (C_2\alpha + C_3 K)^m \, \|M(z)\|^{m-1-k} \,. \end{aligned} \tag{1.4}$$

Since the right side of (1.4) is continuous in z and vanishes at $z = 0$, there exists $\psi(\beta) > 0$ such that

$$\sup_{z\in D_{\psi(\beta)}} \sup_{\epsilon\geq 0} \|A_\epsilon(z)^m - A_\epsilon^m\| \leq 2^{-1}\beta^m.$$

Applying this to (1.3), for each $\epsilon \geq 0$ and $z \in D_{\psi(\beta)}$ we have

$$\|A_\epsilon(z)^m f\| \leq \beta^m \, \|f\| + C_3 K^m \, |f| \,.$$

We can use Lemma 1.2.4 to iterate this inequality, obtaining (KL3) for $\{A_\epsilon(z)^m\}_{\epsilon\geq 0}$ for each $z \in D_{\psi(\beta)}$ with coefficients uniform in z. Standard arguments imply that (KL3) also holds for $\{A_\epsilon(z)\}_{\epsilon\geq 0}$ for each $z \in D_{\psi(\beta)}$ with suitable modified, but still uniform in z, coefficients After possibly shrinking $\psi(\beta)$ we may assume that $D_{\psi(\beta)} \subseteq U$, which finishes the proof. □

Lemmas 1.2.3, 1.2.4 and 1.2.5 verify that, under the hypotheses of Theorem 1.2.2, the families of operators $\{A_\epsilon(z)\}_{\epsilon\geq 0}$ satisfy (KL) uniformly in z on every sufficiently small compact neighbourhood of the origin. The next result, which is a standard application of analytic perturbation theory for linear operators [92, 82, 65, 58], provides a description of the spectrum of $A_0(z)$ for z close to 0. We provide an outline of the proof in our setting.

Lemma 1.2.6. *Assume the hypotheses of Theorem 1.2.2 and let $\beta \in (\alpha, |\lambda_0|)$. There exists $\theta(\beta) \in (0, \psi(\beta))$ and maps $\lambda_0 : D_{\theta(\beta)} \to \mathbb{C}$, $\Pi_0 : D_{\theta(\beta)} \to L(E)$ and*

$N_0 : D_{\theta(\beta)} \to L(E)$ such that for each $z \in D_{\theta(\beta)}$ the operator $A_0(z)$ has quasi-compact decomposition $A_0(z) = \lambda_0(z)\Pi_0(z) + N_0(z)$ and

$$\max\left\{\beta, \sup_{z\in D_{\theta(\beta)}} \rho(N_0(z))\right\} < \inf_{z\in D_{\theta(\beta)}} |\lambda_0(z)|.$$

Proof. As $z \mapsto A_0(z)$ is analytic and A_0 is quasi-compact with decomposition $A_0 = \lambda_0\Pi_0 + N_0$, it is standard that there exists $\theta(\beta) > 0$, and analytic maps $\lambda_0 : D_{\theta(\beta)} \to \mathbb{C}$, $\Pi_0 : D_{\theta(\beta)} \to L(E)$ and $N_0 : D_{\theta(\beta)} \to L(E)$ such that $A_0(z)$ is a simple quasi-compact operator with decomposition $A_0(z) = \lambda_0(z)\Pi_0(z) + N_0(z)$ for each $z \in D_{\theta(\beta)}$. By possibly shrinking $\theta(\beta)$ we may assume that $\theta(\beta) \le \psi(\beta)$. Since $z \mapsto \lambda_0(z)$ is analytic and $\beta < |\lambda_0(0)|$, after shrinking $\theta(\beta)$ we may guarantee that $\beta < \inf_{z\in D_{\theta(\beta)}} |\lambda_0(z)|$. Furthermore, as the spectral radius is upper-semicontinuous as a function of the operator [65, IV.3.1], we may further shrink $\theta(\beta)$ so that $\sup_{z\in D_{\theta(\beta)}} \rho(N_0(z)) < \inf_{z\in D_{\theta(\beta)}} |\lambda_0(z)|$. □

A close examination of the proof of Theorem 1.1.3 reveals that in order to apply the theory in [71] 'uniformly', and thereby obtain Theorem 1.2.2, one needs a uniform bound for the norms of the resolvents of the twisted operators $A_0(z)$. For this we need some notation: for $A \in L(E)$, $\delta > 0$ and $r > \rho_{ess}(A)$ define

$$V_{\delta,r}(A) = \{\omega \in \mathbb{C} : |\omega| \le r \text{ or } \operatorname{dist}(\omega, \sigma(A)) \le \delta\}.$$

Noting that $\sigma(A) \subseteq V_{\delta,r}(A)$, let

$$J_{\delta,r}(A) = \sup\left\{\left\|(\omega - A)^{-1}\right\| : \omega \in \mathbb{C} \setminus V_{\delta,r}(A)\right\}.$$

Lemma 1.2.7. *Assume the hypotheses of Theorem 1.2.2. Let $\beta \in (\alpha, |\lambda_0|)$ and recall $\theta(\beta)$ from Lemma 1.2.6. For every compact $V \subseteq D_{\theta(\beta)}$, $\delta > 0$ and $r > \sup_{z\in D_{\theta(\beta)}} \rho(N_0(z))$ we have*

$$\sup_{z\in V} J_{\delta,r}(A_0(z)) < \infty.$$

Proof. For every $z \in D_{\theta(\beta)}$ and $\omega \in \mathbb{C}\setminus\sigma(A_0(z))$ let $R(\omega, z) = (\omega - A_0(z))^{-1}$ denote the resolvent of $A_0(z)$ at ω. Fix $z \in V$. Recall from Lemma 1.2.6 that $A_0(z)$ is a simple quasi-compact operator with decomposition $A_0(z) = \lambda_0(z)\Pi_0(z) + N_0(z)$. As

$\lambda_0(z)$ is an isolated simple eigenvalue of $A_0(z)$, the partial-fraction decomposition of the resolvent [65, III-(6.32)] yields

$$R(\omega, z) = (\omega - \lambda_0(z))^{-1}\Pi_0(z) + S(\omega, z), \tag{1.5}$$

where for each $\omega \in \{\lambda_0(z)\} \cup (\mathbb{C} \setminus \sigma(A_0(z)))$ the operator

$$S(\omega, z) := \lim_{\omega' \to \omega} R(\omega', z)(\mathrm{Id} - \Pi_0(z))$$

is the reduced resolvent of $A_0(z)$ with respect to $\lambda_0(z)$ at ω [65, III-(6.30), III-(6.31)]. We have

$$\sup_{\omega \in \mathbb{C} \setminus V_{\delta,r}(A_0(z))} \|R(\omega, z)\| \le \delta^{-1} \sup_{z \in V} \|\Pi_0(z)\| + \sup_{\omega \in \mathbb{C} \setminus V_{\delta,r}(A_0(z))} \|S(\omega, z)\|. \tag{1.6}$$

Since $z \mapsto \Pi_0(z)$ is analytic on V, which is compact, we have $\sup_{z \in V} \|\Pi_0(z)\| < \infty$. Hence, to complete the proof it suffices to bound $\sup_{\omega \in \mathbb{C} \setminus V_{\delta,r}(A_0(z))} \|S(\omega, z)\|$ uniformly in $z \in V$. As noted immediately after [65, III-(6.25)], when restricted to the image of $\mathrm{Id} - \Pi_0(z)$ the operator $S(\omega, z)$ coincides with the resolvent of $A_0(z)(\mathrm{Id} - \Pi_0(z)) = N_0(z)$ at ω. Since $S(\omega, z)$ vanishes on the image of $\Pi_0(z)$, this implies that $S(\omega, z) = (\omega - N_0(z))^{-1}(\mathrm{Id} - \Pi_0(z))$. As $r > \sup_{z \in D_{\theta(\beta)}} \rho(N_0(z))$, for $\omega \in \mathbb{C} \setminus D_r$ the standard Neumann series[3] for the resolvent [65, I-(5.10)] yields

$$S(\omega, z) = \left(\sum_{k=0}^{\infty} \omega^{-k-1} N_0(z)^k \right) (\mathrm{Id} - \Pi_0(z)). \tag{1.7}$$

By the spectral radius formula, the upper-semicontinuity of the spectral radius and the continuity of $z \mapsto N_0(z)$, we have that for any fixed $s \in \left(\sup_{z \in D_{\theta(\beta)}} \rho(N_0(z)), r \right)$ there exists $H > 0$ such that for every $n \in \mathbb{N}$ and $z \in V$ we have $\|N_0(z)^n\| \le H s^n$. Using (1.7) we therefore have

$$\begin{aligned}
\sup_{\omega \in \mathbb{C} \setminus V_{\delta,r}(A_0(z))} \|S(\omega, z)\| &\le \sup_{\omega \in \mathbb{C} \setminus B(0,r)} \|S(\omega, z)\| \\
&\le \sup_{\omega \in \mathbb{C} \setminus B(0,r)} \left(\sum_{k=0}^{\infty} \left|\omega^{-k-1}\right| \left\|N_0(z)^k\right\| \right) \|\mathrm{Id} - \Pi_0(z)\| \\
&\le \frac{H}{r} \left(\sum_{k=0}^{\infty} \left(\frac{s}{r}\right)^k \right) \left(\sup_{z \in V} \|\mathrm{Id} - \Pi_0(z)\| \right),
\end{aligned}$$

[3] While the formula in [65, I-(5.10)] is only given for the finite-dimensional case, it also holds in the current setting by the same arguments.

which is finite as $z \mapsto \operatorname{Id} - \Pi_0(z)$ is analytic on V and $s < r$. The proof is completed by recalling (1.6) and the definition of $J_{\delta,r}(A_0(z))$. $\square$

The proof of Theorem 1.2.2. Fix $\beta \in (\alpha, |\lambda_0|)$, recall $\theta(\beta)$ from Lemma 1.2.6, and let $V \subseteq D_{\theta(\beta)}$ be compact. Out of necessity, we construct $\lambda_\epsilon(z), \Pi_\epsilon(z)$ and $N_\epsilon(z)$ for z in a larger compact set whose interior contains V. As V is compact there exists some $\gamma \in (0, \theta(\beta))$ such that $V \subseteq D_\gamma$. By Lemmas 1.2.3, 1.2.4 and 1.2.5, for each $z \in \overline{D_\gamma}$ the family of operators $\{A_\epsilon(z)\}_{\epsilon \geq 0}$ satisfies (KL) with data

$$\tau_\gamma, C_{1,\gamma}, C_{2,\gamma}, C_{3,\gamma}, K_{1,\gamma}, K_{2,\gamma} \text{ and } \beta, \tag{1.8}$$

uniform in z over $\overline{D_\gamma}$.

By Lemma 1.2.6 there exists $r > \max\{\beta, \sup_{z\in D_{\theta(\beta)}} \rho(N_0(z))\}$ and $\delta > 0$ such that $r + \delta < \inf_{z\in\overline{D_\gamma}} |\lambda_0(z)|$. It follows that

$$\{\lambda_0(z)\} = \{\omega \in \sigma(A_0(z)) : |\omega - \lambda_0(z)| < \delta\}.$$

for every $z \in \overline{D_\gamma}$. Hence, by [71, Theorem 1 and the inequality (10)] for each $z \in \overline{D_\gamma}$ there exists $\epsilon_{\delta,r,z} > 0$ such that for $\epsilon \in [0, \epsilon_{\delta,r,z})$ and $\omega \in \mathbb{C}$, with $|\omega - \lambda_0(z)| = \delta$, the operator $(\omega - A_\epsilon(z))^{-1}$ is bounded and so the spectral projection

$$\Pi_\epsilon(z) = \frac{1}{2\pi i} \int_{|\omega-\lambda_0(z)|=\delta} (\omega - A_\epsilon(z))^{-1} \, d\omega \tag{1.9}$$

is a well-defined element of $L(E)$. From the definitions of ϵ_0 and ϵ_1 in the proof of [71, Corollary 1], and the definition [71, (13)] we see that $\epsilon_{\delta,r,z}$ may be chosen independently of $z \in \overline{D_\gamma}$ as (KL) is satisfied for each $\{A_\epsilon(z)\}_{\epsilon\geq 0}$ with data (1.8) independent of $z \in \overline{D_\gamma}$ and as

$$\sup_{z\in\overline{D_\gamma}} J_{\delta,r}(A_0(z)) < \infty, \tag{1.10}$$

which is a consequence of Lemma 1.2.7. The same argument applied to [71, part (3) of Corollary 1] implies that there exists $\epsilon_V \in (0, \epsilon_{\delta,r,z})$ and $\delta_\gamma > 0$ such that if $\delta \in (0, \delta_\gamma)$ then $\operatorname{rank}(\Pi_\epsilon(z)) = \operatorname{rank}(\Pi_0(z)) = 1$ for every $\epsilon \in [0, \epsilon_V)$ and $z \in \overline{D_\gamma}$. Since $\operatorname{rank}(\Pi_\epsilon(z)) = 1$, each $A_\epsilon(z)$ has a simple eigenvalue $\lambda_\epsilon(z)$ such that $|\lambda_\epsilon(z) - \lambda_0(z)| < \delta$. By [71, (10)] we have $\sigma(A_\epsilon(z)) \subseteq \{\omega \in \mathbb{C} : |\omega - \lambda_0(z)| <$

δ or $|\omega| < r\}$. Since $\{\omega \in \sigma(A_\epsilon(z)) : |\omega - \lambda_0(z)| < \delta\} = \{\lambda_\epsilon(z)\}$, it follows that $\sigma(A_\epsilon(z)) \setminus \{\lambda_\epsilon(z)\} \subseteq D_r$. Hence,

$$\mathrm{Id} - \Pi_\epsilon(z) = \frac{1}{2\pi i} \int_{|\omega|=r} (\omega - A_\epsilon(z))^{-1} \, \mathrm{d}\omega. \tag{1.11}$$

Defining $N_\epsilon(z) = A_\epsilon(z)(\mathrm{Id} - \Pi_\epsilon(z))$, for every $z \in \overline{D_\gamma}$ and $\epsilon \in [0, \epsilon_V)$ we therefore have that $A_\epsilon(z)$ is a simple quasi-compact operator with decomposition $A_\epsilon(z) = \lambda_\epsilon(z)\Pi_\epsilon(z) + N_\epsilon(z)$.

We will now show that for each $\epsilon \in [0, \epsilon_V)$ the maps $z \mapsto \lambda_\epsilon(z)$, $z \mapsto \Pi_\epsilon(z)$, and $z \mapsto N_\epsilon(z)$ are analytic on D_γ. As the contour in the integral in (1.11) is fixed, it is well-known that $z \mapsto \mathrm{Id} - \Pi_\epsilon(z)$ is analytic on D_γ [65, VIII.1.3 Theorem 1.7]. Hence $z \mapsto \Pi_\epsilon(z)$ is analytic on D_γ. The map $z \mapsto N_\epsilon(z)$ is analytic on D_γ as $z \mapsto A_\epsilon(z)$ is analytic and by the definition of $N_\epsilon(z)$. That $z \mapsto \lambda_\epsilon(z)$ is analytic on D_γ follows from $\lambda_\epsilon(z)$ having algebraic multiplicity 1 and by the discussion in [65, II.1.8]. Hence $z \mapsto \lambda_\epsilon(z), z \mapsto \Pi_\epsilon(z)$ and $z \mapsto N_\epsilon(z)$ are analytic on V as they may be extended to analytic maps on an open subset of $\mathbb{C}$ that contains V, namely D_γ.

We now confirm that the required Hölder estimate holds for the various spectral data using (KL) for $\{A_\epsilon(z)\}_{\epsilon \geq 0}$ and with uniform data as in (1.9). By [71, Corollary 1], there exists $H_{\delta,r,z} > 0$ such that $|||\Pi_\epsilon(z) - \Pi_0(z)||| \leq H_{\delta,r,z}\tau_V(\epsilon)^{\eta(V)}$ for every $\epsilon \in [0, \epsilon_V)$ and $z \in \overline{D_\gamma}$, where

$$\eta(V) = \frac{\ln(r/\beta)}{\ln(\max\{K_{1,\gamma}, K_{2,\gamma}\}/\beta)}.$$

Recalling the bound (1.10) and that (1.8) is independent of $z \in \overline{D_\gamma}$, we conclude from the proof of [71, Corollary 1] that $H_{\delta,r,z}$ can be chosen independently of $z \in \overline{D_\gamma}$. Moreover, by Lemma 1.2.3 we have $\tau_V(\epsilon) = \sup_{z \in \overline{D_\gamma}} \|M(z)\| \, \tau(\epsilon)$. Hence, if we set $O_0' = H_{\delta,r,z} \sup_{z \in \overline{D_\gamma}} \|M(z)\|^{\eta(V)}$ then for all $\epsilon \in [0, \epsilon_V)$ we have

$$\sup_{z \in \overline{D_\gamma}} |||\Pi_\epsilon(z) - \Pi_0(z)||| \leq O_0' \tau(\epsilon)^{\eta(V)}. \tag{1.12}$$

By definition we have

$$(\lambda_0(z) - \lambda_\epsilon(z))\Pi_0(z) = (\lambda_\epsilon(z) - A_\epsilon(z))(\Pi_\epsilon(z) - \Pi_0(z)) + (A_0(z) - A_\epsilon(z))\Pi_0(z),$$

and so

$$|\lambda_0(z) - \lambda_\epsilon(z)| \, |||\Pi_0(z)||| \leq (|\lambda_\epsilon(z)| + |A_\epsilon(z)|) |||\Pi_\epsilon(z) - \Pi_0(z)||| \\ + |||A_0(z) - A_\epsilon(z)||| \, \|\Pi_0(z)\| \,. \tag{1.13}$$

For every $\epsilon \in [0, \epsilon_V)$ and $z \in \overline{D_\gamma}$ we have $|A_\epsilon(z)| \leq K_{1,\gamma}$ and $|\lambda_\epsilon(z)| \leq |\lambda_0(0)| + \delta$. Provided that γ is sufficiently small, which we may guarantee by shrinking $\theta(\beta)$, by the analyticity of $z \mapsto \Pi_0(z)$ we have that $\sup_{z \in \overline{D_\gamma}} \|\Pi_0(z)\| < \infty$ and that

$$\inf_{z \in \overline{D_\gamma}} |||\Pi_0(z)||| \geq |||\Pi_0(0)||| - \sup_{z \in \overline{D_\gamma}} \|\Pi_0(z) - \Pi_0(0)\| > 0.$$

By the estimates in the previous two sentences, Lemma 1.2.3 and (1.12) it follows that for every $z \in \overline{D_\gamma}$ and $\epsilon \in [0, \epsilon_V)$ we have

$$\begin{aligned} &\left(\frac{|\lambda_\epsilon(z)| + |A_\epsilon(z)|}{|||\Pi_0(z)|||} \right) |||\Pi_\epsilon(z) - \Pi_0(z)||| + \frac{\|\Pi_0(z)\|}{|||\Pi_0(z)|||} |||A_0(z) - A_\epsilon(z)||| \\ &\leq \left(\frac{(K_{1,\gamma} + \delta + |\lambda_0(0)|) O_0' + \sup_{z \in \overline{D_\gamma}} \|\Pi_0(z)\|}{\inf_{z \in \overline{D_\gamma}} |||\Pi_0(z)|||} \right) \tau(\epsilon)^{\eta(V)} \\ &:= O_0'' \tau(\epsilon)^{\eta(V)} < \infty, \end{aligned}$$

which, when applied to (1.13), yields

$$\sup_{z \in \overline{D_\gamma}} |\lambda_0(z) - \lambda_\epsilon(z)| \leq O_0'' \tau(\epsilon)^{\eta(V)}.$$

Examining the proof of [71, Corollary 2] and using the same arguments as before, we similarly find a constant O_0''' such that

$$\sup_{z \in \overline{D_\gamma}} |||N_\epsilon(z) - N_0(z)||| \leq O_0''' \tau(\epsilon)^{\eta(V)}.$$

Since $V \subseteq \overline{D_\gamma}$, the required uniform Hölder estimate for the undifferentiated spectral data holds on V with $O_0 = \max\{O_0', O_0'', O_0'''\}$ (i.e. we obtain the conclusion of Theorem 1.2.2 in the case where $n = 0$). For every compact subset of D_γ one derives a uniform Hölder estimate for the nth derivative of $z \mapsto \lambda_\epsilon(z)$, $z \mapsto \Pi_\epsilon(z)$ and $z \mapsto N_\epsilon(z)$ by a standard application of Cauchy's integral formula along the contour ∂D_γ. In particular, we obtain the required uniform Hölder estimate on V, which concludes the proof of Theorem 1.2.2. $\square$

1.3 On the robustness of (KL)

In this section we discuss a result on the robustness of the condition (KL) to perturbations that are simultaneously small in the operator norms $\|\cdot\|$ and $|\cdot|$. This result will be useful in Chapter 5, where we will use it to reduce the proof of (KL) for some classes of perturbations into a number of more manageable cases. The flow of the proof is similar to that of Lemmas 1.2.3, 1.2.4 and 1.2.5.

Proposition 1.3.1. *Suppose that $\{A_\epsilon\}_{\epsilon\geq 0}$ satisfies (KL) and $\{B_\epsilon\}_{\epsilon\in[0,\epsilon_1)} \subseteq L(E)$ satisfies $B_0 = 0$, $\lim_{\epsilon\to 0}\|B_\epsilon\| = 0$ and $\sup_{\epsilon\in[0,\epsilon_1)}|B_\epsilon| < \infty$. Then there exists $\epsilon_2 \in (0,\epsilon_1)$ so that $\{A_\epsilon + B_\epsilon\}_{\epsilon\in[0,\epsilon_2)}$ satisfies (KL).*

Proof. We prove (KL1), (KL2) and (KL3) separately.

(KL1) As $\|B_\epsilon\| \to 0$, there exists $\epsilon' > 0$ so that $\sup_{\epsilon\in[0,\epsilon')}\|B_\epsilon\| < \infty$. As $\{A_\epsilon\}_{\epsilon\geq 0}$ satisfies (KL1), $\{B_\epsilon\}_{\epsilon\in[0,\epsilon')}$ is bounded in $L(E)$, $\|B_\epsilon\| \to 0$, and

$$|||A_\epsilon + B_\epsilon - A_0||| \leq |||A_\epsilon - A_0||| + \|B_\epsilon\|,$$

it is clear that (KL1) is satisfied.

(KL2) As $\{A_\epsilon\}_{\epsilon\geq 0}$ satisfies (KL2) and $\sup_{\epsilon\in[0,\epsilon_1)}|B_\epsilon| < \infty$, we have

$$\sup_{\epsilon\in[0,\epsilon_1)}|A_\epsilon + B_\epsilon| \leq C_1 K + \sup_{\epsilon\in[0,\epsilon_1)}|B_\epsilon| < \infty.$$

The required bound follows by iterating this inequality.

(KL3) By expanding $(A_\epsilon + B_\epsilon)^n$, applying a counting argument, and using (KL3) for $\{A_\epsilon\}_{\epsilon\geq 0}$ we have

$$\begin{aligned}\|(A_\epsilon + B_\epsilon)^n f\| &\leq \|A_\epsilon^n f\| + \sum_{k=0}^{n-1}\frac{n!}{k!(n-k)!}\|A_\epsilon\|^k\|B_\epsilon\|^{n-k}\|f\| \\ &\leq C_2\alpha^n\|f\| + C_3K_2^n|f| + 2^n\sum_{k=0}^{n-1}\|A_\epsilon\|^k\|B_\epsilon\|^{n-k}\|f\|.\end{aligned} \tag{1.14}$$

Let $\beta \in (\alpha, 1)$ and choose $n \in \mathbb{N}$ so that $C_1\alpha^n < \beta^n$. As (KL3) holds for $\{A_\epsilon\}_{\epsilon\geq 0}$, it follows that $\{A_\epsilon\}_{\epsilon\geq 0}$ is bounded in $L(E)$. Hence, as $\|B_\epsilon\| \to 0$, there exists $\epsilon'' > 0$

so that for every $\epsilon \in [0, \epsilon'')$ we have

$$C_1 \alpha^n + 2^n \sum_{k=0}^{n-1} \|A_\epsilon\|^k \|B_\epsilon\|^{n-k} \le \beta^n.$$

By applying (1.3.1) to (1.14) for all $\epsilon \in [0, \epsilon'')$ we have

$$\|(A_\epsilon + B_\epsilon)^n f\| \le \beta^n \|f\| + C_3 K_2^n |f|. \tag{1.15}$$

Using (KL2) for $\{A_\epsilon + B_\epsilon\}_{\epsilon \in [0, \epsilon_1)}$ one may iterate (1.15) to obtain for all $\epsilon \in [0, \epsilon'')$ and $k \in \mathbb{Z}^+$ that

$$\left\|(A_\epsilon + B_\epsilon)^{nk} f\right\| \le \beta^{nk} \|f\| + C_4 K_3^{nk} |f|, \tag{1.16}$$

where $C_4, K_3 > 0$ are independent of k and ϵ. Since $\sup_{\epsilon \in [0, \epsilon'')} \|A_\epsilon + B_\epsilon\|$ is finite one easily obtains (KL3) from (1.16). $\square$

Chapter 2

Stability of statistical laws

In this chapter we consider the stability of parameters of statistical limit laws for sequences of weakly-dependent random variables when said laws are obtained via the naive Nagaev-Guivarc'h spectral method. In Section 2.1 we review the details of the method, and recall how it may be used to obtain a central limit theorem (CLT) or large deviation principle (LDP). In Section 2.2 we state and prove our main result for this chapter (Theorem 2.2.1), which is a stability result for the variance of a CLT and the rate function of a LDP when these limit laws were obtained via the Nagaev-Guivarc'h method in a setting compatible with the perturbation theory developed in Chapter 1. Lastly, in Section 2.3 we also provide a explicit, computationally-tractable formula for the variance of a CLT that holds under the same hypotheses as Theorem 2.2.1, which will be used to estimate the variance of a CLT for dynamical systems in later chapters.

2.1 Review of the Nagaev-Guivarc'h method

The naive Nagaev-Guivarc'h spectral method is an approach to obtaining statistical limit laws for weakly-dependent sequences of random variables when the dependence structure of these random variables is 'coded' by an analytic family of simple quasi-compact operators. Importantly, the parameters of the resulting limit laws are often completely determined by the behaviour of the leading eigenvalue of the aforementioned analytic family of operators, which will allow us to use spectral stability theory to prove the stability of these parameters.

In this section we recall how a CLT and LDP may be obtained via the naive Nagaev-Guivarc'h method. These results only constitute a small component of a large, developing literature. We refer the interested reader to [53] for a comprehensive overview of the application of the method to dynamical systems, and to [58, 92, 56, 82] for historical context on the method's development.

In what follows let $\{Y_k\}_{k\in\mathbb{N}}$ is a sequence of real-valued random variables with partial sums $S_n = \sum_{k=0}^{n-1} Y_k$ satisfying $\lim_{n\to\infty} \mathbb{E}(S_n)/n = 0$.

Theorem 2.1.1 (Central limit theorem [53, Theorem 2.4])**.** *If there exist a Banach space E, operator-valued map $A : I \to L(E)$, where I is a real open neighbourhood of 0, and $\zeta \in E$, $\nu \in E^*$ such that:*

1. *$A(0)$ is a simple quasi-compact operator with $\rho(A(0)) = 1$.*
2. *The mapping $t \mapsto A(t)$ is $\mathcal{C}^2$ as a map into $(L(E), \|\cdot\|)$.*
3. *$\mathbb{E}(e^{itS_n}) = \nu(A(t)^n\zeta)$ for all $n \in \mathbb{N}$ and $t \in I$.*

Then $\{Y_k\}_{k\in\mathbb{N}}$ satisfies a CLT: there exists $\sigma^2 \geq 0$ such that $S_n/\sqrt{n}$ converges in distribution to a $N(0, \sigma^2)$ random variable as $n \to \infty$.

Theorem 2.1.2 (Large deviation principle [36, Remark 2.3])**.** *If there exist a Banach space E, operator-valued map $A : I \to L(E)$, where I is a real open neighbourhood of 0, and $\zeta \in E$, $\nu \in E^*$ such that:*

1. *$A(0)$ is a simple quasi-compact operator with $\rho(A(0)) = 1$.*
2. *The mapping $t \mapsto A(t)$ is $\mathcal{C}^1$ as a map into $(L(E), \|\cdot\|)$ and $t \mapsto \ln \rho(A(t))$ is strictly convex in some neighbourhood of 0.*
3. *$\mathbb{E}(e^{tS_n}) = \nu(A(t)^n\zeta)$ for all $n \in \mathbb{N}$ and $t \in I$.*

Then $\{Y_k\}_{k\in\mathbb{N}}$ satisfies a LDP: there exists a non-negative, continuous and convex rate function $r : J \to \mathbb{R}$, where J is an open neighbourhood of 0, such that for every $\epsilon \in J \cap (0, \infty)$ we have

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathrm{Prob}(S_n \geq n\epsilon) = -r(\epsilon).$$

Remark 2.1.3. As stated, Theorem 2.1.2 differs slightly to the result in [36, Remark 2.3], however a straightforward modification of the arguments from [36] readily yields Theorem 2.1.2. In particular, see Lemma 2.1.5.

Both the CLT and LDP are parameterised, and under the settings of Theorems 2.1.1 and 2.1.2 these parameters are determined by the spectral data of $A(t)$ as follows. As $t \mapsto A(t)$ is $\mathcal{C}^k$ $(k = 1, 2)$ and $A(0)$ is a simple quasi-compact operator, by [53, Proposition 2.3] there exists $\theta > 0$ and $\mathcal{C}^k$ maps $\lambda : (-\theta, \theta) \to \mathbb{C}$, $\Pi : (-\theta, \theta) \to L(E)$, and $N : (-\theta, \theta) \to L(E)$ such that $A(t)$ is a simple quasi-compact with decomposition $A(t) = \lambda(t)\Pi(t) + N(t)$ for $t \in (-\theta, \theta)$. In this case the variance of the CLT is

$$\sigma^2 = \lambda^{(2)}(0). \tag{2.1}$$

After possibly shrinking θ one may show that $\lambda(z) > 0$ and $\nu(\Pi(z)\zeta) \neq 0$ for all $z \in (-\theta, \theta)$ (see Lemma 2.1.5), in which case the rate function of the LDP is

$$r(s) = \sup_{t\in(-\theta,\theta)} (st - \ln \lambda(t)). \tag{2.2}$$

Moreover, due to the strict convexity and continuous differentiability of $t \mapsto \ln \lambda(t)$ on $(-\theta, \theta)$, and the application of the local Gartner-Ellis Theorem [58, Lemma XIII.2] used to obtain Theorem 2.1.2, we have that the domain of the rate function is

$$\left(\frac{\lambda'(-\theta)}{\lambda(-\theta)}, \frac{\lambda'(\theta)}{\lambda(\theta)} \right). \tag{2.3}$$

In Theorem 2.1.1 the characteristic function of S_n is encoded by $\nu(A(t)^n\zeta)$ whilst in Theorem 2.1.2 it is the moment-generating function of S_n that is encoded. These settings are frequently unified by the following condition, which implies both Theorem 2.1.1 and 2.1.2 and is frequently verified by applications of the naive Nagaev-Guivarc'h method to dynamical systems.

Definition 2.1.4. *Suppose that $\{Y_k\}_{k\in\mathbb{N}}$ is a sequence of real random variables on a common probability space (Ω, m) with partial sums $S_n = \sum_{k=0}^{n-1} Y_k$ satisfying $\lim_{n\to\infty} \mathbb{E}(S_n)/n = 0$. We say that $\{Y_k\}_{k\in\mathbb{N}}$ satisfies (NG) if there exists a Banach space $(E, \|\cdot\|)$, $\zeta \in E$, $\nu \in E^*$, and an analytic operator-valued map $A : U \mapsto L(E)$, where $U \subseteq \mathbb{C}$ is an open neighbourhood of 0, such that $A(0)$ is a simple quasi-compact operator, $\rho(A(0)) = 1$, $t \mapsto \rho(A(t))$ is strictly convex in some real neighbourhood of 0, and*

$$\mathbb{E}(e^{zS_n}) = \nu(A(z)^n\zeta) \tag{2.4}$$

for every $z \in U$ and $n \in \mathbb{N}$. When $\{Y_k\}_{k\in\mathbb{N}}$ satisfies (NG) as in the previous sentence we shall say it has coding (A, ζ, ν).

We finish this section by substantiating the claims about the choice of θ in the definition of the rate function in (2.2).

Lemma 2.1.5. *Under the hypotheses of Theorem 2.1.2, if θ is sufficiently small then $\lambda(z) > 0$ and $\nu(\Pi(z)\zeta) \neq 0$ for $z \in (-\theta, \theta)$.*

Proof. We start by proving that $\nu(\Pi(0)\zeta) = 1$. By (2.4) and the quasi-compact decomposition of $A(0)$ we have

$$1 = \lambda(0)^n \nu(\Pi(0)\zeta) + \nu(N(z)^n \zeta).$$

Hence, as $|\lambda(0)| = \rho(A(0)) = 1$, we have

$$\nu(\Pi(0)\zeta) = \lambda(0)^{-n} - \lambda(0)^{-n}\nu(N(0)^n\zeta).$$

Since $\rho(N(0)) < 1$ by taking n to infinity we see that $\lim_{n\to\infty} \lambda(0)^{-n}$ exists and is equal to $\nu(\Pi(0)\zeta)$. Since $|\lambda(0)| = 1$ it follows that $\nu(\Pi(0)\zeta) = \lambda(0) = 1$. Since $z \mapsto \Pi(z)$ is $\mathcal{C}^1$ on a neighbourhood of 0 it follows that by shrinking θ we may guarantee that $\nu(\Pi(0)\zeta) \neq 0$ for $z \in (-\theta, \theta)$.

We will now show that $\lambda(z) > 0$ for $z \in (-\theta, \theta)$. Let S^1 denote the circle, which may be identified with $(-\pi, \pi]$, and let $\mathrm{Arg} : \mathbb{C} \to S^1$ denote the principal argument. For $z \in (-\theta, \theta)$ and $n \in \mathbb{Z}^+$ we have by (2.4) and the quasi-compact decomposition of $A(z)$ that

$$\begin{aligned} 0 &= \mathrm{Arg}\left(\mathbb{E}(e^{zS_n})\right) \\ &= \mathrm{Arg}\left(\lambda(z)^n \nu\left(\Pi(z)\zeta + \lambda(z)^{-n} N(z)^n \zeta\right)\right) \\ &= n\,\mathrm{Arg}(\lambda(z)) + \mathrm{Arg}\left(\nu(\Pi(z)\zeta) + \lambda(z)^{-n}\nu\left(N(z)^n\zeta\right)\right). \end{aligned} \tag{2.5}$$

Since $\nu(\Pi(z)\zeta) \neq 0$ and $\rho(N(z)) < \lambda(z)$ for $z \in (-\theta, \theta)$, if we divide (2.5) by n and then let $n \to \infty$ we find that $\mathrm{Arg}(\lambda(z)) = 0$. Since $\lambda(z) \neq 0$ for $z \in (-\theta, \theta)$ we therefore obtain the required claim. $\square$

2.2 A stability result for statistical laws obtained via the Nagaev-Guivarc'h method

The main result for this chapter is the following.

Theorem 2.2.1. *Let $\{Y_k\}_{k\in\mathbb{N}}$ be a sequence of real random variables satisfying (NG) with coding (A, ζ, ν). Suppose that $|\cdot|$ is a second norm on E so that the closed, unit ball in $(E, \|\cdot\|)$ is $|\cdot|$-pre-compact and that there exists a compactly $|\cdot|$-bounded twist $M : U \to L(E)$ such that $A(z) = A(0)M(z)$ for every $z \in U$. If $\{A_\epsilon\}_{\epsilon\geq 0}$ satisfies (KL), where $A_0 = A(0)$, then there exists $\theta, \epsilon' > 0$ and, for every $\epsilon \in [0, \epsilon')$, analytic maps $\lambda_\epsilon : D_\theta \to \mathbb{C}$, $\Pi_\epsilon : D_\theta \to L(E)$ and $N_\epsilon : D_\theta \to L(E)$ such that $A_\epsilon(z)$ is a simple quasi-compact operator with decomposition $A_\epsilon(z) = \lambda_\epsilon(z)\Pi_\epsilon(z) + N_\epsilon(z)$ for* *every $\epsilon \in [0, \epsilon')$ and $z \in D_\theta$. Moreover, we have stability of the parameters of the CLT and LDP for $\{Y_k\}_{k\in\mathbb{N}}$ in the following sense:*

1. *The variance is stable: $\lim_{\epsilon\to 0} \lambda_\epsilon^{(2)}(0) = \sigma^2$.*
2. *The rate function is stable: for each sufficiently small compact subset U of the domain of the rate function r there exists an interval $V \subseteq (-\theta, \theta)$ so that*

$$\lim_{\epsilon\to 0} \sup_{z\in V}(sz - \ln|\lambda_\epsilon(z)|) = r(s)$$

uniformly on U.

We split the proof of Theorem 2.2.1 into two parts. First we prove the easier claims regarding the spectral properties of $A_\epsilon(z)$ and the stability of the variance. The stability of the rate function requires some more work, and so we must prepare a few lemmas before attempting the remainder of the proof.

Part 1 of the proof of Theorem 2.2.1. By Theorem 1.2.2 there exists $\theta, \epsilon' > 0$ and, for each $\epsilon \in [0, \epsilon')$, maps $z \mapsto \lambda_\epsilon(z), z \mapsto \Pi_\epsilon(z)$ and $z \mapsto N_\epsilon(z)$ as required by Theorem 2.2.1. Moreover, by Theorem 2.2.1 we have

$$\lim_{\epsilon\to 0} \lambda_\epsilon^{(2)}(0) = \lambda_0^{(2)}(0) = \sigma^2,$$

which is the claimed stability of the variance. □

We now turn to the proof of stability of the rate function, for which we shall assume that θ is small enough so that the conclusions of Lemma 2.1.5 hold. We begin with the definition of the convex conjugation (also known as the Legendre-Fenchel transform).

Definition 2.2.2. *The* convex conjugate *of a function $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, is the function $f^* : \mathbb{R} \to \mathbb{R}$ defined by*

$$f^*(y) = \sup_{x\in I} (xy - f(x)).$$

It is clear that the rate function r of the LDP is exactly the convex conjugate of $z \mapsto \ln \lambda_0(z)$ restricted to (2.3). Naively, one might define approximate rate functions by taking the convex conjugate of $z \mapsto \ln|\lambda_\epsilon(z)|$; this approach is somewhat messy since the resulting approximations have substantially different asymptotic behaviour. However, this obstruction is ultimately not too large and only some diligent bookkeeping is required to prove the claimed convergence. Specifically, for each closed interval $V \subseteq (-\theta,\theta)$ and $\epsilon \in [0,\epsilon')$ let $\Lambda_{\epsilon,V} : V \to \mathbb{R}$ be defined by $\Lambda_{\epsilon,V}(z) = \ln|\lambda_\epsilon(z)|$. We notice from the proof of Theorem 1.2.2 that $\inf_{z\in(-\theta,\theta),\epsilon\in[0,\epsilon')}|\lambda_\epsilon(z)| > 0$ and so $\Lambda^*_{\epsilon,V}$ is well defined.

Lemma 2.2.3. *For every closed interval $V \subseteq (-\theta,\theta)$ we have*

$$\lim_{\epsilon\to 0}\sup_{y\in\mathbb{R}}\left|\Lambda^*_{\epsilon,V}(y) - \Lambda^*_{0,V}(y)\right| = 0.$$

Proof. For every $\epsilon \in [0,\epsilon')$ we have

$$|\Lambda_{0,V}(z) - \Lambda_{\epsilon,V}(z)| \le \left(\sup_{z\in V,\epsilon\in[0,\epsilon')}|\lambda_\epsilon(z)|^{-1}\right)|\lambda_\epsilon(z) - \lambda_0(z)| := \eta_\epsilon(z).$$

Let $y \in \mathbb{R}$. Since V is compact and $z \mapsto zy - \Lambda_{0,V}(z)$ is continuous on V there exists $z_y \in V$ such that $\Lambda^*_{0,V}(y) = z_y y - \Lambda_{0,V}(z_y)$. Hence,

$$\left|\Lambda_{\epsilon,V}(z_y) - (z_y y - \Lambda^*_{0,V}(y))\right| = |\Lambda_{\epsilon,V}(z_y) - \Lambda_{0,V}(z_y)| \le \eta_\epsilon(z_y),$$

and so by the definition of convex conjugation one has

$$\Lambda^*_{0,V}(y) \le \eta_\epsilon(z_y) + z_y y - \Lambda_{\epsilon,V}(z_y) \le \eta_\epsilon(z_y) + \Lambda^*_{\epsilon,V}(y).$$

The same argument yields the same bound with $\Lambda^*_{\epsilon,V}$ and $\Lambda^*_{0,V}$ swapped. Thus

$$\sup_{y\in\mathbb{R}}\left|\Lambda^*_{0,V}(y) - \Lambda^*_{\epsilon,V}(y)\right| \le \sup_{z\in V}\eta_\epsilon(z). \tag{2.6}$$

By Theorem 1.2.2 we have $\inf_{z\in V,\epsilon\in[0,\epsilon')}|\lambda_\epsilon(z)| > 0$ and $\lambda_\epsilon(z) \to \lambda_0(z)$ uniformly on V as $\epsilon \to 0$. We therefore obtain the required claim by taking the limit as $\epsilon \to 0$ in (2.6). □

While Lemma 2.2.3 confirms that $\Lambda^*_{\epsilon,V}$ and $\Lambda^*_{0,V}$ are close, it is not enough to conclude that $\Lambda^*_{0,V}$ and r are close. To clarify this relationship we will use the fact that $z \mapsto \ln(\lambda_0(z))$ is convex on $(-\theta,\theta)$, which is the content of the next lemma. While the strict convexity of $z \mapsto \ln\rho(A_0(z))$ (recall that $\rho(A_0(z)) = \lambda_0(z)$ for $z \in (-\theta,\theta)$) in a real neighbourhood of 0 was a requirement of (NG), the proof fo this next lemma reveals that it is only the strictness of the convexity that is non-trivial, and that the restriction of $z \mapsto \ln\rho(A_0(z))$ to $\mathbb{R}$ is convex whenever the coding relationship (2.4) holds.

Lemma 2.2.4. *The map $z \mapsto \ln \lambda_0(z)$ is convex on $(-\theta, \theta)$.*

Proof. For $z \in (-\theta, \theta)$ we have by (2.4), the quasi-compact decomposition of $A_0(z)$ and the fact that $\nu(\Pi(z)\zeta) \neq 0$ that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{E}(e^{zS_n}) &= \lim_{n\to\infty} \frac{1}{n} \ln \left|\mathbb{E}(e^{zS_n})\right| \\ &= \lim_{n\to\infty} \ln \lambda_0(z) + \frac{1}{n} \ln \left(\left|\nu\left(\Pi(z)\zeta + \lambda_0(z)^{-n} N_0(z)^n \zeta\right)\right|\right) \\ &= \ln \lambda_0(z).\end{aligned} \tag{2.7}$$

It is a standard result from probability theory that the moment-generating function of a real-random variable is log-convex. Since the limit of log-convex functions is also log-convex, the required claim then follows from (2.7). □

We may now finish the proof of Theorem 2.2.1.

Part 2 of the proof of Theorem 2.2.1. Let $\Lambda_0 : (-\theta, \theta) \to \mathbb{R}$ be defined by $\Lambda_0(z) = \ln \lambda_0(z)$, and note that $r = \Lambda_0^*$ on the interval

$$\Lambda_0'((-\theta, \theta)) = (\Lambda_0'(-\theta), \Lambda_0'(\theta)) = \left(\frac{\lambda'(-\theta)}{\lambda(-\theta)}, \frac{\lambda'(-\theta)}{\lambda(-\theta)}\right).$$

Since Λ_0' is continuous on $(-\theta, \theta)$ for any compact $U \subseteq \Lambda_0'((-\theta, \theta))$ there exists a closed interval $V \subseteq (-\theta, \theta)$ such that $U \subseteq \Lambda_0'(V)$. By the definition of the convex conjugate we have

$$\Lambda_{0,V}^*(y) = \sup_{z\in V} (yz - \Lambda_0(z)). \tag{2.8}$$

As a consequence of Lemma 2.2.4, for each $y \in \mathbb{R}$ the function $z \mapsto yz - \Lambda_{0,V}(z)$ is concave on $(-\theta, \theta)$. By differentiating, we therefore see that if $y \in \Lambda_0'(V)$ then the supremum in (2.8) is attained by some z satisfying $\Lambda_0'(z) = y$. Hence, for $y \in \Lambda_0'(V)$ we have

$$\Lambda_{0,V}^*(y) = y(\Lambda_0')^{-1}(y) - \Lambda_0(\Lambda_0')^{-1}(y)).$$

The same argument shows that the same formula holds for r on $\Lambda_0'(V)$, and so the $r = \Lambda_{0,V}^*$ on $\Lambda_0'(V)$. Since Λ_0' is monotonic and continuous we have that $\Lambda_0'(V)$ is a closed interval, and so by Lemma 2.2.3 we have

$$\lim_{\epsilon\to 0} \Lambda_{\epsilon,V}^*(y) = \Lambda_{0,V}^*(y)$$

uniformly on $\Lambda_0'(V)$. Since $r = \Lambda_{0,V}^*$ on $\Lambda_0'(V)$ and $U \subseteq \Lambda_0'(V)$ we obtain the required claim. $\square$

2.3 An explicit formula for the variance

In Chapters 3, 4 and 5 we will apply Theorem 2.2.1 in the case where A_ϵ is a numerical approximation of A, which will provide a rigorous method for approximating σ^2 (with $\lambda_\epsilon^{(2)}(0)$). Rather than computing a second derivative numerically, in these settings we are able to obtain an explicit, computationally-tractable expression for $\lambda_\epsilon^{(2)}(0)$ in terms of the (undifferentiated) spectral data of $A_\epsilon(0)$. The key extra hypothesis that we require is that the twist has a particular form: there exists $G \in L(E)$ such that $M(z) = e^{zG}$ for every z sufficiently close to 0.

Proposition 2.3.1. *Assume the hypotheses of Theorem 2.2.1 and that there exists $G \in L(E)$ such that $M(z) = e^{zG}$ for all sufficiently small z. Let θ denote the constant produced by Theorem 2.2.1. If for any $z \in D_\theta$ and sufficiently small $\epsilon \geq 0$ we choose $v_{\epsilon,z} \in E$ and $\varphi_{\epsilon,z} \in E^*$ so that $\Pi_\epsilon(z)f = \varphi_{\epsilon,z}(f)v_{\epsilon,z}$, then*[1]

$$\begin{aligned}\lambda_\epsilon^{(2)}(z) &= \lambda_\epsilon(z)\varphi_{\epsilon,z}(G^2 v_{\epsilon,z}) \\ &\quad + 2\lambda_\epsilon(z)\varphi_{\epsilon,z}(G(\lambda_\epsilon(z) - A_\epsilon(z))^{-1}A_\epsilon(z)(\mathrm{Id} - \Pi_\epsilon(z))G(v_{\epsilon,z})).\end{aligned} \tag{2.9}$$

Proof. Differentiating the identity $(\lambda_\epsilon(z) - A_\epsilon(z))\Pi_\epsilon(z) = 0$ once with respect to z yields

$$\lambda_\epsilon^{(1)}(z)\Pi_\epsilon(z) = A_\epsilon^{(1)}(z)\Pi_\epsilon(z) - (\lambda_\epsilon(z) - A_\epsilon(z))\Pi_\epsilon^{(1)}(z), \tag{2.10}$$

while differentiating a second time yields

$$\begin{aligned}\lambda_\epsilon^{(2)}(z)\Pi_\epsilon(z) =& A_\epsilon^{(2)}(z)\Pi_\epsilon(z) - 2(\lambda_\epsilon^{(1)}(z) - A_\epsilon^{(1)}(z))\Pi_\epsilon^{(1)}(z) \\ &- (\lambda_\epsilon(z) - A_\epsilon(z))\Pi_\epsilon^{(2)}(z).\end{aligned} \tag{2.11}$$

As $\Pi_\epsilon(z)(\lambda_\epsilon(z) - A_\epsilon(z)) = 0$, by applying $\Pi_\epsilon(z)$ on the left of (2.10) we obtain

$$\lambda_\epsilon^{(1)}(z)\Pi_\epsilon(z) = \Pi_\epsilon(z)A_\epsilon^{(1)}(z)\Pi_\epsilon(z). \tag{2.12}$$

Similarly,

$$\lambda_\epsilon^{(2)}(z)\Pi_\epsilon(z) = \Pi_\epsilon(z)A_\epsilon^{(2)}(z)\Pi_\epsilon(z) - 2\Pi_\epsilon(z)(\lambda_\epsilon^{(1)}(z) - A_\epsilon^{(1)}(z))\Pi_\epsilon^{(1)}(z). \tag{2.13}$$

[1] We note that while $\lambda_\epsilon(z) - A_\epsilon(z)$ is not invertible on E it is invertible on the image of $\mathrm{Id} - \Pi_\epsilon(z)$, which is an invariant subspace under $A_\epsilon(z)$, and so (2.9) is well defined. The details are clear from the proof and the citations therein.

As $\lambda_\epsilon(z)$ is an isolated simple eigenvalue, by [65, II-(2.14)] we have[2]

$$\Pi_\epsilon^{(1)}(z) = \Pi_\epsilon(z)A_\epsilon^{(1)}(z)S_\epsilon(z) + S_\epsilon(z)A_\epsilon^{(1)}(z)\Pi_\epsilon(z), \tag{2.14}$$

where $S_\epsilon(z) = (\lambda_\epsilon(z) - A_\epsilon(z))^{-1}(\mathrm{Id} - \Pi_\epsilon(z))$. Note that $\Pi_\epsilon(z)S_\epsilon(z) = S_\epsilon(z)\Pi_\epsilon(z) = 0$. Applying (2.14) to (2.13), we find that

$$\begin{aligned}\lambda_\epsilon^{(2)}(z)\Pi_\epsilon(z) =& \Pi_\epsilon(z)A_\epsilon^{(2)}(z)\Pi_\epsilon(z) - 2\lambda_\epsilon^{(1)}(z)\Pi_\epsilon(z)\Pi_\epsilon^{(1)}(z) + 2\Pi_\epsilon(z)A_\epsilon^{(1)}(z)\Pi_\epsilon^{(1)}(z)\\ =& \Pi_\epsilon(z)A_\epsilon^{(2)}(z)\Pi_\epsilon(z) - 2\lambda_\epsilon^{(1)}(z)\Pi_\epsilon(z)A_\epsilon^{(1)}(z)S_\epsilon(z)\\ &+ 2\Pi_\epsilon(z)A_\epsilon^{(1)}(z)(\Pi_\epsilon(z)A_\epsilon^{(1)}(z)S_\epsilon(z) + S_\epsilon(z)A_\epsilon^{(1)}(z)\Pi_\epsilon(z)).\end{aligned}$$

Applying $\Pi_\epsilon(z)$ on the right then yields

$$\lambda_\epsilon^{(2)}(z)\Pi_\epsilon(z) = \Pi_\epsilon(z)A_\epsilon^{(2)}(z)\Pi_\epsilon(z) + 2\Pi_\epsilon(z)A_\epsilon^{(1)}(z)S_\epsilon(z)A_\epsilon^{(1)}(z)\Pi_\epsilon(z). \tag{2.15}$$

Recall that the $A_\epsilon(z) = A_\epsilon M(z)$, where $M(z) = e^{zG}$. For each $n \in \mathbb{N}$ we therefore have

$$A_\epsilon^{(n)}(z) = A_\epsilon M^{(n)}(z) = A_\epsilon M(z)G^n = A_\epsilon(z)G^n. \tag{2.16}$$

As $\Pi_\epsilon(z)(f) = \varphi_{\epsilon,z}(f)v_{\epsilon,z}$, the $v_{\epsilon,z}$ and $\varphi_{\epsilon,z}$ are eigenvectors of $A_\epsilon(z)$ and $A_\epsilon(z)^*$, respectively, for the eigenvalue $\lambda_\epsilon(z)$. Moreover, as $\Pi_\epsilon(z)$ is a projection, we have $\varphi_{\epsilon,z}\Pi_\epsilon(z) = \varphi_{\epsilon,z}$ and $\Pi_\epsilon(z)v_{\epsilon,z} = v_{\epsilon,z}$. Using (2.16), and then applying $\varphi_{\epsilon,z}$ on the left and $v_{\epsilon,z}$ on the right to (2.15), we obtain

$$\lambda_\epsilon^{(2)}(z) = \lambda_\epsilon(z)\left(\varphi_{\epsilon,z}(G^2 v_{\epsilon,z}) + 2\varphi_{\epsilon,z}(G(\lambda_\epsilon(z) - A_\epsilon(z))^{-1}(\mathrm{Id} - \Pi_\epsilon(z))A_\epsilon(z)G(v_{\epsilon,z}))\right).$$

We obtain the required statement upon noting that $\mathrm{Id} - \Pi_\epsilon(z)$ and $A_\epsilon(z)$ commute. □

Remark 2.3.2. The expression (2.9) provides an alternative approach for proving the stability of the variance, which has been exploited previously (e.g. in [54]): each of the terms on the right side of (2.9) may be approximated using Theorem 1.1.3. In contrast, our proof for the stability of the rate function requires uniform control of $\lambda_\epsilon(z)$ for z in a real neighbourhood of 0, for which the theory developed in [71] is insufficient and so a result such as Theorem 1.2.2 is required.

[2] We note that the sign discrepancy between [65, II-(2.14)] and (2.14) is due to an additional factor of -1 in the definition of the resolvent in [65].

We will use a slightly different formula to approximate $\lambda_\epsilon^{(2)}(0)$ in Chapters 3, 4 and 5. In these chapters we start with a compact, $\mathcal{C}^\infty$, Riemannian manifold M, possibly with boundary, a Banach space $(E, \|\cdot\|)$ such that $\mathcal{C}^\infty(M, \mathbb{C}) \hookrightarrow E \hookrightarrow \mathcal{C}^\infty(M, \mathbb{C})^*$, and consider Markov[3] numerical perturbations A_ϵ. Since A_ϵ is a Markov operator we may take $\varphi_{\epsilon,0}$ to be the map $f \mapsto \int f \, dm$, which implies that $\int v_{\epsilon,0} \, dm = 1$. Moreover, $1 \in \sigma(A_\epsilon)$ since $\varphi_{\epsilon,0}$ is an eigenvector of A_ϵ for the eigenvalue 1. Since A_ϵ is a simple quasi-compact operator for sufficiently small ϵ, Theorem 1.1.3 implies that $\lambda_\epsilon(0) = 1$ for all sufficiently small ϵ. Evaluating (2.9) with $z = 0$ then yields

$$\lambda_\epsilon^{(2)}(0) = \int G^2 v_{\epsilon,0} + 2G(\mathrm{Id} - A_\epsilon)^{-1} A_\epsilon (\mathrm{Id} - \Pi_\epsilon) G v_{\epsilon,0} \, dm, \tag{2.17}$$

which is the approximation used in the computation of the variance. We note that when $\epsilon = 0$ the expression (2.17) is equal to the expression for $\lambda_0^{(2)}(0)$ from [58, Corollary III.11], which also contains an alternative derivation.

[3]That is A_ϵ preserves the $\|\cdot\|$-completion of the positive cone in $\mathcal{C}^\infty(M, \mathbb{C})$ and integrals with respect to the Riemannian measure m on M.

Chapter 3

Application to piecewise expanding interval maps

In this chapter we demonstrate the utility of the theory developed in Chapters 1 and 2 through an application to a classical example: piecewise expanding interval maps. For such maps we obtain stability of the variance and rate function under standard classes of perturbations, including perturbations arising from the Ulam numerical scheme [75]; the specifics details are contained in Section 3.1. In Section 3.2 we focus on the application of Ulam's method, which we then use to compute rigorous approximations of the variance and rate function, respectively. While the computation of the rate function is, to the best of our knowledge, new, there are existing methods for approximating the variance for one-dimensional expanding maps. We argue in Section 3.2.1 that our method applies to a large class of irregular examples, and is the more efficient than existing methods with equivalent scope.

3.1 Lasota-Yorke maps and functions of bounded variation

The aim of this section is to describe how the naive Nagaev-Guivarc'h method and Theorem 2.2.1 apply to piecewise expanding interval maps. This literature concerning this setting is substantial and so rather than give a full account of the subject we will frequently direct the reader to various references for further details (of particular note are [20, 10]).

Definition 3.1.1. *We call $T : [0,1] \to [0,1]$ a Lasota-Yorke map if there is a sequence $0 = a_0 < a_1 < \cdots < a_r = 1$ and $\gamma > 1$ such that for each $i = 1, \dots, r$ we have*

1. *T is C^1 on (a_{i-1}, a_i) with a C^1 extension to $[a_{i-1}, a_i]$;*
2. *$\left| T'|_{(a_{i-1},a_i)} \right| \geq \gamma$; and*
3. *$\left| T'|_{(a_{i-1},a_i)} \right|^{-1} \in \mathrm{BV}$.*

Let Leb denote Lebesgue measure on $[0,1]$. If $T:[0,1]\to[0,1]$ is a Lasota-Yorke map then we define the Perron-Frobenius operator $\mathcal{L}$ associated to T via duality: for every $f\in L^1(\mathrm{Leb}), h\in L^\infty(\mathrm{Leb})$ we set

$$\int \mathcal{L}(f)\cdot h\,\mathrm{d}\,\mathrm{Leb} = \int f\cdot(h\circ T)\,\mathrm{d}\,\mathrm{Leb}\,. \tag{3.1}$$

Rather than considering the action of $\mathcal{L}$ on $L^1(\mathrm{Leb})$ for our purposes it is more productive to consider it as an operator on the space of complex-valued functions of bounded variation on $[0,1]$, denoted BV. The following result, which is a paraphrasing of the results proved in [20, Section 7.2] together with [20, Theorem 8.2.2], makes this claim precise. Recall that BV is a complex Banach algebra when equipped with the norm $\|\cdot\|_{\mathrm{BV}} = \mathrm{Var}(\cdot) + \|\cdot\|_{L^1(\mathrm{Leb})}$, and that an absolutely continuous invariant probability measure (ACIP) is a T-invariant measure ν such that $\nu \ll \mathrm{Leb}$.

Proposition 3.1.2. *If T is a Lasota-Yorke map then $\mathcal{L}$ is a bounded, quasi-compact operator on* BV *with spectral radius 1 and 1 is an eigenvalue of $\mathcal{L}$. Moreover, if T is topologically mixing then $\mathcal{L}$ is a simple quasi-compact operator on* BV *and T has a unique ACIP μ such that $\frac{\mathrm{d}\mu}{\mathrm{d}\,\mathrm{Leb}}$ is the unique fixed point of $\mathcal{L}$ in* BV*.*

For real-valued $g\in\mathrm{BV}$ one may consider the sequence of random variables $\{g\circ T^k\}_{k=0}^\infty$ on the probability space $([0,1],\mathrm{Leb})$. We can deduce a CLT and LDP for such sequences by verifying (NG) and then applying Theorems 2.1.1 and 2.1.2; the details of the former claim are contained in the following proposition, which summarises the content of [20, Section 8.5]. But first we note that if $g\in\mathrm{BV}$ then the map $M_g:\mathbb{C}\to L(\mathrm{BV})$, defined by $M_g(z)(f)=e^{zg}f$, is well-defined by virtue of BV being a Banach algebra. Moreover, it is clear that $z\mapsto M_g(z)$ is analytic and that $M_g(0)=\mathrm{Id}$, and so M_g is a twist (as per Definition 1.2.1). Recall that $g\in L^2(\mu)$ is said to be a $L^2(\mu)$-coboundary with respect to T if there exists $\phi\in L^2(\mu)$ such that $g=\phi-\phi\circ T$.

Proposition 3.1.3. *Suppose that $T:[0,1]\to[0,1]$ is a topologically mixing Lasota-Yorke map with unique ACIP μ and that $g\in\mathrm{BV}$ is real-valued, satisfies $\int g\,\mathrm{d}\mu=0$ and is not a $L^2(\mu)$-coboundary with respect to T. Let $\varphi:\mathrm{BV}\to\mathbb{C}$ denote the map $f\mapsto\int f\,\mathrm{d}\,\mathrm{Leb}$ and for $z\in\mathbb{C}$ let $\mathcal{L}(z)$ be defined by $\mathcal{L}(z)f=(\mathcal{L}\circ M_g(z))(f)=\mathcal{L}(e^{zg}f)$. Then $z\mapsto\mathcal{L}(z)$ is analytic (in the operator norm on $L(\mathrm{BV})$) and $\{g\circ T^k\}_{k=0}^\infty$, when considered on the probability space $([0,1],\mathrm{Leb})$, satisfies (NG) with coding $(z\mapsto\mathcal{L}(z),\varphi,\frac{\mathrm{d}\mu}{\mathrm{d}\,\mathrm{Leb}})$. In particular, if $S_n:=\sum_{k=0}^{n-1}g\circ T^k$ then for every $z\in\mathbb{C}$*

we have

$$\mathbb{E}_{\mathrm{Leb}}\left(e^{zS_n}\right) = \int \mathcal{L}(z)^n \left(\frac{\mathrm{d}\mu}{\mathrm{d}\,\mathrm{Leb}}\right) \mathrm{d}\,\mathrm{Leb}\,.$$

With g as in Proposition 3.1.3 one therefore has a CLT and LDP for $\{g \circ T^k\}_{k=0}^{\infty}$ on the probability space $([0,1], \mathrm{Leb})$ per Theorems 2.1.1 and 2.1.2. Hence we obtain a variance $\sigma_g^2 \geq 0$ and rate function r_g associated to the CLT and LDP, respectively. Since g is not a $L^2(\mu)$-coboundary with respect to T one may deduce that $\sigma_g^2 > 0$ (see e.g. the proof of [79, Lemma 2.5.]). To prove the stability of these parameters we aim to apply Theorem 2.2.1, for which we require the verification of (KL) and that the map $z \mapsto \mathcal{L}(z)$ is induced by a twist. The latter condition is obvious given the definition of $z \mapsto \mathcal{L}(z)$, and so it remains to verify (KL). Helly's selection theorem states that the closed unit ball in BV is $\|\cdot\|_{L^1(\mathrm{Leb})}$-pre-compact, and so we may take $\|\cdot\|_{L^1(\mathrm{Leb})}$ to be the weak norm $|\cdot|$. Since $\mathcal{L}$ is Markov one obtains (KL2) (for this particular operator) and by standard arguments, as in [20, Section 5.2], one obtains a Lasota-Yorke inequality (i.e. (KL3)) for $\mathcal{L}$ too. Thus we obtain the stability of the σ_g^2 and r_g as a corollary to Theorem 2.2.1 and Proposition 3.1.3.

Theorem 3.1.4. *Let $T : [0,1] \to [0,1]$ be a topologically mixing Lasota-Yorke map, μ be T's unique ACIP, $\mathcal{L}_0$ be the Perron-Frobenius operator induced by T, $\{\mathcal{L}_\epsilon\}_{\epsilon \geq 0}$ be a family of operators satisfying (KL) and $g \in \mathrm{BV}$ be a real-valued observable such that $\int g \,\mathrm{d}\mu = 0$. There exists $\theta, \epsilon' > 0$ so that for each $\epsilon \in [0, \epsilon')$ and $z \in D_\theta$ the operator $\mathcal{L}_\epsilon(z)$ is quasi-compact and simple with leading eigenvalue $\lambda_\epsilon(z)$ depending analytically on z. Moreover, we have stability of the following statistical data associated to T and $\{g \circ T^k\}_{k\in\mathbb{N}}$ as $\epsilon \to 0$:*

1. *The variance is stable: $\lim_{\epsilon\to 0} \lambda_\epsilon^{(2)}(0) = \sigma_g^2$.*
2. *The rate function is stable: for every compact subset U of the domain of r_g there exists a closed interval V such that*

$$\lim_{\epsilon\to 0} \sup_{z\in V}(sz - \ln|\lambda_\epsilon(z)|) = r_g(s)$$

uniformly for $s \in U$.

For the remainder of this section we detail three specific examples of perturbations for which (KL), and therefore also Theorem 3.1.4, hold. In what follows we take $\mathcal{L}_0$ to be the Perron-Frobenius operator associated to a topologically mixing Lasota-Yorke map T with $|T'| \geq 2$.

NP: *Numerical approximations of $\mathcal{L}$ by some finite-rank operator, such as in Ulam's method [75] (see also [50, 68]). Let $\mathbb{E}_n$ be the conditional expectation operator induced by the uniform partition of $[0,1]$ into elements of diameter $1/n$ and let $\mathcal{L}_{1/n} = \mathbb{E}_n \circ \mathcal{L}$. Then $\{\mathcal{L}_{1/n}\}_{n\in\mathbb{Z}^+\cup\{\infty\}}$ satisfies (KL) (see the discussion in Section 3.2 for details).*

SP: *Stochastic perturbations that arise via convolution of the Perron-Frobenius operator with an appropriate bistochastic, nonnegative kernel $K_\epsilon(x,y)$:*

$$\mathcal{L}_\epsilon f(x) = \int (\mathcal{L}f)(y) K_\epsilon(y,x)\, \mathrm{d}\,\mathrm{Leb}(y).$$

If the measure $K_\epsilon d\,\mathrm{Leb}^2$ converges weakly to $d\,\mathrm{Leb}^2$ on the diagonal of $[0,1]$ and $\{K_\epsilon\}_{\epsilon>0}$ satisfies mild monotonicity conditions then $\{\mathcal{L}_\epsilon\}_{\epsilon\geq 0}$ satisfies (KL) by [68, Corollary 17].

DP: *Deterministic perturbations of T in an appropriate metric space. For example, in a "Skorohod"-type metric in the case of piecewise expanding maps on the interval [68, Section 3]:*

$$d(T,T_\epsilon) := \inf \left\{ \delta > 0 \ : \begin{array}{l} \exists U \subset [0,1] \text{ such that } \mathrm{Leb}(U) > 1-\delta; \\ \exists \text{ a diffeomorphism } h : [0,1] \to [0,1], \\ \text{such that } T\big|_U = T_\epsilon \circ h\big|_U, \text{ and } \forall x \in U, \\ |h(x) - x| < \delta,\ |(1/h'(x)) - 1| < \delta \end{array} \right\}. \tag{3.2}$$

It is shown in [68, §3] that if $\lim_{\epsilon\to 0} d(T,T_\epsilon) = 0$ and $\{T_\epsilon\}_{\epsilon>0}$ is appropriately regular as $\epsilon \to 0$ (see [68, Remark 15]) then the family of operators $\{\mathcal{L}_\epsilon\}_{\epsilon\geq 0}$ satisfies (KL).

Corollary 3.1.5. *The conclusion of Theorem 3.1.4 holds for the the numerical, stochastic, and deterministic perturbations ((NP), (SP), and (DP) resp.) and so we obtain the corresponding approximation and stability of the variance and rate function under these perturbations.*

Remark 3.1.6. We note that the stability of the variance for (DP) and (NP) have been obtained before in [70] and [7], respectively. The stability of the rate function is new in this setting.

3.2 Application to Ulam's method

In this section we will compute rigorous approximations of the variance and rate function for an example Lasota-Yorke map. The map we consider for the remainder of this chapter is the following non-Markov, piecewise affine map (with $a = 2.1$):

$$T_a(x) = \begin{cases} ax, & 0 \le x < 1/4; \\ -a(x-1/2), & 1/4 \le x < 1/2; \\ -a(x-1/2)+1, & 1/2 \le x < 3/4; \\ a(x-1)+1, & 3/4 \le x \le 1. \end{cases} \tag{3.3}$$

By standard arguments one obtains a Lasota-Yorke inequality for $\mathcal{L}$ for any $\alpha > 2.1^{-1}$ and thus $\mathcal{L}$ is quasi-compact. Moreover, it is clear from the graph of T_a (Figure 3.1, upper left) that forward images of any interval $I \subset [0,1]$ eventually cover all of $[0,1]$; thus, T_a is topologically mixing.

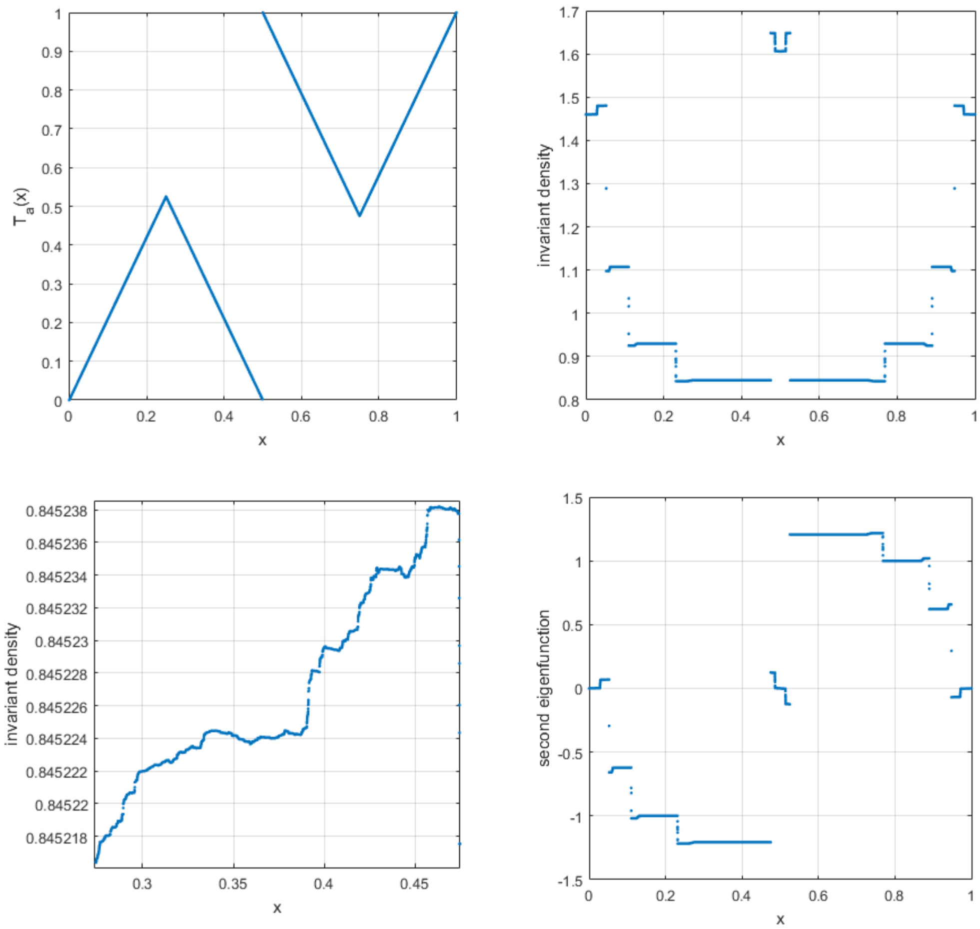

Figure 3.1: Graph of T_a (upper left), an approximation of the invariant density with $n = 25000$ (upper right), a zoom of an apparent "flat" section of the invariant density showing fine structure (lower left), and an approximation of the second eigenfunction with eigenvalue $\lambda_2 \approx 0.8079$ (lower right).

Note that the dynamics of T_a for $a \gtrapprox 2$ has infrequent transitions between the left and right halves of the unit interval; in such a situation, these sets are sometimes called almost-invariant sets [31, 30]. We select observables g (taking values approximately in the range $[-1, 1]$) that emphasise this structure to varying extents, and illustrate the combined effects of the dynamics and the observable on variances and rate functions; see Figure 3.2 for graphs of the various g. For example, the Birkhoff sums of $g(x) = \chi_{[0,1/2]} - \chi_{(1/2,1]}$ will typically take longer to converge because of frequent long sequences of similar g values (in this case either 1 or -1). On the other hand, the observable $g(x) = \cos(2\pi x) - 0.0614$ is not strongly correlated with the almost-invariant dynamics and one expects a more rapid convergence of Birkhoff sums. These arguments are reflected in the table of variances, Table 3.1, and the graph of rate functions, Figure 3.2 (right).

Our choice of numerical scheme is dictated by the class of map. As was outlined in the previous section, because we are considering general (non-Markov) Lasota-Yorke maps the natural choice of Banach space is BV, with the weak and strong norms being the L^1 and BV norms, respectively. Since the eigenfunctions of $\mathcal{L}$ can be discontinuous (see Figure 3.1 upper right and lower right), we use locally supported functions for our approximation space, and in particular, locally constant functions, leading to the well-known Ulam scheme [104]. If, on the other hand, we restricted ourselves to globally differentiable, full-branched maps (a smaller and better-behaved class), then it would be natural to work with C^r functions and use a globally supported basis consisting of Chebyshev polynomials, if the phase space is an interval, or trigonometric polynomials (or Fourier modes), if the phase space is a circle. By exploiting the smoothness of the map these bases could produce commensurately more accurate estimates.

For a partition of $[0, 1]$ into subintervals $I_1, \ldots, I_n$, setting $\mathcal{B}_n = \mathrm{span}\{\chi_{I_i} : 1 \leq i \leq n\}$, we define the conditional expectation operator $\mathbb{E}_n : L^1([0, 1]) \to \mathcal{B}_n$ by

$$\mathbb{E}_n f = \sum_{i=1}^{n} \frac{\int_{I_i} f \, d\,\mathrm{Leb}}{\mathrm{Leb}(I_i)} \chi_{I_i}. \tag{3.4}$$

It is well known (e.g. [75]) that the matrix representation of $\mathbb{E}_n \mathcal{L}$ on $\mathcal{B}_n$ is

$$P_{ij} = \frac{\mathrm{Leb}(I_i \cap T^{-1} I_j)}{\mathrm{Leb}(I_j)}, \tag{3.5}$$

under multiplication on the left. In our experiments, we use equipartitions of $[0,1]$ of increasing cardinality n. Putting $\epsilon = 1/n$, we set $\mathcal{L}_\epsilon = \mathcal{L}_{1/n} = \mathbb{E}_n\mathcal{L}$. The property (KL1) is satisfied; see e.g. the discussion in §16–18 [68]. The operators $\mathcal{L}_{1/n}$ are Markov for every n and therefore satisfy (KL2) and are positive. The expectation operator $\mathbb{E}_n$ reduces variation, and thus (KL3) is also satisfied. Our estimate of the twisted operator $\mathcal{L}(z)$ will be the operator $\mathcal{L}_{1/n}(z) := \mathcal{L}_{1/n}M_{g_n}(z)$, where $g_n = \sum_{i=1}^{n} g((i-1/2)/n)\chi_{I_i}$ takes the value of g at the midpoint[1] of each I_i, $i = 1, \ldots, n$.

3.2.1 Estimating the variance

We numerically evaluate the expression (2.17) for $\lambda^{(2)}(0)$ by taking G to be the multiplication operator induced by g. The term $(\mathrm{Id} - \mathcal{L}_\epsilon)^{-1}\mathcal{L}_\epsilon(gv_{\epsilon,0})$ is numerically determined by solving the single linear system of equations $(\mathrm{Id} - \mathcal{L}_\epsilon)v'(0) = \mathcal{L}_\epsilon(gv_{\epsilon,0})$ for the unknown $v'(0)$ (i.e. $\frac{d}{dz}v\big|_{z=0}$), restricting $v'(0)$ to the codimension 1 subspace of zero-Lebesgue-mean functions. The MATLAB function for computing the variance is given in Listing 3.1.

Ulam	Variance estimates for observable $g(x)$			
subintervals	$\cos(2\pi x) - 0.0614$	$2x-1$	$\sin(2\pi x)$	$\chi_{[0,1/2]} - \chi_{(1/2,1]}$
200	0.51057	4.3355	6.5368	17.006
1000	0.50496	4.2886	6.4959	16.859
5000	0.50430	4.2871	6.4950	16.855
25000	0.50396	4.2860	6.4936	16.851

Table 3.1: Computed variances for four different observables and at four different Ulam resolutions.

For a transformation $T : [0,1] \to [0,1]$ let `P` denote a row-stochastic Ulam matrix constructed on an equi-partition of $[0,1]$ in MATLAB's `sparse` format. To compute an estimate of the variance `v` for the observable $g(x) = \sin(2\pi x)$, use:

```
obs=@(x)sin(2*pi*x);
[v,~,~,~,~] = variance(P,obs);
```

[1]Strictly speaking, Ulam's method for twisted transfer operators will involve integrals of g, which can be numerically evaluated. We have chosen the above midpoint approximation of g for computational convenience; note that the midpoint rule is the same order of accuracy as the trapezoidal method of numerical quadrature and often slightly more accurate (errors are about a factor 1/2 smaller). We additionally computed the values in Table 3.1 with an "exact" implementation of Ulam and the errors due to the midpoint estimate of g were several orders of magnitude smaller than the errors due to the overall Ulam discretisation.

There have been a number of prior rigorous numerical estimates of variance for interval maps. Bahsoun *et al.* [7], Pollicott *et al.* [63], and Wormell [106] develop algorithms that output an interval in which the variance is guaranteed to lie. Bahsoun *et al.* applies to general Lasota-Yorke maps, uses Ulam's method, and employs a "brute force" approach of taking high powers of $\mathcal{L}_{1/n}$ to achieve convergence. The method of [63] applies to real analytic expanding (full-branch) maps with real analytic observables, and is based on evaluations on all periodic orbits up to a certain order. Wormell [106] applies to full-branched, C^3 expanding maps and uses an approach most similar to ours, with computations in Chebyshev/Fourier bases. In each of these papers, an interval containing the variance of the Lanford map $T(x) = 2x + x(1-x)/2 \pmod 1$ for the observable $g(x) = x^2$ is obtained. The latter two papers, exploiting the analyticity of the map T and observable g can achieve more accurate estimates for the same computational effort.

In comparison to [7] we can avoid raising the very sparse matrix $\mathcal{L}_{1/n}$ to high powers (in the full-branch Lanford map studied in [7] $\mathcal{L}_{1/n}^{112}$ is computed). We exploit the differentiability properties of the spectral data with respect to the twist parameter (which exist even for general Lasota-Yorke maps) and preserve the high degree of sparseness of $\mathcal{L}_{1/n}$, which is quickly destroyed by taking powers. We only need to solve a single sparse linear equation to obtain an estimate for $\lambda^{(2)}(0)$, which is related to the equation solved in [106]. In comparison to [63] and [106] we can treat general Lasota-Yorke maps, via the flexible choice of a locally supported basis, however, as explained above, for smoother classes of maps as in [63, 106], one should adapt the basis accordingly as the Ulam basis will not be competitive with specialised approaches. Our variance estimates rigorously converge to the true value as $n \to \infty$; and while it is likely possible to provide an "interval of guarantee", as in the above methods, we have not pursued this here.

Listing 3.1: This function centres the observable *obs* (defined by an anonymous MATLAB function e.g. the code snippet in Section 3.2.1) and computes the required first and second derivatives at zero to estimate the variance.

```
function [ddLam,v,dv,dlam,ddlam]=variance(P,obs),

%P is a row-stochastic matrix
%obs is a pre-defined anonymous function representing the
    observable
%lam is the leading eigenvalue
```

```
%v is the leading eigenfunction
%dv is dv/dtheta, where theta is the twist parameter
%dlam is dlam/dtheta
%ddlam is d^2lam/dtheta^2
%ddLam is d^2Lam/dtheta^2

%% find v and normalise appropriately
n=size(P,1);
phi=ones(n,1)/n;
[v,~]=eigs(P',1);
v=v/sum(v)*n;

%% centre observable g
x=[1/(2*n):1/(n):1-1/(2*n)]';
g=obs(x);
g=g-g'*v/n;  %ensure g has mean zero by subtracting the
   mean

%% estimate dlam and dv using 1.*v=1 and 1.*dv=0

A=[P'-speye(n)  -v;  ones(1,n) 0];
b=[-P'*(g.*v);  0];
y=A\b;
dv=y(1:n);
dlam=y(n+1);

%% compute d^2lam/dtheta^2 and d^2Lam/dtheta^2

ddlam=((g.^2)'*v+2*g'*dv)/n;
ddLam=(ddlam-dlam^2);
```

3.2.2 *Estimating the rate function*

For a fixed value of s, we estimate $r_g(s) = -\min_z(\ln\lambda(z) - zs)$ by applying MATLAB's built-in unconstrained function minimising routine `fminunc` to the function $f(z) = \ln\lambda(z) - sz$. We use the default quasi-newton algorithm option for `fminunc` (we found the trust-region algorithm used slightly more iterates) and supply an expression for the first derivative of $f(z)$ with respect to z, namely $\phi(z)(g\lambda(z)v(z))-s$ (here $\phi(z)$ and $v(z)$ are the leading left and right, respectively, eigenvectors of $\mathcal{L}(z)$); all other settings are the defaults. Each evaluation of $f(z)$ requires the computation of $\lambda(z)$ (we obtain $v(z)$ at the same time) and each evaluation of $f'(z)$ requires an additional computation of $\phi(z)$. These two eigencomputations are made by simply repeatedly iterating $v(0)$ and $\phi(0)$ with $\mathcal{L}(z)$ and $\mathcal{L}(z)^*$ (and normalising), respectively until the change in the estimated eigenvalue is below a tolerance (we used 5×10^{-12}). This is relatively efficient because the Ulam matrix approximation of $\mathcal{L}(z)$ is very sparse, and we found this is also faster than using MATLAB's built-in `eigs` routine to find the single leading eigenvalue. We estimate $r_g(s)$ on a grid of s values (in our experiments s ranges from 0 to 0.8 in steps of 0.01), stepping from one grid point to the next. We use the previous optimal z as the initial seed for the quasi-newton algorithm to find the optimal z for the next s grid point, and found this choice results in slightly fewer quasi-newton steps than choosing a fixed initialisation.

For a transformation $T : [0,1] \to [0,1]$ let `P` denote a row-stochastic Ulam matrix constructed on an equipartition of $[0,1]$ in MATLAB's `sparse` format. To compute estimates of the rate function $r_g(s)$, for the observable $g(x) = \sin(2\pi x)$, at $s \in [0, 0.8]$ spaced 0.01 apart, and store these estimates in a vector `r`, use:

```
s=0:.01:.8;
obs=@(x)sin(2*pi*x);
[r,~] = rate_function(s,P,obs);
```

The necessary MATLAB functions are given in Listings 3.2, 3.3 and 3.4, which appear at the end of this section. To run the above code to compute these 81 values of the rate function takes[2] approximately 1, 4, and 12 seconds for Ulam matrices of sizes 1000, 5000, and 25000, respectively. We use the same set of four observables g as in the variance computations (Figure 3.2 left). The corresponding rate functions are shown in Figure 3.2 right).

[2]On a 7th-generation intel core i5 processor.

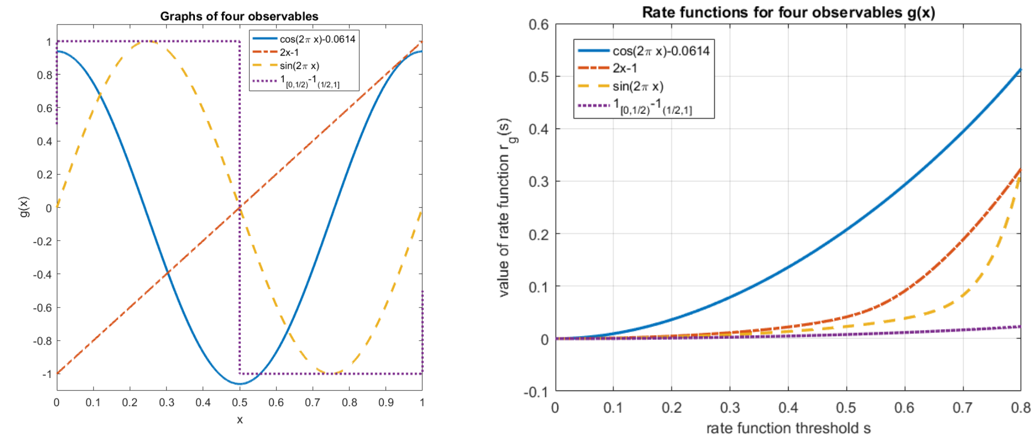

Figure 3.2: Graphs of four different observables (left) and corresponding rate functions (right), computed with $n = 1000$.

Note that the four observables g yield rate functions of increasingly lower value (higher likelihood of large deviations occurring). This corresponds to the correlation between the value of the observable and the almost-invariant sets $[0, 1/2], [1/2, 1]$. The observable $g(x) = \cos(2\pi x) - 0.0614$ is not particularly correlated with the almost-invariant sets and thus large deviations have low probability. On the other hand, the observables $2x - 1$ and $\sin(2\pi x)$ have moderate correlation with the almost-invariant sets and large deviations have an increased probability of occurring (interestingly, there is a crossover of these two rate functions around $s = 0.8$). The observable $g(x) = \chi_{[0,1/2]} - \chi_{(1/2,1]}$ is very strongly correlated with the almost-invariant sets and we see a correspondingly small rate function. Figure 3.3 shows the decrease in errors relative to $n = 25000$ for the calculations using $n = 200, 1000$, and 5000, typically with somewhat larger errors for larger thresholds s, as expected.

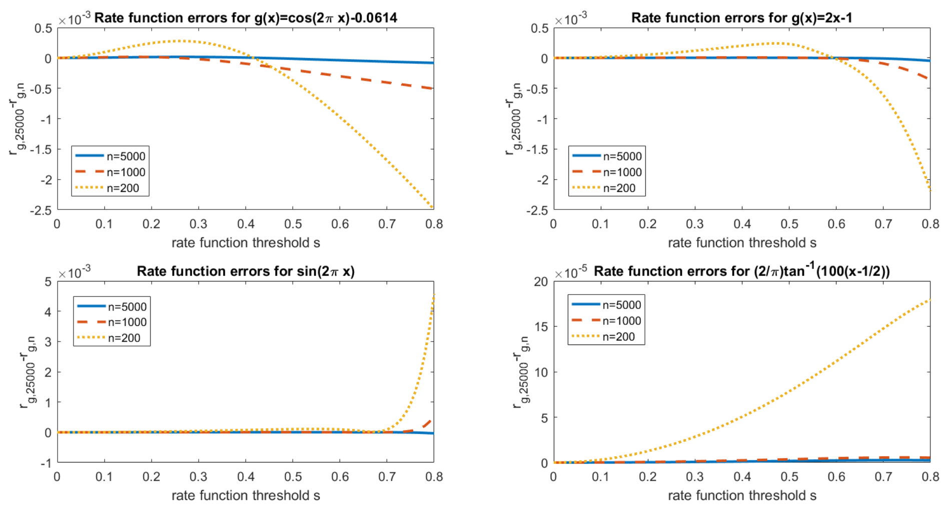

Figure 3.3: Differences between rate function estimates for $n = 200, 1000, 5000$ and the rate function estimate using $n = 25000$.

We are not aware of prior rigorous numerical methods for estimating rate functions for deterministic dynamics. Prior work on estimating rate functions includes [64, 91], which use the Legendre transform but not the spectral approach we use here.

Finally, we note that the rate of escape from the interval $[0, 1/2]$ can be estimated via the observable $g(x) = \chi_{[0,1/2]} - \chi_{(1/2,1]}$ by computing the rate function for a threshold very close to 1. With $n = 1000$, and an s threshold of $1 - 10^{-15}$, one obtains a rate function value of 0.04879016416945 (in this experiment, the optimality tolerance in `rate_function.m` was decreased to 10^{-13}). Alternatively, computing the negative logarithm of the leading eigenvalue of the transfer operator restricted to the interval $[0, 1/2]$ (this is particularly straightforward with an Ulam basis, see e.g. [6, 18]), one obtains 0.04879016416943. Thus, the rate function calculation and the escape rate calculation are consistent up to 13 decimal places for an Ulam matrix of size $n = 1000$ (note we are not claiming accuracy of the true values up to this precision).

Listing 3.2: This function centres the observable *obs* (defined by an anonymous MATLAB function e.g. the code snippet in Section 3.2.2), and performs the required minimisation to evaluate the rate function at points specified in the vector *s*.

```
function [r,optz] = rate_function(s,P,obs)

%P is a row-stochastic matrix
%obs is a pre-defined anonymous function representing the observable
%s is a vector of arguments of the rate function

%% calc acim for centering observable.
[v0,~]=eigs(P',1);
v0=v0/sum(v0);

%% set up objects to pass to legendre_function.m
n=length(P);
[I,J,V]=find(P);
xpts=(I-.5)/n;
xptsorig=1/(2*n):1/n:1-1/(2*n);
```

```
gmean=obs(xptsorig)*v0;

%% set up arrays for r and optz and set optimisation
   options
r=zeros(length(s),1);
optz=r;
options = optimoptions('fminunc','Algorithm','quasi-
   newton','SpecifyObjectiveGradient',true,'
   OptimalityTolerance',1e-6);

%% initial seed point for minimisation
z0=0;

%% evaluate rate function at points specified in s
for i=1:length(s),
    minfun=@(z)legendre_function(z,P,v0,s(i),obs,n,I,J,V,
       xpts,gmean);
    [optz(i),r(i),~,~]=fminunc(minfun,z0,options);
    z0=optz(i);   %use previous optimum for next
       initialisation.
end

r=-r;
```

Listing 3.3: This function evaluates the “Legendre function” (the function to be minimised) and its derivative. This requires twisting the matrix P by z and then computing the leading eigenvalue and eigenvector of the twisted matrix.

```
function [f,df] = legendre_function(z,P,v0,s,fun,n,I,J,V,
   xpts,gmean)

%evaluate 'legendre' function with fixed parameter s,
   maximising over z.

%% twist P by z
gvec=z*(fun(xpts)-gmean);
Vtwist=V.*exp(gvec);
```

```
Ptwist=sparse(I,J,Vtwist);

%% Calculate objective f
[v,lam]=powermethod(Ptwist,v0,1);
f=log(lam)-z*s;

%% Calculate gradient df
if nargout > 1 % gradient required
    v=v/sum(v);
    [phi,lam]=powermethod(Ptwist',ones(n,1),1);
    phi=phi/(phi'*v);
    gvecbasic=fun(1/(2*n):1/n:1-1/(2*n))-gmean;
    dlam=lam*phi'*(gvecbasic'.*v);
    df=dlam/lam-s;
end
```

Listing 3.4: Estimation of the leading eigenvalue and eigenvector by repeated iteration.

```
function [v1,lam1]=powermethod(P,v0,lam0),

%P is a row stochastic matrix
%v0 is an initial (guessed) eigenvector
%lam0 is an initial (guessed) eigenvalue)

v0=v0/sum(v0);
v1=P'*v0;
lam1=sum(v1);
while abs(lam1-lam0)>1e-15,
    lam0=lam1;
    v0=v1/lam1;
    v1=P'*v0;
    lam1=sum(v1);
end
```

Chapter 4

Application to multidimensional piecewise expanding maps

This chapter generalises the setting of the last by moving from one dimension to many: we will apply the theory of Chapters 1 and 2 to the class of multidimensional piecewise expanding maps studied by Saussol in [94]. We start in Section 4.1 by reviewing the functional analytic setting used in [94], describing how it is compatible with (KL), and how the maps in question satisfy a CLT and LDP for quasi-Hölder observables under the Nagaev-Guivarc'h method (as is shown in [1]). Hence we have all the ingredients required to apply Theorem 2.2.1, and we finish the section by deducing the stability of the variance and rate function to perturbations satisfying (KL). We then develop two new examples of perturbations satisfying (KL) in this setting: stochastic perturbations and numerical perturbations arising via Ulam's method. That these perturbations satisfy (KL) is shown in Sections 4.2 and 4.3, respectively.

4.1 Quasi-Hölder spaces for multidimensional piecewise expanding maps

The aim of this section is to describe how the naive Nagaev-Guivarc'h method and Theorem 2.2.1 apply to multidimensional piecewise expanding maps considered in [94]. Let $m \geq 2$ and denote by Leb the Lebesgue measure on $\mathbb{R}^m$. Let $X \subseteq \mathbb{R}^m$ be compact and satisfy $\overline{\mathrm{int}(X)} = X$. We denote by $B(x, r)$ the open ball in $\mathbb{R}^m$ centred at x of radius r, and by $B(Y, r)$ the open r-neighbourhood of a set $Y \subseteq \mathbb{R}^m$. Without loss of generality we may assume that $\mathrm{Leb}(X) = 1$.

Definition 4.1.1 ([94, Section 2])**.** *We say that $T : X \to X$ is a multidimensional piecewise expanding map if there exists a countable family of disjoint open sets*

$\{U_i \subseteq X\}$, sets $\{V_i\}$ and maps $T_i : V_i \to \mathbb{R}^m$ such that $\overline{U_i} \subseteq V_i$ and for some $\beta \in (0,1]$ and $\eta_0 > 0$ one has:

1. *For each i, $T\big|_{U_i} = T_i\big|_{U_i}$ and $B(TU_i, \eta_0) \subseteq T_i(V_i)$.*
2. *For all i, $T_i \in \mathcal{C}^1(V_i)$, T_i is injective and $T_i^{-1} \in \mathcal{C}^1(T_iV_i)$. Moreover the determinant of each T_i^{-1} is uniformly Hölder: for all i, $\eta \le \eta_0$, $z \in T_iV_i$ and $x, y \in B(z, \eta) \cap T_iV_i$ we have*

$$\left|\det D_xT_i^{-1} - \det D_yT_i^{-1}\right| \le c_\eta \left|\det D_zT_i^{-1}\right| \eta^\beta,$$

   for some $c_\eta > 0$.
3. $\mathrm{Leb}\,(X \setminus \bigcup_i U_i) = 0$.
4. *There exists $s = s(T) < 1$ such that for all $u, v \in TV_i$ for which $d(u,v) \le \eta_0$ we have $d(T_i^{-1}u, T_i^{-1}v) \le sd(u,v)$*
5. *Let $G(\eta, \eta_0) := \sup_x G(x, \eta, \eta_0)$ where*

$$G(x, \eta, \eta_0) = \sum_i \frac{\mathrm{Leb}(T_i^{-1}B(\partial TU_i, \eta) \cap B(x, (1-s)\eta_0)}{\mathrm{Leb}(B(x, (1-s)\eta_0)}$$

   and assume that the map γ, which is defined by

$$\gamma(\eta) = s^\beta + 2\eta^\beta \sup_{\delta \le \eta} \delta^{-\beta} G(\delta, \eta),$$

   satisfies $\sup_{\eta \le \eta_0} \gamma(\eta) < 1$.

If $T : X \to X$ is a multidimensional piecewise expanding map then the Perron-Frobenius operator $\mathcal{L}$ associated to T is a well-defined bounded operator on $L^1(X)$, and is a.e. given by

$$(\mathcal{L}f)(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|\det D_yT|}.$$

As in the last chapter, rather than consider $\mathcal{L}$ as an operator on $L^1(X)$ it is more fruitful to consider it as an operator on a Banach space of more regular functions, namely the space of quasi-Hölder functions whose definition we now recall from [1, Section 4] and [94]. Suppose $A \subseteq \mathbb{R}^m$ is Borel. For each $f \in L^1(\mathbb{R}^m)$ the oscillation of f over A is defined to be

$$\mathrm{osc}\,(f, A) := \operatorname*{ess\,sup}_{(y_1, y_2) \in A^2} |f(y_1) - f(y_2)|\,,$$

where the essential supremum is taken with respect to the product measure Leb^2 on A^2. For every $f \in L^1(\mathbb{R}^m)$ and $\eta > 0$ the function $x \mapsto \operatorname{osc}(f, B(x,\eta))$ is well defined and lower-semicontinuous, and therefore also measurable. Let

$$\begin{aligned} |f|_\beta &= \sup_{0<\eta\leq\eta_0} \eta^{-\beta} \int_{\mathbb{R}^m} \operatorname{osc}(f, B(x,\eta))\, \mathrm{d}x \quad \text{and,} \\ \mathbb{V}_\beta(\mathbb{R}^m) &= \{f \in L^1(\mathbb{R}^m) : |f|_\beta < \infty\}. \end{aligned}$$

The space of quasi-Hölder functions on X is defined to be

$$\mathbb{V}_\beta(X) = \{f \in \mathbb{V}_\beta(\mathbb{R}^m) : \operatorname{supp}(f) \subseteq X\},$$

and is a Banach space when endowed with the norm $\|\cdot\|_\beta = \|\cdot\|_{L^1} + |\cdot|_\beta$ [94, 69].

Proposition 4.1.2 ([94, Theorem 5.1, Proposition 5.1])**.** *If T is a multidimensional piecewise expanding map then $\mathcal{L}$ is a bounded, quasi-compact operator on $\mathbb{V}_\beta(X)$ with spectral radius 1 and 1 is an eigenvalue of $\mathcal{L}$. Moreover, if T is topologically mixing then $\mathcal{L}$ is a simple quasi-compact operator on $\mathbb{V}_\beta(X)$ and T has a unique ACIP μ such that $\frac{\mathrm{d}\mu}{\mathrm{d}\,\mathrm{Leb}}$ is the unique fixed point of $\mathcal{L}$ in $\mathbb{V}_\beta(X)$.*

As in the last chapter we consider the sequence of random variables $\{g \circ T^k\}_{k=0}^\infty$ on the probability space (X, Leb), where $g \in \mathbb{V}_\beta(X)$ is an appropriate real-valued observable. In [1] it is shown that a CLT and LDP hold for such sequences via a version of the Nagaev-Guivarc'h method. In particular, the coding hypothesis (NG) follows readily from the proofs in [1] and so in our framing of the Nagaev-Guivarc'h method the laws follow from Theorems 2.1.1 and 2.1.2. As in the last chapter, the coding is induced by a twist: since $\mathbb{V}_\beta(X)$ is a Banach algebra ([94, Proposition 3.4]) for $g \in \mathbb{V}_\beta(X)$ the map $M_g : \mathbb{C} \to L(\mathbb{V}_\beta(X))$ defined by $M_g(z)(f) = e^{zg} f$, is a well-defined, analytic twist (per Definition 1.2.1).

Proposition 4.1.3. *Suppose that $T : X \to X$ is a topologically mixing multidimensional expanding map with unique ACIP μ and that $g \in \mathbb{V}_\beta(X)$ is real-valued, satisfies $\int g \, \mathrm{d}\mu = 0$ and is not a $L^2(\mu)$-coboundary with respect to T. Let $\varphi : \mathbb{V}_\beta(X) \to \mathbb{C}$ denote the map $f \mapsto \int f \, \mathrm{d}\,\mathrm{Leb}$ and for $z \in \mathbb{C}$ let $\mathcal{L}(z)$ be defined by $\mathcal{L}(z)f = (\mathcal{L} \circ M_g(z))(f) = \mathcal{L}(e^{zg}f)$. Then $z \mapsto \mathcal{L}(z)$ is analytic (in the operator norm on $L(\mathbb{V}_\beta(X))$) and $\{g \circ T^k\}_{k=0}^\infty$, when considered on the probability space (X, Leb), satisfies (NG) with coding $(z \mapsto \mathcal{L}(z), \varphi, \frac{\mathrm{d}\mu}{\mathrm{d}\,\mathrm{Leb}})$. In particular, if*

$S_n := \sum_{k=0}^{n-1} g \circ T^k$ *then for every* $z \in \mathbb{C}$ *we have*

$$\mathbb{E}_{\mathrm{Leb}}\left(e^{zS_n}\right) = \int \mathcal{L}(z)^n \left(\frac{\mathrm{d}\mu}{\mathrm{d}\,\mathrm{Leb}}\right) \mathrm{d}\,\mathrm{Leb}\,.$$

Remark 4.1.4. In [1] it is not shown that $t \mapsto \rho(\mathcal{L}(t))$ is strictly convex in a real neighbourhood of 0. However, the proof of Proposition 4.1.3 is much the same as that of Proposition 3.1.3, and in particular the claimed strict convexity follows the observations that $\rho(\mathcal{L}(t))$ equals the leading eigenvalue $\lambda(t)$ of $\mathcal{L}(t)$ for t small enough, that $\lambda''(0) = \sigma^2$, and that g not being an $L^2(\mu)$-coboundary with respect to T implies that $\sigma^2 > 0$ (for this last claim see [79, Lemma 2.5]).

With g as in Proposition 4.1.3 one therefore has a CLT and LDP for $\{g \circ T^k\}_{k=0}^{\infty}$ on the probability space $([0,1], \mathrm{Leb})$ with associated variance σ_g^2 and rate function r_g. Since $z \mapsto \mathcal{L}(z)$ is induced by a twist, in order to deduce the stability of these parameters via an application of Theorem 2.2.1 it remains to verify (KL). We will now gather some important properties of $\mathbb{V}_\beta(X)$, and of $\mathcal{L}$ acting on $\mathbb{V}_\beta(X)$, for this purpose. The first result shows that we may take $\|\cdot\|_{L^1(\mathrm{Leb})}$ to be the weak norm $|\cdot|$ for $\mathbb{V}_\beta(X)$.

Proposition 4.1.5 ([94, Proposition 3.3])**.** *The closed, unit ball in* $\mathbb{V}_\beta(X)$ *is pre-compact with respect to* $\|\cdot\|_{L^1(\mathrm{Leb})}$*.*

Since $\mathcal{L}$ is Markov one obtains (KL2) for $\mathcal{L}$. In later sections we will prove that (KL3) holds for certain perturbations of $\mathcal{L}$ by using the fact that $\mathcal{L}$ satisfies a Lasota-Yorke inequality itself, as confirmed by the following proposition.

Proposition 4.1.6 ([94, Lemma 4.1])**.** *Provided that* η_0 *is small enough, there exists* $\gamma < 1$ *and* $D < \infty$ *such that for each* $f \in \mathbb{V}_\beta(X)$ *we have*

$$|\mathcal{L}f|_\beta \le \gamma\, |f|_\beta + D\, \|f\|_{L^1}\,.$$

Remark 4.1.7. In [94] the space $\mathbb{V}_\beta(X)$ consists of real-valued functions only and so the proof of [94, Lemma 4.1] only applies to real-valued $f \in \mathbb{V}_\beta(X)$. Examining the proof of [94, Lemma 4.1], we note that the same conclusion holds for complex-valued f after minor modifications to the arguments. In particular, the essential infimum in [94, Proposition 3.2 (iii)] must be replaced by an essential supremum and consequently the resulting essential supremum term that appears when bounding $R_i^{(1)}(x)$ must be bounded by $|f(y_i)| + \operatorname{osc}(f, B(y_i, s\epsilon))$. The rest of the argument holds *mutatis mutandis*.

Remark 4.1.8. We will always assume that η_0 is small enough so that Proposition 4.1.6 holds.

We have therefore confirmed that it is possible for (KL) to hold for perturbations to $\mathcal{L}$. Thus we obtain the stability of the σ_g^2 and r_g as a corollary to Theorem 2.2.1 and Proposition 4.1.3.

Theorem 4.1.9. *Let $T : X \to X$ be a topologically mixing multidmensional piecewise expanding map, μ be T's unique ACIP, $\mathcal{L}_0$ be the Perron-Frobenius operator induced by T, $\{\mathcal{L}_\epsilon\}_{\epsilon \geq 0}$ be a family of operators satisfying (KL) and $g \in \mathrm{BV}$ be a real-valued observable such that $\int g \, \mathrm{d}\mu = 0$. There exists $\theta, \epsilon' > 0$ so that for each $\epsilon \in [0, \epsilon')$ and $z \in D_\theta$ the operator $\mathcal{L}_\epsilon(z)$ is quasi-compact and simple with leading eigenvalue $\lambda_\epsilon(z)$ depending analytically on z. Moreover, we have stability of the following statistical data associated to T and $\{g \circ T^k\}_{k \in \mathbb{N}}$ as $\epsilon \to 0$:*

1. *The variance is stable: $\lim_{\epsilon \to 0} \lambda_\epsilon^{(2)}(0) = \sigma_g^2$.*
2. *The rate function is stable: for every compact subset U of the domain of r_g there exists a closed interval V such that*

$$\lim_{\epsilon \to 0} \sup_{z \in V} (sz - \ln |\lambda_\epsilon(z)|) = r_g(s)$$

uniformly for $s \in U$.

Remark 4.1.10. More can be said in this context. In particular, in Theorem 4.1.9 one has stability of the ACIP in the following sense: for sufficiently small ϵ, $\mathcal{L}_\epsilon$ has an eigenvector v_ϵ with $\int v_\epsilon \, \mathrm{d}\,\mathrm{Leb} = 1$ and so that $\lim_{\epsilon \to 0} \left\| v_\epsilon - \frac{\mathrm{d}\mu}{\mathrm{d}\,\mathrm{Leb}} \right\|_{L^1} = 0$. This claim follows from [38, Proposition 2.4, Remark 2.5], whose hypotheses are verified due to the convergence of eigenprojections in Theorem 1.2.2.

In the sections that follow we will develop some perturbations to $\mathcal{L}$ that satisfy (KL), and to which Theorem 4.1.9 can therefore be applied. Before doing so we finish this section by recalling one last result on the quasi-Hölder space on the continuity of inclusion of $\mathbb{V}_\beta(X)$ into $L^\infty(m)$.

Proposition 4.1.11 ([94, Proposition 3.4])**.** *Let ν_m denote the measure of the m-dimensional unit ball i.e. $\nu_m = \mathrm{Leb}(B(0,1))$. For every $f \in \mathbb{V}_\beta(X)$ we have*

$$\|f\|_{L^\infty} \leq \frac{1}{\nu_m \eta_0^m} (\eta_0^\beta |f|_\beta + \|f\|_{L^1}) \leq \frac{\max\{1, \eta_0^\beta\}}{\nu_m \eta_0^m} \|f\|_\beta \, .$$

4.2 Stability under stochastic perturbations

We now consider the case where the Perron-Frobenius operator is perturbed by convolution with a stochastic kernel, which naturally arise when considering i.i.d. additive perturbations to a fixed system (see e.g. the exposition in [10, Section 2.7]).

Definition 4.2.1. *We say that $\{q_\epsilon\}_{\epsilon>0} \subseteq L^1(\mathbb{R}^m)$ approximates the identity if*

1. *Each q_ϵ is non-negative and satisfies $\|q_\epsilon\|_{L^1} = 1$.*
2. *For every $\delta > 0$ we have*

$$\lim_{\epsilon \to 0} \int_{|x| \geq \delta} q_\epsilon(x) \,\mathrm{d}x = 0.$$

Suppose $\{q_\epsilon\}_{\epsilon>0} \subseteq L^1(\mathbb{R}^m)$ approximate the identity. We define the corresponding stochastically perturbed Perron-Frobenius operators by $\mathcal{L}_\epsilon = (q_\epsilon * \mathcal{L})\chi_X$ i.e. for $f \in \mathbb{V}_\beta(X)$ we have

$$(\mathcal{L}_\epsilon f)(x) = \begin{cases} \int_X (\mathcal{L}f)(y) q_\epsilon(x-y) \,\mathrm{d}y & x \in X, \\ 0 & x \notin X. \end{cases}$$

Our main result for this section is the following.

Theorem 4.2.2 (Stability of statistical parameters under stochastic perturbations)**.** *Suppose that $T : X \to X$ be a topologically mixing multidmensional piecewise expanding map with Perron-Frobenius operator $\mathcal{L}_0$. Let $\{q_\epsilon\}_{\epsilon>0}$ approximate the identity, and let $\{\mathcal{L}_\epsilon\}_{\epsilon \geq 0}$ be the corresponding stochastically perturbed Perron-Frobenius operators. If*

$$\left(1 + \frac{1}{\nu_m \eta_0^{m-\beta}} \sup_{0<\eta\leq\eta_0} \eta^{-\beta} \operatorname{Leb}(B(\partial X, \eta))\right) < \frac{1}{\gamma},$$

where the constant γ is from Proposition 4.1.6, then $\{\mathcal{L}_\epsilon\}_{\epsilon\geq0}$ satisfies (KL). Consequently, the conclusion of Theorem 4.1.9 holds.

Remark 4.2.3. It is not obvious when one might have

$$\sup_{0<\eta\leq\eta_0} \eta^{-\beta} \operatorname{Leb}(B(\partial X, \eta)) < \infty.$$

This is the case if, for example, X is convex [55, Theorem 6.6].

We prove Theorem 4.2.2 by showing that $\{\mathcal{L}_\epsilon\}_{\epsilon\geq0}$ satisfies (KL).

Proposition 4.2.4 ((KL1) for stochastic perturbations)**.** *If $\{q_\epsilon\}_{\epsilon>0}$ approximates the identity and $\{\mathcal{L}_\epsilon\}_{\epsilon\geq 0}$ denotes the corresponding stochastically perturbed Perron-Frobenius operators, then*

$$\lim_{\epsilon\to 0} |||\mathcal{L}_\epsilon - \mathcal{L}_0||| = 0.$$

Proof. We have

$$|||\mathcal{L}_\epsilon - \mathcal{L}_0||| \leq \sup_{\|f\|_\beta = 1} \int_X \int_{\mathbb{R}^m} |(\mathcal{L}f)(x-y) - (\mathcal{L}f)(x)|\, q_\epsilon(y)\, \mathrm{d}y\, \mathrm{d}x. \tag{4.1}$$

Let $\delta \in (0, \eta_0)$. We break the inner integral in (4.1) into two parts: the component where $y \in B(0,\delta)$ and the component where $y \in \mathbb{R}^m \setminus B(0,\delta)$. If $f \in \mathbb{V}_\beta(X)$ then by the definition of $\|\cdot\|_\beta$ we have

$$\begin{aligned}\int_X \int_{B(0,\delta)} |(\mathcal{L}f)(x-y) - (\mathcal{L}f)(x)|\, q_\epsilon(y)\, \mathrm{d}y\, \mathrm{d}x &\leq \int_X \operatorname{osc}(\mathcal{L}f, B(x,\delta))\, \mathrm{d}x \\ &\leq \delta^\beta \|\mathcal{L}f\|_\beta .\end{aligned} \tag{4.2}$$

On the other hand, Proposition 4.1.11 we have

$$\begin{aligned}\int_X \int_{\mathbb{R}^m\setminus B(0,\delta)} &|(\mathcal{L}f)(x-y) - (\mathcal{L}f)(x)|\, q_\epsilon(y)\, \mathrm{d}y\, \mathrm{d}x \\ &\leq 2\|\mathcal{L}f\|_{L^\infty} \int_X \int_{\mathbb{R}^m\setminus B(0,\delta)} q_\epsilon(y)\, \mathrm{d}y\, \mathrm{d}x \\ &\leq 2\frac{\max\{1, \eta_0^\beta\}}{\nu_m \eta_0^m} \|\mathcal{L}f\|_\beta \int_{\mathbb{R}^m\setminus B(0,\delta)} q_\epsilon(y)\, \mathrm{d}y.\end{aligned} \tag{4.3}$$

By combining (4.1), (4.2) and (4.3) we obtain

$$|||\mathcal{L}_\epsilon - \mathcal{L}_0||| \leq \delta^\beta \|\mathcal{L}\|_\beta + 2\frac{\max\{1, \eta_0^\beta\}}{\nu_m \eta_0^m} \|\mathcal{L}\|_\beta \int_{\mathbb{R}^m\setminus B(0,\delta)} q_\epsilon(y)\, \mathrm{d}y.$$

As $\{q_\epsilon\}_{\epsilon>0}$ is an approximation to the identity, taking $\epsilon \to 0$ yields

$$\limsup_{\epsilon\to 0} |||\mathcal{L}_\epsilon - \mathcal{L}_0||| \leq \delta^\beta \|\mathcal{L}\|_\beta .$$

We conclude the proof by recalling that δ may be chosen to be arbitrarily small. □

The proof of (KL2) follows from the observation that each $\mathcal{L}_\epsilon$ is Markov on $L^1(X)$, and so is therefore also a contraction on $L^1(X)$.

Proposition 4.2.5 ((KL2) for stochastic perturbations)**.** *If $\{q_\epsilon\}_{\epsilon>0}$ approximates the identity and $\{\mathcal{L}_\epsilon\}_{\epsilon\geq 0}$ denotes the corresponding stochastically perturbed Perron-Frobenius operators, then for each $\epsilon \geq 0$ and $n \in \mathbb{Z}^+$ we have $\|\mathcal{L}_\epsilon^n\|_{L^1} \leq 1$. In particular, $\{\mathcal{L}_\epsilon\}_{\epsilon\geq 0}$ satisfies (KL2).*

We now pursue (KL3), which requires the following preparatory lemma.

Lemma 4.2.6. *If $\{q_\epsilon\}_{\epsilon>0}$ approximates the identity and $\{\mathcal{L}_\epsilon\}_{\epsilon\geq 0}$ denotes the corresponding stochastically perturbed Perron-Frobenius operators, then for every $\epsilon > 0$ and $f \in \mathbb{V}_\beta(X)$ we have*

$$|\mathcal{L}_\epsilon f|_\beta \leq \left(1 + \frac{1}{\nu_m \eta_0^{m-\beta}} \sup_{0<\eta\leq\eta_0} \eta^{-\beta} \operatorname{Leb}(B(\partial X, \eta))\right) |\mathcal{L} f|_\beta + \frac{1}{\nu_m \eta_0^m} \left(\sup_{0<\eta\leq\eta_0} \eta^{-\beta} \operatorname{Leb}(B(\partial X, \eta))\right) \|f\|_{L^1}.$$

Proof. Fix $\epsilon > 0$ and $\eta \in (0, \eta_0]$. Since

$$\operatorname{osc}(\mathcal{L}_\epsilon f, B(x,\eta)) = \operatorname*{ess\,sup}_{y_1,y_2\in B(x,\eta)} |(q_\epsilon * \mathcal{L})(y_1)\chi_X(y_1) - (q_\epsilon * \mathcal{L})(y_2)\chi_X(y_2)|,$$

we consider three (not necessarily distinct) cases when bounding $\operatorname{osc}(\mathcal{L}_\epsilon f, B(x,\eta))$. Depending on how many of the characteristic function terms contribute to the essential supremum, we either have $\operatorname{osc}(\mathcal{L}_\epsilon f, B(x,\eta)) = 0$,

$$\operatorname{osc}(\mathcal{L}_\epsilon, B(x,\eta)) = \operatorname*{ess\,sup}_{y_1\in B(x,\eta)} |(q_\epsilon * \mathcal{L})(f)(y_1)|, \tag{4.4}$$

or

$$\operatorname{osc}(\mathcal{L}_\epsilon f, B(x,\eta)) = \operatorname{osc}(q_\epsilon * (\mathcal{L} f), B(x,\eta)). \tag{4.5}$$

As the support of f is a subset of X, if $x \in \mathbb{R}^m \setminus B(X,\eta)$ then $\operatorname{osc}(\mathcal{L}_\epsilon f, B(x,\eta)) = 0$. By a similar argument, if (4.4) holds, (4.5) does not hold, and $\operatorname{osc}(\mathcal{L}_\epsilon f, B(x,\eta)) \neq 0$, then $x \in B(\partial X, \eta)$. Hence,

$$\int_{\mathbb{R}^m} \operatorname{osc}(\mathcal{L}_\epsilon f, B(x,\eta))\, \mathrm{d}x \leq \int_{\mathbb{R}^m} \operatorname{osc}(q_\epsilon * \mathcal{L} f, B(x,\eta))\, \mathrm{d}x + \int_{B(\partial X,\eta)} \operatorname*{ess\,sup}_{y_1\in B(x,\eta)} |(q_\epsilon * \mathcal{L} f)(y_1)|\, \mathrm{d}x. \tag{4.6}$$

We now bound the quantity (4.4). As $\|\mathcal{L}\|_{L^1} \leq 1$ and by Proposition 4.1.11 we have

$$\begin{aligned}|(q_\epsilon * \mathcal{L}f)(y_1)| &= \left| \int_{\mathbb{R}^m} q_\epsilon(y)(\mathcal{L}f)(y_1 - y)\,\mathrm{d}y \right| \\ &\leq \|\mathcal{L}f\|_\infty \leq \frac{1}{\nu_m \eta_0^m}(\eta_0^\beta \, |\mathcal{L}f|_\beta + \|f\|_{L^1}).\end{aligned}$$

Hence,

$$\int_{B(\partial X,\eta)} \operatorname*{ess\,sup}_{y_1 \in B(x,\eta)} |(q_\epsilon * \mathcal{L}f)(y_1)| \,\mathrm{d}x \leq \frac{\mathrm{Leb}(B(\partial X,\eta))}{\nu_m \eta_0^m}(\eta_0^\beta \, |\mathcal{L}f|_\beta + \|f\|_{L^1}). \tag{4.7}$$

Alternatively, to bound (4.5) we note that

$$\begin{aligned}\mathrm{osc}(q_\epsilon * (\mathcal{L}f), B(x,\eta)) &= \operatorname*{ess\,sup}_{y_1,y_2 \in B(x,\eta)} \left| \int_{\mathbb{R}^m} q_\epsilon(y)((\mathcal{L}f)(y_1 - y) - (\mathcal{L}f)(y_2 - y))\,\mathrm{d}y \right| \\ &\leq \int_{\mathbb{R}^m} q_\epsilon(y) \mathrm{osc}\,(\mathcal{L}f, B(x-y,\eta))\,\mathrm{d}y.\end{aligned}$$

By changing variables and applying Fubini-Tonelli we obtain

$$\begin{aligned}\int_{\mathbb{R}^m} \mathrm{osc}\,(q_\epsilon * (\mathcal{L}f), B(x,\eta))\,\mathrm{d}x &\leq \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} q_\epsilon(y)\mathrm{osc}\,(\mathcal{L}f, B(x-y,\eta))\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} q_\epsilon(x-y)\,\mathrm{d}x \right) \mathrm{osc}\,(\mathcal{L}f, B(y,\eta))\,\mathrm{d}y \qquad (4.8) \\ &\leq \int_{\mathbb{R}^m} \mathrm{osc}\,(\mathcal{L}f, B(y,\eta))\,\mathrm{d}y.\end{aligned}$$

Applying (4.7) and (4.8) to (4.6) yields

$$\begin{aligned}\int_{\mathbb{R}^m} \mathrm{osc}\,(\mathcal{L}_\epsilon f, B(x,\eta))\,\mathrm{d}x &\leq \int_{\mathbb{R}^m} \mathrm{osc}\,(\mathcal{L}f, B(x,\eta))\,\mathrm{d}x \\ &\quad + \frac{\mathrm{Leb}(B(\partial X,\eta))}{\nu_m \eta_0^m}(\eta_0^\beta \, |\mathcal{L}f|_\beta + \|f\|_{L^1}).\end{aligned}$$

Thus,

$$|\mathcal{L}_\epsilon f|_\beta \leq |\mathcal{L}f|_\beta + \frac{1}{\nu_m \eta_0^m} \left(\sup_{0<\eta\leq\eta_0} \eta^{-\beta}\,\mathrm{Leb}(B(\partial X,\eta)) \right) (\eta_0^\beta \, |\mathcal{L}f|_\beta + \|f\|_{L^1}),$$

which is the required bound. □

Proposition 4.2.7 ((KL3) for stochastic perturbations)**.** *Under the hypotheses of Theorem 4.2.2 there exists $\alpha \in (0,1)$ and $C_3 > 0$ such that for every $\epsilon \geq 0$,*

$f \in \mathbb{V}_\beta(X)$ and $n \in \mathbb{Z}^+$ we have

$$\|\mathcal{L}_\epsilon^n f\|_\beta \leq \alpha^n \|f\|_\beta + C_3 \|f\|_{L^1}.$$

Proof. By Lemma 4.2.6, and Propositions 4.1.6 and 4.2.5, for each $\epsilon > 0$ and $f \in \mathbb{V}_\beta(X)$ we have

$$|\mathcal{L}_\epsilon f|_\beta \leq \alpha |f|_\beta + C \|f\|_{L^1} \tag{4.9}$$

where

$$\alpha = \gamma \left(1 + \frac{\eta_0^\beta}{\nu_m \eta_0^m} \sup_{0<\eta\leq\eta_0} \eta^{-\beta} \operatorname{Leb}(B(\partial X, \eta))\right)$$

and

$$C = D + \frac{D\eta_0^\beta + 1}{\nu_m \eta_0^m} \left(\sup_{0<\eta\leq\eta_0} \eta^{-\beta} \operatorname{Leb}(B(\partial X, \eta))\right).$$

Note that $\alpha < 1$ by the hypotheses of Theorem 4.2.2. By iterating (4.9) and applying Proposition 4.2.5, for each $n \in \mathbb{Z}^+$ we have

$$|\mathcal{L}_\epsilon^n f|_\beta \leq \alpha^n |f|_\beta + \frac{C}{1-\alpha} \|f\|_{L^1}.$$

Finally, recalling the definition of $\|\cdot\|_\beta$ and then applying Proposition 4.2.5 again yields

$$\|\mathcal{L}_\epsilon^n f\|_\beta \leq \alpha^n \|f\|_\beta + \left(1 + \frac{C}{1-\alpha}\right) \|f\|_{L^1}.$$

□

Proof of Theorem 4.2.2. By Lemma 4.2.6 each $\mathcal{L}_\epsilon$ is in $L(\mathbb{V}_\beta(X))$. By Propositions 4.2.4, 4.2.5 and 4.2.7 the family of operators $\{\mathcal{L}_\epsilon\}_{\epsilon\geq 0}$ satisfies (KL). The required result then follows from Theorem 4.1.9. □

4.3 Stability under Ulam perturbations

We will now prove that the numerical approximations of the Perron-Frobenius operator arising from Ulam's method satisfy (KL) on $\mathbb{V}_\beta(X)$. As a corollary to Theorem 4.1.9 we then obtain the stability of the rate function and variance with respect to Ulam's method. While (KL2) is simple for Ulam's method, (KL1) and (KL3) require significantly more work. In particular, the proof of (KL3) is quite long and depends critically on the geometry of the partitions inducing the Ulam approximations, so we defer its proof to Section 4.3.1. The class of partitions we consider is the following.

Definition 4.3.1. *For $\kappa \geq 1$ let $\mathcal{P}(\kappa)$ be the collection of finite measurable partitions Q of X satisfying the following conditions:*

1. *The elements of Q are compact and convex polytopes with non-empty interiors.*
2. *For every $I \in Q$ we have $\operatorname{diam}(Q) \leq \kappa \operatorname{diam}(B_I)$, where B_I is a ball of maximal volume inscribed in I and $\operatorname{diam}(Q) = \max_{J \in Q} \operatorname{diam}(J)$.*

Each $Q \in \mathcal{P}(\kappa)$ induces a conditional expectation operator $\mathbb{E}_Q$ on $L^1(X)$ that is given by

$$\mathbb{E}_Q f = \sum_{I \in Q} \hat{f}_I \chi_I,$$

where $\hat{f}_K$ denotes the expected value of f on a Borel set K i.e.

$$\hat{f}_K = \frac{1}{\operatorname{Leb}(K)} \int_K f \, \mathrm{d} \operatorname{Leb}.$$

We adopt the convention that $\hat{f}_K = 0$ if $\operatorname{Leb}(K) = \infty$. As in Section 3.2 the Ulam approximation of $\mathcal{L}$ induced by Q is the operator $\mathcal{L}_Q := \mathbb{E}_Q \mathcal{L}$. If $\{Q_\epsilon\}_{\epsilon > 0} \subseteq \mathcal{P}(\kappa)$ is a sequence of partitions then we define the corresponding sequence of perturbed Perron-Frobenius operators by $\mathcal{L}_\epsilon = \mathbb{E}_{Q_\epsilon} \mathcal{L}$, with $\mathcal{L}_0 := \mathcal{L}$.

Theorem 4.3.2 (Stability of statistical parameters under Ulam approximations)**.** *Suppose that $T : X \to X$ be a topologically mixing multidmensional piecewise expanding map with Perron-Frobenius operator $\mathcal{L}_0$. Let $\{Q_\epsilon\}_{\epsilon > 0} \subseteq \mathcal{P}(\kappa)$ be such that $\lim_{\epsilon \to 0} \operatorname{diam}(Q_\epsilon) = 0$, and let $\{\mathcal{L}_\epsilon\}_{\epsilon \geq 0}$ be the corresponding Ulam approximations of the Perron-Frobenius operator. If*

$$2m \left(1 + \frac{2\kappa}{\sqrt[m]{\frac{3}{2}} - 1} \right)^{\beta} < \frac{1}{\gamma},$$

where the constant γ is from Proposition 4.1.6, then there exists $\epsilon_0 > 0$ such that $\{\mathcal{L}_\epsilon\}_{0 \leq \epsilon \leq \epsilon_0}$ satisfies (KL). Consequently, the conclusion of Theorem 4.1.9 holds.

Example 4.3.3. Let $X = [0,1]^m$. For each $n \in \mathbb{Z}^+$ the set

$$Q_n = \left\{ \frac{X + b}{n} : b \in (\mathbb{Z} \cap [0, n))^m \right\}$$

is a measurable partition of X consisting of hypercubes congruent to $[0, 1/n]^m$. It is straightforward to show that $Q_n \in \mathcal{P}\left(\sqrt{m}\right)$ for every $n \in \mathbb{Z}^+$. Thus, Theorem 4.3.2

applies to any multidmensional piecewise expanding map $T : X \to X$ such that

$$2m\left(1+\frac{2\sqrt{m}}{\sqrt[m]{\frac{3}{2}}-1}\right)^{\beta} < \frac{1}{\gamma}.$$

Proposition 4.3.4 ((KL1) for Ulam approximations)**.** *If $\{Q_\epsilon\}_{\epsilon>0} \subseteq \mathcal{P}(\kappa)$ satisfies $\lim_{\epsilon\to 0}\operatorname{diam}(Q_\epsilon) = 0$, then there exists $\epsilon_1 > 0$ such that for every $\epsilon \in [0, \epsilon_1]$ we have*

$$|||\mathcal{L}_\epsilon - \mathcal{L}_0||| \le 2\operatorname{diam}(Q_\epsilon)^{\beta}\, \|\mathcal{L}\|_{\beta}\,.$$

In particular $\{\mathcal{L}_\epsilon\}_{0\le\epsilon\le\epsilon_1}$ satisfies (KL1).

Proof. The statement is clearly true for $\epsilon = 0$, so we may assume that $\epsilon > 0$. One has

$$|||\mathbb{E}_{Q_\epsilon} - \mathrm{Id}||| = \sup_{\|f\|_\alpha=1}\sum_{I\in Q_\epsilon}\int_I\left|\hat{f}_I - f\right|\,\mathrm{d}\,\mathrm{Leb}\,. \tag{4.10}$$

Fix $I \in Q_\epsilon$. Let f_r and f_i be the real and imaginary parts of $f \in \mathbb{V}_\beta(X)$, respectively. By linearity of integration and the triangle inequality we have

$$\int_I\left|\hat{f}_I - f\right|\,\mathrm{d}\,\mathrm{Leb} \le \int_I\left|(\hat{f}_r)_I - f_r\right|\,\mathrm{d}\,\mathrm{Leb} + \int_I\left|(\hat{f}_i)_I - f_i\right|\,\mathrm{d}\,\mathrm{Leb}\,.$$

Applying this to (4.10) yields

$$|||\mathbb{E}_{Q_\epsilon} - \mathrm{Id}||| \le 2\sup\left\{\sum_{I\in Q_\epsilon}\int_I\left|\hat{f}_I - f\right|\,\mathrm{d}\,\mathrm{Leb} : \substack{f \text{ is real valued}\\ \text{and } \|f\|_\alpha=1}\right\}. \tag{4.11}$$

Let $f \in \mathbb{V}_\beta(X)$ be real valued and fix $x \in I$. Then for almost every $y_1, y_2 \in I$ we have

$$|f(y_1) - f(y_2)| \le \operatorname{osc}\left(f, B(x, \operatorname{diam}(Q_\epsilon))\right).$$

Taking the expectation with respect to y_1 over I we find that

$$\left|\hat{f}_I - f(y_2)\right| \le \frac{1}{\mathrm{Leb}(I)}\int_I|f(y_1) - f(y_2)|\,\mathrm{d}y_1 \le \operatorname{osc}\left(f, B(x, \operatorname{diam}(Q_\epsilon))\right),$$

for almost every $y_2 \in I$. Taking the expectation with respect to both y_2 and x over I we then obtain

$$\int_I\left|\hat{f}_I - f\right|\,\mathrm{d}\,\mathrm{Leb} \le \int_I \operatorname{osc}\left(f, B(x, \operatorname{diam}(Q_\epsilon))\right)\,\mathrm{d}x.$$

As $\lim_{\epsilon\to 0}\operatorname{diam}(Q_\epsilon) = 0$ there exists $\epsilon_1 > 0$ such that $\operatorname{diam}(Q_\epsilon) \le \eta_0$ for all $\epsilon \in (0, \epsilon_1]$. Recalling the definition of $\|\cdot\|_\beta$, for each $\epsilon \in (0, \epsilon_1]$ we have

$$\sum_{I\in Q_\epsilon} \int_I \left| \hat{f}_I - f \right| \, \mathrm{d}\,\mathrm{Leb} \le \int_{\mathbb{R}^m} \operatorname{osc}\left(f, B(x, \operatorname{diam}(Q_\epsilon))\right) \, \mathrm{d}x \le \operatorname{diam}(Q_\epsilon)^\beta \, \|f\|_\beta \, ,$$

which, when applied to (4.11), then yields

$$|||\mathbb{E}_{Q_\epsilon} - \mathrm{Id}||| \le 2 \operatorname{diam}(Q_\epsilon)^\beta.$$

We conclude the proof by noting that $|||\mathcal{L}_\epsilon - \mathcal{L}||| \le |||\mathbb{E}_{Q_\epsilon} - \mathrm{Id}||| \, \|\mathcal{L}\|_\beta$. □

The proof of (KL2) in this case follows easily from the fact that both $\mathcal{L}$ and $\mathbb{E}_{Q_\epsilon}$ are Markov operators on $L^1(X)$.

Proposition 4.3.5 ((KL2) for Ulam approximations). *If $\{Q_\epsilon\}_{\epsilon>0} \subseteq \mathcal{P}(\kappa)$ is a sequence of partitions, then for each $\epsilon \ge 0$ and $n \in \mathbb{Z}^+$ we have $\|\mathcal{L}_\epsilon^n\|_{L^1} \le 1$. In particular, $\{\mathcal{L}_\epsilon\}_{\epsilon\ge 0}$ satisfies (KL2).*

We will now verify (KL3) for the Ulam approximations. The main technical requirements are the following two lemmas, which we defer the proofs of to Section 4.3.1.

Lemma 4.3.6. *If $\{Q_\epsilon\}_{\epsilon>0} \subseteq \mathcal{P}(\kappa)$ satisfies $\lim_{\epsilon\to 0}\operatorname{diam}(Q_\epsilon) = 0$ then there exists $\epsilon_2 > 0$ such that*

$$\sup_{0<\epsilon\le\epsilon_2} |\mathbb{E}_{Q_\epsilon}|_\beta \le 2m \left(1 + \frac{2\kappa}{\sqrt[m]{\frac{3}{2}} - 1}\right)^\beta .$$

Lemma 4.3.7. *If $Q \in \mathcal{P}(\kappa)$ then $\mathbb{E}_Q \in L(\mathbb{V}_\beta(X))$.*

With these lemmas in hand we obtain (KL3).

Proposition 4.3.8 ((KL3) for Ulam approximations). *Under the hypotheses of Theorem 4.3.2 there exists $\alpha \in (0, 1)$ and $C_3 > 0$ such that for all $f \in \mathbb{V}_\beta(X)$, $n \in \mathbb{Z}^+$ and $\epsilon \in [0, \epsilon_2]$ we have*

$$\|\mathcal{L}_\epsilon^n f\|_\beta \le \alpha^n \|f\|_\beta + C_3 \|f\|_{L^1} .$$

Proof. By Proposition 4.1.6 we have

$$|\mathcal{L}f|_\beta \le \gamma \, |f|_\beta + D \, \|f\|_{L^1} ,$$

where $\gamma < 1$ and $D < \infty$. Let ϵ_2 be as in Lemma 4.3.6. If $\epsilon \in [0, \epsilon_2]$ then

$$|\mathcal{L}_\epsilon f|_\beta \leq \left(\sup_{0 < \epsilon \leq \epsilon_2} |\mathbb{E}_{Q_\epsilon}|_\beta \right) |\mathcal{L} f|_\beta \leq \alpha |f|_\beta + C \|f\|_{L^1}, \tag{4.12}$$

where

$$\alpha = \gamma 2m \left(1 + \frac{2\kappa}{\sqrt[m]{\frac{3}{2}} - 1} \right)^\beta \text{ and } C = 2mD \left(1 + \frac{2\kappa}{\sqrt[m]{\frac{3}{2}} - 1} \right)^\beta .$$

Note that $\alpha < 1$ by the hypotheses of Theorem 4.3.2. The remainder of the proof is the same as that of Proposition 4.2.7, with Proposition 4.3.5 being used in place of Proposition 4.2.5. □

Proof of Theorem 4.3.2. With ϵ_1 as in Proposition 4.3.4 and ϵ_2 as in Proposition 4.3.8, set $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$. By Lemma 4.3.7 and Propositions 4.3.4, 4.3.5 and 4.3.8 the family of operators $\{\mathcal{L}_\epsilon\}_{0 \leq \epsilon \leq \epsilon_0}$ satisfies (KL). The required result then follows from Theorem 4.1.9. □

4.3.1 The proofs of Lemmas 4.3.6 and 4.3.7

Before discussing our strategy for proving Lemmas 4.3.6 and 4.3.7 we must discuss the relationship between the space $\mathbb{V}_\beta(X)$ and the seminorm $|\cdot|_\beta$. It is noted in [69] that while $\mathbb{V}_\beta(X)$ is independent of η_0, the seminorm $|\cdot|_\beta$ is obviously not. However, changing η_0 preserves the topology induced by the relevant seminorm, which will be critical to proofs in this section. The following lemma gives the relevant bounds.

Lemma 4.3.9. *For $\zeta > 0$ and $f \in L^1(\mathbb{R}^m)$ let*

$$|f|_{\beta,\zeta} = \sup_{0 < \eta \leq \zeta} \eta^{-\beta} \int_{\mathbb{R}^m} \operatorname{osc}(f, B(x, \eta)) \, \mathrm{d}x.$$

If $0 < t \leq s$ then

$$|\cdot|_{\beta,t} \leq |\cdot|_{\beta,s} \leq S(t,s) |\cdot|_{\beta,t},$$

where $S(t,s)$ denotes the minimal number of balls of radius t required to cover (up to a set of measure 0) a ball of radius s.

Proof. The inequality $|\cdot|_{\beta,t} \leq |\cdot|_{\beta,s}$ is trivial. Let $f \in L^1(\mathbb{R}^m)$. If

$$\sup_{0 < \eta \leq t} \eta^{-\beta} \int_{\mathbb{R}^m} \operatorname{osc}(f, B(x, \eta)) \, \mathrm{d}x = \sup_{0 < \eta \leq s} \eta^{-\beta} \int_{\mathbb{R}^m} \operatorname{osc}(f, B(x, \eta)) \, \mathrm{d}x, \tag{4.13}$$

then, as $S(t,s) \geq 1$, we clearly have $|f|_{\beta,s} \leq S(t,s)\,|f|_{\beta,t}$. Alternatively, if (4.13) does not hold then

$$\sup_{0<\eta\leq t} \eta^{-\beta} \int_{\mathbb{R}^m} \operatorname{osc}\left(f, B(x,\eta)\right) \, \mathrm{d}x < \sup_{t<\eta\leq s} \eta^{-\beta} \int_{\mathbb{R}^m} \operatorname{osc}\left(f, B(x,\eta)\right) \, \mathrm{d}x.$$

By the definition of $S(t,s)$ there exists $\{c_i\}_{i=1}^{S(t,s)} \subseteq \mathbb{R}^m$ and a set N of measure 0 such that

$$B(x,s) \setminus (N+x) \subseteq \bigcup_{i=1}^{S(t,s)} B(x+c_i,t)$$

for every $x \in \mathbb{R}^m$. Hence, for any $\eta \in (t,s]$ and $x \in \mathbb{R}^m$ we have

$$\operatorname{osc}\left(f, B(x,\eta)\right) \leq \operatorname{osc}\left(f, B(x,s)\right) \leq \sum_{i=1}^{S(t,s)} \operatorname{osc}\left(f, B(x+c_i,t)\right).$$

After integrating, taking the supremum and applying the definition of $|\cdot|_{\beta,t}$ we obtain

$$\begin{aligned} \sup_{t<\eta\leq s} \eta^{-\beta} \int_{\mathbb{R}^m} \operatorname{osc}\left(f, B(x,\eta)\right) \, \mathrm{d}x &\leq S(t,s) t^{-\beta} \int_{\mathbb{R}^m} \operatorname{osc}\left(f, B(x,t)\right) \, \mathrm{d}x \\ &\leq S(t,s)\,|f|_{\beta,t}, \end{aligned}$$

completing the proof. □

We obtain Lemmas 4.3.6 and 4.3.7 as corollaries to the following result.

Proposition 4.3.10. *If $Q \in \mathcal{P}(\kappa)$ satisfies $\operatorname{diam}(Q) < \eta_0$, then*

$$|\mathbb{E}_Q|_\beta \leq S(\eta_0 - \operatorname{diam}(Q), \eta_0) \left(1 + \frac{2\kappa}{\sqrt[m]{\frac{3}{2}} - 1}\right)^\beta. \tag{4.14}$$

We prove Proposition 4.3.10 by using Lemma 4.3.9 to extend a bound for $\sup_{|f|_\beta=1} |\mathbb{E}_Q f|_{\beta,\eta_0-\operatorname{diam}(Q)}$ to a bound for $|\mathbb{E}_Q|_\beta$. We do this by combining two bounds for

$$\eta^{-\beta} \int \operatorname{osc}\left(\mathbb{E}_Q f, B(x,\eta)\right) \mathrm{d}\,\mathrm{Leb}$$

over $\eta \in (0, \eta_0 - \operatorname{diam}(Q)]$:

1. We obtain the 'big' η bound by scaling the η-balls in $\operatorname{osc}\left(\cdot, B(\cdot,\eta)\right)$ up to $\eta + \operatorname{diam}(Q)$ balls. This bound is useful for large η, but grows unboundedly as η vanishes. We obtain this bound in Lemma 4.3.12.

2. We obtain the 'small' η bound by using the geometry of the elements of Q to quantify the decay of the measure of the support of $\operatorname{osc}(\mathbb{E}_Q f, B(\cdot, \eta))$ as η vanishes. This bound is used for η arbitrarily close to 0. Obtaining this bound is more complicated and is developed in Lemmas 4.3.13, 4.3.14 and 4.3.15.

Before proving these bounds we derive an expression for $\int \operatorname{osc}(\mathbb{E}_Q f, B(x, \eta))\, \mathrm{d}\,\mathrm{Leb}$.

Lemma 4.3.11. *Let $Q \in \mathcal{P}(\kappa)$, define $Q' = Q \cup \{X^c\}$, and for each $\eta > 0$ and $x \in \mathbb{R}^m$ let*

$$N(x, \eta) = \{J \in Q' : B(x, \eta) \cap J \neq \emptyset\}.$$

For each $\eta > 0$, $f \in \mathbb{V}_\beta(X)$ and $S \subseteq Q'$ let

$$M_S(f) = \max_{J,K \in S} \left| \hat{f}_J - \hat{f}_K \right| \quad \text{and} \quad A_{S,\eta} = \{x \in \mathbb{R}^m : N(x, \eta) = S\}.$$

Then each $A_{S,\eta}$ is measurable, and for every $x \in \mathbb{R}^m$ we have

$$\operatorname{osc}(\mathbb{E}_Q f, B(x, \eta)) = M_{N(x,\eta)}(f). \tag{4.15}$$

Hence,

$$\int_{\mathbb{R}^m} \operatorname{osc}(\mathbb{E}_Q f, B(x, \eta))\, \mathrm{d}x = \sum_{S \subseteq Q'} \mathrm{Leb}(A_{S,\eta}) M_S(f). \tag{4.16}$$

Proof. For every $J \in Q$ the equality

$$\{x \in \mathbb{R}^m : B(x, \eta) \cap J \neq \emptyset\} = \bigcup_{y \in J} B(y, \eta),$$

implies that both sets are open, and therefore measurable. Recalling from Definition 4.3.1 that Q is finite and noting the equality

$$A_{S,\eta} = \left(\bigcap_{J \in S} \{x \in \mathbb{R}^m : B(x, \eta) \cap J \neq \emptyset\} \right) \bigcap \left(\bigcap_{K \in Q' \setminus S} \{x \in \mathbb{R}^m : B(x, \eta) \cap K = \emptyset\} \right),$$

we conclude that each $A_{S,\eta}$ is measurable. Considering the definition of $N(x, \eta)$, we note that the family of sets $\{A_{S,\eta} : S \subseteq Q'\}$ partitions $\mathbb{R}^m$. Hence,

$$\int_{\mathbb{R}^m} \operatorname{osc}(\mathbb{E}_Q f, B(x, \eta))\, \mathrm{d}x = \sum_{S \subseteq Q'} \int_{A_{S,\eta}} \operatorname{osc}(\mathbb{E}_Q f, B(x, \eta))\, \mathrm{d}x,$$

where we note that the sum on the right side is well defined as only finitely many terms are ever non-zero. Thus, in order to prove (4.16) it suffices to prove (4.15). If $N(x,\eta) = S$ then

$$(\mathbb{E}_Q f)(B(x,\eta)) = \left\{ \hat{f}_J : J \cap B(x,\eta) \neq \emptyset \right\} = \left\{ \hat{f}_J : J \in S \right\},$$

which is finite as Q' is finite. By applying the definition of osc, we find that

$$\operatorname{osc}\left(\mathbb{E}_Q f, B(x,\eta)\right) = \max_{J,K\in S} \left| \hat{f}_J - \hat{f}_K \right| = M_S(f),$$

which is exactly (4.15). □

We may now obtain the 'big' η bound.

Lemma 4.3.12. *If $Q \in \mathcal{P}(\kappa)$ then for each $f \in \mathbb{V}_\beta(X)$ and $\eta > 0$ we have*

$$\int_{\mathbb{R}^m} \operatorname{osc}\left(\mathbb{E}_Q f, B(x,\eta)\right) \,\mathrm{d}x \leq \int_{\mathbb{R}^m} \operatorname{osc}\left(f, B(x,\eta + \operatorname{diam}(Q))\right) \,\mathrm{d}x. \tag{4.17}$$

Furthermore, if $\operatorname{diam}(Q) < \eta_0$ *and* $\eta \in (0, \eta_0 - \operatorname{diam}(Q)]$ *then*

$$\eta^{-\beta} \int_{\mathbb{R}^m} \operatorname{osc}\left(\mathbb{E}_Q f, B(x,\eta)\right) \,\mathrm{d}x \leq \left(1 + \frac{\operatorname{diam}(Q)}{\eta}\right)^\beta |f|_\beta \,. \tag{4.18}$$

Proof. Fix $x \in \mathbb{R}^m$. By Lemma 4.3.11, we have

$$\operatorname{osc}\left(\mathbb{E}_Q f, B(x,\eta)\right) = \max_{J,K\in N(x,\eta)} \left| \hat{f}_J - \hat{f}_K \right|.$$

Suppose that $y \in I$ for some $I \in N(x,\eta) \setminus \{X^c\}$. By the definition of $N(x,\eta)$ there exists $z \in I$ such that $|z - x| < \eta$. Since $|z - y| \leq \operatorname{diam}(Q)$ we have $|y - x| < \eta + \operatorname{diam}(Q)$. Hence

$$\bigcup_{I \in N(x,\eta)\setminus\{X^c\}} I \subseteq B(x, \eta + \operatorname{diam}(Q)). \tag{4.19}$$

Now suppose that $J, K \in N(x,\eta)$. In the case where $J, K \in N(x,\eta) \setminus \{X^c\}$ the inclusion (4.19) implies that for almost every $(y_1, y_2) \in J \times K$ we have

$$|f(y_1) - f(y_2)| \leq \operatorname{osc}\left(f, B(x, \eta + \operatorname{diam}(Q))\right).$$

By taking expectations with respect to y_1 over J and y_2 over K we obtain

$$\left|\hat{f}_J - \hat{f}_K\right| \leq \operatorname{osc}\left(f, B(x, \eta + \operatorname{diam}(Q))\right). \tag{4.20}$$

Alternatively, if one of J or K is equal to X^c then

$$\begin{aligned} \left|\hat{f}_J - \hat{f}_K\right| = \max\left\{\left|\hat{f}_J\right|, \left|\hat{f}_K\right|\right\} &\leq \max_{I \in N(x,\eta) \setminus X^c} \left|\hat{f}_I\right| \\ &\leq \operatorname*{ess\,sup}_{y \in B(x, \eta + \operatorname{diam}(Q))} |f(y)|, \end{aligned} \tag{4.21}$$

where we obtain the last inequality by using (4.19). Noting that the set $B(x, \eta + \operatorname{diam}(Q)) \cap X^c$ has non-zero measure, we have

$$\operatorname*{ess\,sup}_{y \in B(x, \eta + \operatorname{diam}(Q))} |f(y)| \leq \operatorname{osc}\left(f, B(x, \eta + \operatorname{diam}(Q))\right). \tag{4.22}$$

By combining (4.21) and (4.22) we obtain (4.20) for the case where one of J or K is equal to X^c. As J and K were arbitrary elements of $N(x, \eta)$ this implies that

$$\operatorname{osc}\left(\mathbb{E}_Q f, B(x, \eta)\right) = \max_{I,J \in N(x,\eta)} \left|\hat{f}_I - \hat{f}_J\right| \leq \operatorname{osc}\left(f, B(x, \eta + \operatorname{diam}(Q))\right).$$

By integrating with respect to x over $\mathbb{R}^m$ we obtain (4.17). We will now prove (4.18). If $\operatorname{diam}(Q) < \eta_0$ and $\eta \in (0, \eta_0 - \operatorname{diam}(Q)]$ then $\eta + \operatorname{diam}(Q) \in (0, \eta_0]$ and so the definition of $|\cdot|_\beta$ implies that

$$\int_{\mathbb{R}^m} \operatorname{osc}\left(f, B(x, \eta + \operatorname{diam}(Q))\right) \mathrm{d}x \leq (\eta + \operatorname{diam}(Q))^\beta |f|_\beta.$$

Thus

$$\begin{aligned} \eta^{-\beta} \int_{\mathbb{R}^m} \operatorname{osc}\left(\mathbb{E}_Q f, B(x, \eta)\right) \mathrm{d}x &\leq \eta^{-\beta} \int_{\mathbb{R}^m} \operatorname{osc}\left(f, B(x, \eta + \operatorname{diam}(Q))\right) \mathrm{d}x \\ &\leq \left(1 + \frac{\operatorname{diam}(Q)}{\eta}\right)^\beta |f|_\beta. \end{aligned}$$

□

We will now pursue the 'small' η bound.

Lemma 4.3.13. *Let $Q \in \mathcal{P}(\kappa)$ and let $S \subseteq Q'$ satisfy $|S| > 1$. If $I \in S \setminus \{X^c\}$ and $\eta > 0$ then $A_{S,\eta} \subseteq B(\partial I, \eta)$.*

Proof. The claim is trivially true if $A_{S,\eta}$ is empty, henceforth we assume that it is not. Let $x \in A_{S,\eta}$. We distinguish between two cases: either $x \in I$ or $x \notin I$. Suppose that $x \in I$. As $|S| > 1$ and $N(x, \eta) = S$ there exists some $J \in Q' \setminus \{I\}$ such that $B(x, \eta) \cap J \neq \emptyset$. Actually, as the closure of the interior of J is J, we have $B(x, \eta) \cap \operatorname{int}(J) \neq \emptyset$. In this case let $y \in B(x, \eta) \cap \operatorname{int}(J)$; as J and I are convex elements of a measurable partition we have $J \cap I \subseteq \partial J \cap \partial I$ and so $y \notin I$. Alternatively, if $x \notin I$, then let $y \in B(x, \eta) \cap I$, which is non-empty by a similar argument. In both cases we have a pair of points in $A_{S,\eta}$: one in I and the other not. Recalling that elements of Q' have non-empty interior and then considering the line segment that joins x and y, it is straightforward to verify that there exists some $z \in \partial I$ on this line segment. Clearly $|x - z| < \eta$ and so $x \in B(\partial I, \eta)$, which completes the proof. □

Lemma 4.3.14. *Let $Q \in \mathcal{P}(\kappa)$. If $\eta > 0$ and $S \subseteq Q'$ is such that $|S| > 1$ and* $\operatorname{Leb}(A_{S,\eta}) > 0$, *then for each $f \in \mathbb{V}_\beta(X)$ we have*

$$M_S(f) \leq \max_{\substack{J,K \in S \\ J \neq K}} \left(\int_J \frac{\operatorname{osc}\left(f, B(x, \eta + \operatorname{diam}(Q))\right)}{\operatorname{Leb}(J)} \, \mathrm{d}x \right.$$
$$\left. + \int_K \frac{\operatorname{osc}\left(f, B(x, \eta + \operatorname{diam}(Q))\right)}{\operatorname{Leb}(K)} \, \mathrm{d}x \right).$$

Proof. Let $J, K \in S$ be partition elements satisfying

$$M_S(f) = \left| \hat{f}_J - \hat{f}_K \right|.$$

We may assume that $J \neq K$, as this case does not contribute to the maximum. Let us first consider the case where $X^c \in \{J, K\}$; without loss of generality let $K = X^c$. For every $j \in J$ we have $J \subseteq B(j, \eta + \operatorname{diam}(Q))$. Hence, as $B(j, \eta + \operatorname{diam}(Q)) \cap X^c$ has non-empty interior, and therefore non-zero measure, for almost every $j, j' \in J$ and $k' \in B(j, \eta + \operatorname{diam}(Q)) \cap X^c$ we have

$$|f(j')| = |f(j') - f(k')| \leq \operatorname{osc}\left(f, B(j, \eta + \operatorname{diam}(Q))\right).$$

Taking expectations with respect to j' and j over J yields

$$M_S(f) = \left| \hat{f}_J \right| \leq \int_J \frac{\operatorname{osc}\left(f, B(x, \eta + \operatorname{diam}(Q))\right)}{\operatorname{Leb}(J)} \, \mathrm{d}x,$$

which implies the required conclusion. Alternatively suppose that neither J nor K is equal to X^c. Fix $j \in J$ and $k \in K$. For any $j' \in J$ we have $|j - j'| \leq \mathrm{diam}(Q)$ and so $j' \in B(j, \eta + \mathrm{diam}(Q))$. Similarly, for every $k' \in K$ we have $k' \in B(k, \eta + \mathrm{diam}(Q))$. As $\mathrm{Leb}(A_{S,\eta}) > 0$, we know that $A_{S,\eta} \neq \emptyset$. For $z \in A_{S,\eta}$ the intersection $B(z, \eta) \cap J$ is non-empty and so $z \in B(j, \eta + \mathrm{diam}(Q))$. Similarly, $z \in B(k, \eta + \mathrm{diam}(Q))$. Hence, for almost every $j' \in J$ and $k' \in K$,

$$\begin{aligned} |f(j') - f(k')| &\leq |f(j') - f(z)| + |f(k') - f(z)| \\ &\leq \mathrm{osc}\left(f, B(j, \eta + \mathrm{diam}(Q))\right) + \mathrm{osc}\left(f, B(k, \eta + \mathrm{diam}(Q))\right). \end{aligned}$$

By taking the expectation with respect to j' over J and k' over K, we find

$$\left|\hat{f}_J - \hat{f}_K\right| \leq \mathrm{osc}\left(f, B(j, \eta + \mathrm{diam}(Q))\right) + \mathrm{osc}\left(f, B(k, \eta + \mathrm{diam}(Q))\right). \tag{4.23}$$

Since (4.23) holds for every $j \in J$ and $k \in K$, we may take expectations again to obtain

$$\left|\hat{f}_J - \hat{f}_K\right| \leq \int_J \frac{\mathrm{osc}\left(f, B(x, \eta + \mathrm{diam}(Q))\right)}{\mathrm{Leb}(J)}\, \mathrm{d}x + \int_K \frac{\mathrm{osc}\left(f, B(x, \eta + \mathrm{diam}(Q))\right)}{\mathrm{Leb}(K)}\, \mathrm{d}x.$$

We obtain the required inequality by taking the maximum over all distinct pairs of $J, K \in S$. □

Combining the previous two results yields the 'small' η bound.

Lemma 4.3.15. *Let $Q \in \mathcal{P}(\kappa)$. If $\mathrm{diam}(Q) < \eta_0$, $\eta \in (0, \eta_0 - \mathrm{diam}(Q)]$ and $f \in \mathbb{V}_\beta(X)$ then*

$$\eta^{-\beta} \int_{\mathbb{R}^m} \mathrm{osc}\left(\mathbb{E}_Q f, B(x, \eta)\right)\, \mathrm{d}x \leq \left(\max_{I \in Q} \frac{\mathrm{Leb}(B(\partial I, \eta))}{\mathrm{Leb}(I)}\right)\left(1 + \frac{\mathrm{diam}(Q)}{\eta}\right)^\beta |f|_\beta.$$

Proof. By Lemma 4.3.11 we have

$$\int_{\mathbb{R}^m} \mathrm{osc}\left(\mathbb{E}_Q f, B(x, \eta)\right)\, \mathrm{d}x = \sum_{S \subseteq Q'} \mathrm{Leb}(A_{S,\eta}) M_S(f). \tag{4.24}$$

Let $G = \{S \subseteq Q' : |S| > 1, \mathrm{Leb}(A_{S,\eta}) > 0\}$. Since $\mathrm{Leb}(A_{S,\eta})M_S(f) = 0$ if $S \notin G$ we may restrict the sum in (4.24) to $S \in G$ i.e.

$$\int_{\mathbb{R}^m} \mathrm{osc}\left(\mathbb{E}_Q f, B(x, \eta)\right)\, \mathrm{d}x = \sum_{S \in G} \mathrm{Leb}(A_{S,\eta}) M_S(f). \tag{4.25}$$

Applying Lemma 4.3.14 to each of the terms in (4.25) yields

$$\begin{aligned}&\int_{\mathbb{R}^m} \operatorname{osc}\left(\mathbb{E}_Q f, B(x,\eta)\right)\,\mathrm{d}x \\ &\leq \sum_{S\in G} \mathrm{Leb}(A_{S,\eta}) \max_{J,K\in S, J\neq K}\Bigg(\int_J \frac{\operatorname{osc}\left(f, B(x,\eta+\mathrm{diam}(Q))\right)}{\mathrm{Leb}(J)}\,\mathrm{d}x \\ &\qquad\qquad + \int_K \frac{\operatorname{osc}\left(f, B(x,\eta+\mathrm{diam}(Q))\right)}{\mathrm{Leb}(K)}\,\mathrm{d}x\Bigg).\end{aligned} \tag{4.26}$$

By rearranging the terms in (4.26) to sum over elements of Q we obtain

$$\begin{aligned}&\int_{\mathbb{R}^m} \operatorname{osc}\left(\mathbb{E}_Q f, B(x,\eta)\right)\mathrm{d}x \\ &\qquad \leq \sum_{I\in Q} \frac{\sum_{S\in G, I\in S}\mathrm{Leb}(A_{S,\eta})}{\mathrm{Leb}(I)} \int_I \operatorname{osc}\left(f, B(x,\eta+\mathrm{diam}(Q))\right)\,\mathrm{d}x \\ &\qquad \leq \left(\max_{I\in Q}\frac{\sum_{S\in G, I\in S}\mathrm{Leb}(A_{S,\eta})}{\mathrm{Leb}(I)}\right)\int_{\mathbb{R}^m} \operatorname{osc}\left(f, B(x,\eta+\mathrm{diam}(Q))\right)\,\mathrm{d}x,\end{aligned}$$

where we omit the case of $I = X^c$, as it does not contribute to the sum. Since the sets $\{A_{S,\eta}\}_{S\subseteq Q'}$ are disjoint, Lemma 4.3.13 implies that

$$\sum_{S\in G, I\in S} \mathrm{Leb}(A_{S,\eta}) \leq \mathrm{Leb}(B(\partial I,\eta)).$$

Thus,

$$\begin{aligned}&\int_{\mathbb{R}^m} \operatorname{osc}\left(\mathbb{E}_Q f, B(x,\eta)\right)\mathrm{d}x \\ &\qquad \leq \left(\max_{I\in Q}\frac{\mathrm{Leb}(B(\partial I,\eta))}{\mathrm{Leb}(I)}\right)\int_{\mathbb{R}^m} \operatorname{osc}\left(f, B(x,\eta+\mathrm{diam}(Q))\right)\,\mathrm{d}x.\end{aligned}$$

The required inequality follows by applying the definition of $|\cdot|_\beta$. □

Before proving Proposition 4.3.10 we require a technical lemma for an inequality from convex geometry. For $U, V \subseteq \mathbb{R}^m$ the Minkowski sum of U and V is denoted by $U + V$ and equal to $\{u + v : u \in U, v \in V\}$; for basic properties we refer to [55, Section 6.1].

Lemma 4.3.16. *If I is a compact convex polytope then for every $\eta > 0$ we have*

$$\mathrm{Leb}(B(\partial I,\eta)\cap I) \leq \mathrm{Leb}(B(\partial I,\eta)\cap I^c).$$

Proof. Let Leb_{m-1} denote $m-1$ dimensional Lebesgue measure. By Steiner's formula [55, Theorem 6.6] there exists a polynomial p_I with positive coefficients and of degree m such that $\mathrm{Leb}(B(I,\eta)) = p_I(\eta)$. The constant coefficient of p_I is clearly $\mathrm{Leb}(I)$, while the coefficient of the linear term is $\mathrm{Leb}_{m-1}(\partial I)$ i.e. the surface area of I. Note that $\mathrm{Leb}(B(\partial I,\eta)\cap I^c) = p_I(\eta) - \mathrm{Leb}(I)$. We aim to prove that $\mathrm{Leb}(B(\partial I,\eta)\cap I) \leq \eta\,\mathrm{Leb}_{m-1}(\partial I)$. Since p_I has degree greater than or equal to 2 and positive coefficients, it would then follow that

$$\mathrm{Leb}(B(\partial I,\eta)\cap I) \leq \eta\,\mathrm{Leb}_{m-1}(\partial I) \leq p_I(\eta) - \mathrm{Leb}(I) \leq \mathrm{Leb}(B(\partial I,\eta)\cap I^c),$$

which would complete the proof.

Let $\mathcal{F}(I)$ denote the set of set of facets of I. Clearly

$$\mathrm{Leb}_{m-1}(\partial I) = \sum_{F\in\mathcal{F}(I)} \mathrm{Leb}_{m-1}(F).$$

Let $y \in B(\partial I,\eta)\cap I$ and denote by F the (possibly not unique) facet in $\mathcal{F}(I)$ that minimises the distance from y to ∂I. Let x be the point on F attaining said minimum. If $x-y$ is not normal to F then the ball $B(y,|x-y|)$ is not tangent to F and so there exists $z \in B(y,|x-y|)\cap I^c$. The line segment from y to z must intersect ∂I at some point that is strictly closer to y than x, which contradicts x minimising the distance from y to ∂I. Hence, $x-y$ must be normal to F and so $y \in F + [0,\eta]n_F$, where n_F is the inward facing unit normal vector to F. This implies that

$$B(\partial I,\eta)\cap I \subseteq \bigcap_{F\in\mathcal{F}(I)} F + [0,\eta]n_F$$

and so $\mathrm{Leb}(B(\partial I,\eta)\cap I) \leq \eta\sum_{F\in\mathcal{F}(I)} \mathrm{Leb}_{m-1}(F) = \eta\,\mathrm{Leb}_{m-1}(\partial I)$ as required. □

The proof of Proposition 4.3.10. We begin by bounding

$$\sup_{|f|_\beta=1} |\mathbb{E}_Q f|_{\beta,\eta_0-\mathrm{diam}(Q)}\,.$$

Let $b:\mathbb{R}\to\mathbb{R}$ be defined by

$$b(\eta) = \left(1+\frac{\mathrm{diam}(Q)}{\eta}\right)^\beta.$$

By taking the minimum of the bounds in Lemmas 4.3.12 and 4.3.15 we have

$$|\mathbb{E}_Q f|_{\beta,\eta_0-\operatorname{diam}(Q)} \leq \sup_{0<\eta\leq\eta_0} \min\left\{\max_{I\in Q} \frac{\operatorname{Leb}(B(\partial I,\eta))}{\operatorname{Leb}(I)}, 1\right\} b(\eta)\,|f|_\beta\,. \tag{4.27}$$

We will now bound $\max_{I\in Q} \frac{\operatorname{Leb}(B(\partial I,\eta))}{\operatorname{Leb}(I)}$. Lemma 4.3.16 implies that for any $I \in Q$ we have

$$\frac{\operatorname{Leb}(B(\partial I,\eta))}{\operatorname{Leb}(I)} \leq \frac{2\operatorname{Leb}(B(\partial I,\eta)\cap I^c)}{\operatorname{Leb}(I)}.$$

Noting that $B(\partial I,\eta)\cap I^c = B(I,\eta)\setminus I$ and $B(I,\eta) = I + B(0,\eta)$, we obtain

$$\frac{\operatorname{Leb}(B(\partial I,\eta))}{\operatorname{Leb}(I)} \leq 2\frac{\operatorname{Leb}(I+B(0,\eta)) - \operatorname{Leb}(I)}{\operatorname{Leb}(I)}. \tag{4.28}$$

Let B_I be a ball inscribed in I of maximal volume. Then, by scaling and possibly translating by some vector $v_I \in \mathbb{R}^m$, we find that $B(0,1) \subseteq \frac{2}{\operatorname{diam}(B_I)}I + v_I$. Consequently

$$\operatorname{Leb}\left(I + B(0,\eta)\right) \leq \operatorname{Leb}\left(I + \frac{2\eta}{\operatorname{diam}(B_I)}I\right) = \left(1+\frac{2\eta}{\operatorname{diam}(B_I)}\right)^m \operatorname{Leb}(I). \tag{4.29}$$

Applying (4.29) to (4.28), and recalling that $1/\operatorname{diam}(B_I) \leq \kappa/\operatorname{diam}(Q)$ (since $Q \in \mathcal{P}(\kappa)$), we find that

$$\frac{\operatorname{Leb}(B(\partial I,\eta))}{\operatorname{Leb}(I)} \leq 2\left(1+\frac{2\eta}{\operatorname{diam}(B_I)}\right)^m - 2 \leq 2\left(1+\frac{2\kappa\eta}{\operatorname{diam}(Q)}\right)^m - 2. \tag{4.30}$$

By applying (4.30) to (4.27) we obtain

$$|\mathbb{E}_Q f|_{\beta,\eta_0-\operatorname{diam}(Q)} \leq \sup_{0<\eta\leq\eta_0} \min\left\{\left(2\left(1+\frac{2\kappa\eta}{\operatorname{diam}(Q)}\right)^m - 2\right) b(\eta), b(\eta)\right\} |f|_\beta\,. \tag{4.31}$$

It is clear that b is monotonically decreasing. Note that

$$2\left(\left(1+\frac{2\kappa\eta}{\operatorname{diam}(Q)}\right)^m - 1\right)b(\eta) = 2\eta^{-\beta}\left(\left(1+\frac{2\kappa\eta}{\operatorname{diam}(Q)}\right)^m - 1\right)(\eta+\operatorname{diam}(Q))^\beta. \tag{4.32}$$

The map $\eta \mapsto (\eta+\operatorname{diam}(Q))^\beta$ is clearly monotonically increasing on $(0,\eta_0]$. As $m \geq 2$ and $\beta \in (0,1]$, the map

$$\eta \mapsto \eta^{-\beta}\left(\left(1+\frac{2\kappa\eta}{\operatorname{diam}(Q)}\right)^m - 1\right)$$

is monotonically increasing on $(0, \eta_0]$ too. Thus the left side of (4.32) is monotonically increasing. Since both b and the left side of (4.32) are continuous on $(0, \eta_0]$, b is monotonically decreasing and the left side of (4.32) is monotonically increasing, it follows that if $\eta' \in (0, \infty)$ solves

$$2\left(1 + \frac{2\kappa\eta'}{\operatorname{diam}(Q)}\right)^m - 2 = 1, \tag{4.33}$$

then

$$\sup_{0<\eta\leq\eta_0} \min\left\{\left(2\left(1 + \frac{2\kappa\eta}{\operatorname{diam}(Q)}\right)^m - 2\right) b(\eta), b(\eta)\right\} \leq b(\eta').$$

Solving (4.33) yields

$$\frac{\operatorname{diam}(Q)}{\eta'} = \frac{2\kappa}{\sqrt[m]{\frac{3}{2}} - 1}.$$

By substituting this into (4.31) we obtain the bound

$$|\mathbb{E}_Q f|_{\beta, \eta_0 - \operatorname{diam}(Q)} \leq \left(1 + \frac{2\kappa}{\sqrt[m]{\frac{3}{2}} - 1}\right)^\beta |f|_\beta \, .$$

Applying Lemma 4.3.9 yields the required bound. □

With Proposition 4.3.10 in hand we may now prove Lemmas 4.3.6 and 4.3.7.

Proof of Lemma 4.3.6. As $\lim_{\epsilon\to 0} \operatorname{diam}(Q_\epsilon) = 0$ there exists $\epsilon_2 > 0$ such that for every $\epsilon \in (0, \epsilon_2]$ we have $\operatorname{diam}(Q_\epsilon) < \eta_0$ and

$$1 + \operatorname{diam}(Q_\epsilon)/(\eta_0 - \operatorname{diam}(Q_\epsilon)) < \sqrt{m/(m-1)}.$$

By [17, Section 8.5, page 236], this implies

$$S(1, 1 + \operatorname{diam}(Q_\epsilon)/(\eta_0 - \operatorname{diam}(Q_\epsilon))) = S(\eta_0 - \operatorname{diam}(Q_\epsilon), \eta_0) \leq 2m.$$

The desired conclusion follows by Proposition 4.3.10. □

Proof of Lemma 4.3.7. If $\operatorname{diam}(Q) < \eta_0$ then $|\mathbb{E}_Q|_\beta < \infty$ by Proposition 4.3.10. Alternatively, if $\operatorname{diam}(Q) \geq \eta_0$, then repeatedly applying Lemma 4.3.9 yields

$$\begin{aligned}
|\mathbb{E}_Q|_\beta &= \sup_{|f|_\beta \leq 1} |\mathbb{E}_Q f|_\beta \\
&\leq \sup\{|\mathbb{E}_Q f|_{\beta, 2\operatorname{diam}(Q)} : |f|_{\beta, 2\operatorname{diam}(Q)} \leq S(\eta_0, 2\operatorname{diam}(Q))\} \\
&\leq S(\eta_0, 2\operatorname{diam}(Q))\, |\mathbb{E}_Q|_{\beta, 2\operatorname{diam}(Q)}\,,
\end{aligned}$$

which is finite by Proposition 4.3.10 applied to the seminorm $|\cdot|_{\beta, 2\operatorname{diam}(Q)}$ (i.e. when $\eta_0 = 2\operatorname{diam}(Q)$). In either case we have $|\mathbb{E}_Q|_\beta < \infty$ and so, as $\|\mathbb{E}_Q\|_{L^1} = 1$, we have $\|\mathbb{E}_Q\|_\beta < \infty$ too. As Q partitions X, for every $f \in \mathbb{V}_\beta(X)$ the support of $\mathbb{E}_Q f$ is a subset of X. Hence $\mathbb{E}_Q f \in \mathbb{V}_\beta(X)$ for every $f \in \mathbb{V}_\beta(X)$ and so $\mathbb{E}_Q \in L(\mathbb{V}_\beta(X))$. $\square$

Chapter 5

Application to Anosov maps

It is classical that topologically transitive Anosov diffeomorphisms have a unique SRB measure and satisfy a CLT and LDP for sufficiently smooth observables; such results were first established by using Markov partitions to reduce to the case of subshifts of finite type (see e.g. [85, 88] for proofs of the CLT and LDP in this way). In this chapter we apply the theory of Chapters 1 and 2 to deduce the stability of these statistical properties: the SRB measure, the variance of the CLT and rate function of the LDP. In particular, in Section 5.1 we use the functional analytic setup of Gouëzel and Liverani [54] to confirm that $\{g \circ T^k\}_{k\in\mathbb{N}}$ satisfies (NG) for a $\mathcal{C}^{r+1}$ Anosov map T and appropriate observables g, where $r > 1$. Theorem 2.2.1 then yields stability of the variance and rate function to perturbations of type (KL). For the remainder of the chapter we develop some classes of perturbations satisfying (KL) for Anosov maps on tori. In Section 5.2 we propose a numerical method for approximating the statistical data of an Anosov map that uses a combination of mollification and Fourier approximation. In Section 5.3 we then consider perturbations arising from non-local stochastic perturbations, which include numerical approximation via a Fejér kernel method. Having verified (KL) for these perturbations, in Section 5.4 we then use these methods to compute estimates of various statistical properties for a perturbation of Arnold's cat map.

5.1 Anisotropic Banach spaces adapted to Anosov maps

We begin by reviewing the functional analytic setup of [54], before moving on to discussing how (NG) may be verified for Anosov maps. For the sake of exposition we defer all proofs to the end of the section. Let $m > 1$ and X be a m-dimensional, $\mathcal{C}^\infty$, compact, connected Riemannian manifold and $T \in \mathcal{C}^{r+1}(X, X)$ be an Anosov map for some $r > 1$. In [54] the metric on X is replaced by an adapted metric $x \mapsto \langle \cdot, \cdot \rangle_x$ such that T exhibits strict contraction and expansion in the stable and unstable directions, respectively. More specifically, if $x \mapsto \|\cdot\|_x$ denotes the norm induced by the adapted metric, and $x \mapsto E_s(x)$ and $x \mapsto E_u(x)$ denote the stable and unstable bundles, respectively, associated to T, then there exists $\nu_u^{-1}, \nu_s \in (0,1)$ such that

$$\sup_{x \in X} \left\| D_x T|_{E^s(x)} \right\|_x \le \nu_s \quad \text{and} \quad \sup_{x \in X} \left\| D_x T^{-1}|_{E^u(x)} \right\|_x \le \nu_u^{-1}.$$

Following [21, Proposition 5.2.2], we review the construction of such a metric in Section 5.3.2 as the specific choice of metric will later simplify some arguments. Let Ω denote both the Riemannian measure on X induced by the adapted metric and the linear functional $f \mapsto \int f \, d\Omega$. The transfer operator $\mathcal{L} : \mathcal{C}^r(X, \mathbb{R}) \to \mathcal{C}^r(X, \mathbb{R})$ associated with T is defined by

$$\int (\mathcal{L}h) \cdot u \, d\Omega = \int h \cdot (u \circ T) \, d\Omega, \tag{5.1}$$

where $u, h \in \mathcal{C}^r(X, \mathbb{R})$. For $\mathcal{L}$ to have 'good' spectral properties it is necessary to consider it as an operator on an appropriately chosen anisotropic Banach space. We now describe the construction of such a space from [54]. Core to this construction is a set Σ of 'admissible leaves': small submanifolds of bounded curvature that are uniformly close to the stable directions of T; see [54, Section 3] for the full definition. For each $W \in \Sigma$ we denote the collection of $\mathcal{C}^r$ vector fields that are defined on a neighbourhood of W by $\mathcal{V}^r(W)$, and by $\mathcal{C}_0^q(W, \mathbb{R})$ the set of functions in $\mathcal{C}^q(W, \mathbb{R})$ that vanish on a neighbourhood of ∂W. For $h \in \mathcal{C}^r(X, \mathbb{R})$, $q > 0$, $p \in \mathbb{N}$ with $p \le r$ let[1]

$$\|h\|_{p,q}^- = \sup_{W \in \Sigma} \sup_{\substack{v_1, \ldots v_p \in \mathcal{V}^r(W) \\ |v_i|_{\mathcal{C}^r} \le 1}} \sup_{\substack{\varphi \in \mathcal{C}_0^q(W, \mathbb{R}) \\ |\varphi|_{\mathcal{C}^q} \le 1}} \int_W (v_1 \ldots v_p h) \cdot \varphi \, d\Omega.$$

Then

$$\|h\|_{p,q} = \sup_{0 \le k \le p} \|h\|_{k,q+k}^- = \sup_{p' \le p, q' \ge q+p'} \|h\|_{p',q'}^- \tag{5.2}$$

is a norm on $\mathcal{C}^r(X, \mathbb{R})$. Denote by $B^{p,q}$ the completion of $\mathcal{C}^r(X, \mathbb{R})$ under this norm. As the naive Nagaev-Guivarc'h method requires a complex Banach space, we consider the complexification $B_{\mathbb{C}}^{p,q}$ of the spaces $B^{p,q}$. When endowed with the norm[2]

$$\|h_r + ih_i\|_{p,q} = \max\{\|h_r\|_{p,q}, \|h_i\|_{p,q}\}, \tag{5.3}$$

[1] In an abuse of notation we also let Ω denote the induced Riemannian measure on the submanifold W.

[2] We abuse notation and denote the norm on $B_{\mathbb{C}}^{p,q}$ by $\|\cdot\|_{p,q}$.

$B^{p,q}_{\mathbb{C}}$ is a complex Banach space. It is on this space that the operator $\mathcal{L}$ is quasi-compact.

Theorem 5.1.1 ([54, Theorem 2.3])**.** *If $p \in \mathbb{Z}^+$ and $q > 0$ satisfy $q + p < r$ then the operator $\mathcal{L} : B^{p,q}_{\mathbb{C}} \to B^{p,q}_{\mathbb{C}}$ is bounded with spectral radius one. In addition, $\mathcal{L}$ is quasi-compact with $\rho_{\mathrm{ess}}(\mathcal{L}) \subseteq \{\omega \in \mathbb{C} : |\omega| \le \max\{\nu_u^{-p}, \nu_s^q\}\}$. Moreover, the eigenfunctions corresponding to eigenvalues of modulus 1 are distributions of order 0, i.e. measures. If the map is topologically transitive, then 1 is a simple eigenvalue and no other eigenvalues of modulus one are present.*

Remark 5.1.2. Some clarification is required about the sense in which one may consider an element of $B^{p,q}_{\mathbb{C}}$ to be a measure. Let $\mathcal{D}'_q(X)$ denote the distributions of order q on X. Each $h \in \mathcal{C}^r(X, \mathbb{C})$ induces an element of $\mathcal{D}'_q(X)$ defined by $h_{\mathcal{D}'_q(X)} : \varphi \in \mathcal{C}^q(X, \mathbb{C}) \mapsto \int_X h \cdot \varphi \, \mathrm{d}\Omega$. As is described in [54, Section 4] the mapping $h \mapsto h_{\mathcal{D}'_q(X)}$ is continuous with respect to $\|\cdot\|_{p,q}$, and so may be extended to all of $B^{p,q}_{\mathbb{C}}$ by taking limits. By [54, Proposition 4.1] this mapping is an injection, and so we will say an element of $B^{p,q}_{\mathbb{C}}$ is a probability measure exactly when it induces a probability measure in $\mathcal{D}'_q(X)$.

Remark 5.1.3. Recall that a T-invariant probability measure is said to be an SRB measure if it has absolutely continuous conditional measures on unstable manifolds. It is well-known that topologically transitive $\mathcal{C}^2$ Anosov diffeomorphism possess a unique SRB measure, which we shall denote by μ. In view of the preceding theorem and remark we deduce that the unique fixed point of $\mathcal{L}$ in $B^{p,q}_{\mathbb{C}} \cap \{h : \Omega(h) = 1\}$ maps to μ under the injection from $B^{p,q}_{\mathbb{C}}$ to $\mathcal{D}_q(X)'$. We will also denote this fixed point by μ.

The first main technical result of this section is Proposition 5.1.5, which verifies (NG) in the present setting. As $1 \in B^{p,q}_{\mathbb{C}}$, for any $g \in \mathcal{C}^r(X, \mathbb{R})$ we may define e^{zg} by the power series $\sum_{k=0}^{\infty} z^k g^k / k!$. We define $M_g : \mathbb{C} \to L(B^{p,q}_{\mathbb{C}})$ by setting $M_g(z)(h) = e^{zg} h$ for $h \in \mathcal{C}^r(X, \mathbb{C})$ and then passing to $B^{p,q}_{\mathbb{C}}$ by density.

Proposition 5.1.4. *Let $p \in \mathbb{Z}^+$, $q > 0$ satisfy $p + q < r$. If $g \in \mathcal{C}^r(X, \mathbb{R})$ and $M_g : \mathbb{C} \to L(B^{p,q}_{\mathbb{C}})$ is defined by $M_g(z)(f) = e^{zg} f$, then M_g is a compactly $\|\cdot\|_{p-1,q+1}$-bounded twist.*

Recall that $g \in \mathcal{C}^r(X, \mathbb{R})$ is called a $L^2(\mu)$-coboundary with respect to T if there exists a $\phi \in L^2(\mu)$ so that $g = \phi - \phi \circ T$

Proposition 5.1.5. *Suppose that $T \in \mathcal{C}^{r+1}(X, X)$, $r > 1$, is a topologically transitive Anosov map with unique SRB measure μ and that $g \in \mathcal{C}^r(X, \mathbb{R})$ satisfies $\int g \, \mathrm{d}\mu = 0$ and is not an $L^2(\mu)$-coboundary with respect to T. Let $p \in \mathbb{Z}^+$ and $q > 0$*

satisfy $q+p<r$ and for $z \in \mathbb{C}$ let $\mathcal{L}(z)$ be defined by $\mathcal{L}(z)f = (\mathcal{L} \circ M_g(z))(f) = \mathcal{L}(e^{zg}f)$. Then $z \mapsto \mathcal{L}(z)$ is analytic (in the operator norm on $L(B^{p,q})$) and if $\zeta \in B^{p,q}$ is a probability measure then $\{g \circ T^k\}_{k\in\mathbb{N}}$, when considered on the probability space (X, ζ), satisfies (NG) with coding $(z \mapsto \mathcal{L}(z), \Omega, \zeta)$. In particular, if $S_n := \sum_{k=0}^{n-1} g \circ T^k$ then for every $z \in \mathbb{C}$ we have

$$\mathbb{E}_\zeta\left(e^{zS_n}\right) = \int \mathcal{L}(z)^n \zeta \, \mathrm{d}\Omega.$$

With Proposition 5.1.5 in hand, by Theorems 2.1.1 and 2.1.2 we immediately obtain a CLT and LDP (on appropriate probability spaces) for $\{g \circ T^k\}_{k\in\mathbb{N}}$ whenever $g \in \mathcal{C}^r(X, \mathbb{R})$ satisfies $\int g \, \mathrm{d}\mu = 0$ and is not an $L^2(\mu)$-coboundary. Since $z \mapsto \mathcal{L}(z)$ is induced by a compactly $\|\cdot\|_{p-1,q+1}$-bounded twist, if we can verify (KL) for some class of perturbations then the stability of the variance and rate functions will follow from Theorem 2.2.1. By [54, Lemma 2.1] the unit ball in $B^{p,q}_{\mathbb{C}}$ is relatively compact in $\|\cdot\|_{p-1,q+1}$. The following result from [54] shows that $\mathcal{L}$ satisfies (KL2) and (KL3) in the present setting.

Lemma 5.1.6 ([54, Lemma 2.2])**.** *For each $p \in \mathbb{N}$ and $q \geq 0$ satisfying $p+q<r$, there exist $A_{p,q}, B_{p,q} > 0$ such that, for each $n \in \mathbb{N}$,*

$$\|\mathcal{L}^n h\|_{0,q} \leq \|h\|_{0,q} \quad \forall\, h \in B^{0,q}_{\mathbb{C}},$$

$$\|\mathcal{L}^n h\|_{p,q} \leq A_{p,q} \max\{\nu_s^q, \nu_u^{-p}\}^n \|h\|_{p,q} + \|h\|_{p-1,q+1} \quad \forall\, h \in B^{p,q}_{\mathbb{C}}.$$

Hence it is possible, in principle, to obtain the stability of the variance and rate function for perturbations to Anosov maps via Theorem 2.2.1. While we will devolp more perturbations satisfying (KL) in later sections, for the moment we give an application on the stability of the rate function for deterministic perturbations to topologically transitive Anosov maps, which are known to satisfy (KL) as per [54, Section 17]. This result should be compared to [54, Theorem 2.8, Remark 2.11]. For $y \in \mathbb{R}$ we denote by τ_y the map $x \mapsto x+y$.

Theorem 5.1.7 (Stability of the rate function under deterministic perturbations)**.** *Let $T \in \mathcal{C}^1([0,1], \mathcal{C}^{r+1}(X,X))$ be such that $T(0)$ is a topologically transitive Anosov diffeomorphism. Let $p \in \mathbb{Z}^+$ and $q > 0$ satisfy $p+q<r$. Fix a probability measure $\zeta \in B^{p,q}$ and suppose that $g \in \mathcal{C}^r(X,\mathbb{R})$ satisfies $\int g \, \mathrm{d}\mu = 0$ and is not an $L^2(\mu)$-coboundary. There exists $\epsilon > 0$ and, for each $t \in [0,\epsilon]$, a number A_t and map $r_t : J - A_t \to \mathbb{R}$, where J is an open real neighbourhood of 0, so that*

$\{g \circ T(t)^k - A_t\}_{k\in\mathbb{N}}$ *satisfies a LDP on* (X, ζ) *with rate function* r_t*,* $A_t \to A_0 = 0$ *and* $r_t \circ \tau_{-A_t} \to r_0$ *compactly on* J*.*

We now prove Propositions 5.1.4 and 5.1.5, and Theorem 5.1.7.

The proof of Proposition 5.1.4. It is clear that $M_g(0)$ is the identity. For each $k \in \mathbb{N}$ let $P_k : \mathcal{C}^r(X, \mathbb{C}) \to \mathcal{C}^r(X, \mathbb{C})$ be defined by $P_k f = g^k f$. Multiplication by g is continuous on $B_{\mathbb{C}}^{p,q}$ by [54, Lemma 3.2] and so $P_k \in L(B_{\mathbb{C}}^{p,q})$ for each $k \in \mathbb{N}$. Moreover, as $\|P_k\|_{p,q} \le \|P_1\|_{p,q}^k$, for each $z \in \mathbb{C}$ the series $\sum_{k=0}^{\infty} z^k P_k$ is absolutely convergent in the $\|\cdot\|_{p,q}$ operator norm, with limit $M_g(z)$. Hence $z \mapsto M_g(z)$ is a well-defined analytic map taking values in $L(B_{\mathbb{C}}^{p,q})$. The same argument holds when $B_{\mathbb{C}}^{p,q}$ is replaced with $B_{\mathbb{C}}^{p-1,q+1}$, and so $z \mapsto M_g(z)$ is analytic on $L(B_{\mathbb{C}}^{p-1,q+1})$ too. In particular, it is compactly $\|\cdot\|_{p-1,q+1}$-bounded. □

From the beginning of [54, Section 4], for each $h \in B^{p,q}$ and $\phi \in \mathcal{C}^q(X, \mathbb{R})$ we have

$$\left| \int h\phi \, \mathrm{d}\Omega \right| \le C \|h\|_{p,q} \|\phi\|_{\mathcal{C}^q},$$

for some $C > 0$ independent of h and ϕ. It is straightforward to show that the same inequality holds for $h \in B_{\mathbb{C}}^{p,q}$ and $\phi \in \mathcal{C}^q(X, \mathbb{C})$ (although with a different C, which is inconsequential). Hence, the functional Ω is in $(B_{\mathbb{C}}^{p,q})^*$. Let $\zeta \in B^{p,q}$ be a probability measure (i.e. the image of ζ under the injection from $B^{p,q}$ to $\mathcal{D}_q'(X)$ is a probability measure) and $h \in \mathcal{C}^r(X, \mathbb{C})$. Since $\mathcal{C}^r(X, \mathbb{R})$ is dense in $B^{p,q}$, there exists $\{\zeta_i\}_{i\in\mathbb{Z}^+} \subseteq \mathcal{C}^r(X, \mathbb{R})$ such that $\zeta_i \to \zeta$ in $B^{p,q}$. As $B^{p,q}$ is continuously injected into $\mathcal{D}_q'(X)$ it follows that $\zeta_i \to \zeta_i$ in $\mathcal{D}_q'(X)$. Note that $\varphi \in \mathcal{C}^r(X, \mathbb{R})$ naturally induces a measure, which we will also denote by φ, so that $\int f \, \mathrm{d}\varphi = \int f\varphi \, \mathrm{d}\Omega$ for each Borel measurable function $f : X \to \mathbb{C}$. Hence,

$$\Omega(h\zeta) = \lim_{n\to\infty} \Omega(h\zeta_n) = \lim_{n\to\infty} \int h\zeta_n \, \mathrm{d}\Omega = \lim_{n\to\infty} \int h \, \mathrm{d}\zeta_n = \int h \, \mathrm{d}\zeta. \tag{5.4}$$

Proposition 5.1.8. *Let* $g \in \mathcal{C}^r(X, \mathbb{R})$*,* $S_n(g) = \sum_{k=0}^{n-1} g \circ T^k$ *and* $\zeta \in B^{p,q}$ *be a probability measure. Then for each* $n \in \mathbb{N}$ *and* $z \in \mathbb{C}$ *we have*

$$\int e^{zS_n(g)} \, \mathrm{d}\zeta = \Omega(\mathcal{L}(z)^n \zeta).$$

Proof. For $h \in \mathcal{C}^r(X, \mathbb{C})$ we have $\mathcal{L}h = (h \left|\det T\right|^{-1}) \circ T^{-1}$. It is straightforward to verify that for every $f_1, f_2 \in \mathcal{C}^r(X, \mathbb{C})$ we have $\mathcal{L}(f_1 \circ T \cdot f_2) = f_1 \mathcal{L}(f_2)$. By [54, Lemma 3.2], multiplication by f_1 is continuous on $B_{\mathbb{C}}^{p,q}$. Hence, by passing to the completion we may take $f_2 \in B_{\mathbb{C}}^{p,q}$. Setting $f_1 = e^{zg}$ and $f_2 = \zeta$, and then inductively using this identity, it follows that for each $n \in \mathbb{N}$ and $z \in \mathbb{C}$ we have $\mathcal{L}(z)^n\zeta = \mathcal{L}^n\left(e^{zS_n(g)}\zeta\right)$. Upon integrating, and using (5.4) and that $\mathcal{L}$ preserves Ω-integrals, we have

$$\Omega(\mathcal{L}(z)^n\zeta) = \Omega\left(\mathcal{L}^n\left(e^{zS_n(g)}\zeta\right)\right) = \Omega\left(e^{zS_n(g)}\zeta\right) = \int e^{zS_n(g)}\,\mathrm{d}\zeta.$$

□

The proof of Proposition 5.1.5. The expectation of $g \circ T^k$ with respect to ζ is

$$\int g \circ T^k\,\mathrm{d}\zeta = \Omega\left((g \circ T^k)\cdot\zeta\right) = \Omega\left((\mathcal{L}^k\zeta)\cdot g\right).$$

By our assumptions, $\mathcal{L} = \mathcal{L}(0)$ is a simple quasi-compact operator on $B_{\mathbb{C}}^{p,q}$ with $\rho(\mathcal{L}) = 1$. Let $\mathcal{L}^k = \Pi + N^k$ be the quasi-compact decomposition of $\mathcal{L}^k$. As μ is T-invariant and $\Omega\mathcal{L} = \Omega$, it follows that $\Pi(f) = \Omega(f)\mu$. Hence, as $N^k \to 0$ and ζ is a probability measure, we have $\mathcal{L}^k\zeta = \Omega(\zeta)\mu + N^k \to \mu$ in $B_{\mathbb{C}}^{p,q}$. Using (5.4) and the fact that $\Omega \in (B_{\mathbb{C}}^{p,q})^*$ we have

$$\lim_{k\to\infty}\int g \circ T^k\,\mathrm{d}\zeta = \lim_{k\to\infty}\int g\cdot(\mathcal{L}^k\zeta)\,\mathrm{d}\Omega = \Omega(\mu g) = \int g\,\mathrm{d}\mu = 0.$$

It follows that $\lim_{n\to\infty}\mathbb{E}_\zeta(S_n)/n = 0$. By Proposition 5.1.4 the map $z \mapsto M_g(z)$ is analytic and so $z \mapsto \mathcal{L}(z)$ must be analytic too (both with respect to the operator norm on $L(B_{\mathbb{C}}^{p,q})$). That $z \mapsto \ln\rho(\mathcal{L}(z))$ is strictly convex follows (as in Remark 4.1.4) from the fact that $\rho(\mathcal{L}(t))$ equals the leading eigenvalue $\lambda(t)$ of $\mathcal{L}(t)$ for t in a small real neighbourhood of 0 and as g not being a $L^1(\mu)$-coboundary with respect to T implying that $\lambda''(0) > 0$ (see e.g. [92, Lemma 6]). Thus $\{g \circ T^k\}_{k\in\mathbb{N}}$ satisfies (NG) on the probability space (X, ζ) with coding $(z \mapsto \mathcal{L}(z), \Omega, \zeta)$. □

The proof of Theorem 5.1.7. For $t \in [0,1]$ let $\mathcal{L}_t$ denote the Perron-Frobenius operator induced by $T(t)$. By [54, Theorem 2.3], topological transitivity of $T(0)$ implies that $\mathcal{L}_0$ is a simple quasi-compact operator on $B_{\mathbb{C}}^{p,q}$ with $\rho(\mathcal{L}_0) = 1$. By [54, Section 7], there exists some $t' > 0$ for which $\{\mathcal{L}_t\}_{t\in[0,t']}$ satisfies (KL) on $B_{\mathbb{C}}^{p,q}$ with $|\cdot| = \|\cdot\|_{p-1,q+1}$. Applying Theorem 2.2.1 to $\{g \circ T^k\}_{k\in\mathbb{N}}$, which satisfies (NG) by Proposition 5.1.5, we obtain $\theta > 0$ and $\epsilon \in (0, t')$ so that whenever $t \in [0,\epsilon]$ and $z \in D_\theta$ the operator $\mathcal{L}_t M_g(z)$ is quasi-compact and simple with leading eigenvalue $\lambda_t(z)$. In particular, 1 is a simple eigenvalue of $\mathcal{L}_t$ for $t \in [0,\epsilon]$ and so $T(t)$ has a unique SRB measure μ_t in $B^{p,q}_{\mathbb{C}}$. By [38, Proposition 2.4, Remark 2.5], for each $t \in (0, \epsilon)$ there exists an eigenvector v_t of $\mathcal{L}_t$ associated to the eigenvalue 1 such that $v_t \to \mu$ in $B^{p-1,q+1}_{\mathbb{C}}$ as $t \to 0$. By simplicity of the eigenvalue 1, for sufficiently small t we have $\mu_t = \frac{v_t}{\int v_t \, \mathrm{d}\Omega}$ and so the continuity of $f \mapsto \int f \, \mathrm{d}\Omega$ on $B^{p-1,q+1}_{\mathbb{C}}$ implies that $\mu_t \to \mu$ in $B^{p-1,q+1}_{\mathbb{C}}$ too.

Fix $t \in [0, \epsilon]$. Let $A_t = \int g \, \mathrm{d}\mu_t$ and $g_t = g - A_t$. As multiplication by g is continuous on $B^{p-1,q+1}_{\mathbb{C}}$ [54, Lemma 3.2] and $\Omega \in (B^{p-1,q+1}_{\mathbb{C}})^*$, we have $\lim_{t\to 0} A_t = \lim_{t \to 0} \Omega(g\mu_t) = \Omega(g\mu) = \int g \, \mathrm{d}\mu = 0$. Note that $e^{zA_t}\mathcal{L}_t M_{g_t}(z) = \mathcal{L}_t M_g(z)$ for every $z \in \mathbb{C}$, and so $\mathcal{L}_t M_{g_t}(z)$ is quasi-compact exactly when $\mathcal{L}_t M_g(z)$ is. In particular, $\mathcal{L}_t M_{g_t}(z)$ is a simple quasi-compact operator for every $z \in D_\theta$ with leading eigenvalue $\kappa_t(z) = e^{-zA_t}\lambda_t(z)$. From the material in this section and the last, it is routine to verify that $\{g_t \circ T(t)^k\}_{k\in\mathbb{N}}$ satisfies (NG) on (X, ζ) with coding $(z \mapsto \mathcal{L}_t M_{g_t}(z), \Omega, \zeta)$. Hence, by Theorem 2.1.2 the sequence $\{g_t \circ T(t)^k\}_{k\in\mathbb{N}}$ satisfies a LDP on (X, ζ) with rate function $r_t : J_t \to \mathbb{R}$ defined by

$$r_t(s) = \sup_{z\in(-\theta,\theta)} (sz - \ln|\kappa_t(z)|).$$

Recall from (2.3) that r_t has domain

$$J_t = \left(\frac{\kappa_t'(-\theta)}{\kappa_t(-\theta)}, \frac{\kappa_t'(\theta)}{\kappa_t(\theta)} \right).$$

As $\kappa_t(z) = e^{-zA_t}\lambda_t(z)$, we therefore have $J_t = J_0 - A_t$ and

$$r_t(s) = \sup_{z\in(-\theta,\theta)} ((s + A_t)z - \ln|\lambda_t(z)|).$$

By Theorem 2.2.1, for each compact $U \subseteq J_0$ there is a closed interval $V \subseteq (-\theta, \theta)$ so that the map $s \mapsto \sup_{z\in V}(sz - \ln|\lambda_t(z)|)$ converges uniformly to r_0 on U. Since the map $z \mapsto \ln|\lambda_t(z)|$ is convex on $(-\theta, \theta)$, by the arguments from the proof of Theorem 2.2.1 we have

$$r_t(s - A_t) = \sup_{z\in V}(sz - \ln|\lambda_t(z)|).$$

Hence, $r_t \circ \tau_{-A_t} \to r_0$ compactly on J_0. $\square$

5.2 Approximating the statistical data of Anosov maps

In this section we introduce a scheme for approximating the spectrum of the Perron-Frobenius operator associated to an Anosov map on the m-dimensional torus $\mathbb{T}^m$, which we identify with $\mathbb{R}^m/\mathbb{Z}^m$. The scheme proceeds by convolving the Perron-Frobenius operator with a compactly supported mollifier, and then approximating the smoothened operator using Fourier series. A similar idea is developed in [14], where Ulam's method is considered instead of Fourier series and convergence of the SRB measure and variance are obtained. We note that [14] did not include any computations (as we do, in Section 5.4) nor did they consider the stability of the rate function.

Throughout this section we adopt the setting, assumptions and notation of Section 5.1, and fix $p \in \mathbb{Z}^+$ and $q > 0$ satisfying $p + q < r$. Let Leb denote the normalised Haar measure on $\mathbb{T}^m$. For some $\epsilon_1 > 0$, suppose that the family of stochastic kernels $\{q_\epsilon\}_{\epsilon \in (0,\epsilon_1)} \subseteq \mathcal{C}^\infty(\mathbb{T}^m, \mathbb{R})$ satisfies the following conditions:

(S1) $q_\epsilon \geq 0$ and $\int q_\epsilon \, \mathrm{d}\,\mathrm{Leb} = 1$;

(S2) The support of q_ϵ is contained in $B(0, \epsilon)$.

For such a family we define operators $Q_\epsilon : \mathcal{C}^r(\mathbb{T}^m, \mathbb{C}) \to \mathcal{C}^r(\mathbb{T}^m, \mathbb{C})$ by $Q_\epsilon f = f * q_\epsilon$. Recall that convolution is defined with respect to the Haar measure Leb on $\mathbb{T}^m$, which may differ from the measure Ω that is induced by the adapted metric. It is evident, however, that the Radon-Nikodym derivatives $\frac{\mathrm{d}\,\mathrm{Leb}}{\mathrm{d}\Omega}$ and $\frac{\mathrm{d}\Omega}{\mathrm{d}\,\mathrm{Leb}}$ both exist, and are elements of $\mathcal{C}^\infty(\mathbb{T}^m, \mathbb{R})$. As a consequence we obtain the following characterisation of Q_ϵ:

Lemma 5.2.1. *Q_ϵ extends to a bounded operator $Q_\epsilon : B^{p,q}_{\mathbb{C}} \to \mathcal{C}^\infty(\mathbb{T}^m, \mathbb{C})$. Consequently, Q_ϵ is compact as an element of $L(B^{p,q}_{\mathbb{C}}, \mathcal{C}^k(\mathbb{T}^m, \mathbb{C}))$ for every $k \in \mathbb{Z}^+$. Moreover, Q_ϵ is compact as an element of $L(B^{p,q}_{\mathbb{C}})$ and, for each $k_1, k_2 \in \mathbb{Z}^+$, as an element of $L(\mathcal{C}^{k_1}(\mathbb{T}^m, \mathbb{C}), \mathcal{C}^{k_2}(\mathbb{T}^m, \mathbb{C}))$.*

Let $\mathcal{L}_0 = \mathcal{L}$ and, for each $\epsilon \in (0, \epsilon_1)$, let $\mathcal{L}_\epsilon = Q_\epsilon \mathcal{L}_0$, which is in $L(B^{p,q}_{\mathbb{C}})$ by virtue of the previous lemma.

Lemma 5.2.2. *There exists $\epsilon_2 \in (0, \epsilon_1)$ so that $\{\mathcal{L}_\epsilon\}_{\epsilon \in [0,\epsilon_2)}$ satisfies (KL) on $B^{p,q}_{\mathbb{C}}$ with $|\cdot| = \|\cdot\|_{p-1,q+1}$.*

By Lemma 5.2.1, for each $\epsilon \in (0, \epsilon_1)$ the operator $\mathcal{L}_\epsilon$ is compact and, for every $k \in \mathbb{Z}^+$, maps the unit ball of $B^{p,q}_{\mathbb{C}}$ into a bounded subset of $\mathcal{C}^k(\mathbb{T}^m, \mathbb{C})$. For this reason $\mathcal{L}_\epsilon$ may be approximated with Fourier series for each $\epsilon > 0$. For $\ell =$

$(\ell_1, \ldots, \ell_d) \in \mathbb{Z}^m$ we set $\|k\|_\infty = \max_i |\ell_i|$ and $\|\ell\|_1 = \sum_i |\ell_i|$. For each $n \in \mathbb{Z}^+$ define $\Pi_n : \mathcal{C}(\mathbb{T}^m, \mathbb{C}) \to \mathcal{C}(\mathbb{T}^m, \mathbb{C})$ by

$$(\Pi_n f)(x) = \sum_{\substack{\ell \in \mathbb{Z}^m \\ \|\ell\|_\infty \leq n}} \hat{f}(\ell) e^{2\pi i \langle x, \ell \rangle},$$

where $\hat{f}$ denotes the Fourier transform[3] of f. For every $\epsilon \in (0, \epsilon_1)$ and $n \in \mathbb{Z}^+$ let $\mathcal{L}_{\epsilon,n} = \Pi_n \mathcal{L}_\epsilon$. To simplify our notation, we set $\mathcal{L}_{\epsilon,\infty} = \mathcal{L}_\epsilon$. Our main technical result for this section is the following.

Proposition 5.2.3. *There exists $\epsilon_3 \in (0, \epsilon_2)$ and a map $N : [0, \epsilon_3) \to \mathbb{N} \cup \{\infty\}$ with $N^{-1}(\infty) = \{0\}$, so that for any map $n : [0, \epsilon_3) \to \mathbb{N} \cup \{\infty\}$ with $n \geq N$ the family of operators $\{\mathcal{L}_{\epsilon,n(\epsilon)}\}_{\epsilon \in [0,\epsilon_3)}$ satisfies (KL) on $B^{p,q}_{\mathbb{C}}$ with $|\cdot| = \|\cdot\|_{p-1,q+1}$.*

Remark 5.2.4. By Proposition 5.2.3 we may apply the results in [71] (see Section 1.1 for a review) to $\{\mathcal{L}_{\epsilon,n(\epsilon)}\}_{\epsilon \in [0,\epsilon_3)}$. A careful examination of the proof of Proposition 5.2.3 shows that the α term in (KL3) for $\{\mathcal{L}_{\epsilon,n(\epsilon)}\}_{\epsilon \in [0,\epsilon_3)}$ may be taken to be any number in $(\max\{\nu_s^q, \nu_u^{-p}\}, 1)$. Hence, all the isolated eigenvalues of $\mathcal{L}$ with modulus strictly greater than $\max\{\nu_s^q, \nu_u^{-p}\}$ are approximated by eigenvalues of $\mathcal{L}_{\epsilon,n(\epsilon)}$, with error vanishing as $\epsilon \to 0$. When such an eigenvalue of $\mathcal{L}$ is simple, we additionally have that the corresponding eigenprojection and eigenvector are approximated by those of $\mathcal{L}_{\epsilon,n(\epsilon)}$ in $|||\cdot|||$ and $|\cdot|$, respectively (see [38, Proposition 2.4, Remark 2.5] for an extension of this idea).

Remark 5.2.5. From the proof of Proposition 5.2.3 it is clear that asymptotic behaviour of $n(\epsilon)$ as $\epsilon \to 0$ is determined by the family of stochastic kernels $\{q_\epsilon\}_{\epsilon>0}$. By restricting to a specific family of kernels one could more explicitly describe the dependence of ϵ on $n(\epsilon)$. This is exactly what is done in [14, Section 2.6]: in their scheme it is shown that $n(\epsilon) = O(\epsilon^{-r})$ for an exponent $r > 0$ that may be estimated in terms of dynamics, where n represents the maximum size of a polytope used in an Ulam discretisation, and ϵ represents the dilation of a fixed stochastic kernel q (i.e $q_\epsilon(x) := \epsilon^{-k} q(\epsilon^{-1} x)$). See [14, Section 2.6] for more details.

Propositions 5.2.3 and 5.1.5 allow us to apply Theorem 2.2.1 to obtain the stability of the invariant measure, variance and rate function. We note that Anosov diffeomorphisms on tori are automatically topologically transitive [66, Proposition 18.6.5], and so T has a unique SRB measure μ, and 1 is both a simple eigenvaue of $\mathcal{L}$ and the only eigenvalue of $\mathcal{L}$ of modulus 1 ([54, Theorem 2.3]).

[3]Specifically, $\hat{f}(\ell) = \int_{\mathbb{T}^m} f(x) e^{-2\pi i x \cdot \ell} \, \mathrm{d}\,\mathrm{Leb}(x)$.

Theorem 5.2.6. *Suppose that $T \in \mathcal{C}^{r+1}(\mathbb{T}^m, \mathbb{T}^m)$, $r > 1$, is a topologically transitive Anosov diffeomorphism with unique SRB measure μ, and that $g \in \mathcal{C}^r(\mathbb{T}^m, \mathbb{R})$ satisfies $\int g \, \mathrm{d}\mu = 0$ and is not a $L^2(\mu)$-coboundary with respect to T. Let N, n satisfy the conditions of Proposition 5.2.3. There exists $\theta, \epsilon' > 0$ so that for each $\epsilon \in [0, \epsilon')$ and $z \in D_\theta$ the operator $\mathcal{L}_{\epsilon,n(\epsilon)}(z)$ is quasi-compact and simple with leading eigenvalue $\lambda_\epsilon(z)$ depending analytically on z. Moreover, we have stability of the following statistical data associated to T and $\{g \circ T^k\}_{k\in\mathbb{N}}$ as $\epsilon \to 0$:*

1. *The invariant measure is stable: there exists eigenvectors $v_\epsilon \in B^{p,q}_{\mathbb{C}}$ of $\mathcal{L}_{\epsilon,n(\epsilon)}$ for the eigenvalue $\lambda_\epsilon(0)$ for which $\lim_{\epsilon\to 0} \|v_\epsilon - \mu\|_{p-1,q+1} = 0$.*
2. *The variance is stable: $\lim_{\epsilon\to 0} \lambda_\epsilon^{(2)}(0) = \sigma^2$.*
3. *The rate function is stable: For each sufficiently small compact subset U of the domain of the rate function r there exists an interval $V \subseteq (-\theta, \theta)$ so that*

$$\lim_{\epsilon\to 0} \sup_{z\in V} (sz - \log |\lambda_\epsilon(z)|) = r(s)$$

uniformly on U.

Remark 5.2.7. In Section 5.4 we aim to estimate the statistical properties of an Anosov map T using Lemma 5.2.2, Proposition 5.2.3 and Theorem 5.2.6. However, these results concern the stability of the spectral data of the transfer operator associated to an Anosov map T on the m-dimensional torus $\mathbb{T}^m$ *equipped with an adapted metric*. In particular, this operator, say $\mathcal{L}_\Omega$, is defined by duality with respect to the adapted Riemannian measure Ω. From a computational perspective one would much rather approximate the transfer operator $\mathcal{L}_{\text{Leb}}$ that is defined by duality with respect to Leb, the Haar probability measure on $\mathbb{T}^m$, since this removes the need to compute any quantities that depend on the adapted metric. Luckily, the relationship between these operators (and their twists) is simple: they are conjugate and therefore have the same spectrum (see Proposition 5.4.1). Hence an approximation of the spectrum of $\mathcal{L}_{\text{Leb}}(z)$ is also an approximation of the spectrum of $\mathcal{L}_\Omega(z)$. However, it is not clear from the proofs in this section that if $\{\Pi_{n(\epsilon)} Q_\epsilon \mathcal{L}_\Omega\}_{\epsilon\in[0,\epsilon')}$ satisfies (KL) due to Proposition 5.2.3 then so too does $\{\Pi_{n(\epsilon)} Q_\epsilon \mathcal{L}_{\text{Leb}}\}_{\epsilon\in[0,\epsilon')}$ i.e. a numerical scheme that is valid for $\mathcal{L}_\Omega$ may not be valid for $\mathcal{L}_{\text{Leb}}$. In Section 5.4.4 we show that this obstruction does not occur, at least not in the current setting; the relevant results are Propositions 5.4.1, 5.4.2 and 5.4.3.

The remainder of this section is dedicated to the proofs of the aforementioned results.

The proof of Lemma 5.2.1. Let $\epsilon \in (0, \epsilon_1)$, $\ell \in \mathbb{N}^m$ and $f \in \mathcal{C}^r(\mathbb{T}^m, \mathbb{C})$. Denote $\frac{\partial^{\|\ell\|_1}}{\partial x_1^{\ell_1} \dots \partial x_m^{\ell_m}}$ by ∂_ℓ. From the beginning of [54, Section 4] the map $h \mapsto \int h \, \mathrm{d}\Omega$ is bounded on $B^{p,q}_{\mathbb{C}}$. Moreover, multiplication by $\mathcal{C}^r$ functions is bounded on $B^{p,q}_{\mathbb{C}}$ by [54, Lemma 3.2]. These facts, together with standard properties of convolutions, imply that there exists a constant C independent of f, ℓ and ϵ such that

$$\begin{aligned}
\sup_{x \in \mathbb{T}^m} |(\partial_\ell(q_\epsilon * f))(x)| &= \sup_{x \in \mathbb{T}^m} \left| \int (\partial_\ell q_\epsilon)(x-y) f(y) \, \mathrm{d}\,\mathrm{Leb}(y) \right| \\
&= \sup_{x \in \mathbb{T}^m} \left| \int (\partial_\ell q_\epsilon)(x-y) f(y) \frac{\mathrm{d}\,\mathrm{Leb}}{\mathrm{d}\Omega}(y) \, \mathrm{d}\Omega(y) \right| \\
&\le C \left\| f \frac{\mathrm{d}\,\mathrm{Leb}}{\mathrm{d}\Omega} \right\|_{p,q} \|\partial_\ell q_\epsilon\|_{\mathcal{C}^q} .
\end{aligned}$$

Since $\frac{\mathrm{d}\,\mathrm{Leb}}{\mathrm{d}\Omega} \in \mathcal{C}^\infty(\mathbb{T}^m, \mathbb{R})$, using the continuity of multiplication by $\mathcal{C}^r$ functions again yields

$$\|Q_\epsilon f\|_{\mathcal{C}^{\|\ell\|_1}} \le C' \left\| \frac{\mathrm{d}\,\mathrm{Leb}}{\mathrm{d}\Omega} \right\|_{\mathcal{C}^r} \|f\|_{p,q} \|q_\epsilon\|_{\mathcal{C}^{q+\|\ell\|_1}}$$

for some appropriate constant C'. Hence, Q_ϵ extends to a bounded operator $Q_\epsilon : B^{p,q}_{\mathbb{C}} \to \mathcal{C}^k(\mathbb{T}^m, \mathbb{C})$ for every $k \in \mathbb{Z}^+$. It follows that Q_ϵ also extends to a bounded operator $Q_\epsilon : B^{p,q}_{\mathbb{C}} \to \mathcal{C}^\infty(\mathbb{T}^m, \mathbb{C})$. As bounded sets in $\mathcal{C}^\infty(\mathbb{T}^m, \mathbb{C})$ are compact in $\mathcal{C}^k(\mathbb{T}^m, \mathbb{C})$ for every $k \in \mathbb{Z}^+$, each operator $Q_\epsilon : B^{p,q}_{\mathbb{C}} \to \mathcal{C}^k(\mathbb{T}^m, \mathbb{C})$ is therefore compact. That $Q_\epsilon : B^{p,q}_{\mathbb{C}} \to B^{p,q}_{\mathbb{C}}$ is compact follows from the continuous embedding of $\mathcal{C}^r(\mathbb{T}^m, \mathbb{C})$ into $B^{p,q}_{\mathbb{C}}$ [54, Remark 4.3]. It is then standard that $Q_\epsilon \in L(\mathcal{C}^{k_1}(\mathbb{T}^m, \mathbb{C}), \mathcal{C}^{k_2}(\mathbb{T}^m, \mathbb{C}))$ for each $k_1, k_2 \in \mathbb{Z}^+$. □

The proof of Lemma 5.2.2. For each $y \in \mathbb{T}^m$ let $T_y : \mathbb{T}^m \to \mathbb{T}^m$ be defined by $T_y(x) = T(x) + y$, and let $\mathcal{L}_{T_y}$ denote the transfer operator associated with T_y (defined by duality as in (5.1)). Let $h \in \mathcal{C}^r(\mathbb{T}^m, \mathbb{C})$ and $x, y \in \mathbb{T}^m$. Let $\tau_y : \mathbb{T}^m \to \mathbb{T}^m$ denote the translation map induced by y. As $D_x T_y = (D_{T(x)} \tau_y)(D_x T)$ we have

$$\begin{aligned}
(\mathcal{L}h)(x-y) &= \left((h \, |\det DT|^{-1}) \circ T^{-1} \right)(x-y) \\
&= \left(h \circ T_y^{-1} \right)(x) \left| \det D_{T_y^{-1}(x)} T_y \right|^{-1} \left| \det (D_{x-y} \tau_y)^{-1} \right|^{-1} \\
&= (\mathcal{L}_{T_y} h)(x) \, |\det D_x \tau_{-y}| .
\end{aligned}$$

Hence,

$$
\begin{aligned}
(\mathcal{L}_\epsilon h)(x) &= \int q_\epsilon(y)(\mathcal{L}h)(x-y)\,\mathrm{d}\,\mathrm{Leb}(y) \\
&= \int q_\epsilon(y)\,|\det D_x\tau_{-y}|\,(\mathcal{L}_{T_y}h)(x)\,\mathrm{d}\,\mathrm{Leb}(y).
\end{aligned}
\tag{5.5}
$$

For $\epsilon \in (0,\epsilon_1)$ let A_ϵ be defined by

$$
(A_\epsilon h)(x) = \int q_\epsilon(y)(\mathcal{L}_{T_y}h)(x)\,\mathrm{d}\,\mathrm{Leb}(y).
$$

Set $A_0 = \mathcal{L}$ and, for each $\epsilon \in [0,\epsilon_1)$, let $F_\epsilon = \mathcal{L}_\epsilon - A_\epsilon$. We aim to apply Proposition 1.3.1 to $\{A_\epsilon\}_{\epsilon\in[0,\epsilon_1)}$ and $\{F_\epsilon\}_{\epsilon\in[0,\epsilon_1)}$, which would imply the required result as $\mathcal{L}_\epsilon = A_\epsilon + F_\epsilon$.

We begin by proving that $\{A_\epsilon\}_{\epsilon\in[0,\epsilon_1)}$ satisfies (KL). Note that $\{A_\epsilon\}_{\epsilon\in[0,\epsilon_1)}$ is a perturbation of the kind considered in [54]. Specifically, we take $\mathbb{T}^m$ to be their Ω, the Haar measure Leb to be their μ, and set $g(\omega,x) = q_\epsilon(\omega)$. Therefore, by the discussion between Corollary 2.6 and Theorem 2.7 in [54], there exists some $\epsilon' \in (0,\epsilon_1)$ such that $\{A_\epsilon\}_{\epsilon\in[0,\epsilon')}$ satisfies (KL) on $B^{p,q}_{\mathbb{C}}$ with $|\cdot| = \|\cdot\|_{p-1,q+1}$ provided that

(C1) For a fixed, small, open (in the $\mathcal{C}^{r+1}(\mathbb{T}^m,\mathbb{T}^m)$ topology) neighbourhood U of T we have $T_y \in U$ whenever $y \in B(0,\epsilon')$; and

(C2) $\lim_{\epsilon\to 0}\int q_\epsilon(y)\, d_{\mathcal{C}^{r+1}}(T_y,T)\,\mathrm{d}\,\mathrm{Leb}(y) = 0$.

The condition (C2) is derived from [54, equation (2.5)] by setting $g(\omega,x) = q_\epsilon(\omega)$, and observing that $x \mapsto q_\epsilon(\omega)$ is constant and so has $\mathcal{C}^{p+q}$ norm $|q_\epsilon(\omega)| = q_\epsilon(\omega)$. The other term in [54, equation (2.5)] is 0 since $\int q_\epsilon(\omega)\,\mathrm{d}\,\mathrm{Leb}(\omega) = 1$. As $\mathbb{T}^m$ is compact and $T \in \mathcal{C}^{r+1}(\mathbb{T}^m,\mathbb{T}^m)$, the map $x \mapsto D_x^k T$ is uniformly continuous for each $0 \le k \le r+1$. It then follows from the definition of T_y that

$$
\lim_{\epsilon\to 0}\sup_{y\in B(0,\epsilon)} d_{\mathcal{C}^{r+1}}(T_y,T) = 0.
$$

Recalling that q_ϵ satisfies (S1) and (S2), it is clear that there exists $\epsilon' \in (0,\epsilon_1)$ so that $\{A_\epsilon\}_{\epsilon\in[0,\epsilon')}$ satisfies both (C1) and (C2), and therefore also (KL).

We will now prove that $\{F_\epsilon\}_{\epsilon\in[0,\epsilon')}$ satisfies the requirements of Proposition 1.3.1. For $y \in \mathbb{T}^m$ let $f_y : \mathbb{T}^m \to \mathbb{R}$ be defined by $f_y(x) = 1 - \det D_x\tau_{-y}$. One verifies that

$$(F_\epsilon h)(x) = \int q_\epsilon(y) f_y(x)(\mathcal{L}_{T_y} h)(x) \,\mathrm{d}\,\mathrm{Leb}(y)$$

From the definition of $\|\cdot\|_{p,q}$ and (S1) we obtain

$$\|F_\epsilon h\|_{p,q} \le \int q_\epsilon(y) \left\| f_y \cdot (\mathcal{L}_{T_y} h) \right\|_{p,q} \mathrm{d}\,\mathrm{Leb}(y).$$

As multiplication by $\mathcal{C}^r$ functions is continuous on $B^{p,q}_{\mathbb{C}}$ ([54, Lemma 3.2]), using (S1) there is some $C > 0$ such that

$$\|F_\epsilon\|_{p,q} \le C \sup_{y\in\operatorname{supp} q_\epsilon} \left\|\mathcal{L}_{T_y}\right\|_{p,q} \int q_\epsilon(y) \left\|f_y\right\|_{\mathcal{C}^r} \mathrm{d}\,\mathrm{Leb}(y). \tag{5.6}$$

As mentioned at the beginning of [54, Section 7], the estimates in [54, Lemma 2.2] apply uniformly to every map in U and so there exists some $\eta > 0$ such that $\sup_{y\in B(0,\eta)} \left\|\mathcal{L}_{T_y}\right\|_{p,q} < \infty$. Since $\det D_x\tau_y = \frac{\mathrm{d}\Omega}{\mathrm{d}\,\mathrm{Leb}}(x+y)\frac{\mathrm{d}\,\mathrm{Leb}}{\mathrm{d}\Omega}(x)$, and $\frac{\mathrm{d}\Omega}{\mathrm{d}\,\mathrm{Leb}}, \frac{\mathrm{d}\,\mathrm{Leb}}{\mathrm{d}\Omega} \in \mathcal{C}^\infty(\mathbb{T}^m, \mathbb{R})$, we have $\lim_{y\to 0} f_y = 0$ in $\mathcal{C}^r(\mathbb{T}^m,\mathbb{R})$. Applying these facts and (S1) to (5.6) yields

$$\lim_{\epsilon\to 0} \|F_\epsilon\|_{p,q} \le C \limsup_{\epsilon\to 0} \sup_{y\in B(y,\epsilon)} \|f_y\|_{\mathcal{C}^r} \sup_{y\in B(0,\epsilon)} \left\|\mathcal{L}_{T_y}\right\|_{p,q} = 0. \tag{5.7}$$

The same argument applies when estimating $\|F_\epsilon\|_{p-1,q+1}$, and so there exists $\epsilon'' \in (0,\epsilon')$ so that $\{F_\epsilon\}_{\epsilon\in[0,\epsilon'')}$ satisfies the requirements for Proposition 1.3.1.

Hence Proposition 1.3.1 applies to $\{A_\epsilon\}_{\epsilon\in[0,\epsilon'')}$ and $\{F_\epsilon\}_{\epsilon\in[0,\epsilon'')}$. Namely, there exists $\epsilon_2 \in (0,\epsilon'')$ so that $\{A_\epsilon + F_\epsilon\}_{\epsilon\in[0,\epsilon_2)}$ satisfies (KL). Since $A_\epsilon + F_\epsilon = \mathcal{L}_\epsilon$, this completes the proof. □

We require the following classical result on the convergence of Fourier series on $\mathbb{T}^m$ (see e.g. [90, Proposition 5.6 and the proof of Theorem 5.7]).

Proposition 5.2.8. *For each $k \in \mathbb{N}$ we have $\Pi_n \to \mathrm{Id}$ strongly in $L(\mathcal{C}^{k+\lceil\frac{m+1}{2}\rceil}, \mathcal{C}^k)$.*

The proof of Proposition 5.2.3. By Lemma 5.2.2 the family of operators $\{\mathcal{L}_\epsilon\}_{\epsilon\in[0,\epsilon_2)}$ satisfies (KL) on $B^{p,q}_{\mathbb{C}}$ with $|\cdot| = \|\cdot\|_{p-1,q+1}$. We plan to find $N : (0,\epsilon_2) \to \mathbb{N}$ so that we may apply Proposition 1.3.1 with $A_\epsilon = \mathcal{L}_\epsilon$ and $B_\epsilon = \mathcal{L}_{\epsilon,N(\epsilon)} - \mathcal{L}_\epsilon$.

By Proposition 5.2.8, $\Pi_n \to \mathrm{Id}$ strongly in $L(\mathcal{C}^{r+\lceil\frac{m+1}{2}\rceil}, \mathcal{C}^r)$. As the unit ball of $\mathcal{C}^{r+1+\lceil\frac{m+1}{2}\rceil}$ is compact in $\mathcal{C}^{r+\lceil\frac{m+1}{2}\rceil}$, Proposition 5.2.8, the uniform boundedness principle and standard estimates imply that $\Pi_n \to \mathrm{Id}$ in $L(\mathcal{C}^{r+1+\lceil\frac{m+1}{2}\rceil}, \mathcal{C}^r)$. As $\mathcal{C}^r$ embeds continuously into $B^{p,q}_{\mathbb{C}}$ [54, Remark 4.3], there exists $C > 0$ so that, for each $\epsilon \in [0, \epsilon_2)$ and $n \in \mathbb{N}$, we have

$$\|\mathcal{L}_{\epsilon,n} - \mathcal{L}_\epsilon\|_{p,q} \le C \|\Pi_n - \mathrm{Id}\|_{L\left(\mathcal{C}^{r+1+\lceil\frac{m+1}{2}\rceil}, \mathcal{C}^r\right)} \|Q_\epsilon\|_{L\left(B^{p,q}_{\mathbb{C}}, \mathcal{C}^{r+1+\lceil\frac{m+1}{2}\rceil}\right)} \|\mathcal{L}\|_{p,q}\,.$$

Hence, as $\|Q_\epsilon\|_{L\left(B^{p,q}_{\mathbb{C}}, \mathcal{C}^{r+1+\lceil\frac{m+1}{2}\rceil}\right)}$ is finite by Lemma 5.2.1, for each $\epsilon \in (0, \epsilon_2)$ there exists $N_1(\epsilon)$ so that $\|\mathcal{L}_{\epsilon,n} - \mathcal{L}_\epsilon\|_{p,q} \le \epsilon$ whenever $n > N_1(\epsilon)$. The same argument produces for each $\epsilon \in (0, \epsilon_2)$ an $N_2(\epsilon)$ so that $\|\mathcal{L}_{\epsilon,n} - \mathcal{L}_\epsilon\|_{p-1,q+1} \le \epsilon$ whenever $n \ge N_2(\epsilon)$. To summarise, if $N(\epsilon) := \max\{N_1(\epsilon), N_2(\epsilon)\}$ and $n : (0, \epsilon_2) \to \mathbb{N}$ is such that $n \ge N$, then $\lim_{\epsilon\to 0} \left\|\mathcal{L}_{\epsilon,n(\epsilon)} - \mathcal{L}_\epsilon\right\|_{p,q} = 0$ and

$$\sup_{\epsilon\in(0,\epsilon_2)} \left\|\mathcal{L}_{\epsilon,n(\epsilon)} - \mathcal{L}_\epsilon\right\|_{p-1,q+1} < \infty. \tag{5.8}$$

Hence for each map $n \ge N$ we may apply Proposition 1.3.1 as planned to produce an $\epsilon_n \in (0, \epsilon_2)$ so that $\{\mathcal{L}_{\epsilon,n(\epsilon)}\}_{\epsilon\in[0,\epsilon_n)}$ satisfies (KL) on $B^{p,q}_{\mathbb{C}}$ with $|\cdot| = \|\cdot\|_{p-1,q+1}$. Examining the proof of Proposition 1.3.1, we observe that ϵ_n may be chosen independently of n since $\lim_{\epsilon\to 0} \sup_{\ell\ge N(\epsilon)} \|\mathcal{L}_{\epsilon,\ell} - \mathcal{L}_\epsilon\|_{p,q} = 0$ and

$$\sup_{\epsilon\in(0,\epsilon_2)} \sup_{\ell\ge N(\epsilon)} \|\mathcal{L}_{\epsilon,\ell} - \mathcal{L}_\epsilon\|_{p-1,q+1} < \infty.$$

$\square$

The proof of Theorem 5.2.6. Hence, by Proposition 5.1.5, the sequence $\{g \circ T^k\}_{k\in\mathbb{N}}$ satisfies (NG) on the probability space (X, ζ) with coding $(z \mapsto \mathcal{L}(z), \Omega, \zeta)$. Since $\{\mathcal{L}_{\epsilon,n(\epsilon)}\}_{\epsilon\in[0,\epsilon_3)}$ satisfies (KL), we have verified all the requirements of Theorem 2.2.1, and so all the claims in the statement of Theorem 5.2.6 follow, with the exception of the stability of the invariant measure. This claim follows from [38, Proposition 2.4, Remark 2.5], whose hypotheses are verified due to the convergence of eigenprojections in Theorem 1.2.2. $\square$

5.3 Statistical stability for some Anosov maps under non-local stochastic perturbations

The main goal of this section is to show that the statistical properties of some Anosov maps on the m-dimensional torus $\mathbb{T}^m$ may be approximated using (weighted) Fourier series by realising the Fejér kernel as a stochastic perturbation. More generally, our results expand upon the stability to stochastic perturbations results of [54] (see [54, Theorem 2.7]) by allowing the stochastic kernel to be supported on all of $\mathbb{T}^m$ at the cost of additional requirements on the dynamics. As a consequence of this generalisation, we *no longer require a mollifier* to estimate the spectral and statistical properties of Anosov maps as in Section 5.2, which improves computational potential of our theory.

We adopt the setting, assumptions and notation from Section 5.1 and fix $p \in \mathbb{Z}^+$ and $q > 0$ satisfying $p + q < r$. Our main assumption on the dynamics is that the associated transfer operator $\mathcal{L}$ satisfies

$$\sup_{y \in \mathbb{T}^m} \max\{\|\tau_y \mathcal{L}\|_{p,q}, \|\tau_y \mathcal{L}\|_{p-1,q+1}\} < \infty, \tag{5.9}$$

where τ_y denotes the translation operator[4] induced by $y \in \mathbb{T}^m$. In Section 5.3.1 we show that (5.9) implies (KL) for non-local stochastic perturbations (i.e. for stochastic kernels supported on all of $\mathbb{T}^m$), which then yields the stability of the invariant SRB measure, variance and rate function to such perturbations. In Section 5.3.2 we then provide conditions for a map to satisfy (5.9). For example, Proposition 5.3.6 implies that (5.9) holds for an iterate of T provided that T is close to a hyperbolic linear toral automorphism. Lastly, in Section 5.3.3 we verify these conditions for a family of perturbations to Arnold's cat map.

5.3.1 (KL) for non-local stochastic perturbation

For some $\epsilon_1 > 0$, suppose that $\{q_\epsilon\}_{\epsilon \in (0,\epsilon_1)} \subseteq L^1(\text{Leb})$ is a family of stochastic kernels satisfying (S1) and

(S3) For every $\eta > 0$, we have $\lim_{\epsilon \to 0} \int_{\mathbb{T}^m \setminus B(0,\eta)} q_\epsilon \, d\,\text{Leb} = 0$.

The condition (S3) replaces (S2) from Section 5.2, and consequently allows the support of each q_ϵ to be all of $\mathbb{T}^m$. Also note that we place no regularity requirements

[4] Throughout this section we treat τ_y as both a composition operator and map.

on the kernels q_ϵ. For $\epsilon > 0$ define $\mathcal{L}_\epsilon := q_\epsilon * \mathcal{L}$, and let $\mathcal{L}_0 := \mathcal{L}$. Our main technical result for this section is the following.

Proposition 5.3.1. *If* (5.9) *holds then there exists* $\epsilon_2 \in (0, \epsilon_1)$ *so that* $\{\mathcal{L}_\epsilon\}_{\epsilon\in[0,\epsilon_2)}$ *satisfies (KL) on* $B^{p,q}_{\mathbb{C}}$ *with* $|\cdot| = \|\cdot\|_{p-1,q+1}$.

Remark 5.3.2. The same comments as those in Remark 5.2.4 apply to the family of operators $\{\mathcal{L}_\epsilon\}_{\epsilon\in[0,\epsilon_2)}$. That is, the spectral data associated to the eigenvalues of $\mathcal{L}_0$ with modulus strictly greater than $\max\{\nu_s^q, \nu_u^{-p}\}$ are well approximated by the spectral data of associated to eigenvalues of $\mathcal{L}_\epsilon$, with error vanishing as $\epsilon \to 0$.

As noted before the statement of Theorem 5.2.6, since T is an Anosov diffeomorphism on a torus, $\mathcal{L}$ is a simple quasi-compact operator on $B^{p,q}_{\mathbb{C}}$. Thus, if (5.9) holds then Propositions 5.3.1 and 5.1.5 allow us to apply Theorem 2.2.1 to obtain the stability of the peripheral spectral data, invariant measure, variance and rate function with respect to the class of stochastic perturbations in consideration. The proof is the same as that of Theorem 5.2.6.

Theorem 5.3.3. *Suppose that* $T \in \mathcal{C}^{r+1}(\mathbb{T}^m, \mathbb{T}^m)$*,* $r > 1$*, is a topologically transitive Anosov diffeomorphism with unique SRB measure* μ*, and that the Perron-Frobenius operator* $\mathcal{L}$ *associated to* T *satisfies* (5.9)*. Further suppose that* $g \in \mathcal{C}^r(\mathbb{T}^m, \mathbb{R})$ *satisfies* $\int g \,\mathrm{d}\mu = 0$ *and is not an* $L^2(\mu)$*-coboundary. There exists* $\theta, \epsilon' > 0$ *so that for each* $\epsilon \in [0, \epsilon')$ *and* $z \in D_\theta$ *the operator* $\mathcal{L}_\epsilon(z)$ *is quasi-compact and simple with leading eigenvalue* $\lambda_\epsilon(z)$ *depending analytically on* z*. Moreover, we have stability of the following statistical data associated to* T *and* $\{g \circ T^k\}_{k\in\mathbb{N}}$*:*

1. *The invariant measure is stable: there exists eigenvectors* $v_\epsilon \in B^{p,q}_{\mathbb{C}}$ *of* $\mathcal{L}_\epsilon$ *for the eigenvalue* $\lambda_\epsilon(0)$ *for which* $\lim_{\epsilon\to 0} \|v_\epsilon - \mu\|_{p-1,q+1} = 0$.
2. *The variance is stable:* $\lim_{\epsilon\to 0} \lambda_\epsilon^{(2)}(0) = \sigma^2$.
3. *The rate function is stable: For each sufficiently small compact subset* U *of the domain of the rate function* r_g *there exists an interval* $V \subseteq (-\theta, \theta)$ *so that*

$$\lim_{\epsilon\to 0} \sup_{z\in V}(sz - \log|\lambda_\epsilon(z)|) = r(s)$$

   uniformly on U*.*

A key application of the results in this section is the rigorous approximation of the spectral and statistical data of some Anosov maps using Fourier series. We define the nth 1-dimensional Fejér kernel $K_{n,1}$ on $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ by

$$K_{n,1}(t) = \sum_{\ell=-n}^{n} \left(1 - \frac{|\ell|}{n+1}\right) e^{2\pi i \ell t}.$$

The nth m-dimensional Fejér kernel $K_{n,m}$ on $\mathbb{T}^m$ is defined by

$$K_{n,m}(t) = \prod_{i=1}^{m} K_{n,1}(t_i),$$

where t_i is the ith component of $t \in \mathbb{T}^m$. It is well known that the 1-dimensional Fejér kernels satisfy (S1) and (S3) as $n \to \infty$ (they are a *summability kernel*; see [67, Section 2.2 and 2.5]). It is routine to verify that the m-dimensional kernels consequently satisfy the same conditions, and so we may apply Proposition 5.3.1 with $q_{1/n} = K_{n,m}$. A straightforward computation yields

$$K_{n,m}(t) = \sum_{\substack{\ell \in \mathbb{Z}^m \\ \|\ell\|_\infty \le n}} \prod_{i=1}^{m} \left(1 - \frac{|\ell_i|}{n+1}\right) e^{2\pi i \langle \ell, t\rangle},$$

and, therefore, convolution with the Fejér kernel may be represented using weighted Fourier series:

$$(K_{n,m} * f)(x) = \sum_{\substack{\ell \in \mathbb{Z}^m \\ \|\ell\|_\infty \le n}} \prod_{i=1}^{m} \left(1 - \frac{|\ell_i|}{n+1}\right) \hat{f}(k) e^{2\pi i \langle \ell, x\rangle}. \tag{5.10}$$

By the above considerations, Proposition 5.3.1 and Theorem 5.3.3 we obtain the following stability result for stochastic perturbations induced by the Fejér kernel.

Corollary 5.3.4. *Assume that T, $\mathcal{L}$ and g are as in 5.3.3. For $n \in \mathbb{N}$ let $\mathcal{L}_{1/n} := K_{n,m} * \mathcal{L}$, where $K_{n,m}$ is the m-dimensional Fejér kernel, and let $\mathcal{L}_0 = \mathcal{L}$. There exists $N > 0$ so that the family of operators $\{\mathcal{L}_{1/n}\}_{n \ge N}$ satisfies (KL) on $B^{p,q}_{\mathbb{C}}$ with $|\cdot| = \|\cdot\|_{p-1,q+1}$. Consequently, we have stability of the invariant measure, variance, and rate function associated to T and $\{g \circ T^k\}_{k \in \mathbb{N}}$ as in Theorem 5.3.3.*

The operators $\mathcal{L}_{1/n}$ are finite-dimensional and leave the span of $\{e^{2\pi i \langle \ell, x\rangle} : \|\ell\|_\infty \le n\}$ invariant. Therefore, we could compute all of the spectral data of $\mathcal{L}_{1/n}$ via its matrix representation with respect to the basis $\{e^{2\pi i \langle \ell, x\rangle} : \|\ell\|_\infty \le n\}$ and use Corollary 5.3.4 to estimate the statistical properties of T.

Remark 5.3.5. The comments made in Remark 5.2.7 also apply here: the stability results in Proposition 5.3.1, Theorem 5.3.3 and Corollary 5.3.4 apply to the transfer operator $\mathcal{L}_\Omega$ that is associated to T via duality with respect to Ω, rather than the transfer operator $\mathcal{L}_{\mathrm{Leb}}$ that is associated to T via duality with respect to Leb. In Section 5.4.4 we show that these operators, and their twists, are conjugate, and therefore have the same spectrum, and that if Proposition 5.3.1, Theorem 5.3.3 and Corollary 5.3.4 apply to $\mathcal{L}_\Omega$ then they also hold true when $\mathcal{L}_\Omega$ is replaced by $\mathcal{L}_{\mathrm{Leb}}$.

Proof of Proposition 5.3.1. For each $\eta \in (0, 1/2)$ let s_η be the characteristic function of $B(0, \eta)$. For $\epsilon \geq 0$ let $a_{\epsilon,\eta} = \int q_\epsilon s_\eta \,\mathrm{d}\,\mathrm{Leb}$, $A_{\epsilon,\eta} = a_{\epsilon,\eta}^{-1}(q_\epsilon s_\eta) * \mathcal{L}$ and $B_{\epsilon,\eta} = (q_\epsilon(1 - a_{\epsilon,\eta}^{-1} s_\eta)) * \mathcal{L}$. Set $A_{0,\eta} = \mathcal{L}$ and $B_{0,\eta} = 0$. Our goal is to apply Proposition 1.3.1 by showing that for some $\eta > 0$ there exists $\epsilon' \in (0, \epsilon_1)$ so that $\{A_{\epsilon,\eta}\}_{\epsilon \in [0,\epsilon')}$ satisfies (KL), and then proving that $\{B_{\epsilon,\eta}\}_{\epsilon \in [0,\epsilon')}$ satisfies the necessary requirements of Proposition 1.3.1. As $\mathcal{L}_\epsilon = A_{\epsilon,\eta} + B_{\epsilon,\eta}$, this yields the required statement.

Since $\{a_{\epsilon,\eta}^{-1} q_\epsilon s_\eta\}_{\epsilon \in (0,\epsilon_1)}$ satisfies (S1) the perturbation $\{A_{\epsilon,\eta}\}_{\epsilon \in [0,\epsilon_1)}$ is similar to the (convolution type) perturbation considered in Lemma 5.2.2. We will explain how to modify the proof of Lemma 5.2.2 to obtain the required result. Examining the proof of Lemma 5.2.2, we note that it was not important that the family of kernels was in $\mathcal{C}^\infty$, indeed it is sufficient for the kernels to be contained in $L^1(\mathrm{Leb})$. We must verify the conditions (C1) and (C2), and a different argument is required here since $\{a_{\epsilon,\eta}^{-1} q_\epsilon s_\eta\}_{\epsilon \geq 0}$ does not satisfy (S2). By choosing η sufficiently small we may make the support of every $a_{\epsilon,\eta}^{-1} q_\epsilon s_\eta$ small enough so that for every $\epsilon \in [0, \epsilon_1)$ and $y \in \operatorname{supp} q_\epsilon s_\eta$ the map $T_y(x) := T(x) + y$ is in the set U from (C1) in Lemma 5.2.2. This verifies (C1) in our setting; we will now verify (C2). By (S1) and (S3) for $\{q_\epsilon\}_{\epsilon \geq 0}$ we have $\lim_{\epsilon \to 0} a_{\epsilon,\eta'}^{-1} = 1$ for every $\eta' \in (0, \eta]$. Hence, for every $\eta' \in (0, \eta)$ we have

$$
\begin{aligned}
&\lim_{\epsilon \to 0} \int a_{\epsilon,\eta}^{-1} |q_\epsilon(y) s_\eta(y)| d_{\mathcal{C}^{r+1}}(T_y, T) \mathrm{d}\,\mathrm{Leb}(y) = \lim_{\epsilon \to 0} \int_{B(0,\eta)} q_\epsilon(y) d_{\mathcal{C}^{r+1}}(T_y, T) \,\mathrm{d}\,\mathrm{Leb}(y) \\
&= \lim_{\epsilon \to 0} \left(\int_{B(0,\eta) \setminus B(0,\eta')} q_\epsilon(y) d_{\mathcal{C}^{r+1}}(T_y, T) \,\mathrm{d}\,\mathrm{Leb}(y) + \int_{B(0,\eta')} q_\epsilon(y) d_{\mathcal{C}^{r+1}}(T_y, T) \,\mathrm{d}\,\mathrm{Leb}(y) \right) \\
&\leq \lim_{\epsilon \to 0} \left(\sup_{y \in B(0,\eta) \setminus B(0,\eta')} d_{\mathcal{C}^{r+1}}(T_y, T) \right) \int_{\mathbb{T}^m \setminus B(0,\eta')} q_\epsilon(y) \,\mathrm{d}\,\mathrm{Leb}(y) + \sup_{y \in B(0,\eta')} d_{\mathcal{C}^{r+1}}(T_y, T) \\
&= \sup_{y \in B(0,\eta')} d_{\mathcal{C}^{r+1}}(T_y, T),
\end{aligned}
\tag{5.11}
$$

where we have used both (S1) and (S3). Since $\lim_{\eta'\to 0}\sup_{y\in B(0,\eta')} d_{\mathcal{C}^{r+1}}(T_y, T) = 0$ we obtain (C2) from (5.11). It remains to provide an alternative proof of the convergence of $F_\epsilon \to 0$ from (5.7) i.e. that the operators

$$F_\epsilon : h \mapsto \int a_{\epsilon,\eta}^{-1} q_\epsilon(y) s_\eta(y) f_y(x) (\mathcal{L}_{T_y} h)(x)\,\mathrm{d}\,\mathrm{Leb}(y)$$

converge to 0 in $L(B^{p,q})$ and $L(B^{p-1,q+1})$ as $\epsilon \to 0$, where $f_y = 1 - \det D_x\tau_{-y}$ as in the proof of Lemma 5.2.2. Using (S3) and the fact that $\lim_{\epsilon\to 0} a_{\epsilon,\eta}^{-1} = 1$ we have for each $\eta' \in (0, \eta)$ that

$$\begin{aligned}
&\lim_{\epsilon\to 0} \int a_{\epsilon,\eta}^{-1} q_\epsilon(y) s_\eta(y) \left\| f_y \right\|_{\mathcal{C}^r} \,\mathrm{d}\,\mathrm{Leb}(y) \\
&\quad \le \lim_{\epsilon\to 0} \left(\int_{B(0,\eta')} q_\epsilon(y) \left\| f_y \right\|_{\mathcal{C}^r} \,\mathrm{d}\,\mathrm{Leb}(y) + \int_{B(0,\eta)\setminus B(0,\eta')} q_\epsilon(y) \left\| f_y \right\|_{\mathcal{C}^r} \,\mathrm{d}\,\mathrm{Leb}(y) \right) \\
&\quad \le \sup_{y\in B(0,\eta')} \left\| f_y \right\|_{\mathcal{C}^r} + \lim_{\epsilon\to 0} \left(\int_{B(0,\eta)\setminus B(0,\eta')} q_\epsilon(y)\,\mathrm{d}\,\mathrm{Leb}(y) \right) \sup_{y\in B(0,\eta)} \left\| f_y \right\|_{\mathcal{C}^r} \\
&\quad = \sup_{y\in B(0,\eta')} \left\| f_y \right\|_{\mathcal{C}^r}.
\end{aligned} \tag{5.12}$$

Since $\lim_{y\to 0} \left\| f_y \right\|_{\mathcal{C}^r} = 0$, by letting $\eta' \to 0$ in (5.12) we can conclude that the left side of (5.12) is 0. Recalling that $T_y \in U$ for every $y \in \operatorname{supp} q_\epsilon s_\eta$, it follows that $\sup_{y\in\operatorname{supp} q_\epsilon s_\eta} \left\| \mathcal{L}_{T_y} \right\|_{p,q} < \infty$. Hence

$$\lim_{\epsilon\to 0} \left\| F_\epsilon \right\|_{p,q} \le \lim_{\epsilon\to 0} \sup_{y\in\operatorname{supp} q_\epsilon s_\eta} \left\| \mathcal{L}_{T_y} \right\|_{p,q} \int a_{\epsilon,\eta}^{-1} q_\epsilon(y) s_\eta(y) \left\| f_y \right\|_{\mathcal{C}^r} \,\mathrm{d}\,\mathrm{Leb}(y) = 0,$$

which proves (5.7) in our setting. The comments made in the sentence following (5.7) apply here too. Hence the arguments in Lemma 5.2.2 apply to $\{A_{\epsilon,\eta}\}_{\epsilon\in[0,\epsilon_1)}$, and so $\{A_{\epsilon,\eta}\}_{\epsilon\in[0,\epsilon_1)}$ satisfies (KL) on $B^{p,q}_{\mathbb{C}}$ with $|\cdot| = \|\cdot\|_{p-1,q+1}$.

We will prove that $\lim_{\epsilon\to 0} B_{\epsilon,\eta} = 0$ in both $L(B^{p,q})$ and $L(B^{p-1,q+1})$, as this readily implies the same for $L(B^{p,q}_{\mathbb{C}})$ and $L(B^{p-1,q+1}_{\mathbb{C}})$. Let $h \in \mathcal{C}^r(\mathbb{T}^m, \mathbb{R})$, $k \le p$ be a non-negative integer, $W \in \Sigma$, $\{v_i\}_{i=1}^k \subseteq \mathcal{V}^r(W)$ with $\|v_i\|_{\mathcal{C}^r} \le 1$, and $\varphi \in$

$\mathcal{C}_0^{q+k}(W,\mathbb{R})$ with $\|\varphi\|_{\mathcal{C}^{q+k}} \le 1$. With $u_{\epsilon,\eta} = q_\epsilon(1 - a_{\epsilon,\eta}^{-1}s_\eta)$, we have

$$\begin{aligned}
&\left| \int_W (v_1 \dots v_k)(u_{\epsilon,\eta} * \mathcal{L}h)(x) \cdot \varphi(x)\, \mathrm{d}\Omega(x) \right| \\
&\qquad\qquad \le \int_{\mathbb{T}^m} |u_{\epsilon,\eta}(y)| \left| \int_W (v_1 \dots v_k)(\tau_{-y}\mathcal{L}h)(x) \cdot \varphi(x)\, \mathrm{d}\Omega(x) \right| \mathrm{d}\,\mathrm{Leb}(y) \\
&\qquad\qquad \le \left(\sup_{y \in \mathbb{T}^m} \|\tau_y \mathcal{L}\|_{p,q} \right) \|h\|_{p,q} \int_{\mathbb{T}^m} |u_{\epsilon,\eta}|\, \mathrm{d}\,\mathrm{Leb}(y).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|B_{\epsilon,\eta}\|_{p,q} &\le \left(\sup_{y \in \mathbb{T}^m} \|\tau_y \mathcal{L}\|_{p,q} \right) \int_{\mathbb{T}^m} |u_{\epsilon,\eta}|\, \mathrm{d}\,\mathrm{Leb}(y) \\
&\le \left(\sup_{y \in \mathbb{T}^m} \|\tau_y \mathcal{L}\|_{p,q} \right) \int q_\epsilon(y) \left|1 - a_{\epsilon,\eta}^{-1} s_\eta(y)\right| \mathrm{d}\,\mathrm{Leb}(y).
\end{aligned} \tag{5.13}$$

Since $\lim_{\epsilon \to 0} a_{\epsilon,\eta} = 1$, using (S3) we have

$$\begin{aligned}
&\lim_{\epsilon \to 0} \int q_\epsilon(y) \left|1 - a_{\epsilon,\eta}^{-1} s_\eta(y)\right| \mathrm{d}\,\mathrm{Leb}(y) \\
&\quad \le \lim_{\epsilon \to 0} \left(\int q_\epsilon(y)(1 - s_\eta(y))\, \mathrm{d}\,\mathrm{Leb}(y) + \int q_\epsilon(y) s_\eta \left|1 - a_{\epsilon,\eta}^{-1}\right| \mathrm{d}\,\mathrm{Leb}(y) \right) \\
&\quad \le \lim_{\epsilon \to 0} \left(\int_{\mathbb{T}^m \setminus B(0,\eta)} q_\epsilon(y)\, \mathrm{d}\,\mathrm{Leb}(y) + \left|1 - a_{\epsilon,\eta}^{-1}\right| \right) = 0.
\end{aligned} \tag{5.14}$$

Together (5.9), (5.13) and (5.14) imply that $\lim_{\epsilon \to 0} \|B_{\epsilon,\eta}\|_{p,q} = 0$. The same argument proves that $\lim_{\epsilon \to 0} \|B_{\epsilon,\eta}\|_{p-1,q+1} = 0$, and so there exists $\epsilon' \in (0, \epsilon_1)$ so that $\sup_{\epsilon \in (0,\epsilon')} \|B_{\epsilon,\eta}\|_{p-1,q+1} < \infty$. We have verified the conditions of Proposition 1.3.1, concluding the proof. □

5.3.2 A class of maps satisfying Proposition 5.3.1, Theorem 5.3.3, and Corollary 5.3.4

Our main result for this section, Proposition 5.3.6, gives conditions for T to have an iterate satisfying (5.9). For instance, we will deduce that Proposition 5.3.6 applies to Anosov maps that are sufficiently close to hyperbolic linear toral automorphisms.

In [54, Section 3] the usual Euclidean metric on $\mathbb{T}^m$ is replaced by an equivalent adapted metric. The choice of adapted metric will be crucial to our arguments in this section, so we begin by reviewing the construction of such metrics, following [21, Proposition 5.2.2]. Let $|\cdot|_{\mathbb{R}^m}$ denote the usual Euclidean norm on the tangent space of $\mathbb{T}^m$. Let $E^s(x)$ and $E^u(x)$ denote the stable and unstable directions, respectively, of T at x, and let π_x^s and π_x^u be the projections induced by the splitting $T_x\mathbb{T}^m = E^s(x) \oplus E^u(x)$. Let $m_s = \dim E^s(x)$ and $m_u = \dim E^u(x)$. Since T is Anosov, there exists $C > 0$, $\lambda_s \in (0,1)$, and $\lambda_u > 1$ so that $\left| D_xT^n|_{E^s(x)} \right|_{\mathbb{R}^m} \leq C\lambda_s^n$ and $\left| D_xT^{-n}|_{E^u(x)} \right|_{\mathbb{R}^m} \leq C\lambda_u^{-n}$ for every $n \in \mathbb{N}$. Recall ν_u and ν_s from Section 5.1, and that $\lambda_s < \nu_s < 1 < \nu_u < \lambda_u$. Let N be such that $\max\{\frac{\lambda_s^{N+1}}{\nu_s^{N+1}}, \frac{\nu_u^{N+1}}{\lambda_u^{N+1}}\} < 1/C$. For $v^s \in E^s(x)$ and $v^u \in E^u(x)$ we define

$$\|v^s\|_0 = \sum_{k=0}^{N} \nu_s^{-k} \left| D_xT^k v^s \right|_{\mathbb{R}^m}, \text{ and } \|v^u\|_0 = \sum_{k=0}^{N} \nu_u^{k} \left| D_xT^{-k} v^u \right|_{\mathbb{R}^m}.$$

Note that $\left\| D_xT|_{E^s(x)} \right\|_0 < \nu_s$ and $\left\| D_xT^{-1}|_{E^u(x)} \right\|_0 < \nu_u^{-1}$. For $v \in T_x\mathbb{T}^m$ we define

$$\|v\|_0 = \sqrt{\|\pi_x^s v\|_0^2 + \|\pi_x^u v\|_0^2},$$

and, since $\|\cdot\|_0$ satisfies the parallelogram law, we may recover a metric $\langle \cdot, \cdot \rangle_0$ via the polarisation identity. Note that $E^s(x) \perp E^u(x)$ with respect to $\langle \cdot, \cdot \rangle_0$. However, since $x \mapsto \langle \cdot, \cdot \rangle_0$ is not necessarily smooth (and so $\mathbb{T}^m$ equipped with $\langle \cdot, \cdot \rangle_0$ would not be a $\mathcal{C}^\infty$ Riemannian manifold), for each sufficiently small $\xi > 0$ we instead consider a smooth metric $\langle \cdot, \cdot \rangle_\xi$ (with corresponding norm denoted $\|\cdot\|_\xi$) such that

(M1) $\sup_{x \in \mathbb{T}^m} \sup_{\substack{v,w \in T_x\mathbb{T}^m \\ \|v\|_0, \|w\|_0 \leq 1}} |\langle v, w \rangle_\xi - \langle v, w \rangle_0| < \xi$;

(M2) $\left\| D_xT|_{E^s(x)} \right\|_\xi < \nu_s$ and $\left\| D_xT^{-1}|_{E^u(x)} \right\|_\xi < \nu_u^{-1}$; and

(M3) $E^s(x)$ and $E^u(x)$ are ξ-orthogonal: for $w_s \in E^s(x)$ and $w_u \in E^u(x)$ with $\|w_s\|_\xi, \|w_u\|_\xi \leq 1$ we have $|\langle w_s, w_u \rangle_\xi| < \xi$.

A metric is called adapted if it satisfies (M2). For sufficiently small ξ, metrics satisfying (M1)-(M3) can be constructed by approximating $\langle \cdot, \cdot \rangle_0$ (see e.g. [59]).

Let $\xi \geq 0$. For $x \in \mathbb{T}^m$ we denote by Γ_x^s and Γ_x^u the orthogonal (with respect to $\langle \cdot, \cdot \rangle_\xi$) projections onto $E^s(x)$ and $E^u(x)$, respectively. Although Γ_x^s clearly depends on ξ, we suppress this from our notation. Define

$$C_{\tau,\xi} = \sup_{x,y \in \mathbb{T}^m} \|D_x\tau_y\|_\xi \quad \text{and} \quad \Theta_{T,\xi} = \sup_{x,y \in \mathbb{T}^m} \left\| \Gamma_{x+y}^s D_x\tau_y - (D_x\tau_y)\Gamma_x^s \right\|_\xi. \tag{5.15}$$

Note that both $C_{\tau,\xi}$ and $\Theta_{T,\xi}$ are finite. The key hypothesis for this section's main result is that $C_{\tau,0}^{-1} > \Theta_{T,0}$. Roughly speaking, this condition ensures that translated leaves never lie in the unstable direction of T (recall that leaves are approximately parallel to the stable directions). One way to see this is by computing the quantities

$C_{\tau,0}$ and $\Theta_{T,0}$ when $\|\cdot\|_0$ is the usual Euclidean norm (ignoring the issue of whether the Euclidean norm is adapted). In this case, $C_{\tau,0} = 1$ and $\Theta_{T,0}$ measures the angle between $E^s(x+y)$ and $E^s(x)$. If this angle is everywhere close to 0 then the translate of a leaf will be approximately parallel to the stable direction of T regardless of the translate.

Our main technical result for this section is the following. We note that in our proof we select a specific adapted metric to both streamline our arguments and strengthen our results; although in doing so we impact the definition of the set of leaves Σ, and hence also the spaces $B^{p,q}_{\mathbb{C}}$ and $B^{p-1,q+1}_{\mathbb{C}}$.

Proposition 5.3.6. *If*

$$C_{\tau,0}^{-1} > \Theta_{T,0}, \tag{5.16}$$

then there exists $N \in \mathbb{Z}^+$, an adapted metric $\langle \cdot, \cdot \rangle$, and a set of leaves $\tilde{\Sigma}$, inducing spaces $\tilde{B}^{p,q}_{\mathbb{C}}$ and $\tilde{B}^{p-1,q+1}_{\mathbb{C}}$, so that $\mathcal{L}$ is quasi-compact on $\tilde{B}^{p,q}_{\mathbb{C}}$, with the same spectral data associated to eigenvalues outside of the ball of radius $\max\{\nu_u^{-p}, \nu_s^q\}$ as when considered as an operator on $B^{p,q}_{\mathbb{C}}$, and so that

$$\sup_{y \in \mathbb{T}^m} \max \left\{ \left\| \tau_y \mathcal{L}^N \right\|_{p,q}, \left\| \tau_y \mathcal{L}^N \right\|_{p-1,q+1} \right\} < \infty. \tag{5.17}$$

We make two comments regarding the applicability of Proposition 5.3.6. Firstly, maps satisfying (5.16) exist as $\Theta_{T,0} = 0$ whenever T is a linear hyperbolic toral automorphism. Secondly, the condition (5.16) is open in $\mathcal{C}^{r+1}(\mathbb{T}^m, \mathbb{T}^m)$. To see this, suppose that T satisfies (5.16), and let $\langle \cdot, \cdot \rangle$ be the metric one obtains by applying Proposition 5.3.6 to T. The following comments apply to all T' in a sufficiently small $\mathcal{C}^{r+1}$-neighbourhood of T: T' is Anosov map, $\langle \cdot, \cdot \rangle$ is an adapted metric for T', and the stable and unstable directions for T and T' are everywhere close in the Grassmanian. It follows that T' also satisfies (5.16) provided that it is sufficiently close to T in the $\mathcal{C}^{r+1}$ topology. In Section 5.3.3 we will construct a family of non-linear Anosov diffeomorphisms satisfying 5.16 by following this line of reasoning; in particular, we will consider a family of non-linear perturbations of Arnold's cat map.

The proof of Proposition 5.3.6 occupies the remainder of this section. The idea of the proof is the following: to bound the left side of (5.17) one must control integrals along the translate of a leaf in Σ. Taking large enough powers of $\mathcal{L}$ corresponds to applying powers of T^{-1} to the translated leaf. Since T^{-1} is expansive along the stable direction of T, and contractive along the unstable direction of T, if the translated leaf does not lie anywhere close to the unstable direction of T (e.g. the condition (5.16)) then applying a sufficiently large iterate of T^{-1} will 'pull' the translated leaf towards the stable direction of T so that it may be covered by untranslated leaves. One can then replace the integral over a translated leaf by the sum of integrals over leaves in Σ, which yields the claim. Making this idea rigorous is arduous: we break it into three main steps. In step 1 we begin by constructing the adapted metric $\langle \cdot, \cdot \rangle$. In step 2 we define a set of leaves $\tilde{\Sigma}$, which induce spaces $\tilde{B}^{p,q}_{\mathbb{C}}$ and $\tilde{B}^{p-1,q+1}_{\mathbb{C}}$, and prove the claim in Proposition 5.3.6 regarding the spectral properties of $\mathcal{L}$. Finally, in step 3 we prove that translated leaves may be covered by the image under T^N of finitely many leaves in $\tilde{\Sigma}$ for some large N; this result is the core of the proof, and (5.17) then easily follows from the adaption of arguments from [54]. Steps 2 and 3 lean heavily on the setting in [54, Section 3]. We have maintained the notation used in [54] whenever possible.

Step 1: Constructing the adapted metric $\langle \cdot, \cdot \rangle$. For sufficiently small $\xi \geq 0$, as T is a $\mathcal{C}^{r+1}$ diffeomorphism and $\mathbb{T}^m$ is compact, the quantity $D_\xi := \sup_{x \in \mathbb{T}^m} \|D_x T^{-1}\|_\xi$ is finite. Moreover, as $\|\cdot\|_\xi \to \|\cdot\|_0$ uniformly, it follows that $D_\xi \to D_0$. As $E^s(x)$ and $E^u(x)$ are ξ-orthogonal with respect to $\langle \cdot, \cdot \rangle_\xi$, one easily verifies that $\|\Gamma^s_x \Gamma^u_x\|_\xi = \|\Gamma^u_x \Gamma^s_x\|_\xi < \xi$. What is less obvious, but still true, however, is that

$$\|\Gamma^s_x \Gamma^u_x\|_\xi = \|\Gamma^u_x \Gamma^s_x\|_\xi = \|(\mathrm{Id} - \Gamma^s_x)(\mathrm{Id} - \Gamma^u_x)\|_\xi = \|(\mathrm{Id} - \Gamma^u_x)(\mathrm{Id} - \Gamma^s_x)\|_\xi < \xi.$$

We refer the reader to the proof of Theorem 2 in [23] for details. For $\xi > 0$, $x \in \mathbb{T}^m$ and $\kappa \in (0, 1)$ we define the stable cone by

$$\mathcal{C}_\xi(x, \kappa) = \{u \in T_x\mathbb{T}^m : \|(\mathrm{Id} - \Gamma^s_x)u\|_\xi \leq \kappa \|\Gamma^s_x u\|_\xi\}.$$

The following lemma is classical; we reprove it here to emphasise the quantitative estimate (5.18).

Lemma 5.3.7. *For every $J > 0$ there exists $\xi_J > 0$ so that for every $\xi \in (0, \xi_J)$, $\gamma \in [0, J)$ and $x \in \mathbb{T}^m$ we have*

$$\left(D_x T^{-1}\right) \mathcal{C}_\xi(x, \gamma) \subseteq \mathcal{C}_\xi\left(T^{-1}x, \gamma \nu_s \nu_u^{-1}\right). \tag{5.18}$$

Proof. Let $\xi > 0$ and suppose that $u \in \mathcal{C}_\xi(x,\gamma)$. As $D_xT^{-1}(E^s(x)) = E^s(T^{-1}x)$, it follows that

$$(\mathrm{Id} - \Gamma^s_{T^{-1}(x)})D_xT^{-1}u = (\mathrm{Id} - \Gamma^s_{T^{-1}(x)})D_xT^{-1}(\mathrm{Id} - \Gamma^s_x)u.$$

Consequently,

$$\begin{aligned}\left\|(\mathrm{Id} - \Gamma^s_{T^{-1}(x)})D_xT^{-1}u\right\|_\xi \le{}& \left\|D_xT^{-1}\Gamma^u_x(\mathrm{Id} - \Gamma^s_x)u\right\|_\xi \\ &+ \left\|D_xT^{-1}(\mathrm{Id} - \Gamma^u_x)(\mathrm{Id} - \Gamma^s_x)u\right\|_\xi .\end{aligned}$$

As $\|(\mathrm{Id} - \Gamma^u_x)(\mathrm{Id} - \Gamma^s_x)\|_\xi < \xi$ and $u \in \mathcal{C}_\xi(x,\gamma)$, it follows that

$$\begin{aligned}\left\|(\mathrm{Id} - \Gamma^s_{T^{-1}(x)})D_xT^{-1}u\right\|_\xi &\le \left(\left\|D_xT^{-1}\big|_{E^u(x)}\right\|_\xi + \xi\left\|D_xT^{-1}\right\|_\xi\right)\|(\mathrm{Id} - \Gamma^s_x)u\|_\xi \\ &\le \gamma\left(\left\|D_xT^{-1}\big|_{E^u(x)}\right\|_\xi + \xi\left\|D_xT^{-1}\right\|_\xi\right)\|\Gamma^s_x u\|_\xi .\end{aligned}$$

Similarly,

$$\begin{aligned}\left\|\Gamma^s_{T^{-1}(x)}D_xT^{-1}u\right\|_\xi &\ge \left\|\Gamma^s_{T^{-1}(x)}D_xT^{-1}\Gamma^s_x u\right\|_\xi - \left\|\Gamma^s_{T^{-1}(x)}D_xT^{-1}(\mathrm{Id} - \Gamma^u_x)(\mathrm{Id} - \Gamma^s_x)u\right\|_\xi \\ &\quad - \left\|\Gamma^s_{T^{-1}(x)}\Gamma^u_{T^{-1}(x)}D_xT^{-1}(\mathrm{Id} - \Gamma^s_x)u\right\|_\xi \\ &\ge \left(\left\|D_xT\big|_{E^s(x)}\right\|_\xi^{-1} - 2\xi\gamma\left\|D_xT^{-1}\right\|_\xi\right)\|\Gamma^s_x u\|_\xi .\end{aligned}$$

For sufficiently small ξ we have $\left\|D_xT^{-1}\big|_{E^u(x)}\right\|_\xi < \nu_u^{-1}$ and $\left\|D_xT\big|_{E^s(x)}\right\|_\xi < \nu_s$. Hence, as $\sup_{x\in\mathbb{T}^m}\|D_xT^{-1}\|_\xi = D_\xi \to D_0$, there exists $\xi_J > 0$ so that for every $\xi \in (0,\xi_J)$ we have

$$\left\|D_xT^{-1}\big|_{E^u(x)}\right\|_\xi + \xi\left\|D_xT^{-1}\right\|_\xi \le \nu_u^{-1},$$

and, for all $\gamma < J$,

$$\left\|D_xT\big|_{E^s(x)}\right\|_\xi^{-1} - 2\xi\gamma\left\|D_xT^{-1}\right\|_\xi \ge \nu_s^{-1}.$$

In view of the above, whenever $\xi \in (0,\xi_J)$ we therefore have

$$\left\|(\mathrm{Id} - \Gamma^s_{T^{-1}(x)})D_xT^{-1}u\right\|_\xi \le \gamma\nu_s\nu_u^{-1}\left\|\Gamma^s_{T^{-1}(x)}D_xT^{-1}u\right\|_\xi$$

for every $u \in \mathcal{C}_\xi(\gamma, x)$. Thus $D_x T^{-1}\mathcal{C}_\xi(x,\gamma) \subseteq \mathcal{C}_\xi\left(T^{-1}x, \gamma\nu_s\nu_u^{-1}\right)$ for every $\xi \in (0, \xi_J)$, $x \in \mathbb{T}^m$ and $\gamma < J$, as required. $\square$

We aim to select ξ so that $C_{\tau,\xi}^{-1} > \Theta_{T,\xi}$, and so that we can apply Lemma 5.3.7 for some appropriate J. As the adapted metric $\langle\cdot,\cdot\rangle_\xi$ uniformly approximates $\langle\cdot,\cdot\rangle_0$, we have $\lim_{\xi\to 0} C_{\tau,\xi}^{-1} = C_{\tau,0}^{-1}$. However, upon examining the definition of $\Theta_{T,\xi}$ we observe that the projections Γ_x^s depend on $\langle\cdot,\cdot\rangle_\xi$, and so the behaviour of $\Theta_{T,\xi}$ as $\xi \to 0$ is not clear. We address this now.

Lemma 5.3.8. *We have* $\lim_{\xi\to 0}\Theta_{T,\xi} = \Theta_{T,0}$.

Proof. Let π_x^s and π_x^u be the projections induced by the direct sum $T_x\mathbb{T}^m = E^s(x) \oplus E^u(x)$. We have

$$\|\Gamma_x^s - \pi_x^s\|_\xi = \|(\Gamma_x^s - \pi_x^s)\pi_x^s\|_\xi + \|(\Gamma_x^s - \pi_x^s)\pi_x^u\|_\xi = \|(\Gamma_x^s - \mathrm{Id})\pi_x^s\|_\xi + \|\Gamma_x^s\pi_x^u\|_\xi .$$

Since $\Gamma_x^s = \mathrm{Id}$ on $E^s(x)$, $\|(\Gamma_x^s - \mathrm{Id})\pi_x^s\|_\xi = 0$. Let $v \in T_x\mathbb{T}^m$. As $E^s(x)$ and $E^u(x)$ are ξ-orthogonal and Γ_x^s is an orthogonal projection (both with respect to $\langle\cdot,\cdot\rangle_\xi$), we have

$$\|\Gamma_x^s\pi_x^u v\|_\xi = \sqrt{|\langle \Gamma_x^s\pi_x^u v, \pi_x^u v\rangle_\xi|} \le \sqrt{\xi\, \|\Gamma_x^s\pi_x^u v\|_\xi \,\|\pi_x^u v\|_\xi} \le \sqrt{\xi}\,\|\pi_x^u\|_\xi .$$

Thus $\|\Gamma_x^s - \pi_x^s\|_\xi \le \sqrt{\xi}\,\|\pi_x^u\|_\xi$. Let $P_\xi = \sup_{x\in\mathbb{T}^m}\|\pi_x^u\|_\xi$. For any $x, y \in \mathbb{T}^m$ the triangle inequality yields

$$\begin{aligned}\left|\left\|\Gamma_{x+y}^s D_x\tau_y - (D_x\tau_y)\Gamma_x^s\right\|_\xi - \left\|\pi_{x+y}^s D_x\tau_y - (D_x\tau_y)\pi_x^s\right\|_\xi\right| &\le \left\|(\Gamma_{x+y}^s - \pi_{x+y}^s)(D_x\tau_y)\right\|_\xi \\ &\quad + \|(D_x\tau_y)(\Gamma_x^s - \pi_x^s)\|_\xi \\ &\le 2\sqrt{\xi}C_{\tau,\xi}P_\xi.\end{aligned}$$

It follows that

$$\left|\Theta_{T,\xi} - \sup_{x,y\in\mathbb{T}^m}\left\|\pi_{x+y}^s D_x\tau_y - (D_x\tau_y)\pi_x^s\right\|_\xi\right| \le 2\sqrt{\xi}C_{\tau,\xi}P_\xi. \tag{5.19}$$

As π_x^u is independent of ξ, we have $P_\xi \to P_0$. Since $E^s(x) \perp E^u(x)$ with respect to $\langle\cdot,\cdot\rangle_0$, the projections π_x^u and π_x^s are orthogonal with respect to $\langle\cdot,\cdot\rangle_0$. Thus the uniform convergence of $\langle \cdot, \cdot \rangle_\xi$ to $\langle \cdot, \cdot \rangle_0$ implies that

$$\lim_{\xi \to 0} \sup_{x,y \in \mathbb{T}^m} \left\| \pi^s_{x+y} D_x \tau_y - (D_x \tau_y) \pi^s_x \right\|_\xi = \sup_{x,y \in \mathbb{T}^m} \left\| \pi^s_{x+y} D_x \tau_y - (D_x \tau_y) \pi^s_x \right\|_0 = \Theta_{T,0}. \tag{5.20}$$

Hence, as $C_{\tau,\xi} \to C_{\tau,0}$, by letting $\xi \to 0$ in (5.19) and applying (5.20) we have

$$\lim_{\xi \to 0} \Theta_{T,\xi} = \lim_{\xi \to 0} \sup_{x,y \in \mathbb{T}^m} \left\| \pi^s_{x+y} D_x \tau_y - (D_x \tau_y) \pi^s_x \right\|_\xi = \Theta_{T,0},$$

as required. $\square$

We now fix, once and for all, the metric that is used in Proposition 5.3.6. As $C_{\tau,0}^{-1} > \Theta_{T,0}$, $C_{\tau,\xi} \to C_{\tau,0}$, and, by Lemma 5.3.8, $\Theta_{T,\xi} \to \Theta_{T,0}$, there exists $\mathcal{E} > 0$ so that

$$\inf_{\xi \in [0,\mathcal{E}]} \left(C_{\tau,\xi}^{-1} - \Theta_{T,\xi} \right) > 0, \quad \text{and} \quad \sup_{\xi \in [0,\mathcal{E}]} \Theta_{T,\xi} < \infty. \tag{5.21}$$

We apply Lemma 5.3.7 with

$$J = 1 + \frac{\sup_{\xi \in [0,\mathcal{E}]} \Theta_{T,\xi}}{\inf_{\xi \in [0,\mathcal{E}]} \left(C_{\tau,\xi}^{-1} - \Theta_{T,\xi} \right)} \tag{5.22}$$

to produce an adapted metric $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\mathcal{E}}$, which may replace the metric defined in [54, Section 3] after possibly shrinking $\mathcal{E}$ further. Until the end of this section we only deal with the metric just constructed, and so we drop references to ξ and $\mathcal{E}$ from our notation.

Step 2: Defining the set of leaves $\tilde{\Sigma}$ and the spaces $\tilde{B}^{p,q}_{\mathbb{C}}$ and $\tilde{B}^{p-1,q+1}_{\mathbb{C}}$. Our task is now to define a set of leaves $\tilde{\Sigma}$ and spaces $\tilde{B}^{p,q}_{\mathbb{C}}$ and $\tilde{B}^{p-1,q+1}_{\mathbb{C}}$ so that the spectral properties of $\mathcal{L}$ on $B^{p,q}_{\mathbb{C}}$ and $\tilde{B}^{p,q}_{\mathbb{C}}$ are identical. It is necessary to understand how leaves are defined; to this end we reproduce material from the beginning of [54, Section 3]. After fixing the metric, in [54] a small $\kappa > 0$ satisfying various properties is fixed; in particular it is required that $D_x T^{-1}$ expands the vectors in $\mathcal{C}(x, \kappa)$ by at least ν_s^{-1}. We now choose a smaller value of κ, as follows. Let $C_\tau = C_{\tau,\mathcal{E}}$ and $\Theta_T = \Theta_{T,\mathcal{E}}$. By (5.21) and (5.22), there exists $\kappa' \leq \min\{\kappa, 1/2\}$ and $\eta > 1$ such that

$$C_\tau^{-1} > \Theta_T(1 + 2\kappa'\eta), \quad \text{and} \quad \frac{\Theta_T + (C_\tau + \Theta_T)2\kappa'\eta}{C_\tau^{-1} - \Theta_T(1 + 2\kappa'\eta)} < J. \tag{5.23}$$

We redefine the original κ to be κ', noting that this does not alter the validity of any arguments in [54] (in [54] it is only requires that κ is sufficiently small, so we are free to make it as small as we require). Two components of this construction appear, at first glance, arbitrary: the constant η and the inequalities (5.23). They appear so that later we may cover translated leaves by the image under some iterate of T by finitely many leaves (Lemma 5.3.10).

As in [54], one may construct finitely many $\mathcal{C}^\infty$ charts $\{\psi_i\}_{i=1}^S$, each respectively defined on $(-r_i, r_i)^m \subseteq \mathbb{R}^m$, so that

(B1) $D_0\psi_i$ is an isometry;
(B2) $(D_0\psi_i)(\mathbb{R}^{m_s} \times \{0\}) = E^s(\psi_i(0))$;
(B3) The $\mathcal{C}^{r+1}$ norms of ψ_i and ψ_i^{-1} are bounded by $1+\kappa$;
(B4) There exists $c_i \in (\kappa, 2\kappa)$ such that the cone

$$\mathcal{C}_i = \{u + v \in \mathbb{R}^m \,|\, u \in \mathbb{R}^{m_s} \times \{0\}, v \in \{0\} \times \mathbb{R}^{m_u}, |v|_{\mathbb{R}^m} \leq c_i \, |u|_{\mathbb{R}^m}\} \quad (5.24)$$

satisfies the following property: for any $x \in (-r_i, r_i)^m$, $\mathcal{C}(\psi_i(x)) \subseteq (D_x\psi_i)\mathcal{C}_i$ and $(D_{\psi_i(x)}T^{-1})(D_x\psi_i)\mathcal{C}_i \subseteq \mathcal{C}(T^{-1}(\psi_i(x)))$; and
(B5) $\mathbb{T}^m$ is covered by $\{\psi_i((-r_i/2, r_i/2)^m)\}_{i=1}^S$.

We require that the charts satisfy the following additional property concerning the distortion of the cones $\mathcal{C}_i$ under $D_x\psi$.

Lemma 5.3.9. *There exist charts $\{\psi_i\}_{i=1}^S$ satisfying (B1)-(B5) and so that for any $x \in (-r_i, r_i)^m$ we have*

$$(D_x\psi_i)\mathcal{C}_i \subseteq \mathcal{C}\left(\psi_i(x), \eta c_i\right),$$

where $\eta > 1$ is the constant appearing in (5.23).

Proof. By compactness it is sufficient to construct for each $y \in \mathbb{T}^m$ a chart $\psi_y : (-r_y, r_y)^m \to \mathbb{T}^m$ that satisfies all of the given requirements and for which $\psi_y(0) = y$. As noted immediately before the statement of [54, Lemma 3.1], for each $y \in \mathbb{T}^m$ one can construct a ψ_y satisfying conditions (B1)-(B5) and so that $\psi_y(0) = y$. Let $\mathcal{C}_y$ denote the corresponding stable cone (from (5.24)) and $c_y \in (\kappa, 2\kappa)$ denote the constant corresponding to c_i. Let $u + v \in \mathcal{C}_y$ so that $u \in \mathbb{R}^{m_s} \times \{0\}$ and $v \in \{0\} \times \mathbb{R}^{m_u}$. Then by (B1) and (B4) we have

$$\left\|(\mathrm{Id} - \Gamma_y^s)D_0\psi_y(u+v)\right\| = \left\|(\mathrm{Id} - \Gamma_y^s)D_0\psi_y v\right\| \leq |v|_{\mathbb{R}^m} \leq c_y \, |u|_{\mathbb{R}^m},$$

and

$$\left\|\Gamma_y^s D_0\psi_y(u+v)\right\| = \left\|\Gamma_y^s D_0\psi_y u\right\| = |u|_{\mathbb{R}^m}.$$

Hence $(D_0\psi_i)\mathcal{C}_y \subseteq \mathcal{C}(y, c_y)$. Hence, using the compactness of the closed unit ball in $\mathbb{R}^m$ and the uniform continuity of ψ_y on $(-r_y, r_y)^m$, $x \mapsto \Gamma_x^s$ on $\mathbb{T}^m$, $u \mapsto \|u\|$ on the tangent space of $\mathbb{T}^m$, $u \mapsto |u|_{\mathbb{R}^m}$ on $(-r_y, r_y)^m$, we may shrink r_y so that $(D_x\psi_i)\mathcal{C}_y \subseteq \mathcal{C}(\psi_y(x), \eta c_y)$ for every $x \in (-r_y, r_y)^m$. Thus ψ_y satisfies all the requirements of the lemma, and we may conclude using the compactness of $\mathbb{T}^m$. □

We are now able to construct our modified set of leaves $\tilde{\Sigma}$ and spaces $\tilde{B}_{\mathbb{C}}^{p,q}$ and $\tilde{B}_{\mathbb{C}}^{p-1,q+1}$, and prove the claim in Proposition 5.3.6 regarding the spectral properties of $\mathcal{L}$. Using the metric $\langle \cdot, \cdot \rangle$ constructed in step 1 of the proof of Proposition 5.3.6, the constant κ defined immediately following (5.23), and the charts from Lemma 5.3.9, we may define the set of leaves $\tilde{\Sigma}$ exactly as in [54, Section 3]. Recall that $p \in \mathbb{Z}^+$ and $q > 0$ satisfy $p + q < r$. In exactly the same way that the set of leaves Σ from [54] induces spaces $B_{\mathbb{C}}^{p,q}$ and $B_{\mathbb{C}}^{p-1,q+1}$ (see [54, Section 3]), our set of leaves $\tilde{\Sigma}$ induces spaces $\tilde{B}_{\mathbb{C}}^{p,q}$ and $\tilde{B}_{\mathbb{C}}^{p-1,q+1}$. The proofs of [54, Lemma 2.2] and [54, Theorem 2.3] hold verbatim for $\mathcal{L}$ on $\tilde{B}_{\mathbb{C}}^{p,q}$. Thus the essential spectral radius of $\mathcal{L}$ on $\tilde{B}_{\mathbb{C}}^{p,q}$ is bounded by $\max\{\nu_u^{-p}, \nu_s^q\}$. As per [54, Remark 2.5], the spectral data of $\mathcal{L}$ associated to eigenvalues outside of the ball of radius $\max\{\nu_u^{-p}, \nu_s^q\}$ are the same on $B_{\mathbb{C}}^{p,q}$ and $\tilde{B}_{\mathbb{C}}^{p,q}$, and in particular the generalised eigenspaces of all such eigenvalues lie in $B_{\mathbb{C}}^{p,q} \cap \tilde{B}_{\mathbb{C}}^{p,q}$.

Step 3: Obtaining the inequality (5.17). Key to establishing (5.17) is the following lemma, which extends [54, Lemma 3.3] to include translated leaves. Throughout this step of the proof we assume the reader is familiar with the definition of the set of leaves $\tilde{\Sigma}$ from [54, Section 3].

Lemma 5.3.10. *There exists $N, M, C > 0$ such that for any $y \in \mathbb{T}^m$ and $W \in \tilde{\Sigma}$, with associated full admissible leaf $\tilde{W}$, there exists $\{W_i\}_{i=1}^m \subseteq \tilde{\Sigma}$ with $m < M$ so that*

1. $T^{-N}(W + y) \subseteq \bigcup_{i=1}^m W_i \subseteq T^{-N}(\tilde{W} + y)$.
2. *There are $\mathcal{C}^{r+1}$ functions $\{\rho_i\}_{i=1}^m$ so that each ρ_i is compactly supported on W_i, $\sum_i \rho_i = 1$ on $T^{-N}(W + y)$, and $\|\rho_i\|_{\mathcal{C}^{r+1}} \leq C$.*

We will now give a brief, non-technical overview of the strategy for proving Lemma 5.3.10. Most of the proof is dedicated to finding leaves $\{W_i\}_{i=1}^m$ which verify the containment

$$T^{-N}(W + y) \subseteq \bigcup_{i=1}^m W_i \subseteq T^{-N}(\tilde{W} + y).$$

Let us consider the case where $y = 0$, which is the subject of [54, Lemma 3.3]. In this case the idea is to use the expansion and regularisation of T^{-1} in the stable direction to prove that $T^{-1}(\tilde{W})$ is locally 'leaf-like'. One then picks certain subsets of $T^{-1}(\tilde{W})$, and proves they are leaves that cover $T^{-1}(W)$. The main issue resulting from translation by a non-zero y is the possibility that the translated leaf will not be 'leaf-like'. Specifically, while all leaves in $\tilde{\Sigma}$ lie approximately parallel to the stable manifold, the translate of a leaf may well lie very near an unstable manifold. Our main hypothesis, the inequality (5.15), will imply that translated leaves do not lie too close to the unstable manifold, which allows for the aforementioned distortion to be corrected using the regularisation of T^{-n} for some large n.

We require some preliminary lemmas before proving Lemma 5.3.10. The following gives an estimate of the distortion that stable cones experience under translation, and is where it is crucial that $C_\tau \Theta_T < 1$.

Lemma 5.3.11. *If $\gamma > 0$ satisfies $C_\tau^{-1} > \Theta_T(1+\gamma)$ then for each $x, y \in \mathbb{T}^m$ we have*

$$(D_x\tau_y)\mathcal{C}(x,\gamma) \subseteq \mathcal{C}\left(x+y, \frac{C_\tau\gamma + \Theta_T(1+\gamma)}{C_\tau^{-1} - \Theta_T(1+\gamma)}\right).$$

Proof. Suppose that $u \in \mathcal{C}(x,\gamma)$. We have

$$\left\|(\mathrm{Id} - \Gamma^s_{x+y})(D_x\tau_y)u\right\| \leq \left\|(\mathrm{Id} - \Gamma^s_{x+y})(D_x\tau_y)\Gamma^s_x u\right\| + \left\|(\mathrm{Id} - \Gamma^s_{x+y})(D_x\tau_y)(\mathrm{Id} - \Gamma^s_x)u\right\|. \tag{5.25}$$

We begin by estimating the first term on the right side of (5.25). By the triangle inequality we have

$$\begin{aligned}\left\|(\mathrm{Id} - \Gamma^s_{x+y})(D_x\tau_y)\Gamma^s_x u\right\| \leq &\left\|(\mathrm{Id} - \Gamma^s_{x+y})\Gamma^s_{x+y}(D_x\tau_y)u\right\| \\ &+ \left\|(\mathrm{Id} - \Gamma^s_{x+y})(\Gamma^s_{x+y}D_x\tau_y - D_x\tau_y\Gamma^s_x)u\right\|.\end{aligned}$$

The first term on the right side is 0, whereas the second term may be estimated using the definition of Θ_T (see (5.15)), yielding

$$\left\|(\mathrm{Id} - \Gamma^s_{x+y})(D_x\tau_y)\Gamma^s_x u\right\| \leq \Theta_T \|u\|.$$

As $u \in \mathcal{C}(x,\gamma)$, we have

$$\left\|(\mathrm{Id} - \Gamma^s_{x+y})(D_x\tau_y)\Gamma^s_x u\right\| \leq \Theta_T(1+\gamma)\|\Gamma^s_x u\|. \tag{5.26}$$

We turn to estimating the second term on the right side of (5.25). Using the definition of C_τ and as $u \in \mathcal{C}(x,\gamma)$, we have

$$\left\|(\mathrm{Id} - \Gamma^s_{x+y}) D_x\tau_y (\mathrm{Id} - \Gamma^s_x) u\right\| \leq \|D_x\tau_y\| \, \|(\mathrm{Id} - \Gamma^s_x) u\| \leq C_\tau \gamma \, \|\Gamma^s_x u\| \, . \tag{5.27}$$

Applying (5.26) and (5.27) to (5.25) yields

$$\left\|(\mathrm{Id} - \Gamma^s_{x+y})(D_x\tau_y) u\right\| \leq (\Theta_T(1+\gamma) + C_\tau\gamma) \, \|\Gamma^s_x u\| \, . \tag{5.28}$$

Alternatively, the reverse triangle inequality yields

$$\left\|\Gamma^s_{x+y} D_x\tau_y u\right\| \geq \|D_x\tau_y\Gamma^s_x u\| - \left\|(\Gamma^s_{x+y} D_x\tau_y - D_x\tau_y\Gamma^s_x) u\right\|$$

Using a similar process as in the estimation of (5.28), we obtain

$$\left\|\Gamma^s_{x+y}(D_x\tau_y) u\right\| \geq \|D_x\tau_y\Gamma^s_x u\| - \Theta_T \, \|u\| \geq (C_\tau^{-1} - \Theta_T(1+\gamma)) \, \|\Gamma^s_x u\| \, . \tag{5.29}$$

By our assumptions $C_\tau^{-1} > \Theta_T(1+\gamma)$, and so we may combine (5.28) and (5.29) to obtain

$$\left\|(\mathrm{Id} - \Gamma^s_{x+y})(D_x\tau_y) u\right\| \leq \frac{C_\tau\gamma + \Theta_T(1+\gamma)}{C_\tau^{-1} - \Theta_T(1+\gamma)} \left\|\Gamma^s_{x+y} D_x\tau_y u\right\| ,$$

as required. □

In the following lemma we show that the distortion of the stable cones experience under translation may be corrected by applying T^{-n} for n large.

Lemma 5.3.12. *Recall the cones $\mathcal{C}_i$ from (B4). There exists $N_1 > 0$ such that for every $x, y \in \mathbb{T}^m$, where $x \in \psi_i\left((-r_i, r_i)^m\right)$, we have*

$$(D_{x+y}T^{-N_1})(D_x\tau_y)(D_{\psi_i^{-1}(x)}\psi_i)\mathcal{C}_i \subseteq \mathcal{C}\left(T^{-N_1}(x+y), \kappa\right) .$$

Moreover, if $T^{-N_1}(x+y) \in \psi_j\left((-r_j, r_j)^m\right)$ then

$$(D_{T^{-N_1}(x+y)}\psi_j^{-1})(D_{x+y}T^{-N_1})(D_x\tau_y)(D_{\psi_i^{-1}(x)}\psi_i)\mathcal{C}_i \subseteq \mathcal{C}_j .$$

Proof. Recall from (B4) that $c_i \leq 2\kappa$. Lemma 5.3.9 implies that $(D_{\psi_i^{-1}(x)}\psi_i)\mathcal{C}_i \subseteq \mathcal{C}\left(x, 2\eta\kappa\right)$. By (5.23) we have $C_\tau^{-1} > \Theta_T(1+2\eta\kappa)$, and so Lemma 5.3.11 yields

$$(D_x\tau_y)(D_{\psi_i^{-1}(x)}\psi_i)\mathcal{C}_i \subseteq \mathcal{C}\left(x+y, \frac{\Theta_T + (C_\tau + \Theta_T)2\kappa\eta}{C_\tau^{-1} - \Theta_T - \Theta_T 2\kappa\eta}\right) .$$

Let $N_1 \in \mathbb{Z}^+$ be large enough so that

$$\nu_s^{N_1} \nu_u^{-N_1} \frac{\Theta_T + (C_\tau + \Theta_T) 2\kappa\eta}{C_\tau^{-1} - \Theta_T - \Theta_T 2\kappa\eta} \leq \kappa.$$

By the definition of the adapted metric $\langle \cdot, \cdot \rangle$ from step 1 of the proof of Proposition 5.3.6, the second inequality in (5.23), and Lemma 5.3.7, it follows that

$$(D_{x+y} T^{-N_1})(D_x \tau_y)(D_{\psi_i^{-1}(x)} \psi_i) \mathcal{C}_i \subseteq \mathcal{C}\left(T^{-N_1}(x+y), \kappa\right).$$

Since all our estimates are uniform in x and y we obtain the first claim. The second claim follows from (B4). □

Recall that T^{-1} is expansive along leaves in $\tilde{\Sigma}$, since all leaves are approximately parallel to the stable direction of T. In the proof of Lemma 5.3.10 we will require that a version of this property holds for translated leaves as well. Up until now we have considered how translation affects the stable cones, and how applying T^{-1} corrects for any distortion in the cones. In the following lemma we apply the same idea to show that T^{-n} is expansive along translated leaves provided that n is sufficiently large.

Lemma 5.3.13. *Let N_1 be as in Lemma 5.3.12 and set $H = \inf_{x \in \mathbb{T}^m} \|D_x T\|$. If $n > N_1$ and $\nu_s^{-n+N_1} H^{-N_1} > 1$ then for any $W \in \tilde{\Sigma}$ and $y \in \mathbb{T}^m$ the map T^{-n} expands distances on $\tilde{W} + y$ by at least $\nu_s^{-n+N_1} H^{-N_1}$.*

Proof. Let ψ_i be a chart whose image contains $\tilde{W}$ and for which the tangent space of $\psi_i^{-1}(\tilde{W})$ is contained in $\mathcal{C}_i$. Suppose $a, b \in \tilde{W} + y$ and that $\gamma : [0,1] \to T^{-n}(\tilde{W} + y)$ is a distance minimizing geodesic from $T^{-n}(a)$ to $T^{-n}(b)$. Define $\gamma_n := T^n \circ \gamma$ and note that γ_n is a differentiable curve from a to b lying in $\tilde{W} + y$. For $n > N_1$ we have

$$\begin{aligned} d_{T^{-n}(\tilde{W}+y)}(T^{-n}(a), T^{-n}(b)) &= \int_0^1 \|D_t \gamma\| \, \mathrm{d}t \\ &= \int_0^1 \left\| (D_{(T^{-N_1} \circ \gamma_n)(t)} T^{-n+N_1})(D_{\gamma_n(t)} T^{-N_1})(D_t \gamma_n) \right\| \, \mathrm{d}t, \end{aligned} \tag{5.30}$$

where N_1 is the constant from Lemma 5.3.12. Since the image of γ_n is a closed submanifold of $\tilde{W} + y$ and the tangent space of $\tilde{W}$ at w is contained in $(D_{\psi^{-1}(w)} \psi) \mathcal{C}_i$, the image of $D_t \gamma_n$ is contained in $(D_{\gamma_n(t) - y} \tau_y)(D_{(\psi_i^{-1} \circ \tau_{-y} \circ \gamma_n)(t)} \psi_i) \mathcal{C}_i$. Thus, by Lemma

5.3.12 we have $(D_{\gamma_n(t)}T^{-N_1})(D_t\gamma_n) \subseteq \mathcal{C}(T^{-N_1}(\gamma_n(t)), \kappa)$. As DT^{-n} expands vectors in stable cones by at least ν_s^{-n} we may bound (5.30) as follows

$$\begin{aligned} d_{T^{-n}(\tilde{W}+y)}(T^{-n}(a), T^{-n}(b)) &\geq \nu_s^{-n+N_1} \int_0^1 \left\|(D_{\gamma_n(t)}T^{-N_1})(D_t\gamma_n)\right\| \, \mathrm{d}t \\ &\geq \nu_s^{-n+N_1} H^{-N_1} \int_0^1 \|D_t\gamma_n\| \, \mathrm{d}t \\ &\geq \nu_s^{-n+N_1} H^{-N_1} d_{\tilde{W}+y}(a,b). \end{aligned}$$

Hence T^{-n} expands distances in $\tilde{W}+y$ by a factor of at least $\nu_s^{-n+N_1}H^{-N_1}$ provided that $n > N_1$ and $\nu_s^{-n+N_1}H^{-N_1} > 1$. □

The following lemma quantifies the regularisation that leaves experience under T^{-1}, and is a strengthening of [54, Lemma 3.1]. Whereas the previous results were concerned with the regularisation of the first derivative of the leaves (via the contraction of stable cones), the forthcoming result concerns the regularisation of the higher derivatives of leaves.

Lemma 5.3.14. *For $L > 0$ and $i \in \{1, \ldots, S\}$ let $G_i(L)$ be the set defined immediately before [54, Lemma 3.1], and let*

$$R(L) := \inf \left\{ L' : \begin{array}{c} (\psi_j^{-1} \circ T^{-1} \circ \psi_i)(W) \in G_j(L') \\ \text{for every } W \in G_i(L) \text{ and } 1 \leq i,j \leq S \end{array} \right\}.$$

For every K sufficiently large the following holds: after possibly refining the charts $\{\psi_i\}_{i=1}^S$ from Lemma 5.3.9, for each $L > 0$ there exists $N(L) \in \mathbb{Z}^+$ so that for each $n \geq N(L)$ we have $R^n(L) \leq K$.

Proof. The finiteness of $R(L)$ and the fact that $R(L) < L$ for L sufficiently large follow from [54, Lemma 3.1]. Our more general claim is a classical consequence of the uniform hyperbolicity of T and the regularisation of the associated graph transform, so we will only sketch the ingredients of the proof.

Suppose that $W \in G_i(L)$ is the graph of $\chi : \overline{B(x, A\delta)} \to (-r_i, r_i)^{m_u}$. As outlined at the beginning of [66, Section 6.4.b], using the exponential map, [66, Lemma 6.2.7] and after possibly refining the set of charts $\{\psi_i\}_{i=1}^S$ so that each r_i is sufficiently small, one may apply the arguments from [66, Theorem 6.2.8] to conclude that $(\psi_j^{-1} \circ T^{-1} \circ \psi_i)(W)$ is the graph of some map $\chi' : U \subseteq (-r_j, r_j)^{m_s} \to (-r_j, r_j)^{m_u}$ (refer to steps 3 and 4 of the proof of [66, Theorem 6.2.8] for context, and to step 5 for the relevant argument). Due to the uniform convergence of the graph transform as outlined in step 5 of the proof of [66, Theorem 6.2.8], there exists some $L' < L$ so that $\|\chi'\|_{\mathcal{C}^{r+1}} \leq L'$ for every such χ provided that L is sufficiently large (i.e. bigger than K). Hence $R(L)$ exists and satisfies $R(L) < L$ for L large enough. Further examining the proof of [66, Theorem 6.2.8] yields the stronger claim that for sufficiently large K we have

$$\sup_{L \geq K} \frac{R(L)}{L} < 1,$$

which immediately yields the required claim. $\square$

Before we prove Lemma 5.3.10 we recall a quantitative version of the inverse function theorem.

Lemma 5.3.15 ([73, XIV §1 Lemma 1.3]). *Let E be a Banach space, $U \subseteq E$ be open, and $f \in \mathcal{C}^1(U, E)$. Assume $f(0) = 0$ and $f'(0) = \mathrm{Id}$. Let $r \geq 0$ and assume that $\overline{B(0,r)} \subseteq U$. Let $s \in (0,1)$, and assume that*

$$\|f'(z) - f'(x)\| \leq s$$

for every $z, x \in \overline{B(0,r)}$. If $y \in E$ and $\|y\| \leq (1-s)r$, then there exists a unique $x \in \overline{B(0,r)}$ such that $f(x) = y$.

The proof of Lemma 5.3.10. Let A be the constant in [54, equation (3.1)], and let δ be the constant defined immediately afterwards. Let $W \in \tilde{\Sigma}$. Denote by $\tilde{W}$ the associated full admissible leaf, and by $\chi : \overline{B(x, A\delta)} \to (-2r_i/3, 2r_i/3)^{m_u}$ the map defining $\tilde{W}$ i.e. $\tilde{W} = \psi_i \circ (\mathrm{Id}, \chi)(\overline{B(x, A\delta)})$. Fix $z \in B(x, \delta)$ and note that $B(z, (A-1)\delta) \subseteq B(x, A\delta)$. For any $n \in \mathbb{Z}^+$ and $y \in \mathbb{T}^m$ let $\ell(n, y)$ be an index for which $T^{-n}(\psi_i(z, \chi(z)) + y) \in \psi_{\ell(n,y)}\left((-r_{\ell(n,y)}/2, r_{\ell(n,y)}/2)^m\right)$ (recall (B5)). Let $\pi^s : \mathbb{R}^m \to \mathbb{R}^{m_s}$ be the projection onto the first m_s components, and $\pi^u : \mathbb{R}^m \to \mathbb{R}^{m_u}$ the projection onto the last m_u components. Note that

$$(\psi_{\ell(N,y)}^{-1} \circ T^{-N} \circ \tau_y \circ \psi_i \circ (\mathrm{Id}, \chi))(B(x, A\delta))$$

is the union of finitely many disjoint, path-connected subsets; let $Q_N \subseteq B(x, A\delta)$ denote the pre-image under $\psi_{\ell(N,y)}^{-1} \circ T^{-N} \circ \tau_y \circ \psi_i \circ (\mathrm{Id}, \chi)$ of the particular subset containing $(\psi_{\ell(N,y)}^{-1} \circ T^{-N} \circ \tau_y \circ \psi_i \circ (\mathrm{Id}, \chi))(z)$. Define $F_N := \psi_{\ell(N,y)}^{-1} \circ T^{-N} \circ \tau_y \circ \psi_i \circ (\mathrm{Id}, \chi)\big|_{Q_N}$. We will show that for sufficiently large N one can use F_N to construct an admissible leaf W_z so that $\psi_i(z) \in T^n(W_z)$ and $T^n(W_z) \subseteq \tilde{W} + y$.

Step I: The invertibility of $\pi^s \circ F_N$ in a neighbourhood of $(\pi^s \circ F_N)(z)$. Let N_1 be the constant from Lemma 5.3.12 and recall $H = \inf_{x\in\mathbb{T}^m} \|D_x T\|$ from Lemma 5.3.13. As remarked in the proof of [54, Lemma 3.3] both ψ_i^{-1} and $\psi_{\ell(N,y)}$ are $(1+\kappa)$-Lipschitz. Let $N_2 > N_1$ be such that $\nu_s^{-n+N_1} H^{-N_1} > 1$ whenever $n > N_2$. By Lemma 5.3.13, if $n > N_2$ then T^{-n} expands distances on $\tilde{W} + y$ by at least $\nu_s^{-n+N_1} H^{-N_1}$. It is clear that τ_y^{-1} is C_τ-Lipschitz by the definition of C_τ, and that $d((\mathrm{Id},\chi)(a), (\mathrm{Id},\chi)(b)) \geq d(a,b)$ for every $a, b \in \overline{B(z, A\delta)}$. Using the above estimates to bound the Lipschitz constant of F_N^{-1} for $N > N_2$ we obtain

$$d(F_N(a), F_N(b)) \geq \nu_s^{-N+N_1} \left(H^{N_1} C_\tau (1+\kappa)^2\right)^{-1} \quad \forall a, b \in Q_N.$$

As in [54, Lemma 3.3] we have $|\pi^s(v)|_{\mathbb{R}^{m_s}} \geq (1 + c^2_{\ell(N,y)})^{-1/2} |v|_{\mathbb{R}^m}$ whenever $v \in \mathcal{C}_{\ell(N,y)}$. Since $\sup_i c_i < 2\kappa$ and, for $N > N_1$, the tangent space of F_N is contained in $\mathcal{C}_{\ell(N,y)}$, for every $N > N_2$ we have

$$d((\pi_s \circ F_N)(a), (\pi_s \circ F_N)(b)) \geq \frac{\nu_s^{-N+N_1}}{H^{N_1} C_\tau (1+\kappa)^2 \sqrt{1+4\kappa^2}}, \tag{5.31}$$

provided that a, b are sufficiently close. Since $\nu_s < 1$ there exists $N_3 \geq N_2$ so that for each $N \geq N_3$ the map $\pi_s \circ F_N$ locally expands distances by at least

$$\frac{\nu_s^{-N+N_1}}{H^{N_1} C_\tau (1+\kappa)^2 \sqrt{1+4\kappa^2}} > \frac{A}{A-1},$$

from which it follows that $D_w(\pi^s \circ F_N)^{-1}$ exists for every $(\pi^s \circ F_N)(w) \in Q_N$ and satisfies

$$\left\|D_w(\pi^s \circ F_N)^{-1}\right\| \leq \nu_s^{N-N_1} (H^{N_1} C_\tau (1+\kappa)^2 \sqrt{1+4\kappa^2}) < \frac{A-1}{A}. \tag{5.32}$$

We will now obtain a lower bound on the size of Q_N. Note that the tangent space of χ being a subset of $\mathcal{C}_i$ implies that that (Id,χ) is $(\sqrt{1+\kappa^2})$-Lipschitz. Let $P = \sup_{x\in\mathbb{T}^m} \|D_x T^{-1}\|$. From these estimates, as well as those in the previous paragraph, we may conclude that F_N is $(P^N C_\tau (1+\kappa)^2 \sqrt{1+\kappa^2})$-Lipschitz. Let

$$L_N := \min\left\{(A-1)\delta, (P^N C_\tau (1+\kappa)^2 \sqrt{1+\kappa^2})^{-1} \min_j r_j/2\right\}.$$

We will prove that $B(z, L_N) \subseteq Q_N$. If $w \in (-r_i, r_i)^{m_s} \cap \overline{B(z, L_N)}$ then $w \in \overline{B(x, A\delta)}$ i.e. w is in the domain of χ. Moreover,

$$\begin{aligned} d((T^{-N} \circ \tau_y \circ \psi_i \circ (\mathrm{Id}, \chi))(w), (T^{-N} \circ \tau_y \circ \psi_i \circ (\mathrm{Id}, \chi))(z)) \\ \leq P^N C_\tau (1+\kappa)\sqrt{1+\kappa^2} \, \|w - z\| \, . \end{aligned} \tag{5.33}$$

Since $F_N(z) \in \left(-r_{\ell(N,y)}/2, r_{\ell(N,y)}/2\right)^m$ it follows that $\psi_{\ell(N,y)}$ is well defined on $B\left(F_N(z), \min_j \frac{r_j}{2}\right)$. Since $\psi_{\ell(N,y)}$ is $(1+\kappa)$-Lipschitz and a bijection (onto its range), we have

$$\psi_{\ell(N,y)}\left(B\left(F_N(z), \min_j \frac{r_j}{2}\right)\right) \supseteq B\left((T^{-N} \circ \tau_y \circ \psi_i \circ (\mathrm{Id}, \chi))(z), \min_j \frac{r_j}{2(1+\kappa)}\right). \tag{5.34}$$

From (5.33) and (5.34) we deduce that if $w \in B(z, L_N)$ then $F_N(w)$ is defined and in $(-r_{\ell(N,y)}, r_{\ell(N,y)})^{m_s}$. Moreover, $F_n(B(z, L_N))$ is path-connected, being the image of a path-connected set under a continuous function, and so $B(z, L_N) \subseteq Q_N$. Let $S_N : B(0, L_N) \to \mathbb{R}^{m_s}$ be defined by

$$S_N(w) = (D_z(\pi^s \circ F_N))^{-1} \cdot ((\pi^s \circ F_N)(w+z) - (\pi^s \circ F_N)(z)) \, .$$

Our goal is to apply Lemma 5.3.15 to S_N, and then deduce the existence of $(\pi^s \circ F_N)^{-1}$ on some neighbourhood of $(\pi^s \circ F_N)(z)$ that is not too small, but this will take some work. For any $a, b \in B(0, L_N)$ we have

$$\begin{aligned} \|D_a S_N - D_b S_N\| &\leq \left\|(D_z(\pi^s \circ F_N))^{-1}\right\| \|D_{a+z}(\pi^s \circ F_N) - D_{b+z}(\pi^s \circ F_N)\| \\ &\leq \left\|(D_z(\pi^s \circ F_N))^{-1}\right\| \|D_{a+z} F_N - D_{b+z} F_N\| \\ &\leq |a - b|_{\mathbb{R}^m} \sup_{w \in Q_N} \left\|D_w^2 F_N\right\| := |a - b|_{\mathbb{R}^m} J_N, \end{aligned} \tag{5.35}$$

where we have used (5.32) and the fact that $w \mapsto D_w \pi^s$ is constant and a contraction. Note that

$$\frac{1 - \sqrt{1 - 8A\delta J_{N_3} \|D_z(\pi^s \circ F_{N_3})^{-1}\|}}{4 J_{N_3}} = \frac{2A\delta \|D_z(\pi^s \circ F_{N_3})^{-1}\|}{1 + \sqrt{1 - 8A\delta J_{N_3} \|D_z(\pi^s \circ F_{N_3})^{-1}\|}}. \tag{5.36}$$

In the definition of $\tilde{\Sigma}$ we may assume that δ is as small as we like. Thus, in view of (5.32) and (5.36), by choosing δ sufficiently small we may guarantee that

$$0 < \frac{1 - \sqrt{1 - 8A\delta J_{N_3} \|D_z(\pi^s \circ F_{N_3})^{-1}\|}}{4 J_{N_3}} < L_{N_3}. \tag{5.37}$$

If (5.37) holds then there exists $s \in (0, L_{N_3})$ so that

$$(1 - 2J_{N_3}s)s \geq A\delta \left\| D_z(\pi^s \circ F_{N_3})^{-1} \right\|. \tag{5.38}$$

To summarise, we have proven the following

1. S_{N_3} is well-defined on $\overline{B(0,s)}$ as $s < L_{N_3}$;
2. $S_{N_3}(0) = 0$ and $S'_N(0) = \mathrm{Id}$; and
3. By (5.35) we have $\|D_a S_{N_3} - D_b S_{N_3}\| \leq 2sJ_{N_3}$ for every $a, b \in \overline{B(0,s)}$.

Thus we may apply Lemma 5.3.15 to S_{N_3} on $\overline{B(0,s)}$ to conclude the existence of an inverse $S_{N_3}^{-1}$ that is defined on $\overline{B(0,(1-2J_{N_3}s)s)}$. Using the definition of S_{N_3} and (5.38) we recover the existence of an inverse $(\pi^s \circ F_{N_3})^{-1}$ on $\overline{B((\pi^s \circ F_{N_3})(z), A\delta)}$.

Step II: The definition and properties of leaves covering $T^{-N}(\tilde{W} + y)$. We may define a map $\chi_0 : \overline{B((\pi^s \circ F_{N_3})(z), A\delta)} \to \mathbb{R}^{m_u}$ by

$$\chi_0 = \pi^u \circ F_{N_3} \circ (\pi^s \circ F_{N_3})^{-1}.$$

Note that the graph of χ_0 is a subset of $F_{N_3}(B(z,s))$ by construction. Since the tangent space of F_{N_3} is contained in $\mathcal{C}_{\ell(N_3,y)}$ it follows that $\|D\chi_0\| \leq c_{\ell(N_3,y)}$. Hence, for $w \in \overline{B((\pi^s \circ F_{N_3})(z), A\delta)}$, we have

$$\|\chi_0((\pi^s \circ F_{N_3})(z)) - \chi_0(w)\| \leq c_{\ell(N_3,y)} A\delta. \tag{5.39}$$

Recall from the line before (5.23) that $\kappa < 1/2$, and from (B4) that $c_{\ell(N_3,y)} < 2\kappa$. Thus, as $A\delta < \min_j r_j/6$ (see the sentence following [54, equation (3.1)]), from (5.39) we have

$$\|\chi_0((\pi^s \circ F_{N_3})(z)) - \chi_0(w)\| < \min_j r_j/6.$$

As $(\pi^s \circ F_{N_3})(z) \in (\frac{-r_{\ell(N_3,y)}}{2}, \frac{r_{\ell(N_3,y)}}{2})^{m_s}$, it follows that $\chi_0(w) \in (\frac{-2r_{\ell(N_3,y)}}{3}, \frac{2r_{\ell(N_3,y)}}{3})^{m_u}$. Thus the image of χ_0 is a subset of $(-2r_{\ell(N_3,y)}/3, 2r_{\ell(N_3,y)}/3)^{m_u}$. Since the $\mathcal{C}^{r+1}$ norm of F_{N_3} may be bounded independently of $y \in \mathbb{T}^m$, $z \in B(x,\delta)$ and $W \in \tilde{\Sigma}$, by the inverse function theorem there exists some absolute Y so that for any χ_0 produced by the construction just carried out we have $\|\chi_0\|_{\mathcal{C}^{r+1}} \leq Y$. Thus the graph of χ_0 belongs to $G_{\ell(N_3,y)}(Y)$ (recall the definition of the sets $G_i(K)$ from [54, Section 3]).

The issue at this stage is that $\|\chi_0\|_{\mathcal{C}^{r+1}}$ may not be bounded by the constant K set in [54, Lemma 3.1] and so may not define a leaf in $\tilde{\Sigma}$. Instead, we show that χ_0 may be covered by the image of higher-regularity leaves under some iterate of

T. The following construction is very similar to the one in the proof of [54, Lemma 3.3].

For each $j \in \mathbb{Z}^+$ we define χ_j inductively as follows, starting with $j = 1$. As the graph of χ_{j-1} is in $G_{\ell(N_3+j-1,y)}(R^{j-1}(Y))$, by Lemma 5.3.14 the image of $\psi^{-1}_{\ell(N_3+j,y)} \circ T^{-1} \circ \psi_{\ell(N_3+j-1,y)} \circ (\mathrm{Id}, \chi_{j-1})$ is in $G_{\ell(N_3+j,y)}(R^j(Y))$ and is therefore the graph of a map χ'_j which contains $\psi^{-1}_{\ell(N_3+j,y)} \circ T^{-(N_3+j)} \circ \tau_y \circ \psi_i \circ (z, \chi(z))$. Using the expansivity of T^{-1} as in [54, Lemma 3.3], one deduces that the domain of χ'_j contains the set

$$\overline{B(\psi^{-1}_{\ell(N_3+j,y)} \circ T^{-(N_3+j)} \circ \tau_y \circ \psi_i \circ (z, \chi(z)), A\delta)}. \tag{5.40}$$

We define χ_j to be the restriction of χ'_j to (5.40). By Lemma 5.3.14 we have $\chi_{N_4} \in G_{\ell(N_3+N_4,y)}(K)$, where N_4 denotes the constant $N(Y)$ given by Lemma 5.3.14 and K is the constant from [54, Lemma 3.1]. Thus the image of

$$\overline{B(\psi^{-1}_{\ell(N_3+N_4,y)} \circ T^{-(N_3+N_4)} \circ \tau_y \circ \psi_i \circ (z, \chi(z)), \delta)}$$

under $\psi_{\ell(N_3+N_4,y)} \circ (\mathrm{Id}, \chi_{N_4})$ is a leaf in $\tilde{\Sigma}$.

Step III: Concluding. We may apply this construction to any $z \in B(x, \delta)$ to produce a leaf $W_z \in \tilde{\Sigma}$ such that $T^{N_3+N_4}(W_z) \subset \tilde{W} + y$. Moreover, the constants N_3 and N_4 are independent of $y \in \mathbb{T}^m$, $z \in B(x, \delta)$ and $W \in \tilde{\Sigma}$. Set $N = N_3 + N_4$. By varying z we observe that the set of such leaves covers $T^{-N}(W + y)$. As in the end of [54, Lemma 3.3], the claim that the number of leaves required to cover $T^{-N}(W+y)$ may be bounded independently of W and y follows from [61, Theorem 1.4.10], as too does the existence of partitions of unity satisfying all of the required properties. Hence we have verified all the conclusions of Lemma 5.3.10. $\square$

By using Lemma 5.3.10 and adapting arguments from [54, Section 6] we may now prove (5.17), which completes the proof of Proposition 5.3.6. Let N be the constant from Lemma 5.3.10. We will show that $\left\|\tau_y \mathcal{L}^N\right\|_{p,q}$ can be bounded independently of y. The argument for bounding $\left\|\tau_y \mathcal{L}^N\right\|_{p-1,q+1}$ is identical. Moreover, we will only derive the inequality (5.17) for the spaces $\tilde{B}^{p,q}$ and $\tilde{B}^{p-1,q+1}$, since it is straightforward to then derive the corresponding inequality for their complexifications. We begin by bounding $\left\|\tau_y \mathcal{L}^N\right\|^-_{0,q}$. Let $h \in \mathcal{C}^r(\mathbb{T}^m, \mathbb{R})$, $W \in \tilde{\Sigma}$, $\varphi \in \mathcal{C}^q_0(W, \mathbb{R})$

satisfy $\|\varphi\|_{\mathcal{C}^q} \leq 1$, and $y \in \mathbb{T}^m$. Let $J_W\tau_{-y}$ denote the Jacobian of $\tau_{-y} : W+y \to W$. Then

$$\int_W \tau_y \mathcal{L}^N h \cdot \varphi \, \mathrm{d}\Omega = \int_{W+y} \mathcal{L}^N h \cdot \varphi \circ \tau_{-y} \cdot J_W\tau_{-y} \, \mathrm{d}\Omega. \tag{5.41}$$

Recall that $\mathcal{L}^N h = (h \left|\det DT^N\right|^{-1}) \circ T^{-N}$ and let $J_{W+y}T^N$ denote the Jacobian of $T^N : T^{-N}(W+y) \to W+y$. Let $\{W_i\}_{i=1}^j$, $\{\rho_i\}_{i=1}^j$ satisfy the conclusion of Lemma 5.3.10. By changing coordinates and applying Lemma 5.3.10 to (5.41) we obtain

$$\begin{aligned}
&\int_{W+y} \mathcal{L}^N h \cdot \varphi \circ \tau_{-y} \cdot J_W\tau_{-y} \, \mathrm{d}\Omega \\
&\qquad = \int_{T^{-N}(W+y)} h \cdot \left|\det DT^N\right|^{-1} \cdot (\varphi \circ \tau_{-y} \circ T^N) \cdot (J_W\tau_{-y}) \circ T^N \cdot J_{W+y}T^N \, \mathrm{d}\Omega \\
&\qquad = \sum_{i=1}^j \int_{W_i} h \cdot \left|\det DT^N\right|^{-1} \cdot (\varphi \circ \tau_{-y} \circ T^N) \cdot (J_W\tau_{-y}) \circ T^N \cdot \rho_i \cdot J_{W+y}T^N \, \mathrm{d}\Omega.
\end{aligned} \tag{5.42}$$

By the definition of $\|\cdot\|_{0,q}$ the final expression in (5.42) is bounded above by

$$\|h\|_{0,q} \sum_{i=1}^j \left\| \left|\det DT^N\right|^{-1} \cdot (\varphi \circ \tau_{-y} \circ T^N) \cdot (J_W\tau_{-y}) \circ T^N \cdot \rho_i \cdot J_{W+y}T^N \right\|_{\mathcal{C}^q(W_i)}.$$

Recall that $q \leq r-1$. Since T is a $\mathcal{C}^r$ diffeomorphism and $\mathbb{T}^m$ is compact, it follows that the $\mathcal{C}^q(W_i)$ norms of $J_{W+y}T^N$ and $\left|\det DT^N\right|^{-1}$ are bounded independently of W and y. Using continuity and compactness, we observe that $\sup_{x,y\in\mathbb{T}^m} \left\|D_x^k \tau_y\right\|$ is finite for every positive integer k. Thus the $\mathcal{C}^q(W_i)$ norm of $\varphi \circ \tau_{-y} \circ T^N$ is bounded independently of y and φ (provided $\|\varphi\|_{\mathcal{C}^q(W_i)} \leq 1$). Similarly, it is clear that the $\mathcal{C}^q(W_i)$ norm of $(J_W\tau_{-y}) \circ T^N$ is bounded independently of W and y. Recall from Lemma 5.3.10 that j and $\sup_i \|\rho_i\|_{\mathcal{C}^q}$ are bounded above independently of W and y. Hence, as $\|\cdot\|_{\mathcal{C}^q(W_i)}$ is sub-multiplicative, there exists some $C_{0,q} > 0$ such that

$$\sum_{i=1}^j \left\| \left|\det DT^N\right|^{-1} \cdot (\varphi \circ \tau_{-y} \circ T^N) \cdot \rho_i \cdot J_{W+y}T^N \right\|_{\mathcal{C}^q(W_i)} \leq C_{0,q}$$

for every choice of W, y and φ with $\|\varphi\|_{\mathcal{C}^q} \leq 1$. Thus, by taking supremums of the terms in (5.41) we have

$$\left\|\tau_y \mathcal{L}^N h\right\|_{0,q}^- \leq C_{0,q} \|h\|_{0,q}. \tag{5.43}$$

We turn to bounding $\left\|\tau_y \mathcal{L}^N h\right\|_{k,q+k}^-$ for $0 < k \leq p$. Let h, W and y be as before. Suppose that $\varphi \in \mathcal{C}_0^{q+k}(W, \mathbb{R})$ satisfies $\|\varphi\|_{\mathcal{C}^{q+k}} \leq 1$ and that $\{v_i\}_{i=1}^k \subseteq \mathcal{V}^r(W)$ is such that $\|v_i\|_{\mathcal{C}^r} \leq 1$. Then

$$\int_W (v_1 \dots v_k)(\tau_y \mathcal{L}^N h) \cdot \varphi \, \mathrm{d}\Omega = \int_W \tau_y (\tilde{v}_{1,y} \dots \tilde{v}_{1,k} \mathcal{L}^N h) \cdot \varphi \, \mathrm{d}\Omega,$$

where $\tilde{v}_{i,y}(x) := (D_{x-y}\tau_y) v_i(x-y)$. With $J_W \tau_{-y}$, $\{W_i\}_{i=1}^j$ and $\{\rho_i\}_{i=1}^j$ as before we have

$$\begin{aligned} \int_W \tau_y (\tilde{v}_{1,y} \dots \tilde{v}_{1,k} \mathcal{L}^N h) \cdot \varphi \, \mathrm{d}\Omega &= \int_{W+y} (\tilde{v}_{1,y} \dots \tilde{v}_{1,k} \mathcal{L}^N h) \cdot \varphi \circ \tau_{-y} \cdot J_W \tau_{-y} \, \mathrm{d}\Omega \\ &= \sum_{i=1}^{j} \int_{T^N(W_i)} (\tilde{v}_{1,y} \dots \tilde{v}_{1,k} \mathcal{L}^N h) \cdot \varphi \circ \tau_{-y} \cdot \rho_i \circ T^{-N} \cdot J_W \tau_{-y} \, \mathrm{d}\Omega. \end{aligned} \tag{5.44}$$

Since the $\mathcal{C}^{q+k}$ norms of $\varphi \circ \tau_{-y}$ and $J_W \tau_{-y}$ are bounded independently of φ, y and W, we may replace $\varphi \circ \tau_{-y} \cdot J_W \tau_{-y}$ by some $\phi \in \mathcal{C}_0^{q+k}(W, \mathbb{R})$ with $\mathcal{C}^{q+k}$ norm bounded independently of φ, y and W. Additionally, the $\mathcal{C}^r$ norm of each $\tilde{v}_{i,y}$ may be bounded independently of y and W due to the $\sup_{x,y \in \mathbb{T}^m} \left\| D_x^\ell \tau_y \right\|$ being finite for each positive integer ℓ. Upon replacing $\varphi \circ \tau_{-y} \cdot J_W \tau_{-y}$ with ϕ, the expression on the right side of (5.44) is exactly in the form of [54, (6.4)]. Using the arguments from [54, Lemma 6.3], one then obtains a bound of the form

$$\left| \int_W (v_1 \dots v_k)(\tau_y \mathcal{L}^N h) \cdot \varphi \, \mathrm{d}\Omega \right| \leq C_{p,q} \|h\|_{p,q} + C_{p-1,q+1} \|h\|_{p-1,q+1},$$

for some $C_{p,q}, C_{p-1,q+1} > 0$ that are independent of h, W, k, y, φ and each v_i. By the definition of $\|\cdot\|_{k,q+k}^-$, and as $\|\cdot\|_{p,q}$ dominates $\|\cdot\|_{p-1,q+1}$, we therefore have

$$\left\| \tau_y \mathcal{L}^N h \right\|_{k,q+k}^- \leq (C_{p,q} + C_{p-1,q+1}) \|h\|_{p,q}. \tag{5.45}$$

The required bound follows by considering (5.43), (5.45) and the definition of $\|\cdot\|_{p,q}$. Thus we have established (5.17), which completes the proof of Proposition 5.3.6.

5.3.3 *Stability for perturbations of the cat map*

We now give a concrete family of maps satisfying the conditions of Proposition 5.3.6. For $\delta \in \mathbb{R}$ let $T_\delta : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by

$$T_\delta(x_1, x_2) = (2x_1 + x_2, x_1 + x_2) + \delta(\cos(2\pi x_1), \sin(4\pi x_2 + 1)).$$

Note that T_0 is Arnold's 'cat map' – a linear hyperbolic toral automorphism – and so it satisfies Proposition 5.3.6. Moreover, since $\delta \mapsto T_\delta$ is smooth it follows that T_δ

is Anosov and satisfies the conditions of Proposition 5.3.6 for sufficiently small δ. The main result of this section is an explicit range of δ for which T_δ is an Anosov diffeomorphism and satisfies (5.17).

Proposition 5.3.16. *If $0 \le \delta < 0.0108$ then T_δ is an Anosov diffeomorphism and satisfies the conditions of Proposition 5.3.6.*

Hence Proposition 5.3.1, Theorem 5.3.3, and Corollary 5.3.4 apply to T_δ whenever $\delta \in [0, 0.0108)$. The proof of Proposition 5.3.16 is broken up into Lemmas 5.3.17, 5.3.19 and 5.3.22. Throughout this section we denote the Euclidean norm on $\mathbb{R}^2$ (and the associated operator norm) by $|\cdot|$, and the usual Euclidean inner product by $\langle \cdot, \cdot \rangle$. We begin by proving that T_δ is a diffeomorphism for sufficiently small δ by using a quantitative version of the inverse function theorem (Lemma 5.3.15).

Lemma 5.3.17. *If $\delta \in [0, 0.0108)$ then T_δ is a diffeomorphism.*

Proof. We have

$$D_{(x_1,x_2)}T_\delta = \begin{pmatrix} 2 - 2\pi\delta\sin(2\pi x_1) & 1 \\ 1 & 1 + 4\pi\delta\cos(4\pi x_2 + 1) \end{pmatrix},$$

and

$$\det D_{(x_1,x_2)}T_\delta = 1 + 8\pi\delta\cos(4\pi x_2+1) - 2\pi\delta\sin(2\pi x_1) - 8\pi^2\delta^2\sin(2\pi x_1)\cos(4\pi x_2+1). \tag{5.46}$$

In particular, $D_{(0,0)}T_\delta$ is invertible if $|\delta| < 1/(8\pi)$. Denote by $\bar{T}_\delta : \mathbb{R}^2 \to \mathbb{R}^2$ the lift of T_δ. Define $S_\delta : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$S_\delta(y_1, y_2) := (D_{(0,0)}T_\delta)^{-1}\left(\bar{T}_\delta(y_1, y_2) - \bar{T}_\delta(0,0)\right).$$

Note that $S_\delta(0,0) = 0$ and $D_{(0,0)}S_\delta = \mathrm{Id}$. For $z \in \mathbb{R}$ let $\widetilde{z}$ denote the equivalence class containing z in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We will estimate

$$\left|D_{(y_1,y_2)}S_\delta - D_{(w_1,w_2)}S_\delta\right| = \left|(D_{(0,0)}T_\delta)^{-1}\left(D_{(\widetilde{y_1},\widetilde{y_2})}T_\delta - D_{(\widetilde{w_1},\widetilde{w_2})}T_\delta\right)\right|.$$

We clearly have

$$\left|D_{(\widetilde{y_1},\widetilde{y_2})}T_\delta - D_{(\widetilde{w_1},\widetilde{w_2})}T_\delta\right| \le 8\pi|\delta|. \tag{5.47}$$

Use the fact that the Frobenius norm dominates the Euclidean operator norm, we have

$$\left|D_{(0,0)}T_\delta^{-1}\right| = \frac{\left|D_{(0,0)}T_\delta\right|}{\left|\det D_{(0,0)}T_\delta\right|} \leq \frac{\sqrt{6+(1+4\pi\delta\cos(1))^2}}{\left|\det D_{(0,0)}T_\delta\right|}. \tag{5.48}$$

Thus, by (5.46), (5.47) and (5.48),

$$\left|D_{(y_1,y_2)}S_\delta - D_{(w_1,w_2)}S_\delta\right| \leq \frac{8\pi\,|\delta|\,\sqrt{6+(1+4\pi\delta\cos(1))^2}}{1-8\pi\delta} := s.$$

If $|\delta| < 0.0108$ then $s < 1$. Then S_δ verifies the conditions of Lemma 5.3.15 and has an inverse $S_\delta : B(0,(1-s)r) \to \mathbb{R}^2$, where $r > 0$. Since there is no dependence on r in the above procedure, we may extend S_δ^{-1} to $\mathbb{R}^2$. Thus $\bar{T}_\delta$ is invertible, and so T_δ is invertible too. It is standard that T_δ^{-1} and T_δ have the same smoothness. $\square$

Let $x \in \mathbb{T}^2$. The eigenvalues of D_xT_0 are $\lambda = \frac{3-\sqrt{5}}{2} < 1$ and $\lambda^{-1} = \frac{3+\sqrt{5}}{2} > 1$. Let $\widetilde{E}^s(x)$ be the span of $\left(1, \frac{-\sqrt{5}-1}{2}\right)$ and $\widetilde{E}^u(x)$ be the span of $\left(1, \frac{\sqrt{5}-1}{2}\right)$. Note that $\widetilde{E}^s(x)$ and $\widetilde{E}^u(x)$ are the eigenspaces of λ and λ^{-1}, respectively. It is trivial that the spaces $\widetilde{E}^u(x)$ and $\widetilde{E}^s(x)$ depend continuously on x, and that $\widetilde{E}^u(x) \oplus \widetilde{E}^s(x) = T_x\mathbb{T}^2$ for every $x \in \mathbb{T}^2$. Let $\Pi^u, \Pi^s : \mathbb{R}^2 \to \mathbb{R}^2$ denote the orthogonal projections onto $\widetilde{E}^u(x)$ and $\widetilde{E}^s(x)$, respectively. Since T_0 is symmetric, $\widetilde{E}^u(x) \perp \widetilde{E}^s(x)$ and so $\mathrm{Id} - \Pi^s = \Pi^u$. For $\alpha > 0$ define

$$K_\alpha^u(x) = \{v \in T_x\mathbb{T}^2 : |\Pi^s v| \leq \alpha\,|\Pi^u v|\}, \text{ and } K_\alpha^s(x) = \{v \in T_x\mathbb{T}^2 : |\Pi^u v| \leq \alpha\,|\Pi^s v|\}.$$

To prove that T_δ is an Anosov diffeomorphism it remains to prove that $\mathbb{T}^2$ is a hyperbolic set for T_δ. We do this by verifying the conditions of the following result, which we have adapted to our setting for simplicity.

Proposition 5.3.18 ([21, Proposition 5.4.3])**.** *If there exists $\alpha > 0$ such that for every $x \in \mathbb{T}^2$ we have*

(A1) $(D_xT_\delta)(K_\alpha^u(x)) \subseteq K_\alpha^u(T_\delta(x))$ and $(D_xT_\delta^{-1})(K_\alpha^s(x)) \subseteq K_\alpha^s(T_\delta^{-1}(x))$; and
(A2) $|D_xT_\delta v| < |v|$ for $v \in K_\alpha^s(x) \setminus \{0\}$ and $\left|D_xT_\delta^{-1}v\right| < |v|$ for $v \in K_\alpha^u(x) \setminus \{0\}$.

Then there are constants $\nu_{u,\delta} > 1$ and $0 < \nu_{s,\delta} < 1$, and for each $x \in \mathbb{T}^2$, subspaces $E_\delta^s(x)$ and $E_\delta^u(x)$ such that

1. *$T_x\mathbb{T}^2 = E_\delta^s(x) \oplus E_\delta^u(x)$;*
2. *$(D_xT_\delta^{-1})(E_\delta^s(x)) = E^s(T_\delta^{-1}(x))$ and $(D_xT_\delta)(E_\delta^u(x)) = E^u(T_\delta(x))$;*
3. *$\left|D_xT_\delta|_{E_\delta^s(x)}\right| \leq \nu_{s,\delta}$ and $\left|D_xT_\delta^{-1}|_{E_\delta^u(x)}\right| \leq \nu_{u,\delta}^{-1}$; and*
4. *$E_\delta^s(x) \subseteq K_\alpha^s(x)$ and $E_\delta^u(x) \subseteq K_\alpha^u(x)$.*

In particular, T_δ is Anosov.

Lemma 5.3.19. *If $\delta \in [0, 0.0108)$ then T_δ is Anosov and the conclusion of Proposition 5.3.18 holds with $\alpha = 0.11872$.*

Proof. We first diagonalise $D_x T_0$ as $R_\theta^{-1} \Lambda R_\theta$, where Λ is a 2×2 diagonal matrix with the vector $[\lambda, 1/\lambda]$ on the diagonal, and R_θ is clockwise rotation by angle $\theta = \tan^{-1}((1-\sqrt{5})/2)$. Note that $D_x T_\delta = D_x T_0 + \Delta_x$ where $\Delta_x : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by the matrix

$$\Delta_x = \begin{pmatrix} -2\pi\delta\sin(2\pi x_1) & 0 \\ 0 & 4\pi\delta\cos(4\pi x_2 + 1) \end{pmatrix}.$$

We use the shorthand $\delta_1 = -2\pi\delta\sin(2\pi x_1)$ and $\delta_2 = 4\pi\delta\cos(4\pi x_2 + 1)$. In order to satisfy the second part of (A1) of Proposition 5.3.18, we require that $\Lambda + R_\theta \Delta_x R_\theta^{-1}$ preserve $\mathcal{K}_\alpha^u := \{(\beta, \gamma)^\top \in \mathbb{R}^2 : |\beta| \leq |\gamma|\,\alpha\}$. One may confirm that

$$R_\theta \Delta_x R_\theta^{-1} = \begin{pmatrix} \delta_1\cos^2\theta + \delta_2\sin^2\theta & (1/2)\sin(2\theta)(\delta_2 - \delta_1) \\ (1/2)\sin(2\theta)(\delta_1 - \delta_2) & \delta_2\cos^2\theta + \delta_1\sin^2\theta \end{pmatrix}. \tag{5.49}$$

Multiplying $\Lambda + R_\theta \Delta_x R_\theta^{-1}$ with the vectors $(\alpha, 1)^\top$ and $(-\alpha, 1)^\top$ we see that a sufficient condition to preserve $\mathcal{K}_\alpha^u$ is that

$$\frac{(\lambda + \delta')\alpha + \delta'}{1/\lambda - \delta' - \delta'\alpha} \leq \alpha,$$

where $\delta' = \max\{\sup_x \delta_1, \sup_x \delta_2\} = 4\pi\delta$. Since $1/\lambda - \delta' - \delta'\alpha > 0$ for $0 \leq \delta \leq 0.1862$ we may rearrange the above in terms of δ to obtain

$$\delta \leq \frac{\alpha(1/\lambda - \lambda)}{4\pi(\alpha+1)^2}. \tag{5.50}$$

Since $\delta \in [0, 0.0108)$, by Lemma 5.3.17 the map T_δ is a diffeomorphism. To satisfy the first part of (A1) of Proposition 5.3.18, using the notation above, we note that

$$(D_x T_\delta)^{-1} = (1/\det(D_x T_\delta)) \begin{pmatrix} 1 + \delta_2 & -1 \\ -1 & 2 + \delta_1 \end{pmatrix}, \tag{5.51}$$

and that for the purposes of cone preservation we need not consider the determinant factor. Therefore, $(D_xT_\delta)^{-1} = (1/\det(D_xT_\delta))((D_xT_0)^{-1} + \Delta'_x)$, where

$$\Delta'_x = \begin{pmatrix} \delta_2 & 0 \\ 0 & \delta_1 \end{pmatrix}. \tag{5.52}$$

The cone preservation condition will be implied by the preservation of $\mathcal{K}^s_\alpha := \{(\gamma,\beta)^\top \in \mathbb{R}^2 : |\beta| \leq |\gamma|\,\alpha\}$ by $D_xT_\delta^{-1}$. Multiplying $\Lambda^{-1} + R_\theta\Delta'_xR_\theta^{-1}$ with the vectors $(1,\alpha)^\top$ and $(1,-\alpha)^\top$ yields an identical set of calculations to those for $\mathcal{K}^u_\alpha$, resulting in the same bound for δ as in (5.50). Substituting $\alpha = 0.11872$ into this bound yields a numerical upper bound for δ of 0.0169, which is larger than the value reported in the proposition statement.

To verify (A2) we demonstrate contraction for elements of $\mathcal{K}^s_\alpha$; the same contractions occur in the original (unrotated) cones $K^s_\alpha(x)$ and $K^u_\alpha(x)$ under D_xT_δ and $D_xT_\delta^{-1}$, respectively. Writing $\Lambda + R_\theta\Delta_xR_\theta^{-1} = \begin{pmatrix} \lambda + a & b \\ c & \lambda^{-1} + d \end{pmatrix}$ and multiplying by the unit vector $(1/\sqrt{1+\beta^2})(1,\beta)^\top$ for $-\alpha \leq \beta \leq \alpha$, the square of the norm of this vector is $((\lambda+a)^2 + b^2\beta^2 + 2(\lambda+a)b\beta + c^2 + (\lambda^{-1}+d)^2\beta^2 + 2c\beta(\lambda^{-1}+d))/(1+\beta^2)$. We require the above expression to be strictly less than 1 for contraction. Grouping terms to obtain a quadratic in β we wish to show

$$\beta^2(b^2 + (\lambda^{-1}+d)^2 - 1) + 2\beta((\lambda+a) + c(\lambda^{-1}+d)) + ((\lambda+a)^2 + c^2 - 1) < 0 \tag{5.53}$$

for $-\alpha \leq \beta \leq \alpha$. This quadratic has a local minimum since $\lambda^{-1} \geq 1 + |d|$ whenever $\delta \in [0, 0.1288)$; therefore the maxima are at $\beta = \pm\alpha$. Using the fact that $\max\{|a|, |b|, |c|, |d|\} \leq \delta'$ one may readily check that contraction occurs at $\beta = \pm\alpha$ for $\delta' \in [0, 0.6734)$ or, equivalently, for $\delta \in [0, 0.0536)$.

The contraction of vectors in $\mathcal{K}^u_\alpha$ under $D_xT_\delta^{-1}$ follows similarly. With the notation above, one easily verifies $1 - (5/2)\delta' - (\delta')^2/2 \leq \det(D_xT_\delta)$. Replacing the two '1's in (5.53) with the factor $1 - (5/2)\delta' - (\delta')^2/2$, one verifies as above that the polynomial (5.53) has positive leading term for $\delta \in [0, 0.1288)$ and is negatively valued for $-\alpha \leq \beta \leq \alpha$ provided that $\delta \in [0, 0.0293)$. Thus vectors in $\mathcal{K}^u_\alpha$ are contracted under $D_xT_\delta^{-1}$ for δ in the advertised range.

As we have verified all the conditions of Proposition 5.3.18 for T_δ whenever $\delta \in [0, 0.0108)$, it follows that T_δ is an Anosov diffeomorphism for δ in the same range. $\square$

To complete the proof of Proposition 5.3.16 it suffices to prove that T_δ satisfies the conditions of Proposition 5.3.6. Denote by π_x^s (resp. π_x^u) the projection onto $E_\delta^s(x)$ along $E_\delta^u(x)$ (resp. $E_\delta^u(x)$ along $E_\delta^s(x)$). Let w_s and w_u be the unit vectors in the rays defined by $\left(1, \frac{-\sqrt{5}-1}{2}\right)$ and $\left(1, \frac{\sqrt{5}-1}{2}\right)$, respectively. Let $w_u(x)$ and $w_s(x)$ be the unit vectors in $E_\delta^s(x)$ and $E_\delta^u(x)$ for which $\langle w_u(x), w_u\rangle > 0$ and $\langle w_s(x), w_s\rangle > 0$. For $v \in \mathbb{R}^2$ we denote by $v^\perp$ the vector obtained by rotating v anticlockwise by $\pi/2$ about the origin. In particular, $w_s^\perp = w_u$ and $w_u^\perp = -w_s$. For $v \in T_x\mathbb{T}^2$ let

$$\|v\|_0 = \sqrt{|\pi_x^s v|^2 + |\pi_x^u v|^2}.$$

We can recover a Riemannian metric from $\|\cdot\|_0$ by the polarisation identity. By Proposition 5.3.18 we have $\left|D_xT_\delta|_{E_\delta^s(x)}\right| \le \nu_{s,\delta}$ and $\left|D_xT_\delta^{-1}|_{E_\delta^u(x)}\right| \le \nu_{u,\delta}^{-1}$. Thus the metric induced by $\|\cdot\|_0$ satisfies (M2) and is adapted. In the following two lemmas we collect some useful inequalities, before proving that T_δ satisfies the conditions of Proposition 5.3.6 for all $\delta \in [0, 0.0108)$ in Lemma 5.3.22. The first such bound follows from basic trigonometry.

Lemma 5.3.20. *If $v_i \in K_\alpha^s(x)$, $i = 1, 2$ with $|v_i| = 1$ and $\langle v_1, v_2\rangle > 0$, then $|v_1 - v_2| \le \frac{2\alpha}{\sqrt{1+\alpha^2}}$. Similar statements hold for $v_i \in K_\alpha^u(x)$.*

Lemma 5.3.21. *If $\alpha < 1$, then for every $v \in T_x\mathbb{T}^2$ we have*

$$\frac{\sqrt{(1-\alpha^2)^2 - \beta(\alpha)}}{1-\alpha^2}\,|v| \le \|v\|_0 \le \frac{\sqrt{(1-\alpha^2)^2 + \beta(\alpha)}}{1-\alpha^2}\,|v|,$$

where

$$\beta(\alpha) := \sqrt{2}\alpha(3\alpha + \sqrt{2+3\alpha^2})\sqrt{1 + \alpha\sqrt{2+3\alpha^2}}.$$

Proof. Let $v \in T_x\mathbb{T}^2$ with $|v| = 1$. By writing $v = \pi_x^s v + \pi_x^u v$ we find that

$$\|v\|_0 = \sqrt{|\pi_x^s v|^2 + |\pi_x^u v|^2} = \sqrt{1 - 2\langle \pi_x^s v, \pi_x^u v\rangle}. \tag{5.54}$$

One verifies that $\pi_x^s v = \frac{\langle w_u(x)^\perp, v\rangle}{\langle w_u(x)^\perp, w_s(x)\rangle} w_s(x)$ and $\pi_x^u v = \frac{\langle w_s(x)^\perp, v\rangle}{\langle w_s(x)^\perp, w_u(x)\rangle} w_u(x)$. Hence

$$|\langle \pi_x^s v, \pi_x^u v\rangle| = \left|\frac{\langle w_s(x), w_u(x)\rangle\langle w_s(x)^\perp, v\rangle\langle w_u(x)^\perp, v\rangle}{\langle w_s(x)^\perp, w_u(x)\rangle\langle w_u(x)^\perp, w_s(x)\rangle}\right|. \tag{5.55}$$

Since $w_s(x) \in K_\alpha^s(x)$ and $w_u(x) \in K_\alpha^u(x)$, basic trigonometry yields

$$|\langle w_s(x), w_u(x)\rangle| \le \cos\left(\frac{\pi}{2} - 2\tan^{-1}(\alpha)\right) = \frac{2\alpha}{1+\alpha^2}. \tag{5.56}$$

Alternatively, as $w_s(x)^\perp \in K^u(x)$,

$$\left|\langle w_s(x)^\perp, w_u(x)\rangle\right| \geq \cos(2\tan^{-1}(\alpha)) = \frac{1-\alpha^2}{1+\alpha^2}. \tag{5.57}$$

The same argument yields the same lower bound for $\left|\langle w_u(x)^\perp, w_s(x)\rangle\right|$. Writing $v = Aw_s(x) + Bw_s(x)^\perp$, we have

$$\langle w_s(x)^\perp, v\rangle\langle w_u(x)^\perp, v\rangle = B(-A + \langle w_u(x)^\perp + w_s(x), v\rangle).$$

Since $w_s(x)$ and $w_s(x)^\perp$ are orthogonal and $|v| = 1$, we have $|B| = \sqrt{1-A^2}$. On the other hand, by Lemma 5.3.20 and Cauchy-Schwarz we have $\left|\langle w_u(x)^\perp + w_s(x), v\rangle\right| \leq \frac{2\alpha}{\sqrt{1+\alpha^2}}$. Upon substituting we have

$$\left|\langle w_s(x)^\perp, v\rangle\langle w_u(x)^\perp, v\rangle\right| \leq \sqrt{1-A^2}\left(|A| + \frac{2\alpha}{\sqrt{1+\alpha^2}}\right). \tag{5.58}$$

We may bound the right side of (5.58) from above by differentiating with respect to A and solving the resulting quadratic equation (noting that we only have to consider the case where $A \geq 0$ due to the symmetry about $A = 0$ in the right side of (5.58)). In particular, (5.58) is maximised when

$$|A| = \frac{-\alpha + \sqrt{2+3\alpha^2}}{2\sqrt{1+\alpha^2}},$$

which, when substituted into (5.58), yields

$$\left|\langle w_s(x)^\perp, v\rangle\langle w_u(x)^\perp, v\rangle\right| \leq \frac{(3\alpha + \sqrt{2+3\alpha^2})\sqrt{1+\alpha\sqrt{2+3\alpha^2}}}{2\sqrt{2}(1+\alpha^2)}. \tag{5.59}$$

Applying (5.56), (5.57) and (5.59) to (5.55) yields

$$|\langle \pi_x^s v, \pi_x^u v\rangle| \leq \frac{\alpha(3\alpha + \sqrt{2+3\alpha^2})\sqrt{1+\alpha\sqrt{2+3\alpha^2}}}{\sqrt{2}(1-\alpha^2)^2}. \tag{5.60}$$

Hence, upon substituting (5.60) into (5.54) we obtain

$$\frac{\sqrt{(1-\alpha^2)^2 - \beta(\alpha)}}{1-\alpha^2}|v| \leq \|v\|_0 \leq \frac{\sqrt{(1-\alpha^2)^2 + \beta(\alpha)}}{1-\alpha^2}|v|$$

as announced. □

Lemma 5.3.22. *In the setting of Lemma 5.3.19 and Proposition 5.3.18 we have* $C_{\tau,0}\Theta_{T_\delta,0} < 1$.

Proof. Since $\alpha < 1$ the conclusion of Lemma 5.3.21 holds. Since $D_x\tau_y$ is an isometry with respect to the usual Riemannian metric on the tangent space of $\mathbb{T}^2$, by Lemma 5.3.21 we have

$$\sup_{\|v\|_0 \leq 1} \|D_x\tau_y v\|_0 \leq \sqrt{\frac{(1-\alpha^2)^2 + \beta(\alpha)}{(1-\alpha^2)^2 - \beta(\alpha)}}.$$

Thus

$$C_{\tau,0} \leq \sqrt{\frac{(1-\alpha^2)^2 + \beta(\alpha)}{(1-\alpha^2)^2 - \beta(\alpha)}}. \tag{5.61}$$

We turn to bounding $\Theta_{T_\delta,0}$. Since π_x^s and π_x^u are complementary orthogonal projections with respect to the inner product associated to $\|\cdot\|_0$ we have

$$\Theta_{T_\delta,0} = \sup_{x,y\in\mathbb{T}^2} \left\|\pi_{x+y}^s D_x\tau_y - (D_x\tau_y)\pi_x^s\right\|_0.$$

By using Lemma 5.3.21 in the same way as when bounding $C_{\tau,0}$ we find that

$$\Theta_{T_\delta,0} \leq \sqrt{\frac{(1-\alpha^2)^2 + \beta(\alpha)}{(1-\alpha^2)^2 - \beta(\alpha)}} \sup_{x,y\in\mathbb{T}^2} \left|\pi_{x+y}^s D_x\tau_y - (D_x\tau_y)\pi_x^s\right|.$$

Let $v \in T_x\mathbb{T}^2$ with $|v| = 1$. Recalling the definition of π^s_x and then applying the triangle and Cauchy-Schwarz inequalities we have

$$
\begin{aligned}
&|(\pi^s_{x+y} D_x\tau_y - (D_x\tau_y)\pi^s_x)v| \\
&\quad = \left| \frac{\langle w_u(x+y)^\perp, v\rangle}{\langle w_u(x+y)^\perp, w_s(x+y)\rangle} w_s(x+y) - \frac{\langle w_u(x)^\perp, v\rangle}{\langle w_u(x)^\perp, w_s(x)\rangle} w_s(x) \right| \\
&\quad \leq \left| \frac{\langle w_u(x+y)^\perp, v\rangle}{\langle w_u(x+y)^\perp, w_s(x+y)\rangle} w_s(x+y) - \frac{\langle w_u(x)^\perp, v\rangle}{\langle w_u(x+y)^\perp, w_s(x+y)\rangle} w_s(x+y) \right| \\
&\qquad + \left| \frac{\langle w_u(x)^\perp, v\rangle}{\langle w_u(x+y)^\perp, w_s(x+y)\rangle} w_s(x+y) - \frac{\langle w_u(x)^\perp, v\rangle}{\langle w_u(x)^\perp, w_s(x)\rangle} w_s(x+y) \right| \\
&\qquad + \left| \frac{\langle w_u(x)^\perp, v\rangle}{\langle w_u(x)^\perp, w_s(x)\rangle} w_s(x+y) - \frac{\langle w_u(x)^\perp, v\rangle}{\langle w_u(x)^\perp, w_s(x)\rangle} w_s(x) \right| \\
&\quad \leq \frac{\left|\langle w_u(x+y)^\perp, v\rangle - \langle w_u(x)^\perp, v\rangle\right|}{|\langle w_u(x+y)^\perp, w_s(x+y)\rangle|} \\
&\qquad + \frac{\left|\langle w_u(x+y)^\perp, w_s(x+y)\rangle - \langle w_u(x)^\perp, w_s(x)\rangle\right|}{|\langle w_u(x+y)^\perp, w_s(x+y)\rangle\langle w_u(x)^\perp, w_s(x)\rangle|} \\
&\qquad + \frac{|w_s(x+y) - w_s(x)|}{|\langle w_u(x)^\perp, w_s(x)\rangle|}.
\end{aligned}
\tag{5.62}
$$

We will bound the various terms on the right side of (5.62). By Lemma 5.3.20 we have

$$
\left|\langle w_u(x+y)^\perp, v\rangle - \langle w_u(x)^\perp, v\rangle\right| \leq \frac{2\alpha}{\sqrt{1+\alpha^2}}, \text{ and } |w_s(x+y) - w_s(x)| \leq \frac{2\alpha}{\sqrt{1+\alpha^2}}.
\tag{5.63}
$$

By definition we have $\langle w_u(x)^\perp, w_s(x)\rangle \leq 0$ for every $x \in \mathbb{T}^2$. More precisely, by using Cauchy-Schwarz and (5.57) we find that

$$
-1 \leq \langle w_u(x)^\perp, w_s(x)\rangle \leq \frac{\alpha^2 - 1}{1+\alpha^2}.
$$

Hence,

$$
\left|\langle w_u(x+y)^\perp, w_s(x+y)\rangle - \langle w_u(x)^\perp, w_s(x)\rangle\right| \leq \frac{\alpha^2}{1+\alpha^2}.
\tag{5.64}
$$

Using (5.64) and (5.57) to bound the second term of (5.62), we obtain

$$
\frac{\left|\langle w_u(x+y)^\perp, w_s(x+y)\rangle - \langle w_u(x)^\perp, w_s(x)\rangle\right|}{|\langle w_u(x+y)^\perp, w_s(x+y)\rangle\langle w_u(x)^\perp, w_s(x)\rangle|} \leq \frac{\alpha^2(1+\alpha^2)}{(1-\alpha^2)^2}.
\tag{5.65}
$$

Using (5.63) and (5.57) to bound the first term of (5.62), (5.64) and (5.57) to bound the second term, and the bound (5.65) yields

$$\begin{aligned}\left|(\pi^s_{x+y}D_x\tau_y - (D_x\tau_y)\pi^s_x)v\right| &\leq \frac{2\alpha\sqrt{1+\alpha^2}}{1-\alpha^2} + \frac{\alpha^2(1+\alpha^2)}{(1-\alpha^2)^2} + \frac{2\alpha\sqrt{1+\alpha^2}}{1-\alpha^2}\\ &\leq \frac{4\alpha(1-\alpha^2)\sqrt{1+\alpha^2}+\alpha^2(1+\alpha^2)}{(1-\alpha^2)^2}.\end{aligned}$$

Thus,

$$\Theta_{T_\delta,0} \leq \left(\sqrt{\frac{(1-\alpha^2)^2+\beta(\alpha)}{(1-\alpha^2)^2-\beta(\alpha)}}\right)\left(\frac{4\alpha(1-\alpha^2)\sqrt{1+\alpha^2}+\alpha^2(1+\alpha^2)}{(1-\alpha^2)^2}\right). \tag{5.66}$$

Combining (5.62) and (5.66) yields

$$C_{\tau,0}\Theta_{T_\delta,0} \leq \left(\frac{(1-\alpha^2)^2+\beta(\alpha)}{(1-\alpha^2)^2-\beta(\alpha)}\right)\left(\frac{4\alpha(1-\alpha^2)\sqrt{1+\alpha^2}+\alpha^2(1+\alpha^2)}{(1-\alpha^2)^2}\right).$$

So if $\alpha < 0.11872$ then $C_{\tau,0}\Theta_{T_\delta,0} < 1$. □

5.4 Estimation of the statistical properties of Anosov maps

In this section we implement the numerical schemes described in Sections 5.2 and 5.3. As in these previous sections we consider a $\mathcal{C}^{r+1}$ Anosov diffeomorphism T on the m-dimensional torus $\mathbb{T}^m$. Before describing these methods we recall a complication that was discussed in Remarks 5.2.7 and 5.3.5. Namely, the numerical methods proposed in Sections 5.2 and 5.3 apply to the Perron-Frobenius operator $\mathcal{L}_\Omega$ of T induced by duality with respect to the Riemannian volume Ω, while it is desirable from a computational perspective to estimate the Perron-Frobenius operator $\mathcal{L}_{\mathrm{Leb}}$ induced by duality with respect to the normalised Haar measure on $\mathbb{T}^m$. Indeed, estimating $\mathcal{L}_\Omega$ would require knowledge of the adapted metric inducing Ω, which itself depends on the dynamics. We address this complication in Section 5.4.4, wherein it is shown that $\mathcal{L}_\Omega$ and $\mathcal{L}_{\mathrm{Leb}}$ (and their twists) are conjugate, and therefore have the same spectrum. In addition, it is shown that whenever one of the perturbations from Sections 5.2 and 5.3 satisfies (KL) when approximating $\mathcal{L}_\Omega$, the same perturbation also satisfies (KL) when used to approximate $\mathcal{L}_{\mathrm{Leb}}$. We defer these technical details to the end of the section, and first describe how our methods may be applied to the perturbed cat maps of Section 5.3.3. The methods we consider are:

Fourier approximation of mollified transfer operators. Proposition 5.2.3 says that if we convolve the (possibly twisted) transfer operator $\mathcal{L}_\Omega$ with a locally supported stochastic kernel (parameterised by ϵ), Fourier approximations (of order $n = n(\epsilon)$) of this mollified transfer operator satisfies (KL) as $\epsilon \to 0$ when considered as a family of operators in $L(B_{\mathbb{C}}^{p,q})$ with weak norm $\|\cdot\|_{p-1,q+1}$. That the same holds for $\mathcal{L}_{\text{Leb}}$ follows from Proposition 5.4.3. The Fourier approximations of $\mathcal{L}_{\text{Leb}}$ are numerically accessible and Theorem 5.2.6 and Proposition 5.4.3 then guarantees convergence of the SRB measure (in the $\|\cdot\|_{p-1,q+1}$ norm), convergence of the variance of a $\mathcal{C}^r$ observable, and uniform convergence of the rate function for $\mathcal{C}^r$ observables, as $\epsilon \to 0$.

Direct Fourier approximation via Fejér kernels. Corollary 5.3.4 states that if we convolve the (possibly twisted) transfer operator $\mathcal{L}_\Omega$ with a Fejér kernel (parameterised by n), this sequence of operators in n satisfies (KL) as $n \to \infty$ when considered as a family of operators in $L(B_{\mathbb{C}}^{p,q})$ with weak norm $\|\cdot\|_{p-1,q+1}$. That the same holds for $\mathcal{L}_{\text{Leb}}$ follows from Proposition 5.4.4. The Fejér kernels directly arise from Fourier projections and this second numerical scheme requires only direct Fourier approximation of the transfer operators $\mathcal{L}_{\text{Leb}}$. Theorem 5.3.3 and Proposition 5.4.4 guarantees convergence of the SRB measure (in the $\|\cdot\|_{p-1,q+1}$ norm), convergence of the variance of a $\mathcal{C}^r$ observable, and uniform convergence of the rate function for $\mathcal{C}^r$ observables, as $n \to \infty$.

For the remainder of this section we will only deal with the operator $\mathcal{L}_{\text{Leb}}$, its twists $\mathcal{L}_{\text{Leb}}(z)$ and approximations of both these two operators i.e. $\mathcal{L}_{\text{Leb},\epsilon}$ and $\mathcal{L}_{\text{Leb},\epsilon}(z)$. To simplify notation we drop the reference to Leb.

5.4.1 General setup

We note that $\mathcal{L}_\epsilon(z)$ arising from both (i) the convolution with a locally supported stochastic kernel q_ϵ and (ii) convolution with a Fejér kernel, can be considered as operators on $L^2(\mathbb{T}^2)$ without changing the spectrum of the operators. A numerical approximation $\mathcal{L}_\epsilon(z)$ of the twisted transfer operator $\mathcal{L}(z)$ can be formed in a number of ways, detailed below, but each of these will be based on Fourier approximation. This is a natural approach as we have a periodic spatial domain and the map and observable are smooth. First, we set up the Fourier function basis and L^2-orthogonal projection of the action of $\mathcal{L}_\epsilon$ on these basis functions. Using the usual L^2 inner product $\langle f, g\rangle = \int_{\mathbb{T}^2} f \cdot \overline{g}\, \mathrm{d}\,\mathrm{Leb}$, for $x \in \mathbb{T}^2$ and $\mathbf{j} \in \mathbb{Z}^2$, define a complex Fourier basis $f_{\mathbf{j}}(x) = e^{2\pi i \mathbf{j}\cdot x}$, so that $g = \sum_{\mathbf{j}\in\mathbb{Z}^2} \langle g, f_{\mathbf{j}}\rangle f_{\mathbf{j}} := \sum_{\mathbf{j}\in\mathbb{Z}^2} \hat{g}(\mathbf{j}) f_{\mathbf{j}}$. To obtain a representation of $\mathcal{L}_\epsilon$ in this basis, we compute:

$$\begin{aligned}
[L(z)_\epsilon]_{\mathbf{kj}} := \langle \mathcal{L}(z)_\epsilon f_{\mathbf{k}}, f_{\mathbf{j}}\rangle &= \int_{\mathbb{T}^2}\left(\int_{\mathbb{T}^2} q_\epsilon(x-y)\mathcal{L}(e^{zg(y)} f_{\mathbf{k}}(y))\,\mathrm{d}\,\mathrm{Leb}(y)\right)\overline{f_{\mathbf{j}}(x)}\,\mathrm{d}\,\mathrm{Leb}(x) \\
&= \int_{\mathbb{T}^2}\left(\int_{\mathbb{T}^2} q_\epsilon(x-Ty)e^{zg(y)} f_{\mathbf{k}}(y)\,\mathrm{d}\,\mathrm{Leb}(y)\right)\overline{f_{\mathbf{j}}(x)}\,\mathrm{d}\,\mathrm{Leb}(x) \\
&= \int_{\mathbb{T}^2}\left(\int_{\mathbb{T}^2} q_\epsilon(x-Ty)\overline{f_{\mathbf{j}}(x)}\,\mathrm{d}\,\mathrm{Leb}(x)\right) e^{zg(y)} f_{\mathbf{k}}(y)\,\mathrm{d}\,\mathrm{Leb}(y) \\
&= \int_{\mathbb{T}^2}\left(\int_{\mathbb{T}^2} q_\epsilon(x)e^{-2\pi i\mathbf{j}\cdot(x+Ty)}\,\mathrm{d}\,\mathrm{Leb}(x)\right) e^{zg(y)} f_{\mathbf{k}}(y)\,\mathrm{d}\,\mathrm{Leb}(y) \\
&= \int_{\mathbb{T}^2} \langle q_\epsilon, f_{\mathbf{j}}\rangle e^{-2\pi i\mathbf{j}\cdot(Ty)} e^{zg(y)} f_{\mathbf{k}}(y)\,\mathrm{d}\,\mathrm{Leb}(y) \\
&= \langle q_\epsilon, f_{\mathbf{j}}\rangle \langle f_{-\mathbf{j}}\circ T\cdot e^{zg}, f_{-\mathbf{k}}\rangle \\
&= \hat{q}_\epsilon(\mathbf{j})\cdot \widehat{(f_{-\mathbf{j}}\circ T\cdot e^{zg})}(-\mathbf{k}) \qquad (5.67)
\end{aligned}$$

Notice that (5.67) only involves Fourier transforms of trivial objects (e.g. composition of a basis function with the map, exponential functions, the stochastic kernel, and the basis functions themselves). To obtain spectral information for $\mathcal{L}_\epsilon(z) : L^2 \to L^2$ we may solve the generalised eigenvalue problem $L_\epsilon(z)v_\epsilon(z) = \lambda_\epsilon(z)v_\epsilon(z)$.

5.4.2 Discrete Fourier transform

To numerically approximate the above Fourier transforms, we first truncate the Fourier modes so that $\mathbf{j} \in \{-n/2+1, \ldots, -1, 0, 1, \ldots, n/2\}^2$, where $n = 2^{n'}$ for some $n' \in \mathbb{Z}^+$. Corresponding to this frequency grid is a regular spatial grid on $\mathbb{T}^2$ of the same cardinality; we call these frequency and spatial grids "coarse grids". The L^2 inner products are estimated using MATLAB's two-dimensional discrete fast Fourier transform (DFT) `fft2` on equispaced spatial and frequency grids with cardinalities $N = 2^{N'}$ for some integer $N' \geq n'$; these grids will be referred to as "fine grids". The DFT is a collocation process, and by using $N \geq n$, we evaluate our functions on a finer spatial grid and produce more accurate estimates of the (lower) frequencies in the coarse grid. One may also think of the DFT as a type of interpolation; for fixed n, as N increases we achieve increasingly accurate estimates of the L^2 inner products. The cardinality n^2 of the coarse grid determines the size of the $n^2 \times n^2$ matrix $L_{\epsilon,n(\epsilon)}(z)$ (if convolving with stochastic kernels) or $L_n(z)$ (if convolving with Fejér kernels), while the cardinality N^2 of the fine grid determines the computation effort put into estimating the inner products via the DFT. In our experiments we will use $n = 32, 64, 128$ and $N = 512$.

The kernel q_ϵ will be either:

1. A stochastic kernel given by an L^1-normalised $\mathcal{C}^\infty$ bump function with support restricted to the disk of radius ϵ centred at 0. The particular bump function we use in the numerics is a well-known transformed version of a Gaussian given by $q_\epsilon(x) = (C/\epsilon^2)\exp(-1/(1-\|x/\epsilon\|^2))$ for $x \in B(0,\epsilon)$, where C is a fixed L^1-normalising constant.
2. The square Fejér kernel of order n. Because of the special form of the square Fejér kernel we have that $\hat{q_n}(\mathbf{j}) = (1-|\mathbf{j}_1|/(n/2+1))(1-|\mathbf{j}_2|/(n/2+1))$, which may be inserted directly into (5.67). Another advantage of the Fejér kernel is that no explicit mollification is required, with the "ϵ" slaved to the coarse resolution n.

In our experiments, given a coarse frequency resolution n, we will try to select ϵ so that the stochastic kernel "matches" the Fejér kernel. We do this by choosing ϵ so that $\min_{\mathbf{j}\in\{-n/2+1,\ldots,n/2\}^2} |\hat{q}_\epsilon(\mathbf{j})| \approx \min_{\mathbf{j}\in\{-n/2+1,\ldots,n/2\}^2} |\hat{q}_n(\mathbf{j})|$.

5.4.3 Numerical results

The specific map $T : \mathbb{T}^2 \circlearrowleft$ on which we carry out our numerics is a small perturbation of a linear toral automorphism:

$$T(x_1, x_2) = (2x_1 + x_2 + 2\delta\cos(2\pi x_1), x_1 + x_2 + \delta\sin(4\pi x_2 + 1)),$$

with $\delta = 0.01$. By Proposition 5.3.16 we have that T is an Anosov diffeomorphism and satisfies the conditions of Proposition 5.3.6, which is required in order to rigorously estimate the statistical properties of T using the Fejér kernel as per Corollary 5.3.4. The observable we use when computing the variance and the rate function is $g(x_1, x_2) = \cos(4\pi x_1) + \sin(2\pi x_2)$, displayed in Figure 5.1.

Estimating the SRB measure

Transitive Anosov systems possess a unique Sinai-Ruelle-Bowen (SRB) measure [107, Theorem 1], which is exhibited by trajectories beginning in a full Lebesgue measure subset of $\mathbb{T}^2$. A trajectory of length 1.5×10^5 initialised at a random location is shown in Figure 5.2. To create a numerical approximation of the SRB measure we compute the leading eigenvector of L_n (the matrix associated with the Fejér kernel). Figure 5.3 illustrates the results of using $n = 128, N = 512$. The left panel of Figure 5.3 is shaded so that higher "density" is indicated by darker shading. Note that this compares very well with the density of points in the trajectory shown

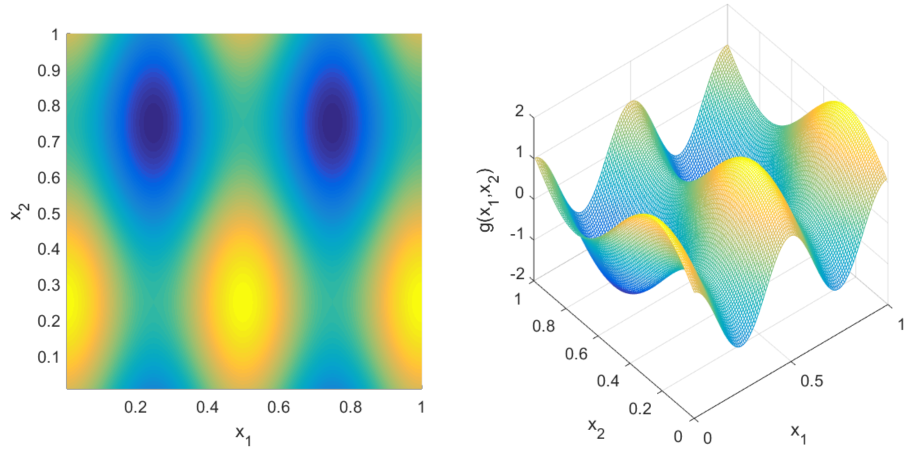

Figure 5.1: Graph of the observable used in the variance and rate function calculations. Left: view from above. Right: view from the side with fine $N \times N$ spatial mesh visible ($N = 512$).

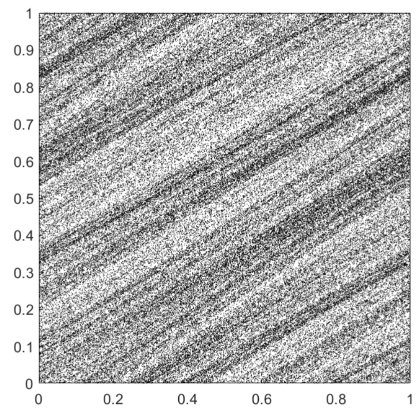

Figure 5.2: A trajectory of length 1.5×10^5 initialised at a random location.

in Figure 5.2, and that Figure 5.3 (left) captures many structures more clearly than the trajectory image. The right panel of Figure 5.3 shows the same image as the left panel, but rotated and with the density plotted along the vertical axis. The high degree of smoothness of the estimate of the SRB measure along unstable directions is evident. Reducing n from 128 to $n = 64$ or $n = 32$ has little effect on the image in unstable directions as these slow oscillations are still well-captured by lower order Fourier modes, but the higher frequency oscillations in stable directions will not be captured as well and the image will be "smoothed" in the stable directions.

As a non-rigorous comparison, we form an Ulam matrix using a 512×512 equipartition of boxes $\{B_1, \ldots, B_{2^{18}}\}$ on $\mathbb{T}^2$. We compute a row-stochastic matrix P_{512} as $[P_{512}]_{ij} = \text{Leb}(B_i \cap T^{-1} B_j) / \text{Leb}(B_i)$, where the entries $[P_{512}]_{ij}$ are estimated

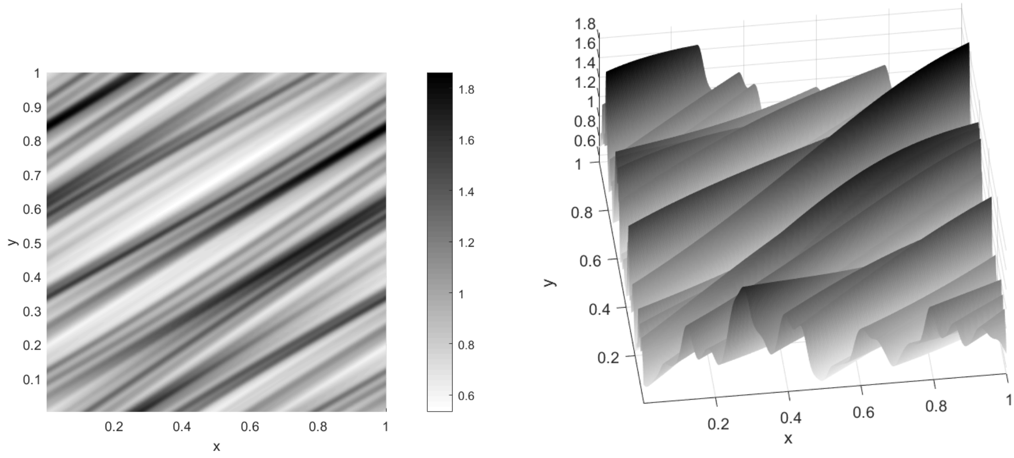

Figure 5.3: Approximations of the SRB measure computed as the leading right eigenvector of L_{128} (Fejér kernel of order 64), using a fine grid cardinality of $N = 512$ to evaluate the Fourier transforms. Left: Darker regions indicate higher "density". Right: The same image rotated and represented in three dimensions with the vertical axis indicating "density".

by uniformly sampling 1600 points in each box and counting the fraction of points initialised in B_i that have their image in B_j. The Ulam estimate of the SRB measure is then obtained as the leading left eigenvector of P_{512}. The images corresponding to Figure 5.3 are shown in Figure 5.4. In comparison to Figure 5.3 two things are

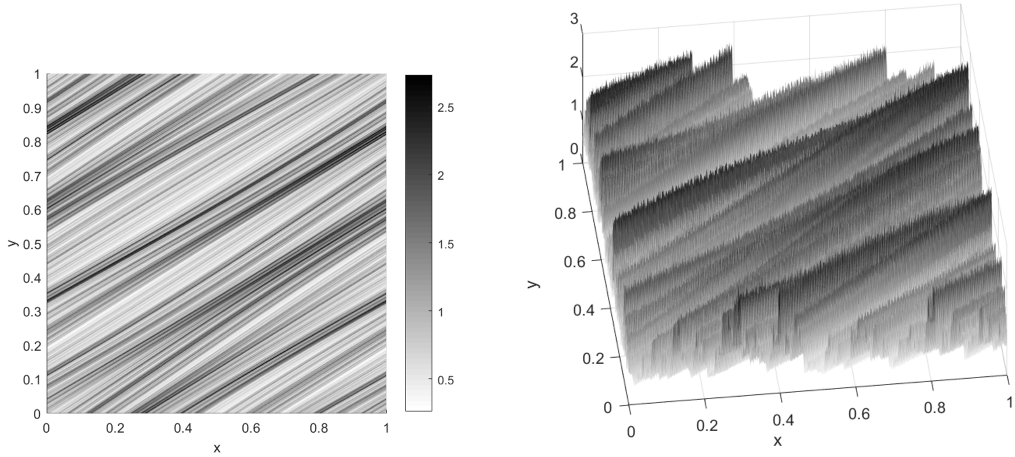

Figure 5.4: Approximations of the SRB measure computed as the leading left eigenvector of P_{512} (Ulam grid of size 512×512). Left: Darker regions indicate higher "density". Right: The same image rotated and represented in three dimensions with the vertical axis indicating "density".

noticeable. Firstly, Figure 5.4(left) appears to produce a finer representation of the SRB measure than Figure 5.3(left), and secondly, the estimate in Figure 5.4(right) is rougher in unstable directions than the estimate in Figure 5.3. Each of these observations is relatively easy to explain at a superficial level through the different approximation bases used. In terms of regularity of approximation basis the Ulam method is very low order (piecewise constant) because it uses a basis of indicator functions on the 512×512 grid. On the other hand, the approximation basis for the Fourier approximation of very high order (analytic). The Ulam basis is thus very flexible and can adapt well to the roughness of the SRB measure in stable directions, but has no apriori smoothness in unstable directions. In contrast, the Fourier basis is less flexible in stable directions, requiring more modes to capture rapid oscillations, but is extremely efficient at approximating smooth functions and easily captures the smooth variation in unstable directions. A recent alternative non-rigorous collocation-based method of SRB measure approximation has been explored in [100] for certain families (Blaschke products) of analytic Anosov maps. In the case of analytic expanding maps, [100] proves that this method produces the true absolutely continuous invariant measure in the limit of increasing numerical resolution.

Estimating the variance

To estimate the variance of a centred observable $g : \mathbb{T}^2 \to \mathbb{R}$ we employ the method described in Section 2.3 of Chapter 2 and Section 3.2.1 of Chapter 3. We use the representation of the (approximate) variance derived from (2.17):

$$\sigma_n^2 := \lambda_n^{(2)}(0) = \int_{\mathbb{T}^2} g^2 v_n(0) + 2g(\mathrm{Id} - \mathcal{L}_n(0))^{-1}\mathcal{L}_n(0)(g v_n(0)) \, \mathrm{d}\,\mathrm{Leb}\,. \tag{5.68}$$

The main difference to the calculations in Chapter 3 is that here we use Fourier approximation, whereas in Chapter 3 we used Ulam's method, which was better suited to the piecewise expanding maps considered there. The key computational component in (5.68) is the solution of a single linear equation to obtain an estimate for $(\mathrm{Id}-\mathcal{L}_n(0))^{-1}\mathcal{L}_n(0)(g v_n(0))$, which is $\frac{d}{dz} v_n(z)$ at $z = 0$. Because our approximate transfer operator $L_n(z)$ is represented in frequency space, we set up and solve this linear equation in frequency space, yielding the DFT of $\frac{d}{dz} v_n(z)$ at $z = 0$. This Fourier transform is then inverted with the inverse DFT to produce the required spatial estimate. Similarly, the DFT of $v_n(0)$ is computed as the leading right eigenvector of $L_n(0)$ and inverted with the inverse DFT to obtain a spatial estimate of $v_n(0)$. These two spatial estimates (analytic functions consisting of a linear combination of Fourier modes) are then evaluated on the fine spatial grid and the integral in (5.68) is computed as a simple Riemann sum over the fine spatial grid. The Ulam-based variance estimates are calculated identically to those in Chapter 3. In Table 5.1 we report variance estimates over a range of coarse grid resolutions to roughly indicate the dependence of the estimates on grid resolution.

Coarse grid resolution n	$n = 32$	$n = 64$	$n = 128$	$n = 256$
Stochastic kernel	0.9359 ($\epsilon = 0.0693$)	0.9342 ($\epsilon = 0.0378$)	0.9337 ($\epsilon = 0.0210$)	
Fejér kernel	0.9447	0.9395	0.9366	
Ulam		0.9320	0.9307	0.9348

Table 5.1: Variance estimates from the two Fourier approximation approaches and Ulam's method. For the two Fourier approximation methods we use a fine frequency grid resolution of $N = 512$ and for Ulam's method we use 1600 sample points per box.

Estimating the rate function

We numerically estimate the rate function $r_g(s) = \sup_{z\in V}(sz - \log|\lambda_\epsilon(z)|)$ for a centred observable $g : \mathbb{T}^2 \to \mathbb{R}$ using the Fejér kernel approach. We create the Fejér kernel estimate $L_n(z)$ of the twisted transfer operator and compute the leading eigenvalue $\lambda_n(z)$ and eigenvector $v_n(z)$ of $L_n(z)$. The leading eigenvector $v_n(z)$ is converted from frequency space to a function on $\mathbb{T}^2$ by evaluating the linear combination (according to the entries of $v_n(z)$) of the n associated Fourier modes on a fine $N \times N$ spatial grid for $N = 512$. For a given s, we are now in a position to find the minimum of $-(sz - \log|\lambda_\epsilon(z)|)$ as a function of z. We used MATLAB's `fminunc` routine (unconstrained function minimisation) with the default quasi-newton option, which takes around four to five iterates to converge to the minimum within a preset tolerance of 10^{-6}. We asked for the values of $r_g(s)$ for s between 0 and 1.8 in steps of 0.1, and initialised the search for the optimal z value using the optimal z from the previous value of s. The results are shown in Figure 5.5 for coarse grids of size $n = 32$ and $n = 64$, with fine grid collocation and function evalution using $N = 512$. Note that the range of g is $[-2, 2]$ (see also Figure 5.1), and that g is already centred with respect to Lebesgue measure on the 2-torus. In the rate function computations we centre g according to our estimate of the SRB measure, but do not expect the range of g to vary significantly. The large values of $r_g(s)$ as s approaches 2 are consistent with this observation.

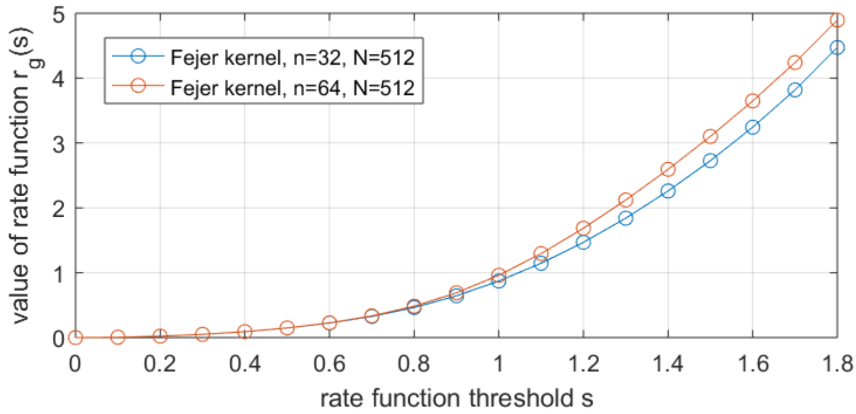

Figure 5.5: Estimates of the rate function r_g using Fejér kernels with $n = 32, 64$ and $N = 512$.

5.4.4 Spectral stability results for $\mathcal{L}_{\mathrm{Leb}}$

In this section we prove the claims made in Remarks 5.2.7 and 5.3.5 regarding the relationship between the operators $\mathcal{L}_\Omega(z)$ and $\mathcal{L}_{\mathrm{Leb}}(z)$, and that the spectral stability results for $\mathcal{L}_\Omega(z)$ from Sections 5.2 and 5.3 imply the spectral stability results for $\mathcal{L}_{\mathrm{Leb}}(z)$.

Proposition 5.4.1. *Let $R : B^{p,q}_{\mathbb{C}} \to B^{p,q}_{\mathbb{C}}$ be defined*

$$Rh = h \cdot \frac{\mathrm{d}\Omega}{\mathrm{d}\,\mathrm{Leb}}.$$

Then $R, R^{-1} \in L(B^{p,q}_{\mathbb{C}}) \cap L(B^{p-1,q+1}_{\mathbb{C}})$ and for every $z \in \mathbb{C}$ we have

$$\mathcal{L}_\Omega(z) = R^{-1} \mathcal{L}_{\mathrm{Leb}}(z) R. \tag{5.69}$$

Hence $\sigma(\mathcal{L}_\Omega(z)) = \sigma(\mathcal{L}_{\mathrm{Leb}}(z))$ for every $z \in \mathbb{C}$.

Proof. The fact that $R \in L(B^{p,q}_{\mathbb{C}}) \cap L(B^{p-1,q+1}_{\mathbb{C}})$ follows from multiplication by $\mathcal{C}^r(\mathbb{T}^m, \mathbb{R})$ functions being continuous on $B^{p,q}_{\mathbb{C}}$ and $B^{p-1,q+1}_{\mathbb{C}}$ [54, Lemma 3.2]. Since $\frac{\mathrm{d}\,\mathrm{Leb}}{\mathrm{d}\Omega} \in \mathcal{C}^r(\mathbb{T}^m, \mathbb{R})$, the same argument implies that R^{-1} exists and is an element of $L(B^{p,q}_{\mathbb{C}})$ and $L(B^{p-1,q+1}_{\mathbb{C}})$.

Let $f, h \in \mathcal{C}^r(\mathbb{T}^m, \mathbb{C})$. By definition we have

$$\begin{aligned}\int \mathcal{L}_\Omega(f) \cdot h \, \mathrm{d}\Omega = \int f \cdot (h \circ T) \, \mathrm{d}\Omega &= \int \left(f \frac{\mathrm{d}\Omega}{\mathrm{d}\,\mathrm{Leb}} \right) \cdot h \circ T \, \mathrm{d}\,\mathrm{Leb} \\ &= \int \mathcal{L}_{\mathrm{Leb}} \left(f \frac{\mathrm{d}\Omega}{\mathrm{d}\,\mathrm{Leb}} \right) \cdot \frac{\mathrm{d}\,\mathrm{Leb}}{\mathrm{d}\Omega} \cdot h \, \mathrm{d}\Omega \\ &= \int (R^{-1} \mathcal{L}_{\mathrm{Leb}} R)(f) \cdot h \, \mathrm{d}\Omega.\end{aligned}$$

Hence $(R^{-1} \mathcal{L}_{\mathrm{Leb}} R) f = \mathcal{L}_\Omega f$ for all $f \in \mathcal{C}^r(\mathbb{T}^m, \mathbb{C})$, and so the same identity holds on $B^{p,q}_{\mathbb{C}}$ by density. The conjugacy relation (5.69) holds for the twisted operators due to the untwisted conjugacy relation and the definition of the twist M_g (see Proposition 5.1.4). One has $\sigma(\mathcal{L}_\Omega(z)) = \sigma(\mathcal{L}_{\mathrm{Leb}}(z))$ immediately from (5.69). □

Proposition 5.4.2. *Let $\{k_\epsilon\}_{\epsilon \in (0,\epsilon_0)} \subseteq L^1(\mathrm{Leb})$ be an $L^1(\mathrm{Leb})$-bounded family. Set $\mathcal{L}_{\Omega,\epsilon} = k_\epsilon * \mathcal{L}_\Omega$ and $\mathcal{L}_{\mathrm{Leb},\epsilon} = k_\epsilon * \mathcal{L}_{\mathrm{Leb}}$. Let $\mathcal{L}_{\Omega,0} = \mathcal{L}_\Omega$ and $\mathcal{L}_{\mathrm{Leb},0} = \mathcal{L}_{\mathrm{Leb}}$. Suppose that $\{\mathcal{L}_{\Omega,\epsilon}\}_{\epsilon \in [0,\epsilon_0)}$ satisfies (KL) and that one of the following conditions holds.*

(K1) For every $\eta > 0$ there exists ϵ_η such that $\operatorname{supp} k_\epsilon \subseteq B(0,\eta)$ for every $\epsilon \in (0, \epsilon_\eta)$.

(K2) $\mathcal{L}_\Omega$ satisfies (5.9) and for every $\eta > 0$ we have $\lim_{\epsilon \to 0} \int_{\mathbb{T}^m \setminus B(0,\eta)} |k_\epsilon| \, \mathrm{d}\,\mathrm{Leb} = 0$

Then $\{\mathcal{L}_{\mathrm{Leb},\epsilon}\}_{\epsilon \in [0,\epsilon_1)}$ satisfies (KL) for some $\epsilon_1 \in (0, \epsilon_0)$.

Proof. We have

$$\mathcal{L}_{\mathrm{Leb},\epsilon} = R \mathcal{L}_{\Omega,\epsilon} R^{-1} + (\mathcal{L}_{\mathrm{Leb},\epsilon} - R \mathcal{L}_{\Omega,\epsilon} R^{-1}) := A_\epsilon + F_\epsilon.$$

We will prove that $\{A_\epsilon\}_{\epsilon \in [0,\epsilon_0)}$ satisfies (KL), and that there exists $\epsilon' \in (0, \epsilon_0)$ such that $\{F_\epsilon\}_{\epsilon \in [0,\epsilon')}$ satisfies the conditions required by Proposition 1.3.1, which will then imply that $\{\mathcal{L}_{\mathrm{Leb},\epsilon}\}_{\epsilon \in [0,\epsilon_1)}$ satisfies (KL) for some $\epsilon_1 \in (0, \epsilon')$.

It is straightforward to confirm that $\{A_\epsilon\}_{\epsilon \in [0,\epsilon_0)}$ satisfies (KL) by using (KL) for $\{\mathcal{L}_{\Omega,\epsilon}\}_{\epsilon \in [0,\epsilon_0)}$, the conjugacy identity (5.69) and the properties of the map R as given in Proposition 5.4.1. For example (KL1) follows from the estimate

$$|||A_\epsilon - \mathcal{L}_{\mathrm{Leb}}||| = \left|\left|\left| R (\mathcal{L}_{\Omega,\epsilon} - \mathcal{L}_\Omega) R^{-1} \right|\right|\right| \le \|R\|_{L(B^{p-1,q+1}_{\mathbb{C}})} |||\mathcal{L}_{\Omega,\epsilon} - \mathcal{L}_\Omega||| \left\| R^{-1} \right\|_{L(B^{p,q}_{\mathbb{C}})}.$$

The proofs of (KL2) and (KL3) follow from similar arguments.

We will now prove that there exists $\epsilon' \in [0, \epsilon_0)$ such that $\{F_\epsilon\}_{\epsilon \in [0,\epsilon')}$ satisfies the conditions required by Proposition 1.3.1. For brevity let $s = \frac{\mathrm{d}\Omega}{\mathrm{d}\,\mathrm{Leb}}$. Let $h \in$

$\mathcal{C}^r(\mathbb{T}^m, \mathbb{R})$, $k \le p$ be a non-negative integer, $W \in \Sigma$, $\{v_i\}_{i=1}^k \subseteq \mathcal{V}^r(W)$ with $\|v_i\|_{\mathcal{C}^r} \le 1$, and $\varphi \in \mathcal{C}_0^{q+k}(W, \mathbb{R})$ with $\|\varphi\|_{\mathcal{C}^{q+k}} \le 1$. Since

$$F_\epsilon R h = R(k_\epsilon * (\mathcal{L}_\Omega h)) - k_\epsilon * (R\mathcal{L}_\Omega h),$$

we have

$$\begin{aligned}(F_\epsilon R h)(x) =& s(x) \int k_\epsilon(y)(\mathcal{L}_\Omega h)(x-y) \,\mathrm{d}\,\mathrm{Leb}(y) \\ &- \int k_\epsilon(y) s(x-y)(\mathcal{L}_\Omega h)(x-y) \,\mathrm{d}\,\mathrm{Leb}(y) \\ =& \int k_\epsilon(y)(\mathcal{L}_\Omega h)(x-y)\left(s(x) - s(x-y)\right) \mathrm{d}\,\mathrm{Leb}(y).\end{aligned}$$

Hence, as multiplication by $\mathcal{C}^r$ functions is continuous on $B_{\mathbb{C}}^{p,q}$ there exists a C_0 such that

$$\begin{aligned}\int_W & v_1 \dots v_k (F_\epsilon R h)(x) \cdot \varphi(x) \,\mathrm{d}x \\ &= \int k_\epsilon(y) \left(\int_W v_1 \dots v_k (\tau_{-y} \mathcal{L}_\Omega h \cdot (s - \tau_{-y} s))(x) \cdot \varphi(x) \,\mathrm{d}x \right) \mathrm{d}\,\mathrm{Leb}(y) \\ &\le \int |k_\epsilon(y)| \left\| \tau_{-y} \mathcal{L}_\Omega h \cdot (s - \tau_{-y} s) \right\|_{p,q} \mathrm{d}\,\mathrm{Leb}(y) \\ &\le C_0 \int |k_\epsilon(y)| \left\| \tau_{-y} \mathcal{L}_\Omega h \right\|_{p,q} \left\| s - \tau_{-y} s \right\|_{\mathcal{C}^r} \mathrm{d}\,\mathrm{Leb}(y),\end{aligned}$$

and so

$$\|F_\epsilon R h\|_{p,q} \le C_0 \int |k_\epsilon(y)| \left\| \tau_{-y} \mathcal{L}_\Omega h \right\|_{p,q} \left\| s - \tau_{-y} s \right\|_{\mathcal{C}^r} \mathrm{d}\,\mathrm{Leb}(y). \tag{5.70}$$

We will bound the right side of (5.70) differently depending on whether (K1) or (K2) holds.

The case where (K1) holds. Recall from (5.5) that $(\tau_{-y}\mathcal{L}_\Omega h)(x) = (\det D_x \tau_{-y}) \cdot (\mathcal{L}_{\Omega, T_y} h)(x)$ where $\mathcal{L}_{\Omega, T_y}$ denotes the transfer operator associated to $T_y := T + y$ by duality with respect to Ω. If we denote $x \mapsto \det D_x \tau_{-y}$ by t_y then $t_y(x) = \frac{\mathrm{d}\Omega}{\mathrm{d}\,\mathrm{Leb}}(x+y) \frac{\mathrm{d}\,\mathrm{Leb}}{\mathrm{d}\Omega}(x)$. Since $\frac{\mathrm{d}\Omega}{\mathrm{d}\,\mathrm{Leb}}, \frac{\mathrm{d}\,\mathrm{Leb}}{\mathrm{d}\Omega} \in \mathcal{C}^r(\mathbb{T}^m, \mathbb{R})$ we have

$$\sup_{y \in \mathbb{T}^m} \|t_y\|_{\mathcal{C}^r} := C_1 < \infty.$$

As noted at the beginning of [54, Section 7], there is a $\mathcal{C}^{r+1}(\mathbb{T}^m, \mathbb{T}^m)$ open neighbourhood U of T such that [54, Lemma 2.2] applies uniformly to every $S \in U$, and so

$$\sup_{S\in U} \|\mathcal{L}_{\Omega,S}\|_{p,q} := C_2 < \infty.$$

Hence, by (K1), there exists an $\epsilon' \in (0, \epsilon_0)$ such that $\left\|\mathcal{L}_{\Omega,T_y}\right\|_{p,q} < C_2$ for every $\epsilon \in (0, \epsilon')$ and $y \in \operatorname{supp} k_\epsilon$. Since multiplication by $\mathcal{C}^r$ functions on $B^{p,q}_{\mathbb{C}}$ [54, Lemma 3.2] is continuous there exists a C_3 independent of h and y such that

$$\|\tau_{-y}\mathcal{L}_\Omega h\|_{p,q} = \left\|t_y \cdot (\mathcal{L}_{\Omega,T_y} h)\right\|_{p,q} \le C_3 \|t_y\|_{\mathcal{C}^r} \left\|\mathcal{L}_{\Omega,T_y} h\right\|_{p,q}.$$

Setting $C' = C_0 C_1 C_2 C_3$ and applying these estimates to (5.70) yields

$$\|F_\epsilon R\|_{p,q} \le C' \left(\int_{\operatorname{supp} k_\epsilon} |k_\epsilon(y)| \, \mathrm{d}\operatorname{Leb}(y) \right) \sup_{y \in \operatorname{supp} k_\epsilon} \|s - \tau_{-y} s\|_{\mathcal{C}^r},$$

provided that $\epsilon \in (0, \epsilon')$. Since $t_y \in \mathcal{C}^\infty(\mathbb{T}^m, \mathbb{R})$, by (K1) we have

$$\lim_{\epsilon\to 0} \sup_{y \in \operatorname{supp} k_\epsilon} \|s - \tau_{-y} s\|_{\mathcal{C}^r} = 0.$$

Recalling that $\{k_\epsilon\}_{\epsilon \in (0,\epsilon_0)}$ is $L^1(\operatorname{Leb})$-bounded, we therefore have

$$\limsup_{\epsilon\to 0} \|F_\epsilon R\|_{p,q} \le C' \limsup_{\epsilon\to 0} \left(\int_{\operatorname{supp} k_\epsilon} |k_\epsilon(y)| \, \mathrm{d}\operatorname{Leb}(y) \right) \sup_{y \in \operatorname{supp} k_\epsilon} \|s - \tau_{-y} s\|_{\mathcal{C}^r} = 0.$$

As $R \in L(B^{p,q}_{\mathbb{C}})$ is invertible it follows that $\lim_{\epsilon\to 0} F_\epsilon = 0$ in $L(B^{p,q}_{\mathbb{C}})$. The same argument can be used to conclude that $\lim_{\epsilon\to 0} F_\epsilon = 0$ in $L(B^{p-1,q+1}_{\mathbb{C}})$, and so $\{F_\epsilon\}_{\epsilon \in [0,\epsilon')}$ satisfies the requirements of Proposition 1.3.1.

The case where (K2) holds. By (5.9) we have

$$\sup_{y\in\mathbb{T}^m} \max\{\|\tau_y \mathcal{L}_\Omega\|_{p,q}, \|\tau_y \mathcal{L}_\Omega\|_{p-1,q+1}\} := C_4 < \infty.$$

Applying this to (5.70) yields

$$\|F_\epsilon R\|_{p,q} \le C_0 C_4 \int |k_\epsilon(y)| \, \|s - \tau_{-y} s\|_{\mathcal{C}^r} \, \mathrm{d}\operatorname{Leb}(y). \tag{5.71}$$

Fix $\eta > 0$. By splitting the integral in (5.71) according to the partition $\mathbb{T}^m = B(0,\eta) \cup (\mathbb{T}^m \setminus B(0,\eta))$ we obtain

$$\int |k_\epsilon(y)| \, \|s - \tau_{-y}s\|_{\mathcal{C}^r} \, \mathrm{d}\,\mathrm{Leb}(y) \leq \|k_\epsilon\|_{L^1} \sup_{y \in B(0,\eta)} \|s - \tau_{-y}s\|_{\mathcal{C}^r} + \sup_{y \notin B(0,\eta)} \|s - \tau_{-y}s\|_{\mathcal{C}^r} \int_{\mathbb{T}^m \setminus B(0,\eta)} |k_\epsilon(y)| \, \mathrm{d}\,\mathrm{Leb}(y).$$

By (K2) we have

$$\limsup_{\epsilon \to 0} \|F_\epsilon R\|_{p,q} \leq C_0 C_4 \left(\sup_{\epsilon \in (0,\epsilon_0)} \|k_\epsilon\|_{L^1} \right) \left(\sup_{y \in B(0,\eta)} \|s - \tau_{-y}s\|_{\mathcal{C}^r} \right), \tag{5.72}$$

where the right side is always finite by virtue of the $L^1(\mathrm{Leb})$ boundedness of $\{k_\epsilon\}_{\epsilon \in (0,\epsilon_0)}$. Since $s \in \mathcal{C}^\infty$, letting $\eta \to 0$ in (5.72) yields $\lim_{\epsilon \to 0} \|F_\epsilon R\|_{p,q} = 0$ in $L(B^{p,q}_{\mathbb{C}})$, which implies that $\lim_{\epsilon \to 0} \|F_\epsilon\|_{p,q} = 0$ by the invertibility of R. As before, the same argument can be used to conclude that $\lim_{\epsilon \to 0} F_\epsilon = 0$ in $L(B^{p-1,q+1}_{\mathbb{C}})$, and so there exists some $\epsilon' \in (0, \epsilon_0)$ such that $\{F_\epsilon\}_{\epsilon \in [0,\epsilon')}$ satisfies the requirements of Proposition 1.3.1. □

Using Proposition 5.4.2 we may now confirm that our spectral stability results for $\mathcal{L}_\Omega$ from Sections 5.2 and 5.3 also apply to $\mathcal{L}_{\mathrm{Leb}}$.

Proposition 5.4.3. *If Proposition 5.2.3 applies to $\mathcal{L}_\Omega$ then it applies to $\mathcal{L}_{\mathrm{Leb}}$ too. Hence Theorem 5.2.6 holds verbatim if $\mathcal{L}_\Omega$ is replaced by $\mathcal{L}_{\mathrm{Leb}}$.*

Proof. Suppose that $\{q_\epsilon\}_{\epsilon \in (0,\epsilon_0)} \subseteq \mathcal{C}^\infty(\mathbb{T}^m, \mathbb{R})$ is a family of kernels satisfying (S1) and (S2). Recall the definition of Q_ϵ from the beginning of Section 5.2. By Lemma 5.2.2 there exists $\epsilon_1 \in (0, \epsilon_0)$ so that $\{Q_\epsilon \mathcal{L}_\Omega\}_{\epsilon \in [0,\epsilon_1)}$ satisfies (KL). Since $\{q_\epsilon\}_{\epsilon \in (0,\epsilon_0)}$ is $L^1(\mathrm{Leb})$-bounded (by (S1)) and satisfies (S2), by Proposition 5.4.2 we may conclude that there exists $\epsilon_2 \in (0, \epsilon_1)$ such that $\{Q_\epsilon \mathcal{L}_{\mathrm{Leb}}\}_{\epsilon \in [0,\epsilon_2)}$ satisfies (KL) too. The proof of Proposition 5.2.3 holds verbatim with $\mathcal{L}_\Omega$ replaced with $\mathcal{L}_{\mathrm{Leb}}$, as does that of Theorem 5.2.6. □

Proposition 5.4.4. *If Proposition 5.3.1 applies to $\mathcal{L}_\Omega$ then it applies to $\mathcal{L}_{\mathrm{Leb}}$ too. Hence Theorem 5.3.3 and Corollary 5.3.4 hold verbatim if $\mathcal{L}_\Omega$ is replaced by $\mathcal{L}_{\mathrm{Leb}}$.*

Proof. Let $\{q_\epsilon\}_{\epsilon \in (0,\epsilon_1)} \subseteq L^1(\mathrm{Leb})$ be a family of stochastic kernels satisfying (S1) and (S3). Since $\mathcal{L}_\Omega$ satisfies (5.9), by Proposition 5.3.1 there exists some $\epsilon_2 \in (0, \epsilon_1)$ such that $\{q_\epsilon * \mathcal{L}_\Omega\}_{\epsilon \in [0,\epsilon_2)}$ satisfies (KL). The family $\{q_\epsilon\}_{\epsilon \in (0,\epsilon_1)}$ is $L^1(\mathrm{Leb})$-bounded by (S1). This, together with (S2) and the fact that (5.9) holds for $\mathcal{L}_\Omega$, means that we can apply Proposition 5.4.2 to conclude that there exists $\epsilon_3 \in (0, \epsilon_2)$ such that $\{q_\epsilon * \mathcal{L}_{\text{Leb}}\}_{\epsilon \in [0, \epsilon_3)}$ satisfies (KL) too. The proofs of Theorem 5.3.3 and Corollary 5.3.4 are the same as before. $\square$

Part II

Random Systems

Chapter 6

Preliminaries and conventions

In this chapter we recall some preliminary theory that will be extensively used throughout Part II of this thesis. Section 6.1 reviews the Grassmannian associated to a Banach space, while Section 6.2 serves as a primer on the theory of Saks spaces.

We now fix some conventions for the remainder of Part II. We shall only consider Banach spaces over $\mathbb{C}$. If X_1 and X_2 are Banach spaces then $L(X_1, X_2)$ denotes the set of bounded operators from X_1 to X_2. When $\|\cdot\|$ is a norm on a vector space X we denote the associated closed-unit ball by either $B_{\|\cdot\|}$ or, when unambiguous, B_X. If X_1 and X_2 are topological vector spaces then we write $X_1 \hookrightarrow X_2$ to mean that X_1 is continuously included into X_2. We denote the spectrum of an operator $A \in L(X)$ by $\sigma(A)$, the spectral radius by $\rho(A)$, and the essential spectral radius by $\rho_{\mathrm{ess}}(A)$. When (Y, d) is a metric space the Borel σ-algebra on Y is denoted $\mathcal{B}_Y$.

6.1 Graphs and the Grassmannian

This section summarises some old, but not particularly well-known, material for the reader's convenience, and has been collated from [65, Chapter IV, §2 and §4], [15, Section 2.1], [43, Section 2], and [87, Appendix A.2]. If $(X, \|\cdot\|)$ is a Banach space then the set of closed subspaces of X is called the Grassmannian of X, and is denoted by $\mathcal{G}(X)$. It is a complete metric space when equipped with the metric

$$d_H(E, F) = \max \left\{ \sup_{\substack{e \in E \\ \|e\|=1}} \inf_{\substack{f \in F \\ \|f\|=1}} \|e - f\| , \sup_{\substack{f \in F \\ \|f\|=1}} \inf_{\substack{e \in E \\ \|e\|=1}} \|e - f\| \right\}.$$

The metric d_H is hard to work with directly. Instead, it is convenient to work with the *gap* between two subspaces:

$$\operatorname{Gap}(E,F) = \sup_{\substack{e\in E\\ \|e\|=1}} \inf_{f\in F} \|e-f\|\,.$$

We can work with the gap in place of d_H due to the following inequality

$$\max\{\operatorname{Gap}(E,F),\operatorname{Gap}(F,E)\} \le d_H(E,F) \le 2\max\{\operatorname{Gap}(E,F),\operatorname{Gap}(F,E)\}. \tag{6.1}$$

We say that $E, F \in \mathcal{G}(X)$ are topologically complementary subspaces if $E+F = X$ and $E\cap F = \emptyset$, in which case we will write $E\oplus F = X$. Denote by $\Pi_{E||F}$ the projection onto E and parallel to F (i.e. the image and kernel of $\Pi_{E||F}$ are E and F, respectively), and note that $\Pi_{E||F} \in L(X)$ by the closed graph theorem. For every $d\in\mathbb{Z}^+$ we denote by $\mathcal{G}_d(X)$ and $\mathcal{G}^d(X)$ the sets of closed d-dimensional and d-codimensional subspaces, respectively. The sets $\mathcal{G}_d(X)$ and $\mathcal{G}^d(X)$ are closed for every $d\in\mathbb{Z}^+$. For each $F\in\mathcal{G}(X)$ the set

$$\mathcal{N}(F) = \{E \in \mathcal{G}(X) : E\oplus F = X\}$$

is open in $\mathcal{G}(X)$, and has a convenient representation in terms of certain charts. Specifically, for any $E\in\mathcal{N}(F)$ we define $\Phi_{E\oplus F} : \mathcal{N}(F) \to L(E,F)$ by

$$\Phi_{E\oplus F}(E') = \left(\Pi_{E||F}\big|_{E'}\right)^{-1} - \operatorname{Id}.$$

We call $\Phi_{E\oplus F}$ the graph representation of $\mathcal{N}(F)$ induced by $E\oplus F$. The following lemma, which is an easy exercise, confirms that $\Phi_{E\oplus F}$ is well-defined.

Lemma 6.1.1. *If $F\in\mathcal{G}(X)$ and $E_1, E_2 \in \mathcal{N}(F)$ then $\Pi_{E_1||F} : E_2 \to E_1$ is invertible.*

We summarise the properties of the graph representation in the next proposition.

Proposition 6.1.2. *If $E\oplus F = X$ then the associated graph representation $\Phi_{E\oplus F}$ is a homeomorphism. Moreover, for every $E'\in\mathcal{N}(F)$ we have*

$$\Pi_{E'||F} = (\operatorname{Id} + \Phi_{E\oplus F}(E'))\Pi_{E||F}, \tag{6.2}$$

and for every $A \in L(E,F)$ we have

$$\Phi_{E\oplus F}^{-1}(A) = (\operatorname{Id} + A)(E). \tag{6.3}$$

The identities (6.2) and (6.3) follow from straightforward algebraic manipulations. That $\Phi_{E\oplus F}$ is a homeomorphism is a consequence of the following two lemmas.

Lemma 6.1.3. *If $E \oplus F = X$ and $A_1, A_2 \in L(E,F)$ then*

$$d_H(\Phi_{E\oplus F}^{-1}(A_1), \Phi_{E\oplus F}^{-1}(A_2)) \leq 2 \left\|\Pi_{E||F}\right\| \|A_1 - A_2\| .$$

Proof. For every $\epsilon > 0$ there exists $u_1 \in \Phi_{E\oplus F}^{-1}(A_1)$ with $\|u_1\| = 1$ such that

$$\operatorname{Gap}(\Phi_{E\oplus F}^{-1}(A_1), \Phi_{E\oplus F}^{-1}(A_2)) \leq \epsilon + \inf_{u_2 \in \Phi_{E\oplus F}^{-1}(A_2)} \|u_1 - u_2\| .$$

Since $A_1 = \left(\Pi_{E||F}\big|_{\Phi_{E\oplus F}^{-1}(A_1)}\right)^{-1} - \operatorname{Id}$, by taking $u = \Pi_{E||F} u_1$ and $u_2 = (\operatorname{Id} + A_2)u$ we get

$$\begin{aligned}\operatorname{Gap}(\Phi_{E\oplus F}^{-1}(A_1), \Phi_{E\oplus F}^{-1}(A_2)) &\leq \epsilon + \|(\operatorname{Id} + A_1)u - (\operatorname{Id} + A_2)u\| \\ &\leq \epsilon + \|A_1 - A_2\| \left\|\Pi_{E||F}\right\| .\end{aligned}$$

Since ϵ was arbitrary the same inequality holds for $\epsilon = 0$. The same bound clearly holds $\operatorname{Gap}(\Phi_{E\oplus F}^{-1}(A_2), \Phi_{E\oplus F}^{-1}(A_1))$, and so we obtain the required inequality from (6.1). ∎

Lemma 6.1.4. *If $E \oplus F = X$ and $E_1, E_2 \in \mathcal{N}(F)$ then*

$$\|\Phi_{E\oplus F}(E_1) - \Phi_{E\oplus F}(E_2)\| \leq \min\left\{\left\|\Pi_{F||E_1}\right\| \left\|\Pi_{E_2||F}\right\|, \left\|\Pi_{F||E_2}\right\| \left\|\Pi_{E_1||F}\right\|\right\} d_H(E_1, E_2).$$

Proof. By (6.2) we have $\Pi_{E_i||F} = (\operatorname{Id} + \Phi_{E\oplus F}(E_i))\Pi_{E||F}$ for $i = 1, 2$, and so $\Pi_{E_i||F} - \Pi_{E||F} = \Phi_{E\oplus F}(E_i)$ on E. Thus

$$\|\Phi_{E\oplus F}(E_1) - \Phi_{E\oplus F}(E_2)\| = \left\|\Pi_{E_1||F} - \Pi_{E_2||F}\right\| = \left\|\Pi_{F||E_1}\Pi_{E_2||F}\right\| \leq \left\|\Pi_{F||E_1}\big|_{E_2}\right\| \left\|\Pi_{E_2||F}\right\| .$$

For $u_i \in E_i$ with $\|u_i\| = 1$, $i \in \{1, 2\}$, we have

$$\left\|\Pi_{F||E_1}\right\| \|u_2 - u_1\| \geq \left\|\Pi_{F||E_1}(u_2 - u_1)\right\| = \left\|\Pi_{F||E_1} u_2\right\| ,$$

Taking the infimum over u_1 and then the supremum over u_2 yields

$$\begin{aligned}\sup_{\substack{u_2\in E_2\\ \|u_2\|=1}} \inf_{\substack{u_1\in E_2\\ \|u_1\|=1}} \|u_1-u_2\| &\geq \left(\left\|\Pi_{F||E_1}\right\|\right)^{-1}\left\|\Pi_{F||E_1}\big|_{E_2}\right\| \\ &\geq \left(\left\|\Pi_{F||E_1}\right\|\left\|\Pi_{E_2||F}\right\|\right)^{-1}\|\Phi_{E\oplus F}(E_1)-\Phi_{E\oplus F}(E_2)\|.\end{aligned}$$

We obtain the required inequality upon noting that the roles of E_1 and E_2 may be swapped in the above argument, and then recalling the definition of d_H. □

Suppose that X_1, X_2 are Banach spaces with $E_i \oplus F_i = X_i$ for $i \in \{1,2\}$, and let $S \in L(X_1, X_2)$ and $d \in \mathbb{Z}^+$. Then S induces natural actions on $\mathcal{G}_d(X_1)$ and $\mathcal{G}(X_2)$, defined by $V_1 \in \mathcal{G}_d(X_1) \mapsto S(V_1) \in \mathcal{G}(X_2)$ and $V_2 \in \mathcal{G}(X_2) \mapsto S^{-1}(V_2) \in \mathcal{G}(X_1)$, respectively. If $\dim(E_1) = d = \dim(E_2)$ then these mappings induce actions on the graph representations of $\mathcal{N}(F_1)$ and $\mathcal{N}(E_2)$. If $U \in L(E_1, F_1)$ is such that $\Pi_{E_2||F_2} S(\mathrm{Id}+U)\big|_{E_1} : E_1 \to E_2$ is invertible then we define the forward graph transform of U by S to be

$$S^*U = \Pi_{F_2||E_2} S(\mathrm{Id}+U)\left(\Pi_{E_2||F_2} S(\mathrm{Id}+U)\big|_{E_1}\right)^{-1},$$

in which case $S^*U \in L(E_2, F_2)$. Alternatively, if $\Pi_{E_2||F_2}(\mathrm{Id}-U\Pi_{F_2||E_2})S : E_1 \to E_2$ is invertible for $U \in L(F_2, E_2)$ then backward graph transform of U by S is defined to be

$$S_*U = \left(\Pi_{E_2||F_2}(\mathrm{Id}-U\Pi_{F_2||E_2})S\big|_{E_1}\right)^{-1}(U\Pi_{F_2||E_2} - \Pi_{E_2||F_2})S.$$

Using Proposition 6.1.2, a quick calculation confirms that S^* and S_* agree with the usual action of an operator on a subspace.

Proposition 6.1.5. *Fix $S \in L(X_1, X_2)$ and suppose that $E_i \oplus F_i = X_i$ for $i = 1, 2$ where $\dim(E_1) = d = \dim(E_2)$ for some $d \in \mathbb{Z}^+$.*

1. *For any $E' \in \mathcal{N}(F_1)$ such that $\Pi_{E_2||F_2} S \Pi_{E'||F_1} : E_1 \to E_2$ is invertible we have*

$$S(E') = \Phi^{-1}_{E_2\oplus F_2}\left(S^*(\Phi_{E_1\oplus F_1}(E'))\right).$$

2. *For any $F' \in \mathcal{N}(E_2)$ such that $\Pi_{E_2||F_2}\Pi_{E_2||F'} S : E_1 \to E_2$ is invertible we have*

$$S^{-1}(F') = \Phi^{-1}_{F_1\oplus E_1}(S_*(\Phi_{F_2\oplus E_2}(F'))).$$

6.2 A Saks space primer

In this section we review the basic theory of Saks spaces, and prove some new results of relevance to dynamical systems. In particular, Saks spaces provide a unified abstract framework for many of the concepts from Part I e.g. the strong and weak norms of Keller-Liverani perturbation, and Lasota-Yorke inequalities. More a thorough introduction we refer the reader to [26, Chapter 1]. Throughout this section X will denote a vector space.

Lemma 6.2.1 ([26, Lemma 3.1])**.** *Let X be a vector space, τ be a locally convex topology on X, and $\|\cdot\|$ be a norm on X. Then the following are equivalent:*

1. *$B_{\|\cdot\|}$ is τ-closed;*
2. *$\|\cdot\|$ is lower-semicontinuous for τ;*
3. *$\|\cdot\| = \sup\{\varphi : \varphi$ is a τ-continuous seminorm with $\varphi \le \|\cdot\|\}$.*

Definition 6.2.2 (Saks space)**.** *Let $(X, \|\cdot\|)$ be a normed space and τ be a Hausdorff locally convex topology on X such that $B_{\|\cdot\|}$ is τ-bounded and any of the conditions from Lemma 6.2.1 are satisfied. Denote by $\gamma[\|\cdot\|, \tau]$ the finest linear topology on X that coincides with τ on $B_{\|\cdot\|}$. The tuple $(X, \|\cdot\|, \tau)$ equipped with the topology $\gamma[\|\cdot\|, \tau]$ is called a Saks space; when clear we simply denote this space by X and the topology by γ. We say that X is complete (resp. compact, pre-compact) if $B_{\|\cdot\|}$ is τ-complete (resp. τ-compact, τ-pre-compact).*

Remark 6.2.3. If X is complete as a Saks space then $(X, \|\cdot\|)$ is a Banach space. The converse is false.

Remark 6.2.4. Sometimes one produces a tuple $(X, \|\cdot\|, \tau)$ satisfying the Definition 6.2.2 bar for the conditions in Lemma 6.2.1. In such a case we could instead consider the Saks spaces $(X, \|\cdot\|', \tau)$, where $\|\cdot\|'$ denotes the Minkowski functional[1] associated to the τ-closure of $\|\cdot\|$. We do not lose any continuous linear maps by performing this procedure [26, Lemma 3.3].

Remark 6.2.5. As outlined in [26, Chapter 1, Section 3.6], there is a canonical completion of a non-complete Saks space $(X, \|\cdot\|, \tau)$. Let $\overline{X}_\tau$ denote the τ-completion of X, and define $\|\cdot\|_\tau$ to be the Minkowski functional of the τ-completion of $B_{\|\cdot\|}$ in $\overline{X}_\tau$. If $\overline{X}$ denotes the linear span of $B_{\|\cdot\|_\tau}$ then $(\overline{X}, \|\cdot\|_\tau, \tau)$ is a complete Saks

[1] If K is a balanced, convex body in a vector space X then the Minkowski functional of K is the map $\rho_K : V \to [0, \infty)$ defined by $\rho_K(x) = \inf\{\lambda \in [0, \infty) : \lambda x \in K\}$. The Minkowski functional of a balanced, convex body is always a seminorm, and if K has non-empty interior then it is also a norm.

space: the Saks space completion of $(X, \|\cdot\|, \tau)$. We refer the reader to [26, Proposition 3.8] and the discussion at the end of [26, Chapter 1, Section 3.6] for further properties of the Saks space completion.

Example 6.2.6. Let $(X, \|\cdot\|)$ be a Banach space, and let τ denote the weak-star topology on X^*. Then $(X^*, \|\cdot\|^*, \tau)$ is a compact Saks space by the Banach-Alaoglu Theorem. Actually, every compact Saks space has this form (see Proposition 6.2.21).

Example 6.2.7. Suppose that $(X_i, \|\cdot\|_i)$, $i \in \{1, 2\}$ are Banach spaces with $X_2 \hookrightarrow X_1$. Hence $B_{\|\cdot\|_2}$ is $\|\cdot\|_1$-bounded, however it may not be the case that $B_{\|\cdot\|_2}$ is $\|\cdot\|_1$-closed. If we let $\|\cdot\|_2'$ denote the Minkowski functional of the $\|\cdot\|_1$-completion of $B_{\|\cdot\|_2}$, then $(X, \|\cdot\|_2', \|\cdot\|_1)$ is a Saks space per Remark 6.2.4. The relevance of this construction to the functional analytic approach to dynamical systems was recognised in [14, Section 2.7]; in addition, in [14] a formula for $\|\cdot\|_2'$ was obtained:

$$\|v\|_2' = \lim_{\delta \to 0} \inf\{\|w\|_2 : \|w - v\|_1 \leq \delta\}. \tag{6.4}$$

For a Saks space $(X, \|\cdot\|, \tau)$ it is possible to give an explicit description of a topological basis for $\gamma[\|\cdot\|, \tau]$. If $(U_n)_{n \in \mathbb{Z}^+}$ denotes a family of absolutely convex τ-open neighbourhoods of 0, then all the sets of the form

$$\bigcup_{n=1}^{\infty} (U_1 \cap B_{\|\cdot\|} + \cdots + U_n \cap 2^{n-1} B_{\|\cdot\|}) \tag{6.5}$$

form a neighbourhood basis about 0 for a locally convex topology on X. By [26, Proposition 1.5], this locally convex topology is the $\gamma[\|\cdot\|, \tau]$ topology.

Remark 6.2.8. Any Banach space $(X, \|\cdot\|)$ induces a Saks space with the structure $(X, \|\cdot\|, \|\cdot\|)$. From the definition of the neighbourhood basis for $\gamma[\|\cdot\|, \|\cdot\|]$, it is clear that the $\|\cdot\|$-topology is equivalent to $\gamma[\|\cdot\|, \|\cdot\|]$.

Let $(X, \|\cdot\|, \tau)$ be a Saks space. Despite X being a locally convex vector space, for practical purposes it is better to forget this characterisation and adopt the following philosophy: provided that one works on $\|\cdot\|$-bounded sets, the topological properties of γ are the same as τ. The following three propositions demonstrate this principle.

Proposition 6.2.9 ([26, Proposition 1.10])**.** *Suppose $(x_n)_{n \in \mathbb{Z}^+} \subseteq X$ and $x \in X$. Then $(x_n)_{n \in \mathbb{Z}^+}$ is γ-convergent to x if and only if $(x_n)_{n \in \mathbb{Z}^+}$ is $\|\cdot\|$-bounded and τ-convergent to x.*

Proposition 6.2.10 ([26, Proposition 1.11, 1.12])**.** *If $V \subseteq X$ then:*

1. *V is γ-bounded if and only if it is $\|\cdot\|$-bounded.*
2. *V is γ-compact (resp. γ-pre-compact) if and only if it is $\|\cdot\|$-bounded and τ-compact (resp. τ-pre-compact).*

Proposition 6.2.11 ([26, Corollary 1.6])**.** *If $(X, \|\cdot\|, \tau)$ and $(X, \|\cdot\|, \tau')$ are Saks spaces then $\gamma[\|\cdot\|, \tau]$ and $\gamma[\|\cdot\|, \tau']$ are equivalent if and only if τ and τ' are equivalent on $B_{\|\cdot\|}$.*

Having described the basic theory of Saks spaces, we mention a few more concrete examples.

Example 6.2.12. Let $\mathcal{P}$ denote the set of strictly increasing finite sequences in S^1 (the circle). Fix $p \in [1, \infty)$. For $f \in L^p(S^1)$ the p-variation of f to be

$$\mathrm{Var}_p(f) = \inf \left\{ \left(\sup_{\{x_i\}_{i=0}^n \in \mathcal{P}} \sum_{i=1}^n |g(x_i) - g(x_{i-1})|^p \right)^{1/p} : g = f \text{ a.e.} \right\}.$$

The set of functions of bounded p-variation on S^1 is

$$\mathrm{BV}_p(S^1) = \{f \in L^1(S^1) : \mathrm{Var}_p(f) < \infty\}.$$

Functions of bounded p-variation have been used to study the statistical properties of piecwise expanding dynamical systems: for $p = 1$ see [9, Chapter 3] or [20], while for general p see [69]. On $\mathrm{BV}_p(S^1)$ the map $f \mapsto \mathrm{Var}_p(f)$ is a seminorm and lower-semicontinuous with respect to $\|\cdot\|_{L^p}$. It follows that $\mathrm{BV}_p(S^1)$ is a Banach space when endowed with the norm $\|\cdot\|_{\mathrm{BV}_p} = \|\cdot\|_{L^p} + \mathrm{Var}_p(\cdot)$, and is also a Saks space with structure $(\mathrm{BV}_p(S^1), \|\cdot\|_{\mathrm{BV}_p}, \|\cdot\|_{L^p})$. In fact, standard arguments show that $(\mathrm{BV}_p(S^1), \|\cdot\|_{\mathrm{BV}_p}, \|\cdot\|_{L^p})$ is a compact Saks space [69].

Example 6.2.13. Fix $p \in [1, \infty)$. The Sobolev space $W^{1,p}(S^1)$ is defined by

$$W^{1,p}(S^1) = \{f \in L^p(S^1) : f' \text{ exists in the weak sense and } \|f'\|_{L^p} < \infty\}.$$

Each $W^{1,p}(S^1)$ becomes a Banach space when equipped with the norm $\|f\|_{W^{1,p}} = \|f'\|_{L^p} + \|f\|_{L^p}$. It is well-known that $W^{1,1}(S^1)$ coincides with the set of absolutely continuous functions on S^1, and that $W^{1,p}(S^1) \subseteq W^{1,1}(S^1)$. Hence every $f \in W^{1,p}(S^1)$ is Riemann integrable. A short calculation then shows that $\mathrm{Var}_p(f) = \|f'\|_{L^p}$ and so $\|f\|_{\mathrm{BV}} = \|f\|_{W^{1,p}}$ for every $f \in W^{1,p}(S^1)$. By Example 6.2.12 it follows that $(W^{1,p}(S^1), \|\cdot\|_{W^{1,p}}, \|\cdot\|_{L^p})$ is a pre-compact Saks space. In fact, it is a straightforward exercise to show that $(\mathrm{BV}_p(S^1), \|\cdot\|_{\mathrm{BV}_p}, \|\cdot\|_{L^p})$ is the Saks space completion of $(W^{1,p}(S^1), \|\cdot\|_{W^{1,p}}, \|\cdot\|_{L^p})$.

An obvious question at this stage is whether $\gamma[\|\cdot\|, \tau]$ is metrisable, since a positive answer would reduce the study of Saks spaces to that of classically studied objects. For interesting examples, however, this is never the case.

Proposition 6.2.14 ([26, Proposition 1.14])**.** *If $\gamma[\|\cdot\|, \tau]$ is metrisable then τ and $\|\cdot\|$ are equivalent, in which case $\gamma[\|\cdot\|, \tau]$ and $\|\cdot\|$ are equivalent too.*

We now specialise to the case where τ is induced by a norm $|\cdot|$. In this case we call $(X, \|\cdot\|, |\cdot|)$ a *normed* Saks space[2]. Normed Saks spaces are also known as two-norm spaces, due to the series of papers by Alexiewicz and Semadeni (see e.g. [2, 3, 4]). If Saks space $(X, \|\cdot\|, |\cdot|)$ is a normed Saks space such that $|\cdot| \leq \|\cdot\|$ then we say $(X, \|\cdot\|, |\cdot|)$ is *normal*. Since $B_{\|\cdot\|}$ is $|\cdot|$-bounded we can make any normed Saks space normal after rescaling either $|\cdot|$ or $\|\cdot\|$.

We turn our attention to continuous linear maps between normed Saks spaces[3]. Let $(X_i, \|\cdot\|_i, |\cdot|_i)$, $i \in \{1, 2\}$, be Saks spaces. Let $L_S(X_1, X_2)$ denote the set of continuous linear operators from X_1 to X_2, $\|\cdot\|$ denote the strong operator norm:

$$\|A\| = \sup_{\|f\|_1 = 1} \|Af\|_2 ,$$

and $|||\cdot|||$ denote the triple norm:

$$|||A||| = \sup_{\|f\|_1 = 1} |Af|_2 .$$

Proposition 6.2.15. *$(L_S(X_1, X_2), \|\cdot\|, |||\cdot|||)$ is a Saks space.*

The following two result are used to prove Proposition 6.2.15.

Proposition 6.2.16 ([26, Proposition 1.9])**.** *Suppose that $B_{\|\cdot\|}$ is τ-metrisable, then a linear mapping from $(X_1, \|\cdot\|, \tau)$ into a topological vector space X_2 is continuous if and only if it is sequentially continuous.*

Lemma 6.2.17. *If $(X_i, \|\cdot\|_i, |\cdot|_i)$, $i \in \{1, 2\}$, are Saks spaces then $L_S(X_1, X_2) \subseteq L(X_1, X_2)$.*

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[2]This terminology should cause no confusion in view of Proposition 6.2.14

[3]Our approach here is based on [25, Chapter 1, Section 3.11] rather than on [26, Chapter 1, Section 3.16], which is the equivalent section in the second edition.

Proof. As A is continuous it maps $\gamma[\|\cdot\|_1, |\cdot|_1]$-bounded sets to $\gamma[\|\cdot\|_2, |\cdot|_2]$-bounded sets. As $B_{\|\cdot\|_1}$ is $\gamma[\|\cdot\|_1, |\cdot|_1]$-bounded it follows that $A(B_{\|\cdot\|_1})$ is $\gamma[\|\cdot\|_2, |\cdot|_2]$-bounded. Proposition 6.2.10 says that the $\gamma[\|\cdot\|_2, |\cdot|_2]$-bounded sets are exactly the $\|\cdot\|_2$-bounded sets, and so $A(B_{\|\cdot\|_1})$ is $\|\cdot\|_2$-bounded. Thus $A \in L(X_1, X_2)$. □

The proof of Proposition 6.2.15. Without loss of generality we may assume that both X_1 and X_2 are normal Saks spaces. By Lemma 6.2.17 we have $L_S(X_1, X_2) \subseteq L(X_1, X_2)$. Since $|||\cdot||| \leq \|\cdot\|$ it follows that $|||\cdot|||$ is finite on $L_S(X_1, X_2)$, and that $B_{\|\cdot\|}$ is $|||\cdot|||$-bounded. It remains to verify one of the conditions from Lemma 6.2.1: we will show that $B_{\|\cdot\|}$ is $|||\cdot|||$-closed. Suppose that $\{A_n\}_{n\in\mathbb{Z}^+} \subseteq B_{\|\cdot\|} \cap L_S(X_1, X_2)$ is a $|||\cdot|||$-convergent sequence with limit $A \in L_S(X_1, X_2)$. For every $\epsilon > 0$ there exists $f_\epsilon \in X_1$ with $\|f_\epsilon\|_1 = 1$ such that $\|A\| \leq \|Af_\epsilon\|_2 + \epsilon$. Since $\lim_{n\to\infty} |||A_n - A||| = 0$ we have $\lim_{n\to\infty} |(A_n - A)f_\epsilon|_2 = 0$. Moreover, $\|A_n f_\epsilon\|_2 \leq 1$ for every $n \in \mathbb{Z}^+$. Since $(X_2, \|\cdot\|_2, |\cdot|_2)$ satisfies each of the conditions in Lemma 6.2.1 we have $\|Af_\epsilon\|_2 \leq 1$, which implies that $\|A\| \leq 1 + \epsilon$ for every $\epsilon > 0$. Hence $\|A\| \leq 1$ and so $A \in B_{\|\cdot\|}$ i.e. $B_{\|\cdot\|}$ is $|||\cdot|||$-closed in $L_S(X_1, X_2)$. It follows that $(L_S(X_1, X_2), \|\cdot\|, |||\cdot|||)$ is a Saks space, as claimed. □

We note that $L_S(X_1, X_2)$ is not necessarily equal to $L(X_1, X_2)$, although the proof of Proposition 6.2.15 also implies that $(L(X_1, X_2), \|\cdot\|, |||\cdot|||)$ is a Saks space. The following proposition gives such a quantitative characterisation of $L_S(X_1, X_2)$, as well as a characterisation of equicontinuous families of operators in $L_S(X_1, X_2)$.

Proposition 6.2.18. *Suppose $(X_i, \|\cdot\|_i, |\cdot|_i)$, $i \in \{1, 2\}$ are Saks spaces, that $\mathcal{A}$ is an index set, and that for each $\alpha \in \mathcal{A}$ there exists a linear map $A_\alpha : X_1 \to X_2$. Then $\{A_\alpha\}_{\alpha\in\mathcal{A}}$ is an equicontinuous subset of $L_S(X_1, X_2)$ if and only if $\{A_\alpha\}_{\alpha\in\mathcal{A}}$ is an equicontinuous subset of $L(X_1, X_2)$ and for every $\eta > 0$ there exists $C_\eta > 0$ such that for every $\alpha \in \mathcal{A}$ and $f \in X_1$ we have*

$$|A_\alpha f|_2 \leq \max\{\eta \|f\|_1, C_\eta |f|_1\}. \tag{6.6}$$

The proof of Proposition 6.2.18 is quite long, and so we defer it to the end of this section. Proposition 6.2.18 allows one to work with the inequality (6.6) in place of open sets of the form in (6.5), which often leads to conceptully simpler proofs, such as that of the following proposition.

Proposition 6.2.19. *Suppose that $(X_i, \|\cdot\|_i, |\cdot|_i)$, $i \in \{1, 2\}$ are Saks spaces. If $(X_2, \|\cdot\|_2, |\cdot|_2)$ is complete then $L_S(X_1, X_2)$ is complete.*

Proof. Suppose $\{A_n\}_{n\in\mathbb{Z}^+} \subseteq L_S(X_1, X_2)$ is $\|\cdot\|$-bounded and $|||\cdot|||$-Cauchy. Then for every $f \in X_1$ the sequence $\{A_n f\}_{n\in\mathbb{Z}^+}$ is $\|\cdot\|_2$-bounded and $|\cdot|_2$-Cauchy, and so there exists $g \in X_2$ such that $A_n f \to g$ in $\gamma[\|\cdot\|_2, |\cdot|_2]$. Define $A : X_1 \to X_2$ by $Af = \lim_{n\to\infty} A_n f$, and note that $|||A - A_n||| \to 0$ due to the fact that $\{A_n\}_{n\in\mathbb{Z}^+} \subseteq L_S(X_1, X_2)$ is $|||\cdot|||$-Cauchy. We will use Proposition 6.2.18 to prove that $A \in L_S(X_1, X_2)$. Since $\|\cdot\|$ is lower-semicontinuous for $|||\cdot|||$ we have

$$\sup_{\substack{f\in X_1\\ \|f\|=1}} \|Af\| \leq \sup_{\substack{f\in X_1\\ \|f\|=1}} \liminf_{n\to\infty} \|A_n f\| \leq \sup_{n\in\mathbb{Z}^+} \|A_n\|,$$

and so $A \in L(X_1, X_2)$. Fix $\eta > 0$. For every $f \in X_1$ and $n \in \mathbb{Z}^+$ we have

$$|Af|_2 \leq |A_n f|_2 + |||A_n - A||| \, \|f\|_1 .$$

By Proposition 6.2.18 and as $\{A_n\}_{n\in\mathbb{Z}^+} \subseteq L_S(X_1, X_2)$, for each $n \in \mathbb{Z}^+$ and $\kappa > 0$ there exists $D_{\kappa,n}$ such that for every $f \in X_1$ we have $|A_n f|_2 \leq \kappa \|f\|_1 + D_{\kappa,n} |f|_1$. Thus

$$|Af|_2 \leq (\kappa + |||A_n - A|||) \|f\|_1 + D_{\kappa,n} |f|_1 .$$

Suppose that n is large enough so that $|||A_n - A||| \leq \eta/4$. Set $\kappa = \eta/4$ and $C_\eta = 2D_{\kappa,n}$. Then

$$|Af|_2 \leq \frac{\eta}{2} \|f\|_1 + D_{\kappa,n} |f|_1 \leq 2 \max\left\{\frac{\eta}{2} \|f\|_1, D_{\kappa,n} |f|_1\right\} = \max\{\eta \|f\|_1, C_\eta |f|_1\}.$$

Hence $A \in L_S(X_1, X_2)$ by Proposition 6.2.18. □

We finish this section with some results on compact Saks spaces. Compact Saks spaces frequently appear in dynamical systems literature due to their use in the Ionescu-Tulcea–Marinescu Theorem, which is also known as Hennion's Theorem due to a later strengthening by Hennion.

Theorem 6.2.20 (A Saks space version of the Ionescu-Tulcea–Marinescu Theorem [57])**.** *Suppose that $(X, \|\cdot\|, |\cdot|)$ is a (pre-)compact Saks space, and that $A \in L(X)$. If there exist sequences of real numbers $\{r_n\}_{n\in\mathbb{Z}^+}$ and $\{R_n\}_{n\in\mathbb{Z}^+}$ such that for each $n \in \mathbb{Z}^+$ and $f \in X$ we have*

$$\|A^n f\| \leq r_n \|f\| + R_n |f|, \tag{6.7}$$

then $\rho_{\mathrm{ess}}(A) \leq \liminf_{n\to\infty} r_n^{1/n}$.

Recall that an operator $A \in L(X)$ is said to be quasi-compact if $\rho_{\text{ess}}(A) < \rho(A)$. One approach to studying the statistical properties of a dynamical system $T : M \to M$, where M is some Riemannian manifold, is by finding a Banach space X on which the Perron-Frobenius operator is quasi-compact and such that $\mathcal{C}^\infty(M) \hookrightarrow X \hookrightarrow (\mathcal{C}^\infty(M))'$. The typical route for proving quasi-compactness is via Theorem 6.2.20 i.e. by endowing X with the structure of a (pre-)compact Saks space and obtaining an appropriate Lasota-Yorke inequality, as in (6.7). This connection prompts some questions about Saks spaces with high relevancy to dynamical systems:

(Q1) What Banach spaces permit the structure of a compact Saks space?

(Q2) Given a Banach space that permits a compact Saks space structure, is this structure unique in any sense?

(Q3) If $(X, \|\cdot\|, |\cdot|)$ is a compact Saks space, to what extent does $\|\cdot\|$ determine $|\cdot|$?

The first question has a very satisfactory answer: a Banach space may be made into a compact Saks space if and only if it has a predual. We state the result for the case where the Banach space permits the structure of a normed compact Saks space and refrain from giving all the relevant definitions (see [26, Proposition 3.9] and [26, Chapter 1] for more details).

Proposition 6.2.21 ([26, Proposition 3.9])**.** *Let $(X, \|\cdot\|, \tau)$ be a Saks space. Then the following are equivalent:*

1. *$B_{\|\cdot\|}$ is compact and metrisable with respect to τ.*
2. *X is the Saks space projective limit of a sequence of finite dimensional Banach spaces.*
3. *X has the form $(F^*, \|\cdot\|, \sigma(F^*, F))$ for some separable Banach space F, where $\sigma(F^*, F)$ denotes the weak-$*$ topology on F^*.*
4. *$B_{\|\cdot\|}$ is compact and normable with respect to τ i.e. there is a norm $|\cdot|$ on X such that $|\cdot|$ and τ are equivalent on $B_{\|\cdot\|}$.*

We answer (Q2) and (Q3) in the following two theorems.

Theorem 6.2.22. *Suppose that $(X, \|\cdot\|)$ is a Banach space, τ is a locally convex topology on X such that $(X, \|\cdot\|, \tau)$ is a compact Saks space and that D is a Hausdorff topological vector space such that $(X, \tau) \hookrightarrow D$. Then $\gamma[\|\cdot\|, \tau]$ is unique up to D i.e. if τ' is a locally convex topology on X such that $(X, \|\cdot\|, \tau')$ is a compact Saks space with $(X, \tau') \hookrightarrow D$ then $\gamma[\|\cdot\|, \tau] = \gamma[\|\cdot\|, \tau']$.*

Proof. We will prove that the set of $\gamma[\|\cdot\|, \tau]$-convergent nets coincides with the set of $\gamma[\|\cdot\|, \tau']$-convergent nets. Suppose that $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ is a $\gamma[\|\cdot\|, \tau]$-convergent net and, for a contradiction, that $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ is not $\gamma[\|\cdot\|, \tau']$-convergent. By continuity, $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ must be convergent in D. Since $(X, \|\cdot\|, \tau')$ is a compact Saks space and $\sup_{\alpha\in\mathcal{A}} \|f_\alpha\| < \infty$ there exists a τ'-convergent sub-net $\{f_\alpha\}_{\alpha\in\mathcal{A}'}$, which accumulates away from the τ-limit of $\{f_\alpha\}_{\alpha\in\mathcal{A}}$. But $\{f_\alpha\}_{\alpha\in\mathcal{A}'}$ can only accumulate in D at the accumulation point of $\{f_\alpha\}_{\alpha\in\mathcal{A}}$, and so the two accumulation points must be the same. □

Theorem 6.2.23. *For a bounded countable family of functionals $\Phi = \{\varphi_n\}_{n\in\mathbb{Z}^+} \in L(X, \mathbb{C})$ we set*

$$\|f\|_\Phi = \sup_{\varphi\in\Phi} |\varphi(f)|$$

and

$$|f|_\Phi = \sum_{n\in\mathbb{Z}^+} 2^{-n} |\varphi_n(f)| .$$

If $(X, \|\cdot\|, |\cdot|)$ is a compact Saks space then there exists $\Phi = \{\varphi_n\}_{n\in\mathbb{Z}^+} \in L(X, \mathbb{C})$ such that $\|\varphi_n\| = 1$ for every $n \in \mathbb{Z}^+$, $\|\cdot\|$ is equivalent to $\|\cdot\|_\Phi$, and $|\cdot|$ is equivalent to $|\cdot|_\Phi$ on $\|\cdot\|$-bounded sets. In particular $\gamma[\|\cdot\|, |\cdot|]$ and $\gamma[\|\cdot\|_\Phi, |\cdot|_\Phi]$ are equivalent.

Proof. By Proposition 6.2.21 there exists a separable Banach space F such that $(X, \|\cdot\|, |\cdot|) = (F^*, \|\cdot\|, \sigma(F^*, F))$. Specifically, $\sigma(F^*, F)$ and $|\cdot|$ are equivalent on $B_{\|\cdot\|}$. Since both $\|\cdot\|_{F^*}$ and $\|\cdot\|$ are stronger than $\sigma(F^*, F)$, which is a Hausdorff topology on F^*, it follows from the closed graph theorem that $\|\cdot\|_{F^*}$ and $\|\cdot\|$ are equivalent (see [101, Proposition 1]). Since F is separable, there exists a $\|\cdot\|_F$-bounded family $\Phi = \{\varphi_n\}_{n\in\mathbb{Z}^+} \subseteq F^* \cap F$ that is $\|\cdot\|_F$-dense in $\partial B_{\|\cdot\|_F}$ and so that $\|\cdot\|_{F^*} = \|\cdot\|_\Phi$. Thus $\|\cdot\|_\Phi$ and $\|\cdot\|$ are equivalent, so it only remains to prove that $|\cdot|$ is equivalent to $|\cdot|_\Phi$ on $B_{\|\cdot\|}$. Suppose that $\{f_k\}_{k\in\mathbb{Z}^+} \subseteq B_{\|\cdot\|}$ is a $|\cdot|$-convergent sequence with limit $f \in B_{\|\cdot\|}$. Fix $\epsilon > 0$. Since $\{f_k\}_{k\in\mathbb{Z}^+} \subseteq B_{\|\cdot\|}$ there exists N such that

$$\sum_{n=N+1}^{\infty} 2^{-n} |\varphi_n(f - f_k)| \le \epsilon.$$

Hence

$$|f - f_k|_\Phi = \sum_{n=1}^{\infty} 2^{-n} |\varphi_n(f - f_k)| \le \sum_{n=1}^{N} 2^{-n} |\varphi_n(f - f_k)| + \epsilon.$$

Viewing $f - f_k$ as an element of F^*, each φ_n as an element of F, and using the equivalence of $\sigma(F^*, F)$ and $|\cdot|$ on $B_{\|\cdot\|}$, we observe that $\limsup_{k\to\infty} |f - f_k|_\Phi \le \epsilon$.

Thus, as ϵ is arbitrary, we have $\lim_{k\to\infty} |f_k - f|_\Phi = 0$. On the other hand, if $\{f_k\}_{k\in\mathbb{Z}^+} \subseteq B_{\|\cdot\|}$ is instead $|\cdot|_\Phi$-convergent then we must have $\varphi_n(f - f_k) \to 0$ for every $n \in \mathbb{Z}^+$. Since Φ is $\|\cdot\|_F$-dense in $\partial B_{\|\cdot\|_F}$, we have $(f - f_k)(x) \to 0$ for every $x \in F$ i.e. $f \to f_k$ in $\sigma(F^*, F)$. Since $\sigma(F^*, F)$ and $|\cdot|$ are equivalent on $B_{\|\cdot\|}$, and $\sigma(F^*, F)$ is metrisable on $B_{\|\cdot\|}$, it follows that $f \to f_k$ in $|\cdot|$. Hence $|\cdot|$ and $|\cdot|_\Phi$ are equivalent on $B_{\|\cdot\|}$, as required. Thus, $\gamma[\|\cdot\|, |\cdot|]$ and $\gamma[\|\cdot\|_\Phi, |\cdot|_\Phi]$ are equivalent by Proposition 6.2.11. $\square$

We finish this section with the proof of Proposition 6.2.18.

The proof of Proposition 6.2.18. We prove the reverse implication first i.e. that for every $\gamma[\|\cdot\|_2, |\cdot|_2]$-open neighbourhood U of 0 there exists a $\gamma[\|\cdot\|_1, |\cdot|_1]$-open neighbourhood V of 0 such that $A_\alpha(V) \subseteq U$ for every $\alpha \in \mathcal{A}$. If U is such a neighbourhood then, after recalling the form of the neighbourhood basis for $\gamma[\|\cdot\|_2, |\cdot|_2]$ from (6.5), we observe that U contains a set of the form

$$\bigcup_{n=1}^{\infty} \left(\sum_{k=1}^{n} 2^{\ell_k} B^{\mathrm{o}}_{|\cdot|_2} \cap 2^{k-1} B_{\|\cdot\|_2} \right)$$

for some sequence $\{\ell_i\}_{i=1}^{\infty} \subseteq \mathbb{R}$, where $B^{\mathrm{o}}_{|\cdot|_1}$ denotes the open unit $|\cdot|_1$-ball. By assumption we have $\sup_\alpha \|A_\alpha\| < \infty$ and so there exists $N \geq 0$ such that $A_\alpha(B_{\|\cdot\|_1}) \subseteq 2^N B_{\|\cdot\|_2}$ for every $\alpha \in \mathcal{A}$. For the moment fix $k > N$, and let us suppose that $f \in 2^{\nu_k} B^{\mathrm{o}}_{|\cdot|_1} \cap 2^{k-N-1} B_{\|\cdot\|_1}$ for some $\nu_k \in \mathbb{R}$. Then $A_\alpha f \in 2^{k-N+N-1} B_{\|\cdot\|_2} = 2^{k-1} B_{\|\cdot\|_2}$ for every $\alpha \in \mathcal{A}$. In addition, by (6.6) we have for every $\eta > 0$ and $\alpha \in \mathcal{A}$ that

$$|A_\alpha f|_2 \leq \max\{\eta 2^{k-N-1}, C_\eta 2^{\nu_k}\},$$

so that if we set $\eta_k = 2^{\ell_k + N - k}$ and take $2^{\nu_k} = C^{-1}_{2^{\ell_k+N-k}} 2^{\ell_k - 1}$ then $|A_\alpha f|_2 < 2^{\ell_k}$ for every $\alpha \in \mathcal{A}$. Thus for every $\alpha \in \mathcal{A}$ and $k \geq 1$ we have

$$A_\alpha(2^{\nu_k} B^{\mathrm{o}}_{|\cdot|_1} \cap 2^{k-N-1} B_{\|\cdot\|_1}) \subseteq 2^{\ell_k} B^{\mathrm{o}}_{|\cdot|_1} \cap 2^{k-1} B_{\|\cdot\|_2}.$$

Set

$$V = \bigcup_{n=1}^{\infty} \left(\sum_{k=N+1}^{n+N} 2^{\nu_k} B^{\mathrm{o}}_{|\cdot|_2} \cap 2^{k-N-1} B_{\|\cdot\|_2} \right),$$

and note that V is open per (6.5). Moreover, for every $\alpha \in \mathcal{A}$ we have

$$\begin{aligned} A_\alpha(V) &= \bigcup_{n=1}^{\infty}\left(\sum_{k=N+1}^{n+N} A_\alpha(2^{\nu_k} B^{\mathrm{o}}_{|\cdot|_2} \cap 2^{k-N-1} B_{\|\cdot\|_2})\right) \\ &\subseteq \bigcup_{n=1}^{\infty}\left(\sum_{k=N+1}^{n+N} 2^{\ell_k} B^{\mathrm{o}}_{|\cdot|_1} \cap 2^{k-1} B_{\|\cdot\|_2}\right) \subseteq U. \end{aligned}$$

Thus $\{A_\alpha\}_{\alpha\in\mathcal{A}}$ is equicontinuous in $L_S(X_1, X_2)$.

We now prove the opposite implication. Let U be a $\gamma[\|\cdot\|_2, |\cdot|_2]$-neighbourhood of 0. Since $\{A_\alpha\}_{\alpha\in\mathcal{A}}$ is equicontinuous in $L_S(X_1, X_2)$, by [95, 4.1] the set $V = \bigcap_{\alpha\in\mathcal{A}} A_\alpha^{-1}(U)$ is $\gamma[\|\cdot\|_1, |\cdot|_1]$-open. As V is $\gamma[\|\cdot\|_1, |\cdot|_1]$-open and $B_{\|\cdot\|_1}$ is $\gamma[\|\cdot\|_1, |\cdot|_1]$-bounded, there exists $\lambda > 0$ such that $B_{\|\cdot\|_1} \subseteq \lambda V$. Hence $A_\alpha(B_{\|\cdot\|_1}) \subseteq \lambda U$ for every $\alpha \in \mathcal{A}$ and so $\bigcup_{\alpha\in\mathcal{A}} A_\alpha(B_{\|\cdot\|_1})$ is $\gamma[\|\cdot\|_2, |\cdot|_2]$-bounded. By Proposition 6.2.10 it follows that $\bigcup_{\alpha\in\mathcal{A}} A_\alpha(B_{\|\cdot\|_1})$ is $\|\cdot\|_2$-bounded i.e. there exists $M > 0$ such that $A_\alpha(B_{\|\cdot\|_1}) \subseteq M B_{\|\cdot\|_2}$ for every $\alpha \in \mathcal{A}$. Hence $\{A_\alpha\}_{\alpha\in\mathcal{A}}$ is bounded in $L(X_1, X_2)$, which implies that $\{A_\alpha\}_{\alpha\in\mathcal{A}}$ is equicontinuous as a subset of $L(X_1, X_2)$.

It remains to prove (6.6). Fix $\eta > 0$. Since $\{A_\alpha\}_{\alpha\in\mathcal{A}}$ is an equicontinuous subset of $L_S(X_1, X_2)$ and $|\cdot|_2$ is $\gamma[\|\cdot\|_2, |\cdot|_2]$-continuous there exists a $\gamma[\|\cdot\|_1, |\cdot|_1]$-neighbourhood of 0, say U_η, such that if $f \in U_\eta$ and $\alpha \in \mathcal{A}$ then $|A_\alpha(f)|_2 < \eta$. By construction U_η contains a set of the form $2^{\ell_\eta} B^{\mathrm{o}}_{|\cdot|_1} \cap B_{\|\cdot\|_1}$ for some $\ell_\eta \in \mathbb{R}$. For any non-zero $f \in X_1$ we have

$$\frac{f}{\max\{2^{-\ell_\eta+1}|f|_1, \|f\|_1\}} \in 2^{\ell_\eta} B^{\mathrm{o}}_{|\cdot|_1} \cap B_{\|\cdot\|_1},$$

and so for every $\alpha \in \mathcal{A}$ we have

$$\left| A_\alpha\left(\frac{f}{\max\{2^{-\ell_\eta+1}|f|_1, \|f\|_1\}}\right)\right|_2 < \eta.$$

In particular, if we set $C_\eta = \eta 2^{-\ell_\eta+1}$ then for every $f \in X_1$ and $\alpha \in \mathcal{A}$ we have

$$|A_\alpha(f)|_2 < \max\{\eta\|f\|_1, C_\eta |f|_1\},$$

as required. $\square$

Chapter 7

Saks space stability for hyperbolic splittings of Lasota-Yorke cocycles

Our main result for this chapter, Theorem 7.1.7, concerns the stability of hyperbolic splittings (Definition 7.1.1) for operator cocycles satisfying a Lasota-Yorke inequality (Definition 7.1.5) and certain Saks space equicontinuity conditions (Definition 7.1.2). In Section 7.1 we state the main definitions and results for the chapter. Section 7.2 contains a number of basic results that are used throughout the proof of Theorem 7.1.7, which is split over Sections 7.3, 7.4 and 7.5.

7.1 Definitions and main results

Let us fix some notation for this chapter. Let Ω be a set, and $\sigma : \Omega \to \Omega$ be an invertible map. For each $\omega \in \Omega$ let $(X_\omega, \|\cdot\|_\omega, |\cdot|_\omega)$ be a normal Saks space, with each $(X_\omega, \|\cdot\|_\omega)$ being a Banach space[1]. We will consider the vector space bundle[2] $\mathbb{X} = \bigsqcup_{\omega\in\Omega}\{\omega\} \times X_\omega$. Let $\pi : \mathbb{X} \to \Omega$ denote the projection onto Ω, and for each $\omega \in \Omega$ let $\tau_\omega : \pi^{-1}(\omega) \to X_\omega$ be defined by $\tau_\omega(\omega, f) = f$. We say that $P : \mathbb{X} \to \mathbb{X}$ is a *bounded linear endomorphism of* $\mathbb{X}$ *covering* σ if $\pi \circ P = \sigma \circ \pi$ and if $f \mapsto \tau_{\sigma(\omega)}(P(\omega, f))$ is in $L(X_\omega, X_{\sigma(\omega)})$ for every $\omega \in \Omega$. We denote the set of all bounded linear endomorphisms of $\mathbb{X}$ covering σ by $\mathrm{End}(\mathbb{X}, \sigma)$. When $n \in \mathbb{N}$, $\omega \in \Omega$, and $P \in \mathrm{End}(\mathbb{X}, \sigma)$ we denote the map $f \mapsto \tau_{\sigma(\omega)}(P(\omega, f))$ by P_ω and set

$$P_\omega^n = \begin{cases} P_{\sigma^{n-1}(\omega)} \circ \cdots \circ P_\omega & \text{if } n \geq 1, \\ \mathrm{Id} & \text{if } n = 0. \end{cases}$$

Clearly $P_\omega^n \in L(X_\omega, X_{\sigma^n(\omega)})$ for every $n \in \mathbb{N}$ and $\omega \in \Omega$. Unless required we will frequently drop the subscript ω from $\|\cdot\|_\omega$ and $|\cdot|_\omega$. We denote the norm on

[1] It will be important later that $L(X_\omega)$ is complete.
[2] Note that we do not endow $\mathbb{X}$ with a topology.

$L(X_\omega, X_{\sigma(\omega)})$ by $\|\cdot\|$, the norm on $L((X_\omega, |\cdot|_\omega), (X_{\sigma(\omega)}, |\cdot|_{\sigma(\omega)}))$ by $|\cdot|$, and the norm on $L((X_\omega, \|\cdot\|_\omega), (X_{\sigma(\omega)}, |\cdot|_{\sigma(\omega)}))$ by $|||\cdot|||$.

Definition 7.1.1. *Suppose that* $P \in \mathrm{End}(\mathbb{X}, \sigma)$*,* $d \in \mathbb{Z}^+$*,* $0 \le \mu < \lambda$*,* $(E_\omega)_{\omega\in\Omega} \in \prod_{\omega\in\Omega} \mathcal{G}_d(X_\omega)$ *and* $(F_\omega)_{\omega\in\Omega} \in \prod_{\omega\in\Omega} \mathcal{G}^d(X_\omega)$*. We say that* $(E_\omega)_{\omega\in\Omega}$ *and* $(F_\omega)_{\omega\in\Omega}$ *form a* (μ, λ, d)*-hyperbolic splitting for* P*, and that* P *has a hyperbolic splitting of index* d*, if there exists constants* $C_\lambda, C_\mu, \Theta > 0$ *such that:*

(H1) *For every* $\omega \in \Omega$ *we have* $E_\omega \oplus F_\omega = X_\omega$ *and*

$$\max\{\left\|\Pi_{F_\omega||E_\omega}\right\|, \left\|\Pi_{E_\omega||F_\omega}\right\|\} \le \Theta. \tag{7.1}$$

(H2) *For each* $\omega \in \Omega$ *we have* $P_\omega E_\omega = E_{\sigma(\omega)}$*. Moreover, for every* $n \in \mathbb{Z}^+$ *and* $f \in E_\omega$ *we have*

$$\|P_\omega^n f\| \ge C_\lambda \lambda^n \|f\|. \tag{7.2}$$

(H3) *For each* $\omega \in \Omega$ *we have* $P_\omega F_\omega \subseteq F_{\sigma(\omega)}$ *and for every* $n \in \mathbb{Z}^+$ *we have*

$$\left\| P_\omega^n|_{F_\omega} \right\| \le C_\mu \mu^n. \tag{7.3}$$

We call $(E_\omega)_{\omega\in\Omega}$ *and* $(F_\omega)_{\omega\in\Omega}$ *the equivariant fast and slow spaces for* P*, respectively.*

We now define the elements of $\mathrm{End}(\mathbb{X}, \sigma)$ that are 'equicontinuous in the Saks space sense'.

Definition 7.1.2 (Saks space continuous endomorphisms)**.** *We call* $P \in \mathrm{End}(\mathbb{X}, \sigma)$ *a* Saks space equicontinuous endomorphism *if* $\sup_{\omega\in\Omega} \|P_\omega\| < \infty$ *and if for each* $\eta > 0$ *there exists* $C_\eta > 0$ *such that for every* $\omega \in \Omega$ *and* $f \in X_\omega$ *we have*

$$|P_\omega f| \le \eta \|f\| + C_\eta |f|. \tag{7.4}$$

We denote the set of all Saks space equicontinuous endomorphisms in $\mathrm{End}(\mathbb{X}, \sigma)$ *by* $\mathrm{End}_S(\mathbb{X}, \sigma)$*.*

Remark 7.1.3. Proposition 6.2.18 justifies the characterisation of the condition in Definition 7.1.2 as an equicontinuity condition. Indeed, when all the spaces $(X_\omega, \|\cdot\|_\omega, |\cdot|_\omega)$ are equal to a fixed space $(X, \|\cdot\|, |\cdot|)$, then $P \in \mathrm{End}_S(\mathbb{X}, \sigma)$ if and only if $\{P_\omega\}_{\omega\in\Omega}$ is equicontinuous in $L_S(X)$.

Remark 7.1.4. $\mathrm{End}_S(\mathbb{X}, \sigma)$ admits an interesting alternative characterisation. For $(f_\omega)_{\omega\in\Omega} \in \prod_{\omega\in\Omega} X_\omega$ let $\|(f_\omega)_{\omega\in\Omega}\|_\infty = \sup_{\omega\in\Omega} \|f_\omega\|_\omega$ and $|(f_\omega)_{\omega\in\Omega}|_\infty = \sup_{\omega\in\Omega} |f_\omega|_\omega$.

The set

$$X_\infty = \left\{ (f_\omega)_{\omega\in\Omega} \in \prod_{\omega\in\Omega} X_\omega : \|(f_\omega)_{\omega\in\Omega}\|_\infty < \infty \right\}$$

is a Banach space when equipped with $\|\cdot\|_\infty$, and a normal Saks space when given the structure $(X_\infty, \|\cdot\|_\infty, |\cdot|_\infty)$. For $P \in \mathrm{End}(\mathbb{X},\sigma)$, one can show that $P \in \mathrm{End}_S(\mathbb{X},\sigma)$ if and only if the map $(f_\omega)_{\omega\in\Omega} \mapsto (P_\omega f_\omega)_{\omega\in\Omega}$ is in $L_S(X_\infty)$.

We will only consider endomorphisms that satisfy a uniform Lasota-Yorke inequality.

Definition 7.1.5 (Lasota-Yorke class)**.** *For $C_1, C_2, r, R \geq 0$ we let $\mathcal{LY}(C_1, C_2, r, R)$ denote the set of $P \in \mathrm{End}(\mathbb{X},\sigma)$ such that for every $\omega \in \Omega$, $f \in X_\omega$ and $n \in \mathbb{Z}^+$ we have*

$$\|P_\omega^n f\| \leq C_1 r^n \|f\| + C_2 R^n |f|. \tag{7.5}$$

Remark 7.1.6. If $P \in \mathcal{LY}(C_1, C_2, r, R)$ admits a (μ, λ, d)-hyperbolic splitting with $\mu > r$ and fast spaces fast spaces $(E_\omega)_{\omega\in\Omega}$ (as in Theorem 7.1.7) then for any $\omega \in \Omega$, $f \in E_\omega \setminus \{0\}$ and $n \in \mathbb{Z}^+$ one has

$$C_\lambda \lambda^n \|f\| \leq C_1 r^n \|f\| + C_2 R^n \|f\|.$$

Since $r < \mu < \lambda$, by taking $n \to \infty$ it follows that $\lambda < R$ Hence $r < R$, and so upon setting where $C_3 = C_1 + C_2$ we have for every $\omega \in \Omega$ and $n \in \mathbb{Z}^+$ that

$$\|P_\omega^n\| \leq C_1 r^n + C_2 R^n \leq C_3 R^n. \tag{7.6}$$

Finally, if $P \in \mathrm{End}_S(\mathbb{X},\sigma)$ then for $\epsilon > 0$ we set

$$\mathcal{O}_\epsilon(P) = \left\{ S \in \mathrm{End}(\mathbb{X},\sigma) : \sup_{\omega\in\Omega} |||P_\omega - S_\omega||| < \epsilon \right\}.$$

The main result of this chapter is the following.

Theorem 7.1.7. *Fix $\mu, \lambda, C_1, C_2, R \geq 0$, with $0 \leq r < \mu < \lambda$, $d \in \mathbb{Z}^+$, $(E_\omega)_{\omega\in\Omega} \in \prod_{\omega\in\Omega} \mathcal{G}_d(X_\omega)$ and $(F_\omega)_{\omega\in\Omega} \in \prod_{\omega\in\Omega} \mathcal{G}^d(X_\omega)$. If $P \in \mathrm{End}_S(\mathbb{X},\sigma) \cap \mathcal{LY}(C_1, C_2, r, R)$ has a (μ, λ, d)-hyperbolic splitting composed of fast spaces $(E_\omega)_{\omega\in\Omega}$ and slow spaces $(F_\omega)_{\omega\in\Omega}$. There exists $\epsilon' > 0$ so that*

1. *If $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon'}(P)$ then S has a hyperbolic splitting of index d.*

2. *If* $(E_\omega^S)_{\omega\in\Omega} \in \prod_{\omega\in\Omega} \mathcal{G}_d(X_\omega)$ *and* $(F_\omega^S)_{\omega\in\Omega} \in \prod_{\omega\in\Omega} \mathcal{G}^d(X_\omega)$ *denote the equivariant fast and slow spaces for* S *then*

$$\sup\left\{\left\|\Pi_{E_\omega^S||F_\omega^S}\right\| : \omega\in\Omega, S\in\mathcal{LY}(C_1,C_2,r,R)\cap\mathcal{O}_{\epsilon'}(P)\right\} < \infty. \tag{7.7}$$

Moreover, for every $\beta\in(0,(\lambda-\mu)/2)$ *and* $\delta>0$ *there exists* $\epsilon_{\beta,\delta}\in(0,\epsilon')$ *and* $C_\beta>0$ *so that if* $S\in\mathcal{LY}(C_1,C_2,r,R)\cap\mathcal{O}_{\epsilon_{\beta,\delta}}(P)$ *then*

1. *We have the estimates*

$$\sup_{\omega\in\Omega}\left|\!\left|\!\left|\Pi_{E_\omega^S||F_\omega^S}-\Pi_{E_\omega||F_\omega}\right|\!\right|\!\right| \le \delta, \tag{7.8}$$

and

$$\sup_{\omega\in\Omega} d_H(F_\omega^S,F_\omega)\le\delta. \tag{7.9}$$

2. *The spaces* $(E_\omega^S)_{\omega\in\Omega}$ *and* $(F_\omega^S)_{\omega\in\Omega}$ *form a* $(\mu+\beta,\lambda-\beta,d)$*-hyperbolic splitting for* S*. More specifically, for every* $\omega\in\Omega$ *and* $n\in\mathbb{Z}^+$ *we have*

$$\left\|S_\omega^n|_{F_\omega^S}\right\|\le C_\beta(\mu+\beta)^n, \tag{7.10}$$

and, for every $v\in E_\omega^S$*, that*

$$\|S_\omega^n v\|\ge C_\beta^{-1}(\lambda-\beta)^n\|v\|. \tag{7.11}$$

Remark 7.1.8. In principle one could compute explicit bounds on the various quantities in the statement of Theorem 7.1.7, such as $\epsilon_{\beta,\delta}$ or the supremum in (7.7). We opted not to pursue such bounds for the sake of simplicity.

Remark 7.1.9. It is possible to obtain an estimate on the distance between E_ω^S and E_ω in the Grassmannian distance on $(X_\omega,|\cdot|_\omega)$ from (7.8) by using [38, Proposition 2.4].

The strategy behind the proof of Theorem 7.1.7 is reminiscent of the usual proof that the class of Anosov maps is open [21, Corollary 5.5.2], and is quite similar to the overall strategy of [16]. We start by collecting some preliminary estimates and results in Section 7.2. In Section 7.3 we construct invariant 'fast' cones of d-dimensional subspaces, defined in terms of the graph representation of the hyperbolic splitting of P, and show that the forward graph transform induced by an iterate of the perturbed cocycle is a contraction mapping on these cones. We then prove that perturbed fast spaces approximate, in a Saks space sense, the unperturbed fast spaces. Once the fast spaces have been constructed we may use similar arguments to construct and prove the stability of the slow spaces, which is the subject of Section 7.4. In Section 7.5 we bring together the results of the previous sections to complete the proof of Theorem 7.1.7.

Remark 7.1.10. For the remainder of this chapter P will refer to an element of $\mathrm{End}_S(\mathbb{X},\sigma)\cap\mathcal{LY}(C_1,C_2,r,R)$ which satisfies all the hypotheses of Theorem 7.1.7.

7.2 Preliminary estimates and lemmata

The following estimate forms the backbone of the proof of Theorem 7.1.7.

Proposition 7.2.1. *For every $\beta\in(0,(\lambda-\mu)/2)$ there exists N_β and for each $n>N_\beta$ an $\epsilon_{n,\beta}>0$ so that if $S\in\mathcal{LY}(C_1,C_2,r,R)\cap\mathcal{O}_{\epsilon_{n,\beta}}(P)$ and $\omega\in\Omega$ then*

$$\left\|S_\omega^n|_{F_\omega}\right\|\le(\mu+\beta)^n, \tag{7.12}$$

and if $v\in E_\omega$ then

$$\left\|\Pi_{E_{\sigma^n(\omega)}||F_{\sigma^n(\omega)}}S_\omega^n v\right\|\ge(\lambda-\beta)^n\left\|v\right\|. \tag{7.13}$$

The proof of Proposition 7.2.1 is split over the following lemmas, all of which are of independent interest.

Lemma 7.2.2. *There exists K such that for every $\omega\in\Omega$ and $v\in X_\omega$ we have*

$$\left\|\Pi_{E_\omega||F_\omega}v\right\|\le K\left|\Pi_{E_\omega||F_\omega}v\right|.$$

Proof. From (7.5) and (7.2) we have

$$C_\lambda\lambda^n\left\|\Pi_{E_\omega||F_\omega}v\right\|\le\left\|P_\omega^n\Pi_{E_\omega||F_\omega}v\right\|\le C_1r^n\left\|\Pi_{E_\omega||F_\omega}v\right\|+C_2R^n\left|\Pi_{E_\omega||F_\omega}v\right|. \tag{7.14}$$

Since $\lambda>r$ there exists N such that $C_\lambda\lambda^N-C_1r^N>0$, from which the claim follows upon rearranging (7.14). □

Lemma 7.2.3. *For every $\eta>0$ there exists $C_\eta>0$ such that for every $\omega\in\Omega$ and $v\in X_\omega$ we have*

$$\left\|\Pi_{E_\omega||F_\omega}v\right\|\le\eta\left\|v\right\|+C_\eta\left|v\right|.$$

Proof. By (7.1), (7.2) and (7.5) we have

$$C_\lambda \lambda^n \left\|\Pi_{E_\omega||F_\omega} v\right\| \leq \left\|P_\omega^n \Pi_{E_\omega||F_\omega} v\right\| = \left\|\Pi_{E_{\sigma^n(\omega)}||F_{\sigma^n(\omega)}} P_\omega^n v\right\| \leq \Theta C_1 r^n \left\|v\right\| + \Theta C_2 R^n \left|v\right|.$$

Since $\lambda > r$, by taking n large enough we may ensure that $\Theta C_1 r^n \lambda^{-n} < \eta$, from which the result follows upon rearranging. □

Remark 7.2.4. Since $|\cdot| \leq \|\cdot\|$, by Lemma 7.2.3 and Proposition 6.2.18 we have $\Pi_{E_\omega||F_\omega}, \Pi_{F_\omega||E_\omega} \in L_S(X_\omega)$ for each ω. It is interesting to note that we did not use the fact that $P \in \mathrm{End}_S(\mathbb{X}, \sigma)$ in the proof of Lemma 7.2.3.

The next lemma is an easy exercise.

Lemma 7.2.5. *If $R, S \in \mathrm{End}_S(\mathbb{X}, \sigma)$ then $RS \in \mathrm{End}_S(\mathbb{X}, \sigma^2)$. Hence if $R \in \mathrm{End}_S(\mathbb{X}, \sigma)$ then $R^n \in \mathrm{End}_S(\mathbb{X}, \sigma^n)$ for every $n \in \mathbb{Z}^+$.*

Lemma 7.2.6. *For every $n \in \mathbb{Z}^+$ and $\epsilon > 0$ there exists $\kappa > 0$ so that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_\kappa(P)$ then*

$$\sup_{\omega \in \Omega} |||P_\omega^n - S_\omega^n||| \leq \epsilon. \tag{7.15}$$

Proof. For $S \in \mathcal{LY}(C_1, C_2, r, R)$ we may write

$$P_\omega^n - S_\omega^n = \sum_{k=0}^{n-1} P_{\sigma^{k+1}(\omega)}^{n-k-1} (P_{\sigma^k(\omega)} - S_{\sigma^k(\omega)}) S_\omega^k.$$

Since $P \in \mathrm{End}_S(\mathbb{X}, \sigma)$, Lemma 7.2.5 and (7.6) imply that for every $\eta > 0$ there exists C_η such that for every $\omega \in \Omega$ and $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_\kappa(P)$ we have

$$\begin{aligned} |||P_\omega^n - S_\omega^n||| &\leq \sum_{k=0}^{n-1} \left(\eta \left\|P_{\sigma^k(\omega)} - S_{\sigma^k(\omega)}\right\| + C_\eta |||P_{\sigma^k(\omega)} - S_{\sigma^k(\omega)}|||\right) \left\|S_\omega^k\right\| \\ &\leq C_3 \sum_{k=0}^{n-1} (2C_3 R\eta + \kappa C_\eta) R^k \leq nC_3 (2C_3 R\eta + \kappa C_\eta) \max\{1, R^n\}. \end{aligned}$$

We obtain (7.15) by choosing η so that $n 2 C_3^2 R\eta \max\{1, R^n\} < \epsilon/2$, and then taking κ small enough so that $nC_3 \kappa C_\eta \max\{1, R^n\} < \epsilon/2$ □

The proof of Proposition 7.2.1. We prove (7.12) and (7.13) separately.

The proof of (7.12). By telescoping we have

$$\begin{aligned}\left\| S_\omega^n|_{F_\omega} \right\| &\leq \left\| P_\omega^n|_{F_\omega} \right\| + \left\| (P_\omega^n - S_\omega^n)|_{F_\omega} \right\| \\ &\leq \left\| P_\omega^n|_{F_\omega} \right\| + \sum_{k=0}^{n-1} \left\| S_{\sigma^{k+1}(\omega)}^{n-k-1} (P_{\sigma^k(\omega)} - S_{\sigma^k(\omega)}) P_\omega^k|_{F_\omega} \right\|.\end{aligned} \tag{7.16}$$

Since $S, P \in \mathcal{LY}(C_1, C_2, r, R)$, by (7.5) and (7.6) we get

$$\left\| (P_\omega^n - S_\omega^n)|_{F_\omega} \right\| \leq \sum_{k=0}^{n-1} \left(2C_1C_3Rr^{n-k-1} + C_2R^{n-k-1} \vert\vert\vert P_{\sigma^k(\omega)} - S_{\sigma^k(\omega)} \vert\vert\vert\right) \left\| P_\omega^k|_{F_\omega} \right\|. \tag{7.17}$$

Combining (7.16) and (7.17), and then applying (7.3) yields

$$\begin{aligned}\left\| S_\omega^n|_{F_\omega} \right\| \leq &C_\mu \mu^n + C_\mu \mu^n \sum_{k=0}^{n-1} \frac{2C_1C_3R}{\mu} \left(\frac{r}{\mu}\right)^{n-k-1} \\ &+ C_\mu \mu^n \sum_{k=0}^{n-1} \frac{C_2}{\mu} \left(\frac{R}{\mu}\right)^{n-k-1} \vert\vert\vert P_{\sigma^k(\omega)} - S_{\sigma^k(\omega)} \vert\vert\vert.\end{aligned}$$

By Lemma 7.2.6 and as $\mu > r$, for any $n \in \mathbb{Z}^+$ there is a $\epsilon_n > 0$ so that if $S \in \mathcal{O}_{\epsilon_n}(P) \cap \mathcal{LY}(C_1, C_2, r, R)$ then

$$\sum_{k=0}^{n-1} \left(\frac{2C_1C_3R}{\mu} \left(\frac{r}{\mu}\right)^{n-k-1} + \frac{C_2}{\mu} \left(\frac{R}{\mu}\right)^{n-k-1} \vert\vert\vert P_{\sigma^k(\omega)} - S_{\sigma^k(\omega)} \vert\vert\vert \right) \leq \frac{3C_1C_3R}{\mu - r} := C'.$$

Thus if $S \in \mathcal{O}_{\epsilon_n}(P) \cap \mathcal{LY}(C_1, C_2, r, R)$ and $\omega \in \Omega$ then $\left\| S_\omega^n|_{F_\omega} \right\| \leq C_\mu(C' + 1)\mu^n$. Setting $N_\beta = \ln(C_\mu C') / \ln(1 + \beta/\mu)$, we therefore get (7.12) whenever $n > N_\beta$, $S \in \mathcal{O}_{\epsilon_n}(P) \cap \mathcal{LY}(C_1, C_2, r, R)$ and $\omega \in \Omega$.

The proof of (7.13). As $|\cdot| \leq \|\cdot\|$, for each $v \in E_\omega$ we have

$$\left\| \Pi_{E_{\sigma^n(\omega)} || F_{\sigma^n(\omega)}} S_\omega^n v \right\| \geq \left| \Pi_{E_{\sigma^n(\omega)} || F_{\sigma^n(\omega)}} P_\omega^n v \right| - \left| \Pi_{E_{\sigma^n(\omega)} || F_{\sigma^n(\omega)}} (P_\omega^n - S_\omega^n) v \right|. \tag{7.18}$$

Using Lemma 7.2.2, (7.2) and the fact that $P_\omega^n v \in E_{\sigma^n(\omega)}$, we get

$$\left| \Pi_{E_{\sigma^n(\omega)} || F_{\sigma^n(\omega)}} P_\omega^n v \right| \geq K^{-1} \|P_\omega^n v\| \geq K^{-1} C_\lambda \lambda^n \|v\|. \tag{7.19}$$

On the other hand, by Lemma 7.2.3 we have for every $\eta > 0$ that

$$\left|\Pi_{E_{\sigma^n(\omega)}||F_{\sigma^n(\omega)}}(P_\omega^n - S_\omega^n)v\right| \leq (2\eta C_3 R^n + C_\eta |||P_\omega^n - S_\omega^n|||) \|v\| \,. \tag{7.20}$$

Applying (7.19) and (7.20) to (7.18) yields

$$\left\|\Pi_{E_{\sigma^n(\omega)}||F_{\sigma^n(\omega)}} S_\omega^n v\right\| \geq \left(K^{-1}C_\lambda \lambda^n - 2\eta C_3 R^n - C_\eta |||P_\omega^n - S_\omega^n|||\right) \|v\| \,.$$

For each n let η be small enough so that $2\eta C_3 K C_\lambda^{-1} R^n < \lambda^n/4$. By Lemma 7.2.6 there is a $\epsilon_{n,\beta}$ so that if $S \in \mathcal{O}_{\epsilon_{n,\beta}}(P) \cap \mathcal{LY}(C_1, C_2, r, R)$ then $KC_\eta C_\lambda^{-1} |||P_\omega^n - S_\omega^n||| < \lambda^n/4$. Thus if $S \in \mathcal{O}_{\epsilon_{n,\beta}}(P) \cap \mathcal{LY}(C_1, C_2, r, R)$ then

$$\left\|\Pi_{E_{\sigma^n(\omega)}||F_{\sigma^n(\omega)}} S_\omega^n v\right\| \geq (2K)^{-1} C_\lambda \lambda^n \|v\| \,.$$

Setting $N_\beta = \ln(2^{-1}K^{-1}C_\lambda)/\ln(1-\beta/\lambda)$, we observe that if $n > N_\beta$ and $S \in \mathcal{O}_{\epsilon_{n,\beta}}(P) \cap \mathcal{LY}(C_1, C_2, r, R)$ then (7.13) holds. $\square$

7.3 Stability of the fast spaces

In this section we will construct perturbed fast spaces $(E_\omega^S)_{\omega\in\Omega} \in \prod_{\omega\in\Omega} \mathcal{G}^d(X_\omega)$ when $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_\epsilon(P)$, and then show that these spaces approximate $(E_\omega)_{\omega\in\Omega}$ in a Saks space sense. We will construct $(E_\omega^S)_{\omega\in\Omega}$ as the fixed point of the forward graph transform of an iterate of S by applying the contraction mapping theorem on certain cones of subspaces. Specifically, for $U = (U_\omega)_{\omega\in\Omega} \in \prod_{\omega\in\Omega} L(F_\omega, E_\omega)$ such that $\Pi_{E_{\sigma^n(\omega)}||F_{\sigma^n(\omega)}} S_\omega^n (\mathrm{Id} + U_\omega)\Big|_{E_\omega}$ is invertible for every $\omega \in \Omega$ we define $(S^n)^*U$ by

$$((S^n)^*U)_\omega = (S_{\sigma^{-n}(\omega)}^n)^* U_{\sigma^{-n}(\omega)},$$

where the forward graph transform has domain $L(E_{\sigma^{-n}(\omega)}, F_{\sigma^{-n}(\omega)})$ and codomain $L(E_\omega, F_\omega)$. For each $\omega \in \Omega$ and $a > 0$ we define

$$\mathcal{C}_{\omega,a} = \{U \in L(E_\omega, F_\omega) : \|U\| \leq a\},$$

and set the fast cone field to be $\mathcal{C}_a = \prod_{\omega\in\Omega} \mathcal{C}_{\omega,a}$. For each $a > 0$ the fast cone field $\mathcal{C}_a$ is a complete metric space with the metric inherited from $\prod_{\omega\in\Omega} L(E_\omega, F_\omega)$. Our first main result for this section is the following.

Proposition 7.3.1. *There are $a_0, \epsilon_0 > 0, n_0 \in \mathbb{Z}^+$ so that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_0}(P)$ then $(S^{n_0})^* \mathcal{C}_{a_0} \subseteq \mathcal{C}_{a_0}$. Moreover, there exists $c_0 \in [0,1)$ such that for every $U, V \in \mathcal{C}_{a_0}$ and $\omega \in \Omega$ we have*

$$\|(S_\omega^{n_0})^*(U_\omega) - (S_\omega^{n_0})^*(V_\omega)\| \le c_0 \|U_\omega - V_\omega\| \tag{7.21}$$

i.e. $(S_\omega^{n_0})^*$ is a contraction mapping on $\mathcal{C}_{a_0}$.

If S satisfies the hypotheses of Proposition 7.3.1 then we let $U^S \in \mathcal{C}_{a_0}$ denote the unique fixed point of $(S^{n_0})^*$ and define $E_\omega^S = \Phi^{-1}_{E_\omega \oplus F_\omega}(U_\omega^S) = (\mathrm{Id} + U_\omega^S)(E_\omega)$. By Proposition 6.1.2 the sequence $(E_\omega^S)_{\omega\in\Omega} \in \prod_{\omega\in\Omega} \mathcal{N}(F_\omega)$ is fixed by S^{n_0} i.e. $S_\omega^{n_0} E_\omega^S = E^S_{\sigma^{n_0}(\omega)}$ for every $\omega \in \Omega$. Our second main result for this section confirms that if ϵ is sufficiently small then $(E_\omega^S)_{\omega\in\Omega}$ satisfies the estimate (7.11) and that $(E_\omega^S)_{\omega\in\Omega}$ and $(E_\omega)_{\omega\in\Omega}$ are close in a Saks space sense.

Proposition 7.3.2. *We have*

$$\sup\left\{ \left\|\Pi_{E_\omega^S || F_\omega}\right\| : \omega \in \Omega, S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_0}(P) \right\} < \infty. \tag{7.22}$$

Moreover, for every $\beta \in (0, (\lambda - \mu)/2)$ and $\delta > 0$ there is $\epsilon_{\beta,\delta} \in (0, \epsilon_0)$ and $C_\beta > 0$ such that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{\beta,\delta}}(P)$ then

$$\sup_{\omega\in\Omega} \left|\left|\left| \Pi_{E_\omega^S || F_\omega} - \Pi_{E_\omega || F_\omega} \right|\right|\right| \le \delta, \tag{7.23}$$

and if, in addition, we have $\omega \in \Omega$, $v \in E_\omega^S$ and $n \in \mathbb{Z}^+$ then

$$\|S_\omega^n v\| \ge C_\beta^{-1} (\lambda - \beta)^n \|v\|. \tag{7.24}$$

We will focus on proving Proposition 7.3.1 first.

Lemma 7.3.3. *Fix $\beta \in (0, (\lambda - \mu)/2)$ and $a > 0$. There exists constants M_β and, for each $n > M_\beta$, $\epsilon_{n,\beta,a} > 0$ such that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{n,\beta,a}}(P)$, $\omega \in \Omega$ and $U \in \mathcal{C}_{\omega,a}$ then $\Pi_{E_{\sigma^n(\omega)} || F_{\sigma^n(\omega)}} S_\omega^n (\mathrm{Id} + U) : E_\omega \to E_{\sigma^n(\omega)}$ is invertible with*

$$\left\| \left(\Pi_{E_{\sigma^n(\omega)} || F_{\sigma^n(\omega)}} S_\omega^n (\mathrm{Id} + U)\Big|_{E_\omega} \right)^{-1} \right\| \le (\lambda - \beta)^{-n}. \tag{7.25}$$

Proof. By Proposition 7.2.1 there exists M_β and, for each $n > M_\beta$, $\epsilon_{n,\beta} > 0$ such that for all $\omega \in \Omega$ and $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{n,\beta}}(P)$ we have

$$\left\| \left(\Pi_{E_{\sigma^n(\omega)} || F_{\sigma^n(\omega)}} S_\omega^n \Big|_{E_\omega} \right)^{-1} \right\| \leq 2(\lambda - \beta)^{-n}. \tag{7.26}$$

On the other hand, since $\Pi_{E_{\sigma^n(\omega)} || F_{\sigma^n(\omega)}} P_\omega^n U = 0$ and by Lemma 7.2.3 we have for every $\eta > 0$ that

$$\begin{aligned} \left\| \Pi_{E_{\sigma^n(\omega)} || F_{\sigma^n(\omega)}} S_\omega^n U \Big|_{E_\omega} \right\| &= \left\| \Pi_{E_{\sigma^n(\omega)} || F_{\sigma^n(\omega)}} (S_\omega^n - P_\omega^n) U \Big|_{E_\omega} \right\| \\ &\leq 2a\eta C_3 R^n + aC_\eta |||S_\omega^n - P_\omega^n|||. \end{aligned} \tag{7.27}$$

By fixing $\eta = \frac{(\lambda-\beta)^n}{4aC_3R^n}$ and applying Lemma 7.2.6 we find $\epsilon_{n,\beta,a} \in (0, \epsilon_{n,\beta})$ such that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{n,\beta,a}}(P)$ then $aC_\eta |||S_\omega^n - P_\omega^n||| \leq (\lambda - \beta)^n/2$. Applying these bounds to (7.27) implies that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{n,\beta,a}}(P)$, $\omega \in \Omega$ and $U \in \mathcal{C}_{\omega,a}$ then

$$\left\| \Pi_{E_{\sigma^n(\omega)} || F_{\sigma^n(\omega)}} S_\omega^n U \Big|_{E_\omega} \right\| \leq (\lambda - \beta)^n. \tag{7.28}$$

By combining (7.26) and (7.28) we confirm that $\Pi_{E_{\sigma^n(\omega)} || F_{\sigma^n(\omega)}} S_\omega^n (\mathrm{Id} + U)\Big|_{E_\omega}$ is invertible, and that the estimate (7.25) holds. □

Lemma 7.3.3 implies that for each $a > 0$ and n sufficiently large there exists $\epsilon_{a,n} > 0$ such that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{a,n}}(P)$, $\omega \in \Omega$ and $U \in \mathcal{C}_{\omega,a}$ then $(S_\omega^n)^* U$ is well defined.

Lemma 7.3.4. *For sufficiently large n there exists $a_n, \epsilon_n > 0$ such that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_n}(P)$ and $\omega \in \Omega$ then $(S_\omega^n)^* \mathcal{C}_{\omega,a_n} \subseteq \mathcal{C}_{\sigma^n(\omega),a_n}$.*

Proof. Fix $\beta \in (0, (\lambda - \mu)/2)$. For $a > 0$ let M_β and $\epsilon_{n,\beta,a}$ denote the constants produced by Lemma 7.3.3. By Lemma 7.3.3, for every $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{n,\beta,a}}(P)$, $\omega \in \Omega$ and $U \in \mathcal{C}_{\omega,a}$ we have $(S_\omega^n)^* U \in L(E_{\sigma^n(\omega)}, F_{\sigma^n(\omega)})$. By the estimate (7.25) and the definition of $(S_\omega^n)^*$ we have

$$\|(S_\omega^n)^* U\| \leq \left\| \Pi_{F_{\sigma^n(\omega)} || E_{\sigma^n(\omega)}} S_\omega^n (\mathrm{Id} + U) \right\| (\lambda - \beta)^{-n}.$$

Let N_β and $\epsilon_{n,\beta}$ denote the constants produced by Proposition 7.2.1 and set $\epsilon_n = \min\{\epsilon_{n,\beta}, \epsilon_{n,\beta,a}\}$. Then for $n > \max\{N_\beta, M_\beta\}$, $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_n}(P)$ and

$U \in \mathcal{C}_{\omega,a}$ we have

$$\left\| \Pi_{F_{\sigma^n(\omega)}||E_{\sigma^n(\omega)}} S_\omega^n(\mathrm{Id}+U) \right\| \leq \Theta \left(\|S_\omega^n\| + a \left\| S_\omega^n|_{F_\omega} \right\| \right) \leq \Theta \left(C_3 R^n + a(\mu+\beta)^n \right),$$

and so

$$\|(S_\omega^n)^* U\| \leq \Theta \left(C_3 \left(\frac{R}{\lambda - \beta} \right)^n + a \left(\frac{\mu+\beta}{\lambda-\beta} \right)^n \right). \tag{7.29}$$

Since $\beta \in (0, (\lambda-\mu)/2)$, it follows from (7.29) that if n is large enough so that $\Theta(\mu+\beta)^n < (\lambda-\beta)^n$ and we set $a_n = \Theta C_3 R^n ((\lambda-\beta)^n - \Theta(\mu+\beta)^n)^{-1}$ then $\|(S_\omega^n)^* U\| \leq a_n$ for every $U \in \mathcal{C}_{\omega,a_n}$ and $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_n}(P)$. □

Lemma 7.3.5. *Suppose that n is large enough so that Lemma 7.3.4 may be applied, and let a_n and ϵ_n denote the produced constants. For any such n there exists $\epsilon' \in (0, \epsilon_n]$, $k \in \mathbb{Z}^+$ and $c \in [0,1)$ such that for every $\omega \in \Omega$, $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon'}(P)$ and $U_1, U_2 \in \mathcal{C}_{\omega,a_n}$ we have*

$$\left\| (S_\omega^{nk})^*(U_1) - (S_\omega^{nk})^*(U_2) \right\| \leq c \|U_1 - U_2\|.$$

Proof. For brevity we set $\Xi_\omega = \Pi_{F_\omega||E_\omega}$ and $\Gamma_\omega = \Pi_{E_\omega||F_\omega}$. By the definition of $(S_\omega^{nk})^*$ we have

$$\begin{aligned}(S_\omega^{nk})^*(U_1) - (S_\omega^{nk})^*(U_2) &= \Xi_{\sigma^{nk}(\omega)} S_\omega^{nk}(U_1 - U_2) \left(\Gamma_{\sigma^{nk}(\omega)} S_\omega^{nk}(\mathrm{Id}+U_1)|_{E_\omega} \right)^{-1} \\ &\quad + ((S_\omega^{nk})^* U_2) \left(\Gamma_{\sigma^{nk}(\omega)} S_\omega^{nk}(U_2 - U_1) \right) \left(\Gamma_{\sigma^{nk}(\omega)} S_\omega^{nk}(\mathrm{Id}+U_1)|_{E_\omega} \right)^{-1}.\end{aligned} \tag{7.30}$$

Fix n large enough so that Lemma 7.3.4 may be applied. If $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_n}(P)$ then for every $k \in \mathbb{Z}^+$ we have $(S_\omega^{nk})^* \mathcal{C}_{\omega,a_n} \subseteq \mathcal{C}_{\sigma^{nk}(\omega),a_n}$, and so $((S_\omega^{nk})^* U_2) \leq a_n$. Thus, (7.30) becomes

$$\begin{aligned}&\left\| (S_\omega^{nk})^*(U_1) - (S_\omega^{nk})^*(U_2) \right\| \\ &\qquad \leq (1+a_n)\Theta \left\| S_\omega^{nk}|_{F_\omega} \right\| \left\| \left(\Gamma_{\sigma^{nk}(\omega)} S_\omega^{nk}(\mathrm{Id}+U_1)|_{E_\omega} \right)^{-1} \right\| \|U_1 - U_2\|.\end{aligned}$$

Fix $\beta \in (0, (\lambda-\mu)/2)$. By Proposition 7.2.1 and Lemma 7.3.3 for every k sufficiently large there exists $\epsilon_k > 0$ such that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{nk,\beta,a_n}}(P)$ then

$$\left\| (S_\omega^{nk})^*(U_1) - (S_\omega^{nk})^*(U_2) \right\| \leq \Theta(1+a_n) \left(\frac{\mu+\beta}{\lambda-\beta} \right)^{nk} \|U_1 - U_2\|. \tag{7.31}$$

By taking k large enough we may ensure that $c := \Theta(1+a_n)(\mu+\beta^{nk})/(\lambda-\beta)^{nk} < 1$, and so we obtain the required inequality from (7.31) upon setting $\epsilon' = \epsilon_k$. $\square$

The proof of Proposition 7.3.1. Suppose that n is large enough so Lemmas 7.3.4 and 7.3.5 may be applied, and let a_n, ϵ', k, and c denote the produced constants. Set $a_0 := a_n$, $n_0 := nk$, $\epsilon_0 := \epsilon'$ and $c_0 := c$. By Lemma 7.3.4 we have $(S^{n_0})^*\mathcal{C}_{a_0} \subseteq \mathcal{C}_{a_0}$ for every $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_0}(P)$. The estimate (7.21) is exactly the content of Lemma 7.3.5. $\square$

We turn to the proof of Proposition 7.3.2. Recall that $U^S \in \mathcal{C}_{a_0}$ denotes the unique fixed point of $(S^{n_0})^*$, and that $E_\omega^S = \Phi_{E_\omega \oplus F_\omega}^{-1}(U_\omega^S) = (\mathrm{Id} + U_\omega^S)(E_\omega)$.

Lemma 7.3.6. *We have*

$$\sup\left\{ \left\| \Pi_{E_\omega^S || F_\omega} \right\| : \omega \in \Omega, S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_0}(P) \right\} < \infty.$$

Proof. By Proposition 6.1.2 we have $\Pi_{E_\omega^S || F_\omega} = (\mathrm{Id} + U_\omega^S)\Pi_{E_\omega || F_\omega}$. Hence, as $U_\omega^S \in \mathcal{C}_{a_0}$, it follows that $\left\|\Pi_{E_\omega^S || F_\omega}\right\| \le \left\|\mathrm{Id} + U_\omega^S\right\| \left\|\Pi_{E_\omega || F_\omega}\right\| \le (1 + a_0)\Theta$. $\square$

Lemma 7.3.7. *For every $\delta > 0$ there exists $\epsilon_\delta \in (0, \epsilon_0]$ so that for every $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\delta}(P)$ we have*

$$\sup_{\omega\in\Omega} \left|\left|\left| \Pi_{E_\omega^S || F_\omega} - \Pi_{E_\omega || F_\omega} \right|\right|\right| \le \delta.$$

Proof. By Proposition 6.1.2 we have $\Pi_{E_\omega^S || F_\omega} = (\mathrm{Id} + U_\omega^S)\Pi_{E_\omega || F_\omega}$, and so

$$\left|\left|\left| \Pi_{E_\omega^S || F_\omega} - \Pi_{E_\omega || F_\omega} \right|\right|\right| \le \left|\left|\left| U_\omega^S \right|\right|\right| \left\| \Pi_{E_\omega || F_\omega} \right\| \le \Theta \left|\left|\left| U_\omega^S \right|\right|\right|. \tag{7.32}$$

For any $k \in \mathbb{Z}^+$ we have

$$\left|\left|\left| U_\omega^S \right|\right|\right| \le \left|\left|\left| (S_{\sigma^{-n_0 k}(\omega)}^{n_0 k})^* 0 \right|\right|\right| + \left\| (S_{\sigma^{-n_0 k}(\omega)}^{n_0 k})^* U_{\sigma^{-n_0 k}(\omega)}^S - (S_{\sigma^{-n_0 k}(\omega)}^{n_0 k})^* 0 \right\|. \tag{7.33}$$

By Proposition 7.3.1 we have

$$\left\| (S_{\sigma^{-n_0 k}(\omega)}^{n_0 k})^* U_{\sigma^{-n_0 k}(\omega)}^S - (S_{\sigma^{-n_0 k}(\omega)}^{n_0 k})^* 0 \right\| \le c_0^k a_0. \tag{7.34}$$

We fix k large enough so that $c_0^k a_0 < \delta/(3\Theta)$. Since $\left(P^{n_0k}_{\sigma^{-n_0k}(\omega)}\right)^* 0 = 0$ and $\Pi_{F_\omega||E_\omega} P^{n_0k}_{\sigma^{-n_0k}(\omega)} \Pi_{E_{\sigma^{-n_0k}(\omega)}||F_{\sigma^{-n_0k}(\omega)}} = 0$, after a short calculation we find that

$$(S^{n_0k}_{\sigma^{-n_0k}(\omega)})^* 0 = \Pi_{F_\omega||E_\omega} \left(S^{n_0k}_{\sigma^{-n_0k}(\omega)} - P^{n_0k}_{\sigma^{-n_0k}(\omega)}\right) \left(\Pi_{E_\omega||F_\omega} P^{n_0k}_{\sigma^{-n_0k}(\omega)}\Big|_{E_{\sigma^{-n_0k}(\omega)}}\right)^{-1}.$$

Hence, by Lemma 7.2.3, Remark 7.2.4 and (7.2), for every $\eta > 0$ there exists $C_\eta > 0$ such that

$$\begin{aligned}\left|\left|\left|(S^{nk}_{\sigma^{-n_0k}(\omega)})^* 0\right|\right|\right| \leq& C_\lambda^{-1}\lambda^{-nk}\eta \left\|S^{nk}_{\sigma^{-n_0k}(\omega)} - P^{nk}_{\sigma^{-n_0k}(\omega)}\right\| \\ &+ C_\lambda^{-1}\lambda^{-nk} C_\eta \left|\left|\left|S^{nk}_{\sigma^{-n_0k}(\omega)} - P^{nk}_{\sigma^{-n_0k}(\omega)}\right|\right|\right| \\ \leq& C_\lambda^{-1}\lambda^{-nk}\left(2\eta C_3 R^{nk} + C_\eta \left|\left|\left|S^{nk}_{\sigma^{-n_0k}(\omega)} - P^{nk}_{\sigma^{-n_0k}(\omega)}\right|\right|\right|\right).\end{aligned}$$

Since k is fixed there exists η such that $2\eta C_3 C_\lambda^{-1} R^{nk}\lambda^{-nk} < \delta/(3\Theta)$. Then, by Lemma 7.2.6, there exists $\epsilon_\delta \in (0, \epsilon_0]$ such that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\delta}(P)$ then

$$C_\eta C_\lambda^{-1}\lambda^{-nk} \left|\left|\left|S^{nk}_{\sigma^{-n_0k}(\omega)} - P^{nk}_{\sigma^{-n_0k}(\omega)}\right|\right|\right| \leq \frac{\delta}{3\Theta}.$$

Thus, if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\delta}(P)$ then $\left|\left|\left|(S^{nk}_{\sigma^{-n_0k}(\omega)})^* 0\right|\right|\right| \leq 2\delta/(3\Theta)$, and so $|||U^S_\omega||| \leq \delta/\Theta$ by (7.33). We obtain the required inequality upon recalling (7.32). $\square$

Lemma 7.3.8. *For each $\beta \in (0, (\lambda - \mu)/2)$ there exists $k_\beta \in \mathbb{Z}^+$ and $\epsilon_\beta > 0$ such that for every $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\beta}(P)$, $\omega \in \Omega$, $U \in \mathcal{C}_{\omega, a_0}$ and $v \in E^S_\omega$ we have*

$$\left\|S^{k_\beta n_0}_\omega v\right\| \geq (\lambda - \beta)^{k_\beta n_0} \|v\|.$$

Proof. Since $|\cdot| \leq \|\cdot\|$ we have

$$\left\|S^{kn_0}_\omega v\right\| \geq \left|S^{kn_0}_\omega v\right| \geq \left|P^{kn_0}_\omega \Pi_{E_\omega||F_\omega} v\right| - \left|\left(S^{kn_0}_\omega - P^{kn_0}_\omega\right)\Pi_{E_\omega||F_\omega} v\right| - \left\|S^{kn_0}_\omega \Pi_{F_\omega||E_\omega} v\right\|. \tag{7.35}$$

Using Lemma 7.2.2 and (7.2) we find that

$$\left|P^{kn_0}_\omega \Pi_{E_\omega||F_\omega} v\right| \geq K^{-1} C_\lambda \lambda^{kn_0} \left\|\Pi_{E_\omega||F_\omega} v\right\|.$$

Since $v \in E_\omega^S = (\mathrm{Id} + U_\omega)(E_\omega)$ and $U_\omega \in \mathcal{C}_{\omega,\epsilon_0}$ we have $\left\|\Pi_{F_\omega || E_\omega} v\right\| \leq a_0 \left\|\Pi_{E_\omega || F_\omega} v\right\|$ and so $(1+a_0)^{-1} \|v\| \leq \left\|\Pi_{E_\omega || F_\omega} v\right\|$. Hence (7.35) becomes

$$\left\|S_\omega^{kn_0} v\right\| \geq \left((1+a_0)^{-1} K^{-1} C_\lambda \lambda^{kn_0} - \Theta \left\|\left| S_\omega^{kn_0} - P_\omega^{kn_0} \right|\right\| - \Theta \left\| S_\omega^{kn_0}|_{F_\omega} \right\|\right) \|v\| . \tag{7.36}$$

Let $k_\beta := k$ be sufficiently large so that

$$(1+a_0)^{-1} K^{-1} C_\lambda \lambda^{k_\beta n_0} \geq 2(\lambda - \beta)^{k_\beta n_0}, \tag{7.37}$$

and so that Proposition 7.2.1 and Lemma 7.2.6 may be applied with $n = k_\beta n_0$ to produce ϵ_β so that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\beta}(P)$ and $\omega \in \Omega$ then

$$\left\| S_\omega^{k_\beta n_0}\big|_{F_\omega} \right\| \leq (\mu + \beta)^{k_\beta n_0} \leq \frac{(\lambda - \beta)^{k_\beta n_0}}{2\Theta} \quad \text{and} \quad \left\|\left| P_\omega^{k_\beta n_0} - S_\omega^{k_\beta n_0} \right|\right\| \leq \frac{(\lambda - \beta)^{k_\beta n_0}}{2\Theta}. \tag{7.38}$$

Applying (7.37) and (7.38) to (7.36) yields the required inequality. □

The proof of Proposition 7.3.2. The estimates (7.22) and (7.23) are proven in Lemmas 7.3.6 and 7.3.7, respectively. Thus to finish the proof it suffices to demonstrate (7.24).

For $\beta \in (0, (\lambda - \mu)/2)$ let k_β and ϵ_β be the constants produced by Lemma 7.3.8. For $n \in \mathbb{Z}^+$ write $n = mn_0 k_\beta + j$ where $m \in \mathbb{N}$ and $j \in \{0, \dots, n_0 k_\beta - 1\}$. For any $\omega \in \Omega$, $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\beta}(P)$ and $v \in E_\omega^S$ we have $S_\omega^{mn_0 k_\beta} v \in E_{\sigma^{mn_0 k_\beta}(\omega)}^S$ and $E_{\sigma^{mn_0 k_\beta}(\omega)}^S = \left(\mathrm{Id} + U_{\sigma^{mn_0 k_\beta}(\omega)}^S\right) E_{\sigma^{mn_0 k_\beta}(\omega)}$. Hence, as $U_{\sigma^{mn_0 k_\beta}(\omega)}^S \in \mathcal{C}_{\sigma^{mn_0 k_\beta}(\omega), a_0}$, by Lemma 7.3.8 we have

$$\left\| S_\omega^{(m+1) n_0 k_\beta} v \right\| = \left\| S_{\sigma^{mn_0 k_\beta}(\omega)}^{n_0 k_\beta} S_\omega^{mn_0 k_\beta} v \right\| \geq (\lambda - \beta)^{n_0 k_\beta} \left\| S_\omega^{mn_0 k_\beta} v \right\| .$$

By repeating this argument we deduce that $\left\| S_\omega^{(m+1) n_0 k_\beta} v \right\| \geq (\lambda - \beta)^{(m+1) n_0 k_\beta} \|v\|$. Therefore, as $R > \lambda - \beta$,

$$\|S_\omega^n v\| \geq \left\| S_{\sigma^{mn_0 k_\beta + j}(\omega)}^{n_0 k_\beta - j} \right\|^{-1} \left\| S_\omega^{(m+1) n_0 k_\beta} v \right\| \geq C_3^{-1} \left(\frac{\lambda - \beta}{R} \right)^{n_0 k_\beta} (\lambda - \beta)^n \|v\| ,$$

and so we obtain the required claim by setting $C_\beta = C_3 \left(\frac{R}{\lambda - \beta}\right)^{n_0 k_\beta}$. □

7.4 Stability of the slow spaces

In this section we will construct and characterise the perturbed slow spaces for $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_\epsilon(P)$ when ϵ is sufficiently small. These perturbed slow spaces will be the fixed point of a backwards graph transform associated to S, although our approach is slightly different to that of the previous section since we may capitalise on the existence of fast spaces for S. Once constructed, we show that the slow spaces are stable in the Grassmannian, and verify the estimate (7.10). Let n_0 and ϵ_0 be the constants produced by Proposition 7.3.1, and suppose that $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_0}(P)$. For $V \in L(F_\omega, E_\omega^S)$ recall that $(S_{\sigma^{-n_0}(\omega)}^{n_0})_* V$ is well-defined if $(\Pi_{E_\omega^S || F_\omega} - V\Pi_{F_\omega || E_\omega^S}) S_{\sigma^{-n_0}(\omega)}^{n_0} : E_{\sigma^{-n_0}(\omega)}^S \to E_\omega^S$ is invertible. Since $S_{\sigma^{-n_0}(\omega)}^{n_0} E_{\sigma^{-n_0}(\omega)}^S = E_\omega^S$ it follows that

$$(\Pi_{E_\omega^S || F_\omega} - V\Pi_{F_\omega || E_\omega^S}) S_{\sigma^{-n_0}(\omega)}^{n_0}\Big|_{E_{\sigma^{-n_0}(\omega)}^S} = S_{\sigma^{-n_0}(\omega)}^{n_0}\Big|_{E_{\sigma^{-n_0}(\omega)}^S},$$

which is always invertible. Hence $(S_{\sigma^{-n_0}(\omega)}^{n_0})_* : L(F_\omega, E_\omega^S) \to L(F_{\sigma^{-n_0}(\omega)}, E_{\sigma^{-n_0}(\omega)}^S)$ is well defined and satisfies

$$(S_{\sigma^{-n_0}(\omega)}^{n_0})_* V = \left(S_{\sigma^{-n_0}(\omega)}^{n_0}\Big|_{E_{\sigma^{-n_0}(\omega)}^S} \right)^{-1} \left(V\Pi_{F_\omega || E_\omega^S} - \Pi_{E_\omega^S || F_\omega} \right) S_{\sigma^{-n_0}(\omega)}^{n_0}.$$

Finally, let $S_*^{n_0} : \prod_{\omega \in \Omega} L(F_\omega, E_\omega^S) \to \prod_{\omega \in \Omega} L(F_\omega, E_\omega^S)$ be defined by

$$(S_*^{n_0} V)_\omega = (S_\omega^{n_0})_* V_{\sigma^{n_0}(\omega)}.$$

Proposition 7.4.1. *There exists $k \in \mathbb{Z}^+$, $c \in [0, 1)$ and $\epsilon_1 \in [0, \epsilon_0)$ such that for any $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_1}(P)$, $\omega \in \Omega$ and $V_1, V_2 \in L(F_\omega, E_\omega^S)$ we have*

$$\left\| (S_{\sigma^{-n_0 k}(\omega)}^{n_0 k})_*(V_1) - (S_{\sigma^{-n_0 k}(\omega)}^{n_0 k})_*(V_2) \right\| \le c \left\| V_1 - V_2 \right\|.$$

Hence $S_*^{n_0 k}$ is a contraction mapping on $\prod_{\omega \in \Omega} L(F_\omega, E_\omega^S)$.

If S satisfies the hypotheses of Proposition 7.4.1 then we denote the unique fixed point of $S_*^{n_0 k}$ by $V^S \in \prod_{\omega \in \Omega} L(F_\omega, E_\omega^S)$, and set $F_\omega^S = \Phi_{F_\omega \oplus E_\omega^S}^{-1}(V_\omega^S) = (\mathrm{Id} + V_\omega^S)(F_\omega)$. Note that, since $S_*^{n_0}$ preserves $\prod_{\omega \in \Omega} L(F_\omega, E_\omega^S)$, we must have $S_*^{n_0} V^S = V^S$. Moreover, by the definition of the graph representation we have $F_\omega^S \in \mathcal{N}(E_\omega^S)$, so that $X = F_\omega^S \oplus E_\omega^S$. Our second main result for this section confirms that the spaces $(F_\omega^S)_{\omega\in\Omega}$ are equivariant slow spaces for S^{n_0k}, and that $(F_\omega^S)_{\omega\in\Omega}$ approximates $(F_\omega)_{\omega\in\Omega}$ in the Grassmannian.

Proposition 7.4.2. *We have*

$$\sup\left\{\left\|\Pi_{E_\omega^S||F_\omega^S}\right\| : \omega \in \Omega, S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_1}(P)\right\} < \infty, \tag{7.39}$$

and $S_\omega^{n_0} F_\omega^S \subseteq F_{\sigma^{n_0}(\omega)}^S$ *for every* $\omega \in \Omega$. *Moreover, for every* $\beta \in (0, (\lambda - \mu)/2)$ *and* $\delta > 0$ *there is* $\epsilon_{\beta,\delta} \in (0, \epsilon_1]$ *and* $C_\beta > 0$ *such that if* $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{\beta,\delta}}(P)$ *then*

$$\sup_{\omega\in\Omega} d_H(F_\omega^S, F_\omega) \leq \delta, \tag{7.40}$$

$$\sup_{\omega\in\Omega} \left|\left|\left|\Pi_{F_\omega^S||E_\omega^S} - \Pi_{F_\omega||E_\omega}\right|\right|\right| \leq \delta, \tag{7.41}$$

and if, in addition, we have $\omega \in \Omega$ *and* $n \in \mathbb{Z}^+$ *then*

$$\left\|S_\omega^n|_{F_\omega^S}\right\| \leq C_\beta(\mu + \beta)^n. \tag{7.42}$$

To fix some notation, we let

$$M := \sup\left\{\left\|\Pi_{E_\omega^S||F_\omega}\right\| : \omega \in \Omega, S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_0}(P)\right\}, \tag{7.43}$$

which is finite by Proposition 7.3.2.

The proof of Proposition 7.4.1. Let $\beta \in (0, (\lambda - \mu)/2)$. By Proposition 7.3.2 there exists $\epsilon_\beta \in (0, \epsilon_0)$ and $C_\beta > 0$ so that for every $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\beta}(P)$ and $k \in \mathbb{Z}^+$ we have

$$\left\|\left(S_{\sigma^{-n_0k}(\omega)}^{n_0k}\Big|_{E_{\sigma^{-n_0k}(\omega)}^S}\right)^{-1}\right\| \leq C_\beta(\lambda - \beta)^{-n_0k}. \tag{7.44}$$

Fix k large enough so that $c := C_\beta(M+1)(\mu+\beta)^{n_0k}/(\lambda-\beta)^{n_0k} < 1$. By Proposition 7.2.1 there exists $\epsilon_{\beta,k} \in (0, \epsilon_\beta)$ so that for $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{\beta,k}}(P)$ we have

$$\left\|S_{\sigma^{-n_0k}(\omega)}^{n_0k}\Big|_{F_{\sigma^{-n_0k}(\omega)}}\right\| \leq (\mu + \beta)^{n_0k}. \tag{7.45}$$

Set $\epsilon_1 := \epsilon_{\beta,k}$. By Proposition 6.1.5 we have for every $k \in \mathbb{Z}^+$ that

$$
\begin{aligned}
&(S^{n_0k}_{\sigma^{-n_0k}(\omega)})_*(V_1) - (S^{n_0k}_{\sigma^{-n_0k}(\omega)})_*(V_2) \\
&\qquad = \left(S^{n_0k}_{\sigma^{-n_0k}(\omega)}\Big|_{E^S_{\sigma^{-n_0k}(\omega)}} \right)^{-1} (V_1 - V_2)\Pi_{F_\omega || E^S_\omega} S^{n_0k}_{\sigma^{-n_0k}(\omega)}\Big|_{F_{\sigma^{-n_0k}(\omega)}}.
\end{aligned}
$$

We obtain the required statement by applying (7.43), (7.44), and (7.45) to the previous equality, and then recalling that $c \in [0, 1)$. □

The proof of Proposition 7.4.2 is broken into a number of lemmas.

Lemma 7.4.3. *For every $\omega \in \Omega$ and $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_1}(P)$ we have $S^{n_0}_\omega F^S_\omega \subseteq F^S_{\sigma^{n_0}(\omega)}$.*

Proof. Since $(S^{n_0}_\omega)_* V^S_{\sigma^{n_0}(\omega)} = V^S_\omega$ we have

$$
\begin{aligned}
S^{n_0}_\omega F^S_\omega &= S^{n_0}_\omega(\operatorname{Id} + V^S_\omega)(F_\omega) \\
&= S^{n_0}_\omega(\operatorname{Id} + (S^{n_0}_\omega)_* V^S_{\sigma^{n_0}(\omega)})(F_\omega) \\
&= S^{n_0}_\omega \left(\operatorname{Id} + \left(S^{n_0}_\omega \Big|_{E^S_\omega} \right)^{-1} \left(V^S_{\sigma^{n_0}(\omega)} \Pi_{F_{\sigma^{n_0}(\omega)} || E^S_{\sigma^{n_0}(\omega)}} - \Pi_{E^S_{\sigma^{n_0}(\omega)} || F_{\sigma^{n_0}(\omega)}} \right) S^{n_0}_\omega \right) F_\omega \\
&= \left(\left(\operatorname{Id} - \Pi_{E^S_{\sigma^{n_0}(\omega)} || F_{\sigma^{n_0}(\omega)}} \right) + V^S_{\sigma^{n_0}(\omega)} \Pi_{F_{\sigma^{n_0}(\omega)} || E^S_{\sigma^{n_0}(\omega)}} \right) S^{n_0}_\omega F_\omega \\
&= (\operatorname{Id} + V^S_{\sigma^{n_0}(\omega)}) \Pi_{F_{\sigma^{n_0}(\omega)} || E^S_{\sigma^{n_0}(\omega)}} S^{n_0}_\omega (F_\omega) \\
&\subseteq (\operatorname{Id} + V^S_{\sigma^{n_0}(\omega)})(F_{\sigma^{n_0}(\omega)}) = F^S_{\sigma^{n_0}(\omega)}.
\end{aligned}
$$

□

Lemma 7.4.4. *We have*

$$
\sup \left\{ \left\| \Pi_{E^S_\omega || F^S_\omega} \right\| : \omega \in \Omega, S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_1}(P) \right\} < \infty. \tag{7.46}
$$

Proof. By Proposition 6.1.2 we have $\Pi_{F^S_\omega || E^S_\omega} = (\operatorname{Id} + V^S_\omega)\Pi_{F_\omega || E^S_\omega}$ and so $\left\| \Pi_{F^S_\omega || E^S_\omega} \right\| \leq (1 + \left\| V^S_\omega \right\|) \left\| \Pi_{F_\omega || E^S_\omega} \right\|$. In view of (7.43) it therefore suffices to bound $\left\| V^S_\omega \right\|$ uniformly in ω and S. By Proposition 7.4.1 we have for every $\omega \in \Omega$ that

$$
\left\| V^S_\omega \right\| \leq \left\| (S^{n_0k}_\omega)_* V^S_{\sigma^{n_0k}(\omega)} - (S^{n_0k}_\omega)_*(0) \right\| + \left\| (S^{n_0k}_\omega)_*(0) \right\| \leq c \left\| V^S_{\sigma^{n_0k}(\omega)} \right\| + \left\| (S^{n_0k}_\omega)_*(0) \right\|,
$$

from which it follows that

$$
\sup_{\omega \in \Omega} \left\| V^S_\omega \right\| \leq (1 - c)^{-1} \sup_{\omega \in \Omega} \left\| (S^{n_0k}_\omega)_*(0) \right\|.
$$

Since

$$(S_\omega^{n_0k})_*(0) = -\left(S_\omega^{n_0k}|_{E_\omega^S}\right)^{-1}\Pi_{E^S_{\sigma^{n_0k}(\omega)}||F_{\sigma^{n_0k}(\omega)}}S_\omega^{n_0k},$$

the bounds used in the proof of Proposition 7.4.1 imply that

$$\left\|(S_\omega^{n_0k})_*(0)\right\| \leq \left\|\left(S_\omega^{n_0k}|_{E_\omega^S}\right)^{-1}\right\|\left\|\Pi_{E^S_{\sigma^{n_0k}(\omega)}||F_{\sigma^{n_0k}(\omega)}}\right\|\left\|S_\omega^{n_0k}|_{F_\omega}\right\| < c.$$

Hence for every $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_1}(P)$ we have $\sup_{\omega\in\Omega}\left\|V_\omega^S\right\| \leq c(1-c)^{-1}$, which completes the proof. □

Lemma 7.4.5. *For all $\delta > 0$ there is $\epsilon_\delta \in (0, \epsilon_1]$ so that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\delta}(P)$ and $\omega \in \Omega$ then $\left\|V_\omega^S\right\| \leq \delta$.*

Proof. Since $V_\omega^S = (S_\omega^{n_0km})_* V^S_{\sigma^{n_0km}(\omega)}$ for every $m \in \mathbb{Z}^+$, we have

$$V_\omega^S = \left(S_\omega^{n_0km}|_{E_\omega^S}\right)^{-1}\left(V^S_{\sigma^{n_0km}(\omega)}\Pi_{F_{\sigma^{n_0km}(\omega)}||E^S_{\sigma^{n_0km}(\omega)}} - \Pi_{E^S_{\sigma^{n_0km}(\omega)}||F_{\sigma^{n_0km}(\omega)}}\right)S_\omega^{n_0km}|_{F_\omega}. \tag{7.47}$$

Fix $\beta \in (0, (\lambda-\mu)/2)$. By Proposition 7.3.2 there exists $\epsilon_\beta \in (0, \epsilon_1]$ and C_β so that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\beta}(P)$ then for every $m \in \mathbb{Z}^+$ we have

$$\left\|\left(S_\omega^{n_0km}|_{E_\omega^S}\right)^{-1}\right\| \leq C_\beta(\lambda-\beta)^{-n_0km}. \tag{7.48}$$

Let N_β be the constant produced by Proposition 7.2.1 and fix $m > N_\beta/(n_0k)$ large enough so that

$$C_\beta\left(\frac{c(1+M)}{1-c} + M\right)\left(\frac{\mu+\beta}{\lambda-\beta}\right)^{n_0km} \leq \delta. \tag{7.49}$$

By Proposition 7.2.1 there is $\epsilon_\delta \in (0, \epsilon_\beta]$ such that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\delta}(P)$ then

$$\left\|S_\omega^{n_0km}|_{F_\omega}\right\| \leq (\mu+\beta)^{mn_0k}. \tag{7.50}$$

Recalling from the proof of Lemma 7.4.4 that $\left\|V_\omega^S\right\| \leq c(1-c)^{-1}$, and then applying (7.48), (7.49) and (7.49) to (7.47) yields the required inequality. □

Lemma 7.4.6. *For all $\delta > 0$ there is $\epsilon_\delta \in (0, \epsilon_1]$ so that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\delta}(P)$ and $\omega \in \Omega$ then $d_H(F_\omega^S, F_\omega) \leq \delta$.*

Proof. Since $\Phi_{F_\omega \oplus E_\omega^S}(F_\omega) = 0$, by Lemma 6.1.3 and (7.43) we have

$$d_H(F_\omega^S, F_\omega) \leq 2 \left\| \Pi_{F_\omega || E_\omega^S} \right\| \left\| V_\omega^S - \Phi_{F_\omega \oplus E_\omega^S}(F_\omega) \right\| \leq 2(M+1) \left\| V_\omega^S \right\|,$$

and so the required inequality follows immediately from Lemma 7.4.5. $\square$

Lemma 7.4.7. *For every $\delta > 0$ there exists $\epsilon_\delta \in (0, \epsilon_1]$ such that for all $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\delta}(P)$ one has*

$$\sup_{\omega \in \Omega} \left\| \left| \Pi_{F_\omega^S || E_\omega^S} - \Pi_{F_\omega || E_\omega} \right| \right\| \leq \delta.$$

Proof. By the triangle inequality we get

$$\left\| \left| \Pi_{F_\omega^S || E_\omega^S} - \Pi_{F_\omega || E_\omega} \right| \right\| \leq \left\| \Pi_{F_\omega^S || E_\omega^S} - \Pi_{F_\omega || E_\omega^S} \right\| + \left\| \left| \Pi_{E_\omega^S || F_\omega} - \Pi_{E_\omega || F_\omega} \right| \right\|. \tag{7.51}$$

By Proposition 6.1.2 and (7.43) we have

$$\left\| \Pi_{F_\omega^S || E_\omega^S} - \Pi_{F_\omega || E_\omega^S} \right\| \leq \left\| V_\omega^S \right\| \left\| \Pi_{F_\omega || E_\omega^S} \right\| \leq (M+1) \left\| V_\omega^S \right\|.$$

Hence by Lemma 7.4.5 there exists $\epsilon_{\delta,1} \in (0, \epsilon_1]$ such that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{\delta,1}}(P)$ then

$$\sup_{\omega \in \Omega} \left\| \Pi_{F_\omega^S || E_\omega^S} - \Pi_{F_\omega || E_\omega^S} \right\| \leq \delta/2. \tag{7.52}$$

On the other hand, by Proposition 7.3.2 there exists $\epsilon_{\delta,2} \in (0, \epsilon_0]$ such that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{\delta,2}}(P)$ then

$$\sup_{\omega \in \Omega} \left\| \left| \Pi_{E_\omega^S || F_\omega} - \Pi_{E_\omega || F_\omega} \right| \right\| \leq \delta/2. \tag{7.53}$$

Upon setting $\epsilon_\delta = \min\{\epsilon_{\delta,1}, \epsilon_{\delta,2}\}$ we may conclude by applying (7.52) and (7.53) to (7.51). $\square$

Lemma 7.4.8. *For $\beta \in (0, (\lambda - \mu)/2)$ there exists $\epsilon_\beta \in (0, \epsilon_1]$ and $m \in \mathbb{Z}^+$ such that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\delta}(P)$ and $\omega \in \Omega$ then*

$$\left\| S_\omega^{n_0 k m} |_{F_\omega^S} \right\| \leq (\mu + \beta)^{n_0 k m}.$$

Proof. We have

$$\left\| S_\omega^{n_0 k m} |_{F_\omega^S} \right\| \leq \left\| S_\omega^{n_0 k m} |_{F_\omega} \right\| + \left\| S_\omega^{n_0 k m} \right\| d_H(F_\omega, F_\omega^S) \leq \left\| S_\omega^{n_0 k m} |_{F_\omega} \right\| + C_3 R^{n_0 k m} d_H(F_\omega, F_\omega^S). \tag{7.54}$$

By Proposition 7.2.1 there exists $\epsilon_{\beta,1} \in (0, \epsilon_1)$ and $m \in \mathbb{Z}^+$ such that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{\beta,1}}(P)$ and $\omega \in \Omega$ then

$$\left\| S_\omega^{n_0 k m}\big|_{F_\omega} \right\| \leq \frac{(\mu+\beta)^{n_0 k m}}{2}. \tag{7.55}$$

By Lemma 7.4.6 there exists $\epsilon_{\beta,2} \in (0, \epsilon_\beta)$ such that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{\beta,2}}(P)$ then

$$\sup_{\omega\in\Omega} d_H(F_\omega, F_\omega^S) \leq (2C_3)^{-1} \left(\frac{\mu+\beta}{R}\right)^{n_0 k m}. \tag{7.56}$$

We obtain the required inequality by setting $\epsilon_\beta = \min\{\epsilon_{\beta,1}, \epsilon_{\beta,2}\}$ and then applying (7.55) and (7.56) to (7.54). $\square$

The proof of Proposition 7.4.2. Lemma 7.4.4 proves (7.39), while it follows from Lemma 7.4.3 that $S_\omega^{n_0} F_\omega^S \subseteq F_{\sigma^{n_0}(\omega)}^S$ for every $\omega \in \Omega$. We get (7.40) and (7.41) from Lemmas 7.4.6 and 7.4.7, respectively.

Thus it remains to prove (7.42), which we will do using Lemma 7.4.8. With the notation of Lemma 7.4.8 set $n_1 = n_0 k m$. For $n \in \mathbb{Z}^+$ write $n = \ell n_1 + j$ where $\ell \in \mathbb{Z}^+$ and $j \in \{0, \ldots, n_1 - 1\}$. By Lemma 7.4.8 and the equivariance of $(F_\omega^S)_{\omega\in\Omega}$ we have for $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\beta}(P)$ that

$$\left\| S_\omega^{\ell n_1}\big|_{F_\omega^S} \right\| \leq \prod_{i=0}^{\ell-1} \left\| S_{\sigma^{(i n_1)}(\omega)}^{n_1}\big|_{F_{\sigma^{(i n_1)}(\omega)}^S} \right\| \leq (\mu+\beta)^{\ell n_1},$$

and so

$$\left\| S_\omega^{n}\big|_{F_\omega^S} \right\| \leq \left\| S_{\sigma^{\ell n_1}(\omega)}^{j} \right\| \left\| S_\omega^{\ell n_1}\big|_{F_\omega^S} \right\| \leq C_3 \left(\frac{R}{\mu+\beta}\right)^j (\mu+\beta)^n.$$

Since $\mu + \beta \leq R$ we obtain (7.42) upon setting $C_\beta = C_3 \left(\frac{R}{\mu+\beta}\right)^{n_1 - 1}$. $\square$

7.5 Completing the proof of Theorem 7.1.7

We have assembled most of the ingredients that are required to complete the proof of Theorem 7.1.7. Indeed, all of the conclusions of Theorem 7.1.7 are verified by Propositions 7.3.2 and 7.4.2, except for the following result.

Proposition 7.5.1. *There exists $\epsilon' \in (0, \epsilon_1)$ such that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon'}(P)$ then the fast spaces $(E_\omega^S)_{\omega\in\Omega} \in \prod_{\omega\in\Omega} \mathcal{G}_d(X_\omega)$ and slow spaces $(F_\omega^S)_{\omega\in\Omega} \in \prod_{\omega\in\Omega} \mathcal{G}^d(X_\omega)$ produced by Propositions 7.3.1 and 7.4.1, respectively, form a hyperbolic splitting of index d for S.*

Proof. Fix $\beta \in (0, (\lambda - \mu)/2)$. By Propositions 7.3.2 and 7.4.2 there exists $\epsilon' > 0$ and C_β such that if $S \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon'}(P)$, $n \in \mathbb{Z}^+$ and $\omega \in \Omega$ then

$$\left\| S_\omega^n|_{F_\omega^S} \right\| \leq C_\beta(\mu + \beta)^n,$$

and if, in addition, $v \in E_\omega^S$ then

$$\|S_\omega^n v\| \geq C_\beta^{-1}(\lambda - \beta)^n.$$

Hence, it suffices to prove that for every $\omega \in \Omega$ we have $S_\omega E_\omega^S = E_{\sigma(\omega)}^S$ and $S_\omega F_\omega^S \subseteq F_{\sigma(\omega)}^S$. We will prove these separately.

The equivariance of $(F_\omega^S)_{\omega\in\Omega}$. If $S_\omega F_\omega^S \not\subseteq F_{\sigma(\omega)}^S$ then there exists $f \in F_\omega^S$ such that $\|f\| = 1$ and $S_\omega f \notin F_{\sigma(\omega)}^S$. Thus $\operatorname{codim}(F_{\sigma(\omega)}^S \oplus \operatorname{span}\{S_\omega f\}) = d - 1$, and so there exists $e \in E_{\sigma(\omega)}^S \cap (F_{\sigma(\omega)}^S \oplus \operatorname{span}\{S_\omega f\})$ with $\|e\| = 1$. Write $e = aS_\omega f + f'$ where a is a scalar and $f' \in F_{\sigma(\omega)}^S$. For every $n \in \mathbb{Z}^+$ we have

$$\frac{(\lambda-\beta)^n}{C_\beta} \leq \left\|S_{\sigma(\omega)}^m e\right\| \leq |a| \left\|S_\omega^{n+1} f\right\| + \left\|S_{\sigma(\omega)}^n f'\right\| \leq C_\beta(\mu+\beta)^n \left(|a| \left(\mu+\beta\right) \|f\| + \|f'\|\right).$$

Since $\lambda - \beta > \mu + \beta$ we obtain a contradiction by taking $n \to \infty$.

The equivariance of $(E_\omega^S)_{\omega\in\Omega}$. If $S_\omega E_\omega^S \neq E_{\sigma(\omega)}^S$ then there exists $e \in E_\omega^S$ such that $\|e\| = 1$ and $S_\omega e \notin E_{\sigma(\omega)}^S$. Recall that for the constant n_0 produced by Propositions 7.3.1 we have $S_{\sigma^{-mn_0}(\omega)}^{mn_0} E_{\sigma^{-mn_0}(\omega)}^S = E_\omega^S$ for every $m \in \mathbb{Z}^+$. Hence, for each $m \in \mathbb{Z}^+$ there is a unique vector $e_m \in E_{\sigma^{-mn_0}(\omega)}^S$ satisfying $S_{\sigma^{-mn_0}(\omega)}^{mn_0} e_m = e$. Since $S_{\sigma^{-mn_0+1}(\omega)}^{mn_0} E_{\sigma^{-mn_0+1}(\omega)}^S = E_{\sigma(\omega)}^S$ we must have $S_{\sigma^{-mn_0}(\omega)} e_m \notin E_{\sigma^{-mn_0+1}(\omega)}^S$. Thus $\dim(E_{\sigma^{-mn_0+1}(\omega)}^S \oplus \operatorname{span}\{S_{\sigma^{-mn_0}(\omega)} e_m\}) = d + 1$, and so there exists $f_m \in (E_{\sigma^{-mn_0+1}(\omega)}^S \oplus \operatorname{span}\{S_{\sigma^{-mn_0}(\omega)} e_m\}) \cap F_{\sigma^{-mn_0+1}(\omega)}^S$ with $\|f_m\| = 1$. Writing $f_m = a_m S_{\sigma^{-mn_0}(\omega)} e_m + g_m$ for some scalar a_m and $g_m \in E_{\sigma^{-mn_0+1}(\omega)}^S$, we have

$$\begin{aligned}
C_\beta(\mu+\beta)^{mn_0-1} &\geq \left\| S_{\sigma^{-mn_0+1}(\omega)}^{mn_0-1} f_m \right\| \\
&= \left\| a_m S_{\sigma^{-mn_0}(\omega)}^{mn_0} e_m + S_{\sigma^{-mn_0+1}(\omega)}^{mn_0-1} g_m \right\| \\
&\geq \max \left\{ \begin{array}{l} \left\| S_{\sigma^{-mn_0+1}(\omega)}^{mn_0-1} g_m \right\| \left\| \Pi_{E_{\sigma(\omega)}^S || \operatorname{span}\{S_\omega e\}} \right\|^{-1}, \\ |a_m| \left\| S_{\sigma^{-mn_0}(\omega)}^{mn_0} e_m \right\| \left\| \Pi_{\operatorname{span}\{S_\omega e\} || E_{\sigma(\omega)}^S} \right\|^{-1} \end{array} \right. \\
&\geq \frac{(\lambda-\beta)^{mn_0-1}}{2C_\beta} \max\left\{ |a_m| \left(\lambda - \beta\right) \|e_m\|, \|g_m\| \right\} \left\| \Pi_{\operatorname{span}\{S_\omega e\} || E_{\sigma(\omega)}^S} \right\|^{-1}.
\end{aligned}$$

Since $f_m = a_m S_{\sigma^{-mn_0}(\omega)} e_m + g_m$ and $\|f_m\| = 1$ we have $1 \leq |a_m| \, C_3 R \, \|e_m\| + \|g_m\|$, and so

$$C_\beta(\mu+\beta)^{mn_0-1} \geq \frac{(\lambda-\beta)^{mn_0-1}}{2C_\beta} \max\left\{(\lambda-\beta)\frac{1-\|g_m\|}{C_3 R}, \|g_m\|\right\} \left\|\Pi_{\mathrm{span}\{S_\omega e\}||E^S_{\sigma(\omega)}}\right\|^{-1}.$$

For any value of $\|g_m\|$ we have

$$\max\left\{(\lambda-\beta)\frac{1-\|g_m\|}{C_3 R}, \|g_m\|\right\} \geq \frac{\lambda-\beta}{C_3 R + \lambda - \beta}.$$

Thus

$$C_\beta(\mu+\beta)^{mn_0-1} \geq \frac{(\lambda-\beta)^{mn_0}}{2C_\beta(C_3 R + \lambda - \beta)} \left\|\Pi_{\mathrm{span}\{S_\omega e\}||E^S_{\sigma(\omega)}}\right\|^{-1},$$

and so we obtain a contradiction by sending $m \to \infty$. $\square$

Chapter 8

Application to random linear systems

In this chapter we will use Theorem 7.1.7 to prove the stability of the Oseledets splitting and Lyapunov exponents of certain random linear systems. In Section 8.1 we recall some notions from multiplicative ergodic theory, discuss their relation to the material of the previous chapter and then state this chapter's main result, Theorem 8.1.8. The proof of Theorem 8.1.8 is divided across Sections 8.2, 8.3 and 8.4. The main ideas of the proof are contained in the former two sections, while Section 8.4 contains some miscellaneous results on the measurability of certain intermediate constructions.

8.1 Definitions and main results

In order to properly formulate our results we need some language from [47] (although we note the existence of alternatives, such as [15, 43]).

Definition 8.1.1. *A separable strongly measurable random linear system is a tuple $\mathcal{Q} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, Q)$ such that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space, $\sigma : \Omega \to \Omega$ is a $\mathbb{P}$-preserving transformation of $(\Omega, \mathcal{F}, \mathbb{P})$, X is a separable Banach space, and the generator $Q : \Omega \to L(X)$ is strongly measurable i.e. for every $x \in X$ the map $\omega \mapsto Q_\omega(x)$ is $(\mathcal{F}, \mathcal{B}_X)$-measurable where $\mathcal{B}_X$ is the Borel σ-algebra on X. We say that $\mathcal{Q}$ has an ergodic invertible base if σ is invertible and $\mathbb{P}$-ergodic.*

Remark 8.1.2. We will frequently use an alternative characterisation of strong measurability from [47, Appendix A]: in the context of Definition 8.1.1 this condition is equivalent to Q being $(\mathcal{F}, \mathcal{S})$-measurable, where $\mathcal{S}$ is the Borel σ-algebra of the strong operator topology on $L(X)$.

Definition 8.1.3. *Let $\mathcal{Q} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, Q)$ be a separable strongly measurable random linear system. Suppose that there exists $k_Q \in \mathbb{Z}^+$, constants $\lambda_{1,Q} > \lambda_{2,Q} >$*

$\cdots > \lambda_{k_Q,Q} > \mu_Q$, a map $F_Q : \Omega \mapsto \mathcal{G}(X)$, and for each $i \in \{1, \ldots, k_Q\}$ a positive integer $d_{i,Q}$ and a map $E_{i,Q} : \Omega \to \mathcal{G}_{d_{i,Q}}(X)$, such that

1. *For a.e. ω we have*

$$X = \left(\bigoplus_{i=1}^{k_Q} E_{i,Q}(\omega) \right) \oplus F_Q(\omega), \tag{8.1}$$

   and each of the projections associated to the decomposition (8.1) is strongly measurable.

2. *For every $i \in \{1, \ldots, k_Q\}$ and a.e. $\omega \in \Omega$ we have $Q_\omega E_{i,Q}(\omega) = E_{i,Q}(\sigma(\omega))$, and for each non-zero $v \in E_{i,Q}(\omega)$ one has*

$$\lim_{n\to\infty} \frac{1}{n} \ln \|Q_\omega^n v\| = \lambda_{i,Q}. \tag{8.2}$$

3. *For a.e. $\omega \in \Omega$ one has $Q_\omega F_Q(\omega) \subseteq F_Q(\sigma(\omega))$ and*

$$\lim_{n\to\infty} \frac{1}{n} \ln \left\| Q_\omega^n |_{F_Q(\omega)} \right\| \leq \mu_Q. \tag{8.3}$$

Then we call (8.1) an Oseledets splitting for $\mathcal{Q}$ of dimension $d = \sum_{i=1}^{k_Q} d_{i,Q}$. The numbers $\{\lambda_{i,Q}\}_{i=1}^{k_Q}$ are called the exceptional Lyapunov exponents of $\mathcal{Q}$, and we say that $d_{i,Q}$ is the multiplicity of $\lambda_{i,Q}$. The spaces $E_{i,Q}(\omega)$ and $F_Q(\omega)$ are called Oseledets subspaces of $\mathcal{Q}$. For convenience we set $\lambda_{k_Q+1,Q} = \mu_Q$. Finally, the Lyapunov exponents of $\mathcal{Q}$ counted with multiplicities is the sequence

$$\lambda_{1,Q}, \ldots, \lambda_{1,Q}, \lambda_{2,Q} \ldots, \lambda_{2,Q}, \lambda_{3,Q}, \ldots, \lambda_{k_Q,Q}, \tag{8.4}$$

where each $\lambda_{i,Q}$ occurs $d_{i,Q}$ times. For $\ell \in \{1, \ldots, d\}$ we set $\gamma_{\ell,Q}$ to be the ℓth element of (8.4) (from left to right).

Remark 8.1.4. It follows from Lemma 8.4.2 that for every $i \in \{1, \ldots, k_Q\}$ the map $\omega \mapsto E_{i,Q}(\omega)$ is $(\mathcal{F}, \mathcal{B}_{\mathcal{G}(X)})$-measurable. It is not clear if the same is true for the slow space $F_Q(\omega)$.

Remark 8.1.5. The existence of an Oseledets splittings may be guaranteed by a multiplicative ergodic theorem. There are now a plethora of such theorems, starting with [86] and being generalised in a number of directions, but for our desired application we are only concerned with semi-invertible multiplicative ergodic theorems on Banach spaces. The semi-invertibility of such a result refers to the requirement that σ is invertible, but that no invertibility assumption is placed on the generator Q. In an infinite-dimensional Banach space there is also a requirement that the random linear system being considered is quasi-compact, which, roughly speaking, implies that the iterates of the cocycle become increasingly close to a compact cocycle. We refer the reader to [47, 48, 15, 43] for precise statements of various semi-invertible multiplicative ergodic theorems. Finally, we note that a semi-invertible multiplicative ergodic theorem for compact cocycles on a continuous field of Banach spaces was recently developed [105], in which case the Banach space is allowed to vary fiber-wise. This setting is quite similar that of Chapter 7, and suggests the possibility of generalising the results of this chapter to cocycles on Banach fields.

To a separable strongly measurable random linear system $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, Q)$ we may associate a canonical bounded linear endomorphism of $\mathbb{X} = \bigsqcup_{\omega\in\Omega}\{\omega\} \times X$, which we also denote by Q, that is defined by

$$Q(\omega, f) = (\sigma(\omega), Q_\omega f).$$

To apply Theorem 7.1.7 we require a hyperbolic splitting for Q when considered as an element of $\mathrm{End}(\mathbb{X}, \sigma)$. The following definition makes precise this requirement in the context of Oseledets splittings.

Definition 8.1.6. *Suppose that $\mathcal{Q} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, Q)$ is a separable strongly measurable random linear system with an Oseledets splitting of dimension d as in Definition 8.1.3. For each $i \in \{1, \dots, k_Q\}$ let $U_{i,Q}(\omega) = \bigoplus_{j\le i} E_{i,Q}(\omega)$ and $V_{i,Q}(\omega) = \left(\bigoplus_{j>i} E_{j,Q}(\omega)\right) \oplus F_Q(\omega)$. We say that $\mathcal{Q}$ has a hyperbolic Oseledets splitting up to the dimension d if there exists a σ-invariant set $\Omega' \subseteq \Omega$ of full $\mathbb{P}$-measure such that for each $i \in \{1, \dots, k_Q\}$ the families of subspaces $\{U_{i,Q}(\omega)\}_{\omega\in\Omega'}$ and $\{V_{i,Q}(\omega)\}_{\omega\in\Omega'}$ form the equivariant fast and slow spaces, respectively, for a hyperbolic splitting of the restriction of Q to $\mathbb{X}' = \bigsqcup_{\omega\in\Omega'}\{\omega\} \times X$ when Q is considered as an element of* $\mathrm{End}(\mathbb{X}, \sigma)$.

Remark 8.1.7. Unpacking the various requirements in Definition 8.1.6, we observe that the Oseledets splitting of $\mathcal{Q}$ being hyperbolic is equivalent to the existence of a σ-invariant set $\Omega' \subseteq \Omega$ of full $\mathbb{P}$-measure, constants $\Theta, C > 0$ and $\eta < 2^{-1}\min_{1\le i\le k_Q}\{\lambda_{i,Q} - \lambda_{i+1,Q}\}$ such that for every $i \in \{1, \dots, k_Q\}$, $\omega \in \Omega'$ and $n \in \mathbb{Z}^+$ we have

$$\max\left\{\left\|\Pi_{U_{i,Q}(\omega)||V_{i,Q}(\omega)}\right\|, \left\|\Pi_{V_{i,Q}(\omega)||U_{i,Q}(\omega)}\right\|\right\} < \Theta, \tag{8.5}$$

$$\left\|Q_\omega^n|_{V_{i,Q}(\omega)}\right\| \le Ce^{n(\lambda_{i+1,Q}+\eta)}, \tag{8.6}$$

and

$$\left\| \left(Q_\omega^n|_{U_{i,Q}(\omega)} \right)^{-1} \right\| \leq C^{-1} e^{-n(\lambda_{i,Q}-\eta)}. \tag{8.7}$$

Before stating our main result for this chapter we require some notation. Suppose that $\mathcal{Q} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, Q)$ is a separable strongly measurable random linear system with Oseledets splitting of dimension d. Rather than indexing the projections onto Oseledets spaces with the index of their Lyapunov exponents, it will be more convenient to state our perturbation results by indexing projections by collections of Lyapunov exponents. If $I \subseteq \mathbb{R}$ is a open interval such that $I \subseteq (\mu_Q, \infty)$ and $\partial I \cap \{\lambda_{i,Q} : 1 \leq i \leq k_Q\} = \emptyset$ then we say that I separates the Lyapunov spectrum of $\mathcal{Q}$. When I separates the Lyapunov spectrum of $\mathcal{Q}$ we may define $\Pi_{I,Q}(\omega) \in L(X)$ to be the projection onto

$$\bigoplus_{i:\lambda_{i,Q} \in I} E_{i,Q}$$

according to the decomposition (8.1). Finally, if $(X, \|\cdot\|, |\cdot|)$ is a Saks space and $\epsilon > 0$ then, as in Chapter 7, we set

$$\mathcal{O}_\epsilon(Q) = \left\{ P : \Omega \mapsto L(X) \,\middle|\, P \text{ is strongly measurable with } \operatorname*{ess\,sup}_{\omega \in \Omega} \|\!|Q_\omega - P_\omega|\!\| < \epsilon \right\}.$$

Our main result for this chapter is the following.

Theorem 8.1.8. *Suppose that $(X, \|\cdot\|, |\cdot|)$ is a Saks space, with $(X, \|\cdot\|)$ being a Banach space, that $\mathcal{Q} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, Q)$ is a separable strongly measurable random linear system with ergodic invertible base and a hyperbolic Oseledets splitting of dimension $d \in \mathbb{Z}^+$, and that $Q \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathrm{End}_S(\mathbb{X}, \sigma)$ for some $C_1, C_2, R > 0$ and $r \in [0, e^{\mu_Q})$. There exists $\epsilon_0 > 0$ such that if $\mathcal{P} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, P)$ is a separable strongly measurable random linear system with $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_0}(Q)$ then $\mathcal{P}$ also has an Oseledets splitting of dimension d. In addition, there exists $c_0 < 2^{-1} \min_{1 \leq i \leq k_Q} \{\lambda_{i,Q} - \lambda_{i+1,Q}\}$ such that each $I_i = (\lambda_{i,Q} - c_0, \max\{\lambda_{i,Q}, \ln(\delta_{1i} R)\} + c_0)$, $i \in \{1, \dots, k_Q\}$, separates the Lyapunov spectrum of $\mathcal{P}$, and the corresponding projections satisfy*

$$\forall i \in \{1, \dots, k_Q\}, \text{ a.e. } \omega \in \Omega \quad \operatorname{rank}(\Pi_{I_i,P}(\omega)) = d_{i,Q}, \tag{8.8}$$

and

$$\sup \left\{ \operatorname*{ess\,sup}_{\omega \in \Omega} \|\Pi_{I_i,P}(\omega)\| : P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_0}(Q), 1 \leq i \leq k_Q \right\} < \infty. \tag{8.9}$$

Moreover, for every $\nu > 0$ there exists $\epsilon_\nu \in (0, \epsilon_0)$ so that if $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\nu}(Q)$ then

$$\sup_{1 \le i \le d} |\gamma_{i,Q} - \gamma_{i,P}| \le \nu, \tag{8.10}$$

$$\sup_{1 \le i \le k_Q} \operatorname*{ess\,sup}_{\omega \in \Omega} |||\Pi_{I_i,Q}(\omega) - \Pi_{I_i,P}(\omega)||| \le \nu, \tag{8.11}$$

and

$$\operatorname*{ess\,sup}_{\omega \in \Omega} d_H(F_{k_Q,Q}(\omega), F_{k_P,P}(\omega)) \le \nu. \tag{8.12}$$

Remark 8.1.9. Contrary to what one might expect given Theorem 7.1.7, in Theorem 8.1.8 we cannot conclude that $\mathcal{P}$ possesses a hyperbolic Oseledets splitting. The obstruction is the following: if $\mathcal{Q}$ has a Lyapunov exponent $\lambda_{j,Q}$ with $d_{j,Q} > 1$ then after perturbing the cocycle one expects the exponent to immediately split into $d_{j,Q}$ distinct exponents. None of the hypotheses of Theorem 8.1.8 may be used to control the angle between the Oseledets spaces for these new Lyapunov exponents, which prevents us from concluding these splittings are hyperbolic. All this is not to say, however, that the Oseledets splitting for $\mathcal{P}$ exhibits no hyperbolicity at all. For each $i \in \{1, \dots, k_Q\}$ let

$$J_i = (\lambda_{i,Q} - c_0, \ln(R) + c_0)$$

and set $U_{i,P}(\omega)$ and $V_{i,P}(\omega)$ to be the image and kernel, respectively, of $\Pi_{J_i,P}(\omega)$. In the course of the proof of Theorem 8.1.8 it will be shown that for each $i \in \{1, \dots, k_Q\}$ there exists a σ-invariant set $\Omega' \subseteq \Omega$ of full $\mathbb{P}$-measure such that $\{U_{i,P}(\omega)\}_{\omega \in \Omega'}$ and $\{V_{i,P}(\omega)\}_{\omega \in \Omega'}$ are the equivariant fast and slow spaces, respectively, for a hyperbolic splitting of P over Ω'. This implies, in particular, that if every Lyapunov exponent of $\mathcal{Q}$ has multiplicity 1 then the Oseledets splitting for $\mathcal{P}$ is hyperbolic.

Remark 8.1.10. Note that $\Pi_{I_i,Q}(\omega)$ is simply the projection onto $E_{i,Q}(\omega)$ according to the Oseledets splitting of $\mathcal{Q}$.

Remark 8.1.11. By possibly rescaling $|\cdot|$, without loss of generality we may assume that the Saks space $(X, \|\cdot\|, |\cdot|)$ in Theorem 8.1.8 is normal.

Remark 8.1.12. Theorem 7.1.7 may be considered a generalisation of Keller-Liverani perturbation theory [71]. Indeed, in the case where Ω is a singleton we obtain a version of the results of [71]. We note that one condition from [71] has been substantially weakened, namely condition (2) from [71] is generalised to the requirement that Q is a Saks space equicontinuous endomorphism (see Proposition 6.2.18, (7.4) and Remark 7.1.3), which we only require for the unperturbed endomorphism Q, and not for any perturbation. In addition, the convergence of the slow spaces in the Grassmannian as in (8.12) is new. We did not pursue Hölder bounds on the $|||\cdot|||$-error between the perturbed and unperturbed projections as in [71]. It is natural to conjecture that the conclusion of Theorem 8.1.8 (and Theorem 7.1.7) could be strengthened to obtain Hölder error bounds in (8.10), (8.11) and (8.12) under the additional assumption that $\operatorname{ess\,sup}_{\omega\in\Omega} |Q_\omega| < \infty$.

The proof of Theorem 8.1.8 is broken into a number of steps. In Section 8.2 we produce an Oseledets splitting of dimension d for $\mathcal{P}$, and then we relate this Oseledets splitting to various hyperbolic splittings produced by Theorem 7.1.7. Once this is done, in Section 8.3 we characterise and then prove the stability of the Lyapunov exponents.

However, before embarking on the proof of Theorem 8.1.8, we will discuss its relation to the [16, Theorem 1.10], to which our result bares a strong resemblance. The primary differences are the following:

1. In [16] it is required that convergence in (8.2) and (8.3) is uniform in ω, while we only require the weaker bounds (8.6) and (8.7).
2. The perturbations in [16] are required to be asymptotically small: (i) each iterate of the perturbed cocycle must converge uniformly in the strong operator topology to the corresponding iterate of the unperturbed cocycle, and (ii) there exists $s \in (\lambda_{k_Q+1,Q}, \lambda_{k_Q,Q})$ and $N \in \mathbb{Z}^+$ such that for every $n > N$ there is $\epsilon(n)$ so that for all $\epsilon \in (0, \epsilon(n))$ and a.e. $\omega \in \Omega$ one has

$$\|Q_\omega^n - P_\omega^n\|_{L(X)} \leq e^{ns}.$$

   We compare (i) to closeness in the Saks space sense in Proposition 8.1.13 and show that our hypotheses are weaker for pre-compact Saks spaces, which is a common setting for Perron-Frobenius operator cocycles. On the other hand, the condition (ii) is not directly comparable to any of our hypotheses, although it is comparable ‘in spirit’ to our requirement that the perturbed cocycle lies in a Lasota-Yorke class: the exponent s plays a similar role to the r term in our Lasota-Yorke inequalities, in that one cannot conclude anything about the stability of any Lyapunov exponents of modulus smaller than s in [16], or $\ln r$ in Theorem 8.1.8.
3. Due to the weaker requirements of our result, our conclusions on the stability of the Oseledets spaces are weaker than that of [16].

4. We require the additional hypotheses that the unperturbed cocycle is a Saks space equicontinuous endomorphism, which presupposes that X admits a Saks space structure. However, (pre-)compact Saks spaces are commonly used to study the statistical properties of dynamical systems via Perron-Frobenius operators, and so these hypotheses are natural for our primary application.

Proposition 8.1.13. *Suppose that $(X, \|\cdot\|, |\cdot|)$ is a (pre-)compact Saks space, and that $\{Q_n\}_{n\in\mathbb{Z}^+} \subseteq L_S(X)$ is an equicontinuous subset of $L_S(X)$ which converges in the strong operator topology to $Q \in L_S(X)$. Then $Q_n \to Q$ in $(L_S(X), \|\cdot\|_{L(X)}, |||\cdot|||)$.*

Proof. That $\{Q_n\}_{n\in\mathbb{Z}^+}$ is bounded in $L(X)$ follows from Proposition 6.2.18. Let $G_\epsilon \subseteq B_{\|\cdot\|}$ be a finite set such that $\inf_{\|f\|=1} \inf_{g\in G} |f - g| \leq \epsilon$. Then

$$|||Q_n - Q||| \leq \sup_{g\in G_\epsilon} \|(Q_n - Q)g\| + \sup_{\|f\|=1} \inf_{g\in G_\epsilon} |(Q_n - Q)(f - g)|$$

Since $\{Q\} \cup \{Q_n\}_{n\in\mathbb{Z}^+}$ is equicontinuous in $L_S(X)$, by Proposition 6.2.18 we have for every $\kappa > 0$ a C_κ such that for every $n \in \mathbb{Z}^+$

$$|||Q_n - Q||| \leq \sup_{g\in G_\epsilon} \|(Q_n - Q)g\| + 2\kappa + C_\kappa \epsilon.$$

Sending $n \to \infty$ yields

$$\limsup_{n\to\infty} |||Q_n - Q||| \leq 2\kappa + C_\kappa \epsilon. \tag{8.13}$$

By first choosing κ to be very small, and then shrinking ϵ appropriately, we may make the right side of (8.13) as small as we like. Hence $\lim_{n\to\infty} |||Q_n - Q||| = 0$. □

8.2 Characterising the perturbed Oseledets splitting

Recall η and Ω' from Remark 8.1.7, and let $\beta_0 > 0$ satisfy

$$\eta + \beta_0 < 2^{-1} \min_{1\leq i\leq k_Q} \lambda_{i,Q} - \lambda_{i+1,Q}.$$

For each $i \in \{1, \ldots, k_Q\}$ we apply Theorem 7.1.7 to Q with respect to the hyperbolic splitting composed of fast spaces $\{U_{i,Q}(\omega)\}_{\omega\in\Omega'}$ and slow spaces $\{V_{i,Q}(\omega)\}_{\omega\in\Omega'}$ to produce $\epsilon_0, C_0, \Theta_0 > 0$ so that if $P \in \mathcal{LY}(C_1, C_2, r, R)$ is strongly measurable and satisfies

$$\sup_{\omega \in \Omega'} |||Q_\omega - P_\omega||| < \epsilon_0 \tag{8.14}$$

then P has a hyperbolic splitting of index $\sum_{i \leq j} d_{j,Q}$ (in the sense of Definition 7.1.1). Moreover, if we denote the fast and slow spaces of these splittings by $\{U_{i,P}(\omega)\}_{\omega \in \Omega'}$ and $\{V_{i,P}(\omega)\}_{\omega \in \Omega'}$, respectively, then for every $n \in \mathbb{Z}^+$, $i \in \{1, \dots, k_Q\}$ and $\omega \in \Omega'$ we have

$$\max\left\{\left\|\Pi_{U_{i,P}(\omega)||V_{i,P}(\omega)}\right\|, \left\|\Pi_{V_{i,P}(\omega)||U_{i,P}(\omega)}\right\|\right\} < \Theta_0, \tag{8.15}$$

$$\left\|P_\omega^n|_{V_{i,P}(\omega)}\right\| \leq C_0 e^{n(\lambda_{i+1,Q}+\eta+\beta_0)}, \tag{8.16}$$

and, for every $v \in U_{i,P}(\omega)$,

$$\|P_\omega^n v\| \geq C_0^{-1} e^{n(\lambda_{i,Q}-\eta-\beta_0)} \|v\|. \tag{8.17}$$

Remark 8.2.1. If, rather than (8.14), we just have that $P \in \mathcal{O}_{\epsilon_0}(Q)$, then we may instead consider the following construction. Let $\Omega_P \in \mathcal{F}$ have full $\mathbb{P}$-measure and satisfy

$$\sup_{\omega \in \Omega_P} |||Q_\omega - P_\omega||| < \epsilon_0.$$

By perhaps replacing Ω_P with $\bigcap_{n \in \mathbb{Z}} \sigma^n(\Omega_P)$ we may assume that Ω_P is σ-invariant. Let $\tilde{P} : \Omega \mapsto L(X)$ be defined by

$$\tilde{P}_\omega = \begin{cases} P_\omega & \text{if } \omega \in \Omega' \cap \Omega_P, \\ Q_\omega & \text{otherwise.} \end{cases}$$

Since $\tilde{P}_\omega = P_\omega$ a.e. and $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete measure space it follows that $\tilde{P}$ is strongly measurable. By construction (8.14) holds with $\tilde{P}$ in place of P, and $\tilde{P} \in \mathcal{LY}(C_1, C_2, r, R)$ since $\Omega' \cap \Omega_P$ is σ-invariant. Thus Theorem 7.1.7 may be applied with $\tilde{P}$, which produces fast spaces $\{U_{i,\tilde{P}(\omega)}\}_{\omega \in \Omega'}$ and slow spaces $\{V_{i,\tilde{P}(\omega)}\}_{\omega \in \Omega'}$ for $\tilde{P}$, which restrict to fast and slow spaces for P when considered on Ω_P. Moreover, we obtain (8.15), (8.16) and (8.17) for P and $\omega \in \Omega_P$ (i.e. for a.e. $\omega \in \Omega$). We will not discuss this technical point any further, and simply carry out of constructions a.e. for P.

For each $i \in \{1, \dots, k_Q\}$ set

$$G_{i,P}(\omega) = \begin{cases} U_{1,P}(\omega) & i = 1, \\ U_{i,P}(\omega) \cap V_{i-1,P}(\omega) & 1 < i \leq k_Q. \end{cases}$$

and

$$H_{i,P}(\omega) = \begin{cases} V_{1,P}(\omega) & i = 1, \\ V_{i,P}(\omega) \oplus U_{i-1,P}(\omega) & 1 < i \leq k_Q. \end{cases}$$

Note that $\dim(G_{i,P}(\omega)) = \operatorname{codim}(H_{i,P}(\omega)) = d_{i,Q}$ and $X = G_{i,P}(\omega) \oplus H_{i,P}(\omega)$ for a.e. ω and each $i \in \{1, \dots, k_Q\}$. Moreover, for a.e. ω we have

$$X = \left(\bigoplus_{1 \leq i \leq k_Q} G_{i,P}(\omega) \right) \oplus V_{k_Q,P}(\omega). \tag{8.18}$$

It is clear that $G_{i,Q}(\omega) = E_{i,Q}(\omega)$, and so we will consider $G_{i,P}(\omega)$ to be perturbation of $E_{i,Q}(\omega)$. Our first main result for this section makes this idea rigorous, and is a straightforward application of Theorem 7.1.7. Later we will see that, in general, $G_{i,P}(\omega)$ is not an Oseledets space for $\mathcal{P}$, but rather a direct sum of finitely many Oseledets spaces of $\mathcal{P}$.

Proposition 8.2.2. *With ϵ_0 as at the beginning of this section, we have*

$$\sup \left\{ \operatorname*{ess\,sup}_{\omega \in \Omega} \left\| \Pi_{G_{i,P}(\omega) || H_{i,P}(\omega)} \right\| : \substack{P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_0}(Q) \\ 1 \leq i \leq k_Q} \right\} \leq \Theta_0^2 < \infty. \tag{8.19}$$

Moreover, for every $\nu > 0$ there exists $\epsilon_\nu \in (0, \epsilon_0)$ so that if $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\nu}(Q)$ and $i \in \{1, \dots, k_Q\}$ then

$$\operatorname*{ess\,sup}_{\omega \in \Omega} \left|\left|\left| \Pi_{E_{i,Q}(\omega) || H_{i,Q}(\omega)} - \Pi_{G_{i,P}(\omega) || H_{i,P}(\omega)} \right|\right|\right| \leq \nu, \tag{8.20}$$

and

$$\operatorname*{ess\,sup}_{\omega \in \Omega} d_H(V_{k_Q,Q}(\omega), V_{k_Q,P}(\omega)) \leq \nu. \tag{8.21}$$

Proof. By (8.15) we have

$$\sup \left\{ \operatorname*{ess\,sup}_{\omega \in \Omega} \left\| \Pi_{U_{i,P}(\omega) || V_{i,P}(\omega)} \right\| : P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_0}(Q), 1 \leq i \leq k_Q \right\} \leq \Theta_0.$$

Since for $1 < i \le k_Q$ we have

$$\Pi_{G_{i,P}(\omega)||H_{i,P}(\omega)} = \Pi_{U_{i,P}(\omega)||V_{i,P}(\omega)} \Pi_{V_{i-1,P}(\omega)||U_{i-1,P}(\omega)}, \tag{8.22}$$

we may therefore bound the left side of (8.19) by $\max\{\Theta_0, \Theta_0^2\} = \Theta_0^2$, since $\Theta_0 \ge 1$ necessarily.

We will now prove (8.20), for which we note that it suffices consider each $i \in \{1, \dots, k_Q\}$ separately. By Theorem 7.1.7 there exists $\epsilon_\nu > 0$ so that if $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\nu}(Q)$ then

$$\operatorname*{ess\,sup}_{\omega \in \Omega} \left|\left|\left| \Pi_{U_{1,Q}(\omega)||V_{1,Q}(\omega)} - \Pi_{U_{1,P}(\omega)||V_{1,P}(\omega)} \right|\right|\right| \le \nu,$$

which yields (8.20) for $i = 1$. Now assume that $1 < i \le k_Q$. If $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_0}(Q)$ and $1 < i \le k_Q$ then by (8.22) we have for a.e. ω that

$$\begin{aligned} &\left|\left|\left| \Pi_{E_{i,Q}(\omega)||H_{i,Q}(\omega)} - \Pi_{G_{i,P}(\omega)||H_{i,P}(\omega)} \right|\right|\right| \\ &\qquad \le \left|\left|\left| \Pi_{U_{i,Q}(\omega)||V_{i,Q}(\omega)} \left(\Pi_{V_{i-1,Q}(\omega)||U_{i-1,Q}(\omega)} - \Pi_{V_{i-1,P}(\omega)||U_{i-1,P}(\omega)} \right) \right|\right|\right| \\ &\qquad\quad + \left|\left|\left| \left(\Pi_{U_{i,Q}(\omega)||V_{i,Q}(\omega)} - \Pi_{U_{i,P}(\omega)||V_{i,P}(\omega)} \right) \Pi_{V_{i-1,P}(\omega)||U_{i-1,P}(\omega)} \right|\right|\right|. \end{aligned} \tag{8.23}$$

Lemma 7.2.3 implies that for every $\kappa > 0$ there exists C_κ such that

$$\begin{aligned} &\left|\left|\left| \Pi_{U_{i,Q}(\omega)||V_{i,Q}(\omega)} \left(\Pi_{V_{i-1,Q}(\omega)||U_{i-1,Q}(\omega)} - \Pi_{V_{i-1,P}(\omega)||U_{i-1,P}(\omega)} \right) \right|\right|\right| \\ &\qquad \le \kappa \left\| \Pi_{V_{i-1,Q}(\omega)||U_{i-1,Q}(\omega)} - \Pi_{V_{i-1,P}(\omega)||U_{i-1,P}(\omega)} \right\| \\ &\qquad\quad + C_\kappa \left|\left|\left| \Pi_{V_{i-1,Q}(\omega)||U_{i-1,Q}(\omega)} - \Pi_{V_{i-1,P}(\omega)||U_{i-1,P}(\omega)} \right|\right|\right| \\ &\qquad \le 2\kappa\Theta_0 + C_\kappa \left|\left|\left| \Pi_{V_{i-1,Q}(\omega)||U_{i-1,Q}(\omega)} - \Pi_{V_{i-1,P}(\omega)||U_{i-1,P}(\omega)} \right|\right|\right|. \end{aligned}$$

Thus from (8.23) we obtain

$$\begin{aligned} &\left|\left|\left| \Pi_{E_{i,Q}(\omega)||H_{i,Q}(\omega)} - \Pi_{G_{i,P}(\omega)||H_{i,P}(\omega)} \right|\right|\right| \\ &\qquad \le 2\kappa\Theta_0 + C_\kappa \left|\left|\left| \Pi_{V_{i-1,Q}(\omega)||U_{i-1,Q}(\omega)} - \Pi_{V_{i-1,P}(\omega)||U_{i-1,P}(\omega)} \right|\right|\right| \\ &\qquad\quad + \Theta_0 \left|\left|\left| \Pi_{U_{i,Q}(\omega)||V_{i,Q}(\omega)} - \Pi_{U_{i,P}(\omega)||V_{i,P}(\omega)} \right|\right|\right|. \end{aligned} \tag{8.24}$$

Fix $\kappa = \frac{\nu}{4\Theta_0}$. By Theorem 7.1.7, there is $\epsilon_\nu \in (0, \epsilon_0)$ so that if $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\nu}(Q)$ and $i \in \{1, \dots, k_Q\}$ then

$$\operatorname*{ess\,sup}_{\omega \in \Omega} \left|\left|\left| \Pi_{U_{i,Q}(\omega)||V_{i,Q}(\omega)} - \Pi_{U_{i,P}(\omega)||V_{i,P}(\omega)} \right|\right|\right| \le \frac{\nu}{2(C_\kappa + \Theta_0)}. \tag{8.25}$$

Thus by applying (8.25) to (8.24) we obtain (8.20). Finally, we obtain (8.21) due to our application of Theorem 7.1.7 with respect to the hyperbolic splitting of X into equivariant fast spaces $\{U_{k_Q,Q}(\omega)\}_{\omega\in\Omega'}$ and slow spaces $\{V_{k_Q,Q}(\omega)\}_{\omega\in\Omega'}$. □

The second main result of this section confirms that the perturbed cocycle $\mathcal{P}$ has an Oseledets splitting, and that this Oseledets splitting refines the splitting in (8.18).

Proposition 8.2.3. *With ϵ_0 as in Proposition 8.2.2, if $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_0}(Q)$ then $\mathcal{P}$ has an Oseledets splitting of dimension d and if for each $i \in \{1, \dots, k_Q\}$ we set*

$$S(i) = \left\{ j : \sum_{1\le \ell < i} d_{\ell,Q} < \sum_{1\le \ell\le j} d_{\ell,P} \le \sum_{1\le\ell\le i} d_{\ell,Q} \right\}$$

then for a.e. ω we have

$$G_{i,P}(\omega) = \bigoplus_{j\in S(i)} E_{j,P}(\omega), \tag{8.26}$$

and $F_P(\omega) = V_{k_Q,P}(\omega)$.

The idea behind the proof of Proposition 8.2.3 is rather simple: since each family $\{G_{i,P}(\omega)\}_{\omega\in\Omega}$ consists of $d_{i,Q}$-dimensional subspaces and is invariant under the action of P we are essentially in the setting of the classical multiplicative ergodic theorem of Oseledets [86]. Unfortunately, actualising this idea requires the strong measurability of several constructions, the proofs of which are rather tedious. As such, many of the purely technical proofs have been deferred to Section 8.4.

Lemma 8.2.4. *For every $i \in \{1, \dots, k_Q\}$ the map $\omega \mapsto \Pi_{U_{i,P}(\omega)||V_{i,P}(\omega)}$ is strongly measurable.*

Proof. From the construction of $\{U_{i,P}(\omega)\}_{\omega\in\Omega}$ in Proposition 7.3.1 and by Proposition 6.1.2 there is $n_0 \in \mathbb{Z}^+$ such that almost uniformly we have

$$\Pi_{U_{i,P}(\omega)||V_{i,Q}(\omega)} = \lim_{m\to\infty} \left(\mathrm{Id} + (P^{mn_0}_{\sigma^{-mn_0}(\omega)})^*(0)\right) \Pi_{U_{i,Q}(\omega)||V_{i,Q}(\omega)},$$

where the graph transform $(P^{mn_0}_{\sigma^{-mn_0}(\omega)})^*$ maps $L(U_{i,Q}(\sigma^{-mn_0}(\omega)), V_{i,Q}(\sigma^{-mn_0}(\omega)))$ to $L(U_{i,Q}(\omega), V_{i,Q}(\omega))$. By [47, Lemma A.5] the map $\omega \mapsto P^{mn_0}_{\sigma^{-mn_0}(\omega)}$ is strongly measurable for each m. Hence, as $\omega \mapsto \Pi_{U_{i,Q}(\sigma^{-mn_0}(\omega))||V_{i,Q}(\sigma^{-mn_0}(\omega))}$ and $\omega \mapsto \Pi_{U_{i,Q}(\omega)||V_{i,Q}(\omega)}$

are strongly measurable, by Proposition 8.4.1 the map

$$\omega \mapsto \left(\operatorname{Id} + (P^{mn_0}_{\sigma^{-mn_0}(\omega)})^*(0)\right) \Pi_{U_{i,Q}(\omega)||V_{i,Q}(\omega)}$$

is strongly measurable for every $m \in \mathbb{Z}^+$. By Proposition 7.3.2 we have

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \left\|\Pi_{U_{i,P}(\omega)||V_{i,Q}(\omega)}\right\| < \infty,$$

and so Lemma 8.4.7 implies that $\omega \mapsto \Pi_{U_{i,P}(\omega)||V_{i,Q}(\omega)}$ is strongly measurable.

From the construction of $\{V_{i,P}(\omega)\}_{\omega\in\Omega}$ in Proposition 7.4.1 and by Proposition 6.1.2 there exists $n_1 \in \mathbb{Z}^+$ such that almost uniformly we have

$$\Pi_{U_{i,P}(\omega)||V_{i,P}(\omega)} = \lim_{m\to\infty} \left(\Pi_{U_{i,P}(\omega)||V_{i,Q}(\omega)} - (P^{mn_1}_{\omega})_*(0)\Pi_{V_{i,Q}(\omega)||U_{i,P}(\omega)}\right),$$

where the backwards graph transform $(P^{mn_1}_{\omega})_*$ maps $L(V_{i,Q}(\sigma^{mn_1}(\omega)), U_{i,P}(\sigma^{mn_1}(\omega)))$ to $L(V_{i,Q}(\omega), U_{i,P}(\omega))$. As $\omega \mapsto \Pi_{U_{i,P}(\sigma^{mn_1}(\omega))||V_{i,Q}(\sigma^{mn_1}(\omega))}$ and $\omega \mapsto \Pi_{U_{i,P}(\omega)||V_{i,Q}(\omega)}$ are strongly measurable, by Proposition 8.4.1 the map

$$\omega \mapsto \Pi_{U_{i,P}(\omega)||V_{i,Q}(\omega)} - (P^{mn_1}_{\omega})_*(0)\Pi_{V_{i,Q}(\omega)||U_{i,P}(\omega)}$$

is strongly measurable for all m. By (8.15) we have $\operatorname{ess\,sup}_{\omega\in\Omega} \left\|\Pi_{U_{i,P}(\omega)||V_{i,P}(\omega)}\right\| < \infty$, and so $\omega \mapsto \Pi_{U_{i,P}(\omega)||V_{i,P}(\omega)}$ is strongly measurable by Lemma 8.4.7. □

Lemma 8.2.5. *For each $i \in \{1, \ldots, k_Q\}$ the map $\omega \mapsto \Pi_{G_{i,P}(\omega)||H_{i,P}(\omega)}$ is strongly measurable.*

Proof. The cases where $i = 1$ is covered by Lemma 8.2.4. For $1 < i \leq k_Q$ we have

$$\Pi_{G_{i,P}(\omega)||H_{i,P}(\omega)} = \Pi_{U_{i,P}(\omega)||V_{i,P}(\omega)}\Pi_{V_{i-1,P}(\omega)||U_{i-1,P}(\omega)}, \tag{8.27}$$

and so $\omega \mapsto \Pi_{G_{i,P}(\omega)||H_{i,P}(\omega)}$ is strongly measurable by Lemma 8.2.4 and [47, Lemma A.5]. □

A key tool in the proof of Proposition 8.2.3 is the following result on the existence of measurable change of basis maps;a similar construction is carried out in [76, Chapter 7]. We defer the proof to Section 8.4.

Lemma 8.2.6. *If $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space, X is a separable Banach space, $d \in \mathbb{Z}^+$ and $\omega \mapsto \Pi_\omega$ is a strongly measurable map such that each Π_ω is rank-d*

projection and $\operatorname{ess\,sup}_{\omega\in\Omega}\|\Pi_\omega\|<\infty$ *then for every* $\epsilon>0$ *there exists a strongly measurable map* $A:\Omega\to L(X,\mathbb{C}^d)$ *such that* $A_\omega|_{\Pi_\omega(X)}:\Pi_\omega(X)\to\mathbb{C}^d$ *is a bijection,* $\ker(A_\omega)=\ker(\Pi_\omega)$ *for a.e.* ω*, and the map* $\omega\mapsto\left(A_\omega|_{\Pi_\omega(X)}\right)^{-1}$ *is strongly measurable. Moreover,*

$$\operatorname*{ess\,sup}_{\omega\in\Omega}\left\|A_\omega|_{\Pi_\omega(X)}\right\|\le\left(\frac{2}{1-\epsilon}\right)^{d-1},\quad \textit{and}\quad \operatorname*{ess\,sup}_{\omega\in\Omega}\left\|\left(A_\omega|_{\Pi_\omega(X)}\right)^{-1}\right\|\le\sqrt{d}. \tag{8.28}$$

The proof of Proposition 8.2.3. For each $i\in\{1,\dots,k_Q\}$ let $A_{i,\omega}$ denote the map produced by applying Lemma 8.2.6 to $\omega\mapsto\Pi_{G_{i,P}(\omega)||H_{i,P}(\omega)}$ with ϵ very small, and set

$$P_{i,\omega}=A_{i,\sigma(\omega)}P_\omega\left(A_{i,\omega}|_{G_{i,P}(\omega)}\right)^{-1}.$$

Then $\mathcal{P}_i=(\Omega,\mathcal{F},\mathbb{P},\sigma,\mathbb{C}^{d_{i,Q}},\omega\mapsto P_{i,\omega})$ is a separable strongly measurable random linear system with an ergodic invertible base. Moreover, by (8.17) and the estimates in Lemma 8.2.6 for every $n\in\mathbb{Z}^+$ and $i\in\{1,\dots,k_Q\}$ we have

$$\begin{aligned}\left\|\left(P_{i,\omega}^n\right)^{-1}\right\|&\le\|A_{i,\omega}\|\left\|\left(P_\omega^n|_{G_{i,P}(\omega)}\right)^{-1}\right\|\left\|\left(A_{i,\sigma^n(\omega)}|_{G_{i,P}(\sigma^n(\omega))}\right)^{-1}\right\|\\&\le C_0\sqrt{d}\left(\frac{2}{1-\epsilon}\right)^{d-1}e^{-n(\lambda_{i,Q}-\eta-\beta_0)}.\end{aligned}\tag{8.29}$$

On the other hand, by (8.16) and the estimates in Lemma 8.2.6 for every $n\in\mathbb{Z}^+$ and $1<i\le k_Q$ we have

$$\begin{aligned}\left\|P_{i,\omega}^n\right\|&\le\left\|A_{i,\sigma^n(\omega)}\right\|\left\|P_\omega^n|_{G_{i,P}(\omega)}\right\|\left\|\left(A_{i,\omega}|_{G_{i,P}(\omega)}\right)^{-1}\right\|\\&\le C_0\sqrt{d}\left(\frac{2}{1-\epsilon}\right)^{d-1}e^{n(\lambda_{i,Q}+\eta+\beta_0)},\end{aligned}\tag{8.30}$$

while for $i=1$ we have

$$\left\|P_{i,\omega}^n\right\|\le C_0C_3\sqrt{d}\left(\frac{2}{1-\epsilon}\right)^{d-1}R^n.\tag{8.31}$$

Thus $\ln^+\left\|P_{i,\omega}^{\pm1}\right\|\in L^1(\Omega,\mathcal{F},\mathbb{P})$, and so by Oseledets' Multiplicative Ergodic Theorem [86], each $\mathcal{P}_i$ has an Oseledets splitting of dimension $d_{i,Q}$ given by

$$\mathbb{C}^d=\bigoplus_{j=1}^{k_{P_i}}E_{j,P_i}(\omega).\tag{8.32}$$

By pulling back these Oseledets spaces to X we obtain for each $i \in \{1, \dots, k_Q\}$ and a.e. ω the splitting

$$G_{i,P}(\omega) = \bigoplus_{j=1}^{k_{P_i}} \left(A_{i,\omega}|_{G_{i,P}(\omega)} \right)^{-1} E_{j,P_i}(\omega),$$

and so in view of (8.18) we have

$$X = \left(\bigoplus_{i=1}^{k_Q} \left(\bigoplus_{j=1}^{k_{P_i}} \left(A_{i,\omega}|_{G_{i,P}(\omega)} \right)^{-1} E_{j,P_i}(\omega) \right) \right) \oplus V_{k_Q,P}(\omega). \tag{8.33}$$

Let $k_P = \sum_{i=1}^{k_Q} k_{P_i}$. For $1 \leq \ell \leq k_P$ set $h(\ell) = \max\{\sum_{i=1}^{t} k_{P_i} : \sum_{i=1}^{t} k_{P_i} \leq \ell\}$, $g(\ell) = \ell - h(\ell)$ and

$$E_{\ell,P}(\omega) = \left(A_{h(\ell),\omega}|_{G_{h(\ell),P}(\omega)} \right)^{-1} E_{g(\ell),P_{h(\ell)}}(\omega).$$

If we set $F_P(\omega) = V_{k_Q,P}(\omega)$ then we may rewrite (8.33) as

$$X = \left(\bigoplus_{\ell=1}^{k_P} E_{\ell,P}(\omega) \right) \oplus F_P(\omega). \tag{8.34}$$

We claim that (8.34) is an Oseledets splitting for $\mathcal{P}$ of dimension d.

The strong measurability of the Oseledets projections. The projection onto $F_P(\omega)$ according to (8.34) is strongly measurable by Lemma 8.2.4. The projection onto each $E_{\ell,P}(\omega)$ according to the decomposition (8.33) is given by

$$\left(A_{h(\ell),\omega}|_{G_{h(\ell),P}(\omega)} \right)^{-1} \Pi_{g(\ell),h(\ell),\omega} A_{h(\ell),\omega} \Pi_{G_{h(\ell),P}(\omega)||H_{h(\ell),P}(\omega)},$$

where $\Pi_{g(\ell),h(\ell),\omega}$ denotes the projection onto $E_{g(\ell),P_{h(\ell)}}(\omega)$ according to the splitting in (8.32). Thus the projection onto $E_{\ell,P}(\omega)$ according to the decomposition (8.34), being the composition of strongly measurable maps, is strongly measurable by [47, Lemma A.5].

The properties of the fast Oseledets spaces. It is easily checked that for any $\ell \in \{1, \dots, k_P\}$ and a.e. ω we have

$$\begin{aligned}
P_\omega^n(E_{\ell,P}(\omega)) &= P_\omega^n\left(\left(A_{h(\ell),\omega}|_{G_{h(\ell),P}(\omega)}\right)^{-1} E_{g(\ell),P_{h(\ell)}}(\omega)\right) \\
&= \left(A_{h(\ell),\sigma^n(\omega)}|_{G_{h(\ell),P}(\sigma^n(\omega))}\right)^{-1} E_{g(\ell),P_{h(\ell)}}(\sigma^n(\omega)) = E_{\ell,P}(\sigma^n(\omega)).
\end{aligned}$$

In addition, due to the bounds (8.28) we have for a.e. ω and every non-zero $v \in E_{\ell,P}(\omega)$ that $A_{h(\ell),\omega} v \in E_{g(\ell),P_{h(\ell)}}(\omega)$ and so

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \ln \|P_\omega^n v\| \\
&\quad = \lim_{n\to\infty} \frac{1}{n} \ln \left\| \left(A_{h(\ell),\sigma^n(\omega)}|_{G_{h(\ell),P}(\sigma^n(\omega))}\right)^{-1} \left(P_{h(\ell),\omega}^n|_{E_{g(\ell),P_{h(\ell)}}(\omega)}\right) A_{h(\ell),\omega} v \right\| \\
&\quad = \lambda_{g(\ell),P_{h(\ell)}}.
\end{aligned}$$

The ordering of the Lyapunov exponents. For every $\ell \in \{1, \dots, k_P\}$ we set $\lambda_{\ell,P} = \lambda_{g(\ell),P_{h(\ell)}}$ so that $\lambda_{\ell,P}$ is the Lyapunov exponent associated to $\{E_{\ell,P}(\omega)\}_{\omega\in\Omega}$. Clearly $\lambda_{\ell_1,P} < \lambda_{\ell_2,P}$ whenever $\ell_1 > \ell_2$ and $h(\ell_1) = h(\ell_2)$, since then $g(\ell_1) > g(\ell_2)$ and so $\lambda_{\ell_1,P} = \lambda_{g(\ell_1),P_{h(\ell_1)}} < \lambda_{g(\ell_2),P_{h(\ell_2)}} = \lambda_{\ell_2,P}$. On the other hand, if $\ell_1 > \ell_2$ and $h(\ell_1) \neq h(\ell_2)$ then since $\eta + \beta_0 < 2^{-1} \min_{1\le i\le k_Q}\{\lambda_{i,Q} - \lambda_{i+1,Q}\}$ we may use (8.29) and (8.30) to conclude that

$$\lambda_{\ell_1,P} \le \lambda_{h(\ell_1),Q} + \eta + \beta_0 < \lambda_{h(\ell_2),Q} - \eta - \beta_0 \le \lambda_{\ell_2,P}.$$

Thus $\lambda_{1,P} > \lambda_{2,P} > \cdots > \lambda_{k,P}$.

The properties of the slow Oseledets spaces. That $P_\omega F_P(\omega) \subseteq F(\sigma(\omega))$ a.e. follows from our application of Theorem 7.1.7 in the construction of $V_{k_Q,P}(\omega)$. By (8.30) we have a.e. that

$$\lim_{n\to\infty} \frac{1}{n} \ln \left\| P_\omega^n|_{V_{k_Q,P}(\omega)} \right\| := \mu_P \le \lambda_{k_Q+1,Q} + \eta + \beta_0. \tag{8.35}$$

By (8.29) we get $\lambda_{k_P,P} > \lambda_{k_Q,Q} - \eta - \beta_0$. Since $\eta + \beta_0 < 2^{-1} \min_{1\le i\le k_Q}\{\lambda_{i,Q} - \lambda_{i+1,Q}\}$ it follows that $\mu_P < \lambda_{k_P,P}$.

The identity (8.26). If we set $s(i) = \sum_{t=1}^{i-1} k_{P_t}$ then

$$G_{i,P}(\omega) = \bigoplus_{j=1}^{k_{P_i}} E_{j+s(i),P}(\omega). \tag{8.36}$$

Then for $j \in \{1, \ldots, k_{P_i}\}$ we have

$$\begin{aligned} \sum_{1 \le \ell \le j+s(i)} d_{\ell,P} &= \left(\sum_{1 \le \ell \le s(i)} d_{\ell,P} \right) + \left(\sum_{1 \le \ell \le j} d_{s(i)+\ell,P} \right) \\ &= \left(\sum_{1 \le t < i} \sum_{1 \le m \le k_{P_t}} d_{s(t)+m,P} \right) + \left(\sum_{1 \le \ell \le j} d_{s(i)+\ell,P} \right). \end{aligned} \tag{8.37}$$

Since $d_{s(t)+m,P} = d_{m,P_t}$ we get

$$\sum_{1 \le t < i} \sum_{1 \le m \le k_{P_t}} d_{s(t)+m,P} = \sum_{1 \le t < i} d_{t,Q}, \quad \text{and} \quad 0 < \sum_{1 \le \ell \le j} d_{s(i)+\ell,P} < d_{i,Q}. \tag{8.38}$$

Thus by combining (8.37) and (8.38) we see that $j + s(i) \in S(i)$. Running our argument in reverse, we observe that if $\ell \in S(i)$ then $h(\ell) = s(i)$ and so $\ell = g(\ell) + s(i)$ with $g(\ell) \in \{1, \ldots, k_{P_i}\}$. Thus we obtain (8.26) by re-indexing the direct sum (8.36). $\square$

The first part of the proof of Theorem 8.1.8. If $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_0}(Q)$ then, as per Proposition 8.2.3, $\mathcal{P}$ has an Oseledets splitting of dimension d. Set $c_0 = \beta_0 + \eta$. From the proof of Proposition 8.2.3, and in particular the estimates (8.29), (8.30) and (8.31), we have for every $i \in \{1, \ldots, k_Q\}$ that

$$\{\lambda_{j,P} : j \in S(i)\} \subseteq I_i = (\lambda_{i,Q} - c_0, \max\{\lambda_{i,Q}, \ln(\delta_{i1} R)\} + c_0). \tag{8.39}$$

Moreover, by (8.35) and the ensuing discussion, we have $I_i \subseteq (\mu_P, \infty)$ for each i. Thus, $\partial I_{i_1} \cap \partial I_{i_2} = \emptyset$ whenever $i_1 \neq i_2$. As for every $j \in \{1, \ldots, k_P\}$ we have $j \in S(i)$ for some $i \in \{1, \ldots, k_Q\}$, it follows that $\partial I_i \cap \{\lambda_{j,P} : 1 \le j \le k_P\} = \emptyset$ for every i. Hence each I_i separates the Lyapunov spectrum of $\mathcal{P}$. In view of Proposition 8.2.3 we therefore have $\Pi_{I_i,P}(\omega) = \Pi_{G_{i,P}(\omega) || H_{i,P}(\omega)}$, and so we obtain (8.8) upon recalling that $\dim(G_{i,P}(\omega)) = d_{i,Q}$ for a.e. $\omega \in \Omega$. We get (8.9) and (8.11) from (8.19) and (8.20), respectively, in Proposition 8.2.2. Finally, as $V_{k_Q,Q}(\omega) = F_Q(\omega)$ and $V_{k_P,P} = F_P(\omega)$ we obtain (8.12) from (8.21) in Proposition 8.2.2. $\square$

8.3 Convergence of the Lyapunov exponents

In this section we focus on the proving the estimate (8.10). A key tool in our proof will be a generalisation of the determinant to operators on Banach spaces, which we will use to access the Lyapunov exponents of $\mathcal{P}$. When $E \in \mathcal{G}(X)$ is finite dimensional we denote by m_E the Haar measure on E, normalised so that $m_E(B_E)$ has the measure of the $\dim(E)$-dimensional Euclidean unit ball. For each $d \in \mathbb{Z}^+$ we define a map $\det : L(X) \times \mathcal{G}_d(X) \to \mathbb{R}$ by

$$\det(A, E) := \det(A|E) = \frac{m_{AE}(A(B_E))}{m_E(B_E)}. \tag{8.40}$$

We refer the reader to [15, Section 2.2] for an overview of the basic properties of the determinant.

Lemma 8.3.1. *Recall ϵ_0 from Proposition 8.2.2. If $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_0}(Q)$, $n \in \mathbb{Z}^+$ and $\ell \in \{1, \dots, k_P\}$ then the maps*

$$\omega \mapsto \ln \det(P_\omega^n | E_{\ell,P}(\omega)),\ \omega \mapsto \ln \left\| P_\omega^n|_{E_{\ell,P}(\omega)} \right\| \text{ and } \omega \mapsto \ln \left\| \left(P_\omega^n|_{E_{\ell,P}(\omega)} \right)^{-1} \right\|^{-1}$$

are $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable and in $L^1(\Omega, \mathcal{F}, \mathbb{P})$.

Proof. Fix n and ℓ. Define $\psi : \Omega \to L(X) \times \mathcal{G}_{d_{\ell,P}}(X)$ by $\psi(\omega) = (P_\omega^n, E_{\ell,P}(\omega))$. The map $\omega \mapsto P_\omega^n$ is strongly measurable by [47, Lemma A.5]. On the other hand, the projection onto $E_{\ell,P}(\omega)$ is strongly measurable since it is an Oseledets space, and so $\omega \mapsto E_{\ell,P}(\omega)$ is $(\mathcal{F}, \mathcal{B}_{\mathcal{G}(X)})$-measurable by Lemma 8.4.2. Thus ψ is $(\mathcal{F}, \mathcal{S} \times \mathcal{B}_{\mathcal{G}(X)})$-measurable. That $\omega \mapsto \ln \det(\psi(\omega))$ is $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable follows from Proposition 8.4.8, while the $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurability of $\omega \mapsto \ln \left\| P_\omega^n|_{E_{\ell,P}(\omega)} \right\|$ is a consequence of [47, Lemma B.16]. To see that $\omega \mapsto \ln \left\| \left(P_\omega^n|_{E_{\ell,P}(\omega)} \right)^{-1} \right\|$ is measurable we note that

$$\left\| \left(P_\omega^n|_{E_{\ell,P}(\omega)} \right)^{-1} \right\| = \left\| \left(\Pi_{E_{\ell,P}(\sigma^n(\omega))} P_\omega^n|_{E_{\ell,P}(\omega)} \right)^{-1} \Pi_{E_{\ell,P}(\sigma^n(\omega))} \Big|_{E_{\ell,P}(\sigma^n(\omega))} \right\|, \tag{8.41}$$

where $\Pi_{E_{\ell,P}(\sigma^n(\omega))}$ denotes the projection onto $E_{\ell,P}(\sigma^n(\omega))$ according to the Oseledets decomposition for $\mathcal{P}$. The map $\omega \mapsto \left(\Pi_{E_{\ell,P}(\sigma^n(\omega))} P_\omega^n|_{E_{\ell,P}(\omega)} \right)^{-1} \Pi_{E_{\ell,P}(\sigma^n(\omega))}$ is $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable by Proposition 8.4.6. Hence the right side of (8.41) is $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable by [47, Lemma B.16], which of course implies that the left side of (8.41) is $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable.

Since

$$\left\| \left(P_\omega^n|_{E_{\ell,P}(\omega)} \right)^{-1} \right\|^{-1} B_{E_{\ell,P}(\sigma^n(\omega))} \subseteq P_\omega^n B_{E_{\ell,P}(\omega)} \subseteq \left\| P_\omega^n|_{E_{\ell,P}(\omega)} \right\| B_{E_{\ell,P}(\sigma^n(\omega))},$$

we have

$$\ln \left\| \left(P_\omega^n|_{E_{\ell,P}(\omega)} \right)^{-1} \right\|^{-1} \leq \frac{1}{d_{\ell,P}} \ln \det(P_\omega^n|E_{\ell,P}(\omega)) \leq \ln \left\| P_\omega^n|_{E_{\ell,P}(\omega)} \right\|. \tag{8.42}$$

By (8.17) and Proposition 8.2.3 we have

$$C_0^{-1} e^{n(\lambda_{k_Q,Q}-\eta-\beta_0)} \leq \left\| \left(P_\omega^n|_{E_{\ell,P}(\omega)} \right)^{-1} \right\|^{-1}. \tag{8.43}$$

On the other hand, since $P \in \mathcal{LY}(C_1, C_2, r, R)$ we have

$$\left\| P_\omega^n|_{E_{\ell,P}(\omega)} \right\| \leq C_3 R^n. \tag{8.44}$$

As $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, by combining (8.42), (8.43) and (8.44) we get that $\omega \mapsto \ln \det(P_\omega^n|E_{\ell,P}(\omega))$, $\omega \mapsto \ln \left\| P_\omega^n|_{E_{\ell,P}(\omega)} \right\|$ and $\omega \mapsto \ln \left\| \left(P_\omega^n|_{E_{\ell,P}(\omega)} \right)^{-1} \right\|^{-1}$ are all contained in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. □

Proposition 8.3.2. *Recall ϵ_0 from Proposition 8.2.2. For all $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_0}(Q)$, $n \in \mathbb{Z}^+$ and $\ell \in \{1, \dots, k_P\}$ we have*

$$\lambda_{\ell,P} = \frac{1}{n d_{\ell,P}} \int_\Omega \ln \det(P_\omega^n|E_{\ell,P}(\omega)) \, d\mathbb{P}, \tag{8.45}$$

$$= \lim_{m\to\infty} \frac{1}{m} \int_\Omega \ln \left\| P_\omega^m|_{E_{\ell,P}(\omega)} \right\| \, d\mathbb{P}, \tag{8.46}$$

$$= \lim_{m\to\infty} \frac{1}{m} \int_\Omega \ln \left\| \left(P_\omega^m|_{E_{\ell,P}(\omega)} \right)^{-1} \right\|^{-1} \, d\mathbb{P}. \tag{8.47}$$

Proof. Recalling (8.42) from the proof of Lemma 8.3.1, we have for every $j \in \mathbb{Z}^+$ that

$$\int \ln \left\| \left(P_\omega^{nj}|_{E_{\ell,P}(\omega)} \right)^{-1} \right\|^{-1} d\mathbb{P} \leq \frac{\int \ln \det(P_\omega^{nj}|E_{\ell,P}(\omega)) \, d\mathbb{P}}{d_{\ell,P}} \leq \int \ln \left\| P_\omega^{nj}|_{E_{\ell,P}(\omega)} \right\| d\mathbb{P}.$$

Hence it suffices to prove that

$$\limsup_{m\to\infty} \frac{1}{m}\int_\Omega \ln \left\| P_\omega^m|_{E_{\ell,P}(\omega)} \right\| \,\mathrm{d}\mathbb{P} \le \lambda_{\ell,P} \le \liminf_{m\to\infty} \frac{1}{m}\int_\Omega \ln \left\| \left(P_\omega^m|_{E_{\ell,P}(\omega)} \right)^{-1} \right\|^{-1} \mathrm{d}\mathbb{P}, \tag{8.48}$$

and that

$$\frac{1}{njd_{\ell,P}} \int \ln\det(P_\omega^{nj}|E_{\ell,P}(\omega))\,\mathrm{d}\mathbb{P} = \frac{1}{nd_{\ell,P}}\int_\Omega \ln\det(P_\omega^n|E_{\ell,P}(\omega))\,\mathrm{d}\mathbb{P}. \tag{8.49}$$

The identity (8.49). Since the determinant is multiplicative [15, Proposition 2.13] we have

$$\int \ln\det(P_\omega^{nj}|E_{\ell,P}(\omega))\,\mathrm{d}\mathbb{P} = \sum_{i=0}^{j-1}\int \ln\det(P_{\sigma^{ni}(\omega)}^n|E_{\ell,P}(\sigma^{ni}(\omega)))\,\mathrm{d}\mathbb{P}. \tag{8.50}$$

Since $\mathbb{P}$ is σ-invariant, for $i \in \{0,\dots,j-1\}$ we have

$$\int \ln\det(P_{\sigma^{ni}(\omega)}^n|E_{\ell,P}(\sigma^{ni}(\omega)))\,\mathrm{d}\mathbb{P} = \int \ln\det(P_\omega^n|E_{\ell,P}(\omega))\,\mathrm{d}\mathbb{P}. \tag{8.51}$$

Combining (8.50) and (8.51) yields (8.49).

The first inequality in (8.48). By Lemma 8.3.1 we have

$$\left\{\omega \mapsto \ln \left\| P_\omega^m|_{E_{\ell,P}(\omega)} \right\| \right\}_{m\in\mathbb{Z}^+} \subseteq L^1(\Omega,\mathcal{F},\mathbb{P}).$$

Since $\left\{\omega \mapsto \ln \left\| P_\omega^m|_{E_{\ell,P}(\omega)} \right\| \right\}_{m\in\mathbb{Z}^+}$ is subadditive with respect to σ and as σ is $\mathbb{P}$-ergodic, by Kingman's subadditive ergodic theorem we have for a.e. ω that

$$\lim_{m\to\infty}\frac{1}{m}\ln \left\| P_\omega^m|_{E_{\ell,P}(\omega)} \right\| = \lim_{m\to\infty}\frac{1}{m}\int_\Omega \ln \left\| P_\omega^m|_{E_{\ell,P}(\omega)} \right\| \,\mathrm{d}\mathbb{P}. \tag{8.52}$$

Fix a normalised Auerbach basis $\{v_i\}_{i=1}^{d_{\ell,P}}$ for $E_{\ell,P}(\omega)$, and for each m let $v_m \in E_{\ell,P}(\omega)$ satisfy $\|w_m\| = 1$ and $\left\| P_\omega^m|_{E_{\ell,P}(\omega)} \right\| = \|P_\omega^m w_m\|$. If we write $w_m = \sum_{i=1}^{d_{\ell,P}} a_{i,m}v_i$ then

$$\left\| P_\omega^m|_{E_{\ell,P}(\omega)} \right\| \le \left(\max_{i\in\{1,\dots,d_{\ell,P}\}} \|P_\omega^m v_i\| \right) \sum_{i=1}^{d_{\ell,P}} |a_{i,m}|.$$

Since $\{v_i\}_{i=1}^{d_{\ell,P}}$ is Auerbach, by [87, Corollary A.7] we have $\sum_{i=1}^{d_{\ell,P}} |a_{i,m}| \leq d_{\ell,P}$, and so

$$\lim_{m\to\infty} \frac{1}{m} \ln \left\| P_\omega^m|_{E_{\ell,P}(\omega)} \right\| \leq \max_{i\in\{1,\dots,d_{\ell,P}\}} \left(\limsup_{m\to\infty} \frac{1}{m} \ln \left(\|P_\omega^m v_i\|\right) \right) = \lambda_{\ell,P},$$

which, in view of (8.52), yields the first inequality in (8.48).

The second inequality in (8.48). The proof is very similar to the one in the previous paragraph. By Lemma 8.3.1 we have

$$\left\{\omega \mapsto \ln \left\| \left(P_\omega^m|_{E_{\ell,P}(\omega)} \right)^{-1} \right\| \right\}_{m\in\mathbb{Z}^+} \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P}).$$

Since $\left\{\omega \mapsto \ln \left\| \left(P_\omega^m|_{E_{\ell,P}(\omega)} \right)^{-1} \right\| \right\}_{m\in\mathbb{Z}^+}$ is subadditive with respect to σ, and as σ is invertible and $\mathbb{P}$-ergodic, by Kingman's subadditive ergodic theorem we have for a.e. ω that

$$\lim_{m\to\infty} \frac{1}{m} \ln \left\| \left(P_{\sigma^{-m}(\omega)}^m \Big|_{E_{\ell,P}(\sigma^{-m}(\omega))} \right)^{-1} \right\| = \lim_{m\to\infty} \frac{1}{m} \int \ln \left\| \left(P_\omega^m|_{E_{\ell,P}(\omega)} \right)^{-1} \right\| d\mathbb{P}. \tag{8.53}$$

Fix a normalised Auerbach basis $\{v_i\}_{i=1}^{d_{\ell,P}}$ for $E_{\ell,P}(\omega)$, and for each m let $w_m \in E_{\ell,P}(\omega)$ satisfy $\|w_m\| = 1$ and

$$\left\| \left(P_{\sigma^{-m}(\omega)}^m \Big|_{E_{\ell,P}(\sigma^{-m}(\omega))} \right)^{-1} \right\| = \left\| \left(P_{\sigma^{-m}(\omega)}^m \Big|_{E_{\ell,P}(\sigma^{-m}(\omega))} \right)^{-1} w_m \right\|.$$

If we write $w_m = \sum_{i=1}^{d_{\ell,P}} a_{i,m} v_i$ then

$$\left\| \left(P_{\sigma^{-m}(\omega)}^m \Big|_{E_{\ell,P}(\sigma^{-m}(\omega))} \right)^{-1} \right\| \leq \left(\max_{i\in\{1,\dots,d_{\ell,P}\}} \left\| \left(P_{\sigma^{-m}(\omega)}^m \Big|_{E_{\ell,P}(\sigma^{-m}(\omega))} \right)^{-1} v_i \right\| \right) \sum_{i=1}^{d_{\ell,P}} |a_{i,m}|.$$

Since $\{v_i\}_{i=1}^{d_{\ell,P}}$ is Auerbach, by [87, Corollary A.7] we have $\sum_{i=1}^{d_{\ell,P}} |a_{i,m}| \leq d_{\ell,P}$, and so

$$\begin{aligned}\lim_{m\to\infty} \frac{1}{m} \ln &\left\| \left(P_{\sigma^{-m}(\omega)}^m \Big|_{E_{\ell,P}(\sigma^{-m}(\omega))} \right)^{-1} \right\| \\ &\leq \max_{i\in\{1,\dots,d_{\ell,P}\}} \limsup_{m\to\infty} \frac{1}{m} \ln \left(\left\| \left(P_{\sigma^{-m}(\omega)}^m \Big|_{E_{\ell,P}(\sigma^{-m}(\omega))} \right)^{-1} v_i \right\| \right) \\ &= -\lambda_{\ell,P},\end{aligned}$$

which, in view of (8.53), yields the second inequality in (8.48). $\square$

Throughout the proof of Theorem 8.1.8 we will use the following corollary of Lemma 7.2.2, which is obtained by applying Lemma 7.2.2 to $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_0}(Q)$ with the hyperbolic splitting consisting of fast spaces $\{U_{k_Q,P}(\omega)\}_{\omega\in\Omega}$ and slow spaces $\{V_{k_Q,P}(\omega)\}_{\omega\in\Omega}$.

Lemma 8.3.3. *Recall ϵ_0 from Proposition 8.2.2. There exists $K > 0$ so that for every $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_0}(Q)$, a.e. ω and every $v \in \bigoplus_{i=1}^{k_P} E_{i,P}(\omega)$ we have $\|v\| \leq K\,|v|$.*

We may now finish the proof of Theorem 8.1.8. For the sake of brevity, throughout the proof we use $\Pi_{E_{i,Q}(\omega)}$ to denote the projection onto $E_{i,Q}(\omega)$ according to the Oseledets splitting of Q, and $\Pi_{G_{i,P}(\omega)}$ to denote the projection onto $G_{i,P}(\omega)$ according to the splitting in (8.18).

The second part of the proof of Theorem 8.1.8. It remains to prove (8.10). Let $\ell \in \{1, \dots, d\}$ and note that it suffices to produce for each $\nu > 0$ a $\epsilon_{\nu,\ell}$ such that if $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{\nu,\ell}}(Q)$ then $|\gamma_{\ell,P} - \gamma_{\ell,Q}| \leq \nu$. By Proposition 8.2.3 we have $\gamma_{\ell,Q} = \lambda_{i,Q}$ for some $i \in \{1, \dots, k_Q\}$ and $\gamma_{\ell,P} = \lambda_{j,P}$ for some $j \in S(i)$. Recalling (8.42) from the proof of Lemma 8.3.1 we have for each $n \in \mathbb{Z}^+$ and a.e. ω that

$$\frac{1}{n}\ln\left\|\left(P_\omega^n|_{E_{j,P}(\omega)}\right)^{-1}\right\|^{-1} \leq \frac{1}{nd_{j,P}}\ln\det(P_\omega^n|E_{j,P}(\omega)) \leq \frac{1}{n}\ln\left\|P_\omega^n|_{E_{j,P}(\omega)}\right\|.$$

Since $|\cdot| \leq \|\cdot\|$ and by Lemma 8.3.3, we get

$$\ln\left(\inf_{\substack{v\in E_{j,P}(\omega)\\ \|v\|\leq 1}} |P_\omega^n v|\right) \leq d_{j,P}^{-1}\det(P_\omega^n|E_{j,P}(\omega)) \leq \ln K + \ln\left(\sup_{\substack{v\in E_{j,P}(\omega)\\ \|v\|\leq 1}} |P_\omega^n v|\right). \tag{8.54}$$

The rest of the proof will run as follows: we will first pursue some technical bounds, which we will then use to obtain lower and upper bounds in terms of $\lambda_{i,Q}$ for the left-most and right-most terms, respectively, in (8.54).

Some technical bounds. For every $v \in E_{j,P}(\omega) \cap B_{\|\cdot\|}$ we have

$$\begin{aligned}\left|\frac{\left|Q^n_\omega \Pi_{E_{i,Q}(\omega)} v\right|}{|P^n_\omega v|} - 1\right| &= |P^n_\omega v|^{-1} \left|\left|Q^n_\omega \Pi_{E_{i,Q}(\omega)} v\right| - |P^n_\omega v|\right| \\ &\le \frac{\left|\left|\left| Q^n_\omega (\mathrm{Id} - \Pi_{E_{i,Q}(\omega)})\big|_{E_{j,P}(\omega)}\right|\right|\right| + |||Q^n_\omega - P^n_\omega|||}{|P^n_\omega v|}.\end{aligned} \tag{8.55}$$

By (8.17), Lemma 8.3.3, and as $\eta + \beta_0 < 2^{-1} \min_{1 \le i \le k_Q}\{\lambda_{i,Q} - \lambda_{i+1,Q}\}$, we have

$$|P^n_\omega v|^{-1} \le K \left\|P^n_\omega v\right\|^{-1} \le K C_0 e^{-n\lambda_{k_Q+1,Q}}. \tag{8.56}$$

By Lemma 7.2.5 we have $Q^n \in \mathrm{End}_S(\mathbb{X}, \sigma)$, and so for every $n \in \mathbb{Z}^+$ and $\kappa > 0$ there exists $C_{\kappa,n}$ such that

$$\begin{aligned}\left|\left|\left| Q^n_\omega (\mathrm{Id} - \Pi_{E_{i,Q}(\omega)})\big|_{E_{j,P}(\omega)}\right|\right|\right| &= \left|\left|\left| Q^n_\omega (\Pi_{G_{i,P}(\omega)} - \Pi_{E_{i,Q}(\omega)})\right|\right|\right| \\ &\le \kappa \left\|\Pi_{G_{i,P}(\omega)} - \Pi_{E_{i,Q}(\omega)}\right\| \\ &\quad + C_{\kappa,n} \left|\left|\left|\Pi_{G_{i,P}(\omega)} - \Pi_{E_{i,Q}(\omega)}\right|\right|\right|,\end{aligned} \tag{8.57}$$

where we also used the fact that $\Pi_{G_{i,P}(\omega)}\big|_{E_{j,P}(\omega)} = \mathrm{Id}\big|_{E_{j,P}(\omega)}$. From the proof of Proposition 8.2.2 we have $\left\|\Pi_{G_{i,P}(\omega)} - \Pi_{E_{i,Q}(\omega)}\right\| \le 2\Theta_0^2$. Thus, by applying (8.56), (8.57) to (8.55) we obtain

$$\left|\frac{\left|Q^n_\omega \Pi_{E_{i,Q}(\omega)} v\right|}{|P^n_\omega v|} - 1\right| \le K C_0 e^{-n\lambda_{k_Q+1,Q}} \left(2\kappa\Theta_0^2 + C_{\kappa,n} \left|\left|\left|\Pi_{G_{i,P}(\omega)} - \Pi_{E_{i,Q}(\omega)}\right|\right|\right| + |||Q^n_\omega - P^n_\omega|||\right).$$

Fix $\gamma > 0$ and take $\kappa = \gamma K^{-1} e^{n\lambda_{k_Q+1,Q}}/(4\Theta_0^2 C_0)$. By Propositions 7.2.6 and 8.2.2, for every $n \in \mathbb{Z}^+$ there exists $\epsilon_{\gamma,n} > 0$ so that if $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{\gamma,n}}(Q)$ then

$$K C_0 e^{-n\lambda_{k_Q+1,Q}} \left(C_{\kappa,n} \left|\left|\left|\Pi_{G_{i,P}(\omega)} - \Pi_{E_{i,Q}(\omega)}\right|\right|\right| + |||Q^n_\omega - P^n_\omega|||\right) \le \frac{\gamma}{2},$$

and so

$$\left|\frac{\left|Q^n_\omega \Pi_{E_{i,Q}(\omega)} v\right|}{|P^n_\omega v|} - 1\right| \le \gamma.$$

Hence

$$\left|\ln\left(|P^n_\omega v|\right) - \ln\left(\left|Q^n_\omega \Pi_{E_{i,Q}(\omega)} v\right|\right)\right| \le \max\left\{\ln(1+\gamma), -\ln(1-\gamma)\right\} := e(\gamma). \tag{8.58}$$

We finish this part of the proof by deriving a lower bound for $\|\Pi_{E_{i,Q}(\omega)}v\|$ when $v \in E_{j,P}(\omega) \cap B_{\|\cdot\|}$. By Lemma 8.3.3 and as $E_{j,P}(\omega) \subseteq G_{i,P}(\omega)$ we have

$$\begin{aligned}\|\Pi_{E_{i,Q}(\omega)}v\| \geq |\Pi_{E_{i,Q}(\omega)}v| \geq |\Pi_{G_{i,P}(\omega)}v| - |||\Pi_{G_{i,P}(\omega)} - \Pi_{E_{i,Q}(\omega)}||| \\ \geq K^{-1} - |||\Pi_{G_{i,P}(\omega)} - \Pi_{E_{i,Q}(\omega)}|||.\end{aligned}$$

Thus by Proposition 8.2.2 there exists some ϵ' such that if $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon'}(Q)$ then $\|\Pi_{E_{i,Q}(\omega)}v\| \geq 1/(2K)$. We assume that $\epsilon_{\gamma,n} \leq \epsilon'$ without loss of generality.

An upper bound for the right side of (8.54). From (8.58), Proposition 8.2.2, and as $\|\Pi_{E_{i,Q}(\omega)}v\| \neq 0$ for $v \in E_{j,P} \cap B_{\|\cdot\|}$, we get

$$\begin{aligned}\ln\left(\sup_{\substack{v\in E_{j,P}(\omega)\\ \|v\|\leq 1}} |P_\omega^n v|\right) &\leq \sup_{\substack{v\in E_{j,P}(\omega)\\ \|v\|\leq 1}} \left(\ln\left(\frac{\|Q_\omega^n \Pi_{E_{i,Q}(\omega)}v\|}{\|\Pi_{E_{i,Q}(\omega)}v\|}\right) + \ln\left(\|\Pi_{E_{i,Q}(\omega)}v\|\right)\right) + e(\gamma) \\ &\leq \ln\left(\left\|Q_\omega^n|_{E_{i,Q}}\right\|\right) + \ln(\Theta_0^2) + e(\gamma).\end{aligned}$$

From (8.54) we deduce that if $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{\gamma,n}}(Q)$ then

$$\frac{1}{nd_{j,P}} \int \det(P_\omega^n | E_{j,P}(\omega))\, d\mathbb{P} \leq \frac{1}{n}\int \ln\left(\left\|Q_\omega^n|_{E_{i,Q}}\right\|\right) d\mathbb{P} + \frac{1}{n}\left(\ln(K\Theta_0^2) + e(\gamma)\right).$$

Applying Proposition 8.3.2 we see that for every ν we may take n to be very large and γ sufficiently small to produce ϵ_{ν_1} (depending on γ and n) so that if $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\nu_1}(Q)$ then

$$\lambda_{j,P} = \frac{1}{nd_{j,P}} \int \det(P_\omega^n | E_{j,P}(\omega))\, d\mathbb{P} \leq \lambda_{i,Q} + \nu.$$

Hence

$$\gamma_{\ell,P} - \gamma_{\ell,Q} = \lambda_{j,P} - \lambda_{i,Q} \leq \nu. \tag{8.59}$$

A lower bound for the left side of (8.54). From (8.58), Lemma 8.3.3, and as $\left\|\Pi_{E_{i,Q}(\omega)}v\right\| \geq 1/(2K)$ we get

$$\ln\left(\inf_{\substack{v\in E_{j,P}(\omega)\\ \|v\|\leq 1}} |P_\omega^n v|\right) \geq \inf_{\substack{v\in E_{j,P}(\omega)\\ \|v\|\leq 1}} \left(\ln\left(\frac{\left|Q_\omega^n \Pi_{E_{i,Q}(\omega)}v\right|}{\left\|\Pi_{E_{i,Q}(\omega)}v\right\|}\right) + \ln\left(\left\|\Pi_{E_{i,Q}(\omega)}v\right\|\right)\right) - e(\gamma)$$
$$\geq \ln\left(\left\|\left(Q_\omega^n|_{E_{i,Q}(\omega)}\right)^{-1}\right\|^{-1}\right) - \ln(K) - \ln(2K) - e(\gamma).$$

Thus by (8.54) for $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{\gamma,n}}(Q)$ we have

$$\frac{1}{nd_{j,P}}\int \det(P_\omega^n|E_{j,P}(\omega))\, d\mathbb{P} \geq \frac{1}{n}\int \ln\left(\left\|\left(Q_\omega^n|_{E_{i,Q}(\omega)}\right)^{-1}\right\|^{-1}\right) d\mathbb{P} - \frac{\ln(2K^2)+e(\gamma)}{n}.$$

Applying Proposition 8.3.2 as in the previous step, we see that for every $\nu > 0$ we may take n to be very large and γ sufficiently small to produce ϵ_{ν_2} (depending on γ and n) so that if $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_{\nu_2}}(Q)$ then $\lambda_{j,P} \geq \lambda_{i,P} - \nu$. Hence

$$\gamma_{\ell,Q} - \gamma_{\ell,P} = \lambda_{i,Q} - \lambda_{\ell,P} \leq \nu. \tag{8.60}$$

Setting $\nu = \min\{\nu_1, \nu_2\}$, and then combining (8.59) and (8.60) yields $|\gamma_{\ell,P} - \gamma_{\ell,Q}| \leq \nu$ for $P \in \mathcal{LY}(C_1, C_2, r, R) \cap \mathcal{O}_{\epsilon_\nu}(Q)$. As discussed at the beginning of the proof, this suffices to prove (8.10), which completes the proof of Theorem 8.1.8. □

8.4 Technical proofs for Sections 8.2 and 8.3

In this section we prove some technical results on the existence, continuity and measurability of certain maps used in the proof of Theorem 8.1.8.

Throughout this section $(X, \|\cdot\|)$ will denote a separable Banach space. Note that, as X is separable, when restricted to the bounded sets of $L(X)$ the strong operator topology is metrisable. In addition, the map $(S, T) \mapsto S \circ T$ is continuous with respect to this restricted topology. We will use this fact frequently throughout this section. Let $\Delta_d = \{A \in L(X) : A^2 = A \text{ and } \operatorname{rank}(A) = d\}$ denote the space of bounded d-dimensional projections on X, and set

$$\Lambda_d = \left\{(A, \Pi_1, \Pi_2) \in L(X) \times \Delta_d \times \Delta_d \;\middle|\; \Pi_2 A|_{\Pi_1(X)} : \Pi_1(X) \to \Pi_2(X) \text{ is invertible}\right\}.$$

Let $\Gamma^* : \Lambda_d \to \Delta_d$ be defined by

$$\Gamma^*(A, \Pi_1, \Pi_2) = (\mathrm{Id} + A^*(0))\Pi_2,$$

where A^* is understood as the forward graph transform from $L(\Pi_1(X), \ker(\Pi_1))$ to $L(\Pi_2(X), \ker(\Pi_2))$, and let $\Gamma_* : \Lambda_d \to \Delta_d$ be defined by

$$\Gamma_*(A, \Pi_1, \Pi_2) = \Pi_1 - A_*(0)(\mathrm{Id} - \Pi_1),$$

where A_* is understood as the backward graph transform from $L(\ker(\Pi_2), \Pi_2(X))$ to $L(\ker(\Pi_1), \Pi_1(X))$.

The first result we will focus on proving is the following.

Proposition 8.4.1. *Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space, X is a separable Banach space, and $Y : \Omega \mapsto L(X)^3$ is a $(\mathcal{F}, \mathcal{S}^3)$-measurable map with $Y(\Omega) \subseteq \Lambda_d$. Then $\Gamma^* \circ Y$ and $\Gamma_* \circ Y$ are $(\mathcal{F}, \mathcal{S})$-measurable.*

We require a number of intermediary lemmas before proving Proposition 8.4.1.

Lemma 8.4.2. *The map $\Psi : \Delta_d \to \mathcal{G}_d(X)$ defined by $\Psi(\Pi) = \Pi(X)$ is continuous with respect to the strong operator topology on Δ_d.*

Proof. Fix a normalised Auerbach basis $\{v_i\}_{i=1}^d$ for $\Psi(\Pi)$ i.e. a normalised basis such that

$$\forall i \in \{1, \dots, d\} \quad \mathrm{dist}(v_i, \mathrm{span}\{v_j : j \neq i\}) = 1.$$

For $\eta > 0$ set

$$S_\eta := \Delta_d \cap \left(\bigcap_{i=1}^{d} \{\Pi' \in L(X) : \|(\Pi - \Pi')v_i\| < \eta\} \right),$$

and note that each S_η is open in the strong operator topology. Set $\epsilon = 2^{-d-2}$ and let $NB_d^\epsilon(X)$ denote the set of ϵ-nice bases for d-dimensional subspaces of X (see [47, Definition 2] for the relevant definition). Note that $\{v_i\}_{i=1}^d \in NB_d^\epsilon(X)$. By [47, Lemma B.8] there exists $\eta > 0$ so that if $\{w_i\}_{i=1}^d$ satisfies $\sup_i \|v_i - w_i\| < \eta'$ then $\{w_i\}_{i=1}^d \in NB_d^\epsilon(X)$ too. Hence if $\Pi' \in S_\eta$ then $\{\Pi' v_i\}_{i=1}^d$ is a ϵ-nice basis for $\Pi'(X)$. Moreover, the map $\Pi' \mapsto \{\Pi' v_i\}_{i=1}^d$ is continuous from S_η to $NB_d^\epsilon(X)$. Thus by [47, Corollary B.6] the map $\Pi' \mapsto \mathrm{span}\{\Pi' v_i\}_{i=1}^d = \Pi(X)$ is continuous from S_η to $\mathcal{G}_d(X)$. $\square$

Lemma 8.4.3. *For every $s > 0$ the set $\Lambda_d \cap (sB_{L(X)})^3$ is open in the restriction of the strong operator topology to $(sB_{L(X)})^3$.*

Proof. It suffices to prove that if $\{(A_\epsilon, \Pi_{1,\epsilon}, \Pi_{2,\epsilon})\}_{\epsilon>0} \subseteq (sB_{L(X)})^3$ converges strongly to $(A_0, \Pi_{1,0}, \Pi_{2,0}) \in \Lambda_d \cap (sB_{L(X)})^3$ then

$$\limsup_{\epsilon\to 0} \left\| \left(\Pi_{2,\epsilon} A_\epsilon \big|_{\Pi_{1,\epsilon}(X)} \right)^{-1} \right\| \le \left\| \left(\Pi_{2,0} A_0 \big|_{\Pi_{1,0}(X)} \right)^{-1} \right\|. \tag{8.61}$$

By definition, for each $v \in \Pi_{1,\epsilon}(X)$ with $\|v\| = 1$ there exists $w \in \Pi_{1,0}(X)$ with $\|w\| = 1$ such that $\|v - w\| \le d_H(\Pi_{1,\epsilon}(X), \Pi_{1,0}(X))$. Hence, for every such v and w,

$$\begin{aligned} \|\Pi_{2,\epsilon} A_\epsilon v\| &\ge \|\Pi_{2,\epsilon} A_\epsilon w\| - \|\Pi_{2,\epsilon} A_\epsilon (v - w)\| \\ &\ge \|\Pi_{2,\epsilon} A_\epsilon w\| - s^2 d_H(\Pi_{1,\epsilon}(X), \Pi_{1,0}(X)). \end{aligned} \tag{8.62}$$

Focusing on the first term yields

$$\|\Pi_{2,\epsilon} A_\epsilon w\| \ge \left\| \left(\Pi_{2,0} A_0 \big|_{\Pi_{1,0}(X)} \right)^{-1} \right\|^{-1} - \left\| \left(\Pi_{2,0} A_0 - \Pi_{2,\epsilon} A_\epsilon \right) \Pi_{1,0} \right\|. \tag{8.63}$$

Hence $\|(\Pi_{2,0} A_0 - \Pi_{2,\epsilon} A_\epsilon) \Pi_{1,0}\| \to 0$, since $\Pi_{1,0}(X)$ is finite-dimensional and therefore has a compact unit ball. Combining this fact with (8.62) and (8.63) yields a lower bound for $\|\Pi_{2,\epsilon} A_\epsilon v\|$ that is uniform in $v \in \Pi_{1,\epsilon}(X)$ with $\|v\| = 1$. By applying Lemma 8.4.2 we see that $\lim_{\epsilon\to 0} \Pi_{1,\epsilon}(X) = \Pi_{1,0}(X)$ in $\mathcal{G}_d(X)$, and so this lower bound converges to $\left\| \left(\Pi_{2,0} A_0 \big|_{\Pi_{1,0}(X)} \right)^{-1} \right\|^{-1}$ as $\epsilon \to 0$. □

Lemma 8.4.4. *Let $\Xi : \Lambda_d \to L(X)$ be defined by*

$$\Xi(A, \Pi_1, \Pi_2) = \left(\Pi_2 A \big|_{\Pi_1(X)} \right)^{-1} \Pi_2.$$

For each $s > 0$ the restriction of Ξ to $\Lambda_d \cap (sB_{L(X)})^3$ is continuous in the strong operator topology.

Proof. Fix $(A_0, \Pi_{1,0}, \Pi_{2,0}) \in \Lambda_d \cap (sB_{L(X)})^3$. By (8.61) there exists a neighbourhood $U \subseteq \Lambda_d \cap (sB_{L(X)})^3$ of $(A_0, \Pi_{1,0}, \Pi_{2,0})$ that is open in the strong operator topology and such that

$$\sup \left\{ \|\Xi(A, \Pi_1, \Pi_2)\| : (A, \Pi_1, \Pi_2) \in U \right\} < \infty. \tag{8.64}$$

Thus $\Xi(U)$ is bounded in $L(X)$, and $L(X)$ is therefore metrisable on $\Xi(U)$. Hence, to prove that Ξ is continuous at $(A_0, \Pi_{1,0}, \Pi_{2,0})$ it is sufficient to show that if $\{(A_\epsilon, \Pi_{1,\epsilon}, \Pi_{2,\epsilon})\}_{\epsilon>0} \subseteq \Lambda_d \cap (sB_{L(X)})^3$ converges to $(A_0, \Pi_{1,0}, \Pi_{2,0}) \in \Lambda_d \cap (sB_{L(X)})^3$ then $\Xi(A_\epsilon, \Pi_{1,\epsilon}, \Pi_{2,\epsilon}) \to \Xi(A_0, \Pi_{1,0}, \Pi_{2,0})$. Fix $v \in X$ with $\|v\| = 1$. We have

$$\begin{aligned} &\|\Xi(A_0, \Pi_{1,0}, \Pi_{2,0})v - \Xi(A_\epsilon, \Pi_{1,\epsilon}, \Pi_{2,\epsilon})v\| \\ &\qquad \leq \left\| \Pi_{1,\epsilon} \left(\Pi_{2,0} A_0|_{\Pi_{1,0}(X)} \right)^{-1} \Pi_{2,0} v - \left(\Pi_{2,\epsilon} A_\epsilon|_{\Pi_{1,\epsilon}(X)} \right)^{-1} \Pi_{2,\epsilon} v \right\| \\ &\qquad\quad + \left\| (\mathrm{Id} - \Pi_{1,\epsilon}) \Pi_{1,0} \left(\Pi_{2,0} A_0|_{\Pi_{1,0}(X)} \right)^{-1} \Pi_{2,0} v \right\|. \end{aligned} \tag{8.65}$$

We of course have

$$\begin{aligned} &\lim_{\epsilon \to 0} \left\| (\mathrm{Id} - \Pi_{1,\epsilon}) \Pi_{1,0} \left(\Pi_{2,0} A_0|_{\Pi_{1,0}(X)} \right)^{-1} \Pi_{2,0} v \right\| \\ &\quad = \left\| (\mathrm{Id} - \Pi_{1,0}) \Pi_{1,0} \left(\Pi_{2,0} A_0|_{\Pi_{1,0}(X)} \right)^{-1} \Pi_{2,0} v \right\| = 0. \end{aligned} \tag{8.66}$$

On the other hand we have

$$\begin{aligned} &\Pi_{1,\epsilon} \left(\Pi_{2,0} A_0|_{\Pi_{1,0}(X)} \right)^{-1} \Pi_{2,0} v - \left(\Pi_{2,\epsilon} A_\epsilon|_{\Pi_{1,\epsilon}(X)} \right)^{-1} \Pi_{2,\epsilon} v \\ &\quad = \Xi(A_\epsilon, \Pi_{1,\epsilon}, \Pi_{2,\epsilon}) \left((\Pi_{2,\epsilon} A_\epsilon \Pi_{1,\epsilon} - \Pi_{2,0} A_0 \Pi_{1,0}) \, \Xi(A_0, \Pi_{1,0}, \Pi_{2,0}) - \Pi_{2,\epsilon}(\mathrm{Id} - \Pi_{2,0}) \right) v. \end{aligned}$$

Hence, as

$$\limsup_{\epsilon \to 0} \| \left((\Pi_{2,\epsilon} A_\epsilon \Pi_{1,\epsilon} - \Pi_{2,0} A_0 \Pi_{1,0}) \, \Xi(A_0, \Pi_{1,0}, \Pi_{2,0}) - \Pi_{2,\epsilon}(\mathrm{Id} - \Pi_{2,0}) \right) v \| = 0,$$

by applying (8.61) from the proof of Lemma 8.4.3 we have

$$\limsup_{\epsilon \to 0} \left\| \Pi_{1,\epsilon} \left(\Pi_{2,0} A_0|_{\Pi_{1,0}(X)} \right)^{-1} \Pi_{2,0} v - \left(\Pi_{2,\epsilon} A_\epsilon|_{\Pi_{1,\epsilon}(X)} \right)^{-1} \Pi_{2,\epsilon} v \right\| = 0 \tag{8.67}$$

We obtain the required claim by applying (8.66) and (8.67) to (8.65). □

Lemma 8.4.5. *For every $s > 0$ the maps $\Gamma^*|_{(sB_{L(X)})^3}$ and $\Gamma_*|_{(sB_{L(X)})^3}$ are continuous with respect to the strong operator topology.*

Proof. We will just prove that $\Gamma^*|_{(sB_{L(X)})^3}$ is continuous, since essentially the same proof applies to $\Gamma_*|_{(sB_{L(X)})^3}$. Fix $(A_0, \Pi_{1,0}, \Pi_{2,0}) \in (sB_{L(X)})^3$. An argument similar to that at the beginning of Lemma 8.4.4 shows that there is a neighbourhood

$U \subseteq \Lambda_d \cap (sB_{L(X)})^3$ of $(A_0, \Pi_{1,0}, \Pi_{2,0})$ that is open in the strong operator topology and such that $\Gamma^*(U)$ is bounded. Therefore, to show that Γ^* is continuous at $(A_0, \Pi_{1,0}, \Pi_{2,0})$ it suffices to prove that if $\{(A_\epsilon, \Pi_{1,\epsilon}, \Pi_{2,\epsilon})\}_{\epsilon>0} \subseteq \Lambda_d \cap (sB_{L(X)})^3$ converges to $(A_0, \Pi_{1,0}, \Pi_{2,0})$ then $\Gamma^*(A_\epsilon, \Pi_{1,\epsilon}, \Pi_{2,\epsilon}) \to \Gamma^*(A_0, \Pi_{1,0}, \Pi_{2,0})$. By the definition of the forward graph transform we have

$$\begin{aligned}\Gamma^*(A_\epsilon, \Pi_{1,\epsilon}, \Pi_{2,\epsilon}) - \Gamma^*(A_0, \Pi_{1,0}, \Pi_{2,0}) = \Pi_{2,\epsilon} + (\mathrm{Id} - \Pi_{2,\epsilon})A_\epsilon \left(\Pi_{2,\epsilon} A_\epsilon|_{\Pi_{1,\epsilon}(X)}\right)^{-1} \Pi_{2,\epsilon} \\ - \Pi_{2,0} - (\mathrm{Id} - \Pi_{2,0})A_0 \left(\Pi_{2,0} A_0|_{\Pi_{1,0}(X)}\right)^{-1} \Pi_{2,0}.\end{aligned}$$

Applying Lemmas 8.4.3 and 8.4.4, and (8.61) yields

$$(\mathrm{Id} - \Pi_{2,\epsilon})A_\epsilon \left(\Pi_{2,\epsilon} A_\epsilon|_{\Pi_{1,\epsilon}(X)}\right)^{-1} \Pi_{2,\epsilon} \to (\mathrm{Id} - \Pi_{2,0})A_0 \left(\Pi_{2,0} A_0|_{\Pi_{1,0}(X)}\right)^{-1} \Pi_{2,0}$$

in the strong operator topology, which completes the proof. □

The proof of Proposition 8.4.1. We will just prove that $\Gamma^* \circ Y$ is $(\mathcal{F}, \mathcal{S})$-measurable, since the same proof works for $\Gamma_* \circ Y$. Let $U \subseteq L(X)$ be open in the strong operator topology. Then

$$\begin{aligned}(\Gamma^* \circ Y)^{-1}(U) &= Y^{-1}\left(\bigcup_{n\in\mathbb{Z}^+} (nB_{L(X)})^3 \cap (\Gamma^*)^{-1}(U)\right) \\ &= Y^{-1}\left(\bigcup_{n\in\mathbb{Z}^+} \left(\Gamma^*|_{(nB_{L(X)})^3}\right)^{-1}(U)\right).\end{aligned} \tag{8.68}$$

By Lemma 8.4.5, the map $\Gamma^*|_{(nB_{L(X)})^3}$ is continuous in the strong operator topology for every $n \in \mathbb{Z}^+$, and so $\left(\Gamma^*|_{(nB_{L(X)})^3}\right)^{-1}(U) = U_n \cap \Lambda_d \cap (nB_{L(X)})^3$ for some $U_n \in B_{L(X)}^3$ that is open in the strong operator topology. Since $Y(\Omega) \subseteq \Lambda_d$ we have

$$Y^{-1}(U_n \cap \Lambda_d \cap (nB_{L(X)})^3) = Y^{-1}(U_n \cap (nB_{L(X)})^3). \tag{8.69}$$

Since $(nB_{L(X)})^3$ is a separable metric space, for each $n \in \mathbb{Z}^+$ there exists countably many rectangles $\{R_{i,n} \times P_{i,n} \times Q_{i,n}\}_{i\in\mathbb{Z}^+} \subseteq (B_{L(X)})^3$ such that $R_{i,n}$, $P_{i,n}$, and $Q_{i,n}$ are open in the strong operator topology on $B_{L(X)}$ and so that

$$U_n \cap (nB_{L(X)})^3 = \bigcup_{i\in\mathbb{Z}^+} (R_{i,n} \cap nB_{L(X)}) \times (P_{i,n} nB_{L(X)}) \times (Q_{i,n} \cap nB_{L(X)}). \tag{8.70}$$

By [47, Lemma A.2] we have $nB_{L(X)} \in \mathcal{S}$, and so $U_n \cap (nB_{L(X)})^3 \in \mathcal{S}^3$, being the countable union of sets in $\mathcal{S}^3$ by (8.70). Since Y is $(\mathcal{F}, \mathcal{S}^3)$-measurable, by (8.68), (8.69) and (8.70) we may conclude that $(\Gamma^* \circ Y)^{-1}(U) \in \mathcal{F}$ i.e $\Gamma^* \circ Y$ is $(\mathcal{F}, \mathcal{S})$-measurable. □

By arguing in the same way, we can deduce the measurability of the map Ξ from Lemma 8.4.4.

Proposition 8.4.6. *Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space, X is a separable Banach space, and that $Y : \Omega \mapsto L(X)^3$ is $(\mathcal{F}, \mathcal{S}^3)$-measurable with $Y(\Omega) \subseteq \Lambda_d$. Then $\Xi \circ Y$ is $(\mathcal{F}, \mathcal{S})$-measurable.*

Proof. The proof is identical to that of Proposition 8.4.1, but with Lemma 8.4.4 used in place of Lemma 8.4.5. □

Lemma 8.4.7. *Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space, X is a separable Banach space, that $\{f_n\}_{n \in \mathbb{Z}^+}$ is a sequence of strongly $(\mathcal{F}, \mathcal{S})$-measurable functions, and that $f : \Omega \to L(X)$ with $f_n \to f$ almost uniformly and $\operatorname{ess\,sup}_{\omega \in \Omega} \|f\| < \infty$. Then f is $(\mathcal{F}, \mathcal{S})$-measurable.*

Proof. Let $r > \operatorname{ess\,sup}_{\omega \in \Omega} \|f\|$. By changing each f_n on a set of measure 0 we may assume that $\limsup_{n \to \infty} \sup_{\omega \in \Omega} \|f_n(\omega)\| \leq r$ and that there exists $g : \Omega \to rB_{L(X)}$ with $f = g$ a.e. and such that $f_n \to g$ uniformly. Since $f_n \to g$ uniformly there exists $N > 0$ such that $f_n(\Omega) \subseteq rB_{L(X)}$ for every $n > N$. By [47, Lemma A.2] we have $rB_{L(X)} \in \mathcal{S}$ and so f_n is $(\mathcal{F}, \mathcal{S}_r)$-measurable for $n > N$, where $\mathcal{S}_r$ denotes the Borel σ-algebra associated to the restriction of the strong operator topology to $rB_{L(X)}$. Since X is separable, the strong operator topology on $rB_{L(X)}$ is metrisable. Thus g is $(\mathcal{F}, \mathcal{S}_r)$-measurable, being the pointwise limit of measurable functions with values in a metric space. For $U \subseteq L(X)$ that is open in the strong operator topology we have $U \cap rB_{L(X)} \in \mathcal{S}_r$, and so $g^{-1}(U) = g^{-1}(U \cap rB_{L(X)}) \in \mathcal{F}$. Thus g is $(\mathcal{F}, \mathcal{S})$-measurable. Since $f = g$ a.e. we have that f is $(\mathcal{F}, \mathcal{S})$-measurable too. □

We now prove the Lemma 8.2.6, which concerned the existence of measurable change-of-basis maps with controlled distortion.

The proof of Lemma 8.2.6. By Lemma 8.4.2 the map $\omega \mapsto \Pi_\omega(X)$ is measurable. By [76, Corollary 39] for every $\epsilon > 0$ there exists measurable maps $e_i : \Omega \to X$,

$1 \leq i \leq d$ such that $\operatorname{span}\{e_1(\omega), \ldots, e_d(\omega)\} = \Pi_\omega(X)$ and for each $1 \leq i \leq d-1$ we have $\|e_i(\omega)\| = 1$ and

$$\operatorname{dist}(e_i(\omega), \operatorname{span}\{e_j(\omega) : j > i\}) \geq 1 - \epsilon.$$

Let $\{\nu_i(\omega)\}_{i=1}^d$ denote the dual basis to $\{e_i(\omega)\}_{i=1}^d$ in $\Pi_\omega(X)$. For every ω we have $\nu_i(\omega)e_j(\omega) = \delta_{ij}$ and so each $\omega \mapsto \nu_i(\omega)e_j(\omega)$ is measurable. For $(a_1, \ldots, a_d) \in \mathbb{Q}^d + i\mathbb{Q}^d$ set $\psi(a_1, \ldots, a_d)(\omega) = \sum_{i=1}^d a_i e_i(\omega)$. We of course have

$$\|\nu_i(\omega)\| = \sup_{\substack{(a_1,\ldots,a_d) \in \mathbb{Q}^d + i\mathbb{Q}^d \\ (a_1,\ldots,a_d) \neq (0,\ldots,0)}} \frac{|\nu_i(\omega)(\psi(a_1, \ldots, a_d)(\omega))|}{|\psi(a_1, \ldots a_d)(\omega)|},$$

and that each of the maps

$$\omega \mapsto \frac{|\nu_i(\omega)(\psi(a_1, \ldots, a_d)(\omega))|}{|\psi(a_1, \ldots, a_d)(\omega)|}$$

is measurable. Hence $\omega \mapsto \|\nu_i(\omega)\|$ is measurable, being the supremum of countably many measurable maps. By [76, Proposition 40], each ν_i may be extended to a strongly measurable map $\nu_i : \Omega \to L(X, \mathbb{C})$ without increasing $\|\nu_i(\omega)\|$. Define $\phi_\omega : X \to \mathbb{C}^d$ by

$$\phi_\omega v = (\nu_1(\omega)(v), \ldots, \nu_d(\omega)(v)),$$

and set $A_\omega = \phi_\omega \Pi_\omega$. We clearly have $\ker(A_\omega) = \ker(\Pi_\omega)$, and that $A_\omega\big|_{\Pi_\omega(X)}$ is a bijection. The map $\omega \mapsto \phi_\omega$ is strongly measurable as each of component maps $\omega \mapsto \nu_i(\omega)$ is strongly measurable, and so $\omega \mapsto A_\omega$ is strongly measurable, due to it being the composition of strongly measurable maps [47, Lemma A.5]. Moreover, we have

$$\left(A_\omega\big|_{\Pi_\omega(X)}\right)^{-1}(a_1, \ldots, a_d) = \sum_{i=1}^d a_i e_i(\omega),$$

which implies that $\left(A_\omega\big|_{\Pi_\omega(X)}\right)^{-1}$ is strongly measurable.

We may now prove the estimates in (8.28). For the second estimate we simply note that if $v \in \Pi_\omega$ then

$$\|v\| \leq \sum_{i=1}^d |\nu_i(\omega)v| \leq \sqrt{d} \left(\sum_{i=1}^d |\nu_i(\omega)v|^2 \right)^{1/2} = \sqrt{d}\, \|A_\omega v\|.$$

Obtaining the first estimate in (8.28) is more involved. For every $v \in \Pi_\omega(X)$ one has

$$\|v\| \geq \max_{1\leq i\leq d}\left\{\left\|\Pi_{\operatorname{span}\{e_i(\omega)\}||\operatorname{span}\{e_j(\omega):j\neq i\}}v\right\|\left\|\Pi_{\operatorname{span}\{e_i(\omega)\}||\operatorname{span}\{e_j(\omega):j\neq i\}}\right\|^{-1}\right\}. \quad (8.71)$$

For each $i \in \{1,\dots,d\}$ set $\Pi_{i,\omega} = \Pi_{\operatorname{span}\{e_i(\omega)\}||\operatorname{span}\{e_j(\omega):j>i\}}$ and $\Gamma_{i,\omega} = \operatorname{Id} - \Pi_{i,\omega}$. Note that

$$\Pi_{\operatorname{span}\{e_i(\omega)\}||\operatorname{span}\{e_j(\omega):j\neq i\}} = \Pi_{i,\omega}\left(\prod_{j=1}^{i-1}\Gamma_{i-j,\omega}\right),$$

and so $\left\|\Pi_{\operatorname{span}\{e_i(\omega)\}||\operatorname{span}\{e_j(\omega):j\neq i\}}\right\| \leq 2^{i-1}\prod_{j=1}^{i}\|\Pi_{j,\omega}\|$. In addition we have for each $i \in \{1,\dots,d-1\}$ that

$$\begin{aligned}\|\Pi_{i,\omega}\| = \sup_{v\in\operatorname{span}\{e_j(\omega):j\geq i\}}\frac{\|\Pi_{i,\omega}(v)\|}{\|v\|} &= \sup_{v'\in\operatorname{span}\{e_j(\omega):j>i\}}\frac{\|e_i(\omega)\|}{\|e_i(\omega)-v'\|}\\ &= \operatorname{dist}(e_i(\omega),\operatorname{span}\{e_j(\omega):j>i\})^{-1} \leq (1-\epsilon)^{-1},\end{aligned}$$

while it is clear that $\|\Pi_{d,\omega}\| = 1$. Thus, for every $i \in \{1,\dots,d\}$ we have

$$\left\|\Pi_{\operatorname{span}\{e_i(\omega)\}||\operatorname{span}\{e_j(\omega):j\neq i\}}\right\| \leq 2^{d-1}(1-\epsilon)^{-d+1}.$$

Since $\Pi_{\operatorname{span}\{e_i(\omega)\}||\operatorname{span}\{e_j(\omega):j\neq i\}}v = \nu_i(\omega)(v)$, from (8.71) we obtain

$$\|v\| \geq \left(\frac{2}{1-\epsilon}\right)^{d-1}\max\{\nu_i(\omega)(v) : 1\leq i\leq d\} \geq \left(\frac{2}{1-\epsilon}\right)^{d-1}\|A_\omega v\|.$$

□

Our final main result for this appendix concerns the measurability of the determinant map, which is crucial for the proof of stability of Lyapunov exponents in Section 8.3. We refer the reader to [15, Section 2.2] for an overview of the basic properties of the determinant.

Proposition 8.4.8. *Suppose that $(\Omega,\mathcal{F},\mathbb{P})$ is a Lebesgue space, X is a separable Banach space, and that $Y : \Omega \mapsto L(X)\times\mathcal{G}_d(X)$ is a $(\mathcal{F},\mathcal{S}\times\mathcal{B}_{\mathcal{G}_d(X)})$-measurable map. Then $\omega \mapsto \det(Y(\omega))$ is $(\mathcal{F},\mathcal{B}_{\mathbb{R}})$-measurable.*

Lemma 8.4.9. *Suppose that X is a separable Banach space. For every $d \in \mathbb{Z}^+$ and $s > 0$ the map $\det : L(X)\times\mathcal{G}_d(X) \to \mathbb{R}$ is continuous with respect to strong operator topology and the usual Grassmannian topology on $\mathcal{G}_d(X)$ when restricted to $sB_{L(X)}\times\mathcal{G}_d(X)$.*

Proof. It suffices to prove that if $\{(A_n, E_n)\}_{n\in\mathbb{Z}^+} \subseteq sB_{L(X)} \times \mathcal{G}_d(X)$ converges to (A, E) then $\det(A_n|E_n) \to \det(A|E)$. Let $F \in \mathcal{G}^d(X)$ be such that $E \oplus F = X$. Since $E_n \to E$ and $\mathcal{N}(F)$ is open in $\mathcal{G}_d(X)$, without loss of generality we may assume that $E_n \oplus F = X$ for every n. Moreover, by Proposition 6.1.2 we have $\Pi_{E_n||F} \to \Pi_{E||F}$ in the operator norm topology. Since $\{A_n\}_{n\in\mathbb{Z}^+}$ is bounded in $L(X)$ it follows that $A_n\Pi_{E_n||F} \to A$ in the strong operator topology.

The case where $\det(A|E) = 0$. If $\det(A_n|E) = 0$ eventually holds for all large $n \in \mathbb{Z}^+$ then we are done. Otherwise we may pass to a subsequence such that $\limsup_{n\to\infty} \det(A_n|E_n)$ is unchanged and $\det(A_n|E) \neq 0$ for every $n \in \mathbb{Z}^+$. In particular, we may assume that $A_n\big|_{E_n}$ is injective for every n. Since $\det(A|E) = 0$ there exists $f \in \ker(A\big|_E) \setminus \{0\}$. Let G be a complementary subspace for $\operatorname{span}\{f\}$ in E. Let $f_n = \Pi_{E_n||F} f$ and $G_n = \Pi_{E_n||F} G$. As $E_n, E \in \mathcal{F}$ by Lemma 6.1.1 we have that $\Pi_{E_n||F}\big|_E$ is invertible. Thus $E_n = \operatorname{span}\{f_n\} \oplus G_n$ and

$$\Pi_{\operatorname{span}\{f_n\}||G_n} = \Pi_{E_n||F}\Pi_{\operatorname{span}\{f\}||G}\left(\Pi_{E_n||F}\big|_E\right)^{-1}.$$

Hence as $E_n \to E$ we have $\limsup_{n\to\infty} \left\|\Pi_{\operatorname{span}\{f_n\}||G_n}\right\| \leq \left\|\Pi_{\operatorname{span}\{f\}||G}\right\| < \infty$. Since each A_n is injective, by [15, Lemma 2.15] there exists $C_d > 0$ such that

$$\det(A_n|E_n) \leq C_d \det(A_n|\operatorname{span}\{f_n\}) \det(A_n|G_n) \left\|\Pi_{\operatorname{span}\{f_n\}||G_n}\right\|. \tag{8.72}$$

On one hand we have $\det(A_n|\operatorname{span}\{f_n\}) \leq \left\|A_n\Pi_{E_n||F} f\right\| \to 0$, while on the other we have $\det(A_n|G_n) \leq \|A_n\|^{d-1}$. Thus by (8.72) we have $\det(A_n|E_n) \to 0 = \det(A|E)$, as required.

The convergence of A_nE_n *to* AE. Henceforth we shall assume that $\det(A|E) \neq 0$, and so $A\big|_E$ has trivial kernel. A quickly calculation verifies that

$$\sup_{\substack{f\in E\\ \|Af\|=1}} \operatorname{dist}(Af, A_nE_n) \leq \left\|(A - A_n\Pi_{E_n||F})\big|_E\right\| \left\|\left(A\big|_E\right)^{-1}\right\|.$$

It follows that $\lim_{n\to\infty} \left\|(A - A_n\Pi_{E_n||F})\big|_E\right\| = 0$ since E is finite-dimensional and therefore has compact unit ball. Hence $\lim_{n\to\infty} \operatorname{Gap}(AE, A_nE_n) = 0$. By [65, IV §2, Corollary 2.6] it follows that $\dim(AE) \leq \dim(A_nE_n)$ for sufficiently large n. Since $A\big|_E$ has trivial kernel we have $\dim(AE) = \dim(E) = \dim(E_n) \geq \dim(A_nE_n)$ and so $\dim(AE) = \dim(A_nE_n)$. By [15, Lemma 2.6] we therefore have $\operatorname{Gap}(A_nE_n, AE) \to 0$, and so $A_nE_n \to AE$ in $\mathcal{G}_d(X)$ by (6.1).

The case where $\det(A|E) \neq 0$. Let $F' \in \mathcal{G}^d(X)$ be such that $AE \oplus F' = X$. Since $A_nE_n \to AE$, without loss of generality we may assume that $A_nE_n \oplus F' = X$ for every n and that $\Pi_{A_nE_n||F'} \to \Pi_{AE||F'}$. By Lemma 6.1.1 the map $\Pi_{A_nE_n||F'}\big|_{AE}$ is invertible, and so the pushforward of m_{AE} under $\Pi_{A_nE_n||F'}$ is a well defined, translation invariant measure on A_nE_n. Since the Haar is unique up to scaling we get

$$m_{A_nE_n} = \frac{m_{A_nE_n}(B_{A_nE_n})}{m_{AE}(\left(\Pi_{A_nE_n||F'}\big|_{AE}\right)^{-1}(B_{A_nE_n}))}(m_{AE} \circ \left(\Pi_{A_nE_n||F'}\big|_{AE}\right)^{-1}).$$

For notational convenience we set $\Gamma_n = \left(\Pi_{A_nE_n||F'}\big|_{AE}\right)^{-1}$. By (8.40) one has

$$\begin{aligned}&|m_E(B_E)|\,|\det(A|E) - \det(A_n|E_n)| \\ &= \left| m_{AE}(AB_E) - \frac{m_{A_nE_n}(B_{A_nE_n})}{m_{AE}(\Gamma_n B_{A_nE_n})} \frac{m_{AE}(\Gamma_n A_n(B_{E_n}))}{m_{AE}(\Gamma_n A_n \Pi_{E_n||F} B_E)} m_{AE}(\Gamma_n A_n \Pi_{E_n||F} B_E) \right|.\end{aligned} \tag{8.73}$$

As $\left\|\Pi_{A_nE_n||F'}\big|_{AE}\right\|^{-1} B_{AE} \subseteq \Gamma_n(B_{A_nE_n}) \subseteq \|\Gamma_n\| B_{AE}$ we have

$$\left\|\Pi_{A_nE_n||F'}\big|_{AE}\right\|^{-d} m_{AE}(B_{AE}) \leq m_{AE}(\Gamma_n(B_{A_nE_n}))) \leq \|\Gamma_n\|^d m_{AE}(B_{AE}). \tag{8.74}$$

Since $A_nE_n \to AE$ we have $\|\Gamma_n - \mathrm{Id}\| \to 0$ by Proposition 6.1.2 and the definition of the graph representation of $\mathcal{N}(F')$. Applying the facts that $m_{AE}(B_{AE}) = m_{A_nE_n}(B_{A_nE_n})$, $\left\|\Pi_{A_nE_n||F'}\big|_{AE}\right\| \to 1$, and $\|\Gamma_n\| \to 1$ to (8.74) yields

$$\lim_{n\to\infty} \frac{m_{A_nE_n}(B_{A_nE_n})}{m_{AE}(\Gamma_n(B_{A_nE_n}))} = 1. \tag{8.75}$$

By a similar argument we find that

$$\lim_{n\to\infty} \frac{m_{AE}(\Gamma_n A_n \Pi_{E_n||F} B_E)}{m_{AE}(\Gamma_n A_n B_{E_n})} = 1. \tag{8.76}$$

Note that

$$\frac{\left|m_{AE}(A(B_E)) - m_{AE}(\Gamma_n A_n \Pi_{E_n||F}(B_E))\right|}{|m_E(B_E)|} = \left|\det(A|E) - \det(\Gamma_n A_n \Pi_{E_n||F}|E)\right|.$$

Since $\Gamma_n A_n \Pi_{E_n||F}\big|_E \in L(E, AE)$ for every n, we have $\Gamma_n A_n \Pi_{E_n||F}\big|_E \to A$ in the operator norm on $L(E, AE)$. Hence by [15, Lemma 2.20] we have

$$\lim_{n\to\infty} \det(\Gamma_n A_n \Pi_{E_n||F}|E) = \det(A|E).$$

Combining this with (8.73), (8.75) and (8.76) completes the proof. □

The proof of Proposition 8.4.8. The proof is similar to that of Proposition 8.4.1, but we include it for completeness. For every open $U \in \mathbb{R}$ we have

$$\{\omega : \det(Y(\omega)) \in U\} = Y^{-1}\left(\bigcup_{n\in\mathbb{Z}^+} \{(A, E) \in nB_{L(X)} \times \mathcal{G}_d(X) : \det(A|E) \in U\}\right). \tag{8.77}$$

By Lemma 8.4.9 for each $n \in \mathbb{Z}^+$ there exists a set $U_n \in nB_{L(X)} \times \mathcal{G}_d(X)$ that is open in the product of the relative strong operator topology and the Grassmannian topology, and such that

$$U_n = \left\{(A, E) \in nB_{L(X)} \times \mathcal{G}_d(X) : \det(A|E) \in U\right\}.$$

Since $nB_{L(X)}$ and $\mathcal{G}_d(X)$ are both are separable metric spaces (for the later claim see [47, Lemma B.11]) we may write $U_n \cap (nB_{L(X)} \times \mathcal{G}_d(X))$ as the union of countably many rectangles $\{(R_{n,i} \cap nB_{L(X)}) \times Q_{n,i}\}_{i\in\mathbb{Z}^+}$, where $R_{n,i}$ is open in the strong operator topology on $L(X)$ and $Q_{n,i}$ is open in the Grassmannian topology. We have $nB_{L(X)} \in \mathcal{S}$ by [47, Lemma A.2], and so $R_{n,i} \in \mathcal{S}$ for every $i, n \in \mathbb{Z}^+$. It follows that each U_n is a countable union of sets in $\mathcal{S} \times \mathcal{B}_{\mathcal{G}_d(X)}$, and so $U_n \in \mathcal{S} \times \mathcal{B}_{\mathcal{G}_d(X)}$ for every n. Thus $\bigcup_{n\in\mathbb{Z}^+} U_n \in \mathcal{S} \times \mathcal{B}_{\mathcal{G}_d(X)}$. Since Y is $(\mathcal{F}, \mathcal{S} \times \mathcal{B}_{\mathcal{G}_d(X)})$-measurable, it follows that the left side (8.77) must be in $\mathcal{F}$. Thus $\omega \mapsto \det(Y(\omega))$ is $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable, as required. □

Chapter 9

Application to random dynamical systems

In this chapter we demonstrate the application of Theorems 7.1.7 and 8.1.8 to cocycles of Perron-Frobenius operators associated to random dynamical systems consisting of $\mathcal{C}^k$ expanding maps on S^1, with $k \geq 2$. We will consider two types of perturbations to such maps: fiber-wise 'deterministic' perturbations to the random dynamics[1], and perturbations that arise via numerical approximations of the Perron-Frobenius cocycle. Section 9.1 contains the main definitions and results for this chapter, while the proofs of our main results are deferred to Section 9.2.

9.1 Definitions and main results

Fix a Lebesgue probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and an invertible, $\mathbb{P}$-ergodic map $\sigma : \Omega \to \Omega$. We will consider random dynamical systems taking values in the following sets.

Definition 9.1.1. *For $k \geq 2$, $\alpha \in (0, 1)$ and $K > 0$ we set*

$$\mathrm{LY}_k(\alpha, K) = \left\{ T \in \mathcal{C}^k(S^1, S^1) : \inf |T'| \geq \alpha^{-1} \text{ and } d_{\mathcal{C}^k}(T, 0) \leq K \right\}.$$

We say that $\mathcal{T} : \Omega \to \mathrm{LY}_k(\alpha, K)$ is measurable if it is measurable with respect to $\mathcal{F}$ and the Borel σ-algebra on $\mathcal{C}^k(S^1, S^1)$.

Suppose $k \geq 2$, $\alpha \in (0, 1)$, and $K > 0$. A measurable map $\mathcal{T} : \Omega \to \mathrm{LY}_k(\alpha, K)$ induces a random dynamical system (RDS) over σ whose trajectories are of the form

$$x, \mathcal{T}_\omega(x), \mathcal{T}_\omega^2(x), \dots, \mathcal{T}_\omega^n(x), \dots$$

[1] In this rather unfortunate oxymoron, a fiber-wise 'deterministic' perturbation simply means that the random maps are fiber-wise perturbed to nearby maps.

where $\mathcal{T}_\omega := \mathcal{T}(\omega)$ and $\mathcal{T}_\omega^n := \mathcal{T}_{\sigma^n(\omega)} \circ \cdots \circ \mathcal{T}_\omega$. To study the statistical properties of such a RDS one is lead to study the associated Perron-Frobenius operator cocycle. If $T \in \mathrm{LY}_k(\alpha, K)$ then we denote by $\mathcal{L}_T : L^1(S^1) \to L^1(S^1)$ the associated Perron-Frobenius operator, which is defined by duality via

$$\int \mathcal{L}_T f \cdot g \,\mathrm{d}\,\mathrm{Leb} = \int f \cdot g \circ T \,\mathrm{d}\,\mathrm{Leb} \quad \forall f \in L^1(S^1), g \in L^\infty(S^1). \tag{9.1}$$

In a slight abuse of notation, whenever $\mathcal{T} : \Omega \to \mathrm{LY}_k(\alpha, K)$ is measurable, we denote by $\mathcal{L}_{\mathcal{T}} : \Omega \to L(L^1)$ the map defined by $\mathcal{L}_{\mathcal{T}}(\omega) = \mathcal{L}_{\mathcal{T}(\omega)}$. The Perron-Frobenius operator cocycle associated to $\mathcal{T}$ is given by:

$$(n, \omega) \in \mathbb{N} \times \Omega \mapsto \mathcal{L}_{\sigma^n(\omega)} \circ \cdots \circ \mathcal{L}_\omega \in L(L^1(S^1)).$$

As one might expect given the deterministic case, studying the Perron-Frobenius cocycle on $L^1(S^1)$ yields little information about the quenched statistical properties of the RDS. Instead, the regularity of maps in $\mathrm{LY}_k(\alpha, K)$ suggests that we should consider how their Perron-Frobenius operators act on objects with some smoothness rather than on $L^1(S^1)$. For $k \in \mathbb{N}$ the Sobolev space $W^{k,1}(S^1)$ is defined by

$$W^{k,1}(S^1) = \left\{ f \in L^p(S^1) : \begin{array}{r} f^{(\ell)} \text{ exists in the weak sense and} \\ \left\| f^{(\ell)} \right\|_{L^1} < \infty \text{ for each } 0 \le \ell \le k \end{array} \right\}.$$

Each $W^{k,1}(S^1)$ becomes a Banach space when equipped with the norm

$$\|f\|_{W^{k,1}} = \|f\|_{L^1} + \left\| f^{(k)} \right\|_{L^1}.$$

For each $k \geq 1$ the embedding of $W^{k,1}(S^1)$ into $W^{k-1,1}(S^1)$ is compact by the Rellich–Kondrachov Theorem. Moreover, $\left\| f^{(k)} \right\|_{L^1} = \mathrm{Var}(f^{(k-1)})$ and so by following the arguments in Examples 6.2.12 and 6.2.13 we conclude that $\|\cdot\|_{W^{k,1}}$ is upper-semicontinuous with respect to $\|\cdot\|_{W^{k-1,1}}$. Thus $(W^{k,1}(S^1), \|\cdot\|_{W^{k,1}}, \|\cdot\|_{W^{k-1,1}})$ is a pre-compact Saks space, and therfore compatible with the perturbation theory developed in Chapters 7 and 8. We remind the reader that each $W^{k,1}(S^1)$ is separable as a Banach space.

Proposition 9.1.2. *If $\mathcal{T} : \Omega \to \mathrm{LY}_k(\alpha, K)$ is measurable for $k \geq 2$, $\alpha \in (0, 1)$ and $K > 0$, then $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \mathcal{L}_{\mathcal{T}})$ is a separable strongly measurable random linear system with ergodic invertible base. Moreover $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \mathcal{L}_{\mathcal{T}})$ has an Oseledets splitting of dimension $d \geq 1$ with $\lambda_{1,\mathcal{T}} = 0$.*

We will now make precise our first type of perturbation. For measurable maps $\mathcal{S}, \mathcal{T} : \Omega \to \mathrm{LY}_k(\alpha, K)$ we set

$$d_{k-1}(\mathcal{S}, \mathcal{T}) = \operatorname*{ess\,sup}_{\omega \in \Omega} d_{\mathcal{C}^{k-1}}(\mathcal{S}(\omega), \mathcal{T}(\omega)).$$

For $\epsilon > 0$ and measurable $\mathcal{T} : \Omega \to \mathrm{LY}_k(\alpha, K)$ we set

$$\mathcal{O}_{\epsilon,k,\alpha,K}(\mathcal{T}) = \left\{ \mathcal{S} : \Omega \to \mathrm{LY}_k(\alpha, K) \,\middle|\, \mathcal{S} \text{ is measurable with } d_{k-1}(\mathcal{T}, \mathcal{S}) \le \epsilon \right\}.$$

The next result concerns the stability of the Oseledets splitting and Lyapunov exponents of cocycles of Perron-Frobenius operator associated to maps in $\mathrm{LY}_k(\alpha, K)$ under perturbations which are small in the d_{k-1} metric. We adopt the notation of Chapter 8, aside from frequently replacing $\mathcal{L}_\mathcal{T}$ (resp. $\mathcal{L}_\mathcal{S}$) with $\mathcal{T}$ (resp. $\mathcal{S}$) in various subscripts.

Theorem 9.1.3. *Fix $k \ge 2$, $\alpha \in (0,1)$ and $K > 0$, and suppose that $\mathcal{T} : \Omega \to \mathrm{LY}_k(\alpha, K)$ is measurable, and that $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \mathcal{L}_\mathcal{T})$ admits a hyperbolic Oseledets splitting of dimension d with $(k-1)\ln\alpha < \mu_{\mathcal{L}_\mathcal{T}}$. There exists $\epsilon > 0$ such that if $\mathcal{S} \in \mathcal{O}_{\epsilon,k,\alpha,K}(\mathcal{T})$ then $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \mathcal{L}_\mathcal{S})$ has an Oseledets splitting of dimension d. In addition, there exists $c_0, R_0 > 0$ such that each $I_i = (\lambda_{i,\mathcal{T}} - c_0, \max\{\lambda_{i,\mathcal{T}}, \ln(\delta_{1i}R_0)\} + c_0)$, $i \in \{1, \dots, k_\mathcal{T}\}$, separates the Lyapunov spectrum of $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \mathcal{L}_\mathcal{S})$, and the corresponding projections satisfy*

$$\forall i \in \{1, \dots, k_\mathcal{T}\},\ \text{a.e. } \omega \in \Omega \quad \operatorname{rank}(\Pi_{I_i,\mathcal{S}}(\omega)) = d_{i,\mathcal{T}},$$

and

$$\sup \left\{ \operatorname*{ess\,sup}_{\omega \in \Omega} \|\Pi_{I_i,\mathcal{S}}(\omega)\|_{L(W^{k-1,1})} \,\middle|\, \mathcal{S} \in \mathcal{O}_{\epsilon,k,\alpha,K}(\mathcal{T}), 1 \le i \le k_\mathcal{T} \right\} < \infty.$$

Moreover, for every $\beta > 0$ there exists $\epsilon_\beta > 0$ so that if $\mathcal{S} \in \mathcal{O}_{\epsilon_\beta,k,\alpha,K}(\mathcal{T})$ then

$$\sup_{1 \le i \le d} |\gamma_{i,\mathcal{T}} - \gamma_{i,\mathcal{S}}| \le \beta,$$

$$\sup_{1 \le i \le k_\mathcal{T}} \operatorname*{ess\,sup}_{\omega \in \Omega} \|\Pi_{I_i,\mathcal{T}}(\omega) - \Pi_{I_i,\mathcal{S}}(\omega)\|_{L(W^{k-1,1}, W^{k-2,1})} \le \beta,$$

and

$$\operatorname*{ess\,sup}_{\omega \in \Omega} d_H(F_\mathcal{T}(\omega), F_\mathcal{S}(\omega)) \le \beta.$$

Our second application concerns the numerical approximation of the Oseledets splitting and Lyapunov exponents associated to a Perron-Frobenius operator cocycle. For each $n \in \mathbb{Z}^+$ the nth Fejér kernel $J_n : S^1 \to \mathbb{C}$ is defined by

$$J_n(t) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{2\pi i k t}.$$

Convolution with the nth Fejér kernel corresponds to taking the Cesàro average of the first $n+1$ partial Fourier series, so that for each $f \in L^1(S^1)$ one has

$$(J_n * f)(x) = \frac{1}{n+1} \sum_{\ell=0}^{n} \left(\sum_{j=-\ell}^{\ell} \hat{f}(j) e^{2\pi i j x} \right) = \sum_{\ell=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \hat{f}(\ell) e^{2\pi i \ell x},$$

where $\hat{f}(\ell) = \int f(x) e^{-2\pi i \ell x} \, \mathrm{d}\,\mathrm{Leb}$. The following proposition, which is well-known, summarises the relevant properties of the Fejér kernel in our setting.

Proposition 9.1.4. *For $n \in \mathbb{Z}^+$ let $\mathcal{J}_n : L^1(S^1) \to L^1(S^1)$ denote the operator defined by*

$$\mathcal{J}_n(f) = J_n * f.$$

For every $n, k \in \mathbb{Z}^+$ the operator $\mathcal{J}_n$ is Markov[2] *and restricts to a contraction in $L(W^{k,1}(S^1))$. In addition, if $k \geq 1$ then*

$$\lim_{n \to \infty} \|\mathcal{J}_n - \mathrm{Id}\|_{L(W^{k,1}, W^{k-1,1})} = 0. \tag{9.2}$$

When $\mathcal{T} : \Omega \to \mathrm{LY}_k(\alpha, K)$ is measurable and $n \in \mathbb{Z}^+$ we define $\mathcal{L}_{\mathcal{T},n} : \Omega \to L(W^{k-1,1}(S^1))$ by $\mathcal{L}_{\mathcal{T},n}(\omega) = \mathcal{J}_n \mathcal{L}_{\mathcal{T}}(\omega)$. Note that each $\mathcal{L}_{\mathcal{T},n}(\omega)$ has finite rank and preserves the span of $\{e^{2\pi i \ell x} : -n \leq \ell \leq n\}$. Hence, by a constant change of basis we may view $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \mathcal{L}_{\mathcal{T},n})$ as a matrix cocycle on $\mathbb{C}^{2n+1}$. One could then use this matrix representation to approximate the Oseledets splitting and Lyapunov exponents of the original cocycle by computing the singular value decomposition of very large iterates of the matrix cocycle, as in [44] or [84]. While a completely rigorous proof of convergence for such an algorithm is outside of the scope of this thesis, we believe that the following theorem is a substantial step in the direction of such a result.

Theorem 9.1.5. *Fix $k \geq 2$, $\alpha \in (0,1)$ and $K > 0$. If $\mathcal{T} : \Omega \to \mathrm{LY}(\alpha, K)$ is measurable and $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \mathcal{L}_{\mathcal{T}})$ admits a hyperbolic Oseledets splitting*

[2] That is, the positive cone in $W^{k,1}(S^1)$ is invariant under $\mathcal{J}_n$, and $\mathcal{J}_n$ preserves integrals.

of dimension d with $(k-1)\ln\alpha < \mu_{\mathcal{L}_{\mathcal{T}}}$, then there exists N such that if $n > N$ then $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \mathcal{L}_{\mathcal{T},n})$ admits an Oseledets splitting of dimension d. In addition, there exists $c_0, R_0 > 0$ such that each $I_i = (\lambda_{i,\mathcal{T}} - c_0, \max\{\lambda_{i,\mathcal{T}}, \ln(\delta_{1i}R_0)\} + c_0)$, $i \in \{1, \dots, k_{\mathcal{T}}\}$, separates the Lyapunov spectrum of $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{1,1}(S^1), \mathcal{L}_{\mathcal{T},n})$, and the corresponding projections satisfy

$$\forall i \in \{1, \dots, k_{\mathcal{T}}\},\ a.e.\ \omega \in \Omega \quad \operatorname{rank}(\Pi_{I_i, \mathcal{L}_{\mathcal{T},n}}(\omega)) = d_{i,\mathcal{T}},$$

and

$$\sup \left\{ \operatorname*{ess\,sup}_{\omega\in\Omega} \left\| \Pi_{I_i, \mathcal{L}_{\mathcal{T},n}}(\omega) \right\|_{L(W^{k-1,1})} \middle| n > N, 1 \le i \le k_{\mathcal{T}} \right\} < \infty.$$

In addition, for every $\beta > 0$, there exists $N_\beta > N$ such that if $n > N_\beta$ then

$$\sup_{1\le i\le d} \left| \gamma_{i,\mathcal{L}_{\mathcal{T}}} - \gamma_{i,\mathcal{L}_{\mathcal{T},n}} \right| \le \beta,$$

$$\sup_{1\le i\le k_{\mathcal{T}}} \operatorname*{ess\,sup}_{\omega\in\Omega} \left\| \Pi_{I_i, \mathcal{L}_{\mathcal{T}}}(\omega) - \Pi_{I_i, \mathcal{L}_{\mathcal{T},n}}(\omega) \right\|_{L(W^{k-1,1}, W^{k-2,1})} \le \beta,$$

and

$$\operatorname*{ess\,sup}_{\omega\in\Omega} d_H(F_{\mathcal{L}_{\mathcal{T}}}(\omega), F_{\mathcal{L}_{\mathcal{T},n}}(\omega)) \le \beta.$$

Before proving the results described thus far we describe some concrete settings in which they may be applied. We note that the chief difficulty in applying Theorems 9.1.3 and 9.1.5 is not proving the existence of an Oseledets splitting (recall Proposition 9.1.2), but verifying that the splitting is hyperbolic.

Example 9.1.6. Fix $k \ge 2$, $\alpha \in (0,1)$ and $K > 0$. For each $T \in \mathrm{LY}_k(\alpha, K)$ we may consider the constant random dynamical system given by $\omega \mapsto T$. In this case the associated Perron-Frobenius operator $\mathcal{L}_T$ is quasi-compact[3] on $W^{k-1,1}(S^1)$ with $\rho_{\mathrm{ess}}(\mathcal{L}_T) \le \alpha^{k-1} < \rho(\mathcal{L}_T) = 1$. It follows that $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \omega \mapsto \mathcal{L}_T)$ admits a hyperbolic Oseledets splitting with $\mu > \ln\alpha^{k-1}$: for any such $\mu \in (\ln\alpha^{k-1}, 0)$ the fast Oseledets spaces are just direct sums of the eigenspaces of $\mathcal{L}_T$ associated to eigenvalues of modulus greater than e^μ (of which there are finitely many), and the Lyapunov exponents are $\{\ln|\lambda| : \lambda \in \sigma(\mathcal{L}_T), |\lambda| > \ln\mu\}$. We refer the reader to [72] and [99] for examples of expanding maps on S^1 with non-trivial eigenvalues with modulus in $(\alpha^{k-1}, 1)$, and note that the different choice of Banach

[3] We remind the reader of the proof. One bounds the essential spectral radius by using Theorem 6.2.20 and Proposition 9.2.3. Since $\mathcal{L}_T$ preserves integrals we have $\rho(\mathcal{L}_T) \ge 1$. If $\rho(\mathcal{L}_T) > 1$ then $\mathcal{L}_T$ has an eigenvalue of modulus greater than 1 on $W^{k-1,1}(S^1)$, which must also be an eigenvalue for $\mathcal{L}_T$ on $L^1(S^1)$; but $\mathcal{L}_T$ is a contraction on $L^1(S^1)$ and so no such eigenvalue can exist.

space in either paper is inconsequential due to [11, Section A.2]. We may therefore apply Theorem 9.1.3 to $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \omega \mapsto \mathcal{L}_T)$ with perturbation $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \mathcal{L}_\mathcal{S})$ whenever $\mathcal{S} : \Omega \mapsto \mathrm{LY}_k(\alpha, K)$ is measurable and such that $\operatorname{ess\,sup}_{\omega\in\Omega} d_{\mathcal{C}^{k-1}}(\mathcal{S}(\omega), T)$ is sufficiently small.

Example 9.1.7. If, in the setting of Example 9.1.6, there exists $\mu \in (\ln \alpha^{k-1}, 0)$ such that for every $r > \ln \mu$ the set $\{\lambda \in \sigma(\mathcal{L}_T) : |\lambda| = r\}$ contains at most a single element, then every Lyapunov exponent of $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \omega \mapsto \mathcal{L}_T)$ has multiplicity one. Thus, by Remark 8.1.9, it follows that if $\mathcal{S} : \Omega \mapsto \mathrm{LY}_k(\alpha, K)$ is measurable and $\operatorname{ess\,sup}_{\omega\in\Omega} d_{\mathcal{C}^{k-1}}(\mathcal{S}(\omega), T)$ is sufficiently small then the Oseledets splitting for $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \mathcal{L}_\mathcal{S})$ that is produced by Theorem 9.1.3 is hyperbolic. Thus both Theorem 9.1.3 and Theorem 9.1.5 may be applied to $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \mathcal{L}_\mathcal{S})$.

9.2 Proofs

We begin by pursuing the proof of Proposition 9.1.2, which requires some preparatory results. Many of these results are well-known, but do not have full proofs in a collected place and so we reproduce the details here. The next proposition summarises the basic properties of Perron-Frobenius operators associated to maps in $\mathrm{LY}_k(\alpha, K)$.

Proposition 9.2.1. *There exists $C_{k-1,\alpha,K} > 0$ such that for every $T \in \mathrm{LY}_k(\alpha, K)$ and $f \in W^{k-1,1}(S^1)$ we have*

$$\|\mathcal{L}_T f\|_{W^{k-1,1}} \leq \alpha^{k-1} \|f\|_{W^{k-1,1}} + C_{k-1,\alpha,K} \|f\|_{W^{k-2,1}} . \tag{9.3}$$

Hence $\{\mathcal{L}_T : T \in \mathrm{LY}_k(\alpha, K)\}$ is an equicontinuous subset of $L_S(W^{k-1,1})$, where $W^{k-1,1}(S^1)$ has the Saks space structure $(W^{k-1,1}(S^1), \|\cdot\|_{W^{k-1,1}}, \|\cdot\|_{W^{k-2,1}})$.

To prove Proposition 9.2.1 we need the following lemma, whose's proof is an easy exercise and widely known (see e.g. [9, the proof of Theorem 2.5]).

Lemma 9.2.2. *For every $k \in \mathbb{N}$ there exists multinomials $G_{k,\ell} : \mathbb{R}^k \to \mathbb{R}$, $\ell \in \{0, \dots, k\}$, such that for every $T \in \mathrm{LY}_k(\alpha, K)$ and $f \in W^{k,1}$ we have*

$$(\mathcal{L}_T f)^{(k)} = \mathcal{L}_T \left((T')^{-2k} \sum_{\ell=0}^{k} G_{k,\ell}(T', \dots, T^{(k+1)}) \cdot f^{(\ell)} \right) . \tag{9.4}$$

Moreover $G_{k,k}(x_1, \dots, x_{k+1}) = x_1^k$.

The proof of Proposition 9.2.1. For brevity denote $G_{k-1,\ell}(T',\ldots,T^{(k)})$ by $G_{k-1,\ell,T}$. By Lemma 9.2.2 and as $\mathcal{L}_T$ is Markov we have

$$\begin{aligned}\left\|(\mathcal{L}_T f)^{(k-1)}\right\|_{L^1} &= \int \left|\mathcal{L}_T\left((T')^{-2(k-1)}\sum_{\ell=0}^{k-1} G_{k-1,\ell,T}\cdot f^{(\ell)}\right)\right| \,\mathrm{d}\,\mathrm{Leb} \\ &\leq \int \frac{\left|f^{(k-1)}\right|}{|T'|^{k-1}}\,\mathrm{d}\,\mathrm{Leb} + \sum_{\ell=0}^{k-2}\int |T'|^{-2(k-1)}\left|G_{k-1,\ell,T}\right|\left|f^{(\ell)}\right|\,\mathrm{d}\,\mathrm{Leb} \\ &\leq \alpha^{k-1}\left\|f\right\|_{W^{k-1,1}} + \left(\sum_{0=\ell}^{k-1}\alpha^{2(k-1)}D_{k-2,\ell}\left\|G_{k-1,\ell,T}\right\|_{L^\infty}\right)\left\|f\right\|_{W^{k-2,1}},\end{aligned}$$

where $D_{k-2,\ell}$ denotes the norm of the embedding of $W^{k-2,1}(S^1)$ into $W^{\ell,1}(S^1)$. Let

$$C_{k-1,\alpha,K} = 1 + \sup_{T\in \mathrm{LY}_k(\alpha,K)}\left(\sum_{0=\ell}^{k-1}\alpha^{2(k-1)}D_{k-2,\ell}\left\|G_{k-1,\ell,T}\right\|_{L^\infty}\right),$$

and note that $C_{k-1,\alpha,K} < \infty$. Since $\mathcal{L}_T$ is Markov on $L^1(S^1)$ we therefore have

$$\left\|\mathcal{L}_T f\right\|_{W^{k-1,1}} = \left\|\mathcal{L}_T f\right\|_{L^1} + \left\|(\mathcal{L}_T f)^{(k-1)}\right\|_{L^1} \leq \alpha^{k-1}\left\|f\right\|_{W^{k-1,1}} + C_{k-1,\alpha,K}\left\|f\right\|_{W^{k-2,1}},$$

which yields (9.3). Since the embedding of $W^{k-1,1}(S^1)$ into $W^{k-2,1}(S^1)$ is bounded, (9.3) also implies that $\{\mathcal{L}_T : T \in \mathrm{LY}_k(\alpha,K)\}$ is a bounded subset of $L(W^{k-1,1}(S^1))$. For $k > 2$ the same argument shows that $\{\mathcal{L}_T : T \in \mathrm{LY}_k(\alpha,K)\}$ is a bounded subset of $L(W^{k-2,1}(S^1))$, while if $k = 2$ then $\{\mathcal{L}_T : T \in \mathrm{LY}_k(\alpha,K)\}$ is bounded in $L(W^{k-2,1}(S^1))$ since each $\mathcal{L}_T$ is Markov on $W^{k-2,1}(S^1) = L^1(S^1)$. That $\{\mathcal{L}_T : T \in \mathrm{LY}_k(\alpha,K)\}$ is an equicontinuous subset of $L_S(W^{k-1,1}(S^1))$ then follows from Proposition 6.2.18. $\square$

Proposition 9.2.3. *Let $\mathbb{W}^{k-1} = \bigsqcup_{\omega\in\Omega}\{\omega\}\times W^{k-1,1}$. There exists $R_{k-1,\alpha,K} \geq 1$ and $A_{k-1,\alpha,K} > 0$ such that if $\mathcal{T} : \Omega \to \mathrm{LY}_k(\alpha,K)$ is measurable then $\mathcal{L}_{\mathcal{T}} \in \mathcal{LY}(1, A_{k-1,\alpha,K}, \alpha^{k-1}, R_{k-1,\alpha,K}) \cap \mathrm{End}_S(\mathbb{W}^{k-1},\sigma)$.*

Proof. That $\mathcal{L}_{\mathcal{T}} \in \mathrm{End}_S(\mathbb{W}^{k-1},\sigma)$ follows trivially from Proposition 9.2.1. Let

$$R_{k-1,\alpha,K} = \max\{1, \sup\{\left\|\mathcal{L}_T\right\|_{L(W^{k-2,1})} : T \in \mathrm{LY}_k(\alpha,K)\}\},$$

and note that $R_{k-1,\alpha,K}$ is finite by Proposition 9.2.1. By Proposition 9.2.1, for every $f \in W^{k-1,1}(S^1)$, $\omega \in \Omega$ and $n \in \mathbb{Z}^+$ we have

$$\begin{aligned}\left\|(\mathcal{L}_{\mathcal{T}(\sigma^n(\omega))} \circ \cdots \circ \mathcal{L}_{\mathcal{T}(\omega)})f\right\|_{W^{k-1,1}} \leq &\alpha^{k-1} \left\|(\mathcal{L}_{\mathcal{T}(\sigma^{n-1}(\omega))} \circ \cdots \circ \mathcal{L}_{\mathcal{T}(\omega)})f\right\|_{W^{k-1,1}} \\ &+ C_{k-1,\alpha,K} R_{k-1,\alpha,K}^{n-1} \left\|f\right\|_{W^{k-2,1}}.\end{aligned}$$

Iterating the above inequality yields

$$\begin{aligned}&\left\|(\mathcal{L}_{\mathcal{T}(\sigma^n(\omega))} \circ \cdots \circ \mathcal{L}_{\mathcal{T}(\omega)})f\right\|_{W^{k-1,1}} \\ &\qquad \leq \alpha^{n(k-1)} \left\|f\right\|_{W^{k-1,1}} + \frac{C_{k-1,\alpha,K}}{R_{k-1,\alpha,K}} R_{k-1,\alpha,K}^{n} \left(\sum_{j=0}^{n-1} \left(\frac{\alpha^{k-1}}{R_{k-1,\alpha,K}}\right)^j\right) \left\|f\right\|_{W^{k-2,1}} \\ &\qquad \leq \alpha^{n(k-1)} \left\|f\right\|_{W^{k-1,1}} + \frac{C_{k-1,\alpha,K} R_{k-1,\alpha,K}^{n}}{R_{k-1,\alpha,K} - \alpha^{k-1}} \left\|f\right\|_{W^{k-2,1}}.\end{aligned}$$

We obtain the claim upon setting $A_{k-1,\alpha,K} = C_{k-1,\alpha,K}(R_{k-1,\alpha,K} - \alpha^{k-1})^{-1}$. □

The proof of Proposition 9.1.2. To show that $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \mathcal{L}_{\mathcal{T}})$ is a separable strongly measurable random linear system with ergodic invertible base it suffices to the map $\Delta : \mathrm{LY}_k(\alpha, K) \to L(W^{k-1,1})$ defined by $\Delta(T) = \mathcal{L}_T$ is measurable with respect to the Borel σ-algebras on $\mathcal{C}^2(S^1, S^1)$ and $L(W^{k-1,1})$, where the later space is equipped with the strong operator topology. We will do this by showing that Δ is continuous: for every $f \in W^{k-1,1}$ we will show that $\|(\mathcal{L}_T - \mathcal{L}_S)f\|_{W^{k-1,1}} \to 0$ as $d_{\mathcal{C}^k}(T, S) \to 0$. By [10, (C1) of Lemma 2.4], for every $g \in \mathcal{C}^{k-1}(S^1)$ we have $\|(\mathcal{L}_T - \mathcal{L}_S)f\|_{\mathcal{C}^{k-1}} \to 0$ as $d_{\mathcal{C}^k}(T, S) \to 0$. Fix $f \in W^{k-1,1}$ and for each $\epsilon > 0$ let $f_\epsilon \in \mathcal{C}^{k-1}(S^1)$ satisfy $\|f - f_\epsilon\|_{W^{k-1,1}} \leq \epsilon$. Then

$$\begin{aligned}\|(\mathcal{L}_T - \mathcal{L}_S)f\|_{W^{k-1,1}} &\leq \|(\mathcal{L}_T - \mathcal{L}_S)f_\epsilon\|_{W^{k-1,1}} + \|(\mathcal{L}_T - \mathcal{L}_S)(f - f_\epsilon)\|_{W^{k-1,1}} \\ &\leq \|(\mathcal{L}_T - \mathcal{L}_S)f_\epsilon\|_{C^{k-1}} + 2\epsilon \sup_{T \in \mathrm{LY}_k(\alpha,K)} \|\mathcal{L}_T\|_{L(W^{k-1,1})} \\ &\to 2\epsilon \sup_{T \in \mathrm{LY}_k(\alpha,K)} \|\mathcal{L}_T\|_{L(W^{k-1,1})},\end{aligned}$$

as $d_{\mathcal{C}^2}(T, S) \to 0$. The set $\{\mathcal{L}_T : T \in \mathrm{LY}_k(\alpha, K)\}$ is bounded in $L(W^{k-1,1})$ by Proposition 9.2.1, and so we obtain the required claim by sending $\epsilon \to 0$.

We will sketch the proof that $\mathcal{P} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \mathcal{L}_{\mathcal{T}})$ has an Oseledets splitting. We aim to verify the hypotheses of [47, Theorem 2.10] i.e. that the index of compactness $\kappa_{\mathcal{P}}^*$ of $\mathcal{P}$ is less than the maximal Lyapunov exponent

$\lambda^*_{\mathcal{P}}$ of $\mathcal{P}$ (see [47, Definition 2.3]). Our proof of this roughly follows the argument laid out in [47, Lemma 3.16]. Recall from Proposition 9.2.3 that $\mathcal{L}_{\mathcal{T}} \in \mathcal{LY}(1, A_{k-1,\alpha,K}, \alpha^{k-1}, R_{k-1,\alpha,K})$. By [47, Lemma C.5], it follows that $\kappa^*_{\mathcal{P}} \leq \ln \alpha^{k-1} < 0$. On the other hand, since each $\mathcal{L}_{\mathcal{T}(\omega)}$ is Markov, we have

$$\left\|(\mathcal{L}_{\mathcal{T}(\sigma^n(\omega))} \circ \cdots \circ \mathcal{L}_{\mathcal{T}(\omega)})1\right\|_{W^{k-1,1}} \geq \left\|(\mathcal{L}_{\mathcal{T}(\sigma^n(\omega))} \circ \cdots \circ \mathcal{L}_{\mathcal{T}(\omega)})1\right\|_{L^1} = 1,$$

and so $\lambda_{\mathcal{P}} \geq 0$. Thus $\mathcal{P}$ has an Oseledets splitting with $\lambda_{1,\mathcal{T}} = \lambda^*_{\mathcal{P}} \geq 0$ by [47, Theorem 2.10]. We will now show that $\lambda_{1,\mathcal{T}} = 0$. Let $\mathcal{P}' = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{1,1}(S^1), \mathcal{L}_{\mathcal{T}})$, and note that the arguments of the previous paragraph imply that $\mathcal{P}'$ has an Oseledets splitting with $\kappa^*_{\mathcal{P}'} \leq \ln \alpha < \lambda^*_{\mathcal{P}'}$ and that $\lambda^*_{\mathcal{P}'} \geq 0$. However, the Lasota-Yorke inequality obtained for $\mathcal{P}'$ from Proposition 9.2.3 has $R_{k-1,\alpha,K} = 1$, and so

$$\sup_{n \in \mathbb{Z}^+} \sup_{\omega \in \Omega} \left\|(\mathcal{L}_{\mathcal{T}(\sigma^n(\omega))} \circ \cdots \circ \mathcal{L}_{\mathcal{T}(\omega)})\right\|_{L(W^{1,1})} < \infty,$$

which implies that $\lambda^*_{\mathcal{P}'} \leq 0$. Thus $\lambda^*_{\mathcal{P}'} = 0$. In the language of [49, Appendix A], $\mathcal{P}$ is a dense restriction of $\mathcal{P}'$. As $\lambda^*_{\mathcal{P}'} \geq \max\{\kappa^*_{\mathcal{P}'}, \kappa^*_{\mathcal{P}}\}$ by [49, Theorem 37] we have $\lambda^*_{\mathcal{P}'} = \lambda^*_{\mathcal{P}} = 0$. □

Proposition 9.2.4. *There exists $Q_{k,\alpha,K} > 0$ such that for every $S, T \in \mathrm{LY}_k(\alpha, K)$ we have*

$$\|\mathcal{L}_T - \mathcal{L}_S\|_{L(W^{k-1,1}, W^{k-2,1})} \leq Q_{k,\alpha,K} d_{\mathcal{C}^{k-1}}(S, T).$$

Proof. The case where $k = 2$ is known: upon recalling from Example 6.2.13 that $\|f\|_{\mathrm{BV}} = \|f\|_{W^{1,1}}$ for $f \in W^{1,1}$, the result is given by [78, Example 3.1]. We therefore focus on the case where $k > 2$, although we use the $k = 2$ case during our argument. Let $f \in W^{k-2,1}(S^1)$ and $g \in L^\infty(S^1)$. For brevity, if $R \in \mathrm{LY}_k(\alpha, K)$ then we will write $G_{k-2,\ell,R}$ in place of $G_{k-2,\ell}(R', \ldots, R^{(k-1)})$. By Lemma 9.2.2 we have

$$\begin{aligned}
&\int (\mathcal{L}_T f - \mathcal{L}_S f)^{(k-2)} g \,\mathrm{d}\,\mathrm{Leb} \\
&\quad = \int \left(\mathcal{L}_T \left(\sum_{\ell=0}^{k-2} \frac{G_{k-2,\ell,T} \cdot f^{(\ell)}}{(T')^{2(k-2)}} \right) - \mathcal{L}_S \left(\sum_{\ell=0}^{k-2} \frac{G_{k-2,\ell,S} \cdot f^{(\ell)}}{(S')^{2(k-2)}} \right) \right) \cdot g \,\mathrm{d}\,\mathrm{Leb} \\
&\quad = \int (\mathcal{L}_T - \mathcal{L}_S) \left(\sum_{\ell=0}^{k-2} \frac{G_{k-2,\ell,T} \cdot f^{(\ell)}}{(T')^{2(k-2)}} \right) \cdot g \,\mathrm{d}\,\mathrm{Leb} \\
&\qquad + \sum_{\ell=0}^{k-2} \int \left(\frac{G_{k-2,\ell,T}}{(T')^{2(k-2)}} - \frac{G_{k-2,\ell,S}}{(S')^{2(k-2)}} \right) \cdot f^{(\ell)} \cdot g \circ S \,\mathrm{d}\,\mathrm{Leb}\,.
\end{aligned} \tag{9.5}$$

To bound the first term we apply the inequality for $k = 2$, which is valid since $d_{\mathcal{C}^1}(S,T) \leq d_{\mathcal{C}^{k-1}}(S,T)$, yielding

$$\begin{aligned}\int (\mathcal{L}_T - \mathcal{L}_S) \left(\sum_{\ell=0}^{k-2} \frac{G_{k-2,\ell,T} \cdot f^{(\ell)}}{(T')^{2(k-2)}} \right) g \,\mathrm{d}\,\mathrm{Leb} \\ \leq Q_{2,\alpha,K} \left\| g \right\|_{L^\infty} \left(\sum_{\ell=0}^{k-2} \left\| \frac{G_{k-2,\ell,T} \cdot f^{(\ell)}}{(T')^{2(k-2)}} \right\|_{W^{1,1}} \right) d_{\mathcal{C}^{k-1}}(S,T).\end{aligned}$$

If we let $D_{k-1,\ell}$ denote the norm of the embedding of $W^{k-1,1}(S^1)$ into $W^{\ell,1}(S^1)$ and define $Z_{k,\ell,\alpha,K} := Z$ by

$$Z = \sup_{T \in \mathrm{LY}_k(\alpha,K)} \Big((D_{k-1,\ell} + \alpha 2(k-2)K + D_{k-1,\ell+1}) \left\| G_{k-2,\ell,T} \right\|_{L^\infty} + D_{k-1,\ell} \left\| G'_{k-2,\ell,T} \right\|_{L^\infty} \Big),$$

then by the product rule, the definition of $\left\| \cdot \right\|_{W^{1,1}}$ and as $\alpha < 1$ we have

$$\sum_{\ell=0}^{k-2} \left\| \frac{G_{k-2,\ell,T} \cdot f^{(\ell)}}{(T')^{2(k-2)}} \right\|_{W^{1,1}} \leq \sum_{\ell=0}^{k-2} Z_{k,\ell,\alpha,K} \left\| f \right\|_{W^{k-1,1}}.$$

Thus

$$\begin{aligned}\int (\mathcal{L}_T - \mathcal{L}_S) \left(\sum_{\ell=0}^{k-2} \frac{G_{k-2,\ell,T} \cdot f^{(\ell)}}{(T')^{2(k-2)}} \right) g \,\mathrm{d}\,\mathrm{Leb} \\ \leq Q_{2,\alpha,K} \left(\sum_{\ell=0}^{k-2} Z_{k,\ell,\alpha,K} \right) \left\| g \right\|_{L^\infty} \left\| f \right\|_{W^{k-1,1}} d_{\mathcal{C}^{k-1}}(S,T).\end{aligned} \tag{9.6}$$

On the other hand,

$$\begin{aligned}\int \left(\frac{G_{k-2,\ell,T}}{(T')^{2(k-2)}} - \frac{G_{k-2,\ell,S}}{(S')^{2(k-2)}} \right) \cdot f^{(\ell)} \cdot g \circ S \,\mathrm{d}\,\mathrm{Leb} \\ \leq D_{k-1,\ell} \left\| g \right\|_{L^\infty} \left\| f \right\|_{W^{k-1,1}} \left\| \left(\frac{G_{k-2,\ell,T}}{(T')^{2(k-2)}} - \frac{G_{k-2,\ell,S}}{(S')^{2(k-2)}} \right) \right\|_{L^\infty}.\end{aligned}$$

Since each of the multinomials $G_{k-2,\ell}$ is Lipschitz on $[-K,K]^{k-1}$ and $G_{k-2,\ell,T}$ (resp. $G_{k-2,\ell,S}$) only contains derivatives of T (resp. S) of order less than $k-1$, for each $\ell \in \{0, \dots, k-2\}$ there exists $V_{k,\ell}$ such that for every $S, T \in \mathrm{LY}_k(\alpha, K)$ we have

$$\left\| \left(\frac{G_{k-2,\ell,T}}{(T')^{2(k-2)}} - \frac{G_{k-2,\ell,S}}{(S')^{2(k-2)}} \right) \right\|_{L^\infty} \leq V_{k,\ell} d_{\mathcal{C}^{k-1}}(S,T).$$

It follows that

$$\begin{aligned}\sum_{\ell=0}^{k-2}\int\left(\frac{G_{k-2,\ell,T}}{(T')^{2(k-2)}}-\frac{G_{k-2,\ell,S}}{(S')^{2(k-2)}}\right)\cdot f^{(\ell)}\cdot g\circ S\,\mathrm{d}\,\mathrm{Leb}&\\ \leq D_{k-1,\ell}\left(\sum_{\ell=0}^{k-2}V_{k,\ell}\right)\|g\|_{L^\infty}\|f\|_{W^{k-1,1}}\,d_{\mathcal{C}^{k-1}}(S,T).&\end{aligned}\tag{9.7}$$

Applying (9.6) and (9.7) to (9.5), and then taking the supremum over $g\in L^\infty(S^1)$ with $\|g\|_{L^\infty}=1$ yields

$$\left\|(\mathcal{L}_T f-\mathcal{L}_S f)^{(k-2)}\right\|_{L^1}\leq\left(\sum_{\ell=0}^{k-2}D_{k-1,\ell}V_{k,\ell}+Q_{2,\alpha,K}Z_{k,\ell,\alpha,K}\right)\|f\|_{W^{k-1,1}}\,d_{\mathcal{C}^{k-1}}(S,T).$$

Thus, by using the case where $k=2$ again, we obtain

$$\begin{aligned}&\|\mathcal{L}_T-\mathcal{L}_S\|_{L(W^{k-1,1},W^{k-2,1})}\\ &\qquad\leq Q_{2,\alpha,K}D_{k-1,1}d_{\mathcal{C}^{k-1}}(S,T)+\left(\sum_{\ell=0}^{k-2}D_{k-1,\ell}V_{k,\ell}+Q_{2,\alpha,K}Z_{k,\ell,\alpha,K}\right)d_{\mathcal{C}^{k-1}}(S,T),\end{aligned}$$

as required. □

The proof of Theorem 9.1.3. By assumption $(\Omega,\mathcal{F},\mathbb{P},\sigma,W^{k-1,1}(S^1),\mathcal{L}_\mathcal{T})$ is a separable strongly measurable random linear system with ergodic invertible base and a hyperbolic Oseledets splitting of dimension d. We have $\mathcal{L}_\mathcal{T}\in\mathrm{End}_S(\mathbb{W}^{k-1},\sigma)\cap\mathcal{LY}(1,A_{k-1,\alpha,K},\alpha^{k-1},R_{k-1,\alpha,K})$ by Proposition 9.2.3, and so all the requirements of Theorem 8.1.8 are verified for $(\Omega,\mathcal{F},\mathbb{P},\sigma,W^{k-1,1}(S^1),\mathcal{L}_\mathcal{T})$. For measurable $\mathcal{S}:\Omega\to\mathrm{LY}_k(\alpha,K)$ we get that $(\Omega,\mathcal{F},\mathbb{P},\sigma,W^{k-1,1}(S^1),\mathcal{L}_\mathcal{S})$ is also a separable strongly measurable random linear system by Proposition 9.1.2 and by Proposition 9.2.3 we have $\mathcal{L}_\mathcal{S}\in\mathcal{LY}(1,A_{k-1,\alpha,K},\alpha^{k-1},R_{k-1,\alpha,K})$. For any $\epsilon>0$ we may ensure that

$$\operatorname*{ess\,sup}_{\omega\in\Omega}\left\|\mathcal{L}_{\mathcal{T}(\omega)}-\mathcal{L}_{\mathcal{S}(\omega)}\right\|_{L(W^{k-1,1},W^{k-2,1})}\leq\epsilon,$$

by making $d_{k-1}(\mathcal{T},\mathcal{S})$ small and then using Proposition 9.2.4. Thus, we obtain the conclusion of Theorem 8.1.8 for the perturbation $(\Omega,\mathcal{F},\mathbb{P},\sigma,W^{k-1,1}(S^1),\mathcal{L}_\mathcal{S})$, as required. □

Proof of Proposition 9.1.4. It is well known that the Fejér kernels approximate the identity [67, Section 2.2 and 2.5]. Thus $\mathcal{J}_n$ is Markov, as claimed. Since convolution and differentiation commute, for each $f \in W^{k,1}(S^1)$ we have

$$\|\mathcal{J}_n f\|_{W^{k,1}} = \|\mathcal{J}_n f\|_{L^1} + \left\|\mathcal{J}_n(f^{(k)})\right\|_{L^1} \leq \|f\|_{L^1} + \left\|f^{(k)}\right\|_{L^1} = \|f\|_{W^{k,1}},$$

and so $\mathcal{J}_n$ restricts to a contraction in $L(W^{k,1}(S^1))$. For $x, y \in S^1$ define $g_{x,y} : S^1 \to \mathbb{R}$ by

$$g_{x,y} = \begin{cases} \chi_{x-y,x} & y < x, \\ \chi_{x,x-y} & x < y. \end{cases}$$

Using Fubini-Tonelli and the fact that every $f \in W^{1,1}$ is absolutely continuous, we have

$$\begin{aligned}
\|\mathcal{J}_n f - f\|_{L^1} &= \int\int J_n(y) \left|f(x-y) - f(x)\right| \,\mathrm{d}\,\mathrm{Leb}(y) \,\mathrm{d}\,\mathrm{Leb}(x) \\
&\leq \int\int\int J_n(y) g_{x,y}(z) \left|f'(z)\right| \,\mathrm{d}\,\mathrm{Leb}(z) \,\mathrm{d}\,\mathrm{Leb}(y) \,\mathrm{d}\,\mathrm{Leb}(x) \\
&= \left(\int J_n(y) \left|y\right| \,\mathrm{d}\,\mathrm{Leb}(y)\right) \|f\|_{W^{1,1}}.
\end{aligned}$$

Since $\{J_n\}_{n\in\mathbb{Z}^+}$ approximates the identity we have $\int J_n(y) \left|y\right| \,\mathrm{d}\,\mathrm{Leb}(y) \to 0$ as $n \to \infty$, which yields (9.2) for $k = 1$. The claim for $k > 1$ follows from the case where $k = 1$, the fact that differentiation commutes with $\mathcal{J}_n$, and the fact that $W^{k,1}(S^1)$ continuously embeds into $W^{k-1,1}(S^1)$ and $W^{1,1}(S^1)$. □

The proof of Theorem 9.1.5. The proof is very similar to that of Theorem 9.1.3. By assumption $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \mathcal{L}_{\mathcal{T}})$ is a separable strongly measurable random linear system with ergodic invertible base and a hyperbolic Oseledets splitting of dimension $d \in \mathbb{Z}^+$. We have $\mathcal{L}_{\mathcal{T}} \in \mathrm{End}_S(\mathbb{W}^{k-1}, \sigma) \cap \mathcal{LY}(1, A_{k-1,\alpha,K}, \alpha^{k-1}, R_{k-1,\alpha,K})$ by Proposition 9.2.3, and so all the requirements of Theorem 8.1.8 are verified for $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \mathcal{L}_{\mathcal{T}})$. Since the composition of strongly measurable maps is strongly measurable ([47, Lemma A.5]), and the constant map $\omega \mapsto \mathcal{J}_n$ is strongly measurable, for each $n \in \mathbb{Z}^+$ we have that $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{k-1,1}(S^1), \mathcal{L}_{\mathcal{T},n})$ is a separable strongly measurable random linear system. Since $\mathcal{J}_n$ is a contraction on $W^{k-1,1}$, from the Lasota-Yorke inequality (9.3) we have for every $f \in W^{k-1,1}(S^1)$, $n \in \mathbb{Z}^+$ and $\omega \in \Omega$ that

$$\left\|\mathcal{J}_n \mathcal{L}_{\mathcal{T}(\omega)} f\right\|_{W^{k-1,1}} \leq \alpha^{k-1} \|f\|_{W^{k-1,1}} + C_{k-1,\alpha,K} \|f\|_{W^{k-2,1}}.$$

By using the fact that $\mathcal{J}_n$ is a contraction on $W^{k-2,1}(S^1)$ and repeating the argument made in Proposition 9.2.3, we deduce that $\mathcal{L}_{\mathcal{T},n} \in \mathcal{LY}(1, A_{k-1,\alpha,K}, \alpha^{k-1}, R_{k-1,\alpha,K})$ for every $n \in \mathbb{Z}^+$. Thus, after using (9.2) from Proposition 9.1.4 we obtain the conclusion of Theorem 8.1.8 for the perturbation $(\Omega, \mathcal{F}, \mathbb{P}, \sigma, W^{1,1}(S^1), \mathcal{L}_{\mathcal{T},n})$, as required. $\square$

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