Evolutionary method in the design of robust control systems

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Evolutionary Method in the Design of Robust Control Systems

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8 April 2011

Statement of Originality

I hereby declare that this submission is my own work and to the best of my knowledge it contains no materials previously published or written by another person, or substantial proportions of material which have been accepted for the award of any other degree or diploma at UNSW or any other educational institution, except where due acknowledgement is made in the thesis. Any contribution made to the research by others, with whom I have worked at UNSW or elsewhere, is explicitly acknowledged in the thesis. I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in the project's design and conception or in style, presentation and linguistic expression is acknowledged.

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Hendra Gunawan Harno
8 April 2011

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5. H. G. Harno and I. R. Petersen, "Nonlinear robust H^∞ control via a stable decentralized nonlinear output feedback controller," in *Proceedings of the 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, Shanghai, China, December 16–18, 2009, pp. 1638–1644.

6. H. G. Harno and I. R. Petersen, "Decentralized state feedback robust H^∞ control using an evolutionary optimization approach," in *Proceedings of the 2009 European Control Conference*, Budapest, Hungary, August 23–26, 2009, pp. 2887–2892.

7. H. G. Harno and I. R. Petersen, "Robust H^∞ stabilization of a nonlinear uncertain system via a stable nonlinear output feedback controller," in *Proceedings of the 2009 American Control Conference*, St. Louis, Missouri, USA, June 10–12, 2009, pp. 3693–3699.

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Canberra, 8 April 2011
Hendra Gunawan Harno

Abstract

In this thesis, we present new systematic methods to synthesize non-decentralized and decentralized robust feedback control systems for classical and quantum dynamical systems. For the decentralized case, we assume that the interconnections between subsystems are known and thus, we do not treat them as uncertainties. We employ a differential evolution (DE) algorithm to solve nonconvex nonlinear constrained optimization problems arising in the feedback control syntheses for those systems. As a class of evolutionary algorithms, the DE algorithm is equipped with variation operators: mutation and recombination, and selection operator. In addition, we also apply a penalty-based fitness test procedure as a link between the DE algorithm and the particular controller design algorithm being considered.

Regarding classical systems, we are concerned with robust H^∞ control for a class of nonlinear uncertain systems via a stable nonlinear output feedback controller. Structured uncertainties and nonlinearities in the system are required to satisfy integral quadratic constraints and global Lipschitz conditions, respectively. Applying this controller, we aim to achieve closed loop absolute stability with a specified disturbance attenuation level. The controller is constructed using stabilizing solutions to algebraic Riccati equations parameterized by scaling constants associated with the uncertainties and nonlinearities. A decentralized version of this control problem is also considered.

For quantum systems, we deal with coherent quantum feedback control for a class of quantum systems represented in terms of linear quantum stochastic differential equations. Synthesis algorithms are provided to construct physically realizable quantum controllers, which are used to solve quantum entanglement and quantum robust H^∞ control problems. In particular, we are interested in synthesizing a strict bounded real quantum robust H^∞ controller for an uncertain

quantum system. This quantum controller is applied to obtain a strict bounded real closed loop quantum system with a specified disturbance attenuation level. The controller matrices are formed using stabilizing solutions to complex algebraic Riccati equations parameterized by scaling constants corresponding to all uncertainties in the quantum system. The same type of quantum controller is used to solve a decentralized quantum robust H^∞ control problem.

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Chapter 1

Introduction

This chapter is intended not only to describe the background and motivation for all topics discussed in this thesis, but also to highlight the main contributions of our research. That is, we aim to present new systematic methods to synthesize non-decentralized and decentralized robust feedback control systems for both classical and quantum dynamical systems. An introduction to the notions of feedback control considered is thus briefly provided in Section 1.1. Then, a short discussion on evolutionary optimization methods is presented in Section 1.2 where we also describe our motivation for applying this method to solve our control problems. Moreover, specific considerations and features of our main results from Chapter 3 to Chapter 8 are presented in separate sub-sections within Section 1.3.

1.1 Feedback Control

In this section, we present a brief overview of the basic concepts of feedback control. Our purpose is to provide a general understanding of what we are concerned with in our research to develop algorithms to synthesize robust feedback controllers for both classical and quantum dynamical systems.

1.1.1 Basic concepts

Feedback control is a common strategy used to ensure stability and performance when a dynamical system is performing certain tasks. This strategy is implemented in a closed loop scheme as shown in Figure 1.1, where the system being

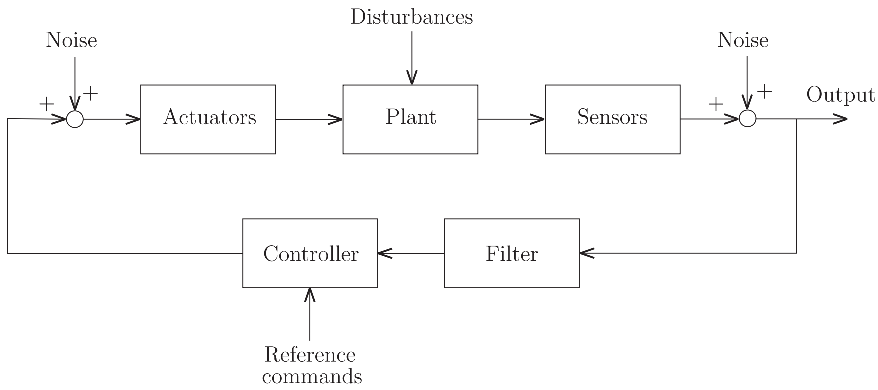


Figure 1.1: Closed loop feedback control; see [1].

controlled (plant) is interconnected and interacts with other systems so that the plant behaves as intended. In this case, sensors are used to fully or partially capture information about the plant dynamics, but unfortunately, the sensor outputs are usually contaminated by noise. We then feed back the acquired information through a filter with which noise effects are removed considerably. Having the filter outputs as controller inputs, we employ an algorithm to generate control commands, which drive the plant via actuators to compensate any deviation from a desired operating point with particular properties.

Often, we can make the plant work well by only applying an open loop control without feedback. This approach, however, is at times unrealistic and risky because the plant may be unstable and is not free from perturbations such as exogenous disturbances, a changing environment and undesirable noise. Another concern is that, due to incomplete knowledge of the plant dynamics and modeling errors, we often have an uncertain plant model based on which a controller is designed. Thus, it is essential to synthesize a feedback control mechanism, which not only achieves prescribed control objectives effectively and efficiently, but also is robust against the uncertainties and perturbations.

Applying feedback control is certainly beneficial to cope with the concerns mentioned above, but we should also be mindful of its potential drawbacks; e.g., see [1, 9]. As shown in Figure 1.1, feedback control usually requires additional components: sensors and filters, which in turn augment the dimensionality and complexity of the closed loop system. It is also inevitable that sensors are accompanied with noise, which reduces measurement accuracy. Moreover, a carelessly designed feedback system may give rise to oscillations and instability. All these issues indicate that feedback cannot be realized arbitrarily and is not merely about having a closing loop. Thus, appropriate systematic methods are necessary to design feedback control systems, which are capable of providing robust stability and performance; e.g., see [10].

1.1.2 Classical feedback control

The idea to build feedback control systems originated for more than twenty centuries ago; e.g., see [11, 12]. One of the first significant inventions in industrial era was a flyball governor, which was successfully used to control the speed of a James Watt's steam engine in the 18th century. Feedback control applications at the present time are much more pervasive and ubiquitous in various technological artifacts such as computers, automobiles, aircraft, robots, telescopes and chemical processes. We can even find feedback control applied to non-engineering systems such as biological systems, ecosystems and economics; e.g., see [1]. We thus could say that feedback control has become an indispensable aspect in technology development; e.g., see [13].

Along with the above applications, feedback control theory has also achieved great advancements, which provide various mathematical methods to rigorously analyze and synthesize feedback control systems. Among others, several well-known controller design methods are proportional-integral-derivative (PID) control, linear quadratic Gaussian (LQG) control and H^∞ control; e.g., see [14–17]. These methods were in fact developed to design feedback controllers for dynamical systems governed by the principles of non-quantum physics. In this sense, we thus consider them as classical feedback control methods.

In this thesis, we apply a linear robust H^∞ control method as described in [2] to synthesize a controller, which absolutely stabilizes an uncertain system with a specified disturbance attenuation level (see Chapter 3 – Chapter 5). The class of uncertain systems being considered is illustrated in Figure 1.2. This model is referred to as a linear fractional transformation (LFT), where a nominal plant is separated from all uncertainties $\phi_1(\cdot), \phi_2(\cdot), \ldots, \phi_k(\cdot)$ in feedback interconnections; e.g., see [18]. In order to be admissible, each uncertainty $\phi_j(\cdot)$ (for all $j = 1, 2, \ldots, k$) is required to satisfy an integral quadratic constraint (IQC):

$$\int_0^\infty \|\xi_j(t)\|^2 \, dt \leq \int_0^\infty \|\zeta_j(t)\|^2 \, dt + d_j, \quad d_j \geq 0 \tag{1.1}$$

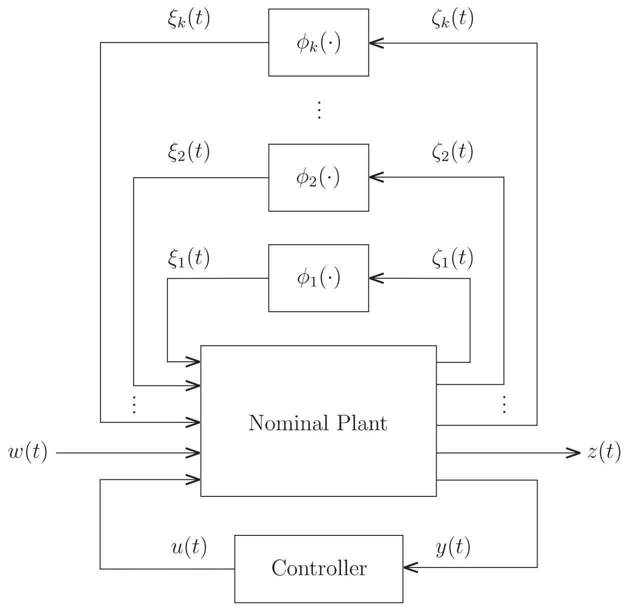


Figure 1.2: An uncertain system with structured uncertainties; see [2].

provided these integrals exist; see [2]. Note that $\|\cdot\|$ is the Euclidean norm.

Characterizing the structured uncertainties in terms of IQCs, we can include not only time-varying, norm bounded uncertainties, but also dynamic nonlinear uncertainties, which may appear due to unmodeled dynamics. In this case, the class of uncertainties satisfying IQCs of the form (1.1) is richer than that of uncertainties satisfying only norm bound conditions. Moreover, a natural stability notion for an uncertain system satisfying IQCs is referred to as absolute stability, which directly ensures the stability of the system trajectories and also implies asymptotic stability; see [2].

From a game viewpoint, a robust H^∞ controller can be view as a *minimizing player*, which provides absolute stability and maintains the closed loop system at a specified performance level in the presence of uncertainties. In contrast, each uncertainty in the plant can be considered as a *maximizing player*, which tends to impair the performance of the closed loop system; see [16]. This perspective allows us to treat the robust control problem as a constrained minimax optimization problem. Moreover, using S-procedure results, we can convert the given constrained optimization problem into an unconstrained optimization problem by applying scaling constants corresponding to the IQCs; see [2]. In this case, the S-procedure is indeed analogous to a Lagrange multiplier method in convex optimization; e.g., see [19]. However, the scaling constants used in the S-procedure are such that there exist stabilizing solutions to algebraic Riccati equations parameterized by the constants. Riccati equations are involved because the unconstrained optimization problem is treated as an H^∞ control problem corresponding to a scaled system. In this regard, the absolute stabilization problem is thus related to the H^∞ control problem; see [2].

An optimal H^∞ controller is the one that minimizes an induced norm $\|T_{wz}\|_\infty$ of a linear map from the disturbance input $w(t)$ to the controlled output $z(t)$ of the closed loop system with zero initial condition (see Figure 1.2). Here, $\|T_{wz}\|_\infty$ is defined as

$$\|T_{wz}\|_\infty := \sup_{w(\cdot)\in L_2[0,\infty), \|w(\cdot)\|_2\neq 0} \frac{\|z(\cdot)\|_2}{\|w(\cdot)\|_2} \tag{1.2}$$

where $\|\cdot\|_2$ is the L_2-norm. Despite its optimality, finding this controller is often difficult and expensive from a numerical perspective; e.g., see [18]. Thus, in practice, we are usually more interested in having a suboptimal controller, which guarantees a certain performance level

$$\|T_{wz}\|_\infty < \gamma, \quad \gamma > 0 \tag{1.3}$$

of the closed loop system; e.g., see [2, 18].

1.1.3 Quantum feedback control

Rigorous theoretical studies on the topic of quantum control started about three decades ago and were documented in some initial publications e.g., [20–22]. These results serve as important references in the quantum control literature and form the foundation for later investigations. Although quantum control is a relatively young discipline, it has set a path for quantum mechanics to move from abstract scientific concepts into real applications. This allows us to employ quantum mechanics as a fundamental framework not only to explain quantum phenomena in our physical world, but also to govern future quantum technology development. Currently, the need for comprehensive theoretical and experimental research on quantum control systems grows rapidly and becomes more compelling. This rising interest is particularly driven by recent advancements in high-precision and nanoscale engineering, which undoubtedly lead us into quantum realm; e.g., see [23, 24].

Quantum feedback control has been acknowledged as a key factor, which will play a significant role in the development of quantum technology and quantum information. Potential applications of quantum control can be found in many areas such as quantum metrology, quantum computation, quantum communication, quantum optical interferometry and quantum electronics; e.g., see [23–26]. Such pervasive applications pose immense challenges to the quantum control discipline to build a unified framework, which can be used in analysis and synthesis of quantum control systems; e.g., see [27–29]. In particular, there has been significant interest in focusing research on developing mathematical models and feedback control methods for linear quantum systems; e.g., see [30–34]. This particular quantum system is commonly found in the field of quantum optics (e.g., see [35–37]), which widely appears in the above applications.

Keeping up with this situation, many researchers have exploited knowledge of classical control theory to formulate systematic methods to analyze and synthesize quantum control systems; e.g., see [38]. In particular, the development of quantum control theory has greatly benefited from the well-established classical stochastic control theory; e.g., see [39]. It has also been found that experience in the classical domain appears to provide a better comprehension of the nature of quantum mechanical problems; e.g., see [24, 40–42]. Although the classical stochastic control is a sensible starting point, it should be noted that this approach is based on classical probability theory, which is considered as a special case of noncommutative or quantum probability theory; e.g., see [43–46].

Unlike a classical stochastic system, the physical variables in a quantum system do not commute. This feature prevents us from having effective and accurate simultaneous observation of a quantum system due to the limitations imposed by Heisenberg's uncertainty principle; e.g., see [41]. Thus, the non-commutative property presents a challenging obstacle if we want to regulate a quantum plant using closed loop feedback control and classical measurements. Also, we cannot perform classical measurements without disturbing the quantum plant and destroying quantum information; e.g., see [25, 40, 47]. Such delicate circumstances then motivate the emergence of quantum measurement and filtering theory, which becomes an essential foundation for developing quantum feedback control methods; e.g., see [33, 46, 48–53].

Despite the fact that a quantum system is non-commutative, the inclusion of classical measurement devices in a quantum feedback control system is often useful as we can still partially observe the dynamics of a quantum plant. Applying classical measurement indeed allows us to exploit the state-of-the-art of the classical control methods to stabilize and provide good performance to the closed loop quantum system; e.g., see [24, 40–42, 47, 54–61]. In this case, the measurement outputs are processed and fed into a classical controller, which can be built using analog or digital electronics. The controller outputs are then applied to the quantum plant through actuators. This approach, in particular, forms an important class of linear quantum control systems involving classical feedback loop; e.g., see [33, 34, 62–65].

To circumvent the quantum mechanical limitations of the system being controlled, we can avoid applying classical measurement to the quantum plant and allow the controller to be a quantum system instead. This approach gives rise to another type of quantum control methods referred to as coherent quantum feedback control. In this case, the quantum controller is directly interconnected with the quantum plant; e.g., see [30, 31, 66–73]. The use of coherent quantum feedback control can be more appealing than classical feedback control in some applications due to its ease of implementation. Also, a coherent quantum controller operates on higher speed and potentially provides better performance than a classical controller; e.g., see [34]. These promising features have raised our interest in developing systematic methods to synthesize two classes of linear quantum controllers. In the first class, we are concerned with constructing a quantum controller, which has the same structure as the quantum LQG controller presented in [72] (see Chapter 6). Then, in the second class, we aim to build a robust quantum controller, which is capable of attenuating exogenous disturbance inputs in the presence of uncertainties in the quantum plant, via an H^∞ control approach as described in [70, 73] (see Chapter 7 and Chapter 8).

To apply the coherent quantum feedback control strategy, physical realizability of the quantum controller is an important concern. This is because the quantum controller is required to exhibit meaningful dynamics at all times according to the principles of quantum mechanics. Thus, the representation of the quantum controller cannot be arbitrary and is not as flexible as that of a classical controller, but has to satisfy physical realizability conditions; e.g., see [70, 72, 73].

Recently, the results in [74–76] provide a systematic method to physically construct a linear quantum controller using serial connections of optical devices such as optical cavities, optical amplifiers, optical squeezers, beam splitters and phase shifters. To apply the method in [74], the quantum controller should be represented by linear quantum stochastic differential equations (QSDEs), which is decomposed into real and complex quadratures as considered in [70, 72]. This implies that the linear QSDEs of the quantum controller will have real coefficients and the order of the quantum controller will be twice that of the quantum plant.

Another approach is given by [77] to physically construct a linear quantum controller using only passive optical elements such as optical cavities, beam splitters and phase shifters. The parameters of such a quantum controller can be synthesized using the method in [73]. Even though the method in [77] is simpler than that in [74], the former is only applicable to a particular class of quantum systems represented by linear QSDEs with complex coefficients and in terms only of annihilation operators; see [73]. Further generalization of the physical realizability conditions in [73] is given by [78], which provides necessary and sufficient conditions for the physical realizability of the linear complex QSDEs written in terms of annihilation and creation operators. These conditions can be related to those in [70] via unitary transformation.

1.2 Evolutionary Method

In this section, we briefly describe the circumstances which lead to the emergence of evolutionary methods as numerical problem solvers. We then discuss general properties of evolutionary algorithms and particular applications of evolutionary methods in feedback control systems for both classical and quantum dynamical systems. Moreover, we also explain our motivation for applying an evolutionary approach in our research.

1.2.1 Background

There has been a long history of numerical method development in order to solve various real-world optimization problems found in, for example, the economy, the environment, transportation, communications and manufacturing industry. As scientific knowledge progresses, we become more able to improve existing methods and to create new computational tool in software and hardware. This increases our opportunity to obtain better solutions with more efficient methods. Also, the introduction of more powerful computers continuously speeds up and increases our capacity for problem solving using optimization methods; e.g., see [79].

Alongside this advancement, real-world optimization problems have escalated in scale and complexity during the last century. This undoubtedly lifts the demand on experts to deliver problem solvers (algorithms) with more reliable, efficient and versatile features. The primary goal is to obtain a good quality solution within a reasonable time frame so as to achieve optimality for the problem being considered. Moreover, parallelism also becomes a rising trend in both the problem and methodology domains. Ultimately, these scenarios pose immense challenges to mathematicians, computer scientists and engineers to yield high performing optimization algorithms; e.g., see [79, 80].

We are hardly able to keep up with the rapid growth in the need for sound problem-specific optimization algorithms because of the time it takes to conduct comprehensive studies; e.g., see [79]. Despite these limitations, a promising breakthrough was offered by the so-called evolutionary algorithm (EA) when its first prototypes appeared in the late 1950s; e.g., see [81, 82]. This algorithm is commonly referred to as a numerical algorithm inspired by the notions of survival of the fittest and genetic evolution in biological science; e.g., see [79, 80, 83–85].

Historically, there were four main streams in the course of early EA development, namely evolutionary programming (EP; see [86]), evolution strategies (ES; see [87, 88]), genetic algorithms (GA; see [89, 90]) and genetic programming (GP; see [91]). A more complete overview of early EA history can be found in e.g., see [85, 92] and the references therein. Despite the differences among these EAs, they have common operators, namely mutation, recombination and selection, which then become the basic ingredients of other variants of EA developed later; e.g., see [79]. These EA operators are usually governed by certain stochastic processes. Thus, EA can be classified as a stochastic search method as shown in Figure 1.3. In addition to those operators, EA also needs a routine to evaluate the fitness of all candidate solutions to a problem under consideration. The outcome of this routine is fitness information associated with each candidate solution and is then exploited by the EA operators to perform their tasks.

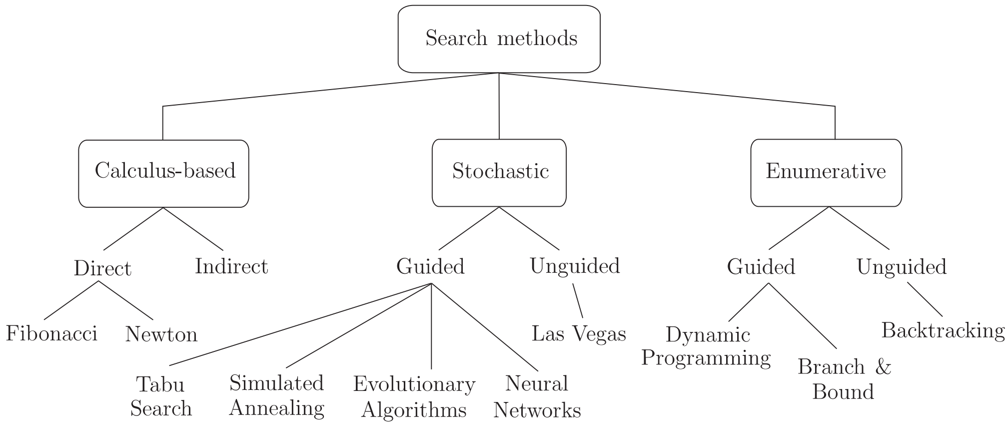


Figure 1.3: Search method classification; see [3].

1.2.2 EA properties

EA becomes attractive because of its favorable features, which may be absent from many analytical and conventional numerical optimization solvers; e.g., see [79, 80, 84, 85, 93]. In the following, we list some of advantageous EA properties:

1. EA accommodates many different types of data structure such as binary, integer, real, symbolic or finite state values. This gives EA a large degree of versatility and flexibility as a solver for various classes of optimization problems.

2. EA works with a population of individuals, which serve as potential candidate solutions to a given problem. These individuals are generated randomly and spread over a search space defined by boundary constraints on the decision variables. This feature liberates EA from dependency on suitable initial points to start its evolution and also enables EA to cope with a large and discontinuous search space. Moreover, operating on a population, EA is then able to provide a set of feasible solutions in one run and is compatible with parallel computing because each individual can be assessed independently.

3. EA is equipped with variation operators: mutation and recombination, which maintain population diversity at a high level. This makes EA much less susceptible to stagnation and premature convergence, especially when EA is used to solve nonconvex problems. Moreover, EA also brings about

competition among candidate solutions, which is triggered by a selection operator based on the fitness information of each individual in the population. In the selection process, EA is allowed to capitalize on low fitness and/or infeasible individuals to guide an evolutionary search toward a desired global solution. This is also useful to increase population diversity, especially at an early stage evolution.

4. EA yields Pareto-optimal solutions when it is used to handle constrained multi-objective problems where there may be conflicting objectives. Moreover, EA can also be implemented together with other numerical methods in order to enhance the performance of both sides as we can exploit good properties of all parties. This will result in a hybrid EA.

In spite of the useful EA characteristics, it is wise not to take them for granted when implementing EA. There are indeed some technical issues to which users should pay attention because they may greatly affect EA behavior. Among other things, these issues concern population diversity, constraint handling and parameter setting, which are left for discussion in Chapter 2. Furthermore, EA usually takes a long computational time to accomplish a whole evolution cycle. This is due to its stochastic evolutionary nature and the fact that all individuals in the population have to undergo fitness evaluation in each cycle. In particular, this concern is critical if EA is applied to real-time applications with fast dynamics. An alternative way to speed up the computation process is by applying parallel computing to which EA is adaptable; e.g., see [85, 94].

1.2.3 Existing EA-based control applications

In the literature, we can find a multitude of EA applications in different fields, including engineering; e.g., see [95–99]. In particular, automatic control engineering has benefited from EA development in the areas of single- and multi-objective optimization; e.g., see [94, 100–103]. Here, we mention some controller synthesis and analysis algorithms based on evolutionary methods as follows:

- Linear control such as PID, LQG and H^∞ control: e.g., see [104–115];
- Nonlinear control such as Lyapunov's direct method, sliding mode, nonlinear dynamic inversion and nonlinear PD control: e.g., see [116–123];

- Stochastic control such as Monte Carlo probabilistic and stochastic traffic control: e.g., see [124–128];
- Adaptive control: e.g., see [129–131];
- Evolutionary control: e.g., see [132–135];
- Quantum control: e.g., see [136–139]

All of these control system algorithms are mainly constructed using ES, GA and GP, which are part of the four major classes of EAs. The pervasive success of EAs in control applications give further confirmation that EAs are reliable and effective in dealing with difficult engineering problems. This obviously becomes an impetus to enhance existing EAs and to find new evolutionary methods with better performance and efficiency. As a result, one of the recent EAs to appear is the differential evolution (DE) algorithm, which was initially designed to solve real-valued problems; e.g., see [4, 140].

Since its first appearance, DE has drawn a lot of attention from EA experts and users because it tends to outperform many preceding EAs. Hence, the DE algorithm immediately became popular as it is also relatively simple and easy to implement. When applied to continuous space optimization problems, DE is a formidable competitor for ES and GA; e.g., see [4]. However, this is not to suggest that DE is the most superior EA for any kind of problem (e.g., see [141]) because, according to *no free lunch* theorem, such an EA simply does not exist; see [142]. A more detail discussion about DE and its components is presented in Chapter 2.

Recently, DE has also been applied to various types of linear and nonlinear control applications for both classical and quantum dynamical systems; e.g., see [143–151]. It is demonstrated in those references that the DE approach is relatively more successful and efficient as compared to other EAs when searching for an optimal real-valued solution. This fact indeed confirms previous empirical studies about the performance of the DE algorithm in comparison with other EAs; e.g., see [4]. Considering the capability of the DE algorithm, we are confident to apply this algorithm to solve optimization problems arising in our research to develop new methods for synthesizing robust feedback controllers.

1.2.4 EA-based robust control methods

We wish to clarify our motivation to develop EA-based algorithms for constructing robust feedback control systems. We begin with specifying the classes of control problems being considered. In our research, we consider both classical (Chapter 3 – Chapter 5) and quantum (Chapter 6 – Chapter 8) dynamical systems. We assume that each dynamical system has structured uncertainties, except the quantum system in Chapter 6.

We refer to the robust H^∞ control methods presented in [2] to synthesize non-decentralized and decentralized robust H^∞ controllers for classical uncertain systems in Chapter 3 – Chapter 5. Using these controllers, we aim to absolutely stabilize the closed loop uncertain system with a specified disturbance attenuation level. In the controller synthesis algorithms, we assign a scaling constant to each structured uncertainty in the uncertain system so that the constrained robust H^∞ control problems can be transformed into the unconstrained standard H^∞ control problems. These formulations then lead to solutions, which are given in terms of stabilizing solutions to algebraic Riccati equations parameterized by the scaling constants.

In Chapter 6, we deal with a class of quantum systems with real coefficient QSDEs as considered in [70, 72]. In this case, we are concerned with stabilizing a closed loop quantum system with a physically realizable linear quantum controller. The physical realizability condition for the quantum controller is then represented by a complex algebraic Riccati equation. We also allow particular control objectives to be incorporated in the quantum controller design algorithm in addition to the closed loop stability and physically realizability requirements.

Meanwhile, in Chapter 7 and Chapter 8, we are concerned with a class of uncertain quantum systems with complex coefficient QSDEs as considered in [73]. Here, we aim to synthesize non-decentralized and decentralized quantum H^∞ controllers, which robustly stabilize the closed loop uncertain quantum systems with a specified disturbance attenuation level. Here, the idea of introducing scaling constants to the structured uncertainties is also applied when designing the quantum controllers. This approach allows us to apply the results in [73] to obtain solutions to the given quantum control problems. The quantum controllers are then constructed in terms of stabilizing solutions to complex algebraic Riccati equations parameterized by the scaling constants. Also, we require the resulting quantum controllers to be physically realizable.

In order to solve the controller design problems presented in Chapter 3 – Chapter 8, we consider them as nonconvex nonlinear constrained optimization problems. From a numerical computation perspective, it is difficult to obtain optimal solutions for such problems due to the functional nonsmoothness inherently arising in our control problems. This issue has been discussed in e.g., [152–154] with an observation that regular linear matrix inequality (LMI) and algebraic Riccati equation techniques are often inapplicable to solve these problems. For some cases, there have been numerical algorithms proposed by [72, 155–157] to solve this type of controller design problems based on a rank constrained LMI approach given in [158]. However, those algorithms tend to result in complicated formulations and strongly depend on a suitable initial point in order to be effective. Furthermore, it has also been shown in [153, 159] that numerous conventional optimization methods based on, for example, quadratic programming (e.g., see [160]), augmented Lagrangian (e.g., see [161, 162]), gradient sampling (e.g., see [163]) and bilinear matrix inequality (e.g., see [164, 165]) techniques often encounter numerical difficulties or even fail when used to handle particular nonconvex H^∞ synthesis problems.

In fact, when dealing with a nonconvex problem, a feasible initial point is often difficult to obtain prior to numerical iteration. This is because we have very little information about a numerical environment where the optimization takes place. Unfortunately, there is no reliable systematic method to find such an initial point that can efficiently direct an iteration process toward an optimal solution. Also, in the approaches mentioned above, we can only evaluate one point as a potential candidate solution in one iteration cycle. If the given point is infeasible, it is immediately discarded and the iteration process is terminated. A new initial point is then required to begin new iteration cycles. Even if a feasible solution is found, premature convergence is likely to happen. In other words, it is very unlikely to escape from a local optimum because the iteration process only relies on information from the current search path. Hence, those approaches tend to have low success rates and often lead to unsatisfactory solutions if they are applied to solve difficult nonconvex problems.

Based on this observation, we are motivated to find another approach, which allow us to solve the controller design problems above with less dependence on a particular initial point and in a more straightforward manner. Having these concerns and considering the properties of EA discussed in sub-Section 1.2.2, we then propose to apply an evolutionary optimization approach to design feedback control systems as described in Chapter 3 – Chapter 8. In particular, we opt to use the DE algorithm, that is drift-free $DE/rand/1/either-or$ (see Chapter 2), as it is simple and reliable in solving constrained optimization problems with real-valued decision variables. Note that our approach can also be applied to other related previous works in [2, 6, 8, 70, 73, 166] for which computational algorithms have yet to be provided.

1.3 Thesis Organization

We now outline the organization of this thesis. In Chapter 2, we briefly discuss general structure and technical issues associated with an EA and basic elements of the DE algorithm. As mentioned previously, robust H^∞ controller synthesis algorithms for classical uncertain systems are presented in Chapter 3 – Chapter 5. In Chapter 6, we discuss the construction of a physically realizable quantum controller for a quantum system represented in terms of real and complex quadratures. Then, in Chapter 7 and Chapter 8, we introduce algorithms for designing physically realizable robust quantum H^∞ controllers for linear complex uncertain quantum systems. Conclusions and potential future work are given in Chapter 9. Moreover, to emphasize the significance of our main results, we describe the background, main ideas and contributions from Chapter 3 to Chapter 8 in the following sub-sections.

1.3.1 Chapter 3: Nonlinear robust H^∞ control

An unstable controller is not desirable in many applications because it is sensitive to actuator and sensor failures and to plant uncertainties and nonlinearities. It may also lead to implementation problems and impair the closed loop system performance in tracking reference signals and rejecting disturbances; e.g., see [167–169]. These concerns have motivated many control system experts to develop algorithms for constructing stable controllers. Such a control problem is usually referred to as a strong stabilization problem.

Here, we mention some relevant previous results which address this particular control problem. In the literature, there have been characterizations of strong stabilizability for both single-input single-output (SISO) and multiple-input multiple-output (MIMO) systems based on, for example, a parity interlacing property (e.g., see [170,171]) and a notion of stable range (e.g., see [172,173]). Also, the inclusion of a strong stabilization requirement into the H^∞ control framework has led to various parameterization techniques; e.g., see [167–169,174–177]. However, they deal only with linear control problems and do not consider robustness issues in controller design.

There is a similar interest in the use of a stable nonlinear controller for robust stabilization of nonlinear uncertain systems. Although many nonlinear controller synthesis methods are available in the literature such as [178–181], they do not necessarily lead to a stable nonlinear controller. This fact has motivated us to propose a new approach to solve a nonlinear robust H^∞ control problem via a stable nonlinear output feedback controller. A related discrete-time approach can be found in [182] for the finite-time horizon case without a controller stability requirement. Also, a method to construct a stable linear robust H^∞ output feedback controller has been presented in [7], but it requires that any known nonlinearity in the system be treated as an uncertainty. In contrast to the latter approach, we do not directly consider the known nonlinearity as an uncertainty in order to obtain a stable nonlinear controller, which can provide better disturbance attenuation performance; e.g., see [6].

In Chapter 3, we consider a class of nonlinear uncertain systems where admissible uncertainties and nonlinearities are described by integral quadratic constraints (IQCs) and global Lipschitz conditions (GLCs), respectively. A system of this type has also been considered in [6] for a guaranteed cost control problem, but without a controller stability requirement. There are two main ideas underlying our approach to synthesize a stable nonlinear controller for this system. Firstly, we modify a standard IQC approach to robust H^∞ control by adding a copy of each known plant nonlinearity to the controller so that it is able to exploit the nonlinearity. This technique has also been used in the designs of linear parameter varying controllers (e.g., see [183]), fault detection systems (e.g., see [184]), nonlinear observers (e.g., see [185]) and nonlinear observer-based controllers (e.g., see [186]). Secondly, all known nonlinearities and their copies are combined into the plant and characterized by extra IQCs derived from the GLCs. Thus, we can apply the method in [187] to achieve an absolutely stable closed loop system with a specified disturbance attenuation level.

In order to guarantee controller stability, we first solve a state feedback control problem and then introduce an additional uncertainty to form an artificial uncertain system. For a certain value of the additional uncertainty, the artificial uncertain system reduces to the original uncertain system. If a suitable controller for the artificial uncertain system exists, then the same controller also solves the absolute stabilization problem with a specified disturbance attenuation level for the original uncertain system. Moreover, for another value of the additional uncertainty, the artificial uncertain system reduces to a specific open loop system such that any suitable controller must be stable. This approach, therefore, provides only sufficient conditions and gives rise to conservatism to some extent due to the additional uncertainty; e.g., see [7].

Our algorithm to construct the stable nonlinear controller involves stabilizing solutions to algebraic Riccati equations parameterized by scaling constants corresponding to all IQCs. With this formulation, we introduce nonconvex nonlinear constraints to the nonlinear H^∞ controller synthesis problem. To find a solution to this problem, we may apply the rank constrained LMI method, which was also used in [157] to construct a stable linear H^∞ controller. However, this approach tends to be complicated and adds more constraints to the original controller design algorithm, which makes the problem even harder to solve. Thus, to obtain a more straightforward solution and to avoid unnecessary complication, we transform this problem into a constrained nonlinear optimization problem and then solve it using an evolutionary optimization method, namely the DE algorithm as given in Chapter 2.

1.3.2 Chapter 4: Decentralized state feedback robust H^∞ control

In real-world control applications, one of the great challenges is how to control a large-scale system efficiently. The large-scale system may have a particular structure and limited communications between subsystems so that it is not always possible to share complete information with the entire control system. This imposes constraints on the controller structure in order to accomplish a control task with a prescribed performance level; e.g., see [188]. A reasonable approach to cope with this situation is to apply a decentralized robust control strategy, which allows us to obtain a control system with reduced complexity. This approach is advantageous from a computation and implementation perspectives. Nevertheless, we should also beware that a simple architecture usually leads to a loss of performance because it relies only on limited (or even inaccurate) information; see [189]. Another important aspect of these problems is to consider how easy it is to implement a decentralized control system; e.g., see [190]. All of these concerns make the problem of designing a decentralized robust controller for a large-scale system an interesting research area.

It is very common to view a large-scale system as consisting of interconnected subsystems and that their interactions are sometimes uncertain. Thus, in many previous results (e.g., see [191–193]), the interconnections between subsystems are considered as sources of uncertainties in addition to uncertainties in each subsystem. Naturally, the main goal is to design a decentralized robust controller that is also capable of alleviating interconnection effects when attaining a prescribed performance level. To achieve this goal, one can take an approach that either ignores or takes into account the interconnections in the controller design; e.g., see [194–196]. In both paradigms, the interconnections are seen as the parties which contribute to the performance degradation of the overall system. A contrasting viewpoint to the previous one is that the interconnection structure may have a positive contribution to the decentralized control system, especially when the number of decentralized controllers is less than the number of subsystems; e.g., see [197].

As the main contribution of Chapter 4, we present a new method for designing a decentralized state feedback robust H^∞ controller, which can exploit the interconnections between subsystems. Here, we assume that the interconnections are fully (or partially) known and hence, we do not treat them as uncertainties. Instead, we neglect off-diagonal blocks in a full state feedback gain matrix and consider them as uncertainties; e.g., see [198]. This approach yields a block-diagonal state feedback control law, which is also robust against perturbations in the controller itself. Thus, this idea is also related to non-fragile controller design methods; e.g., see [199–201].

Here, we are concerned with a class of large-scale linear uncertain systems in which the uncertainties are described by IQCs. This uncertainty description is more general than those in e.g., [202–206]. Thus, we wish to construct a decentralized state feedback robust H^∞ controller that is capable of absolutely stabilizing the closed loop system while achieving a certain disturbance attenuation level. This idea is based on the results in [187] and therefore, the resulting controller design algorithm involves solving an algebraic Riccati equation parameterized by scaling constants. A stabilizing solution to the Riccati equation is then used to construct the decentralized state feedback controller.

The scaling constants are associated with the system uncertainties and norm bounds on the size of the neglected off-diagonal blocks of the controller gain matrix as in [156, 198] for a guaranteed cost control problem. This formulation leads to a nonconvex nonlinear optimization problem, which is often difficult to solve using regular optimization methods. In the approach of [156], a nonconvex optimization problem also arose and was solved using a rank constrained LMI method; see [158]. However, in Chapter 4, we are solving an H^∞ control problem rather than a guaranteed cost control problem and it turns out that the rank constrained LMI approach can no longer be easily applied. Moreover, we may reformulate this optimization problem in terms of bilinear matrix inequality (BMI), which can be solved using an algorithm based on, for example, homotopy methods (e.g., see [207]), cross decomposition (e.g., see, [208]), a separation procedure (e.g., see [209]) and cone complementarity linearization (e.g., see [160, 210]). However, like the rank constrained LMI algorithm, the success of the BMI approach is dependent on feasible initial parameters, which is usually unknown beforehand, to determine a suitable path toward a desired optimal solution. Considering these facts, we thus employ an evolutionary optimization approach, namely the DE algorithm (see Chapter 2), to find an optimal solution to the decentralized controller synthesis problem. In doing so, we need to satisfy nonconvex nonlinear constraints related to the parameterized Riccati equation.

1.3.3 Chapter 5: Decentralized nonlinear robust H^∞ control

There have been a great variety of algorithms proposed to synthesize decentralized robust control systems, which provide sufficient margins of stability and performance robustness for a large-scale dynamical system. The strategy used in each of these algorithms is very much dependent on the information structure available for feedback, the control objective and also perturbations in the system model. These conditions have led to either a linear or a nonlinear decentralized controller design algorithm, which is only suitable to solve a robust control problem for a particular class of large-scale system. Thus, to confine the scope of our discussion, we only focus on a decentralized nonlinear robust H^∞ control problem for a large-scale nonlinear uncertain system where admissible uncertainties and nonlinearities satisfy IQCs and GLCs, respectively; e.g., see [6]. This particular interest originates from what we have done in Chapter 3 and Chapter 4.

A brief survey about some relevant previous research on the topic of decentralized control for large-scale nonlinear uncertain systems is given as follows. Both nonlinear state feedback and output feedback decentralized controllers can be constructed by applying methods such as Lyapunov-based nonlinear control (e.g., see [211, 212]), nonlinear adaptive control (e.g., see [213–215]), nonlinear H^∞ (L_2-gain) control (e.g., see [216,217]), observer-based nonlinear control (e.g., see [218–223]) and sliding mode control (e.g., see [224]). Apart from these nonlinear control techniques, there is also interest in applying linear feedback control to large-scale dynamical systems where the nonlinearities appear only in the interconnections. This latter approach can be applied using methods such as feedback domination with a growth condition (e.g., see [225, 226]), a Luenberger observer-controller synthesized using solutions to algebraic Riccati equations (e.g., see [227]) and LMIs (e.g., see [228,229]), and linear dynamic controllers (e.g., see [230]).

In those references, interconnections between subsystems are mainly considered as uncertainties and assumed to induce performance degradation of the large-scale system. This paradigm leads to a decentralized controller, which is capable of mitigating interconnection effects while achieving a prescribed control objective. This approach is suitable for decentralized control problems with unknown interconnections. Moreover, the stability of the decentralized dynamic controller does not appear in the discussions of the references above and hence, is not taken into account in the proposed controller design algorithms. This indicates that the issue of the controller stability have been less important in the decentralized nonlinear control literature. However, strong stabilization has been considered in some decentralized linear control problems; e.g., see [166, 231, 232].

These observations then inspire us to look at decentralized nonlinear control problems from a different perspective. In this case, we believe that the interconnections can be exploited for control purposes if we have sufficient knowledge of how the subsystems are interconnected. This is beneficial whenever we can only assign direct feedback control action to a limited number of subsystems. Adopting this idea in our approach, we thus do not consider the interconnections as uncertainties, but rather as useful information to facilitate robust stabilization with a certain performance level for the large-scale system (see Chapter 4). In addition, we also include the requirement in Chapter 3 that the decentralized nonlinear controllers must be stable.

Therefore, as the main contribution of Chapter 5, we present a new method to solve a robust H^∞ control problem for a large-scale nonlinear uncertain system using a stable decentralized nonlinear output feedback controller. In this method, the discrepancies between non-decentralized and decentralized controllers constitute a set of nonlinear error systems, which are required to be absolutely stable and considered as additional uncertainties; e.g., see [166]. Moreover, to guarantee the stability of the decentralized controllers, we first solve a state feedback H^∞ control problem and then introduce another additional uncertainty to form an artificial uncertain system. The latter system is used to synthesize an artificial stable output feedback controller with which we construct the decentralized nonlinear controllers. This approach, however, inherently gives rise to some conservatism; e.g., see [7].

A solution to the decentralized nonlinear control problem is given in terms of stabilizing solutions to algebraic Riccati equations, which are parameterized by scaling constants corresponding to all of the uncertainties involved. This brings nonconvexity into play when solving this problem. Thus, a numerical algorithm used to compute the solution has to cope with the nonconvex nonlinear constraints emerging in the proposed method. This motivates us to apply the DE algorithm in Chapter 2 to find an optimal solution to a constrained nonlinear optimization problem arising in our control problem.

1.3.4 Chapter 6: Coherent control of linear quantum systems

In Chapter 6, we are interested in developing a computational method to synthesize a linear coherent quantum feedback controller for a class of linear quantum systems represented in terms of QSDEs with real coefficients; see [70, 72]. The quantum controller has the same structure as a coherent quantum LQG controller in [72] and is required to satisfy the algebraic physical realizability condition given in [72]. Apparently, the coherent quantum control problem with such a constraint is nonconvex. Solving this problem numerically is a challenging task because its solution is often difficult to obtain using conventional optimization methods.

The authors of [72] propose to use a rank constrained LMI approach (see [158]) to synthesize a physically realizable coherent quantum LQG controller. This approach, however, is applied only to solve a relaxed feasibility version of the original quantum control problem. Therefore, the original quantum control problem, which is a nonconvex nonlinear optimization problem, remains to be solved. Moreover, the success of the rank constrained LMI method is strongly dependent on a suitable initial point, which is usually unknown, to begin a numerical iteration. In addition, when dealing with higher order quantum systems, the approach of [72] will lead to excessively complicated rank constrained LMI problems.

These concerns have motivated us to develop a more straightforward algorithm to solve the linear coherent quantum control problem reliably. Thus, as the main contribution of Chapter 6, we propose to use the DE algorithm described in Chapter 2, under which the given quantum control problem is reformulated as a constrained nonlinear optimization problem. In particular, the physically realizable condition is transformed into a complex algebraic Riccati equation, which serves as one of equality constraints. This optimization approach provides us a framework to solve various linear quantum control problems, which include the coherent quantum H^∞ and LQG control as presented in [70, 72]. Thus, we are quite flexible to determine a particular objective function to optimize and the constraints involved.

To demonstrate our DE-based approach, we carry out a case study involving entanglement enhancement for a quantum optical network using a coherent quantum controller. The quantum network consists of two optical parametric amplifiers (OPAs), which are directly interconnected through an optical field and driven by quantum noise. An initial entanglement of this network is then enhanced by applying a coherent quantum controller in place of the optical field interconnecting the two OPAs. This case study is motivated by the results in [233] where the authors successfully increase and preserve the entanglement level of a quantum optical network using both a homodyne detector and a classical feedback control gain. Also, we note the significance of entanglement as a unique physical phenomenon in quantum mechanics, which becomes a fundamental resource required in quantum information processing.

Although the notion of quantum entanglement seems to be less than fully comprehended, it is true that entanglement is useful to increase speed and security of some applications in, for example, quantum cryptography, quantum computation, quantum teleportation and quantum communication; e.g., see [25, 234, 235]. However, entanglement is susceptible to decoherence because of inevitable interaction between a quantum network and its environment. It is indeed a challenging problem to prevent entanglement from decaying or disappearing so that we can exploit entanglement for application purposes; e.g., see [236, 237]. This has given rise to the emergence of various control techniques to generate, preserve and restore entanglement within a quantum network; e.g., see [61, 233, 238–244].

There are many ways to measure the entanglement level depending on properties of the quantum states; e.g., see [245, 246]. In our case, we have Gaussian quantum states, and thus, the entanglement level can be measured in terms of the logarithmic negativity as a function of the covariance matrix of the quantum network; e.g., see [233, 247–249]. The covariance matrix can be obtained as a solution of a Lyapunov equation associated with the quantum network. Then, applying our DE-based algorithm to synthesize the coherent quantum controller, we can consider the logarithmic negativity as an objective function to maximize while satisfying an entanglement criterion. This in turn leads to a physically realizable quantum controller, which is capable of stabilizing the quantum network with an enhanced entanglement level. It should also be noted that the logarithmic negativity is a nonconvex functional; see [250].

1.3.5 Chapter 7: Coherent quantum robust H^∞ control

In practice, it is often assumed that a quantum system is subject to perturbations generated by noises, exogenous disturbances and uncertainties (e.g., model mismatch and unknown dynamics). This situation imposes an essential task on a quantum feedback controller to maintain sufficient margins of stability and performance of the closed loop system in the presence of those perturbations. Thus, as in the classical control theory, robustness against perturbations also becomes a central issue in quantum control studies; e.g., see [64, 70, 251–255]. Taking perturbations into account, we then represent them explicitly in a mathematical model of the quantum system being controlled.

In Chapter 7, we consider coherent quantum robust H^∞ control for a class of linear complex quantum stochastic systems with norm-bounded structured uncertainties. The dynamics of an uncertain quantum system in this class is determined only by annihilation operators and described in terms of QSDEs with complex coefficients; e.g., see [33, 73]. The aim of applying coherent quantum robust H^∞ control strategy is to achieve a strict bounded real closed loop uncertain quantum system with a specified disturbance attenuation level. It is certainly possible to solve this quantum control problem based on the quantum H^∞ control methods presented in [70, 73] by lumping all uncertainties in the quantum system into a single unstructured uncertainty. However, these methods may result in a conservative quantum H^∞ controller, which is not always stable and strict bounded real. Hence, the quantum controller may not be physically realizable.

As the main contribution of Chapter 7, we thus propose a new method to construct a stable and strict bounded real coherent quantum H^∞ controller, which is guaranteed to be physically realizable. The coherent quantum H^∞ controller is of the same order as that of the plant. The underlying main idea of our approach is to introduce an additional uncertainty to form an artificial uncertain quantum system, based on which the desired quantum controller is designed. This idea is taken from the classical control approach in [8]. The additional uncertainty has specific properties that for one particular uncertainty value, the artificial uncertain quantum system reduces to the original uncertain quantum system and thus, any suitable coherent quantum H^∞ controller will also lead to the satisfaction of the H^∞ control objective for the original uncertain quantum system. Also, for another uncertainty value, the artificial uncertain quantum system reduces to a particular open loop configuration such that the coherent quantum H^∞ controller must be stable and strict bounded real, and hence, is physically realizable. Thus, we only provide sufficient conditions to construct such a coherent quantum H^∞

controller. Also, it should be noted that the inclusion of the additional uncertainty will introduce some extra conservatism in the quantum controller design process.

To reduce controller conservatism, we introduce scaling parameters to exploit the structure of the uncertainties. Along with the formulation in [73], this approach leads to a solution to the quantum robust H^∞ control problem, which is given in terms of stabilizing solutions to parameterized complex algebraic Riccati equations. This implies that we have to deal with nonconvex nonlinear constraints, which are difficult to satisfy when solving the given quantum control problem using conventional optimization methods. Thus, we apply the DE algorithm in Chapter 2, which is a reliable way to find an optimal solution to a difficult nonconvex optimization problem.

The class of quantum systems under consideration includes quantum optical systems with purely passive optical elements such as optical cavities, beam-splitters and phase-shifters; e.g., see [73]. Thus, to demonstrate the efficacy of our DE-based algorithm, we consider an example of stabilizing a quantum optical system using a quantum H^∞ controller, which cannot be solved using the method in [73]. We also show that the resulting quantum controller can be physically constructed using an algorithm in [77] as a cascade of n generalized m-mirror cavities using the passive optical components.

1.3.6 Chapter 8: Decentralized coherent quantum robust H^∞ control

Given the current progress of quantum technology development, we would expect large-scale quantum systems in future applications of quantum information and computation; e.g., see [23,234,256]. When a quantum system is of high dimension and complexity, applying a non-decentralized coherent quantum controller may not be feasible due to the high cost of computation and implementation. These issues have also been considered in Chapter 4 and Chapter 5 for the classical decentralized control system. Meanwhile, rigorous investigation on how to control large-scale quantum systems in a decentralized manner is still in its infancy. This situation provides us an opportunity to develop new methods to design a decentralized coherent quantum control system.

In Chapter 8, we consider decentralized coherent quantum robust H^∞ control for a class of large-scale linear complex quantum stochastic systems with norm-bounded structured uncertainties. The dynamics of a quantum system in this class is determined only by annihilation operators and represented in terms of QSDEs with complex coefficients as in Chapter 7. Moreover, the H^∞ control objective is to obtain a closed loop uncertain quantum system, which is strict bounded real with a specified disturbance attenuation level.

The underlying main idea to synthesize the decentralized quantum controller is similar to that in Chapter 4 and Chapter 5. In this case, we view a large-scale quantum system as consisting of interconnected quantum subsystems. We then assume that interconnections between quantum subsystems are known and hence, they are not treated as sources of uncertainties. Instead, we neglect the off-diagonal parts of the transfer function matrix of a non-decentralized quantum controller and consider them as additional uncertainties to the quantum plant; e.g., see [166]. This implies that the non-decentralized quantum controller has to be stable. Moreover, this approach also allows the decentralized quantum controller to exploit the interconnections, which is very useful when the number of controllers is less than that of the subsystems; e.g., see [197].

We then propose two systematic methods to synthesize decentralized quantum controllers. A direct extension of the results in [73] lead to the first method, but it does not immediately yield a physically realizable decentralized quantum robust H^∞ controller. Therefore, we must check the physical realizability of the controller before it can be implemented. We are indeed able to guarantee the decentralized quantum controller to be physically realizable by applying the results in Chapter 7. This approach then gives rise to the second method and hence, yields a stable and strict bounded real decentralized quantum controller. However, the use of additional artificial uncertainty in the second method introduces some extra conservatism to the controller design process. Also, we need more constraints to ensure that the physical realizability condition is satisfied.

In both methods, we apply scaling constants to exploit the structure of the uncertainties involved in the quantum system in order to reduce conservatism of the resulting controller. Thus, solutions to the decentralized quantum control problem are given in terms of stabilizing solutions to parameterized complex algebraic Riccati equations. This formulation leads to nonconvex nonlinear constraints, which are difficult to satisfy when we solve the given control problem.

We then employ the DE algorithm in Chapter 2 to compute all scaling parameters necessary for constructing the decentralized quantum controllers.

To demonstrate the DE-based controller design algorithms, we consider an example of a quantum optical network for each method. We also show that for a particular network, the first method fails to yield a physically realizable decentralized quantum controller, whereas the second method succeeds in doing so. Moreover, using an algorithm in [77], the resulting decentralized quantum controller can be physically constructed as a cascade of n generalized m-mirror cavities using passive optical components such as optical cavities, beam-splitters and phase-shifters.

Chapter 2

Differential Evolution

In this chapter, we present a brief description of the notion of evolutionary algorithms together with some important technical issues. We then provide a short discussion about a particular class of evolutionary algorithm, namely the differential evolution algorithm. The latter is applied to solve various controller design problems for both classical and quantum dynamical systems in subsequent chapters. Moreover, along with the characterization of principal operators of the differential evolution algorithm, we also touch on the issues of constraint handling and parameter setting.

2.1 Evolutionary Algorithm

The evolutionary algorithm (EA) operates on a population of candidate solutions, which is generated randomly according to a certain probability distribution function. Thus, the EA is considered as a population-based stochastic numerical solver. An initial population with sufficient diversity is necessary to begin a numerical evolution because it affects the success of the EA in returning a desirable solution. In general, a candidate solution consists of decision variables, which can be represented in various data formats such as a binary string, a real-valued vector, a symbolic expression, a finite state machine and a tree structure; e.g., see [79, 80]. In some cases, it may also be a mixture of these possibilities; e.g., see [85,257]. It should be noted, however, that not all representations are suitable for a particular EA.

Given a population of candidate solutions, it is natural that they may not fit

Algorithm 2.1 Evolutionary Algorithm

```
Initialization: randomly generate an initial population
Fitness evaluation
while Termination criterion is not satisfied do
    Parent selection
    Recombination
    Mutation
    Fitness evaluation
    Select the next generation candidate solutions
end while
Return
```

equally into the numerical environment defined by the constraints and/or objective function involved. An EA then requires an appropriate fitness assessment as a routine to rate the quality of each candidate solution and as a link to a problem under consideration. The fitness information is very important as it will affect the quality of the new population formed through evolution and selection processes. The higher the fitness of a candidate solution is, the more probable it is chosen to propagate its genetic codes (parameter values) to the next generation.

A new candidate solution (offspring) is produced by recombining two or more candidate solutions (parents) chosen from the current population. There are many ways to choose the parents randomly such as fitness proportional selection, ranking selection, tournament selection and stochastic universal sampling; e.g., see [79]. Due to the random selection, it is then possible that one of the parents has a poor fitness. However, this is beneficial in the early stage evolution as it keeps the population diversity high so as to avoid premature convergence. Also, an offspring may undergo mutation where its genetic codes are perturbed randomly. Both recombination and mutation follow particular probability distribution functions. Having recombined and mutated the population, we also need to assess the fitness of its members. It is logical to expect that the offspring will have a better fitness rate as compared to their parents. Thus, recombination and mutation is usually referred to as an adaptation process to match with the environment; e.g., see [79]. Now, on the basis of their fitness quality, all members in the parent population have to compete with those in the offspring population in order to survive and be selected as members of a new population. The evolution processes are repeated until a termination criterion is satisfied. A general pseudocode of EA is illustrated in Algorithm 2.1; e.g., see [79, 258, 259].

2.1.1 Evolutionary algorithm and optimization

One of the most popular EA applications is to solve an optimization problem with constraints; e.g., see [258, 260–262]. In general, this problem can be formulated as follows (e.g., see [19, 263, 264]): Find an optimal solution $\vartheta^\star$ to solve

$$\min_{\vartheta} \mathsf{f}(\vartheta) \tag{2.1}$$

subject to

$$\begin{aligned} \mathsf{g_j}(\vartheta) &= 0, \quad \text{for } \mathsf{j} = 1, 2, \ldots, \mathsf{a}; \\ \mathsf{h_k}(\vartheta) &\leq 0, \quad \text{for } \mathsf{k} = 1, 2, \ldots, \mathsf{b} \end{aligned} \tag{2.2}$$

where $\vartheta \in \mathbf{C}^n$; $\mathsf{f} : \mathbf{C}^n \to \mathbf{R}$ is an objective function to be minimized; $\mathsf{g_j} : \mathbf{C}^n \to \mathbf{C}^{p\times q}$ is an equality constraint function; and $\mathsf{h_k} : \mathbf{C}^n \to \mathbf{C}^{r\times s}$ is an inequality constraint function. Here, a and b are the total number of equality and inequality constraints, respectively. In practice, the optimization problem (2.1), (2.2) does not always have nice properties from a numerical perspective so that this problem is considered as a hard problem to solve.

The EA is widely used to solve various difficult optimization problems (from a conventional perspective) with a satisfactory success rate. This is particularly true when we are dealing with an optimization problem under unfavorable circumstances such as a non-differentiable and time-varying objective function, a disjoint and nonconvex feasible space, a multimodal and noisy landscape, highly nonlinear and discontinuous constraints and an uncertain model; e.g., see [79, 80, 92, 94, 265, 266]. Although a rigorous mathematical analysis of EA is difficult to obtain (e.g., see [79,85,267]), it is often empirically evident that the EA is capable of delivering a good solution. This is often acceptable when obtaining a global optimal solution is arduous and computationally difficult, if not impossible, using, for example, fixed-point methods, bracketing methods, gradient-based methods or even stochastic direct search methods; e.g., see [79, 80, 261].

In real-world applications, the EA is recognized as a flexible and versatile approach toward an optimization problem at hand (e.g., see [79, 80, 94]), which is also evident in our results from Chapter 3 to Chapter 8. These characteristics are beneficial for practitioners who might have to deal with a complex problem without an appropriate mathematical formulation and yet, a reasonably good solution is required. Hence, the EA is a legitimate choice, especially when a proven exact solver does not exist. On the other hand, if we are solving a convex optimization problem and an efficient problem-specific method is available, then applying an EA is not recommended as it is very unlikely for the EA to return a better result; e.g., see [79].

Considering recent advancements in computational technology (software and hardware), we can further exploit the capabilities and properties of EA when implemented through parallel computing for solving a large-scale optimization problems with high complexity; e.g., see [3, 102, 268–271]. If such complex problems can be divided into several sub-problems and efficient exact methods are available for some of them, it is possible to combine an EA with those methods in order to obtain a better solution. This combination then leads to a hybrid EA; e.g., see [272, 273]. Another celebrated application of EA is for solving multi-objective optimization problems; e.g., see [269, 271, 274, 275]. In practice, we may find that one objective function is in conflict with other ones. In this case, the EA will result in a so-called Pareto front, which is a set of all non-dominated solutions defining the best trade-off among all objective functions.

2.1.2 Technical issues

Despite the favorable properties and versatility of EA, it has some technical issues about which we need to be cautious. If we fail to anticipate their influence, EA performance will be degraded. This indicates that EA implementation is by no means trivial, although it is acknowledged as a generic numerical solver. Such concerns are worth considering no matter which type of EA is used. Here, we mention several critical aspects by which the performance of EA is affected:

1. It is important to ensure that the initial population has sufficient diversity, which the recombination and mutation operators should maintain during the numerical evolution. If the initial population lacks of this diversity, we could find that the EA converges prematurely due to stagnation in a small search space. Conversely, an overly large population size and/or search space is used at the expense of computational time; e.g., see [80, 257].

2. Since *a priori* information about a search space is often not available, we should aim for a balance between exploration and exploitation. Otherwise, the search process could either be too long due to an extensive exploration over the search space or be trapped in a local optimum due to an intense exploitation in the neighborhood of some good candidate solutions. Despite their importance, it is not immediately clear which operators of an EA are responsible for these tasks. However, in general, many users would agree that exploration is done by the recombination and mutation operators, while exploitation is done by the selection operator; e.g., see [85, 276].

3. Regarding the stochastic nature of EA, defining suitable probability distribution functions to generate an initial population and to govern the recombination, mutation and selection operators is somewhat delicate. A careless choice of these functions and their parameters can lead to a poor solution or even no solution at all. Therefore, it is important to have an appropriate representation of the distribution functions because the recombination and mutation characteristics are related to the data structure of the candidate solutions; e.g., see [80, 84, 277].

4. Many practical problems are inherently constrained. If an EA is applied to solve such problems, it should be realized that the EA machinery does not anticipate how to handle the constraints properly. This concern is crucial as it determines how we examine the fitness of each candidate solution. Hence, a large number of direct and indirect constraint handling techniques have been proposed to restore and to improve EA capabilities in the presence of the constraints; e.g., see [260, 261, 278, 279].

5. The EA performance robustness may be sensitive to the choice of EA parameters such as the population size, the recombination probability, the mutation probability and the number of iteration cycles. It is indeed a time-consuming effort to find a proper set of parameters before starting the numerical evolution. This has become an impetus to the emergence of various parameter control techniques, which change the EA parameters dynamically when the EA is running; e.g., see [280–282].

There are also reciprocal relationships among those issues so that it is necessary to address them effectively at a minimum expense. As it is not easy to find a comprehensive solution to these problems, we need to be mindful regarding the advantages and disadvantages of any technique proposed in the EA literature. In this regard, *no free lunch* principle is at work because no EA performs equally well on every optimization problem; see [142]. However, this opens an opportunity for EA enhancement and development.

2.2 Differential Evolution Algorithm

Solving a general global optimization problem with real-valued variables remains a challenging topic, especially because the size and complexity of such a problem are steadily increasing. We often find that this problem is extremely difficult to solve using conventional optimization methods. In this situation, a direct search approach often turns out to be a useful technique used to find a global optimal solution. In particular, the EA has demonstrated its reliability as one type of stochastic direct search methods.

Among early generation EAs, evolution strategies and genetic algorithms have probably been the most widely used algorithms to solve the real-valued optimization problems, often with remarkable success. Yet, a requirement for a simpler and more powerful EA undoubtedly remains. Thus, in 1995, Kenneth V. Price and Rainer M. Storn promoted a new EA referred to as the differential evolution (DE) algorithm; e.g., see [140]. This invention was motivated by some requirements found in practical minimization tasks as follows:

1. Capable of dealing with ill-conditioned cost functionals;
2. Suitable for parallel computation to solve large problems on an affordable time scale;
3. Only needs a small number of EA parameters, which are easy to choose;
4. Less sensitive to EA parameter variation;
5. Have a reliable convergence behavior.

The DE algorithm is a self-reliant EA as it only employs information from within an initial population to start a numerical evolution. More precisely, it does not rely on any prescribed probability distribution function to drive its machinery, namely the recombination, mutation and selection operators. The DE algorithm only has three parameters, that is, the population size N_P, the mutation factor F and the recombination rate C_R; e.g., see [140]. It is quite simple to choose the values of those parameters and the inventors also claimed that the DE algorithm has a good robustness against their variation. However, later investigations showed that this claim is not accurate and there are indeed some issues regarding their roles in driving the search process; e.g., see [283, 284].

One way to assess the merit of DE is done by comparing its performance with that of other stochastic algorithms, which are assigned to solve a given global optimization problem. It has been reported in [140] that the DE algorithm outperforms other stochastic global optimization methods (e.g., simulated annealing (see [285]) and stochastic differential equations (see [286])) and EAs (e.g., breeder genetic algorithm (see [287]) and an evolutionary algorithm with soft genetic operators (see [288])) when solving some benchmark unconstrained optimization problems. We may also find empirical results in e.g., [4, 289, 290] which show that the DE algorithm exhibits higher success rate and more efficient, robust, accurate and consistent as compared to, for instance, random search algorithm, evolution strategies, particle swarm optimization and genetic algorithms.

Also, there have been investigations into the DE behavior when it is applied to handle constrained optimization problems with nonlinear constraints, mixed-variables and multiple objective cost functions; e.g, see [291–296]. The conclusions are that the DE algorithm is reliable, competitive and more likely to return better results in a consistent manner than those given by other EAs. However, when the DE algorithm is applied to optimize noisy cost functionals, it does not perform as well as a standard EA, which incorporates stochastic properties in its design; e.g., see [141]. Also, many experiments have been done to compare the performance of different DE variants; e.g., see [297–299]. These results offer useful guidance for users to recognize the strengths and weaknesses of the DE algorithm.

The structure of a standard DE algorithm is consistent with the one in Algorithm 2.1. The core DE components, constraint handling and parameter setting are briefly described in the following sub-sections. Moreover, a detailed account on the DE algorithm and its applications can be found in [4].

2.2.1 Population

A candidate solution $\vartheta_{i,j}$ is defined as a D-dimensional vector of real-valued decision variables, which is modeled as follows:

$$\vartheta_{i,j} := \begin{bmatrix} \vartheta_{i,j,1} & \vartheta_{i,j,2} & \cdots & \vartheta_{i,j,D} \end{bmatrix}^T \tag{2.3}$$

where i is the population index ($i = 1, 2, \ldots, G$) and j is the individual index ($j = 1, 2, \ldots, N_P$). The k-th element of $\vartheta_{i,j}$ in (2.3) is confined to an interval

$$L_k \leq \vartheta_{i,j,k} \leq U_k, \quad \text{for } k = 1, 2, \ldots, D \tag{2.4}$$

where L_k is a lower bound and U_k is an upper bound. An initial population of size N_P is generated randomly and distributed uniformly within a search space defined by the lower and upper bounds on all decision variables. Thus, the k-th element of an initial candidate solution $\vartheta_{1,j}$ is given as

$$\vartheta_{1,j,k} = \alpha_{j,k} \left(U_k - L_k\right) + L_k, \quad \forall j, k \tag{2.5}$$

where $\alpha_{j,k}$ is a uniformly distributed random number in $[0, 1]$.

Each individual $\vartheta_{i,j}$ of the i-th population is referred to as a *target* vector. Applying the mutation and recombination operators to a target population, the DE algorithm produces a new potential candidate solution called the *trial* vector. Both the target and trial vectors are then involved in a competition in order to survive as a member of the next generation population. The competition is determined by the selection operator based on the fitness of both vectors.

2.2.2 Mutation and recombination

The DE mutation and recombination on each individual $\vartheta_{i,j}$ can be carried out element-wise. These operations depend on two parameters: the mutation factor $F \in [0, 1]$ and the recombination rate $C_R \in [0, 1]$. Thus, a trial vector $\xi_{i,j}$ is formed according to the following rule:

$$\xi_{i,j,k} = \begin{cases} \vartheta_{i,a,k} + F\left(\vartheta_{i,b,k} - \vartheta_{i,c,k}\right), & \text{if } \beta_{i,j,k} \leq C_R \text{ or } k = \gamma_{i,j,k}; \\ \vartheta_{i,j,k}, & \text{otherwise} \end{cases} \tag{2.6}$$

where a, b, c are random indexes sampled from $\{1, 2, \ldots, N_P\}$ and $j \neq a \neq b \neq c$; $\beta_{i,j,k}$ is a uniformly distributed random number in $[0, 1]$; and $\gamma_{i,j,k}$ is a random index sampled from $\{1, 2, \ldots, D\}$. It is possible that $F > 1$, but this tends to reduce the convergence rate and result in a less reliable solution; see [4]. Moreover, if $\xi_{i,j,k}$ exceeds the lower bound L_k or the upper bound U_k, we regenerate $\xi_{i,j,k}$ as follows:

$$\xi_{i,j,k} = \begin{cases} \vartheta_{i,a,k} + \zeta_{i,j,k}\left(U_k - \vartheta_{i,a,k}\right), & \text{if } \xi_{i,j,k} > U_k\,; \\ \vartheta_{i,a,k} + \zeta_{i,j,k}\left(L_k - \vartheta_{i,a,k}\right), & \text{if } \xi_{i,j,k} < L_k \end{cases} \tag{2.7}$$

where $\zeta_{i,j,k}$ is a uniformly distributed random number in $[0, 1]$.

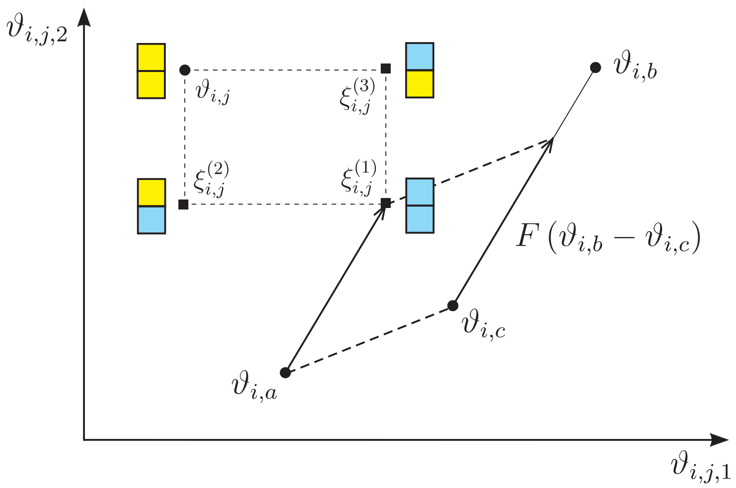


Figure 2.1: The DE mutation and recombination with two-dimensional $\vartheta_{i,j}$ result in three potential trial vectors: $\xi_{i,j}^{(1)}$, $\xi_{i,j}^{(2)}$ or $\xi_{i,j}^{(3)}$; e.g., see [4].

In the rule (2.6), recombination occurs if $\beta_{i,j,k} \leq C_R$ or $k = \gamma_{i,j,k}$. That is, the k-th element $\xi_{i,j,k}$ of the j-th trial vector equals the k-th mutated element $[\vartheta_{i,a,k} + F\left(\vartheta_{i,b,k} - \vartheta_{i,c,k}\right)]$. In this case, $\vartheta_{i,a,k}$ is the k-th element of the a-th base vector $\vartheta_{i,a}$, which is perturbed by the scaled k-th element $F\left(\vartheta_{i,b,k} - \vartheta_{i,c,k}\right)$ of the difference vector $(\vartheta_{i,b} - \vartheta_{i,c})$. We introduce $\gamma_{i,j,k}$ in (2.6) in order to prevent a trial vector $\xi_{i,j}$ from inheriting all elements of a target vector $\vartheta_{i,j}$. Thus, at least one element of $\xi_{i,j}$ is different from those of $\vartheta_{i,j}$. This process is illustrated in Figure 2.1 for two-dimensional mutation and recombination.

The DE algorithm with the mutation and recombination scheme in (2.6) is considered as a standard variant denoted by $DE/rand/1/bin$ (see Algorithm 2.2). In this variant, '*rand*' means that $\vartheta_{i,a}$ is chosen randomly; '1' means that only one difference vector $(\vartheta_{i,b} - \vartheta_{i,c})$ is required for mutation; and '*bin*' means that

Algorithm 2.2 Standard DE algorithm: $DE/rand/1/bin$

Parameter inputs: $N_P, F, C_R, D, G, L_k, U_k$
Initial population: $\vartheta_{1,j,k} = \alpha_{j,k}\left(U_k - L_k\right) + L_k, \quad \forall j, k$
Fitness evaluation of the initial population
Population index: $i = 1$
while $i \leq G$ **do**
    for $j = 1$ to N_P **do**
        Mutation & recombination:
        for $k = 1$ to D **do**
            Random sample: $a, b, c \in \{1, 2, \ldots, N_P\}$ and $j \neq a \neq b \neq c$
            $\xi_{i,j,k} = \begin{cases} \vartheta_{i,a,k} + F\left(\vartheta_{i,b,k} - \vartheta_{i,c,k}\right), & \text{if } \beta_{i,j,k} \leq C_R \text{ or } k = \gamma_{i,j,k}; \\ \vartheta_{i,j,k}, & \text{otherwise} \end{cases}$
            if $\xi_{i,j,k} > U_k$ **then**
                $\vartheta_{i,a,k} + \zeta_{i,j,k}\left(U_k - \vartheta_{i,a,k}\right)$
            end if
            if $\xi_{i,j,k} < L_k$ **then**
                $\vartheta_{i,a,k} + \zeta_{i,j,k}\left(L_k - \vartheta_{i,a,k}\right)$
            end if
        end for
        Fitness evaluation of the i-th trial population
        Select the next generation candidate solutions:
        $\vartheta_{i+1,j} = \begin{cases} \xi_{i,j}, & \text{if } \mathsf{f}(\xi_{i,j}) \leq \mathsf{f}(\vartheta_{i,j}) \\ \vartheta_{i,j}, & \text{otherwise} \end{cases}$
    end for
    $i = i + 1$
end while
Return

a binomial recombination is used. In general, this shorthand notation takes the form of $DE/x/y/z$; see [140]. Here, x denotes how $\vartheta_{i,a}$ is chosen; y denotes the number of difference vectors $(\vartheta_{i,b} - \vartheta_{i,c})$ required for mutation; and z denotes which type of recombination is applied.

2.2.3 Selection

The standard DE algorithm in [140] only aims to handle an unconstrained optimization problem. Thus, a criterion to select the trial vector $\xi_{i,j}$ in favor of the target vector $\vartheta_{i,j}$ (or vice versa) to be a member $\vartheta_{i+1,j}$ of a next generation population is as follows:

$$\vartheta_{i+1,j} = \begin{cases} \xi_{i,j}, & \text{if } \mathsf{f}(\xi_{i,j}) \leq \mathsf{f}(\vartheta_{i,j}) \\ \vartheta_{i,j}, & \text{otherwise} \end{cases} \tag{2.8}$$

where $\mathsf{f}(\cdot)$ is an objective function (to be minimized) of the optimization problem being considered.

If the DE algorithm is applied to solve a constrained optimization problem (e.g., see [292, 294–296, 300, 301]), we then select the trial vector $\xi_{i,j}$ in favor of the target vector $\vartheta_{i,j}$ whenever the following selection criteria are satisfied:

1. $\xi_{i,j}$ is a feasible candidate solution, while $\vartheta_{i,j}$ is an infeasible one; or
2. $\mathsf{f}(\xi_{i,j}) \leq \mathsf{f}(\vartheta_{i,j})$, when both $\xi_{i,j}$ and $\vartheta_{i,j}$ are feasible; or
3. $\xi_{i,j}$ has a smaller number of constraint violations and/or a lower cost due to the violations than $\vartheta_{i,j}$ does, when both $\xi_{i,j}$ and $\vartheta_{i,j}$ are infeasible.

The feasibility of a candidate solution informs us about its fitness with respect to all constraints involved in the optimization.

2.2.4 Constraint handling

When applying the DE algorithm to solve a constrained optimization problem, it is important to pay attention to the constraint handling task as it is related to the selection operator; e.g., see [292]. There are many ways to accomplish this task in order to obtain one or more optimal solution(s); e.g., see [279,300,301]. Through a constraint handling scheme, we acquire information about the fitness of each candidate solution with respect to all constraints. For our controller synthesis application, we employ an indirect constraint handling technique by counting the number v of constraint violations and assigning a penalty function $\mathsf{p}(\cdot)$ to an infeasible candidate solution, where $\mathsf{p} : \mathbf{C}^n \to \mathbf{R}$. The function $\mathsf{p}(\cdot)$ is associated with a constraint $\mathsf{g_j}(\cdot) \leq 0$ or $\mathsf{h_k}(\cdot) = 0$ in (2.2), which is directly violated by a target vector $\vartheta_{i,j}$ or a trial vector $\xi_{i,j}$.

In particular, we apply a static penalty function, which has the following form:

$$\mathsf{p}(\sigma) = (\mathsf{q}(\sigma))^{\mathsf{s}} \tag{2.9}$$

where σ can be a target vector $\vartheta_{i,j}$ or a trial vector $\xi_{i,j}$; and $\mathsf{s} \geq 1$ is a power constant; e.g., see [95, 259, 279]. The function $\mathsf{q}(\cdot)$ is not necessarily the same as $\mathsf{g_j}(\cdot)$ or $\mathsf{h_k}(\cdot)$, but it can be another function, which represents a desired property of $\mathsf{g_j}(\cdot)$ or $\mathsf{h_k}(\cdot)$. In this case, $\mathsf{q}(\cdot)$ can also be interpreted as a metric, which penalizes the distance from an infeasible target vector $\vartheta_{i,j}$ or trial vector $\xi_{i,j}$ to the violated constraint. Moreover, if there is no constraint violation ($\mathsf{v} = 0$), we have

$$\mathsf{p}(\sigma) = \mathsf{f}(\sigma) \tag{2.10}$$

where $\mathsf{f}(\cdot)$ is the objective function in (2.1).

This penalty-based approach is simple, but it involves an extra parameter: power constant s, which may affect the overall DE performance. Thus, in practice, we can take $\mathsf{s} = 2$ or $\mathsf{s} = 3$, because a large s may lead to a high pressure selection process. This is not beneficial during early stage numerical evolution since it tends to reduce population diversity and cause exploration-exploitation imbalance. With these concerns in mind, we realize that there are always advantages and disadvantages associated with each constraint handling technique; e.g., see [261,279]. Note that since the objective function $\mathsf{f}(\sigma)$ in (2.10) (see Chapter 3 – Chapter 8) may also involve power constants, we then have the same concern about them as in the case of the power constant s for $\mathsf{q}(\sigma)$ in (2.9).

We also assume that a constraint violation at a lower level implies those at the higher levels. This assumption is made in order to have an efficient fitness test routine. In fact, the levels in this routine are introduced because we have some technical assumptions to satisfy for each particular controller design method presented in Chapter 3 – Chapter 8.

2.2.5 Drift-free mutation and recombination

As explained in [4, 5], $DE/rand/1/bin$ with $0 < C_R < 1$ is not rotational invariant because its performance is dependent on the orientation of the coordinate system where the candidate solution $\vartheta_{i,j}$ is defined. Thus, the positions of some potential trial vectors will be rotated when rotational misalignment occurs between the coordinate system of the candidate solution $\vartheta_{i,j}$ and that in which an optimization problem is defined (see Figure 2.2). As *a priori* information about the coordinate system orientation is generally unavailable, this drawback may

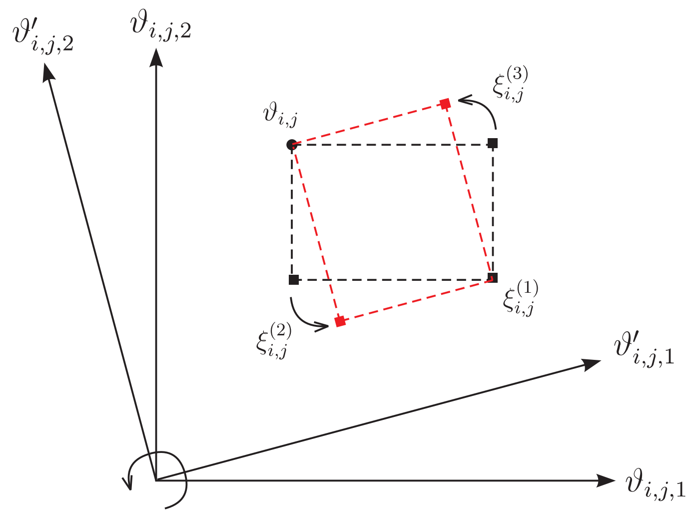


Figure 2.2: For $0 < C_R < 1$, the potential trial vectors $\xi^{(2)}_{i,j}$ and $\xi^{(3)}_{i,j}$ are rotated due to misalignment between the DE coordinates $(\vartheta_{i,j,1}, \vartheta_{i,j,2})$ and the problem coordinates $(\vartheta'_{i,j,1}, \vartheta'_{i,j,2})$; e.g., see [5].

cause $DE/rand/1/bin$ to drift and return a biased solution or no solution at all.

Considering the illustration in Figure 2.2, we notice that $DE/rand/1/bin$ is rotational invariant only when $C_R = 1$, which implies that the numerical variation is only determined by mutation without recombination. In this case, we only have one potential trial vector $\xi^{(1)}_{i,j}$ whose position does not change due to rotation. Setting $C_R = 1$, however, may degrade the performance of $DE/rand/1/bin$ when this algorithm is applied to solve a multimodal optimization problem; e.g., see [4]. Thus, a remedy to the shortcoming of $DE/rand/1/bin$ is given by an alternative DE algorithm: $DE/rand/1/either-or$, which is rotational invariant; see [4]. In the latter algorithm, the standard mutation-binomial recombination scheme in (2.6) is replaced by the mutation-arithmetic recombination scheme:

$$\xi_{i,j} = \begin{cases} \vartheta_{i,j} + F_1 \delta_{b,c}, & \text{if } \ \eta_{i,j} \leq C_M; \\ \vartheta_{i,j} + F_2 \left(\delta_{b,c} - 2\vartheta_{i,j}\right), & \text{otherwise} \end{cases} \tag{2.11}$$

where $\delta_{b,c} := \vartheta_{i,b} - \vartheta_{i,c}$; $F_1, F_2 \in [0,1]$; and $\eta_{i,j}$ is a uniformly distributed random number in $[0,1]$. Note that the recombination rate C_R in (2.6) is also replaced by the mutation rate $C_M \in [0,1]$ in (2.11).

Nevertheless, further investigation on $DE/rand/1/either-or$ shows that this algorithm does not have a center-symmetric distribution, which also causes

Algorithm 2.3 Enhanced DE algorithm: Drift-Free $DE/rand/1/either - or$

Parameter inputs: $N_P, F, C_M, D, G, L_k, U_k$
Initial population: $\vartheta_{1,j,k} = \alpha_{j,k}\left(U_k - L_k\right) + L_k, \quad \forall j, k$
Fitness evaluation of the initial population
Population index: $i = 1$
while $i \leq G$ **do**
  for $j = 1$ to N_P **do**
    Mutation & recombination:
    Random sample: $a, b, c, d \in \{1, 2, \ldots, N_P\}$, $j \neq a \neq b$ and $j \neq c \neq d$
    $\xi_{i,j} = \begin{cases} \vartheta_{i,j} + F\delta_{b,c}, & \text{if } \eta_{i,j} \leq C_M; \\ \vartheta_{i,j} + \sqrt{D}\left(\delta_{b,c} \cdot \varepsilon_{c,d}\right)\varepsilon_{c,d}, & \text{otherwise} \end{cases}$
    for $k = 1$ to D **do**
      if $\xi_{i,j,k} > U_k$ **then**
        $\vartheta_{i,a,k} + \zeta_{i,j,k}\left(U_k - \vartheta_{i,a,k}\right)$
      end if
      if $\xi_{i,j,k} < L_k$ **then**
        $\vartheta_{i,a,k} + \zeta_{i,j,k}\left(L_k - \vartheta_{i,a,k}\right)$
      end if
    end for
    Fitness evaluation of the i-th trial population
    Select the next generation candidate solutions:
    $\vartheta_{i+1,j} = \begin{cases} \xi_{i,j}, & \text{if } \mathsf{f}(\xi_{i,j}) \leq \mathsf{f}(\vartheta_{i,j}) \\ \vartheta_{i,j}, & \text{otherwise} \end{cases}$
  end for
  $i = i + 1$
end while
Return

drift in the numerical variation; see [5]. To cope with this drawback, another modification is done to the arithmetic recombination in (2.11). This results in a drift-free $DE/rand/1/either - or$ (see Algorithm 2.3) in which the trial vector $\xi_{i,j}$ is formed according to the following rule (see [5]):

$$\xi_{i,j} = \begin{cases} \vartheta_{i,j} + F\delta_{b,c}, & \text{if } \eta_{i,j} \leq C_M; \\ \vartheta_{i,j} + \sqrt{D}\left(\delta_{b,c} \cdot \varepsilon_{c,d}\right)\varepsilon_{c,d}, & \text{otherwise} \end{cases} \tag{2.12}$$

where a, b, c, d are random indexes sampled from $\{1, 2, \ldots, N_P\}$; $j \neq a \neq b$; $j \neq c \neq d$; and

$$\varepsilon_{c,d} := \frac{\vartheta_{i,c} + \vartheta_{i,d} - 2\vartheta_{i,j}}{\|\vartheta_{i,c} + \vartheta_{i,d} - 2\vartheta_{i,j}\|}. \tag{2.13}$$

Note that $(\delta_{b,c} \cdot \varepsilon_{c,d})$ is a dot product of $\delta_{b,c}$ and $\varepsilon_{c,d}$. The scheme in (2.12) is immune against drift bias due to coordinate rotation and a distribution, which is not center-symmetric; see [5].

2.2.6 Parameter setting

The setting of the DE parameters (N_P, F, C_R or C_M) is important because it affects the robustness, efficiency and convergence rate of the DE algorithm. Extensive studies on this issue have been conducted in e.g., [283,284,298,302,303]. An inappropriate choice of the DE parameters can lead to inadequate population diversity, stagnation, premature (or slow) convergence and inaccuracy. These are critical risks which may happen due to poor parameter setting. Many approaches, therefore, have been proposed to facilitate DE parameter tuning and parameter control; e.g., see [304–309]. They aims at reducing the occurrence probability of those risks while maintaining high probability of success of the DE algorithm.

To set the DE parameter values for the drift-free $DE/rand/1/either-or$, we follow a strategy proposed in [5]. That is,

$$F = 1; \quad C_M = 0.5; \quad N_p = k\left[(1 - C_M)D^2 + 2C_M D\right] \tag{2.14}$$

where $k \in \{2, 4, 8, \ldots, D\}$. Note that the strategy in (2.14) is presented in [5] as a result of empirical studies about how the variations of the DE parameters (N_P, D, F, C_M) affect the number of function evaluations per success. Applying this strategy, we can start with a small k, and if the results of the numerical evolution are not good enough, we can increase the value of $k \leq D$. If this does not lead to satisfactory results either, we can gradually reduce the value of C_M and then F. Also, we fix the number of iteration cycles G prior to running the DE algorithm as a stopping criterion. Besides this procedure, it is also possible to determine the values of those parameters adaptively as presented in e.g., see [305, 307–310]. This approach allows the DE parameter values and stopping criterion to vary dynamically according to the evolution progress of the population.

Chapter 3

Nonlinear Robust H^∞ Control via a Stable Nonlinear Output Feedback Controller

3.1 Introduction

This chapter aims to present a systematic method to design a stable nonlinear robust H^∞ output feedback controller for a class of nonlinear uncertain systems. The admissible uncertainty and nonlinearity in the system are required to satisfy an integral quadratic constraint and a global Lipschitz condition, respectively; e.g., see [2, 311]. Applying such controller, we aim to obtain an absolutely stable closed loop nonlinear uncertain system with a specified disturbance attenuation level.

There are two underlying main ideas in our method to solve this control problem. Firstly, we include a copy of each known nonlinearity of the plant in the controller in order to enable the controller to exploit the plant nonlinearity (see [6]). Secondly, we use a state feedback gain matrix and introduce an additional uncertainty used to form an artificial uncertain system (see [7]). The purpose of this construction is to guarantee that any suitable output feedback controller for the artificial uncertain system is always stable and also solves the original absolute stabilization problem. The resulting controller matrices are given in terms of the stabilizing solutions of parameterized algebraic Riccati equations.

We propose to apply an evolutionary optimization method, namely the differential evolution (DE) algorithm, to compute all required controller design parameters. This is motivated by the fact that our control problem is subject to a set of both nonconvex and nonlinear constraints, which is often difficult to solve using regular optimization methods.

3.2 Problem Statement

A nonlinear robust H^∞ control problem is presented in this section, which begins by describing a nonlinear uncertain system under consideration and the notions of admissible uncertainty and absolute stabilizability. Additional notation is also defined in order to transform the given nonlinear control problem into a standard form based on the approach in [6].

3.2.1 System description and definitions

We are concerned with a class of nonlinear uncertain systems, which can be represented as follows:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) + \sum_{s=1}^{f} E_{1,s}\xi_s(t) + \sum_{i=1}^{g} E_{2,i}\mu_i(t); \\
z(t) &= C_1 x(t) + D_{12} u(t); \\
\zeta_1(t) &= H_{1,1} x(t) + G_{1,1} u(t); \\
&\vdots \\
\zeta_f(t) &= H_{1,f} x(t) + G_{1,f} u(t); \\
\nu_1(t) &= H_{2,1} x(t) + G_{2,1} u(t); \\
&\vdots \\
\nu_g(t) &= H_{2,g} x(t) + G_{2,g} u(t); \\
y(t) &= C_2 x(t) + D_{21} w(t) + \sum_{s=1}^{f} F_{1,s}\xi_s(t) + \sum_{i=1}^{g} F_{2,i}\mu_i(t). \qquad (3.1)
\end{aligned}$$

The variables involved in the state equations (3.1) are the state $x \in \mathbf{R}^n$, the control input $u \in \mathbf{R}^m$, the disturbance input $w \in \mathbf{R}^p$, the controlled output $z \in \mathbf{R}^q$, the measurement output $y \in \mathbf{R}^l$, the uncertainty input $\xi_s \in \mathbf{R}^{r_s}$, the uncertainty output $\zeta_s \in \mathbf{R}^{h_s}$ (for $s = 1, 2, \ldots, f$), the nonlinearity input $\mu_i \in \mathbf{R}$ and the nonlinearity output $\nu_i \in \mathbf{R}$ (for $i = 1, 2, \ldots, g$). All coefficient matrices in (3.1) are assumed to have compatible dimensions with those of the signals.

The relation between each nonlinearity input $\mu_i(t)$ and nonlinearity output $\nu_i(t)$ is given by the following scalar-valued nonlinear function $\psi_i(\cdot)$:

$$\mu_i(t) = \psi_i\left(\nu_i(t)\right), \quad \forall i = 1, 2, \ldots, g \tag{3.2}$$

where each nonlinear function $\psi_i(\cdot)$ is required to satisfy $\psi_i(0) = 0$. The nonlinear function $\psi_i(\cdot)$ is assumed to be known and is required to satisfy the global Lipschitz condition (GLC)

$$|\psi_i(\nu(t)) - \psi_i(\tilde{\nu}(t))| \leq \beta_i |\nu(t) - \tilde{\nu}(t)|, \quad \beta_i > 0 \tag{3.3}$$

for all pairs of $(\nu(t), \tilde{\nu}(t))$ and $i = 1, 2, \ldots, g$. Meanwhile, each structured uncertainty in the system (3.1) is described as follows:

$$\xi_s(t) = \phi_s\left(t, \zeta_s(t)\right), \quad \forall s = 1, 2, \ldots, f \tag{3.4}$$

where $\phi_s(\cdot)$ may be considered as a nonlinear time-varying and dynamic functional; see [2]. A structured uncertainty is said to be admissible if it satisfies the integral quadratic constraint (IQC) stated in the following definition:

Definition 3.1. (*Integral Quadratic Constraint*; e.g., see [2].) An uncertainty of the form (3.4) is an admissible uncertainty for the system (3.1) if the following conditions hold: Given any locally square integrable control input $u(\cdot)$ and locally square integrable disturbance input $w(\cdot)$, and any corresponding solution to the system (3.1), (3.2), (3.4), let $(0, t_\star)$ be the interval on which the solution exists. Then there exist constants $d_{1,1} \geq 0, \ldots, d_{1,f} \geq 0$ and a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \to t_\star$, $t_k \geq 0$ and

$$\int_0^{t_k} \|\xi_s(t)\|^2 dt \leq \int_0^{t_k} \|\zeta_s(t)\|^2 dt + d_{1,s} \tag{3.5}$$

for all k and for all $s = 1, 2, \ldots, f$. Here, $\|\cdot\|$ denotes the standard Euclidean norm and $\mathbf{L}_2[0, \infty)$ denotes the Hilbert space of square integrable vector valued functions defined on $[0, \infty)$. Note that t_k and $t_\star$ may be equal to infinity. The class of all such admissible uncertainties $\xi(\cdot) = [\xi_1(\cdot), \ldots, \xi_f(\cdot)]$ is denoted by Ξ.

Considering the nonlinear uncertain system (3.1), (3.2), (3.5), we are concerned with solving a problem of absolute stabilization with a specified level of disturbance attenuation in the presence of uncertainties and nonlinearities. Besides closed loop system stability and performance, we are also interested in achieving controller stability. This requirement is motivated by the fact that a stable controller is preferable in many applications because it is much less susceptible to sensor and actuator failures, and to system uncertainties and exogenous disturbances when it is implemented in a real system; e.g., see [7, 167–169]. This particular control problem is often referred to as a strong stabilization problem. Thus, to solve this control problem, we apply a stable nonlinear output feedback controller, which can be represented as follows:

$$\begin{aligned}
\dot{x}_c(t) &= A_c x_c(t) + B_c y(t) + \sum_{i=1}^{g} L_i \tilde{\mu}_i(t); \quad x_c(0) = x_{c_0};\\
u(t) &= C_{c_1} x_c(t);\\
\tilde{\nu}_1(t) &= C_{c_{2,1}} x_c(t);\\
&\vdots\\
\tilde{\nu}_g(t) &= C_{c_{2,g}} x_c(t)
\end{aligned} \tag{3.6}$$

where

$$\tilde{\mu}_i(t) = \psi_i\left(\tilde{\nu}_i(t)\right), \quad \forall i = 1, 2, \ldots, g. \tag{3.7}$$

The presence of (3.7) in the controller state equations (3.6) is to include a copy of each known nonlinearity (3.2) into the controller as depicted in Figure 3.1(a). This is to enable the controller to exploit each known nonlinearity of the nonlinear uncertain system (3.1), (3.2), (3.5).

To solve the problem of constructing a stable nonlinear controller of the form (3.6), (3.7) for the nonlinear uncertain system (3.1), (3.2), (3.5), we use the approaches presented in [7] and [187]. Thus, we first need to reformulate the state equations (3.1) by incorporating the nonlinearities (3.7) into the plant description such that

$$\tilde{y}(t) = \begin{bmatrix} y(t) \\ \tilde{\mu}(t) \end{bmatrix}; \quad \tilde{u}(t) = \begin{bmatrix} u(t) \\ \tilde{\nu}(t) \end{bmatrix}; \tag{3.8}$$

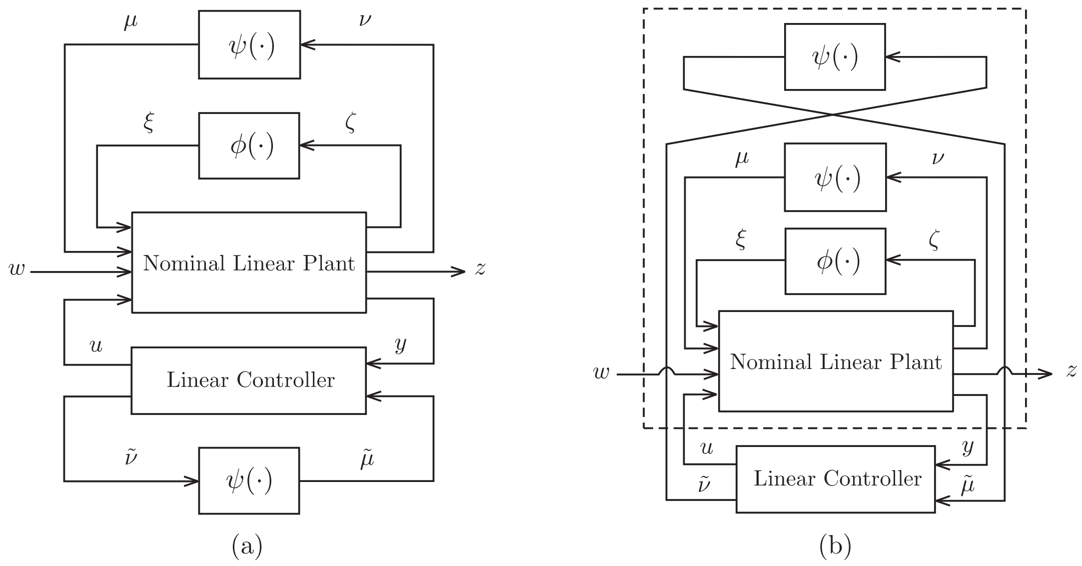


Figure 3.1: (a) Nonlinear uncertain system with nonlinear controller. (b) Nonlinear uncertain system and linear controller with repeated nonlinearity. Here, $\psi(\cdot)$ is a known nonlinearity and $\phi(\cdot)$ is an uncertainty; see [6].

$$\tilde{\mu}(t) = \begin{bmatrix} \tilde{\mu}_1(t) \\ \vdots \\ \tilde{\mu}_g(t) \end{bmatrix}; \quad \tilde{\nu}(t) = \begin{bmatrix} \tilde{\nu}_1(t) \\ \vdots \\ \tilde{\nu}_g(t) \end{bmatrix}; \quad \tilde{C}_c = \begin{bmatrix} C_{c_1} \\ C_{c_{2,1}} \\ \vdots \\ C_{c_{2,g}} \end{bmatrix};$$
$$\tilde{B}_c = \begin{bmatrix} B_c & L_1 & \cdots & L_g \end{bmatrix}. \tag{3.9}$$

and obtain a configuration as shown in Figure 3.1(b). Indeed, this step has also been considered in the controller design method presented in [6] (but without imposing the controller stability requirement). Then, using the expression in (3.8) and (3.9), we can rewrite the controller state equations (3.6) as

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + \tilde{B}_c \tilde{y}(t); \\ \tilde{u}(t) &= \tilde{C}_c x_c(t). \end{aligned} \tag{3.10}$$

This indicates that the problem of controlling the nonlinear uncertain system (3.1), (3.2), (3.5) using the nonlinear controller (3.6), (3.7) is transformed into that of controlling the nonlinear uncertain system (3.1), (3.2), (3.5), (3.7) using the linear controller (3.10). The purpose of applying such controller is to achieve an absolutely stable closed loop nonlinear uncertain system with a specified disturbance attenuation level.

The notion of absolute stabilizability for the nonlinear uncertain system (3.1), (3.2), (3.5) is defined as follows:

Definition 3.2. (*Absolute stabilizability*; e.g., see [2].) The nonlinear uncertain system (3.1), (3.2), (3.5) is said to be absolutely stabilizable with disturbance attenuation level $\gamma > 0$ via a stable nonlinear output feedback controller (3.6), (3.7) if there exists constants $c_1 > 0$ and $c_2 > 0$ such that the following conditions hold:

1. For any initial condition $(x(0), x_c(0))$, any admissible uncertainty inputs $\xi_1(\cdot), \ldots, \xi_f(\cdot)$ and any disturbance input $w(\cdot) \in \mathbf{L}_2[0,\infty)$, we have
$$[x(\cdot), x_c(\cdot), u(\cdot), \xi_1(\cdot), \ldots, \xi_f(\cdot)] \in \mathbf{L}_2[0,\infty) \tag{3.11}$$
(hence, $t_\star = \infty$) and
$$\begin{aligned} &\|x(\cdot)\|_2^2 + \|x_c(\cdot)\|_2^2 + \|u(\cdot)\|_2^2 + \sum_{s=1}^{f} \|\xi_s(\cdot)\|_2^2 \\ &\leq c_1 \left[\|x(0)\|^2 + \|x_c(0)\|^2 + \|w(\cdot)\|_2^2 + \sum_{s=1}^{f} d_{1,s} \right]. \end{aligned} \tag{3.12}$$

2. The following H^∞ norm bound condition is satisfied: If $x(0) = 0$ and $x_c(0) = 0$, then for $w(\cdot) \in \mathbf{L}_2[0,\infty)$ and $\xi_s(\cdot) \in \Xi$ (for all $s = 1, 2, \ldots, f$)
$$\mathcal{J} := \sup_{w(\cdot)} \sup_{\xi_s(\cdot)} \frac{\|z(\cdot)\|_2^2 - c_2 \sum_{s=1}^{f} d_{1,s}}{\|w(\cdot)\|_2^2} < \gamma^2. \tag{3.13}$$

Here, $\|q(\cdot)\|_2$ denotes the $\mathbf{L}_2[0,\infty)$ norm of a function $q(\cdot)$. That is, $\|q(\cdot)\|_2^2 := \int_0^\infty \|q(t)\|^2 dt$.

3.2.2 Robust H^∞ control

As we intend to apply the results of [7] and [187], the nonlinearities (3.2) and their copies (3.7) need to be characterized using IQCs. For this purpose, we refer to the GLCs (3.3), which imply that

$$\begin{aligned}
(\mu_i(t) - \tilde{\mu}_i(t))^2 &\leq \beta_i^2 (\nu_i(t) - \tilde{\nu}_i(t))^2 ; \\
(\mu_i(t))^2 &\leq \beta_i^2 (\nu_i(t))^2 ; \\
(\tilde{\mu}_i(t))^2 &\leq \beta_i^2 (\tilde{\nu}_i(t))^2
\end{aligned} \tag{3.14}$$

for all $i = 1, 2, \ldots, g$. It then follows from (3.14) that we have the following IQCs

$$\begin{aligned}
&\int_0^{t_k} (\mu_i(t) - \tilde{\mu}_i(t))^2 \, dt \leq \int_0^{t_k} \beta_i^2 (\nu_i(t) - \tilde{\nu}_i(t))^2 \, dt + d_{2,i}; \\
&\int_0^{t_k} (\mu_i(t))^2 \, dt \leq \int_0^{t_k} \beta_i^2 (\nu_i(t))^2 \, dt + d_{3,i}; \\
&\int_0^{t_k} (\tilde{\mu}_i(t))^2 \, dt \leq \int_0^{t_k} \beta_i^2 (\tilde{\nu}_i(t))^2 \, dt + d_{4,i},
\end{aligned} \tag{3.15}$$

which are to be satisfied for all $i = 1, 2, \ldots, g$; and for all $\{t_k \geq 0\}_{k=1}^{\infty}$. We note that $d_{2,i} \geq 0$, $d_{3,i} \geq 0$ and $d_{4,i} \geq 0$. The extra IQCs (3.15) impose more constraints in addition to those in (3.5). However, the description of the system state equations (3.1) can be simplified as follows:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + \tilde{B}_2 \tilde{u}(t) + \sum_{s=1}^{\tilde{f}} \tilde{E}_s \tilde{\xi}_s(t); \\
z(t) &= C_1 x(t) + \tilde{D}_{12} \tilde{u}(t); \\
\tilde{\zeta}_1(t) &= \tilde{H}_1 x(t) + \tilde{G}_1 \tilde{u}(t); \\
&\vdots \\
\tilde{\zeta}_{\tilde{f}}(t) &= \tilde{H}_{\tilde{f}} x(t) + \tilde{G}_{\tilde{f}} \tilde{u}(t); \\
\tilde{y}(t) &= \tilde{C}_2 x(t) + \tilde{D}_{21} w(t) + \sum_{s=1}^{\tilde{f}} \tilde{F}_s \tilde{\xi}_s(t)
\end{aligned} \tag{3.16}$$

where $\tilde{f} = f + 2g$; $h = \sum_{s=1}^{f} h_s$;

$$\tilde{\xi}(t) = \begin{bmatrix} \xi(t) \\ \mu(t) \\ \tilde{\mu}(t) \end{bmatrix}; \quad \tilde{\zeta}(t) = \begin{bmatrix} \zeta(t) \\ \nu(t) \\ \tilde{\nu}(t) \end{bmatrix}; \quad \xi(t) = \begin{bmatrix} \xi_1(t) \\ \vdots \\ \xi_f(t) \end{bmatrix}; \quad \zeta(t) = \begin{bmatrix} \zeta_1(t) \\ \vdots \\ \zeta_f(t) \end{bmatrix}; \tag{3.17}$$

$$\mu(t) = \begin{bmatrix} \mu_1(t) \\ \vdots \\ \mu_g(t) \end{bmatrix}; \quad \nu(t) = \begin{bmatrix} \nu_1(t) \\ \vdots \\ \nu_g(t) \end{bmatrix}; \quad \tilde{B}_2 = \begin{bmatrix} B_2 & 0_{n\times g} \end{bmatrix}; \quad \tilde{D}_{12} = \begin{bmatrix} D_{12} & 0_{q\times g} \end{bmatrix};$$

$$\tilde{E} = \begin{bmatrix} \tilde{E}_1 & \dots & \tilde{E}_{\tilde{f}} \end{bmatrix} = \begin{bmatrix} E_{1,1} & \cdots & E_{1,f} & E_{2,1} & \cdots & E_{2,g} & 0_{n\times g} \end{bmatrix}; \; \tilde{C}_2 = \begin{bmatrix} C_2 \\ 0_{g\times n} \end{bmatrix};$$

$$\tilde{F} = \begin{bmatrix} \tilde{F}_1 & \dots & \tilde{F}_{\tilde{f}} \end{bmatrix} = \begin{bmatrix} F_{1,1} & \cdots & F_{1,f} & F_{2,1} & \cdots & F_{2,g} & 0_{l\times g} \\ 0_{g\times r_1} & \cdots & 0_{g\times r_f} & 0_{g\times 1} & \cdots & 0_{g\times 1} & I_{g\times g} \end{bmatrix};$$

$$\tilde{H} = \begin{bmatrix} \tilde{H}_1 \\ \vdots \\ \tilde{H}_{\tilde{f}} \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \\ 0_{g\times n} \end{bmatrix}; \quad H_1 = \begin{bmatrix} H_{1,1} \\ \vdots \\ H_{1,f} \end{bmatrix}; \quad H_2 = \begin{bmatrix} H_{2,1} \\ \vdots \\ H_{2,g} \end{bmatrix}; \quad \tilde{D}_{21} = \begin{bmatrix} D_{21} \\ 0_{g\times p} \end{bmatrix};$$

$$\tilde{G} = \begin{bmatrix} \tilde{G}_1 \\ \vdots \\ \tilde{G}_{\tilde{f}} \end{bmatrix} = \begin{bmatrix} G_1 & 0_{h\times g} \\ G_2 & 0_{g\times g} \\ 0_{g\times m} & I_{g\times g} \end{bmatrix}; \quad G_1 = \begin{bmatrix} G_{1,1} \\ \vdots \\ G_{1,f} \end{bmatrix}; \quad G_2 = \begin{bmatrix} G_{2,1} \\ \vdots \\ G_{2,g} \end{bmatrix}. \tag{3.18}$$

We now rewrite all IQCs in (3.5), (3.15) into the following form

$$\int_0^{t_k} \tilde{\xi}(t)^T Q_j \tilde{\xi}(t)\, dt \le \int_0^{t_k} \tilde{\zeta}(t)^T R_j \tilde{\zeta}(t)\, dt + d_j, \quad \forall\, j = 1, 2, \dots, \hat{f} \tag{3.19}$$

and for all $\{t_k \ge 0\}_{k=1}^{\infty}$. Note that $\hat{f} = f + 3g$ and $d_j \ge 0$. Also, the constants β_i^2 corresponding to $\nu_i(t)$ and $\tilde{\nu}_i(t)$ in (3.15) are accordingly included in R_j in (3.19). The set of all admissible uncertainty inputs $\tilde{\xi}(\cdot)$ for the uncertain system (3.16), (3.19) is defined in the same way as in Definition 3.1 and denoted by $\tilde{\Xi}$.

To solve the H^∞ strong stabilization problem for the uncertain system (3.16), (3.19) using the results of [7] and [187], we need to introduce a vector λ of scaling constants $\lambda_j \in \mathbf{R}$, $\lambda_j \ge 0$ (see [187, Theorem 3.1])

$$\lambda := \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{\hat{f}} \end{bmatrix}^T \tag{3.20}$$

so that we can define weighting matrices $\tilde{Q}(\lambda) \ge 0$ and $\tilde{R}(\lambda) \ge 0$, and a constant $\tilde{d}(\lambda) \ge 0$ as functions of λ. That is,

$$\tilde{Q}(\lambda) := \sum_{j=1}^{\hat{f}} \lambda_j Q_j; \quad \tilde{R}(\lambda) := \sum_{j=1}^{\hat{f}} \lambda_j R_j; \quad \tilde{d}(\lambda) := \sum_{j=1}^{\hat{f}} \lambda_j d_j \tag{3.21}$$

where λ belongs to the set

$$\Lambda := \left\{ \lambda \in \mathbf{R}^{\hat{f}} : \lambda_j \geq 0, \ \forall j = 1, 2, \ldots, \hat{f} \right\}. \tag{3.22}$$

This implies that the IQCs (3.19) lead to the satisfaction of an IQC parameterized by scaling constants defined in (3.20). Thus, it follows from (3.19) that

$$\int_0^{t_k} \tilde{\xi}(t)^T \tilde{Q}(\lambda) \tilde{\xi}(t) \, dt \leq \int_0^{t_k} \tilde{\zeta}(t)^T \tilde{R}(\lambda) \tilde{\zeta}(t) \, dt + \tilde{d}(\lambda) \tag{3.23}$$

for all $\{t_k \geq 0\}_{k=1}^{\infty}$. For our purposes, we only consider a subset $\tilde{\Lambda} \subseteq \Lambda$ such that $\tilde{Q}(\lambda) > 0$. Then, for each $\lambda \in \tilde{\Lambda}$, the quantities defined in (3.21) can be written as

$$\tilde{Q}(\lambda) = \bar{Q}(\lambda)^T \bar{Q}(\lambda); \quad \tilde{R}(\lambda) = \bar{R}(\lambda)^T \bar{R}(\lambda) \tag{3.24}$$

where $\bar{Q}(\lambda) = \bar{Q}(\lambda)^T = \tilde{Q}(\lambda)^{\frac{1}{2}} > 0$ and $\bar{R}(\lambda)$ is a rectangular matrix. For convenience, $\bar{R}(\lambda)$ can be chosen as a square matrix such that $\bar{R}(\lambda) = \bar{R}(\lambda)^T = \tilde{R}(\lambda)^{\frac{1}{2}} > 0$. Thus, using the notation in (3.24), the IQC (3.23) can be written in a more compact form as

$$\int_0^{t_k} \|\bar{\xi}(t)\|^2 \, dt \leq \int_0^{t_k} \|\bar{\zeta}(t)\|^2 \, dt + \bar{d}(\lambda) \tag{3.25}$$

for all $\{t_k \geq 0\}_{k=1}^{\infty}$ with

$$\bar{\xi}(t) := \bar{Q}(\lambda) \tilde{\xi}(t); \quad \bar{\zeta}(t) := \bar{R}(\lambda) \tilde{\zeta}(t); \quad \bar{d}(\lambda) := \tilde{d}(\lambda). \tag{3.26}$$

Furthermore, using (3.26), (3.17) and (3.18), we can also represent the system (3.16) as

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + \tilde{B}_2 \tilde{u}(t) + \tilde{E} \bar{Q}(\lambda)^{-1} \bar{\xi}(t); \\
z(t) &= C_1 x(t) + \tilde{D}_{12} \tilde{u}(t); \\
\bar{\zeta}(t) &= \bar{R}(\lambda) \tilde{H} x(t) + \bar{R}(\lambda) \tilde{G} \tilde{u}(t); \\
\tilde{y}(t) &= \tilde{C}_2 x(t) + \tilde{D}_{21} w(t) + \tilde{F} \bar{Q}(\lambda)^{-1} \bar{\xi}(t)
\end{aligned} \tag{3.27}$$

which satisfies the IQC (3.25). Thus, the desired stable nonlinear controller will be constructed based on the uncertain system (3.27), (3.25).

3.3 Stable Nonlinear Controller Design

In this section, we provide sufficient conditions under which a controller of the form (3.10) not only achieves absolute stabilization with a specified disturbance attenuation level $\gamma > 0$ when applied to the uncertain system (3.27), (3.25), but also is stable. To meet this stability requirement, we introduce an additional uncertainty term to form an artificial uncertain system, which is then employed to synthesize the desired controller. Although the additional uncertainty is only required to be an unknown constant uncertain parameter, this approach gives rise to some degree of conservatism. However, this conservatism can be reduced by applying scaling multipliers to exploit the structure of the additional uncertainty; e.g., see [2, 7, 187].

Using a state feedback gain matrix obtained in a preliminary step, the artificial uncertain system is constructed in such a way that for one particular value of the additional uncertainty, the artificial uncertain system reduces to the original nonlinear uncertain system (3.1), (3.2), (3.5). Therefore, if we can find a suitable controller for the artificial uncertain system, this controller will also solve our original problem of absolute stabilization with a specified disturbance attenuation level $\gamma > 0$. Also, for another value of the additional uncertainty, the artificial uncertain system reduces to a particular open loop configuration, which will ensure that the same controller must be stable. Moreover, as we follow standard results of H^∞ control theory in [187], the solution to our nonlinear robust H^∞ strong stabilization problem are obtained in terms of the stabilizing solutions to a pair of parameterized algebraic Riccati equations. This approach results in a controller which has the same order as that of the plant.

3.3.1 State feedback control problem

Solving a state feedback control problem for the uncertain system (3.27), (3.25) is one of preliminary steps required in our approach to synthesize the required stable output feedback controller. This approach aims to guarantee the stability of any suitable controller of the form (3.10), which absolutely stabilizes the artificial uncertain system with specified disturbance attenuation level $\gamma > 0$. Based on S-procedure type results (see [187, Theorem 3.1]), we introduce a scaling constant $\kappa > 0$ corresponding to the IQC (3.25) so that the state equations of the system

(3.27) can be rewritten as

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + \tilde{B}_1\tilde{w}(t) + \tilde{B}_2\tilde{u}(t); \\
\tilde{z}(t) &= \tilde{C}_1 x(t) + \bar{D}_{12}\tilde{u}(t); \\
\tilde{y}(t) &= \tilde{C}_2 x(t) + \bar{D}_{21}\tilde{w}(t)
\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}
&\tilde{w}(t) = \begin{bmatrix} \gamma\, w(t) \\ \sqrt{\kappa}\,\bar{\xi}(t) \end{bmatrix}; \quad \tilde{z}(t) = \begin{bmatrix} z(t) \\ \sqrt{\kappa}\,\bar{\zeta}(t) \end{bmatrix}; \quad \tilde{B}_1 = \begin{bmatrix} \gamma^{-1}B_1 & \sqrt{\kappa}^{-1}\tilde{E}\bar{Q}(\lambda)^{-1} \end{bmatrix}; \\
&\tilde{C}_1 = \begin{bmatrix} C_1 \\ \sqrt{\kappa}\,\bar{R}(\lambda)\tilde{H} \end{bmatrix}; \; \bar{D}_{12} = \begin{bmatrix} \tilde{D}_{12} \\ \sqrt{\kappa}\,\bar{R}(\lambda)\tilde{G} \end{bmatrix}; \; \bar{D}_{21} = \begin{bmatrix} \gamma^{-1}\tilde{D}_{21} & \sqrt{\kappa}^{-1}\tilde{F}\bar{Q}(\lambda)^{-1} \end{bmatrix}.
\end{aligned} \tag{3.29}$$

Using a vector $\lambda = \tilde{\lambda} \in \tilde{\Lambda}$ and a scaling constant $\kappa > 0$ in (3.29), we exploit the structure of the uncertainty in the uncertain system (3.27), (3.25) and transform a constrained robust H^∞ control problem into an unconstrained one such that it can be solved using the results of standard H^∞ control theory; e.g., see [187].

Assumption 3.1. Given a vector $\tilde{\lambda} \in \tilde{\Lambda}$ and constants $\beta_1 \geq 0, \ldots, \beta_g \geq 0, \kappa > 0$, the uncertain system (3.27), (3.25) is assumed to be such that $J = \bar{D}_{12}^T \bar{D}_{12} > 0$.

Lemma 3.1. *Let a vector $\tilde{\lambda} \in \tilde{\Lambda}$ and constants $\beta_1 \geq 0, \ldots, \beta_g \geq 0$ be given. Suppose that the uncertain system* (3.27), (3.25) *satisfies Assumption 3.1, and is absolutely stabilizable with disturbance attenuation level $\gamma > 0$ via an output feedback controller of the form* (3.10) *(but which is not necessarily stable). Then, there exists a constant $\kappa > 0$ such that the algebraic Riccati equation*

$$\begin{aligned}
&\left(A - \tilde{B}_2 J^{-1}\bar{D}_{12}^T\tilde{C}_1\right)^T X + X\left(A - \tilde{B}_2 J^{-1}\bar{D}_{12}^T\tilde{C}_1\right) \\
&\quad + X\left(\tilde{B}_1\tilde{B}_1^T - \tilde{B}_2 J^{-1}\tilde{B}_2^T\right)X + \tilde{C}_1^T\left(I - \bar{D}_{12}J^{-1}\bar{D}_{12}^T\right)\tilde{C}_1 = 0
\end{aligned} \tag{3.30}$$

has a stabilizing solution $X \geq 0$. Moreover, if the condition (3.30) *holds, the uncertain system* (3.27), (3.25) *is absolutely stabilizable with disturbance attenuation level $\gamma > 0$ via the state feedback controller*

$$\tilde{u}(t) = Kx(t) \tag{3.31}$$

where

$$K = \begin{bmatrix} K_u \\ K_{\tilde{\nu}} \end{bmatrix} = -J^{-1}\left(\tilde{B}_2^T X + \bar{D}_{12}^T \tilde{C}_1\right). \tag{3.32}$$

Proof. If the uncertain system (3.27), (3.25) satisfies Assumption 3.1, and is absolutely stabilizable with disturbance attenuation level $\gamma > 0$ via an output feedback controller of the form (3.10), it follows from the proof of Theorem 4.1 in [187] that there exists a constant $\kappa > 0$ such that the controller (3.10) solves the H^∞ control problem defined by the system (3.28) and the H^∞ norm bound condition

$$\tilde{\mathcal{J}} := \sup_{\tilde{w}(\cdot)\in\mathbf{L}_2[0,\infty),x(0)=0,x_c(0)=0} \frac{\|\tilde{z}(\cdot)\|_2^2}{\|\tilde{w}(\cdot)\|_2^2} < 1. \tag{3.33}$$

Then, it follows from a standard result on H^∞ control (e.g., see [312, Theorem 3.3]) that there exists a state feedback control law (3.31) that stabilizes the system (3.28) and leads to the satisfaction of (3.33). Moreover, it follows from the H^∞ control theory (e.g., see [312, Corollary 3.1]) that the Riccati equation (3.30) has a stabilizing solution $X \geq 0$ and that the state feedback control law (3.31), (3.32) stabilizes the system (3.28) and leads to the satisfaction of (3.33). Then, using similar arguments to those in the proof of [187, Theorem 4.1], it follows that the state feedback control law (3.31), (3.32) absolutely stabilizes the uncertain system (3.27), (3.25) with disturbance attenuation level $\gamma > 0$. □

3.3.2 Artificial uncertain system

We now suppose that the vector $\tilde{\lambda} \in \tilde{\Lambda}$ and constants $\beta_1 \geq 0, \ldots, \beta_g \geq 0, \kappa > 0$ as stated in Lemma 3.1 are given. Then, the state feedback gain matrix K defined in (3.32) can be constructed in terms of the stabilizing solution $X \geq 0$ to the algebraic Riccati equation (3.30). Using the matrix K and introducing an additional uncertainty, we form an artificial uncertain system as follows:

$$\begin{aligned}
\dot{x}(t) &= \bar{A}x(t) + B_1 w(t) + \bar{B}_2\tilde{u}(t) + \bar{E}_1\bar{\xi}_1(t) + \bar{E}_2\bar{\xi}_2(t); \\
z(t) &= \bar{C}_1 x(t) + M_1\bar{\xi}_2(t) + \check{D}_{12}\tilde{u}(t); \\
\bar{\zeta}_1(t) &= \bar{H}_1 x(t) + M_2\bar{\xi}_2(t) + \bar{G}_1\tilde{u}(t); \\
\bar{\zeta}_2(t) &= \bar{H}_2 x(t) + \bar{G}_2\tilde{u}(t); \\
\tilde{y}(t) &= \tilde{C}_2 x(t) + \tilde{D}_{21} w(t) + \bar{F}_1\bar{\xi}_1(t) + \bar{F}_2\bar{\xi}_2(t)
\end{aligned} \tag{3.34}$$

where

$$
\begin{aligned}
&\bar{A} = A + \tfrac{1}{2}B_2K_u; \quad \bar{B}_2 = \begin{bmatrix} \tfrac{1}{2}B_2 & 0_{n\times g} \end{bmatrix}; \quad \bar{E}_1 = \tilde{E}\bar{Q}(\lambda)^{-1}; \quad \bar{E}_2 = B_2N^{-1};\\
&\bar{C}_1 = C_1 + \tfrac{1}{2}D_{12}K_u; \quad M_1 = D_{12}N^{-1}; \quad \check{D}_{12} = \begin{bmatrix} \tfrac{1}{2}D_{12} & 0_{q\times g} \end{bmatrix};\\
&\bar{H}_1 = \bar{R}(\lambda)\begin{bmatrix} H_1 + \tfrac{1}{2}G_1K_u \\ H_2 + \tfrac{1}{2}G_2K_u \\ 0_{g\times n} \end{bmatrix}; \quad M_2 = \bar{R}(\lambda)\begin{bmatrix} G_1 \\ G_2 \\ 0_{g\times m} \end{bmatrix}N^{-1};\\
&\bar{G}_1 = \bar{R}(\lambda)\begin{bmatrix} \tfrac{1}{2}G_1 & 0_{h\times g} \\ \tfrac{1}{2}G_2 & 0_{g\times g} \\ 0_{g\times m} & I_{g\times g} \end{bmatrix}; \quad \bar{H}_2 = \tfrac{1}{2}NK_u; \quad \bar{G}_2 = -\tfrac{1}{2}N\begin{bmatrix} I_{m\times m} & 0_{m\times g} \end{bmatrix};\\
&\bar{F}_1 = \tilde{F}\bar{Q}(\lambda)^{-1}; \quad \bar{F}_2 = 0_{(l+g)\times m}.
\end{aligned}
\tag{3.35}
$$

Here, N is any $m \times m$ non-singular scaling matrix and $\lambda = \bar{\lambda} \in \tilde{\Lambda}$. Note that the uncertainty input $\bar{\xi}_1(t)$ and uncertainty output $\bar{\zeta}_1(t)$ are related according to the IQC (3.25) with $\bar{\xi}(t)$ and $\bar{\zeta}(t)$ replaced by $\bar{\xi}_1(t)$ and $\bar{\zeta}_1(t)$, respectively. The IQC (3.25) is also extended to include the additional uncertainty input $\bar{\xi}_2(t)$ and uncertainty output $\bar{\zeta}_2(t)$. That is,

$$
\int_0^{t_k} \|\bar{\xi}_v(t)\|^2\,dt \le \int_0^{t_k} \|\bar{\zeta}_v(t)\|^2\,dt + \bar{d}_v(\lambda) \tag{3.36}
$$

with $\bar{d}_v(\lambda) \ge 0$ for all $v = 1, 2$ and for all $\{t_k \ge 0\}_{k=1}^{\infty}$, and $\bar{\xi}_2(t), \bar{\zeta}_2(t) \in \mathbf{R}^m$.

We consider two special cases of the uncertainty input $\bar{\xi}_2(t)$ in order to show that any suitable output feedback controller of the form (3.10) for the artificial uncertain system (3.34), (3.36) is indeed stable and solves the original control problem as stated in Section 3.2. In this case, the relation between $\bar{\xi}_2(t)$ and $\bar{\zeta}_2(t)$ is described as

$$
\bar{\xi}_2(t) = \Delta\,\bar{\zeta}_2(t) \tag{3.37}
$$

where $\Delta \in \mathbf{R}$ is an unknown constant uncertain parameter satisfying

$$
|\Delta| \le 1. \tag{3.38}
$$

Special case I: $\Delta = 1$. In this case, we have that $\bar{\xi}_2(t) \equiv \bar{\zeta}_2(t) = \frac{1}{2}K_ux(t) - \frac{1}{2}u(t)$. It is clear that this uncertainty input satisfies the IQC (3.36) and that with this $\bar{\xi}_2(t)$, the state equation (3.34) becomes

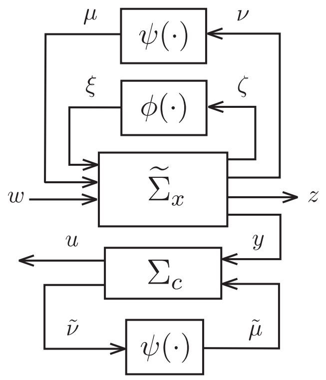
$$\begin{aligned}
\dot{x}(t) &= (A + B_2K_u)x(t) + B_1w(t) + \tilde{E}\bar{Q}(\lambda)^{-1}\bar{\xi}(t); \\
z(t) &= (C_1 + D_{12}K_u)x(t); \\
\bar{\zeta}(t) &= \bar{R}(\lambda)(\tilde{H} + \tilde{G}K)x(t); \\
\tilde{y}(t) &= \tilde{C}_2x(t) + \tilde{D}_{21}w(t) + \tilde{F}\bar{Q}(\lambda)^{-1}\bar{\xi}(t)
\end{aligned} \tag{3.39}$$

where the IQC (3.25) is also satisfied. It then follows from (3.17), (3.18) and (3.26) that we can further decompose the state equations (3.39) as

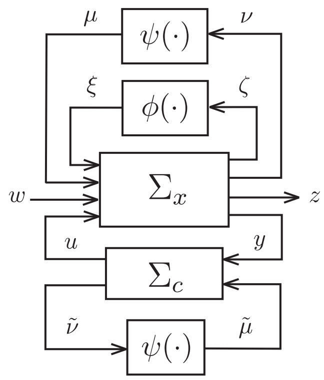
$$\begin{aligned}
\dot{x}(t) &= (A + B_2K_u)x(t) + B_1w(t) + \sum_{s=1}^{f} E_{1,s}\,\xi_s(t) + \sum_{i=1}^{g} E_{2,i}\,\mu_i(t); \\
z(t) &= (C_1 + D_{12}K_u)x(t); \\
\zeta_1(t) &= (H_{1,1} + G_{1,1}K_u)x(t); \\
&\vdots \\
\zeta_f(t) &= (H_{1,f} + G_{1,f}K_u)x(t); \\
\nu_1(t) &= (H_{2,1} + G_{2,1}K_u)x(t); \\
&\vdots \\
\nu_g(t) &= (H_{2,g} + G_{2,g}K_u)x(t); \\
y(t) &= C_2x(t) + D_{21}w(t) + \sum_{s=1}^{f} F_{1,s}\,\xi_s(t) + \sum_{i=1}^{g} F_{2,i}\,\mu_i(t).
\end{aligned} \tag{3.40}$$

where the IQCs (3.5) and GLCs (3.3) are satisfied. The state equations (3.40) represent the closed loop nonlinear uncertain system, which is obtained when the state feedback controller (3.31), (3.32) is applied to the original nonlinear uncertain system (3.1), (3.2), (3.5). Thus, it follows from the construction of the matrix K and Lemma 3.1 that the closed loop nonlinear uncertain system (3.40), (3.2), (3.5) is in fact absolutely stable with disturbance attenuation level $\gamma > 0$.

Through Special case I, we will show that the output feedback controller (3.10) designed using our method is indeed stable. We now suppose that the artificial uncertain system (3.34), (3.36) is absolutely stabilizable with disturbance attenuation level $\gamma > 0$ via the application of any suitable output feedback controller of the form (3.10), which may not be stable. It then follows from Special case



(a) Special case I



(b) Special case II

Figure 3.2: Block diagrams corresponding to Special case I and Special case II; see [7].

I that for this particular additional uncertainty input $\bar{\xi}_2(t)$ and output $\bar{\zeta}_2(t)$, we will obtain a closed loop system which has an open loop configuration as shown in Figure 3.2(a). Moreover, we also notice that the system (3.40) is not affected by the control input $u(t)$, which is the output $u(t)$ of the controller (3.6), (3.7). Here, the block $\widetilde{\Sigma}_x$ is the absolutely stable closed loop system (3.40) and the block Σ_c is the output feedback controller (3.6). As we require the entire closed loop system to be absolutely stable, then the output feedback controller (3.10) must be stable.

Special case II: $\Delta = -1$. From this case, we have that $\bar{\xi}_2(t) \equiv -\bar{\zeta}_2(t) = -\frac{1}{2}K_u x(t) + \frac{1}{2}u(t)$, which satisfies the IQC (3.36) and that the state equations (3.34) becomes (3.27). Based on (3.17), (3.18) and (3.26), it is straightforward to further decompose (3.27) into the system (3.1) with the IQCs (3.5) and GLCs (3.3) satisfied. It is clear that the artificial uncertain system (3.34), (3.36) reduces to the original nonlinear uncertain system (3.1), (3.2), (3.5).

Thus, it follows from this case that when the controller (3.10) is applied to the artificial uncertain system (3.34), (3.36), the resulting closed loop system is equivalent to the one obtained by applying the controller (3.6), (3.7) to the nonlinear uncertain system (3.1), (3.2), (3.5) as shown in Figure 3.2(b). Here, the block Σ_x is the original system (3.1) and the block Σ_c is the controller (3.6). This implies that the controller (3.10) indeed solves the original problem of absolute stabilization with disturbance attenuation level $\gamma > 0$.

As the additional uncertainty satisfying the IQCs (3.36) overbounds the scalar uncertainty (3.37), (3.38), we conclude from both special cases that the controller

(3.10) obtained by applying the method in [187] to the artificial uncertain system (3.34), (3.36) is indeed a stable controller solving the original problem of absolute stabilization with disturbance attenuation level $\gamma > 0$ for the nonlinear uncertain system (3.1), (3.2), (3.5). This fact allows us to apply the method in [187] in order to obtain our main result involving the stabilizing solutions to parameterized algebraic Riccati equations.

3.3.3 Stable nonlinear H^∞ controller

Introducing scaling constants $\tau_1 > 0$ and $\tau_2 > 0$ for the uncertainties satisfying the IQCs (3.36), we can rewrite the state equations (3.34) as follows:

$$\begin{aligned}
\dot{x}(t) &= \bar{A}x(t) + \hat{B}_1\bar{w}(t) + \hat{B}_2\tilde{u}(t);\\
\bar{z}(t) &= \hat{C}_1x(t) + \hat{D}_{11}\bar{w}(t) + \hat{D}_{12}\tilde{u}(t);\\
\tilde{y}(t) &= \tilde{C}_2x(t) + \hat{D}_{21}\bar{w}(t)
\end{aligned} \tag{3.41}$$

where $r = \sum_{s=1}^{f} r_s$; $\tilde{r} = r + 2g$; $\tilde{h} = h + 2g$;

$$\bar{w}(t) = \begin{bmatrix} \gamma\, w(t) \\ \sqrt{\tau_1}\,\bar{\xi}_1(t) \\ \sqrt{\tau_2}\,\bar{\xi}_2(t) \end{bmatrix};\ \bar{z}(t) = \begin{bmatrix} z(t) \\ \sqrt{\tau_1}\,\bar{\zeta}_1(t) \\ \sqrt{\tau_2}\,\bar{\zeta}_2(t) \end{bmatrix};\ \hat{B}_1 = \begin{bmatrix} \gamma^{-1}B_1 & \sqrt{\tau_1}^{-1}\bar{E}_1 & \sqrt{\tau_2}^{-1}\bar{E}_2 \end{bmatrix};$$

$$\hat{B}_2 = \bar{B}_2; \quad \hat{C}_1 = \begin{bmatrix} \bar{C}_1 \\ \sqrt{\tau_1}\,\bar{H}_1 \\ \sqrt{\tau_2}\,\bar{H}_2 \end{bmatrix}; \quad \hat{D}_{11} = \begin{bmatrix} 0_{q\times p} & 0_{q\times \tilde{r}} & \frac{1}{\sqrt{\tau_2}}M_1 \\ 0_{\tilde{h}\times p} & 0_{\tilde{h}\times \tilde{r}} & \sqrt{\frac{\tau_1}{\tau_2}}M_2 \\ 0_{m\times p} & 0_{m\times \tilde{r}} & 0_{m\times m} \end{bmatrix};$$

$$\hat{D}_{12} = \begin{bmatrix} \check{D}_{12} \\ \sqrt{\tau_1}\,\bar{G}_1 \\ \sqrt{\tau_2}\,\bar{G}_2 \end{bmatrix}; \quad \hat{D}_{21} = \begin{bmatrix} \gamma^{-1}\tilde{D}_{21} & \sqrt{\tau_1}^{-1}\bar{F}_1 & \sqrt{\tau_2}^{-1}\bar{F}_2 \end{bmatrix}. \tag{3.42}$$

Since a $\hat{D}_{11}$ term explicitly appears in (3.41), the standard H^∞ control theory cannot immediately be used to obtain a solution to our control problem. Thus, it is necessary to apply a loop shifting transformation so that the $\hat{D}_{11}$ term can be eliminated from (3.41). In order to achieve this, we first need to impose the following assumption in order that our H^∞ control problem is non-singular and well-defined; e.g., see [16, Sections 4.5.1 and 5.5.1] and [18, Section 17.2].

Assumption 3.2. Given a vector $\bar{\lambda} \in \tilde{\Lambda}$, constants $\beta_1 \geq 0, \ldots, \beta_g \geq 0, \tau_1 > 0, \tau_2 > 0$ and any non-singular scaling matrix N, the artificial uncertain system (3.34), (3.36) is assumed to be such that $\hat{D}_{11}\hat{D}_{11}^T < I$.

We can now define

$$\Phi := I - \hat{D}_{11}^T\hat{D}_{11} > 0; \qquad \bar{\Phi} := I - \hat{D}_{11}\hat{D}_{11}^T > 0 \tag{3.43}$$

and also

$$\begin{aligned} \hat{w}(t) &:= \Phi^{\frac{1}{2}}\bar{w}(t) - \Phi^{-\frac{1}{2}}\hat{D}_{11}^T\left(\hat{C}_1 x(t) + \hat{D}_{12}\tilde{u}(t)\right); \\ \hat{z}(t) &:= \bar{\Phi}^{-\frac{1}{2}}\left(\hat{C}_1 x(t) + \hat{D}_{12}\tilde{u}(t)\right). \end{aligned} \tag{3.44}$$

From the relationships in (3.44), it is straightforward to show that

$$\bar{w}(t) = \Phi^{-\frac{1}{2}}\hat{w}(t) + \Phi^{-1}\hat{D}_{11}^T\left(\hat{C}_1 x(t) + \hat{D}_{12}\tilde{u}(t)\right) \tag{3.45}$$

and

$$\|\bar{z}(t)\|_2^2 - \|\bar{w}(t)\|_2^2 \equiv \|\hat{z}(t)\|_2^2 - \|\hat{w}(t)\|_2^2. \tag{3.46}$$

Then, assuming Assumption 3.2 and substituting (3.45) into (3.41), we can rewrite the state equations (3.41) as

$$\begin{aligned} \dot{x}(t) &= \breve{A}x(t) + \breve{B}_1\hat{w}(t) + \breve{B}_2\tilde{u}(t); \\ \hat{z}(t) &= \breve{C}_1 x(t) + \breve{D}_{12}\tilde{u}(t); \\ \tilde{y}(t) &= \breve{C}_2 x(t) + \breve{D}_{21}\hat{w}(t) + \breve{D}_{22}\tilde{u}(t) \end{aligned} \tag{3.47}$$

where

$$\begin{array}{rclcrcl} \breve{A} &=& \bar{A} + \hat{B}_1\hat{D}_{11}^T\bar{\Phi}^{-1}\hat{C}_1; & & \breve{B}_1 &=& \hat{B}_1\Phi^{-\frac{1}{2}}; \\ \breve{B}_2 &=& \hat{B}_2 + \hat{B}_1\hat{D}_{11}^T\bar{\Phi}^{-1}\hat{D}_{12}; & & \breve{C}_1 &=& \bar{\Phi}^{-\frac{1}{2}}\hat{C}_1; \\ \breve{C}_2 &=& \tilde{C}_2 + \hat{D}_{21}\hat{D}_{11}^T\bar{\Phi}^{-1}\hat{C}_1; & & \breve{D}_{12} &=& \bar{\Phi}^{-\frac{1}{2}}\hat{D}_{12}; \\ \breve{D}_{22} &=& \hat{D}_{21}\hat{D}_{11}^T\bar{\Phi}^{-1}\hat{D}_{12}; & & \breve{D}_{21} &=& \hat{D}_{21}\Phi^{-\frac{1}{2}}; \\ \breve{J}_1 &=& \breve{D}_{12}^T\breve{D}_{12}; & & \breve{J}_2 &=& \breve{D}_{21}\breve{D}_{21}^T. \end{array} \tag{3.48}$$

Furthermore, we also define

$$\bar{y}(t) := \tilde{y}(t) - \breve{D}_{22}\tilde{u}(t) \tag{3.49}$$

to eliminate the $\breve{D}_{22}$ term from (3.47). Hence, the state equations (3.47) can be rewritten as

$$\begin{aligned}\dot{x}(t) &= \breve{A}x(t) + \breve{B}_1\hat{w}(t) + \breve{B}_2\tilde{u}(t);\\ \hat{z}(t) &= \breve{C}_1x(t) + \breve{D}_{12}\tilde{u}(t);\\ \bar{y}(t) &= \breve{C}_2x(t) + \breve{D}_{21}\hat{w}(t).\end{aligned} \tag{3.50}$$

Then, the output feedback controller for the system (3.50) is of the form

$$\begin{aligned}\dot{x}_c(t) &= \breve{A}_c x_c(t) + \tilde{B}_c\bar{y}(t);\\ \tilde{u}(t) &= \tilde{C}_c x_c(t).\end{aligned} \tag{3.51}$$

If we interconnect the controller (3.51) to the system (3.50), the resulting closed loop system is required to satisfy the following H^∞ norm bound condition

$$\hat{\mathcal{J}} := \sup_{\hat{w}(\cdot)\in\mathbf{L}_2[0,\infty),x(0)=0,x_c(0)=0} \frac{\|\hat{z}(\cdot)\|_2^2}{\|\hat{w}(\cdot)\|_2^2} < 1. \tag{3.52}$$

A solution to this standard H^∞ control problem is given in terms of solutions to the parameterized algebraic Riccati equations:

$$\begin{aligned}&\left(\breve{A} - \breve{B}_2\breve{J}_1^{-1}\breve{D}_{12}^T\breve{C}_1\right)^T\breve{X} + \breve{X}\left(\breve{A} - \breve{B}_2\breve{J}_1^{-1}\breve{D}_{12}^T\breve{C}_1\right)\\ &\quad + \breve{X}\left(\breve{B}_1\breve{B}_1^T - \breve{B}_2\breve{J}_1^{-1}\breve{B}_2^T\right)\breve{X} + \breve{C}_1^T\left(I - \breve{D}_{12}\breve{J}_1^{-1}\breve{D}_{12}^T\right)\breve{C}_1 = 0;\end{aligned} \tag{3.53}$$

$$\begin{aligned}&\left(\breve{A} - \breve{B}_1\breve{D}_{21}^T\breve{J}_2^{-1}\breve{C}_2\right)\breve{Y} + \breve{Y}\left(\breve{A} - \breve{B}_1\breve{D}_{21}^T\breve{J}_2^{-1}\breve{C}_2\right)^T\\ &\quad + \breve{Y}\left(\breve{C}_1^T\breve{C}_1 - \breve{C}_2^T\breve{J}_2^{-1}\breve{C}_2\right)\breve{Y} + \breve{B}_1\left(I - \breve{D}_{21}^T\breve{J}_2^{-1}\breve{D}_{21}\right)\breve{B}_1^T = 0\end{aligned} \tag{3.54}$$

such that

1. $\breve{A} - \breve{B}_2\breve{J}_1^{-1}\breve{D}_{12}^T\breve{C}_1 + \left(\breve{B}_1\breve{B}_1^T - \breve{B}_2\breve{J}_1^{-1}\breve{B}_2^T\right)\breve{X}$ is Hurwitz;
2. $\breve{A} - \breve{B}_1\breve{D}_{21}^T\breve{J}_2^{-1}\breve{C}_2 + \breve{Y}\left(\breve{C}_1^T\breve{C}_1 - \breve{C}_2^T\breve{J}_2^{-1}\breve{C}_2\right)$ is Hurwitz;
3. The spectral radius $\rho(\breve{X}\breve{Y})$ of the product $\hat{X}\hat{Y}$ is strictly less than one.

The Riccati equations (3.53) and (3.54) are solvable if $\breve{J}_1$ and $\breve{J}_2$ are non-singular matrices. Thus, we need to impose the following assumption; e.g., see [187, 312].

Assumption 3.3. Given a vector $\bar{\lambda} \in \tilde{\Lambda}$, constants $\beta_1 \geq 0, \ldots, \beta_g \geq 0, \tau_1 > 0, \tau_2 > 0$ and any non-singular scaling matrix N, the artificial uncertain system (3.34), (3.36) is assumed to be such that $\breve{J}_1 > 0$ and $\breve{J}_2 > 0$.

Theorem 3.1. *Let vectors $\tilde{\lambda}, \bar{\lambda} \in \tilde{\Lambda}$ and constants $\beta_1 \geq 0, \ldots, \beta_g \geq 0$ be given. Suppose that there exists a constant $\kappa > 0$ such that the uncertain system* (3.27), (3.25) *satisfies Assumption 3.1 and the algebraic Riccati equation* (3.30) *has a stabilizing solution $X \geq 0$, and let*

$$K = -J^{-1}\left(\tilde{B}_2^T X + \bar{D}_{12}^T \tilde{C}_1\right).$$

Also, suppose that there exist a non-singular scaling matrix N and constants $\tau_1 > 0$ and $\tau_2 > 0$ such that the artificial uncertain system (3.34), (3.36) *satisfies Assumptions 3.2 and 3.3 and that both Riccati equations* (3.53) *and* (3.54) *have stabilizing solutions $\breve{X} \geq 0$ and $\breve{Y} \geq 0$, and the spectral radius of the product $\breve{X}\breve{Y}$ satisfies $\rho(\breve{X}\breve{Y}) < 1$. Then the nonlinear uncertain system* (3.1), (3.2), (3.5) *is absolutely stabilizable with disturbance attenuation $\gamma > 0$ via a stable nonlinear output feedback controller of the form* (3.6), (3.7). *Moreover, the controller matrices are given as follows*

$$\begin{aligned}
A_c &= \breve{A}_c - \tilde{B}_c \breve{D}_{22} \tilde{C}_c;\\
\breve{A}_c &= \breve{A} + \breve{B}_2 \tilde{C}_c - \tilde{B}_c \breve{C}_2 + \left(\breve{B}_1 - \tilde{B}_c \breve{D}_{21}\right) \breve{B}_1^T \breve{X};\\
\tilde{B}_c &= \left(I - \breve{Y}\breve{X}\right)^{-1} \left(\breve{Y}\breve{C}_2^T + \breve{B}_1 \breve{D}_{21}^T\right) \breve{J}_2^{-1};\\
\tilde{C}_c &= -\breve{J}_1^{-1}\left(\breve{B}_2^T \breve{X} + \breve{D}_{12}^T \breve{C}_1\right).
\end{aligned} \tag{3.55}$$

Proof. It follows using similar arguments to those in the proof of Theorem 4.1 in [187] that the artificial uncertain system (3.34), (3.36) is absolutely stabilizable with disturbance attenuation level $\gamma > 0$ via a controller of the form (3.10) if and only if there exist constants $\tau_1 > 0$ and $\tau_2 > 0$ such that the controller (3.51) solves the H^∞ control problem defined by (3.50) and (3.52). Moreover, it follows from the results of H^∞ control theory (e.g., see [18, 312]) that the H^∞ control problem defined by (3.50) and (3.52) has a solution if and only if the algebraic Riccati equations (3.53) and (3.54) have stabilizing solutions $\breve{X} \geq 0$ and $\breve{Y} \geq 0$, respectively, such that the spectral radius of the product $\breve{X}\breve{Y}$ satisfies $\rho(\breve{X}\breve{Y}) < 1$. Moreover, if all conditions of the theorem hold, the controller

(3.51), (3.55) will absolutely stabilize the uncertain system (3.34), (3.36) with disturbance attenuation level $\gamma > 0$. Then, using arguments for the two special cases discussed above, it follows that the controller (3.51), (3.55) is indeed stable and absolutely stabilizes the nonlinear uncertain system (3.1), (3.2), (3.5) with disturbance attenuation level $\gamma > 0$. □

3.4 A Differential Evolution Approach

The stable nonlinear controller design method presented in Section 3.3 involves several design parameters, which constitute a vector of decision variables

$$\vartheta := \begin{bmatrix} \gamma & \kappa & \tau_1 & \tau_2 & \tilde{\lambda}^T & \bar{\lambda}^T \end{bmatrix}^T \tag{3.56}$$

where $\vartheta \in \mathbf{R}^{\bar{f}}$, $\bar{f} = 2(f + 3g) + 4$; and $\tilde{\lambda}$, $\bar{\lambda}$ are as defined in (3.20). All elements of ϑ are positive real numbers. As ϑ may have a large dimension, we need to apply an optimization method to determine the value of each element of ϑ in order to satisfy all constraints arising in the controller design algorithm. This condition leads us to formulate our controller design problem as a constrained nonlinear optimization problem stated as follows: Find an optimal vector $\vartheta^\star$ of design parameters to solve

$$\min_{\vartheta} \mathsf{f}(\theta) \tag{3.57}$$

subject to

$$\mathsf{g}_\mathsf{j}(\vartheta) = 0; \quad \mathsf{h}_\mathsf{k}(\vartheta) \leq 0 \tag{3.58}$$

for $\mathsf{j} = 1, 2, \ldots, \mathsf{a}$ and $\mathsf{k} = 1, 2, \ldots, \mathsf{b}$. Here, $\mathsf{f}(\vartheta)$ is an objective function to be minimized; and $\mathsf{g}_\mathsf{j}(\vartheta)$ and $\mathsf{h}_\mathsf{k}(\vartheta)$ are the equality and inequality constraints, respectively.

Considering the controller design algorithm presented in Section 3.3, we in fact deal with an optimization problem that is subject to both nonconvex and nonlinear constraints. To find an optimal solution $\vartheta^\star$ to such a problem, we apply the differential evolution (DE) algorithm as described in Chapter 2. Since we are concerned with an H^∞ control problem, a suitable cost function $\mathsf{f}(\vartheta)$ to be minimized is chosen as

$$\mathsf{f}(\vartheta) = \gamma^\mathsf{n} \tag{3.59}$$

where $\mathsf{n} \geq 1$ is a power constant used to help the DE algorithm to return an optimal $\gamma > 0$. Moreover, the equality constraints are

$$\begin{aligned}
\mathsf{g}_1(\vartheta) &= \left(A - \tilde{B}_2 J^{-1} \bar{D}_{12}^T \tilde{C}_1\right)^T X + X\left(A - \tilde{B}_2 J^{-1} \bar{D}_{12}^T \tilde{C}_1\right) \\
&\quad + X\left(\tilde{B}_1 \tilde{B}_1^T - \tilde{B}_2 J^{-1} \tilde{B}_2^T\right) X + \tilde{C}_1^T \left(I - \bar{D}_{12} J^{-1} \bar{D}_{12}^T\right) \tilde{C}_1 = 0; \\
\mathsf{g}_2(\vartheta) &= \left(\breve{A} - \breve{B}_2 \breve{J}_1^{-1} \breve{D}_{12}^T \breve{C}_1\right)^T \breve{X} + \breve{X}\left(\breve{A} - \breve{B}_2 \breve{J}_1^{-1} \breve{D}_{12}^T \breve{C}_1\right) \\
&\quad + \breve{X}\left(\breve{B}_1 \breve{B}_1^T - \breve{B}_2 \breve{J}_1^{-1} \breve{B}_2^T\right) \breve{X} + \breve{C}_1^T \left(I - \breve{D}_{12} \breve{J}_1^{-1} \breve{D}_{12}^T\right) \breve{C}_1 = 0; \\
\mathsf{g}_3(\vartheta) &= \left(\breve{A} - \breve{B}_1 \breve{D}_{21}^T \breve{J}_2^{-1} \breve{C}_2\right) \breve{Y} + \breve{Y}\left(\breve{A} - \breve{B}_1 \breve{D}_{21}^T \breve{J}_2^{-1} \breve{C}_2\right)^T \\
&\quad + \breve{Y}\left(\breve{C}_1^T \breve{C}_1 - \breve{C}_2^T \breve{J}_2^{-1} \breve{C}_2\right) \breve{Y} + \breve{B}_1 \left(I - \breve{D}_{21}^T \breve{J}_2^{-1} \breve{D}_{21}\right) \breve{B}_1^T = 0
\end{aligned} \tag{3.60}$$

and the inequality constraints are

$$\begin{array}{rclrcl}
\mathsf{h}_1(\vartheta) &=& -\tilde{Q}(\tilde{\lambda}) < 0; & \mathsf{h}_2(\vartheta) &=& -\tilde{R}(\tilde{\lambda}) < 0; \\
\mathsf{h}_3(\vartheta) &=& -J < 0; & \mathsf{h}_4(\vartheta) &=& -X < 0; \\
\mathsf{h}_5(\theta) &=& -\tilde{Q}(\bar{\lambda}) < 0; & \mathsf{h}_6(\theta) &=& -\tilde{R}(\bar{\lambda}) < 0; \\
\mathsf{h}_7(\vartheta) &=& \hat{D}_{11} \hat{D}_{11}^T - I < 0; & \mathsf{h}_8(\vartheta) &=& -\breve{J}_1 < 0; \\
\mathsf{h}_9(\vartheta) &=& -\breve{J}_2 < 0; & \mathsf{h}_{10}(\vartheta) &=& -\breve{X} < 0; \\
\mathsf{h}_{11}(\vartheta) &=& -\breve{Y} < 0; & \mathsf{h}_{12}(\vartheta) &=& \rho(\breve{X}\breve{Y}) - 1 < 0; \\
\mathsf{h}_{13}(\vartheta) &=& e_{\max,r}\left(\mathcal{A}_X\right) < 0; & \mathsf{h}_{14}(\vartheta) &=& e_{\max,r}(\breve{\mathcal{A}}_X) < 0; \\
\mathsf{h}_{15}(\vartheta) &=& e_{\max,r}(\breve{\mathcal{A}}_Y) < 0 & & &
\end{array} \tag{3.61}$$

where $\rho(\mathcal{M})$ and $e_{\max,r}(\mathcal{M})$ denote the spectral radius and the largest real part of the eigenvalues of the matrix $\mathcal{M}$, respectively; and

$$\begin{aligned}
\mathcal{A}_X &:= A - \tilde{B}_2 J^{-1} \bar{D}_{12}^T \tilde{C}_1 + \left(\tilde{B}_1 \tilde{B}_1^T - \tilde{B}_2 J^{-1} \tilde{B}_2^T\right) X; \\
\breve{\mathcal{A}}_X &:= \breve{A} - \breve{B}_2 \breve{J}_1^{-1} \breve{D}_{12}^T \breve{C}_1 + \left(\breve{B}_1 \breve{B}_1^T - \breve{B}_2 \breve{J}_1^{-1} \breve{B}_2^T\right) \breve{X}; \\
\breve{\mathcal{A}}_Y &:= \breve{A} - \breve{B}_1 \breve{D}_{21}^T \breve{J}_2^{-1} \breve{C}_2 + \breve{Y}\left(\breve{C}_1^T \breve{C}_1 - \breve{C}_2^T \breve{J}_2^{-1} \breve{C}_2\right).
\end{aligned} \tag{3.62}$$

The constraints in (3.60) and (3.61) are used to evaluate the fitness of each candidate solution in a population. For a given ϑ, the fitness test proceeds according to the following steps:

1. Compute the eigenvalues of $\tilde{Q}(\tilde{\lambda})$, $\tilde{R}(\tilde{\lambda})$ and J in order to check if the

constraints $\mathsf{h}_1(\vartheta)$, $\mathsf{h}_2(\vartheta)$ and $\mathsf{h}_3(\vartheta)$ are satisfied.

2. Use the constraint $\mathsf{g}_1(\vartheta)$ to obtain a solution X to the Riccati equation (3.30).

3. If the Riccati equation (3.30) has a solution X, we need to verify whether it is a positive definite stabilizing solution through the evaluation of the constraints $\mathsf{h}_4(\vartheta)$ and $\mathsf{h}_{13}(\vartheta)$.

4. Compute the eigenvalues of $\tilde{Q}(\bar{\lambda})$, $\tilde{R}(\bar{\lambda})$, $(\hat{D}_{11}\hat{D}_{11}^T - I)$, $\breve{J}_1$ and $\breve{J}_2$ in order to check if the constraints $\mathsf{h}_5(\theta)$, $\mathsf{h}_6(\theta)$, $\mathsf{h}_7(\theta)$, $\mathsf{h}_8(\theta)$ and $\mathsf{h}_9(\theta)$ are satisfied.

5. Use the constraints $\mathsf{g}_2(\vartheta)$ and $\mathsf{g}_3(\vartheta)$ to obtain solutions $\breve{X}$ and $\breve{Y}$ to the Riccati equations (3.53) and (3.54).

6. If the Riccati equations (3.53) and (3.54) have solutions $\breve{X}$ and $\breve{Y}$, we need to verify whether they are positive definite stabilizing solutions through the evaluation of the constraints $\mathsf{h}_{10}(\theta)$, $\mathsf{h}_{11}(\theta)$, $\mathsf{h}_{12}(\theta)$, $\mathsf{h}_{14}(\theta)$ and $\mathsf{h}_{15}(\theta)$.

7. Compute the spectral radius of the product $\breve{X}\breve{Y}$ to verify if the constraint $\mathsf{h}_{10}(\vartheta)$ is satisfied.

8. Compute the value of the objective function $\mathsf{f}(\vartheta)$ in (3.59).

Through the fitness test routine above, we acquire information above how many constraints have been violated by a candidate solution and how much penalty is incurred. Thus, from the constraints in (3.60) and (3.61), we can derive a set of penalty functions corresponding to those constraints. That is,

$$\begin{aligned}
\mathsf{p}_1(\vartheta) &= |e_{\min}(\tilde{Q}(\tilde{\lambda}))|^{\mathsf{s}_1}; & \mathsf{p}_2(\vartheta) &= |e_{\min}(\tilde{R}(\tilde{\lambda}))|^{\mathsf{s}_2};\\
\mathsf{p}_3(\vartheta) &= |e_{\min}(J)|^{\mathsf{s}_3}; & \mathsf{p}_4(\vartheta) &= \rho(\mathcal{C}_X)^{\mathsf{s}_4};\\
\mathsf{p}_5(\vartheta) &= |e_{\min}(X)|^{\mathsf{s}_5}; & \mathsf{p}_6(\vartheta) &= e_{\max,r}(\mathcal{A}_X)^{\mathsf{s}_6};\\
\mathsf{p}_7(\theta) &= |e_{\min}(\tilde{Q}(\bar{\lambda}))|^{\mathsf{s}_7}; & \mathsf{p}_8(\theta) &= |e_{\min}(\tilde{R}(\bar{\lambda}))|^{\mathsf{s}_8};\\
\mathsf{p}_9(\vartheta) &= e_{\max}(\hat{D}_{11}\hat{D}_{11}^T - I)^{\mathsf{s}_9}; & \mathsf{p}_{10}(\vartheta) &= |e_{\min}(\breve{J}_1)|^{\mathsf{s}_{10}};\\
\mathsf{p}_{11}(\vartheta) &= |e_{\min}(\breve{J}_2)|^{\mathsf{s}_{11}}; & \mathsf{p}_{12}(\vartheta) &= \rho(\mathcal{C}_{\breve{X}})^{\mathsf{s}_{12}};\\
\mathsf{p}_{13}(\vartheta) &= \rho(\mathcal{C}_{\breve{Y}})^{\mathsf{s}_{13}}; & \mathsf{p}_{14}(\vartheta) &= |e_{\min}(\breve{X})|^{\mathsf{s}_{14}};\\
\mathsf{p}_{15}(\vartheta) &= |e_{\min}(\breve{Y})|^{\mathsf{s}_{15}}; & \mathsf{p}_{16}(\vartheta) &= e_{\max,r}(\mathcal{A}_{\breve{X}})^{\mathsf{s}_{16}};\\
\mathsf{p}_{17}(\vartheta) &= e_{\max,r}(\mathcal{A}_{\breve{Y}})^{\mathsf{s}_{17}}; & \mathsf{p}_{18}(\vartheta) &= (\rho(\breve{X}\breve{Y}) - 1)^{\mathsf{s}_{18}};\\
\mathsf{p}_{19}(\vartheta) &= \mathsf{f}(\vartheta)
\end{aligned} \tag{3.63}$$

where $\mathsf{s_r} \geq 1$ for $\mathsf{r} = 1, 2, \ldots, 18$. Here, $e_{\min}(\mathcal{M})$ and $e_{\max}(\mathcal{M})$ denote the smallest and the largest eigenvalue of the symmetric matrix $\mathcal{M}$, respectively. If the matrix $\mathcal{M}$ is required to be positive definite, we assign $|e_{\min}(\mathcal{M})|^{\mathsf{s_r}}$ as a penalty because when this requirement is violated, the matrix $\mathcal{M}$ can be either negative (semi)definite or indefinite. Moreover, $\mathcal{C}_X$, $\mathcal{C}_{\breve{X}}$ and $\mathcal{C}_{\breve{Y}}$ are defined as

$$\begin{aligned}
\mathcal{C}_X &:= \tilde{C}_1^T \left(I - \bar{D}_{12} J^{-1} \bar{D}_{12}^T\right) \tilde{C}_1; \\
\mathcal{C}_{\breve{X}} &:= \breve{C}_1^T \left(I - \breve{D}_{12} \breve{J}_1^{-1} \breve{D}_{12}^T\right) \breve{C}_1; \\
\mathcal{C}_{\breve{Y}} &:= \breve{B}_1 \left(I - \breve{D}_{21}^T \breve{J}_2^{-1} \breve{D}_{21}\right) \breve{B}_1^T.
\end{aligned} \tag{3.64}$$

3.5 Illustrative Examples

In this section, we consider two examples to demonstrate the stable nonlinear controller design method presented in Section 3.3. Through these examples, we will show that if a continuous time version of the method in [182] is applied to the problem under consideration, the resulting nonlinear controller is not necessarily stable. Furthermore, we also show that our stable nonlinear controller allows for better disturbance attenuation performance as compared to its linear counterpart which can be synthesized using the method in [7]. Thus, in order to fit into our nonlinear robust H^∞ control framework, we modify the nominal models of both examples by adding several terms corresponding to disturbance input $w(t)$, controlled output $z(t)$, uncertainty input $\xi(t)$ and output $\zeta(t)$, and nonlinearity input $\mu(t)$ and output $\nu(t)$. All parameters required in the controller design are computed using the DE method as described in Section 3.4.

Example 3.1. The nominal model of the following example taken from [175] is also considered as a benchmark problem in [177]. This example is defined by the following matrices:

$$\begin{aligned}
&A = \begin{bmatrix} -2 & 1.7321 \\ 1.7321 & 0 \end{bmatrix}; \quad B_1 = \begin{bmatrix} 0.1 & -0.1 \\ -0.5 & 0.5 \end{bmatrix}; \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad E_{1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \\
&E_{2,1} = \begin{bmatrix} 0 \\ 1.7 \end{bmatrix}; \; C_1 = \begin{bmatrix} 0.2 & -1 \\ 0 & 0 \end{bmatrix}; \; D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \; H_{1,1} = \begin{bmatrix} 0 & 1 \end{bmatrix}; \; H_{2,1} = \begin{bmatrix} 0 & 1 \end{bmatrix}; \\
&C_2 = \begin{bmatrix} 10 & 11.5470 \end{bmatrix}; \quad D_{21} = \begin{bmatrix} 0.7071 & 0.7071 \end{bmatrix}; \quad F_{1,1} = \begin{bmatrix} 0 & 1 \end{bmatrix}.
\end{aligned} \tag{3.65}$$

The uncertainty and nonlinearity of the nonlinear uncertain system are respectively described as follows:

$$\begin{aligned}\xi(t) &= \begin{bmatrix}\xi_1(t)\\ \xi_2(t)\end{bmatrix} = \begin{bmatrix}\Delta_1\\ \Delta_2\end{bmatrix}\zeta(t);\\ \mu(t) &= \psi(\nu(t)) = \sin\nu(t).\end{aligned} \tag{3.66}$$

Thus, the IQC corresponding to the uncertainty is given by

$$\int_0^\infty \left(\xi_1^2(t) + \xi_2^2(t)\right)\, dt \le \int_0^\infty \zeta^2(t)\, dt + d_1, \quad d_1 \ge 0 \tag{3.67}$$

and the GLC corresponding to the known nonlinearity $\psi(\cdot)$ is given by

$$|\sin\nu(t) - \sin\tilde{\nu}(t)| \le \beta|\nu(t) - \tilde{\nu}(t)|. \tag{3.68}$$

From the first derivative of the nonlinear function $\psi(\nu(\cdot))$ with respect to $\nu(\cdot)$, we obtain

$$\left|\frac{d\sin\nu(\cdot)}{d\nu(\cdot)}\right| = |\cos\nu(\cdot)| \le 1, \quad \forall\nu(\cdot) \in \mathbf{R} \tag{3.69}$$

which implies that $\beta = 1$. Moreover, the corresponding IQCs derived from the GLC (3.68) are as follows:

$$\begin{aligned}&\int_0^\infty (\sin\nu(t) - \sin\tilde{\nu}(t))^2\, dt \le \int_0^\infty (\nu(t) - \tilde{\nu}(t))^2\, dt + d_2;\\ &\int_0^\infty (\sin\nu(t))^2\, dt \le \int_0^\infty \nu^2(t)\, dt + d_3;\\ &\int_0^\infty (\sin\tilde{\nu}(t))^2\, dt \le \int_0^\infty \tilde{\nu}^2(t)\, dt + d_4\end{aligned} \tag{3.70}$$

where $d_2, d_3, d_4 \ge 0$. Note that $\tilde{\mu}(t) = \sin\tilde{\nu}(t)$ represents the copy of the known nonlinearity $\psi(\cdot)$ included in the controller (3.6).

We first solve the nonlinear H^∞ control problem for this system using the continuous time version of [182] to show that this method may not lead to a stable controller. In this case, the required parameters are obtained as

$$\begin{aligned}\gamma &= 3.1375; & \kappa &= 0.1035; & \lambda_1 &= 208.4058;\\ \lambda_2 &= 981.5735; & \lambda_3 &= 207.1877; & \lambda_4 &= 498.0892\end{aligned} \tag{3.71}$$

with $\lambda_1, \ldots, \lambda_4$ are respectively corresponding to the IQCs (3.67) and (3.70). Then, the resulting controller matrices are

$$
\begin{aligned}
A_c &= 10^5 \times \begin{bmatrix} 0.3249 & 0.4270 \\ -3.9584 & -5.2037 \end{bmatrix}; \\
\tilde{B}_c &= 10^4 \times \begin{bmatrix} -0.2939 & -0.4896 \\ 3.5792 & 5.9622 \end{bmatrix}; \\
\tilde{C}_c &= \begin{bmatrix} -12.6141 & -31.3799 \\ 0 & 0.6634 \end{bmatrix}.
\end{aligned} \tag{3.72}
$$

Since the plant nonlinearity (3.66) is represented by a sine function, the stability of the corresponding nonlinear controller (3.6) can be investigated through linearization around the equilibrium point $x_c^* = 0$, that is when $\tilde{\mu}(t) \approx \tilde{\nu}(t)$. In this case, the linearized controller matrix $\bar{A}_c$ is obtained as follows:

$$
\bar{A}_c := A_c + \sum_{i=1}^{g} L_i C_{c2,i} = 10^5 \times \begin{bmatrix} 0.3249 & 0.3945 \\ -3.9584 & -4.8081 \end{bmatrix}. \tag{3.73}
$$

The eigenvalues of $\bar{A}_c$ are $e_1 = 9.9399$ and $e_2 = -4.4833 \times 10^5$, which implies that $\bar{A}_c$ is unstable. Thus, the corresponding nonlinear controller is also unstable.

We now apply our main result in Theorem 3.1 to construct a stable nonlinear output feedback controller for the nonlinear uncertain system (3.65), (3.66), (3.67). The required parameters obtained using the DE method are

$$
\begin{aligned}
&\gamma = 34.7286; \quad \kappa = 0.0806; \quad \tau_1 = 0.4314; \quad \tau_2 = 1.0057; \\
&\lambda_1 = 117.9752; \quad \lambda_2 = 928.1436; \quad \lambda_3 = 216.6302; \quad \lambda_4 = 743.8695
\end{aligned} \tag{3.74}
$$

where $\lambda = \tilde{\lambda} = \bar{\lambda}$. Thus, the resulting controller matrices are obtained as

$$
\begin{aligned}
A_c &= 10^6 \times \begin{bmatrix} -1.3780 & -1.5946 \\ -0.0842 & -0.0974 \end{bmatrix}; \\
\tilde{B}_c &= \begin{bmatrix} 1.3769 \times 10^5 & 1.1773 \times 10^4 \\ 8.4136 \times 10^3 & 721.3153 \end{bmatrix}; \\
\tilde{C}_c &= \begin{bmatrix} -9.0971 & -34.3064 \\ 0 & 0.5551 \end{bmatrix}
\end{aligned} \tag{3.75}
$$

and the corresponding linearized controller matrix $\bar{A}_c$ is

$$\bar{A}_c = 10^6 \times \begin{bmatrix} -1.3780 & -1.5881 \\ -0.0842 & -0.0970 \end{bmatrix} \tag{3.76}$$

with eigenvalues $e_1 = -0.2862$ and $e_2 = -1.4750 \times 10^6$. This indicates that the controller is at least locally stable around the equilibrium point $x_c^* = 0$.

In order to verify that the controller is globally asymptotically stable, we will check the stability of A_c and the H^∞ norm $\|T_{\tilde{\mu}\tilde{\nu}}(s)\|_\infty$ of the transfer function from the nonlinearity input $\tilde{\mu}(t)$ to the nonlinearity output $\tilde{\nu}(t)$. The eigenvalues of A_c are $e_1 = -1.3075$ and $e_2 = -1.4754 \times 10^6$, and the H^∞ norm $\|T_{\tilde{\mu}\tilde{\nu}}(s)\|_\infty = 0.7812$. Thus, using the small gain theorem (e.g., see [18, Section 9.2]), we conclude that the nonlinear controller (3.75) is indeed stable.

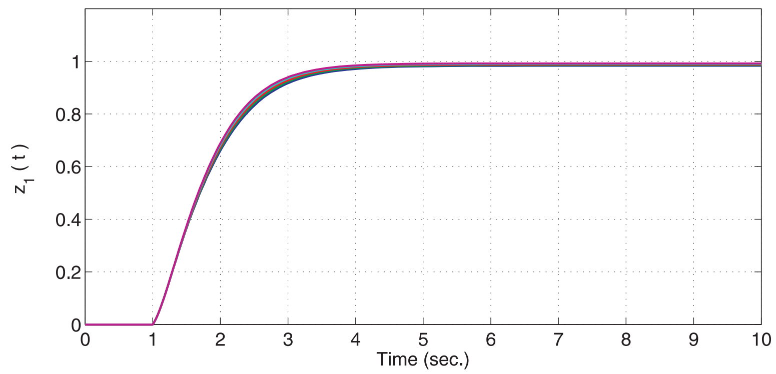


Figure 3.3: The controlled output $z_1(t)$ for different values of Δ_1 and Δ_2.

The performance of the stable nonlinear controller (3.75) is demonstrated through a closed loop simulation using Simulink for several values of $\Delta_1 \in \{-1, -0.5, 0, 0.5, 1\}$ and Δ_2 is set to be equal to Δ_1. The initial conditions of the system and the controller are set to be zero. The closed loop system is perturbed by an exogenous unit step function $w_1(t)$, while $w_2(t) = 0$. The response of the corresponding controlled output $z_1(t)$ is shown in Figure 3.3 and indicates that the stable nonlinear controller (3.75) absolutely stabilizes the closed loop system with a sufficient attenuation against the disturbance input $w_1(t)$.

Example 3.2. In this example, we solve a problem of designing a stable nonlinear controller for a flexible link robot. The nominal model of this system is taken from [313], which has also been considered in [186], and is defined by the following matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{bmatrix}; \; B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \; B_2 = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix}; \; E_{1,1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix};$$

$$E_{2,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3.33 \end{bmatrix}; \quad C_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad H_{1,1} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix};$$

$$H_{2,1} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}; \; C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}; \; D_{21} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}; \; F_{1,1} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \tag{3.77}$$

where the uncertainty and nonlinearity of the system (3.77) are similarly described as in the Example 3.1. Applying the DE approach presented in Section 3.4, we obtain all parameters required for controller design as follows:

$$\begin{aligned} &\gamma = 0.1866; \quad \kappa = 0.0116; \quad \tau_1 = 0.0116; \quad \tau_2 = 2.9542; \\ &\lambda_1 = 14.7055; \; \lambda_2 = 1925.5322; \; \lambda_3 = 74.0920; \; \lambda_4 = 38.3448 \end{aligned} \tag{3.78}$$

where $\lambda = \tilde{\lambda} = \bar{\lambda}$. Thus, the nonlinear controller matrices are given as

$$\begin{aligned} A_c &= \begin{bmatrix} -185.6621 & 9.8266 & 44.0019 & -54.9750 \\ 974.2362 & -134.3833 & -169.6195 & 300.7635 \\ 797.5493 & -16.6491 & -189.7179 & 237.3781 \\ -945.3289 & -85.1382 & 210.8858 & -283.8434 \end{bmatrix}; \\ \tilde{B}_c &= \begin{bmatrix} 11.4628 & -23.9149 & -3.1943 \\ -101.2022 & 203.8707 & 30.6886 \\ -47.8036 & 81.8149 & 14.0482 \\ 60.9064 & 5.4548 & -208754 \end{bmatrix}; \\ \tilde{C}_c &= \begin{bmatrix} -30.1101 & -3.0441 & 8.2684 & -9.0469 \\ 0 & 0 & 0.9807 & 0 \end{bmatrix}. \end{aligned} \tag{3.79}$$

Through the same analysis as in the Example 3.1, the nonlinear controller (3.79) is at least locally stable around the equilibrium point $x_c^* = 0$ since the eigenvalues of $\bar{A}_c$ are

$$\begin{aligned} e_1 &= -614.2753; \quad e_2 = -0.9460; \\ e_3 &= -133.6701; \quad e_4 = -30.9376. \end{aligned} \tag{3.80}$$

Moreover, A_c in (3.79) is also Hurwitz with eigenvalues

$$\begin{aligned} e_1 &= -630.5257; \quad e_2 = -0.9175; \\ e_3 &= -136.7811; \quad e_4 = -25.3824 \end{aligned} \tag{3.81}$$

and the H^∞ norm $\|T_{\tilde{\mu}\tilde{\nu}}(s)\|_\infty = 0.5636$. Thus, using the small gain theorem (e.g., see [18, Section 9.2]), we conclude that the nonlinear controller (3.79) is globally asymptotically stable. The performance of the closed loop system is then simulated using Simulink under similar conditions as in the Example 3.1. That is, we assume that $\xi(t) = \Delta\zeta(t)$, where $\Delta \in \{-1, -0.5, 0, 0.5, 1\}$. The response of the controlled output $z_1(t)$ is shown in Figure 3.4.

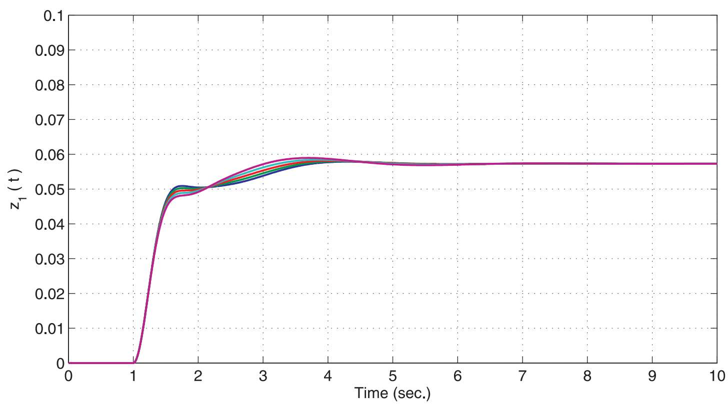


Figure 3.4: The controlled output $z_1(t)$ for different values of Δ.

If we apply the linear controller design method in [7] to this example, the required parameters are obtained as

$$\begin{aligned} \gamma &= 0.2352; \quad \tau_1 = 0.2112; \quad \tau_2 = 2.0932; \\ \tilde{\tau}_1 &= 0.1669; \quad \tilde{\tau}_2 = 2.9655; \quad \tilde{\tau}_3 = 4.3449 \end{aligned} \tag{3.82}$$

and the corresponding controller is defined by the matrices as follows:

$$A_c = 10^4 \times \begin{bmatrix} 0.1142 & 0.0486 & 0.0041 & 0.0268 \\ -0.4892 & -0.2076 & -0.0098 & -0.1141 \\ -0.8237 & -0.3503 & -0.0295 & -0.1929 \\ -3.0739 & -1.3049 & -0.1121 & -0.7206 \end{bmatrix};$$
$$B_c = 10^4 \times \begin{bmatrix} -0.0406 & -0.0420 \\ 0.1676 & 0.1785 \\ 0.2925 & 0.3032 \\ 1.0932 & 1.1289 \end{bmatrix};$$
$$C_c = \begin{bmatrix} -5.4683 & -0.8526 & 0.7404 & -1.5737 \end{bmatrix}. \tag{3.83}$$

The eigenvalues of A_c in (3.83) are

$$\begin{aligned} e_1 &= -8.3973 \times 10^3; & e_2 &= -0.5988; \\ e_3 &= -18.1361 + i17.1584; & e_4 &= -18.1361 - i17.1584 \end{aligned} \tag{3.84}$$

and therefore, the controller is stable. Comparing the value of $\gamma = 0.1866$ obtained with the nonlinear controller (3.79) to that of $\gamma = 0.2352$ obtained with the linear controller (3.83), we notice that the nonlinear controller (3.79) allows for a significant improvement in disturbance attenuation performance.

3.6 Conclusions

We have presented a systematic methodology to design a stable nonlinear robust H^∞ output feedback controller for a class of nonlinear uncertain systems. All admissible uncertainties of the systems are described in terms of IQCs and each known nonlinearity of the systems has to satisfy a GLC. The underlying main idea of our method is to add a copy of each known plant nonlinearity to the linear part of the controller. We then characterize the nonlinearities and their copies with extra IQCs derived from the GLCs. The nonlinear controller is synthesized based on existing results of robust H^∞ control theory applied to an artificial uncertain system, which is formed using a state feedback gain matrix and an additional uncertainty. This approach is used to guarantee that the resulting controller is stable and achieves absolute stability of the closed loop system with a specified disturbance attenuation level. The solution to this control problem involves the stabilizing solutions to algebraic Riccati equations, which are dependent on a set of scaling parameters. We then reformulate the absolute stabilization problem into an optimization problem with those parameters as decision variables and subjected to nonconvex nonlinear constraints. This latter problem is solved using an evolutionary optimization method, namely the DE algorithm.

To demonstrate the merit of the proposed nonlinear controller design method, we have also shown through Example 3.1 that our method indeed results in a stable nonlinear output feedback controller. Such a controller may not be achievable if we apply the nonlinear robust control method in [182] because controller stability is not required. Moreover, as shown in Example 3.2, the stable nonlinear controller synthesized using our method may have better disturbance attenuation performance than that synthesized using the method presented in [7]. This is because we do not directly treat known nonlinearities of the system as uncertainties, but rather exploit them as useful information for control purposes. Despite beneficial features of our method, we note that introducing additional artificial uncertainty to impose controller stability may give rise to some degree of conservatism in the controller design process.

Chapter 4

Decentralized State Feedback Robust H^∞ Control

4.1 Introduction

The main contribution of this chapter is to present a new method for designing a decentralized state feedback robust H^∞ controller for a large-scale linear uncertain system, which consists of interconnected subsystems. In this case, the interconnections between subsystems are not treated as uncertainties. Instead, we neglect the off-diagonal blocks in the controller gain matrix and consider them as uncertainties; e.g., see [156,198], which also use this approach for a guaranteed cost control problem. This approach thus yields a block-diagonal state feedback controller, which is able to exploit the interconnections between subsystems and is also robust against perturbations in the controller itself.

We are concerned with a class of linear uncertain systems in which the uncertainties are described by the integral quadratic constraints. For these systems, we aim to construct a decentralized state feedback robust H^∞ controller that is capable of absolutely stabilizing the corresponding closed loop system while achieving a certain disturbance attenuation level. The solution to this control problem is given in terms of a stabilizing solution to an algebraic Riccati equation parameterized by a set of scaling constants.

The required scaling constants are associated with the system uncertainties and norm bounds on the size of the neglected off-diagonal blocks of the controller gain matrix. This formulation leads to a nonconvex nonlinear optimization problem which is often difficult to solve using regular optimization methods. Thus, we employ an evolutionary optimization approach, namely the differential evolution (DE) algorithm (see Chapter 2), which is a population-based stochastic optimization method. The DE approach is used to find an optimal solution to a suitable objective function for the decentralized controller synthesis problem.

4.2 Problem Statement

We consider a decentralized state feedback robust H^∞ control problem for a large-scale linear uncertain system of the type described in [187]. The state equations of this system are given as follows:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) + \sum_{s=1}^{k} B_{3,s}\xi_s(t);\\
z(t) &= C_1 x(t) + D_{12} u(t);\\
\zeta_1(t) &= F_1 x(t) + G_1 u(t);\\
&\ \vdots\\
\zeta_k(t) &= F_k x(t) + G_k u(t)
\end{aligned} \tag{4.1}$$

where $x(t) \in \mathbf{R}^n$ is the state, $w(t) \in \mathbf{R}^g$ is the disturbance input, $u(t) \in \mathbf{R}^m$ is the control input, $z(t) \in \mathbf{R}^q$ is the controlled output, $\zeta_1(t) \in \mathbf{R}^{h_1}, \ldots, \zeta_k(t) \in \mathbf{R}^{h_k}$ are the uncertainty outputs and $\xi_1(t) \in \mathbf{R}^{r_1}, \ldots, \xi_k(t) \in \mathbf{R}^{r_k}$ are the uncertainty inputs.

All of the uncertainties in this system are described by a set of equations of the form

$$\xi_s(t) = \phi_s\left(t, \zeta_s(\cdot)|_0^t\right), \quad \text{for } s = 1, 2, \ldots, k \tag{4.2}$$

where $\phi(\cdot)$ is a nonlinear time-varying and dynamic functional; see [2]. Those uncertainties are said to be admissible if they satisfy the following integral quadratic constraints (IQCs).

Definition 4.1. (*Integral Quadratic Constraint*; e.g., see [2]) An uncertainty of the form (4.2) is an admissible uncertainty for the system (4.1) if the following conditions hold: Given any locally square integrable control input $u(\cdot)$ and locally square integrable disturbance input $w(\cdot)$, and any corresponding solution to the system (4.1), (4.2), let $(0, t_\star)$ be the interval on which the solution exists. Then there exist constants $d_1 \geq 0, \ldots, d_k \geq 0$ and a sequence $\{t_j\}_{j=1}^{\infty}$ such that $t_j \to t_\star$, $t_j \geq 0$ and

$$\int_0^{t_j} \|\xi_s(t)\|^2 dt \leq \int_0^{t_j} \|\zeta_s(t)\|^2 dt + d_s \tag{4.3}$$

for $s = 1, \ldots, k$ and $\forall j$. Note that t_j and $t_\star$ may be equal to infinity. The class of all such admissible uncertainties $\xi(\cdot) = [\xi_1(\cdot), \ldots, \xi_k(\cdot)]$ is denoted by Ξ.

We assume that the large-scale system (4.1) comprises p interconnected subsystems. Thus, the state vector $x(t) \in \mathbf{R}^n$ can be partitioned into p components as follows:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_p(t) \end{bmatrix} \tag{4.4}$$

where $x_i(t) \in \mathbf{R}^{n_i}$ and $n = \sum_{i=1}^{p} n_i$. If the large-scale uncertain system (4.1), (4.3) is to be controlled using a decentralized state feedback controller, then the decentralized control input $\tilde{u}(t) \in \mathbf{R}^m$ also has p components as follows:

$$\tilde{u}(t) = \begin{bmatrix} \tilde{u}_1(t) \\ \tilde{u}_2(t) \\ \vdots \\ \tilde{u}_p(t) \end{bmatrix} \tag{4.5}$$

where $\tilde{u}_i(t) \in \mathbf{R}^{m_i}$ and $m = \sum_{i=1}^{p} m_i$. Each component $\tilde{u}_i(t)$ is dependent only on the corresponding state component $x_i(t)$ although no assumptions are made concerning the structure of the system matrices A and B_2. Thus, each local controller $\tilde{u}_i(t)$ can be written as

$$\tilde{u}_i(t) = K_{ii} x_i(t), \quad \text{for } i = 1, 2, \ldots, p \tag{4.6}$$

where K_{ii} is the i-th diagonal block of the state feedback gain matrix K. Indeed, the decentralized control input (4.5) can also be considered as a special case of the non-decentralized state feedback control input of the form

$$u(t) = Kx(t) \tag{4.7}$$

such that the matrix K has a block-diagonal structure.

Moreover, when constructing the decentralized state feedback controller (4.5), (4.6), we do not treat interconnections between subsystems as uncertainties. This approach then allows us to exploit any useful structural information of the large-scale uncertain system (4.1), (4.3); e.g., see [197, 229]. The purpose of applying the decentralized state feedback controller (4.5), (4.6) is to achieve an absolutely stable closed loop system with a specified disturbance attenuation level.

Definition 4.2. (*Absolute stabilizability*; e.g., see [2]) The large-scale uncertain system (4.1), (4.3) is said to be absolutely stabilizable with disturbance attenuation level $\gamma > 0$ via the state feedback controller (4.7) if there exists constants $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that the following conditions hold:

1. For any initial condition $x(0)$, any admissible uncertainty inputs $\xi_1(\cdot), \dots, \xi_k(\cdot)$ and any disturbance input $w(\cdot) \in \mathbf{L}_2[0, \infty)$, we have $[x(\cdot), \xi_1(\cdot), \dots, \xi_k(\cdot)] \in \mathbf{L}_2[0, \infty)$ (hence, $t_\star = \infty$) and

$$\|x(\cdot)\|_2^2 + \sum_{s=1}^{k} \|\xi_s(\cdot)\|_2^2 \leq \varepsilon_1 \left[\|x(0)\|^2 + \|w(\cdot)\|_2^2 + \sum_{s=1}^{k} d_s \right]. \tag{4.8}$$

2. The following H^∞ norm bound condition is satisfied: If $x(0) = 0$, then for $w(\cdot) \in \mathbf{L}_2[0, \infty)$ and $\xi_s(\cdot) \in \Xi$ (for all $s = 1, 2, \dots, k$)

$$\mathcal{J} := \sup_{w(\cdot)} \sup_{\xi_s(\cdot)} \frac{\|z(\cdot)\|_2^2 - \varepsilon_2 \sum_{s=1}^{k} d_s}{\|w(\cdot)\|_2^2} < \gamma^2. \tag{4.9}$$

4.3 Decentralized Controller Design

In this section, we present a systematic procedure to synthesize a decentralized state feedback controller (4.5), (4.6) for the large-scale uncertain system (4.1), (4.3). The key idea of our approach is to partition a non-decentralized state feedback gain matrix K according to the partition of $x(t)$ in (4.4) and $\tilde{u}(t)$ in (4.5). That is,

$$K = \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1p} \\ K_{21} & K_{22} & \dots & K_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ K_{p1} & K_{p2} & \dots & K_{pp} \end{bmatrix}. \tag{4.10}$$

The off-diagonal blocks of matrix K are then neglected and treated as additional uncertainties, which are added to the uncertainties in the original uncertain system (4.1), (4.3). This implies that the decentralized state feedback gain matrix $\widetilde{K}$ is constructed by taking only diagonal blocks of the matrix K. That is,

$$\widetilde{K} = \begin{bmatrix} K_{11} & 0 & \dots & 0 \\ 0 & K_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & K_{pp} \end{bmatrix}. \tag{4.11}$$

Accordingly, a sequence of uncertainty matrices is then obtained from the off-diagonal blocks of the matrix K in (4.10) as follows:

$$\begin{aligned} \Delta_1^u &:= \begin{bmatrix} K_{12} & K_{13} & K_{14} & \dots & K_{1p} \end{bmatrix}; \\ \Delta_2^u &:= \begin{bmatrix} K_{21} & K_{23} & K_{24} & \dots & K_{2p} \end{bmatrix}; \\ \Delta_3^u &:= \begin{bmatrix} K_{31} & K_{32} & K_{34} & \dots & K_{3p} \end{bmatrix}; \\ &\vdots \\ \Delta_p^u &:= \begin{bmatrix} K_{p1} & K_{p2} & K_{p3} & \dots & K_{p(p-1)} \end{bmatrix}. \end{aligned} \tag{4.12}$$

Also, we define an additional uncertainty input $\xi_i^u(t)$ and uncertainty output $\zeta_i^u(t)$ corresponding to each Δ_i^u in (4.12) as

$$\xi_i^u(t) := -\Delta_i^u \zeta_i^u(t); \quad \zeta_i^u(t) := F_i^u x(t) \tag{4.13}$$

for $i = 1, 2, \dots, p$. The matrices $F_1^u, F_2^u, \dots, F_p^u$ are expressed as follows:

$$\begin{aligned} F_1^u &= \begin{bmatrix} 0_{\tilde{n}_1 \times n_1} & I_{\tilde{n}_1 \times \tilde{n}_1} \end{bmatrix}; \\ F_l^u &= \begin{bmatrix} I_{\bar{n}_{l-1} \times \bar{n}_{l-1}} & 0_{\bar{n}_{l-1} \times n_l} & 0_{\bar{n}_{l-1} \times \tilde{n}_l} \\ 0_{\tilde{n}_l \times \bar{n}_{l-1}} & 0_{\tilde{n}_l \times n_l} & I_{\tilde{n}_l \times \tilde{n}_l} \end{bmatrix}; \\ F_p^u &= \begin{bmatrix} I_{\bar{n}_{p-1} \times \bar{n}_{p-1}} & 0_{\bar{n}_{p-1} \times n_p} \end{bmatrix} \end{aligned} \tag{4.14}$$

for $l = 2, \dots, p-1$. Here, $\bar{n}_i = \sum_{k=1}^{i} n_k$ and $\tilde{n}_i = n - \bar{n}_i$. Then, using the construction of the matrix $\widetilde{K}$ in (4.11), we can use the relationships in (4.7) and

(4.13) to express the decentralized control input (4.5) as

$$\tilde{u}(t) = \widetilde{K}x(t) = Kx(t) + \sum_{i=1}^{p} J_i^u \xi_i^u(t) \tag{4.15}$$

where

$$J_1^u = \begin{bmatrix} I_{m_1\times m_1} \\ 0_{\tilde{m}_1\times m_1} \end{bmatrix}; \quad J_l^u = \begin{bmatrix} 0_{\bar{m}_{l-1}\times m_l} \\ I_{m_l\times m_l} \\ 0_{\tilde{m}_l\times m_l} \end{bmatrix}; \quad J_p^u = \begin{bmatrix} 0_{\bar{m}_{p-1}\times m_p} \\ I_{m_p\times m_p} \end{bmatrix} \tag{4.16}$$

for $l = 2, \ldots, p-1$. Here $\bar{m}_i = \sum_{k=1}^{i} m_k$ and $\tilde{m}_i = m - \bar{m}_i$.

Now, it follows from the above formulation that if we apply the decentralized state feedback controller (4.15) to the large-scale uncertain system (4.1), (4.3), we will obtain the same closed loop system as if we apply the non-decentralized state feedback controller (4.7) to the following uncertain system:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) + \sum_{i=1}^{p} B_2 J_i^u \xi_i^u(t) + \sum_{s=1}^{k} B_{3,s}\xi_s(t); \\
z(t) &= C_1 x(t) + D_{12}u(t) + \sum_{i=1}^{p} D_{12} J_i^u \xi_i^u(t); \\
\zeta_1(t) &= F_1 x(t) + G_1 u(t) + \sum_{i=1}^{p} G_1 J_i^u \xi_i^u(t); \\
&\vdots \\
\zeta_k(t) &= F_k x(t) + G_k u(t) + \sum_{i=1}^{p} G_k J_i^u \xi_i^u(t); \\
\zeta_1^u(t) &= F_1^u x(t); \\
&\vdots \\
\zeta_p^u(t) &= F_p^u x(t).
\end{aligned} \tag{4.17}$$

Also, for a given matrix K, we define constant $\beta_i > 0$ to bound the size of each additional uncertainty Δ_i^u defined in (4.12). That is,

$$\|\Delta_i^u\|^2 \leq \beta_i, \quad \text{for } i = 1, 2, \ldots, p. \tag{4.18}$$

Here, $\|\cdot\|$ denotes the induced matrix norm and each Δ_i^u is defined as in (4.12). From (4.18), it follows that the uncertainty input $\xi_i^u(t)$ and output $\zeta_i^u(t)$ will satisfy an IQC of the form

$$\int_0^{t_j} \|\xi_i^u(t)\|^2 dt \leq \int_0^{t_j} \beta_i \|\zeta_i^u(t)\|^2 dt + d_i^u \tag{4.19}$$

with $d_i^u \geq 0$ for all $\{t_j \geq 0\}_{j=1}^{\infty}$ and for all $i = 1, 2, \ldots, p$.

In order to construct a state feedback controller for the uncertain system (4.17), (4.3), (4.19) based on the state feedback interpretation of Theorem 4.1 in [187], we first apply S-procedure type results (see [187, Theorem 3.1]) by introducing scaling constants $\tau_1 > 0, \ldots, \tau_{k+p} > 0$ corresponding to the IQCs (4.3), (4.19) so that the state equations (4.17) can be represented as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \tilde{B}_1\tilde{w}(t) + \tilde{B}_2 u(t); \\ \tilde{z}(t) &= \tilde{C}_1 x(t) + \tilde{D}_{11}\tilde{w}(t) + \tilde{D}_{12}u(t) \end{aligned} \tag{4.20}$$

where

$$\begin{aligned}
\tilde{B}_1 &= \begin{bmatrix} \gamma^{-1}B_1 & \tilde{B}_3 & \tilde{B}_2^u \end{bmatrix}; \quad \tilde{B}_2 = B_2; \\
\tilde{B}_3 &= \begin{bmatrix} \sqrt{\tau_1}^{-1}B_{3,1} & \ldots & \sqrt{\tau_k}^{-1}B_{3,k} \end{bmatrix}; \\
\tilde{B}_2^u &= \begin{bmatrix} \sqrt{\tau_{k+1}}^{-1}B_2J_1^u & \ldots & \sqrt{\tau_{k+p}}^{-1}B_2J_p^u \end{bmatrix}; \\
\tilde{C}_1 &= \begin{bmatrix} C_1 \\ \sqrt{\tau_1}\,F_1 \\ \vdots \\ \sqrt{\tau_k}\,F_k \\ \sqrt{\beta_1\tau_{k+1}}\,F_1^u \\ \vdots \\ \sqrt{\beta_p\tau_{k+p}}\,F_p^u \end{bmatrix}; \; \tilde{D}_{11} = \begin{bmatrix} 0_{q\times g} & 0_{q\times r} & \tilde{D}_{12}^u \\ 0_{h\times g} & 0_{h\times r} & \tilde{G}_u \\ 0_{\breve{n}\times g} & 0_{\breve{n}\times r} & 0_{\breve{n}\times m} \end{bmatrix}; \; \tilde{D}_{12} = \begin{bmatrix} D_{12} \\ \sqrt{\tau_1}\,G_1 \\ \vdots \\ \sqrt{\tau_k}\,G_k \\ 0_{\breve{n}\times m} \end{bmatrix}; \\
\tilde{D}_{12}^u &= \begin{bmatrix} \sqrt{\tau_{k+1}}^{-1}D_{12}J_1^u & \ldots & \sqrt{\tau_{k+p}}^{-1}D_{12}J_p^u \end{bmatrix}; \\
\tilde{G}_u &= \begin{bmatrix} \sqrt{\frac{\tau_1}{\tau_{k+1}}}\,G_1J_1^u & \ldots & \sqrt{\frac{\tau_1}{\tau_{k+p}}}\,G_1J_p^u \\ \vdots & \vdots & \vdots \\ \sqrt{\frac{\tau_k}{\tau_{k+1}}}\,G_kJ_1^u & \ldots & \sqrt{\frac{\tau_k}{\tau_{k+p}}}\,G_kJ_p^u \end{bmatrix};
\end{aligned} \tag{4.21}$$

$$\tilde{w}(t) = \begin{bmatrix} \gamma\, w(t) \\ \sqrt{\tau_1}\,\xi_1(t) \\ \vdots \\ \sqrt{\tau_k}\,\xi_k(t) \\ \sqrt{\tau_{k+1}}\,\xi_1^u(t) \\ \vdots \\ \sqrt{\tau_{k+p}}\,\xi_p^u(t) \end{bmatrix}; \quad \tilde{z}(t) = \begin{bmatrix} z(t) \\ \sqrt{\tau_1}\,\zeta_1(t) \\ \vdots \\ \sqrt{\tau_k}\,\zeta_k(t) \\ \sqrt{\beta_1\tau_{k+1}}\,\zeta_1^u(t) \\ \vdots \\ \sqrt{\beta_p\tau_{k+p}}\,\zeta_p^u(t) \end{bmatrix}. \tag{4.22}$$

Here, $h = \sum_{s=1}^{k} h_s$, $r = \sum_{s=1}^{k} r_s$, $\breve{n} = \sum_{i=1}^{p} \tilde{n}_i + \bar{n}_{i-1}$ ($\bar{n}_0 = 0$ and $\tilde{n}_p = 0$).

As a $\tilde{D}_{11}$ term appears in (4.20), we cannot immediately apply the state feedback results of [187] to synthesize a state feedback controller of the form (4.7) for the system (4.20). To eliminate the $\tilde{D}_{11}$ term, it is necessary to impose the following assumption:

Assumption 4.1. Given constants $\tau_1 > 0, \ldots, \tau_{k+p} > 0$, $\beta_1 > 0, \ldots, \beta_p > 0$, the uncertain system (4.17), (4.3), (4.19) is assumed to be such that $\tilde{D}_{11}\tilde{D}_{11}^T < I$.

Now, we are able to apply a loop shifting transformation (e.g., see [16, Sections 4.5.1 and 5.5.1] and [18, Section 17.2]) to the system (4.20) by first defining

$$\begin{aligned} \Phi &:= I - \tilde{D}_{11}^T\tilde{D}_{11} > 0; \\ \bar{\Phi} &:= I - \tilde{D}_{11}\tilde{D}_{11}^T > 0; \\ \bar{w}(t) &:= \Phi^{\frac{1}{2}}\tilde{w}(t) - \Phi^{-\frac{1}{2}}\tilde{D}_{11}^T\left(\tilde{C}_1 x(t) + \tilde{D}_{12}u(t)\right); \\ \bar{z}(t) &:= \bar{\Phi}^{-\frac{1}{2}}\left(\tilde{C}_1 x(t) + \tilde{D}_{12}u(t)\right). \end{aligned} \tag{4.23}$$

From (4.23), it is straightforward to verify that

$$\tilde{w}(t) = \Phi^{-\frac{1}{2}}\bar{w}(t) + \Phi^{-1}\tilde{D}_{11}^T\left(\tilde{C}_1 x(t) + \tilde{D}_{12}u(t)\right) \tag{4.24}$$

and

$$\|\tilde{z}(t)\|_2^2 - \|\tilde{w}(t)\|_2^2 \equiv \|\bar{z}(t)\|_2^2 - \|\bar{w}(t)\|_2^2. \tag{4.25}$$

Hence, the state equations (4.20) can now be rewritten as

$$\begin{aligned} \dot{x}(t) &= \bar{A}x(t) + \bar{B}_1\bar{w}(t) + \bar{B}_2 u(t); \\ \bar{z}(t) &= \bar{C}_1 x(t) + \bar{D}_{12}u(t) \end{aligned} \tag{4.26}$$

where

$$
\begin{aligned}
\bar{A} &= A + \tilde{B}_1 \tilde{D}_{11}^T \bar{\Phi}^{-1} \tilde{C}_1; \\
\bar{B}_1 &= \tilde{B}_1 \Phi^{-\frac{1}{2}}; \\
\bar{B}_2 &= \tilde{B}_2 + \tilde{B}_1 \tilde{D}_{11}^T \bar{\Phi}^{-1} \tilde{D}_{12}; \\
\bar{C}_1 &= \bar{\Phi}^{-\frac{1}{2}} \tilde{C}_1; \\
\bar{D}_{12} &= \bar{\Phi}^{-\frac{1}{2}} \tilde{D}_{12}.
\end{aligned} \tag{4.27}
$$

The results of [187] involve solving an H^∞ control problem corresponding to the system (4.26) and the H^∞ norm bound condition

$$
\bar{\mathcal{J}} := \sup_{\bar{w}(\cdot) \in \mathbf{L}_2[0,\infty), x(0)=0} \frac{\|\bar{z}(\cdot)\|_2^2}{\|\bar{w}(\cdot)\|_2^2} < 1. \tag{4.28}
$$

The solution to the H^∞ control problem defined by (4.26), (4.28) is then given in terms of the stabilizing solution to the parameterized algebraic Riccati equation defined as follows: Let $\tau_1 > 0, \ldots, \tau_{k+p} > 0$, $\beta_1 > 0, \ldots, \beta_p > 0$ be given constants. The Riccati equation is then written as (e.g., see [187])

$$
\begin{aligned}
&(\bar{A} - \bar{B}_2 E^{-1} \bar{D}_{12}^T \bar{C}_1)^T X + X(\bar{A} - \bar{B}_2 E^{-1} \bar{D}_{12}^T \bar{C}_1) \\
&\quad + X\left(\bar{B}_1 \bar{B}_1^T - \bar{B}_2 E^{-1} \bar{B}_2^T\right) X + \bar{C}_1^T (I - \bar{D}_{12} E^{-1} \bar{D}_{12}^T) \bar{C}_1 = 0
\end{aligned} \tag{4.29}
$$

where $E := \bar{D}_{12}^T \bar{D}_{12}$.

We now present our main results relating the Riccati equation (4.29) to the problem of absolute stabilization with a specified disturbance attenuation level via a state feedback controller. Here, we only provide sufficient conditions for absolute stabilization via a decentralized state feedback controller because the main results only hold for a specific realization of the additional uncertainties defined by (4.12), (4.13). To solve the Riccati equation (4.29), we need to impose the following assumption; e.g., see [187, 312].

Assumption 4.2. Given constants $\tau_1 > 0, \ldots, \tau_{k+p} > 0$, $\beta_1 > 0, \ldots, \beta_p > 0$, the uncertain system (4.17), (4.3), (4.19) is assumed to be such that $\bar{D}_{12}^T \bar{D}_{12} > 0$.

Theorem 4.1. *Suppose $\beta_1 > 0, \ldots, \beta_p > 0$ are given constants such that the uncertain system* (4.17), (4.3), (4.19) *is absolutely stabilizable with a specified* *disturbance attenuation level $\gamma > 0$ via a controller of the form* (4.7)

$$u(t) = Kx(t).$$

Then there exists $\tau_1 > 0, \ldots, \tau_{k+p} > 0$ such that conditions in Assumption 4.1 and Assumption 4.2 hold, and the Riccati equation (4.29) *has a stabilizing solution $X \geq 0$. Moreover, if these conditions hold, then the state feedback gain matrix K given by*

$$K = -E^{-1}(\bar{B}_2^T X + \bar{D}_{12}^T \bar{C}_1) \tag{4.30}$$

is such that the resulting closed loop system is absolutely stable with disturbance attenuation $\gamma > 0$.

Proof. If all conditions of the theorem hold, then it follows from the proof of Theorem 4.1 in [187] that there exist constants $\tau_1 > 0, \ldots, \tau_{k+p} > 0$ such that conditions in Assumption 4.1 and Assumption 4.2 hold, and the controller (4.7) solves the H^∞ control problem defined by the system (4.26) and the H^∞ norm bound condition (4.28). Also, standard results on H^∞ control (e.g., see Theorem 3.3 and Corollary 3.1 in [312]) confirm the existence of a stabilizing solution $X \geq 0$ to the Riccati equation (4.29). Moreover, if the Riccati equation (4.29) has a stabilizing solution $X \geq 0$, then the state feedback control law (4.7), (4.30) absolutely stabilizes the uncertain system (4.17), (4.3), (4.19) with disturbance attenuation level $\gamma > 0$ (see the proof of Theorem 4.1 in [187]). □

Theorem 4.2. *Let $\tau_1 > 0, \ldots, \tau_{k+p} > 0$, $\beta_1 > 0, \ldots, \beta_p > 0$ be given constants such that Assumption 4.1 and Assumption 4.2 hold, and the Riccati equation* (4.29) *has a stabilizing solution $X \geq 0$. Also, suppose that the state feedback controller gain matrix K defined by* (4.30) *is such that the neglected off-diagonal blocks of K defined by* (4.12) *satisfy the bounds* (4.18)*. Then the uncertain system* (4.1), (4.3) *is absolutely stabilizable with a specified disturbance attenuation level $\gamma > 0$ via a decentralized state feedback controller of the form* (4.15)

$$\tilde{u}(t) = \widetilde{K}x(t)$$

where $\widetilde{K}$ is the decentralized state feedback gain matrix of the form (4.11) *with block diagonal elements of the full state feedback gain matrix K given by* (4.30)*.*

Proof. If all conditions of the theorem hold, then it follows from Theorem 4.1

that the uncertain system (4.17), (4.3), (4.19) is absolutely stabilizable with disturbance attenuation level $\gamma > 0$ via the state feedback controller of the form (4.7), (4.30). Also, if the full state feedback gain matrix K defined by (4.30) is such that the neglected off-diagonal blocks of K defined by (4.12) satisfy the bounds (4.18), then the uncertainties (4.13) satisfy the IQCs (4.19). Moreover, from the construction of the uncertain system (4.17), (4.3), (4.19), the closed loop system obtained by applying the decentralized controller (4.15) to the original uncertain system (4.1), (4.3) is identical to the closed loop system obtained by applying the controller (4.7) to the uncertain system (4.17), (4.3), (4.19) when the uncertainties (4.13) are applied. Therefore, it follows that the original uncertain system (4.1), (4.3) is absolutely stabilized with a specified disturbance attenuation level $\gamma > 0$ via a decentralized state feedback controller (4.15). $\square$

4.4 A Differential Evolution Approach

From the decentralized controller synthesis method presented in Section 4.3, we can reformulate the decentralized state feedback control problem stated in Section 4.2 as a nonlinear optimization problem with nonconvex constraints. It involves a decision variable

$$\vartheta := \begin{bmatrix} \gamma & \tau_1 & \tau_2 & \ldots & \tau_{k+p} & \beta_1 & \beta_2 & \ldots & \beta_p \end{bmatrix}^T . \tag{4.31}$$

where the dimension of ϑ is $k + 2p + 1$. All elements of ϑ are positive real numbers and they correspond to the parameterized algebraic Riccati equation (4.29). To compute their values, we then propose apply an evolutionary optimization method, namely the differential evolution (DE) algorithm, as described in Chapter 2. Thus, we aim to find an optimal vector $\vartheta^\star$ of design parameters to solve

$$\min_{\vartheta} \mathsf{f}(\vartheta) \tag{4.32}$$

that is subject to

$$\mathsf{g_j}(\vartheta) = 0; \quad \mathsf{h_k}(\vartheta) \leq 0 \tag{4.33}$$

for $\mathsf{j} = 1, 2, \ldots, \mathsf{a}$ and $\mathsf{k} = 1, 2, \ldots, \mathsf{b}$. Here, $\mathsf{f}(\vartheta)$ is an objective function to be minimized, and $\mathsf{g_j}(\vartheta)$ and $\mathsf{h_k}(\vartheta)$ are the equality and inequality constraints, respectively.

As we are concerned with a decentralized robust H^∞ control problem as formulated in Section 4.2 and Section 4.3, a suitable objective function to be minimized is

$$\mathsf{f}(\vartheta) = \eta_0 \gamma^{\kappa_0} + \sum_{i=1}^{p} \eta_i \beta_i^{\kappa_i}. \tag{4.34}$$

This objective function $\mathsf{f}(\vartheta)$ is defined in order to obtain a decentralized state feedback controller that absolutely stabilizes the closed loop system with a specified disturbance attenuation level $\gamma > 0$. Here, $\eta_0, \eta_i \geq 1$ are weighting factors used to set priority between γ and each β_i term in (4.34). The $\sum_{i=1}^{p} \eta_i \beta_i^{\kappa_i}$ term is used to force the DE algorithm to produce a state feedback gain matrix K such that each of its off-diagonal blocks are small. Moreover, the power constants $\kappa_0, \kappa_i \geq 1$ are chosen such that the numerical iteration evolves toward an optimal solution $\vartheta^\star$ at a considerable rate.

In addition to the objective function $\mathsf{f}(\vartheta)$ in (4.34), we also define a set of constraints parameterized by ϑ, which are defined using the assumptions and main results in Section 4.3. In this case, the equality constraint is

$$\begin{aligned} \mathsf{g}_1(\vartheta) = {} & (\bar{A} - \bar{B}_2 E^{-1} \bar{D}_{12}^T \bar{C}_1)^T X + X(\bar{A} - \bar{B}_2 E^{-1} \bar{D}_{12}^T \bar{C}_1) \\ & + X\left(\bar{B}_1 \bar{B}_1^T - \bar{B}_2 E^{-1} \bar{B}_2^T\right) X + \bar{C}_1^T (I - \bar{D}_{12} E^{-1} \bar{D}_{12}^T) \bar{C}_1 = 0 \end{aligned} \tag{4.35}$$

and the inequality constraints are

$$\begin{array}{llll} \mathsf{h}_1(\vartheta) & = \tilde{D}_{11}\tilde{D}_{11}^T - I < 0; & \mathsf{h}_2(\vartheta) & = -E < 0; \\ \mathsf{h}_3(\vartheta) & = -X < 0; & \mathsf{h}_4(\vartheta) & = e_{\max,r}(\mathcal{A}) < 0; \\ \mathsf{h}_{5,i}(\vartheta) & = \|\Delta_i^u\|^2 - \beta_i \leq 0 & & \end{array} \tag{4.36}$$

for $i = 1, 2, \ldots, p$. Here, $e_{\max,r}(\mathcal{A})$ denotes the largest real part of the eigenvalues of the matrix $\mathcal{A}$ defined as follows:

$$\mathcal{A} := \bar{A} - \bar{B}_2 E^{-1} \bar{D}_{12}^T \bar{C}_1 + \left(\bar{B}_1 \bar{B}_1^T - \bar{B}_2 E^{-1} \bar{B}_2^T\right) X. \tag{4.37}$$

Based on the constraints (4.35), (4.36), we then form a penalty-based fitness test procedure through which all candidate solutions in the population are evaluated. Thus, the fitness test proceeds as follows:

1. Compute the eigenvalues of $(\tilde{D}_{11}\tilde{D}_{11}^T - I)$ and E in order to examine if the

constraints $\mathsf{h}_1(\vartheta)$ and $\mathsf{h}_2(\vartheta)$ hold.

2. Compute the solution of the Riccati equation (4.29) using the constraint $\mathsf{g}_1(\vartheta)$ in (4.35).

3. Evaluate the constraints $\mathsf{h}_3(\vartheta)$ and $\mathsf{h}_4(\vartheta)$ in (4.36) to check if the solution X to the Riccati equation (4.29) is a stabilizing positive definite solution;

4. Evaluate the constraints $\mathsf{h}_{5,i}(\vartheta)$ (for $i = 1, 2, \ldots, p$) in (4.36) to check if the i-th off-diagonal block of the matrix K satisfies the norm bound condition in (4.18).

5. Calculate the value of the objective function $\mathsf{f}(\vartheta)$ in (4.34).

Through this routine, we acquire information about how many constraints have been violated by each candidate solution and the accompanying violation cost or the value of the objective function if no constraint violation has occurred. We also assume that a constraint violation in a lower level implies the one(s) in the higher level.

The penalty functions corresponding to the violation of each constraint in (4.35), (4.36) are as follows:

$$\begin{array}{rclrcl} \mathsf{p}_1(\vartheta) &=& e_{\max}(\tilde{D}_{11}\tilde{D}_{11}^T - I)^{\mathsf{n}_1}; & \mathsf{p}_2(\vartheta) &=& |e_{\min}(E)|^{\mathsf{n}_2}; \\ \mathsf{p}_3(\vartheta) &=& \rho(\mathcal{C})^{\mathsf{n}_3}; & \mathsf{p}_4(\vartheta) &=& |e_{\min}(X)|^{\mathsf{n}_4}; \\ \mathsf{p}_5(\vartheta) &=& e_{\max,r}(\mathcal{A})^{\mathsf{n}_5}; & \mathsf{p}_6(\vartheta) &=& \sum_{i=1}^{p} \mathcal{D}_i^{\mathsf{m}_i}; \\ \mathsf{p}_7(\vartheta) &=& \mathsf{f}(\vartheta) & & & \end{array} \tag{4.38}$$

for $\mathsf{n}_\mathsf{j}, \mathsf{m}_i \geq 1$ for $\mathsf{j} = 1, 2, \ldots, 5$ and $i = 1, 2, \ldots, p$. Here, $\rho(\mathcal{M})$ denotes the spectral radius of the matrix $\mathcal{M}$; $e_{\min}(\mathcal{M})$ and $e_{\max}(\mathcal{M})$ denote the smallest and the largest eigenvalue of the symmetric matrix $\mathcal{M}$, respectively. If the matrix $\mathcal{M}$ is required to be positive definite, we assign $|e_{\min}(\mathcal{M})|^{\mathsf{n}_\mathsf{j}}$ as a penalty for the case when this requirement is violated. Here, the matrix $\mathcal{M}$ can be either negative (semi)definite or indefinite. Moreover, $\mathcal{C}$ and $\mathcal{D}_i$ in (4.38) are defined as follows:

$$\begin{aligned} \mathcal{C} &:= \bar{C}_1^T \left(I - \bar{D}_{12} E^{-1} \bar{D}_{12}^T \right) \bar{C}_1; \\ \mathcal{D}_i &:= \begin{cases} \|\Delta_i^u\|^2, & \text{if } \mathsf{h}_{5,i}(\vartheta) \text{ is violated;} \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{4.39}$$

Each penalty function in (4.38) has a positive value and is evaluated only when a violation occurs.

4.5 An Illustrative Example

To demonstrate the controller design method in Section 4.3, we consider a decentralized robust H^∞ control problem for two inverted pendulums interconnected by a spring. The linearized and normalized model of this system is adjusted from that in [230] and is given as follows:

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.1 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 1 \\ 0.9 & 0 & 0.1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} w(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \\ 0 & 0.1 \end{bmatrix} \xi(t); \\
z(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} u(t); \\
\zeta(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t).
\end{aligned} \tag{4.40}$$

Since the system (4.40) consists of two interconnected subsystems, both the state $x(t)$ and the control input $u(t)$ are partitioned into two components ($p = 2$) so that $n_i = 2$ and $m_i = 1$ for $i = 1, 2$. The interconnection between subsystems is assumed to consist of 90% known linear part and 10% unknown nonlinear part. The latter is then considered as uncertainty in the system. The relationship between the uncertainty input $\xi(t)$ and the uncertainty output $\zeta(t)$ in the system (4.40) is represented as

$$\xi(t) = \Delta(\mu)\,\zeta(t), \quad \Delta(\mu) := \mu \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \text{for } -0.5 \le \mu \le 0.5 \tag{4.41}$$

with $\|\Delta(\mu)\|^2 \le 1$ such that the following IQC is satisfied. That is,

$$\int_0^\infty \|\xi(t)\|^2\, dt \le \int_0^\infty \|\zeta(t)\|^2\, dt + d, \quad \forall\, d \ge 0. \tag{4.42}$$

Applying Theorem 4.1 and the DE approach to this problem, we obtain the disturbance attenuation level $\gamma = 0.5181$ and other constants as follows:

$$\tau = \begin{bmatrix} 0.0200 & 0.0153 & 0.0400 & 0.0401 \end{bmatrix}; \quad \beta = \begin{bmatrix} 0.0536 & 0.0903 \end{bmatrix}. \tag{4.43}$$

Given these parameters, we solve the Riccati equation (4.29) and use its stabilizing solution $X \geq 0$ to construct the non-decentralized and decentralized state feedback gain matrices K and $\widetilde{K}$, respectively. That is,

$$\begin{aligned} K &= \begin{bmatrix} -5.3744 & -0.9078 & 0.1178 & 0.1985 \\ 0.2255 & 0.1985 & -5.4123 & -0.9461 \end{bmatrix}; \\ \widetilde{K} &= \begin{bmatrix} -5.3744 & -0.9078 & 0 & 0 \\ 0 & 0 & -5.4123 & -0.9461 \end{bmatrix} \end{aligned} \tag{4.44}$$

where $\|\Delta_1^u\|^2 \leq 0.0536$ and $\|\Delta_2^u\|^2 \leq 0.0903$. It thus follows from Theorem 4.2 that the decentralized state feedback controller $\tilde{u}(t) = \widetilde{K}x(t)$ absolutely stabilizes the uncertain system (4.40), (4.42).

For comparison purpose, we apply the method in [191] to synthesize a decentralized state feedback robust H^∞ controller for the same interconnected pendulums as in (4.40). Here, we consider the interconnection between the two subsystems as uncertainty. Thus, the linearized model of this uncertain system is written as follows:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} w(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(t) \\ &\quad + \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \xi(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \eta(t); \\ z(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} u(t); \\ \zeta(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t) \end{aligned} \tag{4.45}$$

where $\eta(t)$ is an uncertainty input due to interconnection effect on each subsystem. From [191, 230], the uncertainty inputs $\xi(t)$ and $\eta(t)$, and the uncertainty output $\zeta(t)$ in (4.45) are related according to

$$\xi(t) = \Delta\,\zeta(t); \quad \eta(t) = \Delta\,\zeta(t); \quad \text{for } 0 \le \Delta \le 1. \tag{4.46}$$

Thus, the following IQCs

$$\begin{aligned} \int_0^\infty \|\xi(t)\|^2\,dt &\le \int_0^\infty \|\zeta(t)\|^2\,dt + d_1; \\ \int_0^\infty \|\eta(t)\|^2\,dt &\le \int_0^\infty \|\zeta(t)\|^2\,dt + d_2 \end{aligned} \tag{4.47}$$

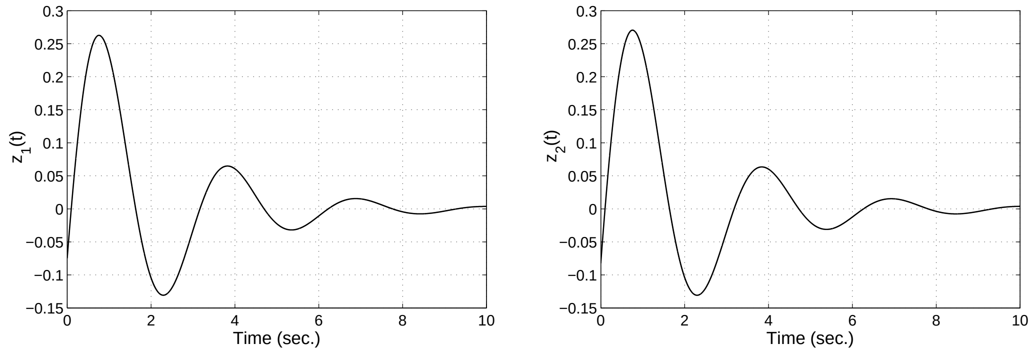
are satisfied for any $d_1, d_2 \ge 0$. Note that the paper [191] does not provide a numerical algorithm for computing the required parameters. Thus, in this example, we also apply the DE method to this problem and obtain

$$\gamma = 0.5308; \quad \tau = \begin{bmatrix} 0.3609 & 0.3526 \end{bmatrix}; \quad \bar{\theta} = \begin{bmatrix} 0.3427 & 0.3592 \end{bmatrix} \tag{4.48}$$

where $\bar{\theta}$ is a vector of scaling constants associated with uncertainty due to interconnection. Given the constants in (4.48), we then use them to compute stabilizing solutions to two Riccati equations of the form (4.29) corresponding to the two inverted pendulums in (4.45). Thus, the decentralized state feedback gain matrices are obtained as follows:

$$\widetilde{K} = 10^3 \times \begin{bmatrix} -8.9995 & -3.8592 & 0 & 0 \\ 0 & 0 & -6.8909 & -2.9499 \end{bmatrix}. \tag{4.49}$$

Having the numerical outcomes above, we notice that our method results in a smaller disturbance attenuation level γ than that given by the method in [191]. This fact confirms that our knowledge about the interconnection between subsystems is useful to improve the disturbance attenuation level γ and to reduce conservatism of the resulting decentralized controller. Moreover, the decentralized state feedback gains in (4.44) are much smaller those in (4.49). This is a consequence of our method which involves reducing the size of the neglected off-diagonal blocks of the non-decentralized state feedback gain matrix K. Small decentralized state feedback gains are indeed preferable in practice because they are less likely to cause control input saturation, noise amplification and instability due to unmodeled dynamics.



(a) (b)

Figure 4.1: Controlled outputs $z_1(t)$ and $z_2(t)$.

To demonstrate the performance of the decentralized state feedback controller $\widetilde{K}$ in (4.44), we simulate the resulting decentralized control system using Simulink for different $\Delta(\mu)$ as defined in (4.41). The closed loop system is only perturbed by an initial condition

$$x(0) = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^T. \tag{4.50}$$

The trajectories of both controlled outputs $z_1(t)$ and $z_2(t)$ are then shown in Figure 4.1. Moreover, the H^∞ norms of the closed loop transfer function matrix $T_{wz}(s)$ from $w(t)$ to $z(t)$ for different values of μ are obtained as follows:

μ	-0.5	-0.2	0	0.2	0.5
$\|T_{wz}(s)\|_\infty$	0.4776	0.4801	0.4818	0.4834	0.4859

(4.51)

and the maximum singular values $\sigma_{\max}$ of the closed loop system are shown in Figure 4.2. It is evident from (4.51) that the decentralized state feedback controller $\widetilde{K}$ in (4.44) results in the closed loop H^∞ norms which are less than the specified value of $\gamma = 0.5181$. These results show that the closed loop system is absolutely stable and the decentralized controller is capable of maintaining good disturbance attenuation performance of the closed loop system in the presence of uncertainties.

Remark 4.1. We compare our method with that in [191] because the latter also applies an IQC-based framework to solve the decentralized state feedback

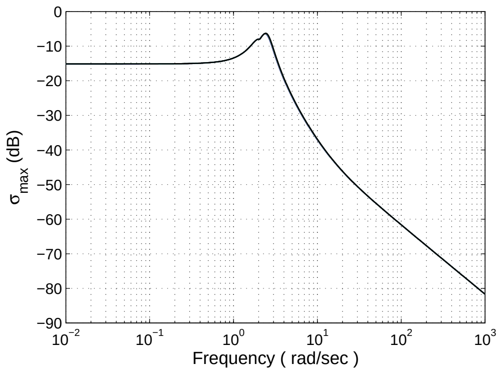


Figure 4.2: Maximum singular value $\sigma_{\max}$.

robust H^∞ control problem. Here, we show that the (partially) known interconnections between subsystems can be exploited to improve the performance of the decentralized controller. Moreover, when there is an unstable subsystem which cannot be controlled directly, our method still can be used to synthesize an absolutely stabilizing decentralized controller provided the interconnections are known. However, this latter case is not solvable using the method in [191] since the interconnections are considered as uncertainties rather than as useful structural information which can be exploited for control purposes.

4.6 Conclusions

We have presented a new method for constructing a decentralized state feedback robust H^∞ controller for a large-scale linear time-invariant uncertain system with IQCs. The main idea is to treat the neglected off-diagonal blocks of the controller gain matrix as uncertainties. This approach enables the controller to exploit the interconnections between subsystems. Moreover, the decentralized controller is required to provide an absolutely stable closed loop system with a specified disturbance attenuation level.

We have also applied an evolutionary optimization approach, namely the DE algorithm, to solve a nonconvex nonlinear optimization problem arising in the proposed decentralized controller synthesis method. The decentralized controller is then constructed using a stabilizing solution to an algebraic Riccati equation, which is dependent on a set of scaling constants. These constants correspond to the system uncertainties and norm bounds of the neglected off-diagonal blocks.

An example with numerical and simulation results is presented in order to demonstrate the efficacy of the proposed decentralized controller design method. Through this example, we also provide a comparison with another relevant IQC-based method as presented in [191]. It is evident that our decentralized state feedback controller has better features as compared to that constructed using the method in [191] whenever the interconnections between subsystems are (partially) known and can be exploited for control purposes.

Chapter 5

Decentralized Nonlinear Robust H^∞ Control

5.1 Introduction

Combining the ideas in Chapter 3 and Chapter 4, we propose a new method to synthesize stable decentralized nonlinear robust H^∞ controllers for a class of large-scale nonlinear uncertain systems. The structured uncertainties and known nonlinearities in the system are characterized in terms of integral quadratic constraints (IQCs) and global Lipschitz conditions (GLCs), respectively. It is common to view a large-scale system as consisting of interconnected subsystems. We assume that the interconnections are well known and are therefore not treated as uncertainties because they may provide useful structural information about the large-scale system being controlled. On the basis of this perspective, we allow the decentralized control system to exploit the interconnections while achieving absolute stability with a specified disturbance attenuation level.

Each decentralized controller is assumed to be dependent only on the measurement output of the corresponding local subsystem although no assumptions are made regarding the structure of the large-scale system. However, we cannot simply ignore the influence of the measurement outputs of other subsystems as in the case of controlling a large-scale nonlinear system using a single non-decentralized nonlinear controller of the same structure. This then results in nonlinear error systems, which originate from the difference between the decentralized and non-decentralized nonlinear controllers. Thus, in constructing the decentralized nonlinear controllers, we consider the nonlinear error system corresponding to each decentralized controller as an additional uncertainty for the original nonlinear plant. Consequently, the decentralized nonlinear controllers are required to be stable and robust against both the plant and additional uncertainties. In fact, decentralized controllers with these features have also been considered in [166] for a linear robust H^∞ control problem.

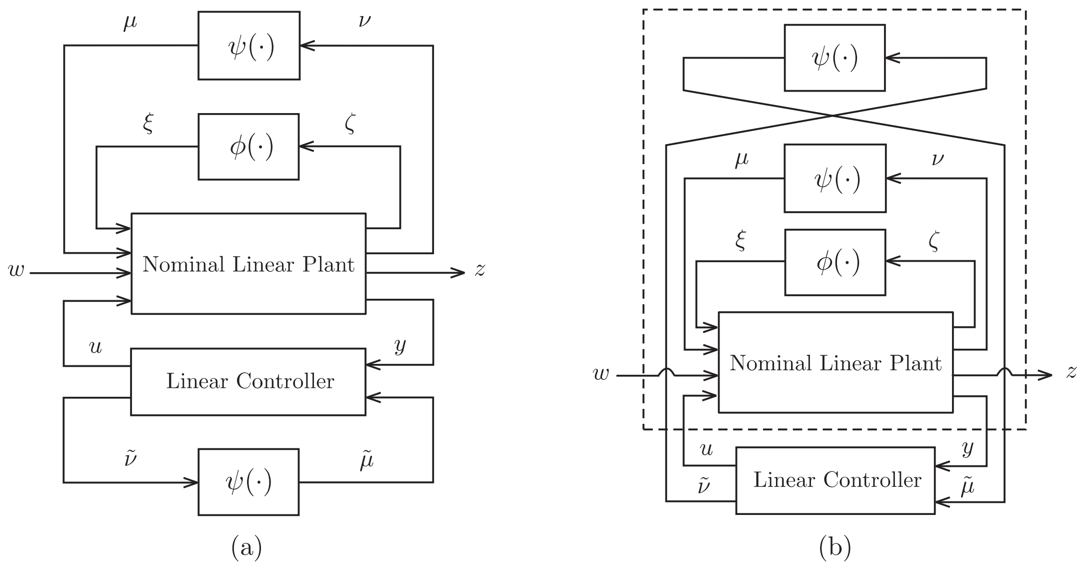


Figure 5.1: (a) Large-scale nonlinear uncertain system with decentralized nonlinear controller. (b) Large-scale nonlinear uncertain system and decentralized linear controller with repeated nonlinearity. Here, $\psi(\cdot)$ is a known nonlinearity and $\phi(\cdot)$ is an uncertainty; see [6].

A particular way to obtain our decentralized nonlinear controllers is through the addition of a copy of the plant nonlinearity to the linear part of the controllers as shown in Figure 5.1(a). The aim of this inclusion is to enable the decentralized controllers to exploit the plant nonlinearity without directly treating it as uncertainty such that we can obtain decentralized nonlinear controllers with better disturbance attenuation performance. As we use the results on linear robust H^∞ control in [187] to solve our decentralized control problem, the copy of the nonlinearity is then incorporated into the plant (see Figure 5.1(b)) and it is required to satisfy extra IQCs derived from the GLCs. Therefore, the solution to the decentralized nonlinear control problem is given in terms of the stabilizing solutions to algebraic Riccati equations, which are dependent on a set of scaling parameters.

This approach to designing decentralized nonlinear controllers only provides us with sufficient conditions. Moreover, it also results in a numerical algorithm, which involves parameterized nonconvex nonlinear constraints. We realize that this type of numerical problem is often difficult to solve using regular optimization methods. Thus, all design parameters are computed using an evolutionary optimization method, namely the differential evolution (DE) algorithm as described in Chapter 2.

5.2 Problem Statement

In this section, we introduce the nonlinear uncertain systems being considered and formulate a corresponding decentralized nonlinear control problem. This formulation includes all necessary notation and definitions, which closely follow those in previous chapters.

5.2.1 System description and definitions

We are concerned with a decentralized nonlinear robust H^∞ control problem for a class of large-scale nonlinear uncertain systems represented as follows:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) + \sum_{s=1}^{f} E_{1,s}\xi_s(t) + \sum_{i=1}^{g} E_{2,i}\mu_i(t), \quad x(0) = x_0;\\
z(t) &= C_1 x(t) + D_{12} u(t);\\
\zeta_1(t) &= H_{1,1} x(t) + G_{1,1} u(t);\\
&\vdots\\
\zeta_f(t) &= H_{1,f} x(t) + G_{1,f} u(t);\\
\nu_1(t) &= H_{2,1} x(t) + G_{2,1} u(t);\\
&\vdots\\
\nu_g(t) &= H_{2,g} x(t) + G_{2,g} u(t);\\
y(t) &= C_2 x(t) + D_{21} w(t) + \sum_{s=1}^{f} F_{1,s}\xi_s(t) + \sum_{i=1}^{g} F_{2,i}\mu_i(t)
\end{aligned} \tag{5.1}$$

where $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^m$ is the control input, $w \in \mathbf{R}^d$ is the disturbance input, $z \in \mathbf{R}^q$ is the controlled output, $y \in \mathbf{R}^l$ is the measurement output, $\xi_1 \in \mathbf{R}^{r_1}, \ldots, \xi_f \in \mathbf{R}^{r_f}$ are the uncertainty inputs, $\zeta_1 \in \mathbf{R}^{h_1}, \ldots, \zeta_f \in \mathbf{R}^{h_f}$ are the uncertainty outputs, $\mu_1 \in \mathbf{R}, \ldots, \mu_g \in \mathbf{R}$ are the nonlinearity inputs, and $\nu_1 \in \mathbf{R}, \ldots, \nu_g \in \mathbf{R}$ are the nonlinearity outputs. All coefficient matrices in (5.1) have compatible dimensions with those of the signals.

The relationship between the nonlinearity input $\mu_i(t)$ and nonlinearity output $\nu_i(t)$ in the system (5.1) is described as

$$\mu_i(t) = \psi_i\left(\nu_i(t)\right), \quad \forall i = 1, 2, \ldots, g \tag{5.2}$$

and satisfies condition $\psi_i(0) = 0$. Each nonlinear function $\psi_i(\cdot)$ is assumed to be known and satisfies the global Lipschitz condition written as

$$|\psi_i(\nu(t)) - \psi_i(\tilde{\nu}(t))| \leq \beta_i |\nu(t) - \tilde{\nu}(t)|, \quad \forall i = 1, 2, \ldots, g \tag{5.3}$$

for all $(\nu(t), \tilde{\nu}(t))$ and $\beta_i \geq 0$. Moreover, the uncertainty input $\xi_s(t)$ and uncertainty output $\zeta_s(t)$ in the system (5.1) are related as follows:

$$\xi_s(t) = \phi_s\left(t, \zeta_s(\cdot)|_0^t\right), \quad \forall s = 1, 2, \ldots, f. \tag{5.4}$$

where $\phi_s(\cdot)$ can be a nonlinear time-varying and dynamic functional; see [2]. The uncertainty input $\xi_s(t)$ is said to be admissible if it satisfies the integral quadratic constraint defined as follows:

Definition 5.1. (*Integral Quadratic Constraint*; e.g., see [2]) An uncertainty of the form (5.4) is an admissible uncertainty for the system (5.1) if the following conditions hold: Given any locally square integrable control input $u(\cdot)$ and locally square integrable disturbance input $w(\cdot)$, and any corresponding solution to the system (5.1), (5.4), let $(0, t_\star)$ be the interval on which the solution exists. Then there exist constants $d_{1,1} \geq 0, \ldots, d_{1,f} \geq 0$ and a sequence $\{t_k\}_{k=1}^\infty$ such that $t_k \to t_\star$, $t_k \geq 0$ and

$$\int_0^{t_k} \|\xi_s(t)\|^2 dt \leq \int_0^{t_k} \|\zeta_s(t)\|^2 dt + d_{1,s} \tag{5.5}$$

for all k and for all $s = 1, 2, \ldots, f$. Note that t_k and $t_\star$ may be equal to infinity.

The class of all admissible uncertainties $\xi(\cdot) = [\xi_1(\cdot), \ldots, \xi_f(\cdot)]$ is denoted by Ξ.

We wish to synthesize a stable decentralized nonlinear controller for the nonlinear uncertain system (5.1), (5.2), (5.5) based on the results in Chapter 3. In this case, the general form of a nonlinear robust H^∞ output feedback controller is given as follows:

$$\begin{aligned}
\dot{x}_c(t) &= Nx_c(t) + My(t) + \sum_{i=1}^{g} L_i \tilde{\mu}_{c_i}(t); \quad x_c(0) = x_{c_0};\\
u(t) &= Kx_c(t);\\
\tilde{\nu}_{c_1}(t) &= P_1 x_c(t);\\
&\vdots\\
\tilde{\nu}_{c_g}(t) &= P_g x_c(t)
\end{aligned} \tag{5.6}$$

where $x_c \in \mathbf{R}^n$ and

$$\tilde{\mu}_{c_i}(t) = \psi_i\left(\tilde{\nu}_{c_i}(t)\right) \quad \text{for } i = 1, 2, \ldots, g. \tag{5.7}$$

Indeed, the nonlinear controller (5.6), (5.7) is of n-th order and is constructed by adding a copy of each nonlinearity (5.2) to the linear part of the controller. The purpose of this inclusion is to enable the controller to exploit the plant nonlinearity while achieving a desired control objective.

It is assumed that the large-scale nonlinear uncertain system (5.1), (5.2), (5.5) consists of p interconnected subsystems. This structure then leads to a decomposition of the measurement output $y(t) \in \mathbf{R}^l$ into p components as follows:

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix} \tag{5.8}$$

where the output $y_j(t) \in \mathbf{R}^{l_j}$, for $j = 1, 2, \ldots, p$, is only available to the controller for the j-th subsystem. Therefore, an output feedback control input $\tilde{u}_j(t)$ for j-th subsystem is assumed to be dependent only on the measurement output $y_j(t)$. This assumption leads to a decentralized control input $\tilde{u}(t) \in \mathbf{R}^m$, which also has

p components as follows:

$$\tilde{u}(t) = \begin{bmatrix} \tilde{u}_1(t) \\ \tilde{u}_2(t) \\ \vdots \\ \tilde{u}_p(t) \end{bmatrix} \tag{5.9}$$

where $\tilde{u}_j(t) \in \mathbf{R}^{m_j}$ for $j = 1, 2, \ldots, p$. The assumption on the relation between $y_j(t)$ and $\tilde{u}_j(t)$ is made regardless of the structure of A, B_2 and C_2 in (5.1).

The structure of j-th decentralized nonlinear output feedback controller is the same as that of the non-decentralized one as shown in (5.6), (5.7). Thus, each decentralized controller can be viewed as a special case of its non-decentralized counterpart. That is,

$$\begin{aligned} \dot{x}_{d_j}(t) &= N x_{d_j}(t) + M_j y_j(t) + \sum_{i=1}^{g} L_i \tilde{\mu}_{d_i}(t); \quad x_{d_j}(0) = x_{d_{j0}} \\ \tilde{u}_j(t) &= K_j x_{d_j}(t); \\ \tilde{\nu}_{d_1}(t) &= P_1 x_{d_j}(t); \\ &\vdots \\ \tilde{\nu}_{d_g}(t) &= P_g x_{d_j}(t) \end{aligned} \tag{5.10}$$

where $x_{d_j} \in \mathbf{R}^n$ and

$$\tilde{\mu}_{d_i}(t) = \psi_i\left(\tilde{\nu}_{d_i}(t)\right) \tag{5.11}$$

for $j = 1, 2, \ldots, p$ and $i = 1, 2, \ldots, g$. The purpose of applying the decentralized controller (5.10), (5.11) to the large-scale nonlinear uncertain system (5.1), (5.2), (5.5) is to obtain an absolutely stable closed loop system with a specified disturbance attenuation level.

Definition 5.2. (*Absolute Stabilizability*; e.g., see [2]) The nonlinear uncertain system (5.1), (5.2), (5.5) is said to be absolutely stabilizable with disturbance attenuation level γ via the decentralized stable output feedback controller (5.10), (5.11) if there exist constants $c_1 > 0$ and $c_2 > 0$ such that the following conditions hold:

1. For any initial condition $[x(0), x_{d_1}(0), \ldots, x_{d_p}(0)]$, any admissible uncertainty inputs $\xi_1(\cdot), \ldots, \xi_f(\cdot)$ and any disturbance input $w(\cdot) \in \mathbf{L}_2[0, \infty)$,

then $\left[x(\cdot), x_{d_1}(\cdot), \ldots, x_{d_p}(\cdot), u(\cdot), \xi_1(\cdot), \ldots, \xi_f(\cdot)\right] \in \mathbf{L}_2[0,\infty)$ (hence, $t_\star = \infty$) and

$$\begin{aligned}\|x(\cdot)\|_2^2+&\sum_{j=1}^{p}\|x_{d_j}(\cdot)\|_2^2 + \|u(\cdot)\|_2^2 + \sum_{s=1}^{f}\|\xi_s(\cdot)\|_2^2\\ &\le c_1\left[\|x(0)\|^2 + \sum_{j=1}^{p}\|x_{d_j}(0)\|^2 + \|w(\cdot)\|_2^2 + \sum_{s=1}^{f} d_{1,s}\right]. \end{aligned} \tag{5.12}$$

2. The following H^∞ norm bound condition is satisfied: If $x(0) = 0$ and $x_{d_1}(0) = \ldots = x_{d_p}(0) = 0$, then for $w(\cdot) \in \mathbf{L}_2[0,\infty)$ and $\xi_s(\cdot) \in \Xi$ (for all $s = 1, 2, \ldots, f$)

$$\mathcal{J} := \sup_{w(\cdot)}\sup_{\xi_s(\cdot)} \frac{\|z(\cdot)\|_2^2 - c_2\sum_{s=1}^{f} d_{1,s}}{\|w(\cdot)\|_2^2} < \gamma^2. \tag{5.13}$$

When constructing a decentralized control system, it is common to consider the interconnections between subsystems as uncertainties in addition to those of the plant under consideration; e.g., see [216, 219, 220, 224]. This is in contrast to our approach where they are assumed to be known and hence, not considered as additional sources of uncertainties for the whole large-scale system. The same assumption has been introduced in Chapter 4. It is based on an observation that in some applications, the interconnections may provide useful structural information for the entire decentralized control system. This paradigm allows the decentralized controllers to exploit the interconnections while achieving the control objective. However, we need to take into account the influence of other components of $y(t)$ apart from $y_j(t)$ on j-th decentralized nonlinear controller. Thus, they are considered to give rise additional uncertainties in the plant. Using this approach, the resulting decentralized controller is expected to be robust against uncertainties in itself and of the plant.

5.2.2 Nonlinear error system

Although the decentralized control input $\tilde{u}_j(t)$ depends only on the measurement output $y_j(t)$, the other components of $y(t)$ may also affect the dynamics of j-th decentralized controller as it is a nonlinear output feedback controller. This situation constitutes a difference between the non-decentralized control input $u_j(t)$ and the decentralized control input $\tilde{u}_j(t)$, which can be described as

$$\Delta u_j(t) := u_j(t) - \tilde{u}_j(t) = K_j \left(x_c(t) - x_{d_j}(t) \right) \tag{5.14}$$

for $j = 1, 2, \ldots, p$. The error signal $\Delta u_j(t)$ is considered as an output of the j-th nonlinear error system defined as follows:

$$\begin{aligned} \begin{bmatrix} \dot{x}_c(t) \\ \dot{x}_{d_j}(t) \end{bmatrix} &= \mathcal{N} \begin{bmatrix} x_c(t) \\ x_{d_j}(t) \end{bmatrix} + \mathcal{M} y(t) + \mathcal{L} \begin{bmatrix} \tilde{\mu}_c(t) \\ \tilde{\mu}_d(t) \end{bmatrix}; \\ \Delta u_j(t) &= \mathcal{K} \begin{bmatrix} x_c(t) \\ x_{d_j}(t) \end{bmatrix}; \\ \begin{bmatrix} \tilde{\nu}_c(t) \\ \tilde{\nu}_d(t) \end{bmatrix} &= \mathcal{P} \begin{bmatrix} x_c(t) \\ x_{d_j}(t) \end{bmatrix} \end{aligned} \tag{5.15}$$

where

$$\begin{aligned} \mathcal{N} &= \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}; \ \mathcal{M} = \begin{bmatrix} M \\ MV_j \end{bmatrix}; \ \mathcal{L} = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}; \ \mathcal{P} = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}; \\ \mathcal{K} &= \begin{bmatrix} Z_j K & -Z_j K \end{bmatrix}; \ M = \begin{bmatrix} M_1 & \cdots & M_p \end{bmatrix}; \ L = \begin{bmatrix} L_1 & \cdots & L_g \end{bmatrix}; \\ K &= \begin{bmatrix} K_1 \\ \vdots \\ K_p \end{bmatrix}; \ P = \begin{bmatrix} P_1 \\ \vdots \\ P_g \end{bmatrix}; \ \tilde{\mu}_c(t) = \begin{bmatrix} \mu_{c_1}(t) \\ \vdots \\ \mu_{c_g}(t) \end{bmatrix}; \ \tilde{\mu}_d(t) = \begin{bmatrix} \mu_{d_1}(t) \\ \vdots \\ \mu_{d_g}(t) \end{bmatrix}; \\ \tilde{\nu}_c(t) &= \begin{bmatrix} \nu_{c_1}(t) \\ \vdots \\ \nu_{c_g}(t) \end{bmatrix}; \quad \tilde{\nu}_d(t) = \begin{bmatrix} \nu_{d_1}(t) \\ \vdots \\ \nu_{d_g}(t) \end{bmatrix}; \quad V_1 = \begin{bmatrix} I_{l_1 \times l_1} & 0_{l_1 \times \tilde{l}_1} \\ 0_{\tilde{l}_1 \times l_1} & 0_{\tilde{l}_1 \times \tilde{l}_1} \end{bmatrix}; \\ V_b &= \begin{bmatrix} 0_{\bar{l}_{b-1} \times \bar{l}_{b-1}} & 0_{\bar{l}_{b-1} \times l_b} & 0_{\bar{l}_{b-1} \times \tilde{l}_b} \\ 0_{l_b \times \bar{l}_{b-1}} & I_{l_b \times l_b} & 0_{l_b \times \tilde{l}_b} \\ 0_{\tilde{l}_b \times \bar{l}_{b-1}} & 0_{\tilde{l}_b \times l_b} & 0_{\tilde{l}_b \times \tilde{l}_b} \end{bmatrix}; \quad V_p = \begin{bmatrix} 0_{\bar{l}_{p-1} \times \bar{l}_{p-1}} & 0_{\bar{l}_{p-1} \times l_p} \\ 0_{l_p \times \bar{l}_{p-1}} & I_{l_p \times l_p} \end{bmatrix}; \\ Z_1 &= \begin{bmatrix} I_{m_1 \times m_1} & 0_{m_1 \times \tilde{m}_1} \end{bmatrix}; \quad Z_b = \begin{bmatrix} 0_{m_b \times \bar{m}_{b-1}} & I_{m_b \times m_b} & 0_{m_b \times \tilde{m}_b} \end{bmatrix}; \\ Z_p &= \begin{bmatrix} 0_{m_p \times \bar{m}_{p-1}} & I_{m_p \times m_p} \end{bmatrix} \end{aligned} \tag{5.16}$$

for $b = 2, 3, \ldots, p-1$ and $j = 1, 2, \ldots, p$. Note that $\bar{l}_j = \sum_{k=1}^{j} l_k$; $\tilde{l}_j = l - \bar{l}_j$; $\bar{m}_j = \sum_{k=1}^{j} m_k$ and $\tilde{m}_j = m - \bar{m}_j$.

The nonlinearity input and output in (5.15) are represented as follows:

$$\rho(t) := \begin{bmatrix} \rho_1(t) \\ \vdots \\ \rho_{\bar{g}}(t) \end{bmatrix} = \begin{bmatrix} \tilde{\mu}_c(t) \\ \tilde{\mu}_d(t) \end{bmatrix}; \quad \eta(t) := \begin{bmatrix} \eta_1(t) \\ \vdots \\ \eta_{\bar{g}}(t) \end{bmatrix} = \begin{bmatrix} \tilde{\nu}_c(t) \\ \tilde{\nu}_d(t) \end{bmatrix} \tag{5.17}$$

with $\bar{g} = 2g$ and therefore,

$$\rho_a(t) = \psi_a(\eta_a(t)), \quad \psi_a(0) = 0, \quad \text{for } a = 1, 2, \ldots, \bar{g}. \tag{5.18}$$

The nonlinear function $\psi_a(\cdot)$ in (5.18) is also required to satisfy the GLC as follows:

$$|\psi_a(\eta) - \psi_a(\tilde{\eta})| \leq \beta_a |\eta - \tilde{\eta}|, \quad \beta_a \geq 0. \tag{5.19}$$

Note that $\beta_i = \beta_{g+i}$ for $i = 1, 2, \ldots, g$. From the GLCs in (5.19), we can define an additional set of IQCs as follows:

$$\int_0^{t_k} (\rho_a(t))^2 \, dt \leq \int_0^{t_k} \beta_a^2 (\eta_a(t))^2 \, dt + d_{5,a} \tag{5.20}$$

with $d_{5,a} \geq 0$, for all $a = 1, 2, \ldots, \bar{g}$ and for all $\{t_k \geq 0\}_{k=1}^{\infty}$. The set of all admissible uncertainty inputs $\rho_a(\cdot)$ is analogously defined as in the Definition 5.1 and denoted by Ξ_e.

Moreover, we define a constant $\delta_j > 0$ associated with the j-th nonlinear error system (5.15), (5.20) such that the following H^∞ norm bound condition

$$\mathcal{J}_{e,j} := \sup_{y(\cdot) \in \mathbf{L}_2[0,\infty)} \sup_{\rho_a(\cdot) \in \Xi_e} \frac{\|\Delta u_j(\cdot)\|_2^2 - \varepsilon_j \sum_{a=1}^{\bar{g}} d_{5,a}}{\|y(\cdot)\|_2^2} < \delta_j^2 \tag{5.21}$$

is satisfied for all $\rho_a(\cdot) \in \Xi_e$, a given constant $\varepsilon_j > 0$, $x_c(0) = 0$ and $x_{d_j}(0) = 0$. The condition (5.21) then imposes an absolute stability constraint on the j-th nonlinear error system (5.15), (5.20) and thus leads to the following lemma:

Lemma 5.1. *Let $\beta_1 \geq 0, \ldots, \beta_g \geq 0$, $\delta_j > 0$ be given constants. Suppose that*

$\mathcal{A}_{e_j}$ is stable and there exist constants $\theta_{j,1} > 0, \ldots, \theta_{j,\bar{g}} > 0$ such that

$$\| \mathcal{C}_{e_j}(sI - \mathcal{A}_{e_j})^{-1}\mathcal{B}_{e_j} \|_\infty < 1, \quad \text{for } j = 1, 2, \ldots, p \tag{5.22}$$

where $\mathcal{A}_{e_j} = \mathcal{N}$;

$$\mathcal{B}_{e_j} = \begin{bmatrix} \delta_j^{-1}M & \sqrt{\theta_{j,1}}^{-1}L_1 & \ldots & \sqrt{\theta_{j,g}}^{-1}L_g & 0 & \ldots & 0 \\ \delta_j^{-1}MV_j & 0 & \ldots & 0 & \sqrt{\theta_{j,g+1}}^{-1}L_1 & \ldots & \sqrt{\theta_{j,\bar{g}}}^{-1}L_g \end{bmatrix};$$

$$\mathcal{C}_{e_j} = \begin{bmatrix} Z_jK & -Z_jK \\ \beta_1\sqrt{\theta_{j,1}}\,P_1 & 0 \\ \vdots & \vdots \\ \beta_g\sqrt{\theta_{j,g}}\,P_g & 0 \\ 0 & \beta_1\sqrt{\theta_{j,g+1}}\,P_1 \\ \vdots & \vdots \\ 0 & \beta_g\sqrt{\theta_{j,\bar{g}}}\,P_g \end{bmatrix}. \tag{5.23}$$

Then, the H^∞ norm bound

$$\tilde{\mathcal{J}}_{e,j} := \sup_{\tilde{y}_j(\cdot)\in\mathbf{L}_2[0,\infty), x_c(0)=0, x_{d_j}(0)=0} \frac{\|\Delta\tilde{u}_j(\cdot)\|_2^2}{\|\tilde{y}_j(\cdot)\|_2^2} < 1 \tag{5.24}$$

is satisfied, where

$$\tilde{y}_j(t) = \begin{bmatrix} \delta_j y(t) \\ \sqrt{\theta_{j,1}}\,\rho_1(t) \\ \vdots \\ \sqrt{\theta_{j,\bar{g}}}\,\rho_{\bar{g}}(t) \end{bmatrix}; \quad \Delta\tilde{u}_j = \begin{bmatrix} \Delta u_j \\ \beta_1\sqrt{\theta_{j,1}}\,\eta_1(t) \\ \vdots \\ \beta_{\bar{g}}\sqrt{\theta_{j,\bar{g}}}\,\eta_{\bar{g}}(t) \end{bmatrix}. \tag{5.25}$$

Moreover, if this condition holds, then the j-th nonlinear error system (5.15), (5.20) *is absolutely stable with disturbance attenuation level $\delta_j > 0$. Hence, the H^∞ norm bound* (5.21) *is satisfied.*

Remark 5.1. The proof of this lemma follows via the same arguments as the sufficiency proof of Theorem 4.1 in [187]. Since the j-th nonlinear error system (5.15), (5.20) is such that $\mathcal{A}_{e_j}$ is stable, then we can always find a constant $\delta_j > 0$ such that (5.22) is satisfied.

5.2.3 Equivalent nonlinear uncertain system

To solve the decentralized nonlinear control problem described in sub-Section 5.2.1 using the stable nonlinear controller design method presented in Chapter 3, we first need to form an equivalent nonlinear uncertain system. For this purpose, we consider the error output $\Delta u_j(t)$ in (5.15) as an additional uncertainty input $\xi_j^u(t)$ to the original nonlinear uncertain system (5.1), (5.2), (5.5). Then, referring to (5.14), we can write $\tilde{u}_j(t)$ as

$$\tilde{u}_j(t) = u_j(t) + \xi_j^u(t) \tag{5.26}$$

for $j = 1, 2, \ldots, p$. According to (5.15), the additional uncertainty input $\xi_j^u(t)$ and uncertainty output $\zeta_j^u(t)$ can then be defined as follows:

$$\begin{aligned} \xi_j^u(t) &:= -\Delta u_j(t); \\ \zeta_j^u(t) &:= y(t) \end{aligned} \tag{5.27}$$

for $j = 1, 2, \ldots, p$. Now, we can rewrite the decentralized control input $\tilde{u}(t)$ as

$$\tilde{u}(t) = u(t) + \sum_{j=1}^{p} J_j^u \xi_j^u(t) \tag{5.28}$$

where, for $b = 2, 3, \ldots, p-1$,

$$\begin{aligned} J_1^u &= \begin{bmatrix} I_{m_1 \times m_1} \\ 0_{\tilde{m}_1 \times m_1} \end{bmatrix}; \\ J_b^u &= \begin{bmatrix} 0_{\bar{m}_{b-1} \times m_b} \\ I_{m_b \times m_b} \\ 0_{\tilde{m}_b \times m_b} \end{bmatrix}; \\ J_p^u &= \begin{bmatrix} 0_{\bar{m}_{p-1} \times m_p} \\ I_{m_p \times m_p} \end{bmatrix}. \end{aligned} \tag{5.29}$$

Applying the decentralized control input $\tilde{u}(t)$ in (5.28) to the nonlinear uncertain system (5.1), (5.2), (5.5), we obtain the same closed loop system as if we apply a non-decentralized control input $u(t)$ to an equivalent nonlinear uncertain system defined as follows:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) + B_2 J_u \xi_u(t) + E_1 \xi(t) + E_2 \mu(t);\\
z(t) &= C_1 x(t) + D_{12} u(t) + D_{12} J_u \xi_u(t);\\
\zeta(t) &= H_1 x(t) + G_1 u(t) + G_1 J_u \xi_u(t);\\
\nu(t) &= H_2 x(t) + G_2 u(t) + G_2 J_u \xi_u(t);\\
\zeta_u(t) &= \bar{I}\left[C_2 x(t) + D_{21} w(t) + F_1 \xi(t) + F_2 \mu(t)\right];\\
y(t) &= C_2 x(t) + D_{21} w(t) + F_1 \xi(t) + F_2 \mu(t)
\end{aligned} \tag{5.30}$$

where

$$\begin{aligned}
\xi(t) &= \begin{bmatrix} \xi_1(t) \\ \vdots \\ \xi_f(t) \end{bmatrix}; \quad \zeta(t) = \begin{bmatrix} \zeta_1(t) \\ \vdots \\ \zeta_f(t) \end{bmatrix}; \quad \mu(t) = \begin{bmatrix} \mu_1(t) \\ \vdots \\ \mu_g(t) \end{bmatrix};\\
\nu(t) &= \begin{bmatrix} \nu_1(t) \\ \vdots \\ \nu_g(t) \end{bmatrix}; \quad \xi_u(t) = \begin{bmatrix} \xi_1^u(t) \\ \vdots \\ \xi_p^u(t) \end{bmatrix}; \quad \zeta_u(t) = \begin{bmatrix} \zeta_1^u(t) \\ \vdots \\ \zeta_p^u(t) \end{bmatrix};\\
J_u &= \begin{bmatrix} J_1^u & \cdots & J_p^u \end{bmatrix}; \; E_1 = \begin{bmatrix} E_{1,1} & \cdots & E_{1,f} \end{bmatrix}; \; E_2 = \begin{bmatrix} E_{2,1} & \cdots & E_{2,g} \end{bmatrix};\\
H_1 &= \begin{bmatrix} H_{1,1} \\ \vdots \\ H_{1,f} \end{bmatrix}; \; H_2 = \begin{bmatrix} H_{2,1} \\ \vdots \\ H_{2,g} \end{bmatrix}; \; G_1 = \begin{bmatrix} G_{1,1} \\ \vdots \\ G_{1,f} \end{bmatrix}; \; G_2 = \begin{bmatrix} G_{2,1} \\ \vdots \\ G_{2,g} \end{bmatrix}; \; \bar{I} = \begin{bmatrix} I_1 \\ \vdots \\ I_p \end{bmatrix};\\
F_1 &= \begin{bmatrix} F_{1,1} & \cdots & F_{1,f} \end{bmatrix}; \quad F_2 = \begin{bmatrix} F_{2,1} & \cdots & F_{2,g} \end{bmatrix};\\
I_j &= I_{l \times l}
\end{aligned} \tag{5.31}$$

for all $j = 1, 2, \ldots, p$. Note that l is the dimension of the measurement output y. Each pair of additional uncertainty input $\xi_j^u(t)$ and output $\zeta_j^u(t)$ as defined in (5.27) has to satisfy an IQC of the form

$$\int_0^{t_k} \|\xi_j^u(t)\|^2 \, dt \leq \int_0^{t_k} \delta_j^2 \|\zeta_j^u(t)\|^2 \, dt + d_j^u \tag{5.32}$$

with $d_j^u \geq 0$, for all $j = 1, 2, \ldots, p$ and for all $\{t_k \geq 0\}_{k=1}^{\infty}$. It then follows from Definition 5.1 that an admissible uncertainty input $\xi_j^u(t)$ belongs to a set Ξ_u.

5.2.4 Robust H^∞ control

Following from the construction in sub-Section 5.2.3, we transform the problem of designing a stable decentralized nonlinear controller of the form (5.10), (5.11) for the original nonlinear uncertain system (5.1), (5.2), (5.5) to that of designing a stable non-decentralized nonlinear controller of the form (5.6), (5.7) for the equivalent nonlinear uncertain system (5.30), (5.2), (5.5), (5.32). Then, in order to construct the stable non-decentralized nonlinear controller (5.6), (5.7) using the results in Chapter 3, we first incorporate all copies of the nonlinearities (5.7) into the plant description (5.30). This step allows us to rewrite the nonlinear controller (5.6), (5.7) as

$$\begin{aligned} \dot{x}_c(t) &= N x_c(t) + \tilde{M}\bar{y}(t); \\ \bar{u}(t) &= \tilde{K} x_c(t) \end{aligned} \tag{5.33}$$

where

$$\begin{aligned} &\tilde{M} := \begin{bmatrix} M & L \end{bmatrix}; \quad \tilde{K} := \begin{bmatrix} K \\ P \end{bmatrix}; \quad \bar{y}(t) := \begin{bmatrix} y(t) \\ \tilde{\mu}(t) \end{bmatrix}; \\ &\bar{u}(t) := \begin{bmatrix} u(t) \\ \tilde{\nu}(t) \end{bmatrix}; \quad \tilde{\mu}(t) = \begin{bmatrix} \tilde{\mu}_1(t) \\ \vdots \\ \tilde{\mu}_g(t) \end{bmatrix}; \quad \tilde{\nu}(t) = \begin{bmatrix} \tilde{\nu}_1(t) \\ \vdots \\ \tilde{\nu}_g(t) \end{bmatrix}. \end{aligned} \tag{5.34}$$

Furthermore, referring to the GLCs (5.3), the nonlinearities (5.2) and their copies (5.7) can be characterized in terms of the IQCs:

$$\begin{aligned} &\int_0^{t_k} (\mu_i(t) - \tilde{\mu}_i(t))^2 \, dt \le \int_0^{t_k} \beta_i^2 (\nu_i(t) - \tilde{\nu}_i(t))^2 \, dt + d_{2,i}; \\ &\int_0^{t_k} (\mu_i(t))^2 \; dt \le \int_0^{t_k} \beta_i^2 (\nu_i(t))^2 \; dt + d_{3,i}; \\ &\int_0^{t_k} (\tilde{\mu}_i(t))^2 \; dt \le \int_0^{t_k} \beta_i^2 (\tilde{\nu}_i(t))^2 \; dt + d_{4,i} \end{aligned} \tag{5.35}$$

with $d_{2,i} \ge 0$, $d_{3,i} \ge 0$ and $d_{4,i} \ge 0$, and for all $i = 1, 2, \ldots, g$ and for all $\{t_k \ge 0\}_{k=1}^{\infty}$.

Using the expressions in (5.34) and (5.35), we can further simplify the equivalent nonlinear uncertain system (5.30), (5.2), (5.5), (5.32) into a linear uncertain system as follows:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + \bar{B}_2\bar{u}(t) + \bar{E}\tilde{\xi}(t);\\
z(t) &= C_1 x(t) + \bar{D}_{12}\bar{u}(t) + \bar{D}_{12}^J\tilde{\xi}(t);\\
\tilde{\zeta}(t) &= \bar{H}x(t) + \bar{G}_w w(t) + \bar{G}\bar{u}(t) + \bar{G}_J\tilde{\xi}(t);\\
\bar{y}(t) &= \bar{C}_2 x(t) + \bar{D}_{21}w(t) + \bar{F}\tilde{\xi}(t)
\end{aligned} \tag{5.36}$$

where $\bar{f} = f + \bar{g} + p$; $r = \sum_{s=1}^{f} r_s$; $\bar{r} = r + \bar{g}$; $h = \sum_{s=1}^{f} h_s$; $\bar{h} = h + \bar{g}$; and

$$\tilde{\xi}(t) = \begin{bmatrix}\xi(t)\\ \mu(t)\\ \tilde{\mu}(t)\\ \xi_u(t)\end{bmatrix};\ \tilde{\zeta}(t) = \begin{bmatrix}\zeta(t)\\ \nu(t)\\ \tilde{\nu}(t)\\ \zeta_u(t)\end{bmatrix};\ \bar{B}_2 = \begin{bmatrix}B_2 & 0_{n\times g}\end{bmatrix};\ \bar{E} = \begin{bmatrix}E_1 & E_2 & 0_{n\times g} & B_2 J_u\end{bmatrix};$$

$$\bar{D}_{12} = \begin{bmatrix}D_{12} & 0_{q\times g}\end{bmatrix};\ \bar{D}_{12}^J = \begin{bmatrix}0_{q\times\bar{r}} & D_{12}J_u\end{bmatrix};\ \bar{G}_w = \begin{bmatrix}0_{\bar{h}\times d}\\ \bar{I}D_{21}\end{bmatrix};$$

$$\bar{H} = \begin{bmatrix}H_1\\ H_2\\ 0_{g\times n}\\ \bar{I}C_2\end{bmatrix};\ \bar{G} = \begin{bmatrix}G_1 & 0_{h\times g}\\ G_2 & 0_{g\times g}\\ 0_{g\times m} & I_{g\times g}\\ 0_{pl\times m} & 0_{pl\times g}\end{bmatrix};\ \bar{G}_J = \begin{bmatrix}0_{h\times r} & 0_{h\times g} & 0_{h\times g} & G_1 J_u\\ 0_{g\times r} & 0_{g\times g} & 0_{g\times g} & G_2 J_u\\ 0_{g\times r} & 0_{g\times g} & 0_{g\times g} & 0_{g\times m}\\ \bar{I}F_1 & \bar{I}F_2 & 0_{pl\times g} & 0_{pl\times m}\end{bmatrix};$$

$$\bar{C}_2 = \begin{bmatrix}C_2\\ 0_{g\times n}\end{bmatrix};\ \bar{D}_{21} = \begin{bmatrix}D_{21}\\ 0_{g\times d}\end{bmatrix};\ \bar{F} = \begin{bmatrix}F_1 & F_2 & 0_{l\times g} & 0_{l\times m}\\ 0_{g\times r} & 0_{g\times g} & I_{g\times g} & 0_{g\times m}\end{bmatrix}. \tag{5.37}$$

The uncertainties of the system (5.36) are represented by rewriting all IQCs in (5.5), (5.35), (5.32) in the following form:

$$\int_0^{t_k} \tilde{\xi}(t)^T Q_\alpha \tilde{\xi}(t)\, dt \le \int_0^{t_k} \tilde{\zeta}(t)^T R_\alpha \tilde{\zeta}(t)\, dt + d_\alpha \tag{5.38}$$

with $d_\alpha \ge 0$, for all $\alpha = 1, 2, \ldots, \hat{f}$ and for all $\{t_k \ge 0\}_{k=1}^{\infty}$. Note that $\hat{f} = f + 3g + p$, and Q_α and R_α are symmetric matrices. Also, the constants δ_j^2 in (5.32) and β_i^2 in (5.35) are accordingly included in R_α in (5.38) corresponding to $\zeta_j^u(t)$, $\nu_i(t)$ and $\tilde{\nu}_i(t)$, respectively. The admissible uncertainty input $\tilde{\xi}(\cdot)$ is an element of $\tilde{\Xi}$ as defined in Definition 5.1.

However, we notice that the IQCs (5.38) are not expressed in the standard form as given in (5.5). Therefore, it is necessary to introduce scaling constants

$\lambda_\alpha \geq 0, \lambda_\alpha \in \mathbf{R}$ corresponding to each IQC in (5.38) as we use the results of [187] to solve our control problem. This approach allows us to define weighting matrices $\tilde{Q}(\lambda) \geq 0$ and $\tilde{R}(\lambda) \geq 0$, and a constant $\tilde{d}(\lambda) \geq 0$ as functions of

$$\lambda := \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{\hat{f}} \end{bmatrix}^T . \tag{5.39}$$

That is,

$$\tilde{Q}(\lambda) := \sum_{\alpha=1}^{\hat{f}} \lambda_\alpha Q_\alpha; \quad \tilde{R}(\lambda) := \sum_{\alpha=1}^{\hat{f}} \lambda_\alpha R_\alpha; \quad \tilde{d}(\lambda) := \sum_{\alpha=1}^{\hat{f}} \lambda_\alpha d_\alpha \tag{5.40}$$

where λ belongs to the set

$$\Lambda := \left\{ \lambda \in \mathbf{R}^{\hat{f}} : \lambda_\alpha \geq 0, \ \forall \alpha = 1, 2, \ldots, \hat{f} \right\}. \tag{5.41}$$

This implies that the IQCs (5.38) lead to the satisfaction of an IQC parameterized by scaling constants as defined in (5.39). It then follows from (5.38) that

$$\int_0^{t_k} \tilde{\xi}(t)^T \tilde{Q}(\lambda) \tilde{\xi}(t) \, dt \leq \int_0^{t_k} \tilde{\zeta}(t)^T \tilde{R}(\lambda) \tilde{\zeta}(t) \, dt + \tilde{d}(\lambda) \tag{5.42}$$

for all $\{t_k \geq 0\}_{k=1}^{\infty}$. Using this formulation, we are particularly interested in a subset $\tilde{\Lambda} \subseteq \Lambda$ such that $\tilde{Q}(\lambda) > 0$. Then, for each $\lambda \in \tilde{\Lambda}$, the weighting matrices defined in (5.40) can be written as

$$\begin{aligned} \tilde{Q}(\lambda) &= \bar{Q}(\lambda)^T \bar{Q}(\lambda); \\ \tilde{R}(\lambda) &= \bar{R}(\lambda)^T \bar{R}(\lambda) \end{aligned} \tag{5.43}$$

where $\bar{Q}(\lambda) = \bar{Q}(\lambda)^T = \tilde{Q}(\lambda)^{\frac{1}{2}} > 0$ and $\bar{R}(\lambda)$ is a rectangular matrix. However, $\bar{R}(\lambda)$ can be chosen as a square matrix such that $\bar{R}(\lambda) = \bar{R}(\lambda)^T = \tilde{R}(\lambda)^{\frac{1}{2}} > 0$. The IQC (5.42) can then be reformulated as

$$\int_0^{t_k} \|\bar{\xi}(t)\|^2 \, dt \leq \int_0^{t_k} \|\bar{\zeta}(t)\|^2 \, dt + \bar{d}(\lambda) \tag{5.44}$$

for all $\{t_k \geq 0\}_{k=1}^{\infty}$ with

$$\bar{\xi}(t) := \bar{Q}(\lambda)\tilde{\xi}(t); \quad \bar{\zeta}(t) := \bar{R}(\lambda)\tilde{\zeta}(t); \quad \bar{d}(\lambda) := \tilde{d}(\lambda). \tag{5.45}$$

Using the expressions in (5.45), the state equations (5.36) can be rewritten as

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + \bar{B}_2 \bar{u}(t) + \bar{E}\bar{Q}(\lambda)^{-1}\bar{\xi}(t); \\
z(t) &= C_1 x(t) + \bar{D}_{12}\bar{u}(t) + \bar{D}_{12}^J \bar{Q}(\lambda)^{-1}\bar{\xi}(t); \\
\bar{\zeta}(t) &= \bar{R}(\lambda)\left[\bar{H}x(t) + \bar{G}_w w(t) + \bar{G}\bar{u}(t) + \bar{G}_J \bar{Q}(\lambda)^{-1}\bar{\xi}(t)\right]; \\
\bar{y}(t) &= \bar{C}_2 x(t) + \bar{D}_{21} w(t) + \bar{F}\bar{Q}(\lambda)^{-1}\bar{\xi}(t).
\end{aligned} \tag{5.46}$$

The stable decentralized nonlinear controller (5.10), (5.11) will then be constructed based on the uncertain system (5.46), (5.44).

5.3 Stable Decentralized Controller Synthesis

Based on the decentralized nonlinear control problem stated in Section 5.2, we present an algorithm to synthesize a stable decentralized nonlinear controller (5.10), (5.11) for the nonlinear uncertain system (5.1), (5.2), (5.5). For this purpose, we apply the results in Chapter 3 where we first solve a state feedback control problem for the uncertain system (5.46), (5.44) using the approach of [187]. The resulting state feedback gain matrix is then used to introduce an additional uncertainty to the uncertain system (5.46), (5.44) in order to form an artificial uncertain system. Any suitable output feedback controller for the artificial uncertain system is guaranteed to be stable and provides absolute stability with a specified disturbance attenuation level for the closed loop system (see [7]).

5.3.1 State feedback control problem

Solving a state feedback control problem for the uncertain system (5.46), (5.44) using the results of [187], we expect that the resulting closed loop system be absolutely stable with a specified disturbance attenuation level $\gamma > 0$. To achieve this goal, we first need to introduce a scaling constant $\kappa > 0$ corresponding to the IQC (5.44). This allows us to rewrite the state equations (5.46) as follows:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + \bar{B}_1 \bar{w}(t) + \bar{B}_2 \bar{u}(t); \\
\bar{z}(t) &= \bar{C}_1 x(t) + \bar{D}_{11}\bar{w}(t) + \tilde{D}_{12}\bar{u}(t); \\
\bar{y}(t) &= \bar{C}_2 x(t) + \tilde{D}_{21}\bar{w}(t)
\end{aligned} \tag{5.47}$$

where

$$\bar{w}(t) = \begin{bmatrix} \gamma\, w(t) \\ \sqrt{\kappa}\,\bar{\xi}(t) \end{bmatrix}; \ \bar{z}(t) = \begin{bmatrix} z(t) \\ \sqrt{\kappa}\,\bar{\zeta}(t) \end{bmatrix}; \ \bar{B}_1(t) = \begin{bmatrix} \gamma^{-1} B_1 & \sqrt{\kappa}^{-1} \bar{E}\bar{Q}(\lambda)^{-1} \end{bmatrix};$$
$$\bar{C}_1 = \begin{bmatrix} C_1 \\ \sqrt{\kappa}\,\bar{R}(\lambda)\bar{H} \end{bmatrix}; \ \bar{D}_{11} = \begin{bmatrix} 0_{q\times d} & \sqrt{\kappa}^{-1} \bar{D}_{12}^J \bar{Q}(\lambda)^{-1} \\ \gamma^{-1}\sqrt{\kappa}\,\bar{R}(\lambda)\bar{G}_w & \bar{R}(\lambda)\bar{G}_J \bar{Q}(\lambda)^{-1} \end{bmatrix};$$
$$\tilde{D}_{12} = \begin{bmatrix} \bar{D}_{12} \\ \sqrt{\kappa}\,\bar{R}(\lambda)\bar{G} \end{bmatrix}; \ \tilde{D}_{21} = \begin{bmatrix} \gamma^{-1}\bar{D}_{21} & \sqrt{\kappa}^{-1}\bar{F}\bar{Q}(\lambda)^{-1} \end{bmatrix}. \tag{5.48}$$

Note that $\lambda = \tilde{\lambda} \in \tilde{\Lambda}$. In (5.47), a $\bar{D}_{11}$ term appears explicitly so that the results of [187] cannot be applied immediately. However, this term can be eliminated via a loop shifting transformation so that (5.47) can be written in a non-singular H^∞ standard form; e.g., see [16, Sections 4.5.1 and 5.5.1] and [18, Section 17.2]. Thus, to perform this transformation, we need to satisfy the following assumption.

Assumption 5.1. Given a vector $\tilde{\lambda} \in \tilde{\Lambda}$ and constants $\beta_1 \geq 0, \ldots, \beta_g \geq 0$, $\delta_1 > 0, \ldots, \delta_p > 0$, $\kappa > 0$, the uncertain system (5.46), (5.44) is such that $\bar{D}_{11}\bar{D}_{11}^T < I$.

On the basis of Assumption 5.1, we define

$$\begin{aligned} \Theta &:= I - \bar{D}_{11}^T\bar{D}_{11} > 0; \quad \bar{\Theta} := I - \bar{D}_{11}\bar{D}_{11}^T > 0; \\ \breve{w}(t) &:= \Theta^{\frac{1}{2}}\bar{w}(t) - \Theta^{-\frac{1}{2}}\bar{D}_{11}^T\left(\bar{C}_1 x(t) + \tilde{D}_{12}\bar{u}(t)\right); \\ \breve{z}(t) &:= \bar{\Theta}^{-\frac{1}{2}}\left(\bar{C}_1 x(t) + \tilde{D}_{12}\bar{u}(t)\right) \end{aligned} \tag{5.49}$$

such that

$$\begin{aligned} \bar{w}(t) &= \Theta^{-\frac{1}{2}}\breve{w}(t) + \Theta^{-1}\bar{D}_{11}^T\left(\bar{C}_1 x(t) + \tilde{D}_{12}\bar{u}(t)\right); \\ \|\bar{z}(t)\|^2 &- \|\bar{w}(t)\|^2 \equiv \|\breve{z}(t)\|^2 - \|\breve{w}(t)\|^2. \end{aligned} \tag{5.50}$$

Applying the relations in (5.49) and (5.50) to (5.47), we then obtain

$$\begin{aligned} \dot{x}(t) &= \check{A}x(t) + \check{B}_1\breve{w}(t) + \check{B}_2\bar{u}(t); \\ \breve{z}(t) &= \check{C}_1 x(t) + \check{D}_{12}\bar{u}(t); \\ \bar{y}(t) &= \check{C}_2 x(t) + \check{D}_{21}\breve{w}(t) + \check{D}_{22}\bar{u}(t) \end{aligned} \tag{5.51}$$

where

$$\begin{array}{rclrcl}\check{A} &=& A+\bar{B}_1\bar{D}_{11}^T\bar{\Theta}^{-1}\bar{C}_1; & \check{B}_1 &=& \bar{B}_1\Theta^{-\frac{1}{2}};\\ \check{B}_2 &=& \bar{B}_2+\bar{B}_1\bar{D}_{11}^T\bar{\Theta}^{-1}\tilde{D}_{12}; & \check{C}_1 &=& \bar{\Theta}^{-\frac{1}{2}}\bar{C}_1;\\ \check{C}_2 &=& \bar{C}_2+\tilde{D}_{21}\bar{D}_{11}^T\bar{\Theta}^{-1}\bar{C}_1; & \check{D}_{12} &=& \bar{\Theta}^{-\frac{1}{2}}\tilde{D}_{12};\\ \check{D}_{22} &=& \tilde{D}_{21}\bar{D}_{11}^T\bar{\Theta}^{-1}\tilde{D}_{12}; & \check{D}_{21} &=& \tilde{D}_{21}\Theta^{-\frac{1}{2}}.\end{array} \tag{5.52}$$

The solution to our state feedback control problem involves a stabilizing solution to a parameterized algebraic Riccati equation, which is solvable if the following assumption is satisfied; e.g., see [187, 312].

Assumption 5.2. Given a vector $\tilde{\lambda} \in \tilde{\Lambda}$ and constants $\beta_1 \geq 0, \ldots, \beta_g \geq 0$, $\delta_1 > 0, \ldots, \delta_p > 0$, $\kappa > 0$, the uncertain system (5.46), (5.44) is assumed to be such that $J = \check{D}_{12}^T\check{D}_{12} > 0$.

Lemma 5.2. *Let a vector* $\tilde{\lambda} \in \tilde{\Lambda}$ *and constants* $\beta_1 \geq 0, \ldots, \beta_g \geq 0$, $\delta_1 > 0, \ldots, \delta_p > 0$ *be given. Also, suppose that the uncertain system* (5.46), (5.44) *satisfies Assumption 5.2, and is absolutely stabilizable with disturbance attenuation level* $\gamma > 0$ *via a controller of the form* (5.33) *(but which is not necessarily stable). Then, there exists a constant* $\kappa > 0$ *satisfying Assumption 5.1 and such that the algebraic Riccati equation*

$$\begin{aligned}&\left(\check{A}-\check{B}_2J^{-1}\check{D}_{12}^T\check{C}_1\right)^T X + X\left(\check{A}-\check{B}_2J^{-1}\check{D}_{12}^T\check{C}_1\right)\\ &\quad + X\left(\check{B}_1\check{B}_1^T-\check{B}_2J^{-1}\check{B}_2^T\right)X+\check{C}_1^T\left(I-\check{D}_{12}J^{-1}\check{D}_{12}^T\right)\check{C}_1=0.\end{aligned} \tag{5.53}$$

has a stabilizing solution $X \geq 0$ *(see [187]). Moreover, the uncertain system* (5.46), (5.44) *is absolutely stabilizable with disturbance attenuation level* $\gamma > 0$ *via the state feedback controller*

$$\bar{u}(t) = \bar{K}x(t) \tag{5.54}$$

where

$$\bar{K} = \begin{bmatrix}\bar{K}_u\\ \bar{K}_{\tilde{\nu}}\end{bmatrix} = -J^{-1}\left(\check{B}_2^TX+\check{D}_{12}^T\check{C}_1\right). \tag{5.55}$$

Proof. The proof of this lemma follows similar arguments to those in the proof of Lemma 3.1 in Chapter 3. □

5.3.2 Artificial uncertain system

Considering Lemma 5.2, we suppose that the vector $\lambda \in \tilde{\Lambda}$ and constants $\beta_1 \geq 0, \ldots, \beta_g \geq 0$, $\delta_1 > 0, \ldots, \delta_p > 0$, $\kappa > 0$ have been obtained such that Assumptions 5.1 and 5.2 are satisfied, and the algebraic Riccati equation (5.53) has a stabilizing solution $X \geq 0$. Then, using the state feedback gain matrix $\bar{K}$ in (5.55) and adding an additional artificial uncertainty, an artificial uncertain system is formed based on the uncertain system (5.46), (5.44) as follows:

$$
\begin{aligned}
\dot{x}(t) &= \bar{A}x(t) + B_1 w(t) + \tilde{B}_2\bar{u}(t) + \tilde{E}_1\bar{\xi}_1(t) + \tilde{E}_2\bar{\xi}_2(t);\\
z(t) &= \tilde{C}_1 x(t) + \tilde{D}_{\bar{u}}\bar{u}(t) + \tilde{D}_J\bar{\xi}_1(t) + W_1\bar{\xi}_2(t);\\
\bar{\zeta}_1(t) &= \tilde{H}_1 x(t) + \tilde{G}_w w(t) + \tilde{G}_1\bar{u}(t) + \tilde{G}_J\bar{\xi}_1(t) + W_2\bar{\xi}_2(t);\\
\bar{\zeta}_2(t) &= \tilde{H}_2 x(t) + \tilde{G}_2\bar{u}(t);\\
\bar{y}(t) &= \bar{C}_2 x(t) + \bar{D}_{21} w(t) + \tilde{F}_1\bar{\xi}_1(t) + \tilde{F}_2\bar{\xi}_2(t)
\end{aligned}
\tag{5.56}
$$

where

$$
\begin{aligned}
&\bar{A} = A + \tfrac{1}{2}B_2\bar{K}_u; \quad \tilde{B}_2 = \begin{bmatrix} \tfrac{1}{2}B_2 & 0_{n\times g}\end{bmatrix}; \quad \tilde{E}_1 = \bar{E}\bar{Q}(\lambda)^{-1}; \quad \tilde{E}_2 = B_2U^{-1};\\
&\tilde{C}_1 = C_1 + \tfrac{1}{2}D_{12}\bar{K}_u;\ \tilde{D}_{\bar{u}} = \begin{bmatrix} \tfrac{1}{2}D_{12} & 0_{q\times g}\end{bmatrix};\ \tilde{D}_J = \bar{D}_{12}^J\bar{Q}(\lambda)^{-1};\ W_1 = D_{12}U^{-1};\\
&\tilde{H}_1 = \bar{R}(\lambda)\begin{bmatrix} H_1 + \tfrac{1}{2}G_1\bar{K}_u \\ H_2 + \tfrac{1}{2}G_2\bar{K}_u \\ 0_{g\times n} \\ \bar{I}C_2 \end{bmatrix}; \ \tilde{G}_1 = \bar{R}(\lambda)\begin{bmatrix} \tfrac{1}{2}G_1 & 0_{h\times g} \\ \tfrac{1}{2}G_2 & 0_{g\times g} \\ 0_{g\times m} & I_{g\times g} \\ 0_{pl\times m} & 0_{pl\times g}\end{bmatrix}; \ \tilde{G}_w = \bar{R}(\lambda)\bar{G}_w;\\
&W_2 = \bar{R}(\lambda)\begin{bmatrix} G_1 \\ G_2 \\ 0_{g\times m} \\ 0_{pl\times m}\end{bmatrix}U^{-1}; \quad \tilde{G}_J = \bar{R}(\lambda)\bar{G}_J\bar{Q}(\lambda)^{-1}; \quad \tilde{H}_2 = \tfrac{1}{2}U\bar{K}_u;\\
&\tilde{G}_2 = -\tfrac{1}{2}U\begin{bmatrix} I_{m\times m} & 0_{m\times g}\end{bmatrix}; \quad \tilde{F}_1 = \bar{F}\bar{Q}(\lambda)^{-1}; \quad \tilde{F}_2 = 0_{(l+g)\times m}.
\end{aligned}
\tag{5.57}
$$

Here, U is any $m\times m$ non-singular scaling matrix and $\lambda = \bar{\lambda} \in \tilde{\Lambda}$. The uncertainty input $\bar{\xi}_1(t)$ and uncertainty output $\bar{\zeta}_1(t)$ are related according to the IQC (5.44) with $\bar{\xi}_1(t) = \bar{\xi}(t)$ and $\bar{\zeta}_1(t) = \bar{\zeta}(t)$. Moreover, the IQC (5.44) is also extended to include the additional artificial uncertainty input $\bar{\xi}_2(t)$ and uncertainty output

$\bar{\zeta}_2(t)$. That is,

$$\int_0^{t_k} \|\bar{\xi}_v(t)\|^2\, dt \le \int_0^{t_k} \|\bar{\zeta}_v(t)\|^2\, dt + \bar{d}_v(\lambda) \tag{5.58}$$

for all $v = 1, 2$ and for all $\{t_k \ge 0\}_{k=1}^{\infty}$, and $\bar{\xi}_2(t), \bar{\zeta}_2(t) \in \mathbf{R}^m$. An output feedback controller of the form (5.33) is then synthesized for the artificial uncertain system (5.56), (5.58). We then consider two special cases to show that any absolutely stabilizing output feedback controller for the artificial uncertain system (5.56), (5.58) is indeed stable and solves the robust H^∞ control problem for the equivalent nonlinear uncertain system (5.30), (5.2), (5.5), (5.32). Thus, we suppose that the relation between $\bar{\xi}_2(t)$ and $\bar{\zeta}_2(t)$ is given by

$$\bar{\xi}_2(t) = \Delta \bar{\zeta}_2(t) \tag{5.59}$$

where $\Delta \in \mathbf{R}$ is an unknown constant uncertain parameter satisfying

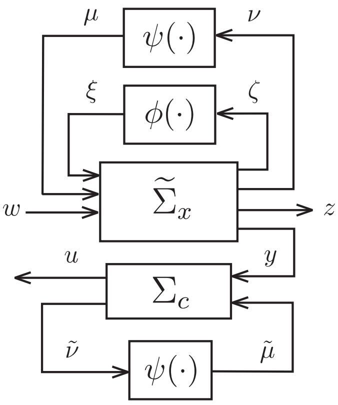
$$|\Delta| \le 1 \tag{5.60}$$

such that the IQC (5.58) holds.

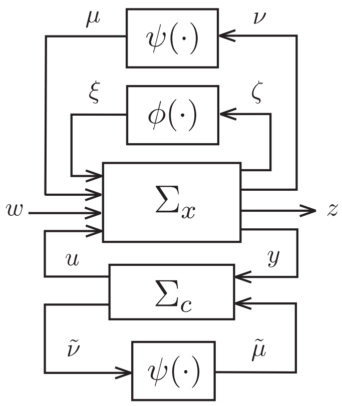
Special case I: $\Delta = 1$. In this case, we have that $\bar{\xi}_2(t) = \bar{\zeta}_2(t) = \frac{1}{2}\bar{K}_u x(t) - \frac{1}{2}u(t)$. Therefore, using the expressions in (5.36), (5.37) and (5.45), it follows that the state equations (5.56) can be decomposed into

$$\begin{aligned}
\dot{x}(t) &= \left(A + B_2\bar{K}_u\right)x(t) + B_1 w(t) + E_1\xi(t) + E_2\mu(t) + B_2 J_u \xi_u(t);\\
z(t) &= \left(C_1 + D_{12}\bar{K}_u\right)x(t) + D_{12}J_u\xi_u(t);\\
\zeta(t) &= \left(H_1 + G_1\bar{K}_u\right)x(t) + G_1 J_u\xi_u(t);\\
\nu(t) &= \left(H_2 + G_2\bar{K}_u\right)x(t) + G_2 J_u\xi_u(t);\\
\zeta_u(t) &= \bar{I}\left[C_2 x(t) + D_{21}w(t) + F_1\xi(t) + F_2\mu(t)\right];\\
y(t) &= C_2 x(t) + D_{21}w(t) + F_1\xi(t) + F_2\mu(t)
\end{aligned} \tag{5.61}$$

where the IQCs (5.5), (5.32) and the GLCs (5.2) are satisfied. We notice that in fact, the state equations (5.61) represent the closed loop system when the state feedback controller (5.54), (5.55) is applied to the uncertain system (5.46), (5.44). Thus, it follows from the construction of the matrix $\bar{K}_u$ in (5.55) and Lemma 5.2 that the system (5.61) is absolutely stable with the disturbance attenuation level $\gamma > 0$.



(a) Special case I



(b) Special case II

Figure 5.2: Block diagrams corresponding to Special case I and Special case II; see [7].

Moreover, for Special case I, if we can find a suitable output feedback controller of the form (5.33) for the artificial uncertain system (5.56), (5.58), the resulting closed loop system will have an open loop configuration as shown in Figure 5.2(a). Apparently, the output $u(t)$ of the controller does not affect the system to be controlled. Here, the block $\widetilde{\Sigma}_x$ is the absolutely stable system (5.61) and the block Σ_c is the output feedback controller (5.33). As we require that the entire closed loop system to be absolutely stable with disturbance attenuation level $\gamma > 0$, then the output feedback controller (5.33) must be stable.

Special case II: $\Delta = -1$. For this case, we have that $\bar{\xi}_2(t) = -\bar{\zeta}_2(t) = -\frac{1}{2}\bar{K}_u x(t) + \frac{1}{2}u(t)$ and the state equations (5.56) reduce to (5.46). Furthermore, using (5.36), (5.37) and (5.45), we are able to decompose (5.46) into the equivalent nonlinear uncertain system (5.30) with the IQCs (5.5), (5.32) and the GLCs (5.2). Thus, if we can find an absolutely stabilizing output feedback controller Σ_c of the form (5.33) for the artificial uncertain system (5.56), (5.58), then the same controller will also absolutely stabilize the equivalent nonlinear uncertain system (5.30), (5.2), (5.5), (5.32) with a specified disturbance attenuation level $\gamma > 0$. The resulting closed loop system is as shown in Figure 5.2(b), where Σ_x denotes the system (5.30).

Given the fact that the additional artificial uncertainty satisfying the IQC (5.58) overbounds the scalar uncertainty (5.59), (5.60), we can infer from both special cases that the nonlinear output feedback controller (5.6), (5.7) is indeed stable and absolutely stabilize the equivalent nonlinear uncertain system (5.30), (5.2), (5.5), (5.32) with a specified disturbance attenuation level $\gamma > 0$. However, we should note that the additional artificial uncertainty may result in some additional conservatism to the controller design process.

5.3.3 Stable output feedback controller

To construct the output feedback controller (5.33), we introduce scaling constants $\tau_1 > 0$ and $\tau_2 > 0$ corresponding to the IQCs (5.58) and then rewrite the state equations (5.56) as

$$\begin{aligned}\dot{x}(t) &= \bar{A}x(t) + \hat{B}_1\hat{w}(t) + \hat{B}_2\bar{u}(t);\\ \hat{z}(t) &= \hat{C}_1x(t) + \hat{D}_{11}\hat{w}(t) + \hat{D}_{12}\bar{u}(t);\\ \bar{y}(t) &= \bar{C}_2x(t) + \hat{D}_{21}\hat{w}(t)\end{aligned} \tag{5.62}$$

where $\hat{r} = \bar{r} + m$;

$$\begin{aligned}\hat{w}(t) &= \begin{bmatrix}\gamma\, w(t)\\ \sqrt{\tau_1}\,\bar{\xi}_1(t)\\ \sqrt{\tau_2}\,\bar{\xi}_2(t)\end{bmatrix}; \quad \hat{z}(t) = \begin{bmatrix} z(t)\\ \sqrt{\tau_1}\,\bar{\zeta}_1(t)\\ \sqrt{\tau_2}\,\bar{\zeta}_2(t)\end{bmatrix};\\ \hat{B}_1 &= \begin{bmatrix}\gamma^{-1}B_1 & \sqrt{\tau_1}^{-1}\tilde{E}_1 & \sqrt{\tau_2}^{-1}\tilde{E}_2\end{bmatrix}; \quad \hat{B}_2 = \tilde{B}_2;\\ \hat{C}_1 &= \begin{bmatrix}\tilde{C}_1\\ \sqrt{\tau_1}\,\tilde{H}_1\\ \sqrt{\tau_2}\,\tilde{H}_2\end{bmatrix}; \quad \hat{D}_{11} = \begin{bmatrix} 0_{q\times d} & \sqrt{\tau_1}^{-1}\tilde{D}_J & \sqrt{\tau_2}^{-1}W_1\\ \gamma^{-1}\sqrt{\tau_1}\,\tilde{G}_w & \tilde{G}_J & \sqrt{\frac{\tau_1}{\tau_2}}\,W_2\\ 0_{m\times d} & 0_{m\times\hat{r}} & 0_{m\times m}\end{bmatrix};\\ \hat{D}_{12} &= \begin{bmatrix}\tilde{D}_{\bar{u}}\\ \sqrt{\tau_1}\,\tilde{G}_1\\ \sqrt{\tau_2}\,\tilde{G}_2\end{bmatrix}; \quad \hat{D}_{21} = \begin{bmatrix}\gamma^{-1}\bar{D}_{21} & \sqrt{\tau_1}^{-1}\tilde{F}_1 & \sqrt{\tau_2}^{-1}\tilde{F}_2\end{bmatrix}.\end{aligned} \tag{5.63}$$

Moreover, we again apply a loop shifting transformation so that the $\hat{D}_{11}$ term in (5.62) can be eliminated and the state equations (5.62) can be transformed into a non-singular H^∞ standard form; e.g., see [16, Sections 4.5.1 and 5.5.1] and [18, Section 17.2].

Assumption 5.3. Given a vector $\bar{\lambda} \in \tilde{\Lambda}$, constants $\beta_1 \geq 0, \ldots, \beta_g \geq 0$, $\delta_1 > 0, \ldots, \delta_p > 0$, $\tau_1 > 0$, $\tau_2 > 0$, and any non-singular scaling matrix U, the artificial uncertain system (5.56), (5.58) is assumed to be such that $\hat{D}_{11}\hat{D}_{11}^T < I$.

Assuming that Assumption 5.3 holds, we can define

$$
\begin{aligned}
\Phi &:= I - \hat{D}_{11}^T \hat{D}_{11} > 0; \\
\bar{\Phi} &:= I - \hat{D}_{11} \hat{D}_{11}^T > 0; \\
\breve{w}(t) &:= \Phi^{\frac{1}{2}} \hat{w}(t) - \Phi^{-\frac{1}{2}} \hat{D}_{11}^T \left(\hat{C}_1 x(t) + \hat{D}_{12} \bar{u}(t) \right); \\
\breve{z}(t) &:= \bar{\Phi}^{-\frac{1}{2}} \left(\hat{C}_1 x(t) + \hat{D}_{12} \bar{u}(t) \right)
\end{aligned}
\tag{5.64}
$$

such that

$$
\begin{aligned}
&\hat{w}(t) = \Phi^{-\frac{1}{2}} \breve{w}(t) + \Phi^{-1} \hat{D}_{11}^T \left(\hat{C}_1 x(t) + \hat{D}_{12} \bar{u}(t) \right); \\
&\|\hat{z}(t)\|^2 - \|\hat{w}(t)\|^2 \equiv \|\breve{z}(t)\|^2 - \|\breve{w}(t)\|^2.
\end{aligned}
\tag{5.65}
$$

Therefore, the state equations (5.62) can be rewritten as

$$
\begin{aligned}
&\dot{x}(t) = \breve{A} x(t) + \breve{B}_1 \breve{w}(t) + \breve{B}_2 \bar{u}(t); \\
&\breve{z}(t) = \breve{C}_1 x(t) + \breve{D}_{12} \bar{u}(t); \\
&\bar{y}(t) = \breve{C}_2 x(t) + \breve{D}_{21} \breve{w}(t) + \breve{D}_{22} \bar{u}(t)
\end{aligned}
\tag{5.66}
$$

where

$$
\begin{array}{rclrcl}
\breve{A} &=& \bar{A} + \hat{B}_1 \hat{D}_{11}^T \bar{\Phi}^{-1} \hat{C}_1 & \breve{B}_1 &=& \hat{B}_1 \Phi^{-\frac{1}{2}}; \\
\breve{B}_2 &=& \hat{B}_2 + \hat{B}_1 \hat{D}_{11}^T \bar{\Phi}^{-1} \hat{D}_{12}; & \breve{C}_1 &=& \bar{\Phi}^{-\frac{1}{2}} \hat{C}_1; \\
\breve{C}_2 &=& \bar{C}_2 + \hat{D}_{21} \hat{D}_{11}^T \bar{\Phi}^{-1} \hat{C}_1; & \breve{D}_{12} &=& \bar{\Phi}^{-\frac{1}{2}} \hat{D}_{12}; \\
\breve{D}_{22} &=& \hat{D}_{21} \hat{D}_{11}^T \bar{\Phi}^{-1} \hat{D}_{12}; & \breve{D}_{21} &=& \hat{D}_{21} \Phi^{-\frac{1}{2}}; \\
\breve{J}_1 &=& \breve{D}_{12}^T \breve{D}_{12}; & \breve{J}_2 &=& \breve{D}_{21} \breve{D}_{21}^T.
\end{array}
\tag{5.67}
$$

Furthermore, the $\breve{D}_{22}$ term in (5.66) is also eliminated by defining

$$
\breve{y}(t) := \bar{y}(t) - \breve{D}_{22} \bar{u}(t).
\tag{5.68}
$$

Hence, the state equations (5.66) can be rewritten as

$$
\begin{aligned}
&\dot{x}(t) = \breve{A} x(t) + \breve{B}_1 \breve{w}(t) + \breve{B}_2 \bar{u}(t); \\
&\breve{z}(t) = \breve{C}_1 x(t) + \breve{D}_{12} \bar{u}(t); \\
&\breve{y}(t) = \breve{C}_2 x(t) + \breve{D}_{21} \breve{w}(t)
\end{aligned}
\tag{5.69}
$$

and the output feedback controller for (5.69) is of the form

$$\begin{aligned}\dot{x}_c(t) &= \breve{N}x_c(t) + \tilde{M}\breve{y}(t);\\ \bar{u}(t) &= \tilde{K}x_c(t).\end{aligned}\tag{5.70}$$

If the controller (5.70) is applied to the system (5.69), the resulting closed loop system is required to satisfy the following H^∞ norm bound condition

$$\breve{\mathcal{J}} := \sup_{\breve{w}(\cdot)\in\mathbf{L}_2[0,\infty),x(0)=0,x_c(0)=0} \frac{\|\breve{z}(\cdot)\|_2^2}{\|\breve{w}(\cdot)\|_2^2} < 1. \tag{5.71}$$

The solution to this standard H^∞ control problem is given in terms of the solutions $\breve{X} \geq 0$ and $\breve{Y} \geq 0$ to the algebraic Riccati equations given as

$$\begin{aligned}&\left(\breve{A} - \breve{B}_2\breve{J}_1^{-1}\breve{D}_{12}^T\breve{C}_1\right)^T \breve{X} + \breve{X}\left(\breve{A} - \breve{B}_2\breve{J}_1^{-1}\breve{D}_{12}^T\breve{C}_1\right)\\ &\quad + \breve{X}\left(\breve{B}_1\breve{B}_1^T - \breve{B}_2\breve{J}_1^{-1}\breve{B}_2^T\right)\breve{X} + \breve{C}_1^T\left(I - \breve{D}_{12}\breve{J}_1^{-1}\breve{D}_{12}^T\right)\breve{C}_1 = 0;\end{aligned}\tag{5.72}$$

$$\begin{aligned}&\left(\breve{A} - \breve{B}_1\breve{D}_{21}^T\breve{J}_2^{-1}\breve{C}_2\right)\breve{Y} + \breve{Y}\left(\breve{A} - \breve{B}_1\breve{D}_{21}^T\breve{J}_2^{-1}\breve{C}_2\right)^T\\ &\quad + \breve{Y}\left(\breve{C}_1^T\breve{C}_1 - \breve{C}_2^T\breve{J}_2^{-1}\breve{C}_2\right)\breve{Y} + \breve{B}_1\left(I - \breve{D}_{21}^T\breve{J}_2^{-1}\breve{D}_{21}\right)\breve{B}_1^T = 0\end{aligned}\tag{5.73}$$

such that

1. $\breve{A} - \breve{B}_2\breve{J}_1^{-1}\breve{D}_{12}^T\breve{C}_1 + \left(\breve{B}_1\breve{B}_1^T - \breve{B}_2\breve{J}_1^{-1}\breve{B}_2^T\right)\breve{X}$ is Hurwitz;
2. $\breve{A} - \breve{B}_1\breve{D}_{21}^T\breve{J}_2^{-1}\breve{C}_2 + \breve{Y}\left(\breve{C}_1^T\breve{C}_1 - \breve{C}_2^T\breve{J}_2^{-1}\breve{C}_2\right)$ is Hurwitz;
3. The spectral radius $\varrho(\breve{X}\breve{Y})$ of the product $\hat{X}\hat{Y}$ is strictly less than one.

To solve the Riccati equations (5.72) and (5.73), we require the following assumption to be satisfied; e.g., see [187, 312].

Assumption 5.4. Given a vector $\bar{\lambda} \in \tilde{\Lambda}$, constants $\beta_1 \geq 0, \ldots, \beta_g \geq 0$, $\delta_1 > 0, \ldots, \delta_p > 0$, $\tau_1 > 0$, $\tau_2 > 0$, and any non-singular scaling matrix U, the artificial uncertain system (5.56), (5.58) is assumed to be such that $\breve{J}_1 > 0$ and $\breve{J}_2 > 0$.

Theorem 5.1. *Let vectors $\tilde{\lambda}, \bar{\lambda} \in \tilde{\Lambda}$ and constants $\beta_1 \geq 0, \ldots, \beta_g \geq 0$, $\delta_1 > 0, \ldots, \delta_p > 0$ be given. Also, suppose that the uncertain system* (5.46), (5.44) *satisfies Assumption 5.2 and that there exists a constant $\kappa > 0$ such that Assumption*

5.1 is satisfied and the Riccati equation (5.53) *has a stabilizing solution* $X \geq 0$*; and let* $\bar{K}$ *be given as in* (5.55)*. Moreover, suppose that there exist constants* $\tau_1 > 0$ *and* $\tau_2 > 0$*, and a non-singular scaling matrix* U *such that Assumption 5.3 and Assumption 5.4 are satisfied and the Riccati equations* (5.72) *and* (5.73) *have stabilizing solutions* $\breve{X} \geq 0$ *and* $\breve{Y} \geq 0$ *such that the spectral radius of the product* $\breve{X}\breve{Y}$ *satisfies* $\varrho(\breve{X}\breve{Y}) < 1$*. Then the equivalent nonlinear uncertain system* (5.30)*,* (5.2)*,* (5.5)*,* (5.32) *is absolutely stabilizable with disturbance attenuation level* $\gamma > 0$ *via a stable nonlinear output feedback controller* (5.6)*,* (5.7)*, where the controller matrices are given as follows:*

$$
\begin{aligned}
N &= \breve{N} - \tilde{M}\breve{D}_{22}\tilde{K};\\
\breve{N} &= \breve{A} + \breve{B}_2\tilde{K} - \tilde{M}\breve{C}_2 + \left(\breve{B}_1 - \tilde{M}\breve{D}_{21}\right)\breve{B}_1^T\breve{X};\\
\tilde{M} &= \left(I - \breve{Y}\breve{X}\right)^{-1}\left(\breve{Y}\breve{C}_2^T + \breve{B}_1\breve{D}_{21}^T\right)\breve{J}_2^{-1};\\
\tilde{K} &= -\breve{J}_1^{-1}\left(\breve{B}_2^T\breve{X} + \breve{D}_{12}^T\breve{C}_1\right).
\end{aligned}
\tag{5.74}
$$

Proof. It follows using similar arguments to those in the proof of Theorem 4.1 in [187] that the uncertain system (5.56), (5.58) is absolutely stablizable with disturbance attenuation level $\gamma > 0$ via a controller of the form (5.33) if and only if there exist constants $\tau_1 > 0$ and $\tau_2 > 0$ such that the controller (5.33) solves the H^∞ control problem defined by (5.62) and the H^∞ norm bound condition

$$
\hat{\mathcal{J}} := \sup_{\hat{w}(\cdot)\in\mathbf{L}_2[0,\infty),x(0)=0,x_c(0)=0} \frac{\|\hat{z}(\cdot)\|_2^2}{\|\hat{w}(\cdot)\|_2^2} < 1. \tag{5.75}
$$

Moreover, using a loop shifting transformation, the H^∞ control problem defined by (5.62), (5.75) has a solution if and only if the Riccati equations (5.72) and (5.73) have stabilizing solutions $\breve{X} \geq 0$ and $\breve{Y} \geq 0$ such that the spectral radius of the product $\breve{X}\breve{Y}$ satisfies $\varrho(\breve{X}\breve{Y}) < 1$. Thus, a controller of the form (5.33) solving the H^∞ control problem (5.62), (5.75) is defined by (5.74).

If all the conditions of the theorem hold, the controller (5.33), (5.74) is absolutely stabilizing with disturbance attenuation $\gamma > 0$ for the artificial uncertain system (5.56), (5.58). Then, from the arguments in the two special cases given above, it follows that the controller (5.33), (5.74), or equivalently (5.6), (5.7), is indeed stable and absolutely stabilizing for the equivalent nonlinear uncertain system (5.30), (5.2), (5.5), (5.32) with disturbance attenuation $\gamma > 0$. □

As described in Section 5.2, the equivalent nonlinear uncertain system (5.30), (5.2), (5.5), (5.32) is formed such that if a stable nonlinear controller of the form (5.6), (5.7) is absolutely stabilizing with disturbance attenuation $\gamma > 0$ and is such that the nonlinear error systems (5.15), (5.20) satisfy the bound (5.21), the stable decentralized nonlinear controller (5.10), (5.11) is also absolutely stabilizing with disturbance attenuation $\gamma > 0$. This argument leads to the following theorem.

Theorem 5.2. *Let vectors $\tilde{\lambda}, \bar{\lambda} \in \tilde{\Lambda}$ and constants $\beta_1 \geq 0, \ldots, \beta_g \geq 0$, $\delta_1 > 0, \ldots, \delta_p > 0$ be given. Also, suppose that there exists a constant $\kappa > 0$ such that Assumptions 5.1 and 5.2 are satisfied and the Riccati equation* (5.53) *has a stabilizing solution $X \geq 0$. Moreover, suppose there exist constants $\tau_1 > 0$ and $\tau_2 > 0$, and a non-singular scaling matrix U such that Assumption 5.3 and Assumption 5.4 are satisfied and the Riccati equations* (5.72) *and* (5.73) *have stabilizing solutions $\breve{X} \geq 0$ and $\breve{Y} \geq 0$ such that the spectral radius of the product $\breve{X}\breve{Y}$ satisfies $\varrho(\breve{X}\breve{Y}) < 1$. Also, suppose that the stable nonlinear controller defined by* (5.6), (5.74) *is such that the nonlinear error systems defined by* (5.15), (5.20) *satisfy the bound* (5.21) *as guaranteed by Lemma 5.1. Then the corresponding stable decentralized nonlinear output feedback controller defined by* (5.10), (5.11) *is absolutely stabilizing with disturbance attenuation level $\gamma > 0$ for the large-scale nonlinear uncertain system* (5.1), (5.2), (5.5).

Proof. If the conditions of the theorem are satisfied, then it follows from Theorem 5.1 that the equivalent nonlinear uncertain system (5.30), (5.2), (5.5), (5.32) is absolutely stabilizable with disturbance attenuation level $\gamma > 0$ via a stable nonlinear controller of the form (5.6), (5.7), (5.74). Moreover, if the controller is such that the nonlinear error systems defined by (5.15), (5.20) satisfy the bound (5.21) as guaranteed by Lemma 5.1, then it follows that the corresponding uncertainties defined by (5.27) satisfy the IQCs (5.32). Also, as described in the construction of the equivalent nonlinear uncertain system (5.30), (5.2), (5.5), (5.32), the closed loop system obtained by applying the decentralized controller (5.10), (5.11), (5.74) to the large-scale nonlinear uncertain system (5.1), (5.2), (5.5) is equivalent to the closed loop system obtained by applying the controller (5.6), (5.7), (5.74) to the equivalent nonlinear uncertain system (5.30), (5.2), (5.5), (5.32) when the uncertainties defined by (5.27) are applied. Hence, it follows that the decentralized nonlinear output feedback controller (5.10), (5.11), (5.74) is stable and absolutely stabilizing for the large-scale nonlinear uncertain system (5.1), (5.2), (5.5) with disturbance attenuation level $\gamma > 0$. □

5.4 A Differential Evolution Approach

The decentralized nonlinear controller design method presented in Section 5.3 involves a set of design parameters corresponding to uncertainties, nonlinearities and disturbance attenuation level. All of these parameters can be collected into a single vector defined as

$$\vartheta := \begin{bmatrix} \gamma & \kappa & \tau_1 & \tau_2 & \tilde{\lambda}^T & \bar{\lambda}^T & \delta \end{bmatrix}^T \tag{5.76}$$

where

$$\delta = \begin{bmatrix} \delta_1 & \delta_2 & \ldots & \delta_p \end{bmatrix}^T \tag{5.77}$$

for $j = 1, 2, \ldots, p$; and $\tilde{\lambda}, \bar{\lambda}$ are as defined in (5.39). All elements of ϑ in (5.76) are positive real numbers and the dimension of ϑ is $2(f + 3g) + 3p + 4$. To determine the values of these parameters, we propose to apply an evolutionary optimization method, namely the differential evolution (DE) algorithm, as described in Chapter 2. This means that we consider the stable decentralized nonlinear controller design problem as the following optimization problem subject to nonconvex nonlinear constraints: Find an optimal solution $\vartheta^\star$ to solve

$$\min_{\vartheta} \mathsf{f}(\vartheta) \tag{5.78}$$

subject to

$$\mathsf{g_j}(\vartheta) = 0; \quad \mathsf{h_k}(\vartheta) \leq 0 \tag{5.79}$$

for $\mathsf{j} = 1, 2, \ldots, \mathsf{a}$ and $\mathsf{k} = 1, 2, \ldots, \mathsf{b}$. Here, a and b are the total number of equality and inequality constraints, respectively.

It follows from Section 5.2 that our control objectives are to achieve an absolutely stable closed loop system with a specified disturbance attenuation level $\gamma > 0$ and to bound the H^∞ norm (5.21) of the nonlinear error systems (5.15), (5.20). Thus, a suitable objective function to be minimized is

$$\mathsf{f}(\vartheta) = \mathsf{c}_0 \gamma^{\mathsf{n}_0} + \sum_{j=1}^{p} \mathsf{c}_j \delta_j^{\mathsf{n}_j} \tag{5.80}$$

where $\mathsf{c}_\mathsf{q} \geq 1$ is a weighting factor and $\mathsf{n}_\mathsf{q} \geq 1$ is a power constant for $\mathsf{q} = 0, 1, 2, \ldots, p$. Moreover, the equality constraints are given by algebraic Riccati equations as follows:

$$\begin{aligned}
\mathsf{g}_1(\vartheta) &= \left(\breve{A} - \breve{B}_2 J^{-1} \breve{D}_{12}^T \breve{C}_1\right)^T X + X \left(\breve{A} - \breve{B}_2 J^{-1} \breve{D}_{12}^T \breve{C}_1\right) \\
&\quad + X \left(\breve{B}_1 \breve{B}_1^T - \breve{B}_2 J^{-1} \breve{B}_2^T\right) X + \breve{C}_1^T \left(I - \breve{D}_{12} J^{-1} \breve{D}_{12}^T\right) \breve{C}_1 = 0; \\
\mathsf{g}_2(\vartheta) &= \left(\breve{A} - \breve{B}_2 \breve{J}_1^{-1} \breve{D}_{12}^T \breve{C}_1\right)^T \breve{X} + \breve{X} \left(\breve{A} - \breve{B}_2 \breve{J}_1^{-1} \breve{D}_{12}^T \breve{C}_1\right) \\
&\quad + \breve{X} \left(\breve{B}_1 \breve{B}_1^T - \breve{B}_2 \breve{J}_1^{-1} \breve{B}_2^T\right) \breve{X} + \breve{C}_1^T \left(I - \breve{D}_{12} \breve{J}_1^{-1} \breve{D}_{12}^T\right) \breve{C}_1 = 0; \\
\mathsf{g}_3(\vartheta) &= \left(\breve{A} - \breve{B}_1 \breve{D}_{21}^T \breve{J}_2^{-1} \breve{C}_2\right) \breve{Y} + \breve{Y} \left(\breve{A} - \breve{B}_1 \breve{D}_{21}^T \breve{J}_2^{-1} \breve{C}_2\right)^T \\
&\quad + \breve{Y} \left(\breve{C}_1^T \breve{C}_1 - \breve{C}_2^T \breve{J}_2^{-1} \breve{C}_2\right) \breve{Y} + \breve{B}_1 \left(I - \breve{D}_{21}^T \breve{J}_2^{-1} \breve{D}_{21}\right) \breve{B}_1^T = 0.
\end{aligned} \tag{5.81}$$

From all assumptions and conditions to be satisfied, we also obtain the inequality constraints as follows:

$$\begin{array}{rclrcl}
\mathsf{h}_1(\vartheta) &=& -\tilde{Q}(\tilde{\lambda}) < 0; & \mathsf{h}_2(\vartheta) &=& -\tilde{R}(\tilde{\lambda}) < 0; \\
\mathsf{h}_3(\vartheta) &=& \bar{D}_{11} \bar{D}_{11}^T - I < 0; & \mathsf{h}_4(\vartheta) &=& -J < 0; \\
\mathsf{h}_5(\vartheta) &=& -X < 0; & \mathsf{h}_6(\vartheta) &=& -\tilde{Q}(\bar{\lambda}) < 0; \\
\mathsf{h}_7(\vartheta) &=& -\tilde{R}(\bar{\lambda}) < 0; & \mathsf{h}_8(\vartheta) &=& \hat{D}_{11} \hat{D}_{11}^T - I < 0; \\
\mathsf{h}_9(\vartheta) &=& -\breve{J}_1 < 0; & \mathsf{h}_{10}(\vartheta) &=& -\breve{J}_2 < 0; \\
\mathsf{h}_{11}(\vartheta) &=& -\breve{X} < 0; & \mathsf{h}_{12}(\vartheta) &=& -\breve{Y} < 0; \\
\mathsf{h}_{13}(\vartheta) &=& \varrho(\breve{X}\breve{Y}) - 1 < 0; & \mathsf{h}_{14}(\vartheta) &=& e_{\max,r}\left(\mathcal{A}_X\right) < 0; \\
\mathsf{h}_{15}(\vartheta) &=& e_{\max,r}(\breve{\mathcal{A}}_X) < 0; & \mathsf{h}_{16}(\vartheta) &=& e_{\max,r}(\breve{\mathcal{A}}_Y) < 0
\end{array} \tag{5.82}$$

and

$$\mathsf{h}_{17,j}(\vartheta) = \| \, \mathcal{C}_{e_j} (sI - \mathcal{A}_{e_j})^{-1} \mathcal{B}_{e_j} \, \|_\infty - 1 < 0 \tag{5.83}$$

for $j = 1, 2, \ldots, p$. Here, we have

$$\mathcal{A}_X := \breve{A} - \breve{B}_2 J^{-1} \breve{D}_{12}^T \breve{C}_1 + \left(\breve{B}_1 \breve{B}_1^T - \breve{B}_2 J^{-1} \breve{B}_2^T\right) X; \tag{5.84}$$

$$\breve{\mathcal{A}}_X := \breve{A} - \breve{B}_2\breve{J}_1^{-1}\breve{D}_{12}^T\breve{C}_1 + \left(\breve{B}_1\breve{B}_1^T - \breve{B}_2\breve{J}_1^{-1}\breve{B}_2^T\right)\breve{X};$$
$$\breve{\mathcal{A}}_Y := \breve{A} - \breve{B}_1\breve{D}_{21}^T\breve{J}_2^{-1}\breve{C}_2 + \breve{Y}\left(\breve{C}_1^T\breve{C}_1 - \breve{C}_2^T\breve{J}_2^{-1}\breve{C}_2\right) \tag{5.85}$$

and $\varrho(\mathcal{G})$ and $e_{\max,r}(\mathcal{G})$ denote the spectral radius and the largest real part of the eigenvalues of the matrix $\mathcal{G}$, respectively.

All constraints in (5.81), (5.82) and (5.83) are included into a fitness test procedure, which is used to evaluate the fitness of each candidate solution ϑ. Thus, for a given ϑ, the fitness test proceeds as follows:

1. Compute the eigenvalues of $\tilde{Q}(\tilde{\lambda})$, $\tilde{R}(\tilde{\lambda})$, $(\bar{D}_{11}\bar{D}_{11}^T - I)$ and J in order to check if the constraints $\mathsf{h}_1(\vartheta)$, $\mathsf{h}_2(\vartheta)$, $\mathsf{h}_3(\vartheta)$ and $\mathsf{h}_4(\vartheta)$ are satisfied.

2. Evaluate the constraint $\mathsf{g}_1(\vartheta)$ to obtain a solution X to the Riccati equation (5.53).

3. If the Riccati equation (5.53) has a solution X, we need to verify whether it is a stabilizing positive definite solution through the evaluation of the constraints $\mathsf{h}_5(\vartheta)$ and $\mathsf{h}_{14}(\vartheta)$.

4. Compute the eigenvalues of $\tilde{Q}(\bar{\lambda})$, $\tilde{R}(\bar{\lambda})$, $(\hat{D}_{11}\hat{D}_{11}^T - I)$, $\breve{J}_1$ and $\breve{J}_2$ in order to check if the constraints $\mathsf{h}_6(\vartheta)$, $\mathsf{h}_7(\vartheta)$, $\mathsf{h}_8(\vartheta)$, $\mathsf{h}_9(\vartheta)$ and $\mathsf{h}_{10}(\vartheta)$ are satisfied.

5. Evaluate the constraints $\mathsf{g}_2(\vartheta)$ and $\mathsf{g}_3(\vartheta)$ to obtain solutions $\breve{X}$ and $\breve{Y}$ to the Riccati equations (5.72) and (5.73).

6. If the Riccati equations (5.72) and (5.73) have solutions $\breve{X}$ and $\breve{Y}$, we need to verify whether they are stabilizing positive definite solutions through the evaluation of the constraints $\mathsf{h}_{11}(\vartheta)$, $\mathsf{h}_{12}(\vartheta)$, $\mathsf{h}_{15}(\vartheta)$ and $\mathsf{h}_{16}(\vartheta)$.

7. Compute the spectral radius of the product $\breve{X}\breve{Y}$ to verify if the constraint $\mathsf{h}_{13}(\theta)$ is satisfied.

8. Compute the decentralized controller matrices (5.74) and check if j-th nonlinear error system satisfies the constraint $\mathsf{h}_{17,j}(\vartheta)$.

9. Calculate the value of the objective function $\mathsf{f}(\vartheta)$ in (5.80).

As a penalty-based fitness test is applied, we have to pay for a penalty incurred for each constraint violation by a candidate solution. Referring to the fitness test procedure above, the penalty functions are then formed according to the constraints in (5.81), (5.82) and (5.83). That is,

$$
\begin{array}{llll}
\mathsf{p}_1(\vartheta) = |e_{\min}(\tilde{Q}(\tilde{\lambda}))|^{\mathsf{s}_1}; & \mathsf{p}_2(\vartheta) = |e_{\min}(\tilde{R}(\tilde{\lambda}))|^{\mathsf{s}_2}; \\
\mathsf{p}_3(\vartheta) = e_{\max}(\bar{D}_{11}\bar{D}_{11}^T - I)^{\mathsf{s}_3}; & \mathsf{p}_4(\vartheta) = |e_{\min}(J)|^{\mathsf{s}_4}; \\
\mathsf{p}_5(\vartheta) = \varrho(\mathcal{C}_X)^{\mathsf{s}_5}; & \mathsf{p}_6(\vartheta) = |e_{\min}(X)|^{\mathsf{s}_6}; \\
\mathsf{p}_7(\vartheta) = e_{\max,r}(\mathcal{A}_X)^{\mathsf{s}_7}; & \mathsf{p}_8(\vartheta) = |e_{\min}(\tilde{Q}(\bar{\lambda}))|^{\mathsf{s}_8}; \\
\mathsf{p}_9(\vartheta) = |e_{\min}(\tilde{R}(\bar{\lambda}))|^{\mathsf{s}_9}; & \mathsf{p}_{10}(\vartheta) = e_{\max}(\hat{D}_{11}\hat{D}_{11}^T - I)^{\mathsf{s}_{10}}; \\
\mathsf{p}_{11}(\vartheta) = |e_{\min}(\breve{J}_1)|^{\mathsf{s}_{11}}; & \mathsf{p}_{12}(\vartheta) = |e_{\min}(\breve{J}_2)|^{\mathsf{s}_{12}}; \\
\mathsf{p}_{13}(\vartheta) = \varrho(\mathcal{C}_{\breve{X}})^{\mathsf{s}_{13}}; & \mathsf{p}_{14}(\vartheta) = \varrho(\mathcal{C}_{\breve{Y}})^{\mathsf{s}_{14}}; \\
\mathsf{p}_{15}(\vartheta) = |e_{\min}(\breve{X})|^{\mathsf{s}_{15}}; & \mathsf{p}_{16}(\vartheta) = |e_{\min}(\breve{Y})|^{\mathsf{s}_{16}}; \\
\mathsf{p}_{17}(\vartheta) = e_{\max,r}(\mathcal{A}_{\breve{X}})^{\mathsf{s}_{17}}; & \mathsf{p}_{18}(\vartheta) = e_{\max,r}(\mathcal{A}_{\breve{Y}})^{\mathsf{s}_{18}}; \\
\mathsf{p}_{19}(\vartheta) = (\varrho(\breve{X}\breve{Y}) - 1)^{\mathsf{s}_{19}}; & \mathsf{p}_{20}(\vartheta) = \sum_{j=1}^{p} \mathcal{D}_j^{\mathsf{m}_j}; \\
\mathsf{p}_{21}(\vartheta) = \mathsf{f}(\vartheta)
\end{array}
\tag{5.86}
$$

where $\mathsf{s}_\mathsf{r}, \mathsf{m}_j \geq 1$ for $\mathsf{r} = 1, 2, \ldots, 19$ and $j = 1, 2, \ldots, p$. Here, $e_{\min}(\mathcal{G})$ and $e_{\max}(\mathcal{G})$ denote the smallest and the largest eigenvalue of the symmetric matrix $\mathcal{G}$, respectively. If the matrix $\mathcal{G}$ is required to be positive definite, we assign $|e_{\min}(\mathcal{G})|^{\mathsf{s}_\mathsf{r}}$ as a penalty because when this requirement is violated, the matrix $\mathcal{G}$ can be either negative (semi)definite or indefinite. Moreover, $\mathcal{C}_X$, $\mathcal{C}_{\breve{X}}$, $\mathcal{C}_{\breve{Y}}$ and $\mathcal{D}_j$ are defined as follows:

$$
\begin{aligned}
\mathcal{C}_X &:= \breve{C}_1^T \left(I - \breve{D}_{12} J^{-1} \breve{D}_{12}^T\right) \breve{C}_1; \\
\mathcal{C}_{\breve{X}} &:= \breve{C}_1^T \left(I - \breve{D}_{12} \breve{J}_1^{-1} \breve{D}_{12}^T\right) \breve{C}_1; \\
\mathcal{C}_{\breve{Y}} &:= \breve{B}_1 \left(I - \breve{D}_{21}^T \breve{J}_2^{-1} \breve{D}_{21}\right) \breve{B}_1^T; \\
\mathcal{D}_j &:= \begin{cases} \| \mathcal{C}_{e_j}(sI - \mathcal{A}_{e_j})^{-1}\mathcal{B}_{e_j} \|_\infty, & \text{if } \mathsf{h}_{15,j}(\vartheta) \text{ in (5.82) is violated;} \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
\tag{5.87}
$$

5.5 An Illustrative Example

To demonstrate the decentralized stable nonlinear controller design method presented in Section 5.3, we consider an example of a nonlinear uncertain system comprising of two inverted pendulums interconnected with a nonlinear spring. This example is adapted from an example presented in [314]. In this case, the nonlinear uncertain system being considered has two subsystems and can be represented as follows:

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.02 & -0.1 & -0.02 & 0 \\ 0 & 0 & 0 & 1 \\ -0.02 & 0 & 0.02 & -0.05 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} w(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(t) \\
&+ \begin{bmatrix} 0 & 0 \\ 0.2 & 0 \\ 0 & 0 \\ 0 & 0.2 \end{bmatrix} \xi(t) + \begin{bmatrix} 0 & 0 \\ 0.98 & 0 \\ 0 & 0 \\ 0 & 0.98 \end{bmatrix} \mu(t); \\
z(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} u(t); \\
\zeta(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t); \\
\nu(t) &= \begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} x(t); \\
y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} w(t)
\end{aligned} \tag{5.88}$$

The interconnection between the two pendulums consists of both linear and nonlinear parts, which are assumed to be known. The uncertainties in the system (5.88) are modeled as follows:

$$\xi(t) = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \zeta(t) \tag{5.89}$$

where $\Delta_s \in \mathbf{R}$, $|\Delta_s| \leq 1$ for all $s = 1, 2$ and hence, the uncertainties (5.89) satisfy the IQCs (5.5) for any $d_{1,s} \geq 0$. Moreover, the nonlinearities in the system (5.88) are given as

$$\mu_i(t) = \psi_i(\nu_i(t)) = \sin \nu_i(t) \tag{5.90}$$

and satisfy the following GLCs

$$|\sin \nu_i(t) - \sin \tilde{\nu}_i(t)| \leq \beta_i \, |\nu_i(t) - \tilde{\nu}_i(t)| \tag{5.91}$$

with $\beta_i \geq 0$ for $i = 1, 2$. From the first derivative of $\sin(\nu_i(\cdot))$ with respect to $\nu_i(\cdot)$, we obtain

$$\left|\frac{d\sin\nu_i(\cdot)}{d\nu_i(\cdot)}\right| = |\cos\nu_i(\cdot)| \leq 1, \quad \forall\nu(\cdot) \in \mathbf{R} \tag{5.92}$$

which implies that $\beta_i = 1$. From the GLCs (5.91), the nonlinearities (5.90) and their copies can be characterized in terms of IQCs as described in (5.35).

We now can synthesize two decentralized nonlinear controllers of the form (5.10) for the nonlinear uncertain system (5.88), (5.89), (5.90) using the decentralized controller design method developed in Section 5.3. The required design parameters were computed using the DE approach presented in Section 5.4 and are then given as follows:

$$\begin{array}{llll}
\gamma = 0.4520; & \kappa = 1.3951 \times 10^{-4}; & \tau_1 = 1.4495 \times 10^{-4}; & \tau_2 = 0.0194; \\
\delta_1 = 0.7245; & \delta_2 = 0.6593; & \tilde{\lambda}_1 = 1603.5095; & \tilde{\lambda}_2 = 992.3272; \\
\tilde{\lambda}_3 = 944.4646; & \tilde{\lambda}_4 = 1615.9850; & \tilde{\lambda}_5 = 142.0642; & \tilde{\lambda}_6 = 357.4983; \\
\tilde{\lambda}_7 = 1746.6423; & \tilde{\lambda}_8 = 731.8955; & \tilde{\lambda}_9 = 1371.3715; & \tilde{\lambda}_{10} = 745.8950; \\
\bar{\lambda}_1 = 1544.7904; & \bar{\lambda}_2 = 701.6927; & \bar{\lambda}_3 = 1877.8732; & \bar{\lambda}_4 = 587.5819; \\
\bar{\lambda}_5 = 163.0875; & \bar{\lambda}_6 = 400.8780; & \bar{\lambda}_7 = 388.7251; & \bar{\lambda}_8 = 71.1934; \\
\bar{\lambda}_9 = 1404.8656; & \bar{\lambda}_{10} = 1279.2498. & &
\end{array} \tag{5.93}$$

Note that (λ_1, λ_2), $(\lambda_3, \ldots, \lambda_8)$ and $(\lambda_9, \lambda_{10})$ correspond to the IQCs (5.5), (5.35) and (5.32), respectively, for $\lambda = \tilde{\lambda}$ and $\lambda = \bar{\lambda}$ in (5.93). Using these parameters, we can compute coefficient matrices of both decentralized nonlinear controllers, which are given as follows:

$$N = \begin{bmatrix} -622.2520 & -25.2206 & 12.7291 & 5.4402 \\ -8026.0342 & -347.0469 & 101.6167 & 70.1597 \\ -59.7382 & -1.3785 & -157.9092 & -3.5601 \\ -1029.8417 & -27.5612 & -1938.9203 & -61.1766 \end{bmatrix}; M_1 = \begin{bmatrix} 467.2367 \\ 5982.3974 \\ 49.7723 \\ 839.2038 \end{bmatrix};$$

$$M_2 = \begin{bmatrix} 11.7370 \\ 207.0494 \\ 127.1357 \\ 1531.0732 \end{bmatrix}; \quad L_1 = \begin{bmatrix} 8.0600 \\ 106.5139 \\ 0.8242 \\ 14.0880 \end{bmatrix}; \quad L_2 = \begin{bmatrix} -0.0306 \\ 0.1770 \\ 1.2636 \\ 17.1505 \end{bmatrix}; \tag{5.94}$$

$$K_1 = \begin{bmatrix} -19.5962 & -5.2182 & 2.1814 & 0.6503 \end{bmatrix};$$
$$K_2 = \begin{bmatrix} 2.1615 & 0.6469 & -17.6790 & -4.6460 \end{bmatrix};$$
$$P_1 = \begin{bmatrix} 0.8285 & 0 & -0.8285 & 0 \end{bmatrix}; \quad P_2 = \begin{bmatrix} -0.8919 & 0 & 0.8919 & 0 \end{bmatrix}. \tag{5.95}$$

In order to show that both decentralized nonlinear controllers are stable, the stability of their equilibrium points: $x^*_{d_1} = 0$ and $x^*_{d_2} = 0$ are examined by following the same steps as those in Section 3.5. As the plant nonlinearities are represented by sine functions, we have $\tilde{\mu}_i(t) \approx \tilde{\nu}_i(t)$ around the equilibrium points. Therefore, we can have

$$\bar{N} := N + \sum_{i=1}^{g} L_i P_i \tag{5.96}$$

where $\bar{N}$ is the linearized controller system matrix around the equilibrium point $x^*_{d_1} = x^*_{d_2} = 0$. The eigenvalues of the matrix $\bar{N}$ are given as follows:

$$e_1 = -938.0871; \quad e_2 = -213.8854; \quad e_3 = -14.4219; \quad e_4 = -14.8412 \tag{5.97}$$

which indicate that both decentralized nonlinear controllers are locally stable around the equilibrium points. Furthermore, to verify that they are globally asymptotically stable, it is necessary to examine the stability of the matrix N and the H^∞ norm $\|T_{\tilde{\mu}_j\tilde{\nu}_j}(s)\|_\infty$ of the transfer function of the j-th decentralized nonlinear controller from the nonlinearity input $\tilde{\mu}_j(t)$ to the nonlinearity output $\tilde{\nu}_j(t)$ (for $j = 1, 2$). The eigenvalues of the matrix N are

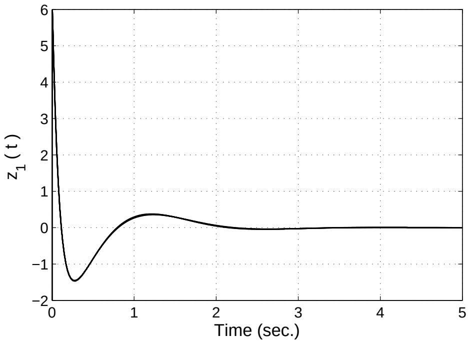
$$e_1 = -944.1716; \quad e_2 = -214.9509; \quad e_3 = -14.4212; \quad e_4 = -14.8411 \tag{5.98}$$

and hence, the matrix N is Hurwitz. Meanwhile,

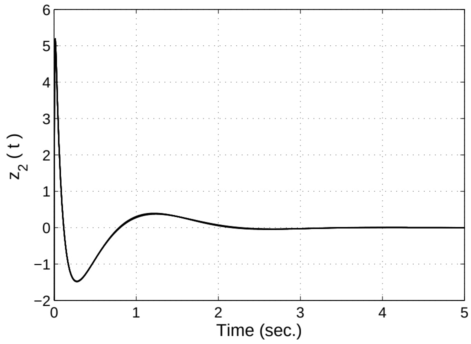
$$\|T_{\tilde{\mu}_1\tilde{\nu}_1}(s)\|_\infty = 0.0064; \quad \|T_{\tilde{\mu}_2\tilde{\nu}_2}(s)\|_\infty = 0.0049. \tag{5.99}$$

Then, using the small gain theorem (e.g., see [18, Section 9.2]), we conclude that both decentralized nonlinear controllers are globally asymptotically stable.

Interconnecting the nonlinear uncertain system (5.88), (5.89), (5.90) and the decentralized nonlinear controllers (5.94), (5.95), we then simulated the resulting closed loop system using Simulink for different values of Δ_1 and Δ_2. In this simulation, we take $\Delta_1 = \Delta_2 = \Delta \in \{-1, -0.5, 0, 0.5, 1\}$. The initial conditions of both subsystems are set to be $x_1(0) = x_2(0) = \begin{bmatrix}-2 & 0\end{bmatrix}^T$ and $w(t) = 0$ for all $t \geq 0$. The time responses of the controlled outputs $z_1(t)$ and $z_2(t)$ depicted in Figure 5.3 show that the decentralized nonlinear controllers (5.94), (5.95) have a good performance in the presence of perturbations.



(a)



(b)

Figure 5.3: Time responses of $z_1(t)$ and $z_2(t)$ for different values of Δ_1 and Δ_2.

For comparison purpose, we apply the method in [166] to synthesize decentralized stable linear robust H^∞ controllers to absolutely stabilize the nonlinear uncertain system (5.88), (5.89), (5.90). In this case, each nonlinearity $\psi_i(\cdot)$ in the system is necessarily considered as an uncertainty and thus, we only need to solve a linear robust H^∞ control problem. This approach, however, may result in decentralized linear controllers with a degraded performance in attenuating exogenous disturbances $w(t)$ as compared to the performance of the decentralized nonlinear controllers (5.94), (5.95). This is indicated by the disturbance attenuation level $\gamma = 0.4932$ obtained using the method in [166] and it is apparently larger than $\gamma = 0.4520$ obtained using our method.

5.6 Conclusions

In this chapter, we have proposed a new method to synthesis stable decentralized nonlinear robust H^∞ controllers for a class of large-scale nonlinear uncertain systems. The admissible uncertainties and nonlinearities in this system are characterized by IQCs and GLCs, respectively. We assume that the large-scale nonlinear system consists of interconnected subsystems and their interconnections are well known. Thus, we do not consider the interconnections between subsystems as uncertainties, but rather as useful structural information on the entire large-scale system. With this approach, the decentralized nonlinear controllers are able to exploit the interconnections although no assumption is made on how all subsystems are interconnected.

The H^∞ control objective is to guarantee an absolute stability for the resulting closed loop large-scale nonlinear uncertain system with a specified disturbance attenuation level. Along with this objective, we require the decentralized nonlinear controllers to be stable. The controller stability requirement is imposed because we consider all nonlinear error systems arising from the discrepancies between the decentralized and the non-decentralized nonlinear controllers as additional uncertainties. These error systems are also required to have bounded H^∞ norms. Thus, the resulting decentralized nonlinear controllers have to be robust against not only the plant uncertainties, but also the additional uncertainties due to the nonlinear error systems.

To guarantee the controller stability, we first solve a state feedback control problem for the given nonlinear uncertain system. Then, using the resulting state feedback gain matrix and introducing an additional artificial uncertainty, we form an artificial uncertain system for which a robust H^∞ output feedback controller is designed. If there exists a suitable controller for this latter system, we argue that this controller must be stable and also solves the original decentralized nonlinear control problem. Therefore, we only provide sufficient conditions for synthesizing the stable decentralized nonlinear controllers as our approach involves particular realization of nonlinear error systems and artificial uncertain system. However, we should note that the additional artificial uncertainty may give rise to some extra conservatism to the controller design process.

The solution to our control problem is then given in terms of the stabilizing solutions to algebraic Riccati equations, which are dependent on a set of scaling parameters. This formulation leads to a nonconvex decentralized nonlinear control problem, which is generally difficult to solve using regular optimization methods. We thus propose to use an evolutionary optimization method, namely the DE algorithm, to compute the required design parameters.

Moreover, we have shown through an example that our method is capable of resulting in a decentralized nonlinear controller which allows for a better performance in attenuating exogenous disturbances as compared to its linear counterpart designed using the method in [166]. This is achievable because we do not directly treat the admissible known nonlinearities in the system as uncertainties so that we can exploit them for control purposes.

Chapter 6

Coherent Control of Linear Quantum Systems

6.1 Introduction

This chapter presents a computational algorithm to solve a coherent quantum feedback control problem for a class of non-commutative linear quantum stochastic systems. It has been pointed out in [70, 72, 73] that a coherent quantum controller must satisfy a physical realizability condition in order to exhibit meaningful dynamic behavior according to quantum mechanical principles. This requirement in turn leads to a coherent quantum controller synthesis problem, which involves a constraint that is naturally nonconvex and nonlinear. From numerical perspective, it is often considered to be difficult to solve this problem. Thus, inspired by the results of [72] on coherent quantum LQG control, we propose to use an optimization approach to solve a coherent quantum feedback control problem. In particular, we apply an evolutionary optimization method, namely the differential evolution (DE) algorithm, as presented in Chapter 2. The main ideas of our approach are to obtain a straightforward and less complicated algorithm and to avoid a critical dependence on a suitable initial point to start a numerical iteration.

We then demonstrate our DE-based approach through a case study on the problem of quantum network entanglement control. This problem has drawn a lot of attention in the quantum information and computation literature. This is due to the fact that entanglement is a fundamental property used in quantum information processing; e.g., see [25, 234, 235]. Thus, entanglement generation, preservation and restoration have been extensively studied and many techniques have been proposed to attain these goals; e.g., see [61, 233, 238–244]. For our purposes, we consider a simple ideal quantum network consisting of two cascaded optical parametric amplifiers interacting through an optical field. This quantum network has been entangled, but we aim to increase its entanglement level using a dynamic coherent quantum controller to replace the simple optical field connection. Thus, the quantum controller is required not only to satisfy the physical realizability condition (e.g., see [70, 72]), but also to drive the controlled quantum system so that an entanglement criterion is satisfied.

There are many ways to measure the entanglement level depending on properties of the quantum states; e.g., see [245, 246]. In our case, we have Gaussian quantum states, and thus, the entanglement level can be measured in terms of the logarithmic negativity as a function of the covariance matrix of the quantum network; e.g., see [233, 247–249]. The covariance matrix can be obtained as a solution of a Lyapunov equation associated with the quantum network. Thus, applying our DE-based algorithm to synthesize the coherent quantum controller, we can consider the logarithmic negativity as an objective function to maximize. This in turn leads to a physically realizable quantum controller, which is capable of stabilizing the quantum network with an enhanced entanglement level. Moreover, it should also be noted that the logarithmic negativity is a nonconvex functional; see [250].

We use the following notation throughout this chapter. If $M = [m_{jk}]$ is a $p \times q$ complex matrix, then M^*, M^T and $M^\dagger$ denote the operation of taking the complex conjugate of each entry of M, the transpose of M, and the complex conjugate transpose of M, respectively. That is, $M^* = [m_{jk}^*]$, $M^T = [m_{kj}]$ and $M^\dagger = [m_{kj}^*] = (M^*)^T$. Moreover, if M is an operator matrix, then M^* denotes the operation of taking the adjoint of each entry of M.

6.2 Linear Quantum Stochastic Systems

As a preliminary discussion, we recall a linear quantum stochastic system modeled in terms of its real and imaginary quadratures, which has been described in [70, 72] on the basis of quantum probability theory; e.g., see [46]. This model is used to represent a non-commutative quantum system, which may consist of both quantum and non-quantum components. Moreover, we also define the physical realizability of this quantum system in terms of the realization of an open quantum harmonic oscillator.

6.2.1 Non-commutative model

The linear non-commutative quantum stochastic system under consideration is described in terms of linear quantum stochastic differential equations (QSDEs) as follows:

$$\begin{aligned} dx(t) &= A\,x(t)\,dt + B\,dw(t); \quad x(0) = x_0; \\ dy(t) &= C\,x(t)\,dt + D\,dw(t) \end{aligned} \tag{6.1}$$

where the matrices A, B, C and D are real matrices belonging to $\mathbf{R}^{n\times n}$, $\mathbf{R}^{n\times n_w}$, $\mathbf{R}^{n_y\times n}$ and $\mathbf{R}^{n_y\times n_w}$, respectively. We assume that n_y is even and $n_w \geq n_y$; see [70]. Moreover, $x(t) := \begin{bmatrix} x_1(t) & \dots & x_n(t) \end{bmatrix}^T$ is an $n \times 1$ vector of non-commutative system variables whose initial condition $x(0)$ satisfies the following commutation relation

$$x(0)x(0)^T - (x(0)x(0)^T)^T = 2i\Theta \tag{6.2}$$

where Θ is a real skew-symmetric canonical commutation matrix defined as

$$\Theta := \mathrm{diag}(J, J, \dots, J). \tag{6.3}$$

Note that diag$(\cdot)$ denotes a block diagonal matrix; the zero matrix in (6.3) is an $m \times m$ matrix with $0 < m \leq n$; and J is a real skew-symmetric matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{6.4}$$

The linear quantum system (6.1) is driven by an $n_w \times 1$ vector of input signals $w(t)$, which is assumed to be decomposed as follows (see [70, 72, 73]):

$$dw(t) = \beta_w(t)\,dt + d\tilde{w}(t). \tag{6.5}$$

Here, $\beta_w(t)$ and $\tilde{w}(t)$ are respectively the self-adjoint adapted process and quantum noise components of $w(t)$. A rigorous description of the notion of adapted process can be found in [43, 44, 46]. The adapted process $\beta_w(t)$ can be used to represent variables of other systems interacting with the quantum system (6.1). This leads to our assumption that $\beta_w(t)$ commutes with $x(t)$ and $\tilde{w}(t)$ for all $t \geq 0$ because they live in different Hilbert spaces; see [70, 72].

The self-adjoint quantum noise $\tilde{w}(t)$ is characterized by the Ito table

$$d\tilde{w}(t)d\tilde{w}(t)^T = F_{\tilde{w}}\, dt \tag{6.6}$$

where the Ito matrix $F_{\tilde{w}}$ is a non-negative Hermitian matrix; e.g., see [44, 49]. It is also common to consider the quantum noise $\tilde{w}(t)$ as a vector of self-adjoint operators belonging to a particular Fock space; e.g., see [44, 50, 70]. The relation in (6.6) leads to the following non-commutative relation

$$d\tilde{w}(t)d\tilde{w}(t)^T - \left(d\tilde{w}(t)d\tilde{w}(t)^T\right)^T = 2T_{\tilde{w}}\, dt \tag{6.7}$$

where $T_{\tilde{w}} := \frac{1}{2}\left(F_{\tilde{w}} - F_{\tilde{w}}^T\right)$ is a Hermitian commutation matrix. As in [70], we assume that $F_{\tilde{w}}$ is canonical

$$F_{\tilde{w}} := I + i\,\mathrm{diag}(J, J, \ldots, J) \tag{6.8}$$

and hence, n_w must be even.

Since the dynamics of the quantum system (6.1) is represented in terms of a linear QSDE, the integral operation with respect to $dw(t)$ is a quantum stochastic integral. This operation then results in an evolution $x(t)$ of the linear quantum system (6.1), which depends only on the past input signal $w(s)$ for $0 \leq s \leq t$. Thus, $x(t)$ is also a quantum adapted process which commutes with the Ito increment $d\tilde{w}(t)$; see [70].

6.2.2 Physical realizability

In order to be physically realizable, the representation $\{A, B, C, D\}$ of the quantum system (6.1) cannot be arbitrary as it is subject to the requirement that it has to preserve the canonical commutation relation

$$x(t)x(t)^T - (x(t)x(t)^T)^T = 2i\Theta, \quad \forall t \geq 0. \tag{6.9}$$

According to Theorem 2.1 in [70], the relation (6.9) is equivalent to

$$iA\Theta + i\Theta A^T + BT_{\tilde{w}}B^T = 0. \tag{6.10}$$

This property is well described by an open physical system unitarily evolving for all time; e.g., see [36, 70]. Thus, we relate the physical realizability of the quantum system (6.1) to the realization of an open quantum harmonic oscillator.

Definition 6.1. (*Open quantum harmonic oscillator*; see [70, Definition 3.1]) The quantum system (6.1) with $\beta_w(t) = 0$ is said to be an open quantum harmonic oscillator if Θ is canonical and there exists a quadratic Hamiltonian $H := \frac{1}{2}x(0)^T Rx(0)$ ($R \in \mathbf{R}^{n\times n}$ is a symmetric Hamiltonian matrix) and a coupling operator $L := \Lambda x(0)$ ($\Lambda \in \mathbf{C}^{n_w\times n}$ is a coupling matrix) such that

$$\begin{aligned} x_k(t) &= U(t)^* \, x_k(0) \, U(t), \quad \forall k = 1, 2, \ldots, n; \\ y_l(t) &= U(t)^* \, w_l(t) \, U(t), \quad \forall l = 1, 2, \ldots, n_y. \end{aligned} \tag{6.11}$$

Here, $\{U(t); t \geq 0\}$ is an adapted process of unitary operators satisfying the following QSDE

$$dU(t) = \left(-iH\,dt - \tfrac{1}{2}L^\dagger L + \begin{bmatrix} -L^\dagger & L^T \end{bmatrix} \Upsilon \, dw(t)\right) U(t), \quad U(0) = I \tag{6.12}$$

where

$$\begin{aligned} \Upsilon &:= \mathsf{P}_{n_w/2} \, \mathrm{diag}_{n_w/2}(\mathsf{M}); \\ \mathsf{M} &= \frac{1}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \end{aligned} \tag{6.13}$$

and $\mathsf{P}_{n_w/2}$ is an $n_w \times n_w$ permutation matrix satisfying

$$\mathsf{P}_{n_w/2}\begin{bmatrix} a_1 & \ldots & a_{n_w} \end{bmatrix}^T = \begin{bmatrix} a_1 & a_3 & \ldots & a_{n_w-1} & a_2 & a_4 & \ldots & a_{n_w} \end{bmatrix}^T. \tag{6.14}$$

Thus, for the quantum system (6.1),

$$A = 2\Theta\left(R + \Im\left(\Lambda^\dagger \Lambda\right)\right); \quad B = 2i\Theta\begin{bmatrix} -\Lambda^\dagger & \Lambda^T \end{bmatrix}\Upsilon; \tag{6.15}$$

$$C = \mathsf{P}_{n_y/2}^T \operatorname{diag}(\Xi, \Xi) \begin{bmatrix} \Lambda + \Lambda^* \\ -i\Lambda + i\Lambda^* \end{bmatrix}; \quad D = \begin{bmatrix} I_{n_y \times n_y} & 0_{n_y \times (n_w - n_y)} \end{bmatrix} \tag{6.16}$$

where $\Xi := \begin{bmatrix} I_{(n_y/2)\times(n_y/2)} & 0_{(n_y/2)\times((n_w/2)-(n_y/2))} \end{bmatrix}$ and $\Im(\cdot)$ denotes an imaginary part of $(\cdot)$.

Definition 6.2. (*Physical realizability*; see [70, Definition 3.3]) The quantum system (6.1) is said to be physically realizable if Θ is canonical and (6.1) represents the dynamics of an open quantum harmonic oscillator.

Lemma 6.1. *(see [70, Theorem 3.4] or [72, Theorem 1]) The quantum system* (6.1) *is physically realizable if and only if*

$$\begin{aligned} &iA\Theta + i\Theta A^T + BT_{\tilde{w}}B^T = 0; \\ &B \begin{bmatrix} I_{n_y \times n_y} \\ 0_{(n_w - n_y)\times n_y} \end{bmatrix} = \Theta C^T \operatorname{diag}_{n_y/2}(J); \quad D = \begin{bmatrix} I_{n_y \times n_y} & 0_{n_y \times (n_w - n_y)} \end{bmatrix} \end{aligned} \tag{6.17}$$

where $T_{\tilde{w}} := i \operatorname{diag}(J, \dots, J)$*. Moreover, if* Θ *is canonical, the Hamiltonian matrix* R *and the coupling matrix* Λ *are uniquely given by*

$$\begin{aligned} R &= \frac{1}{4}\left(-\Theta A + A^T\Theta\right); \\ \Lambda &= -\frac{1}{2} i \begin{bmatrix} 0_{(n_w/2)\times(n_w/2)} & I_{(n_w/2)\times(n_w/2)} \end{bmatrix} (\Upsilon^{-1})^T B^T \Theta. \end{aligned} \tag{6.18}$$

Remark 6.1. Considering a coherent quantum feedback control problem, we do not include in this section a discussion about the degenerate canonical case, which can be found in [70, 72].

6.3 Quantum Feedback Control Problem

We consider a linear quantum system described by the following non-commutative stochastic dynamic model in terms of QSDEs:

$$\begin{aligned} dx(t) &= A\,x(t)\,dt + B\,du(t) + B_w\,dw(t); \quad x(0) = x_0; \\ dy(t) &= C\,x(t)\,dt + D_w\,dw(t) \end{aligned} \tag{6.19}$$

where $x(t)$ is a vector of non-commutative system variables; $w(t)$ is a quantum Wiener process; $u(t)$ is a control input; and $y(t)$ is a system output. The dimensions of $x(t)$, $w(t)$, $u(t)$ and $y(t)$ are compatible with those of the plant coefficient matrices whose entries are real numbers. That is, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times n_u}$, $B_w \in \mathbf{R}^{n\times n_w}$, $C \in \mathbf{R}^{n_y\times n}$ and $D_w \in \mathbf{R}^{n_y\times n_w}$. A detailed account of the linear quantum stochastic system (6.19) can also be found in [70] and [72].

The control input $u(t)$ of the quantum system (6.19) is modeled as

$$du(t) = \beta_u(t)\,dt + d\tilde{u}(t) \tag{6.20}$$

where $\beta_u(t)$ and $\tilde{u}(t)$ denote signal and quantum noise components of $u(t)$, respectively. Moreover, $\beta_u(t)$ is considered as an adapted, self-adjoint process commuting with $x(t)$ and therefore,

$$\beta_u(t)x(t)^T - (x(t)\beta_u(t)^T)^T = 0. \tag{6.21}$$

The quantum noise $\tilde{u}(t)$ is independent of $w(t)$ as they belong to different Fock spaces. The corresponding Ito matrices $F_{\tilde{u}}$ and F_w are non-negative Hermitian matrices. Moreover, it is also assumed that the initial condition $x(0)$ of the quantum system (6.19) is a non-commutative process satisfying (6.2).

For the quantum system (6.19), we intend to construct a dynamic coherent quantum controller, which is also assumed to be a non-commutative quantum stochastic system. Plant-controller coherency means that a closed loop quantum system is formed without direct measurement of the system variables as in the classical control case. A general dynamic quantum controller is written as

$$\begin{aligned} dx_c(t) &= A_K x_c(t)dt + \sum_{j=1}^{2} B_{K_j}\, dw_{K_j}(t) + B_{K_3}\, dy(t); \quad x_c(0) = x_{c_0}; \\ du(t) &= C_K x_c(t)dt + dw_{K_1}(t) \end{aligned} \tag{6.22}$$

where $x_c(t)$ is a vector of self-adjoint operators (controller variables) and each $w_{K_j}(t)$ (for $j = 1, 2$) is a non-commutative quantum Wiener process, which is also independent of $w(t)$. The quantum controller (6.22) is assumed to be of n-th order, and hence $A_K \in \mathbf{R}^{n\times n}$. Also, B_{K_2} has the same dimension as A_K, and B_{K_1} has the same number of columns as the rows of C_K. The initial condition $x_c(0) =$

x_{c_0} of the quantum controller (6.22) is also assumed to be a non-commutative process. Therefore,

$$x_c(0)x_c(0)^T - (x_c(0)x_c(0)^T)^T = 2i\Theta_K \tag{6.23}$$

where Θ_K is a real skew-symmetric commutation matrix of the quantum controller (6.22). Moreover, we assume that there is no initial coupling between the plant and the controller. That is,

$$x(0)x_c(0)^T - (x_c(0)x(0)^T)^T = 0. \tag{6.24}$$

Unlike its classical (non-quantum) controller counterpart, a quantum controller of the form (6.22) has to be physically realizable as the representation in (6.22) does not necessarily lead to a meaningful physical system governed by quantum mechanics principles. This requires the canonical commutation relation

$$x_c(t)x_c(t)^T - (x_c(t)x_c(t)^T)^T = 2i\Theta_K \tag{6.25}$$

to be preserved for all $t \geq 0$; e.g., see [70, 72]. Thus, referring to the notion of physical realizability of a quantum system as in Definition 6.2, the quantum controller (6.22) is said to be physically realizable if it represents the dynamics of an open quantum harmonic oscillator as defined in Definition 6.1.

Consequently, based on Lemma 6.1, the physical realizability condition of the quantum controller (6.22) can be stated as follows:

Corollary 6.1. *(see also [70, 72]) Let Θ_K be a given real skew-symmetric commutation matrix. Then the coherent quantum controller (6.22) is physically realizable if and only if its coefficient matrices: A_K, B_{K_1}, B_{K_2}, B_{K_3} and C_K are such that*

$$A_K\Theta_K + \Theta_K A_K^T + \sum_{j=1}^{3} B_{K_j}\Gamma_j B_{K_j}^T = 0; \tag{6.26}$$

$$B_{K_1} = \Theta_K C_K^T \operatorname{diag}_{n_u/2}(J) \tag{6.27}$$

where $\Gamma_j := \operatorname{diag}_{n_j/2}(J)$ for $j = 1, 2, 3$; and n_j is the dimension of w_{K_j} with $w_{K_3}(t) \equiv y(t)$.

Remark 6.2. The physical realizability condition (6.26), (6.27) given in Corollary 6.1 can be derived from Lemma 6.1 with the following notation:

$$\begin{aligned} A &= A_K; \quad B = \begin{bmatrix} B_{K_1} & B_{K_2} & B_{K_3} \end{bmatrix}; \quad C = C_K; \\ D &= \begin{bmatrix} I_{n_u \times n_u} & 0 \end{bmatrix}; \quad w = \begin{bmatrix} w_{K_1}^T & w_{K_2}^T & y^T \end{bmatrix}^T. \end{aligned} \tag{6.28}$$

Remark 6.3. To construct a coherent quantum controller, we could require that Θ_K is a canonical commutation matrix, which implies that each of its diagonal blocks is equal to J as in (6.3). However, in our approach, we allow Θ_K to be a general skew-symmetric matrix. A suitable similarity transformation can then be applied to the quantum controller (6.22) to obtain Θ_K in canonical or degenerate canonical form; see also [72]. Moreover, the equality condition (6.26) is a nonconvex nonlinear constraint, which poses difficulty if we solve the quantum controller synthesis problem using a regular optimization method. It has also been claimed in [72] that an analytical solution to this problem has not yet been developed.

Interconnecting the quantum controller (6.22) with the open loop quantum system (6.19), we obtain a closed loop quantum system written as

$$d\eta(t) = \mathcal{A}\eta(t)dt + \mathcal{B}d\omega(t) \tag{6.29}$$

where

$$\begin{aligned} \eta(t) &= \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}; \quad \omega(t) = \begin{bmatrix} w(t) \\ w_{K_1}(t) \\ w_{K_2}(t) \end{bmatrix}; \\ \mathcal{A} &= \begin{bmatrix} A & BC_K \\ B_{K_3}C & A_K \end{bmatrix}; \quad \mathcal{B} = \begin{bmatrix} B_w & B & 0 \\ B_{K_3}D_w & B_{K_1} & B_{K_2} \end{bmatrix}. \end{aligned} \tag{6.30}$$

Here, $\beta_u(t) \equiv C_K x_c(t)$ and $\tilde{u}(t) \equiv w_{K_1}(t)$. The closed loop quantum system matrix $\mathcal{A}$ in (6.29) is required to be Hurwitz.

6.4 Coherent Quantum Controller Synthesis

In this section, we present a computational algorithm in order to systematically obtain a solution to the coherent quantum control problem presented in Section

6.3. This is a nonconvex nonlinear problem as the stabilizing coherent quantum controller (6.22) is required to satisfy the physical realizability condition (6.26). This characteristic has also been pointed out, for example, in [72], where a coherent quantum LQG control problem is addressed using the rank constrained LMI algorithm (see [158]). However, this approach tends to become very complicated if it is used to solve a higher dimension quantum control problem. Also, as with many nonconvex problem solvers, the success of the rank constrained LMI approach is strongly dependent on the initial point from which a numerical iteration is started. Then, as an immediate consequence, the numerical iteration tends to become unreliable and less tractable.

Those concerns above have motivated us to propose an alternative method to solve the coherent quantum controller synthesis problem using a population-based stochastic optimization method, namely the differential evolution (DE) algorithm, as described in Chapter 2. Interestingly, it is straightforward to adapt the DE algorithm to the original coherent quantum control problem to be solved. This fact indicates a potential application of DE-based methods to other classes of coherent quantum control problem such as those presented in [70, 72, 73].

Based on the DE approach, we then reformulate the coherent quantum control problem described in Section 6.3 as a constrained nonlinear optimization problem. That is, we want to find an optimal solution $\mathcal{K}^\star$ to solve

$$\min_{\mathcal{K}} \mathsf{f}(\mathcal{K}) \tag{6.31}$$

subject to

$$\mathsf{g}_k(\mathcal{K}) = 0; \quad \mathsf{h}_l(\mathcal{K}) \leq 0 \tag{6.32}$$

for $k = 1, 2, \ldots, \mathsf{a}$ and $l = 1, 2, \ldots, \mathsf{b}$, where a and b are the total number of equality and inequality constraints, respectively. In this case, $f(\mathcal{K})$ is the objective function to be minimized, which is a function of

$$\mathcal{K} := (A_K,\, B_{K_1},\, B_{K_2},\, B_{K_3},\, C_K)\,. \tag{6.33}$$

Moreover, applying the DE approach, we in fact have some flexibility to define the objective function $\mathsf{f}(\mathcal{K})$ in accordance with a particular quantum control problem to be solved. Since the physical realizability condition (6.26) is an essential property of the coherent quantum controller (6.22) and the closed loop quantum system (6.29) is required to be asymptotically stable, we then have

$$\mathsf{g}_1(\mathcal{K}) = A_K\Theta_K + \Theta_K A_K^T + \sum_{j=1}^{3} B_{K_j}\Gamma_j B_{K_j}^T = 0; \tag{6.34}$$

$$\mathsf{h}_1(\mathcal{K}) = e_{\max,r}(\mathcal{A}) < 0 \tag{6.35}$$

where $e_{\max,r}(\mathcal{A})$ denotes the largest real part of the eigenvalues of $\mathcal{A}$. The constraints (6.34) and (6.35) must always be taken into account and satisfied in the synthesis of the quantum controller (6.22).

For a second order quantum system, we only need to solve a quadratic equation to obtain a solution Θ_K to (6.34). However, for a higher order quantum system, this is generally not the case. Hence, we transform (6.34) into a complex algebraic Riccati equation in order to obtain a solution Θ_K to (6.34). We first substitute (6.27) into (6.34) and then multiply by $i = \sqrt{-1}$. This leads to a complex algebraic Riccati equation

$$A_K\Sigma_K + \Sigma_K A_K^T + \Sigma_K C_K^T\widetilde{\Gamma}_1 C_K\Sigma_K + B_{K_2}\widetilde{\Gamma}_2 B_{K_2}^T + B_{K_3}\widetilde{\Gamma}_3 B_{K_3}^T = 0 \tag{6.36}$$

where

$$\Sigma_K := i\Theta_K; \quad \Sigma_K^\dagger = \Sigma_K; \quad \widetilde{\Gamma}_j := i\Gamma_j; \quad \widetilde{\Gamma}_j^\dagger = \widetilde{\Gamma}_j \tag{6.37}$$

for $j = 1, 2, 3$. Thus, we can replace the constraint (6.34) with (6.36). That is,

$$\mathsf{g}_1(\mathcal{K}) = A_K\Sigma_K + \Sigma_K A_K^T + \Sigma_K C_K^T\widetilde{\Gamma}_1 C_K\Sigma_K + B_{K_2}\widetilde{\Gamma}_2 B_{K_2}^T + B_{K_3}\widetilde{\Gamma}_3 B_{K_3}^T = 0. \tag{6.38}$$

To compute the solution to the Riccati equation (6.38), we use an approach based on an invariant subspace of a Hamiltonian matrix corresponding to (6.38), which is defined as

$$\mathcal{H} := \begin{bmatrix} A_K^T & C_K^T\widetilde{\Gamma}_1 C_K \\ -\left(B_{K_2}\widetilde{\Gamma}_2 B_{K_2}^T + B_{K_3}\widetilde{\Gamma}_3 B_{K_3}^T\right) & -A_K \end{bmatrix}. \tag{6.39}$$

Thus, the existence of a solution to the Riccati equation (6.38) is characterized in the following lemma.

Lemma 6.2. *(see [18, Theorem 13.1 and Theorem 13.3]) Let $\mathcal{V} \subset \mathbf{C}^{2n}$ be an n-dimensional invariant subspace of the Hamiltonian matrix* (6.39) *and let $\mathcal{X}, \mathcal{Y} \in$*

$\mathbf{C}^{n\times n}$ *be two complex matrices such that*

$$\mathcal{V} = \mathrm{Im}\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}. \tag{6.40}$$

Then,

1. *If $\mathcal{X}$ is invertible, then $\Sigma_K := \mathcal{Y}\mathcal{X}^{-1}$ is a solution to the Riccati equation* (6.38) *and $\sigma(A_K^T + C_K^T\widetilde{\Gamma}_1 C_K\Sigma_K) = \sigma\left(\mathcal{H}|_{\mathcal{V}}\right)$. Furthermore, the solution Σ_K is independent of a particular choice of bases of $\mathcal{V}$.*

2. *If $e_k + e_l^* \neq 0$, $\forall k, l = 1, 2, \ldots, n$, $e_k, e_l \in \sigma\left(\mathcal{H}|_{\mathcal{V}}\right)$, then $\mathcal{X}^*\mathcal{Y}$ is Hermitian, that is, $\mathcal{X}^*\mathcal{Y} = (\mathcal{X}^*\mathcal{Y})^*$. Moreover, if $\mathcal{X}$ is nonsingular, then $\Sigma_K = \mathcal{Y}\mathcal{X}^{-1}$ is Hermitian.*

Here, $\mathrm{Im}(M)$ *denotes the image of the matrix M; and $\sigma(M)$ denotes a set of eigenvalues of the matrix M.*

Remark 6.4. A particular choice for the bases of the $\mathcal{H}$-invariant subspace $\mathcal{V}$ can be taken as the (generalized) eigenvectors of the Hamiltonian matrix (6.39).

If the conditions in Lemma 6.2 hold, we are assured that there exists a Hermitian solution Σ_K to (6.36) although it is not a unique solution. However, we need to confirm that Σ_K is a purely imaginary solution as defined in (6.37). The following lemma and theorem provide conditions to obtain such a Σ_K.

Lemma 6.3. *(e.g., see [18, Lemma 2.7]) Consider the Lyapunov equation*

$$AX + XA^\dagger = B \tag{6.41}$$

where $A, B \in \mathbf{C}^{n\times n}$. There exists a unique solution $X \in \mathbf{C}^{n\times n}$ to (6.41) *if and only if $e_k(A) + e_l^*(A) \neq 0$ for all $k, l = 1, 2, \ldots, n$.*

Theorem 6.1. *Suppose that the complex Riccati equation* (6.36) *has a Hermitian solution $\Sigma_K = \Phi_K + i\Pi_K$ such that*

$$e_k(\tilde{A}_K) + e_l^*(\tilde{A}_K) \neq 0, \quad \forall k, l = 1, 2, \ldots, n \tag{6.42}$$

where $\tilde{A}_K := A_K - \Pi_K C_K^T \Gamma_1 C_K$. Then, Σ_K is indeed an imaginary solution, that is $\Sigma_K = i\Pi_K$, which satisfies the physical realizability condition (6.26).

Proof. Suppose that a Hermitian matrix $\Sigma_K = \Phi_K + i\Pi_K$ satisfies (6.36), where Φ_K is a real symmetric matrix and Π_K is a real skew-symmetric matrix. Then, substituting Σ_K into (6.36), we obtain

$$\begin{aligned}&\left(A_K - \Pi_K C_K^T \Gamma_1 C_K\right)\Phi_K + \Phi_K\left(A_K - \Pi_K C_K^T \Gamma_1 C_K\right)^T + i\left(A_K \Pi_K + \Pi_K A_K^T\right.\\ &\left.+\Phi_K C_K^T \Gamma_1 C_K \Phi_K - \Pi_K C_K^T \Gamma_1 C_K \Pi_K + B_{K_2}\Gamma_2 B_{K_2}^T + B_{K_3}\Gamma_3 B_{K_3}^T\right) = 0. \quad (6.43)\end{aligned}$$

The left-hand side of (6.43) is equal to zero if and only if its real and imaginary parts are equal to zero. That is,

$$\left(A_K - \Pi_K C_K^T \Gamma_1 C_K\right)\Phi_K + \Phi_K\left(A_K - \Pi_K C_K^T \Gamma_1 C_K\right)^T = 0; \quad (6.44)$$

$$\begin{aligned}&A_K \Pi_K + \Pi_K A_K^T + \Phi_K C_K^T \Gamma_1 C_K \Phi_K\\ &\quad - \Pi_K C_K^T \Gamma_1 C_K \Pi_K + B_{K_2}\Gamma_2 B_{K_2}^T + B_{K_3}\Gamma_3 B_{K_3}^T = 0. \quad (6.45)\end{aligned}$$

It follows from Lemma 6.3 that if condition (6.42) is satisfied, then Φ_K must equal to zero in order that the real part (6.44) holds. Therefore, Σ_K is indeed an imaginary solution, that is, $\Sigma_K = i\Pi_K$. Furthermore, the imaginary part (6.45) will lead to the satisfaction of the physical realizability condition (6.26). That is, $\Theta_K = \Pi_K$. □

Note that $e_k(M)$ denotes the k-th eigenvalue of the matrix M and $e_l^*(M)$ denotes the complex conjugate of the l-th eigenvalue of the matrix M.

Remark 6.5. Although a coherent quantum LQG control problem as described in [72] is not particularly addressed in this chapter, we can also use the DE-based approach to solve this quantum control problem. In this regard, when solving the same example as in [72], our DE-method is capable of returning a cost function value $\mathsf{f}(\mathcal{K}^\star) = 4.0801$, which is smaller than $\mathsf{f}(\mathcal{K}^\star) = 4.1793$ as given in [72]. These numerical results indicate that, apart from being more straightforward, our DE-method is likely to outperform the rank constrained LMI method used in [72]. This fact then becomes an impetus for us to demonstrate the efficacy of our DE-method when it is used to design a coherent quantum controller of the form (6.22) for solving a more general linear quantum control problem with a higher dimension. Thus, in the next section, we apply the DE-method to solve a quantum entanglement control problem, which is considered as one of essential aspects in quantum technology development.

6.5 Case Study: Entanglement Control

We utilize a dynamic coherent quantum controller designed using the method developed in Section 6.4 to increase quantum entanglement level of two cascaded optical parametric amplifiers (OPAs) (e.g., see [37]) interacting through an ideal optical field as shown in Figure 6.1. This particular application is motivated by [233] where the OPAs are referred to as damped optical cavities. The entanglement control mechanism used in [233] is applied to avoid finite-time entanglement sudden-death as well as to enhance entanglement level. It is attained through direct measurement feedback from a homodyne detector to control the dynamic behaviour of both OPAs using a static gain controller. This is in contrast to our approach where we apply a coherent quantum controller without a feedback loop to achieve enhanced entanglement as shown in Figure 6.2.

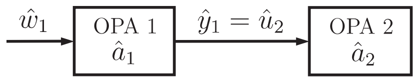


Figure 6.1: An ideal quantum network of two cascaded OPAs.

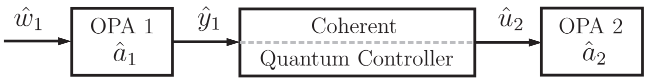


Figure 6.2: Dynamic entanglement control.

The dynamic model of the first OPA is described in terms of complex linear QSDEs (e.g., see [78]):

$$\begin{aligned} d\hat{a}_1(t) &= \chi_1\,\hat{a}_1^*(t)dt - \kappa_1\,\hat{a}_1(t)dt - i\Delta_1\,\hat{a}_1(t)dt + \sqrt{2\kappa_1}\,d\hat{w}_1(t); \\ d\hat{y}_1(t) &= -\sqrt{2\kappa_1}\,\hat{a}_1(t)dt + d\hat{w}_1(t) \end{aligned} \tag{6.46}$$

and that of the second OPA is

$$d\hat{a}_2(t) = \chi_2\,\hat{a}_2^*(t)dt - \kappa_2\,\hat{a}_2(t)dt - i\Delta_2\,\hat{a}_2(t)dt + \sqrt{2\kappa_2}\,d\hat{u}_2(t) \tag{6.47}$$

where, for each OPA, $\hat{a}$ is an annihilation operator with $\hat{a}^*$ as its corresponding creation operator; $\hat{w}$ and $\hat{u}$ are the input signals; $\hat{y}$ is the output signal; $\chi := \alpha + i\beta$ is a complex coupling constant; Δ is a detuning parameter; and κ is the loss rate of the OPA. The dynamic models (6.46) and (6.47) can also be found in [37].

We can decompose the annihilation and creation operators, and the input and output signals in (6.46) and (6.47) into amplitude and phase quadratures as follows:

- Amplitude quadrature:

$$\begin{aligned} q_1 &:= \tfrac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_1^*); \quad & q_2 &:= \tfrac{1}{\sqrt{2}}(\hat{a}_2 + \hat{a}_2^*); \quad & w_{11} &:= \hat{w}_1 + \hat{w}_1^*; \\ y_{11} &:= \hat{y}_1 + \hat{y}_1^*; & u_{21} &:= \hat{u}_2 + \hat{u}_2^*. \end{aligned} \tag{6.48}$$

- Phase quadrature:

$$\begin{aligned} p_1 &:= \tfrac{-i}{\sqrt{2}}(\hat{a}_1 - \hat{a}_1^*); \quad & p_2 &:= \tfrac{-i}{\sqrt{2}}(\hat{a}_2 - \hat{a}_2^*); \quad & w_{12} &:= -i(\hat{w}_1 - \hat{w}_1^*); \\ y_{12} &:= -i(\hat{y}_1 - \hat{y}_1^*); & u_{22} &:= -i(\hat{u}_2 - \hat{u}_2^*). \end{aligned} \tag{6.49}$$

Based on the expressions in (6.48) and (6.49), the dynamic models (6.46) and (6.47) can then be written in the form of QSDEs (6.19) with the system matrices:

$$A = \begin{bmatrix} -\alpha_1 - \kappa_1 & -\beta_1 + \Delta_1 & 0 & 0 \\ -\beta_1 - \Delta_1 & \alpha_1 - \kappa_1 & 0 & 0 \\ 0 & 0 & -\alpha_2 - \kappa_2 & -\beta_2 + \Delta_2 \\ 0 & 0 & -\beta_2 - \Delta_2 & \alpha_2 - \kappa_2 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \sqrt{\kappa_2} & 0 \\ 0 & \sqrt{\kappa_2} \end{bmatrix};$$

$$B_w = \begin{bmatrix} \sqrt{\kappa_1} & 0 \\ 0 & \sqrt{\kappa_1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}; \quad C = \begin{bmatrix} -2\sqrt{\kappa_1} & 0 & 0 & 0 \\ 0 & -2\sqrt{\kappa_1} & 0 & 0 \end{bmatrix}; \quad D_w = I_{2\times 2} \tag{6.50}$$

and the system variable vector x, control input vector u, quantum noise vector w and output vector y are defined as follows:

$$x := \begin{bmatrix} q_1^T & p_1^T & q_2^T & p_2^T \end{bmatrix}^T; \quad u := \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix}; \quad w := \begin{bmatrix} w_{11} \\ w_{12} \end{bmatrix}; \quad y := \begin{bmatrix} y_{11} \\ y_{12} \end{bmatrix}. \tag{6.51}$$

To consider the same example as in [233], we take the parameter values of the quantum system (6.50) to be:

$$\alpha = 0; \quad \beta = -0.4; \quad \kappa_1 = 1; \quad \kappa_2 = 1; \quad \Delta_1 = 0.6; \quad \Delta_2 = 0.6. \tag{6.52}$$

6.5.1 Direct connection

We first examine the quantum entanglement level of two OPAs (6.50), which are directly connected through an ideal optical field. That is, when the output y of the first OPA becomes the control input u to the second OPA as depicted in Figure 6.1, we obtain

$$dx(t) = \bar{\mathcal{A}}\, x(t)\, dt + \bar{\mathcal{B}}\, dw(t) \tag{6.53}$$

where

$$\bar{\mathcal{A}} = \begin{bmatrix} -\alpha_1 - \kappa_1 & -\beta_1 + \Delta_1 & 0 & 0 \\ -\beta_1 - \Delta_1 & \alpha_1 - \kappa_1 & 0 & 0 \\ -2\sqrt{\kappa_1 \kappa_2} & 0 & -\alpha_2 - \kappa_2 & -\beta_2 + \Delta_2 \\ 0 & -2\sqrt{\kappa_1 \kappa_2} & -\beta_2 - \Delta_2 & \alpha_2 - \kappa_2 \end{bmatrix}; \quad \bar{\mathcal{B}} = \begin{bmatrix} \sqrt{\kappa_1} & 0 \\ 0 & \sqrt{\kappa_1} \\ \sqrt{\kappa_2} & 0 \\ 0 & \sqrt{\kappa_2} \end{bmatrix}. \tag{6.54}$$

Definition 6.3. (*Entanglement criterion*) A quantum system

$$dx(t) = A\, x(t)\, dt + B\, dw(t) \tag{6.55}$$

is said to be entangled if there exists a complex vector $d_c = d_r + i d_i$ such that

$$\mathcal{N} := d_c^{\dagger}\, \mathcal{M}\, d_c < 0 \tag{6.56}$$

where

$$\mathcal{M} := V + \frac{i}{2}\Omega; \quad \mathcal{M}^{\dagger} = \mathcal{M}; \quad \Omega := \begin{bmatrix} J & 0 \\ 0 & -J \end{bmatrix} \tag{6.57}$$

and V is the solution to a Lyapunov equation

$$AV + VA^T + BB^T = 0. \tag{6.58}$$

Remark 6.6. Since all eigenvalues of the Hermitian matrix $\mathcal{M}$ in (6.57) are real numbers, the entanglement condition (6.56) implies that at least one of the eigenvalues of $\mathcal{M}$ has to be a negative real number. In this case, the complex vector d_c can be taken as the corresponding eigenvector of $\mathcal{M}$.

Since the system variable vector x in (6.53) is Gaussian, the entanglement of the cascaded OPAs (6.53) can be measured in terms of logarithmic negativity (e.g., see [233, 247–249]):

$$\mathcal{E} := \max\{0, \, -\ln(2\nu)\} \tag{6.59}$$

where

$$V := \begin{bmatrix} V_1 & V_2 \\ V_2^T & V_3 \end{bmatrix}; \quad \nu := \frac{1}{\sqrt{2}}\sqrt{\Psi - \sqrt{\Psi^2 - 4\det(V)}}\,; \tag{6.60}$$

$$\Psi := \det(V_1) + \det(V_3) - 2\det(V_2). \tag{6.61}$$

From (6.59), we infer that if the quantum system (6.53) is entangled, then $\mathcal{E} > 0$. Otherwise, $\mathcal{E} = 0$. The logarithmic negativity (6.59) is defined as a function of the covariance matrix V, which can be obtained as a solution to the following Lyapunov equation (e.g., see [233])

$$\bar{\mathcal{A}}V + V\bar{\mathcal{A}}^T + \bar{\mathcal{B}}\bar{\mathcal{B}}^T = 0. \tag{6.62}$$

Thus, for the cascaded OPAs (6.53), we need to choose suitable parameter values such that $\bar{\mathcal{A}}$ is Hurwitz, $\bar{\mathcal{B}}\bar{\mathcal{B}}^T \geq 0$ and therefore, the Lyapunov equation (6.62) has a unique solution $V > 0$ as stated in the following lemma.

Lemma 6.4. *(see [315, Lemma 12.1]) If $Q \geq 0$ and A is stable, the linear equation*

$$A^T P + PA + Q = 0 \tag{6.63}$$

has a unique solution $P \geq 0$.

With the parameter values in (6.52), the quantum system (6.53) directly connected through the optical field is entangled with

$$\mathcal{N} = -0.0892\,; \quad \mathcal{E} = 0.2256 \tag{6.64}$$

and a corresponding complex vector d_c in (6.56) is

$$\begin{bmatrix} -0.3084 + i0.4196 & 0.5410 - i0.2104 & 0.0827 - i0.2942 & -0.5463 \end{bmatrix}^T. \tag{6.65}$$

6.5.2 Controlled entanglement

It is possible to increase the quantum entanglement level given in (6.64) by applying a coherent quantum controller (6.22) such that the controlled quantum system has a configuration as shown in Figure 6.2. The problem of designing this quantum controller can be considered as an optimization problem and then solved using the DE algorithm described in Chapter 2. Thus, based on Definition 6.3, the objective function to be minimized is taken as

$$\mathsf{f}(\mathcal{K}) = \varrho(\mathcal{M})^{\mathsf{n}} \tag{6.66}$$

where $\varrho(\mathcal{M})$ is the smallest eigenvalue of the matrix $\mathcal{M}$ as in (6.57); and $\mathsf{n} \geq 1$ is a power constant. The constraints related to $\mathsf{f}(\mathcal{K})$ are (6.38), (6.35) and

$$\begin{aligned}
\mathsf{g}_2(\mathcal{K}) &= \mathcal{A}P + P\mathcal{A}^T + \mathcal{B}\mathcal{B}^T = 0;\\
\mathsf{h}_2(\mathcal{K}) &= -P < 0;\\
\mathsf{g}_3(\mathcal{K}) &= \Re(\Sigma_K) = 0;\\
\mathsf{g}_4(\mathcal{K}) &= \Sigma_K - \Sigma_K^\dagger = 0
\end{aligned} \tag{6.67}$$

where the matrices $\mathcal{A}$ and $\mathcal{B}$ are of the form (6.30); and

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}; \quad V = P_{11}. \tag{6.68}$$

Note that $\Re(\Sigma_K) = \Phi_K$ is the real part of matrix Σ_K as in Theorem 6.1.

Considering all constraints involved, we form a fitness test procedure to rate the fitness of each candidate solution $\mathcal{K}$ as follows:

1. Evaluate the stability of the controlled quantum system $\mathcal{A}$ by referring to $\mathsf{h}_1(\mathcal{K})$ in (6.35);

2. Compute the solution Σ_K to the Riccati equation represented by $\mathsf{g}_1(\mathcal{K})$ in (6.38);

3. Evaluate $\mathsf{g}_3(\mathcal{K})$ and $\mathsf{g}_4(\mathcal{K})$ in (6.67) to check if the solution Σ_K is of imaginary Hermitian;

4. Compute the solution P of $\mathsf{g}_2(\mathcal{K})$ in (6.67) and ensure that $\mathsf{h}_2(\mathcal{K})$ in (6.67) is satisfied;

5. Calculate the value of the objective function $\mathsf{f}(\mathcal{K})$ in (6.66).

Along with this fitness test routine, we also form a set of penalty functions, which correspond to the violation of each constraint in (6.38), (6.35) and (6.67). That is,

$$\begin{aligned} \mathsf{p}_1(\mathcal{K}) &= e_{\max,r}(A_K)^{\mathsf{s}_1}; & \mathsf{p}_2(\mathcal{K}) &= |\log(\det(\mathcal{X}))|^{\mathsf{s}_2}; \\ \mathsf{p}_3(\mathcal{K}) &= \ell_{\max}\left(\Re(\Sigma_K)\right)^{\mathsf{s}_3}; & \mathsf{p}_4(\mathcal{K}) &= \ell_{\max}\left(\Sigma_K - \Sigma_K^\dagger\right)^{\mathsf{s}_4}; \\ \mathsf{p}_5(\mathcal{K}) &= e_{\max}\left(\mathcal{B}\mathcal{B}^T\right)^{\mathsf{s}_5}; & \mathsf{p}_6(\mathcal{K}) &= |e_{\min}(P)|^{\mathsf{s}_6}; \\ \mathsf{p}_7(\mathcal{K}) &= \mathsf{f}(\mathcal{K}) \end{aligned} \tag{6.69}$$

where $\mathsf{s}_\mathsf{r} \geq 1$ for $\mathsf{r} = 1, 2, \ldots, 6$. Here, $e_{\max,r}(M)$ denotes the largest real part of the eigenvalues of the matrix M; $e_{\max}(M)$ denotes the largest eigenvalue of a symmetric matrix M; and $\ell_{\max}(M)$ denotes the largest magnitude of all elements of the matrix M. Moreover, we are particularly concerned with the penalty function $\mathsf{p}_2(\mathcal{K})$ in (6.69), which is applied whenever $\mathcal{X}$ is very close to singularity, that is, $0 < |\det(\mathcal{X})| \leq \varepsilon < 1$. In this case, ε is a sufficiently small real positive number and $\mathcal{X}$ is as defined in Lemma 6.2.

We now can apply the DE algorithm solve the quantum entanglement problem for the quantum system (6.50), (6.51), (6.52). The quantum controller matrices are then obtained as follows:

$$\begin{aligned} A_K &= \begin{bmatrix} -907.6167 & -344.0940 & -86.2280 & -97.4048 \\ 236.9691 & -963.4073 & -377.8802 & -78.7616 \\ -145.2300 & 219.1872 & -904.0325 & 392.0398 \\ 10.1165 & -33.0976 & -517.4231 & -987.2990 \end{bmatrix}; \\ B_{K_1} &= \begin{bmatrix} -14.3498 & -81.5469 \\ -4.7526 & 28.2512 \\ -5.7654 & 0.6262 \\ -66.3267 & -12.5105 \end{bmatrix}; \quad B_{K_3} = \begin{bmatrix} 29.3006 & -60.5369 \\ -33.2270 & 28.8097 \\ 18.6475 & 4.8260 \\ -28.5866 & -54.1088 \end{bmatrix}; \\ C_K &= \begin{bmatrix} -3.3050 & -1.9170 & 3.1872 & 17.3733 \\ 10.2118 & -10.3368 & -4.5259 & -1.0752 \end{bmatrix}; \end{aligned} \tag{6.70}$$

$$B_{K_2} = \begin{bmatrix} -12.0745 & -5.7322 & -26.4841 & -15.6115 \\ -6.7239 & -3.2510 & -14.7481 & -8.7313 \\ -13.5743 & -5.2675 & -1.4916 & -3.2254 \\ -7.1492 & -2.9682 & -8.0450 & -5.2909 \end{bmatrix};$$

$$\Theta_K = \begin{bmatrix} 0 & -0.6382 & -0.6076 & -4.6528 \\ 0.6382 & 0 & -0.0264 & 1.7524 \\ 0.6076 & 0.0264 & 0 & 0.1545 \\ 4.6528 & -1.7524 & -0.1545 & 0 \end{bmatrix}. \tag{6.71}$$

Alternatively, if B_{K2} is assumed to be zero, the quantum controller matrices are obtained as:

$$A_K = \begin{bmatrix} -997.6993 & -212.1764 & -38.0295 & -89.1340 \\ 24.6918 & -903.4779 & -181.4780 & -129.7399 \\ -55.2244 & 23.8663 & -925.4836 & -100.2914 \\ -25.7120 & 40.2243 & 47.9460 & -888.8602 \end{bmatrix};$$

$$B_{K_1} = \begin{bmatrix} -3.1086 & 10.7631 \\ 13.9274 & -8.6766 \\ 13.3195 & -3.1036 \\ -9.8868 & 1.3991 \end{bmatrix}; \quad B_{K_3} = \begin{bmatrix} 6.7045 & -7.0180 \\ -14.1398 & -3.9128 \\ -7.7089 & -7.0357 \\ 7.3145 & 6.3358 \end{bmatrix};$$

$$C_K = \begin{bmatrix} -48.8473 & -43.8428 & -59.7859 & -28.7381 \\ -101.7984 & 3.2332 & -83.3746 & -75.7597 \end{bmatrix};$$

$$\Theta_K = \begin{bmatrix} 0 & -0.1264 & -0.1381 & 0.1055 \\ 0.1264 & 0 & 0.0746 & -0.0681 \\ 0.1381 & -0.0746 & 0 & -0.0129 \\ -0.1055 & 0.0681 & 0.0129 & 0 \end{bmatrix}. \tag{6.72}$$

The assumption that $B_{K_2} = 0$ is based on the experience of [72] where B_{K_2} has very small entries. Therefore, it has an insignificant effect on the performance of the controlled quantum system.

In our example, we also find that the entanglement levels of the controlled quantum systems obtained by applying controllers (6.70), (6.71) and (6.72) are the same. That is, in both cases, we obtain

$$\mathcal{N} = -0.1237\,; \quad \mathcal{E} = 0.2944 \tag{6.73}$$

and the corresponding complex vectors d_c are

$$\begin{bmatrix}-0.0669+i0.4613 & -0.4932+i0.1997 & -0.0010+i0.4652 & 0.5322\end{bmatrix}^T;$$
$$\begin{bmatrix}-0.0694+i0.4609 & -0.4921+i0.2023 & -0.0017+i0.4655 & 0.5319\end{bmatrix}^T \quad (6.74)$$

for the quantum controllers (6.70), (6.71) and (6.72), respectively. In particular, from (6.70) and (6.71), we notice that B_{K_2} is not necessarily small or equal to zero. However, this example does not show that any improvement in the objective function can be obtained by using a nonzero B_{K_2}. This is consistent with the example presented in [72].

Another interesting aspect of this example is that we obtain approximately 30% entanglement improvement in terms of logarithmic negativity through the application of a coherent quantum controller as opposed to the direct interconnection of the two OPAs. This result is not surprising because it is hard to drastically enhance the entanglement level of this type of quantum network as reported in [233]. However, our method has shown a potential approach to improve the entanglement level of a realistic quantum network using a dynamic coherent quantum controller.

Remark 6.7. Besides the objective function in (6.66), another possible objective function to be minimized is

$$\mathsf{f}(\mathcal{K}) = (\ln(2\nu))^{\mathsf{n}} \tag{6.75}$$

which is subject to (6.38), (6.35), (6.67) and

$$\mathsf{h}_3(\mathcal{K}) = e_{\min}(\mathcal{M}) < 0 \tag{6.76}$$

where $e_{\min}(\mathcal{M})$ denotes the smallest eigenvalue of the Hermitian matrix $\mathcal{M}$ as defined in (6.57). Having an additional constraint $\mathsf{h}_3(\mathcal{K})$, we thus need to include an additional step in the fitness test routine to ensure that the entanglement criterion (6.56) holds before calculating the value of the objective function (6.75). The penalty function corresponding to the violation of (6.76) is

$$\mathsf{p}_7(\mathcal{K}) = e_{\min}(\mathcal{M})^{\mathsf{s}_7} \tag{6.77}$$

where $\mathsf{s}_7 \geq 1$. Thus, if there is no constraint violation by a candidate solution throughout the fitness test, the objective function (6.75) will be

$$\mathsf{p}_8(\mathcal{K}) = \mathsf{f}(\mathcal{K}). \tag{6.78}$$

Applying this algorithm to solve the same entanglement control problem as the one discussed above, we obtain approximately the same amount of entanglement level (with or without B_{K_2}), that is, $\mathcal{E} = 0.2954$. This fact implies that both algorithms have a comparable performance.

6.6 Conclusions

We have presented a new method to solve a linear coherent quantum control problem based on a DE approach. The solution to this problem involves the solution to a complex algebraic Riccati equation. As a case study, we consider a quantum entanglement control problem for two cascaded OPAs. Applying a suitable coherent quantum controller, we show that the entanglement level can be increased. This result indicates that our method can have potential future applications to realistic quantum networks. Interestingly, with or without the B_{K_2} term in the quantum controller, we obtain the same amount of entanglement in terms of logarithmic negativity. This fact motivates a further investigation on the significance of the inclusion of B_{K_2} in the realization of a dynamic coherent quantum controller.

Chapter 7

Coherent Quantum Robust H^∞ Control via A Strict Bounded Real Quantum Controller

7.1 Introduction

In this chapter, we consider coherent quantum robust H^∞ control for a class of linear complex quantum stochastic systems with norm-bounded structured uncertainties. The corresponding quantum H^∞ control objective is to achieve a strict bounded real closed loop uncertain quantum system with a specified disturbance attenuation level. It is possible to solve this quantum control problem based on the quantum H^∞ control methods presented in [70, 73] by lumping all structured uncertainties into a single unstructured uncertainty. However, this may lead to a conservative quantum H^∞ controller and these methods do not necessarily lead to a stable and strict bounded real quantum controller. Hence, the resulting quantum H^∞ controller may not be physically realizable.

This concern has motivated us to propose a systematic method to construct a stable and strict bounded real coherent quantum H^∞ controller, which is guaranteed to be physically realizable. The underlying main idea of our approach is to introduce an additional uncertainty to form an artificial uncertain quantum system, based on which our quantum controller is to be designed. A similar idea has also been applied in [8] for the classical robust H^∞ control problem. The additional uncertainty has specific properties such that any suitable coherent quantum H^∞ controller solves the original quantum control problem and is also stable and strict bounded real. The resulting quantum controller then must be physically realizable and is of the same order as that of the quantum plant. However, the additional uncertainty also introduces some extra conservatism to this controller design method.

An algorithm to synthesize such a quantum robust H^∞ controller involves finding the stabilizing solutions to complex algebraic Riccati equations parameterized by a non-singular scaling matrix and scaling parameters; e.g., see [8]. The Riccati equations then constitute nonconvex nonlinear constraints, which are often difficult to satisfy, in our quantum control problem. Therefore, to compute a solution, we apply an evolutionary optimization method, namely the differential evolution (DE) algorithm, as presented in Chapter 2.

As both complex and operator matrices are involved in this chapter, we then use the following notation: $M = [m_{jk}]$, $M^* = [m_{jk}^*]$, $M^T = [m_{kj}]$ and $M^\dagger = [m_{kj}^*] = (M^*)^T$ to denote the same operations as those explained in Section 6.1.

7.2 Linear Complex Quantum Systems

In this section, we briefly recall some preliminary results on linear complex quantum systems. Instead of the quadrature representation of a linear quantum system presented in Chapter 6, we are concerned with a linear complex quantum stochastic system modeled in terms of annihilation operators as presented in [73]. This formulation is described using quantum probability theory to characterize the non-commutative nature of the quantum system; e.g., see [46]. Moreover, the physical realizability condition for these quantum systems is presented in relation to the realization of a complex open quantum harmonic oscillator.

7.2.1 Non-commutative model

We now consider a particular class of linear quantum stochastic systems described in terms of linear quantum stochastic differential equations (QSDEs) as follows:

$$\begin{aligned} da(t) &= F\, a(t)\, dt + G\, dw(t); \quad a(0) = a_0; \\ dy(t) &= H\, a(t)\, dt + J\, dw(t) \end{aligned} \tag{7.1}$$

where the matrices F, G, H and J are respectively complex matrices in $\mathbf{C}^{n\times n}$, $\mathbf{C}^{n\times n_w}$, $\mathbf{C}^{n_y\times n}$ and $\mathbf{C}^{n_y\times n_w}$. Moreover, $a(t) := \begin{bmatrix} a_1(t) & \dots & a_n(t) \end{bmatrix}^T$ is an $n \times 1$ vector of annihilation operators; and $w(t)$ is an $n_w \times 1$ vector of input signals. Here, we assume that n_w is equal to n_y. As described in Section 6.2, we also assume that $w(t)$ can be decomposed into two components as follows:

$$dw(t) = \beta_w(t)\,dt + d\nu(t) \tag{7.2}$$

where $\beta_w(t)$ and $\nu(t)$ are respectively adapted process and quantum noise parts of $w(t)$. As the quantum system (7.1) may interact with other systems, the variables of those systems determine the adapted process $\beta_w(t)$. Thus, we assume that $\beta_w(t)$ commutes with $a(t)$ and $\nu(t)$ for all $t \geq 0$ because they are operating on distinct Hilbert spaces; e.g., see [70, 72, 73]. The notion of an adapted process is rigorously described in [43, 44, 46].

Moreover, the quantum noise $\nu(t)$ is a vector of operators on a Fock space, which is characterized by the Ito table

$$d\nu(t)d\nu(t)^\dagger = F_\nu\,dt \tag{7.3}$$

where the Ito matrix $F_{\tilde{w}}$ is a non-negative Hermitian matrix; e.g., see [44, 49]. This leads to a commutation relation of the form

$$d\nu(t)d\nu(t)^\dagger - \left(d\nu(t)^* d\nu(t)^T\right)^T = T_\nu\,dt \tag{7.4}$$

where T_ν is a Hermitian commutation matrix.

Since the dynamic equation of the quantum system (7.1) is expressed as a linear quantum stochastic differential equation, integration with respect to $dw(t)$ is a quantum stochastic integral. Therefore, the evolution of $a(t)$ in (7.1) depends only on $w(s)$ for $0 \leq s \leq t$ and is then adapted. Moreover, the annihilation operator $a(t)$ also commutes with the Ito increment $d\nu(t)$; see [70].

7.2.2 Physical realizability

For a quantum system (7.1), the realization $\{F, G, H, J\}$ cannot be arbitrary because it may not represent meaningful dynamics governed by quantum mechanical principles. In order to be physically realizable, the quantum system (7.1) has to satisfy the following commutation relation

$$a(t)a(t)^\dagger - \left(a(t)^* a(t)^T\right)^T = \Theta \tag{7.5}$$

for all $t \geq 0$, where Θ is a complex commutation matrix; see [70, 73]. Referring to [73, Theorem 4.1], the commutation relation (7.5) is equivalent to

$$F\Theta + \Theta F^\dagger + GT_\nu G^\dagger = 0. \tag{7.6}$$

This condition is satisfied by a complex open quantum harmonic oscillator evolving unitarily; e.g., see [36, 46, 70, 73]. Thus, the physical realizability of the quantum system (7.1) is then defined in relation to the realization of a complex open quantum harmonic oscillator.

If Θ is canonical, we have that $\Theta = I$, $F_\nu = I$ and $T_\nu = I$. In this case, the commutation relation (7.5) holds if and only if

$$F + F^\dagger + GG^\dagger = 0. \tag{7.7}$$

However, if Θ is generalized canonical, we have that Θ is a positive definite Hermitian matrix, $F_\nu = I$ and $T_\nu = I$. In this case, the commutation relation (7.5) holds if and only if

$$F\Theta + \Theta F^\dagger + GG^\dagger = 0. \tag{7.8}$$

For this latter case, we are always be able to find a similarity transformation $\bar{a} = Sa$ such that the transformed quantum system

$$\bar{F} = SFS^{-1}; \quad \bar{G} = SG; \quad \bar{H} = HS^{-1}; \quad \bar{J} = J \tag{7.9}$$

satisfies (7.7) with $\bar{\Theta} = I$, $F_\nu = I$ and $T_\nu = I$; see [73].

A complex open quantum harmonic oscillator is characterized by a Hamiltonian operator $\mathcal{H}$ and a coupling operator $\mathcal{C}$, which are respectively defined as follows:

$$\mathcal{H} := a^\dagger \mathcal{M} a; \quad \mathcal{C} := \Lambda a. \tag{7.10}$$

Here, $\mathcal{M}$ is an $n \times n$ complex Hermitian matrix; and Λ is an $n_w \times n$ complex coupling matrix. Using the quantum Langevin equation and Lindblad generator, the dynamical equations of a complex open quantum harmonic oscillator can be written as follows (e.g., see [73, 316, 317]):

$$
\begin{aligned}
da(t) &= -\Theta\left(i\mathcal{M} + \tfrac{1}{2}\Lambda^\dagger\Lambda\right) a(t)\,dt - \Theta\Lambda^\dagger\,dw(t);\\
dy(t) &= \Lambda\,a(t)\,dt + dw(t).
\end{aligned}
\tag{7.11}
$$

Therefore, matching (7.1) with (7.11), we obtain

$$
F = -\Theta\left(i\mathcal{M} + \tfrac{1}{2}\Lambda^\dagger\Lambda\right); \quad G = -\Theta\Lambda^\dagger; \quad H = \Lambda; \quad J = I. \tag{7.12}
$$

Definition 7.1. (*Physical realizability*; see [73, Definition 5.1 and Definition 5.2]) A linear complex quantum system (7.1) is said to be physically realizable if it satisfies the commutation relation (7.5) for (generalized) canonical Θ and represents the dynamics of a complex open quantum harmonic oscillator (7.11).

Lemma 7.1. *(see [73, Theorem 5.1]) A linear complex quantum system* (7.1) *is physically realizable if and only if there exists* $\Theta = \Theta^\dagger > 0$ *such that* (7.8) *and* (7.12) *hold.*

7.2.3 Bounded real property

We also need to consider the bounded real property of the linear complex quantum system (7.1). This will enable us to relate the physical realizability condition given in Definition 7.1 and Lemma 7.1 to the realization of a coherent quantum H^∞ controller discussed in subsequent sections.

Definition 7.2. (*Dissipativity*; see [70, Definition 4.1] and [73, Definition 6.1]) Given an operator-valued quadratic form

$$
\mathfrak{r}(a, \beta_w) = \frac{1}{2}\begin{bmatrix} a^\dagger & \beta_w^\dagger \end{bmatrix}\mathcal{R}\begin{bmatrix} a \\ \beta_w \end{bmatrix} + \frac{1}{2}\begin{bmatrix} a^T & \beta_w^T \end{bmatrix}\mathcal{R}\begin{bmatrix} a^* \\ \beta_w^* \end{bmatrix} \tag{7.13}
$$

where

$$
\mathcal{R} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \tag{7.14}
$$

is a given complex Hermitian matrix, we say that the quantum system (7.1) is dissipative with supply rate $\mathfrak{r}(a, \beta_w)$ if there exists a positive operator-valued quadratic form

$$\mathcal{V}(a) = \tfrac{1}{2}a^\dagger \mathcal{X} a + \tfrac{1}{2}a^T \mathcal{X} a^* \tag{7.15}$$

(where $\mathcal{X}$ is a positive definite Hermitian matrix) and a constant $\lambda > 0$ such that

$$\langle \mathcal{V}\left(a\left(t\right)\right)\rangle + \int_0^t \langle \mathfrak{r}(a(s), \beta_w(s))\rangle \, ds \leq \langle \mathcal{V}\left(a\left(0\right)\right)\rangle + \lambda t, \quad \forall t > 0 \tag{7.16}$$

for all Gaussian state $\mathfrak{g}$ for the initial variables $a(0)$. Note that we use $\langle\cdot\rangle$ to denote a quantum expectation over all initial variables and noises. We say that the quantum system (7.1) is strictly dissipative if there exists a constant $\varepsilon > 0$ such that the inequality (7.15) holds with the matrix $\mathcal{R}$ replaced by the matrix $\mathcal{R} + \varepsilon I$.

Definition 7.3. (*Bounded real*; see [70, Definition 4.3] and [73, Definition 6.2]) The linear complex quantum system (7.1) is said to be bounded real with disturbance attenuation $\gamma > 0$ if the system (7.1) is dissipative with supply rate

$$\begin{aligned}
\mathfrak{r}(a, \beta_w) &= \frac{1}{2}\left(\beta_z^\dagger \beta_z - \gamma^2 \beta_w^\dagger \beta_w\right) + \frac{1}{2}\left(\beta_z^T \beta_z^* - \gamma^2 \beta_w^T \beta_w^*\right) \\
&= \frac{1}{2}\begin{bmatrix} a^\dagger & \beta_w^\dagger \end{bmatrix} \begin{bmatrix} H^\dagger H & H^\dagger J \\ J^\dagger H & J^\dagger J - \gamma^2 I \end{bmatrix} \begin{bmatrix} a \\ \beta_w \end{bmatrix} \\
&\quad + \frac{1}{2}\begin{bmatrix} a^T & \beta_w^T \end{bmatrix} \begin{bmatrix} H^T H^* & H^T J^* \\ J^T H^* & J^T J^* - \gamma^2 I \end{bmatrix} \begin{bmatrix} a^* \\ \beta_w^* \end{bmatrix}
\end{aligned} \tag{7.17}$$

where $\beta_z := H\,a(t) + J\,\beta_w(t)$. Moreover, The linear complex quantum system (7.1) is said to be strictly bounded real with disturbance attenuation $\gamma > 0$ if the system (7.1) is strictly dissipative with the supply rate (7.17).

Definition 7.4. (*Minimal realization*; see [73, Definition 6.4] and [318]) The quantum system (7.1) is said to be minimal if the following conditions hold:

1. Controllability: $a^\dagger F = \lambda\, a^\dagger$ for some $\lambda \in \mathbf{C}$ and $a^\dagger G = 0$ implies that $a = 0$.

2. Observability: $F\,a = \lambda\, a$ for some $\lambda \in \mathbf{C}$ and $H\,a = 0$ implies that $a = 0$.

Definition 7.5. (*Lossless bounded real*; see [73, Definition 6.3] and [319, Chapter 7]) The linear complex quantum system (7.1) is said to be lossless bounded real if the following conditions hold:

1. F is Hurwitz.

2. The transfer function matrix $Q(s) := H(sI - F)^{-1}G + J$ satisfies

$$Q(i\omega)^\dagger Q(i\omega) = I, \quad \forall\, \omega \in \mathbf{R}. \tag{7.18}$$

Lemma 7.2. *(see [73, Theorem 6.4 and Theorem 6.5] and [319, Theorem 6.4 and Theorem 6.5]) Suppose that the quantum system* (7.1) *has a minimal realization* $\{F, G, H, J\}$*. Then, it is lossless bounded real if and only if there exists* $X = X^\dagger > 0$ *such that*

$$\begin{aligned} XF + F^\dagger X + H^\dagger H &= 0; \\ H^\dagger J &= -XG; \\ J^\dagger J &= I. \end{aligned} \tag{7.19}$$

Furthermore, the minimal realization $\{F, G, H, J\}$ *of the quantum system* (7.1) *is physically realizable if and only if it is lossless bounded real with* $J = I$*.*

Lemma 7.3. *(Complex bounded real lemma I; see [73, Theorem 6.1]) The quantum system* (7.1) *is bounded real with disturbance attenuation* $\gamma > 0$ *if and only if there exists a Hermitian matrix* $X > 0$ *such that*

$$\begin{bmatrix} F^\dagger X + XF + H^\dagger H & G^\dagger X + J^\dagger H \\ XG + H^\dagger J & J^\dagger J - \gamma^2 I \end{bmatrix} \leq 0. \tag{7.20}$$

Lemma 7.4. *(Complex bounded real lemma II; see [73, Theorem 6.2] and [319, Chapter 7]) Suppose that the quantum system* (7.1) *has a minimal realization* $\{F, G, H, J\}$ *and satisfies* $\gamma^2 I - J^\dagger J > 0$ *for* $\gamma > 0$*. Then, the following statements are equivalent:*

1. *The quantum system* (7.1) *is bounded real with disturbance attenuation* γ*.*

2. F *is Hurwitz and* $\|H(sI - F)^{-1}G + J\|_\infty \leq \gamma$*.*

3. *The complex algebraic Riccati equation*

$$F^\dagger X + XF + H^\dagger H + \left(XG + H^\dagger J\right)\left(\gamma^2 I - J^\dagger J\right)^{-1}\left(G^\dagger X + J^\dagger H\right) = 0 \tag{7.21}$$

has a positive definite Hermitian solution X*.*

Lemma 7.5. *(Complex strict bounded real lemma I; see [73, Theorem 6.3]) The quantum system* (7.1) *is strictly bounded real with disturbance attenuation* $\gamma > 0$ *if and only if* $\gamma^2 I - J^\dagger J > 0$ *and there exists a Hermitian matrix* $X > 0$ *such that*

$$\begin{bmatrix} F^\dagger X + XF + H^\dagger H & G^\dagger X + J^\dagger H \\ XG + H^\dagger J & J^\dagger J - \gamma^2 I \end{bmatrix} < 0. \tag{7.22}$$

Lemma 7.6. *(Complex strict bounded real lemma II; see [73, Theorem 6.3], [312, Theorem 2.1] and [320]) For the quantum system* (7.1)*, the following statements are equivalent:*

1. *The quantum system* (7.1) *is strictly bounded real with disturbance attenuation* $\gamma > 0$.

2. F *is Hurwitz and* $\|H(sI - F)^{-1}G + J\|_\infty < \gamma$.

3. $\gamma^2 I - J^\dagger J > 0$ *and the complex algebraic Riccati inequality*

$$F^\dagger \tilde{X} + \tilde{X} F + H^\dagger H + \left(\tilde{X} G + H^\dagger J\right) \left(\gamma^2 I - J^\dagger J\right)^{-1} \left(G^\dagger \tilde{X} + J^\dagger H\right) < 0 \tag{7.23}$$

   has a positive definite Hermitian solution $\tilde{X}$.

4. $\gamma^2 I - J^\dagger J > 0$ *and the complex algebraic Riccati equation*

$$F^\dagger X + XF + H^\dagger H + \left(XG + H^\dagger J\right) \left(\gamma^2 I - J^\dagger J\right)^{-1} \left(G^\dagger X + J^\dagger H\right) = 0 \tag{7.24}$$

   has a stabilizing solution $X \geq 0$.

Furthermore, if these statements hold, then $X < \tilde{X}$.

7.3 Quantum Robust H^∞ Control Problem

In this section, we describe the quantum robust H^∞ control problem for a class of linear uncertain complex quantum systems. Applying a coherent quantum H^∞ controller to solve this control problem, we require this quantum controller to be physically realizable. Also, the resulting closed loop uncertain quantum system has to satisfy an H^∞ control objective. These issues are considered in the following sub-sections.

7.3.1 Linear uncertain complex quantum system

We consider a class of linear complex quantum stochastic systems with structured uncertainties (see [70, 72, 73]):

$$\begin{aligned}
da(t) &= F\, a(t)dt + G_0\, dv(t) + G_1\, dw(t) + G_2\, du(t) + \sum_{j=1}^{k} G_{3,j}\, d\xi_j(t);\ a(0) = a_0;\\
dz(t) &= H_1\, a(t)dt + J_{12}\, du(t);\\
d\zeta_1(t) &= L_1\, a(t)dt + M_1\, du(t);\\
&\vdots\\
d\zeta_k(t) &= L_k\, a(t)dt + M_k\, du(t);\\
dy(t) &= H_2\, a(t)dt + J_{20}\, dv(t) + J_{21}\, dw(t)
\end{aligned} \tag{7.25}$$

where a is an $n \times 1$ vector of the plant annihilation operators; v is an $n_v \times 1$ vector of quantum noises; w is an $n_w \times 1$ vector of disturbance inputs; u is an $n_u \times 1$ vector of control inputs; ξ_j is an $n_{q_j} \times 1$ vector of uncertainty inputs (for $j = 1, 2, \ldots, k$); ζ_j is an $n_{s_j} \times 1$ vector of uncertainty outputs (for $j = 1, 2, \ldots, k$); z is an $n_z \times 1$ vector of controlled outputs; and y is an $n_y \times 1$ vector of 'measurement' outputs. All coefficient matrices in (7.25) are complex matrices, which have compatible dimensions with those of the operators and signals in (7.25). Quantum systems of this form, defined only in terms of annihilation operators, can be used to represent interconnections of linear passive optical components such as optical cavities, beam-splitters and phase-shifters; e.g., see [73, 77].

The disturbance input $w(t)$ and the control input $u(t)$ in (7.25) are represented respectively as

$$dw(t) = \beta_w(t)\, dt + d\nu(t); \tag{7.26}$$

$$du(t) = \beta_u(t)\, dt + d\mu(t) \tag{7.27}$$

where $\beta_w(t)$ and $\beta_u(t)$ are adapted processes; and $d\nu(t)$ and $d\mu(t)$ are the noise parts of (7.26) and (7.27). Meanwhile, $dv(t)$ represents an additional quantum noise in the plant. The quantum noises $dv(t)$, $d\nu(t)$ and $d\mu(t)$ have corresponding Hermitian Ito matrices F_v, F_ν and F_μ, and Hermitian commutation matrices T_v,

T_ν and T_μ, which are assumed to be

$$F_v = F_\nu = F_\mu = I; \tag{7.28}$$

$$T_v = T_\nu = T_\mu = I. \tag{7.29}$$

The j-th structured uncertainty in (7.25) is modeled as an additional unknown linear time-invariant complex quantum stochastic system:

$$\begin{aligned} d\tilde{a}_j(t) &= A_j\, \tilde{a}_j(t)\, dt + B_j\, d\zeta_j(t); \quad \tilde{a}_j(0) = \tilde{a}_{0,j}; \\ d\xi_j(t) &= C_j\, \tilde{a}_j(t)\, dt + D_j\, d\zeta_j(t) \end{aligned} \tag{7.30}$$

with A_j Hurwitz and transfer function matrix

$$\Delta_j(s) = C_j(sI - A_j)^{-1}B_j + D_j \tag{7.31}$$

which is required to satisfy

$$\|\Delta_j(s)\|_\infty \leq 1 \tag{7.32}$$

for all $j = 1, 2, \ldots, k$.

7.3.2 Coherent quantum H^∞ controller

We aim to control the uncertain quantum system (7.25), (7.30), (7.31), (7.32) using a coherent dynamic quantum H^∞ controller, which is assumed to be a non-commutative quantum stochastic system. A general form of this quantum controller can be written as

$$\begin{aligned} dc(t) &= F_c\, c(t)dt + G_{c_0}\, dw_{c_0}(t) + G_{c_1}\, dw_{c_1}(t) + G_c\, dy(t); \quad c(0) = c_0 \\ du(t) &= H_c\, c(t)dt + dw_{c_0}(t) \end{aligned} \tag{7.33}$$

where c is an $n \times 1$ vector of the controller annihilation operators; w_{c_0} and w_{c_1} are respectively $n_{c_0} \times 1$ and $n_{c_1} \times 1$ vectors of non-commutative quantum Wiener processes. For the quantum H^∞ controller (7.33), we assume that the Ito matrices $F_{w_{c_0}}$ and $F_{w_{c_1}}$, and commutation matrices $T_{w_{c_0}}$ and $T_{w_{c_1}}$ of w_{c_0} and w_{c_1} are respectively

$$\begin{aligned} F_{w_{c0}} &= F_{w_{c1}} = I; \\ T_{w_{c0}} &= T_{w_{c1}} = I. \end{aligned} \tag{7.34}$$

Also, at time $t = 0$, it is assumed that $a(0)$ and $\tilde{a}(0)$ commute with $c(0)$. Moreover, the quantum H^∞ controller (7.33) is required to be stable and strict bounded real, which will imply that it is physically realizable. Therefore, referring to [73], we present a physical realizability condition for the quantum H^∞ controller (7.33) in terms of its bounded real property.

Definition 7.6. (*Physical realizability of a quantum controller*; see [73, Definition 7.1]) The realization $\{F_c, G_c, H_c\}$ is said to define a physically realizable quantum controller of the form (7.33) if there exist matrices G_{c_0}, G_{c_1}, H_{c_1} and H_{c_2} such that

$$\begin{aligned} dc(t) &= F_c\, c(t)dt + G_{c_0}\, dw_{c_0}(t) + G_{c_1}\, dw_{c_1}(t) + G_c\, dy(t); \quad c(0) = c_0; \\ \begin{bmatrix} du(t) \\ du_1(t) \\ du_2(t) \end{bmatrix} &= \begin{bmatrix} H_c \\ H_{c_1} \\ H_{c_2} \end{bmatrix} c(t)dt + \begin{bmatrix} dw_{c_0}(t) \\ dw_{c_1}(t) \\ dy(t) \end{bmatrix} \end{aligned} \tag{7.35}$$

is physically realizable according to Definition 7.1 when

$$T_y := J_{20} T_v J_{20}^\dagger + J_{21} T_\nu J_{21}^\dagger = I. \tag{7.36}$$

Lemma 7.7. *(see [73, Theorem 7.2]) Suppose that $\{F_c, G_c, H_c\}$ is a minimal realization of the quantum controller* (7.33). *Then, the quantum controller* (7.33) *is physically realizable if and only if F_c is Hurwitz and*

$$\|H_c(sI - F_c)^{-1}G_c\|_\infty \le 1. \tag{7.37}$$

This implies that the quantum controller (7.33) *is bounded real.*

Remark 7.1. (see [73, Theorem 7.2]) The matrices G_{c_1} and H_{c_1} can be set to zero as the exogenous quantum noise dw_{c_1} term is not needed in the realization of the quantum controller (7.33). Moreover, an immediate consequence of Lemma

7.7 is that a strict bounded real quantum H^∞ controller of the form (7.33) must always be physically realizable, where F_c is Hurwitz and $\|H_c(sI-F_c)^{-1}G_c\|_\infty < 1$.

7.3.3 H^∞ control objective

Interconnecting the quantum controller (7.33) with the uncertain quantum system (7.25), (7.32), we obtain the following closed loop uncertain quantum system

$$\begin{aligned}
\begin{bmatrix} da(t) \\ dc(t) \\ d\tilde{a}(t) \end{bmatrix} &= \begin{bmatrix} (F+G_3DL) & (G_2H_c+G_3DMH_c) & G_3C \\ G_cH_2 & F_c & 0 \\ BL & BMH_c & A \end{bmatrix} \begin{bmatrix} a(t) \\ c(t) \\ \tilde{a}(t) \end{bmatrix} dt \\
&\quad + \begin{bmatrix} G_0 & (G_2+G_3DM) & 0 \\ G_cJ_{20} & G_{c_0} & G_{c_1} \\ 0 & BM & 0 \end{bmatrix} \begin{bmatrix} dv(t) \\ dw_{c_0}(t) \\ dw_{c_1}(t) \end{bmatrix} + \begin{bmatrix} G_1 \\ G_cJ_{21} \\ 0 \end{bmatrix} dw(t); \\
dz(t) &= \begin{bmatrix} H_1 & J_{12}H_c & 0 \end{bmatrix} \begin{bmatrix} a(t) \\ c(t) \\ \tilde{a}(t) \end{bmatrix} dt + \begin{bmatrix} 0 & J_{12} & 0 \end{bmatrix} \begin{bmatrix} dv(t) \\ dw_{c_0}(t) \\ dw_{c_1}(t) \end{bmatrix}
\end{aligned} \tag{7.38}$$

where

$$\begin{aligned}
d\tilde{a}(t) &:= \begin{bmatrix} d\tilde{a}_1(t) \\ d\tilde{a}_2(t) \\ \vdots \\ d\tilde{a}_k(t) \end{bmatrix}; \quad M := \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_k \end{bmatrix}; \quad L := \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_k \end{bmatrix}; \\
A &:= \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}; \quad B := \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_k \end{bmatrix}; \\
C &:= \begin{bmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_k \end{bmatrix}; \quad D := \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_k \end{bmatrix}; \\
G_3 &:= \begin{bmatrix} G_{3,1} & G_{3,2} & \cdots & G_{3,k} \end{bmatrix}.
\end{aligned} \tag{7.39}$$

We require the closed loop uncertain quantum system (7.38) to satisfy the following H^∞ control objective

$$\begin{aligned}&\int_0^t \left\langle z(s)^\dagger z(s) + z(s)^T z(s)^* + \varepsilon\left(\eta(s)^\dagger \eta(s) + \eta(s)^T \eta(s)^*\right)\right\rangle ds \\ &\leq (\gamma^2 - \varepsilon^2)\int_0^t \left\langle \beta_w(s)^\dagger \beta_w(s) + \beta_w(s)^T \beta_w(s)^*\right\rangle ds + \pi_1 + \pi_2 t \end{aligned} \tag{7.40}$$

where $\varepsilon, \pi_1, \pi_2 > 0$ are real constants and

$$d\eta(t) := \begin{bmatrix} da(t)^T & dc(t)^T & d\tilde{a}(t)^T \end{bmatrix}^T. \tag{7.41}$$

This objective is attained if the closed loop quantum system (7.38) is strict bounded real with a specified disturbance attenuation level $\gamma > 0$; see [70, 73].

7.4 Quantum H^∞ Controller Synthesis

An algorithm to construct a coherent quantum H^∞ controller of the form (7.33) has been provided in [70, 73]. However, these algorithms do not guarantee that the resulting quantum controller is stable and strict bounded real, and hence, the controller may not be physically realizable. Thus, we are motivated to provide a new systematic method to synthesize a stable and strict bounded real quantum H^∞ controller based on the approach used in [8]. In this case, we force the quantum H^∞ controller to be physically realizable.

The main idea of our approach is to introduce an additional uncertainty to form an artificial uncertain quantum system based on which the desired coherent quantum H^∞ controller is to be designed. Thus, this approach only provides a sufficient condition such that any suitable quantum controller of the form (7.33) will lead to a strict bounded real closed loop uncertain quantum system with disturbance attenuation $\gamma > 0$ when applied to the original uncertain quantum system (7.25), (7.32). Moreover, the same quantum controller must be stable and strict bounded real when it is applied to a particular open loop uncertain quantum system while achieving the closed loop H^∞ control objective. These properties hold even when the quantum controller is detached from the open loop quantum system; see [8].

In order to apply this idea, we first consider the following uncertain quantum system:

$$
\begin{aligned}
da(t) &= F\,a(t)\,dt + G_0\,dv(t) + G_1\,dw(t) + \sum_{j=1}^{k} G_{2,j}\,d\xi_j(t); \quad a(0) = a_0;\\
dz(t) &= H_1\,a(t)\,dt + J_1\,dw(t) + \sum_{j=1}^{k} J_{2,j}\,d\xi_j(t);\\
d\zeta_1(t) &= H_{2,1}\,a(t)\,dt + K_1\,dw(t) + \sum_{j=1}^{k} L_{1,j}\,d\xi_j(t);\\
&\vdots\\
d\zeta_k(t) &= H_{2,k}\,a(t)\,dt + K_k\,dw(t) + \sum_{j=1}^{k} L_{1,j}\,d\xi_j(t).
\end{aligned}
\tag{7.42}
$$

Here, the j-th structured uncertainty in (7.42) is modeled as an unknown quantum system:

$$
\begin{aligned}
d\bar{a}_j(t) &= \tilde{A}_j\,\bar{a}_j(t)\,dt + \tilde{B}_j\,d\zeta_j(t); \quad \bar{a}_j(0) = \bar{a}_{0,j};\\
d\xi_j(t) &= \tilde{C}_j\,\bar{a}_j(t)\,dt + \tilde{D}_j\,d\zeta_j(t)
\end{aligned}
\tag{7.43}
$$

with $\tilde{A}_j$ Hurwitz and transfer function matrix

$$
\tilde{\Delta}_j(s) = \tilde{C}_j(sI - \tilde{A}_j)^{-1}\tilde{B}_j + \tilde{D}_j \tag{7.44}
$$

which is required to satisfy

$$
\|\tilde{\Delta}_j(s)\|_\infty \leq 1 \tag{7.45}
$$

for all $j = 1, 2, \ldots, k$. Now, we present the following lemma, which is required in subsequent sub-sections.

Lemma 7.8. *Consider the uncertain quantum system* (7.42), (7.43), (7.44), (7.45) *and let* $\tau_1 > 0, \ldots, \tau_k > 0$ *be given constants. Suppose that* F *in* (7.42) *is Hurwitz and the scaled quantum system*

$$
\begin{aligned}
da(t) &= F\,a(t)\,dt + G_0\,dv(t) + \breve{G}_1\,d\breve{w}(t);\\
d\breve{z}(t) &= \breve{H}\,a(t)\,dt + \breve{J}\,d\breve{w}(t)
\end{aligned}
\tag{7.46}
$$

where

$$\breve{G}_1 = \left[\gamma^{-1}\, G_1 \quad \sqrt{\tau_1}^{-1} G_{2,1} \quad \sqrt{\tau_2}^{-1} G_{2,2} \quad \ldots \quad \sqrt{\tau_k}^{-1} G_{2,k}\right];$$

$$\breve{H} = \begin{bmatrix} H_1 \\ \sqrt{\tau_1}\, H_{2,1} \\ \sqrt{\tau_2}\, H_{2,2} \\ \vdots \\ \sqrt{\tau_k}\, H_{2,k} \end{bmatrix}; \quad d\breve{w}(t) = \begin{bmatrix} \gamma\, dw(t) \\ \sqrt{\tau_1}\, d\xi_1(t) \\ \sqrt{\tau_2}\, d\xi_2(t) \\ \vdots \\ \sqrt{\tau_k}\, d\xi_k(t) \end{bmatrix}; \quad d\breve{z}(t) = \begin{bmatrix} dz(t) \\ \sqrt{\tau_1}\, d\zeta_1(t) \\ \sqrt{\tau_2}\, d\zeta_2(t) \\ \vdots \\ \sqrt{\tau_k}\, d\zeta_k(t) \end{bmatrix};$$

$$\breve{J} = \begin{bmatrix} \gamma^{-1}\, J_1 & \sqrt{\tau_1}^{-1} J_{2,1} & \sqrt{\tau_2}^{-1} J_{2,2} & \ldots & \sqrt{\tau_k}^{-1} J_{2,k} \\ \gamma^{-1}\, \sqrt{\tau_1} K_1 & L_{1,1} & \sqrt{\frac{\tau_1}{\tau_2}} L_{1,2} & \ldots & \sqrt{\frac{\tau_1}{\tau_k}} L_{1,k} \\ \gamma^{-1}\, \sqrt{\tau_2} K_2 & \sqrt{\frac{\tau_2}{\tau_1}} L_{2,1} & L_{2,2} & \ldots & \sqrt{\frac{\tau_2}{\tau_k}} L_{2,k} \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma^{-1}\, \sqrt{\tau_k} K_k & \sqrt{\frac{\tau_k}{\tau_1}} L_{k,1} & \sqrt{\frac{\tau_k}{\tau_2}} L_{k,2} & \ldots & L_{k,k} \end{bmatrix} \tag{7.47}$$

is such that $\breve{J}\breve{J}^\dagger < I$ *and strict bounded real with*

$$\left\| \breve{H}(sI - F)^{-1} \breve{G}_1 + \breve{J} \right\|_\infty < 1. \tag{7.48}$$

Then, the uncertain quantum system (7.42), (7.43), (7.44), (7.45) *is strict bounded real with disturbance attenuation* $\gamma > 0$.

Proof. The proof of this lemma follows from the proofs of Proposition 8.6 ((i) to (iii)) and Proposition 9.9 ((c) to (a)) in [321], and also from [73, Theorem 7.2] (Lemma 7.6). That is, suppose that all conditions in the lemma are satisfied and

$$\left\| \breve{H}(sI - F)^{-1} \breve{G}_1 + \breve{J} \right\|_\infty = \left\| \bar{\mathcal{T}} \left[\bar{H}(sI - F)^{-1} \bar{G}_1 + \bar{J} \right] \bar{\mathcal{T}}^{-1} \right\|_\infty < 1 \tag{7.49}$$

where

$$\bar{\mathcal{T}} := \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & \sqrt{\tau_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\tau_k} \end{bmatrix}; \quad \bar{H} := \begin{bmatrix} H_1 \\ H_{2,1} \\ \vdots \\ H_{2,k} \end{bmatrix}; \quad \bar{J} := \begin{bmatrix} \gamma^{-1} J_1 & J_{2,1} & \cdots & J_{2,k} \\ \gamma^{-1} K_1 & L_{1,1} & \cdots & L_{1,k} \\ \vdots & \vdots & & \vdots \\ \gamma^{-1} K_k & L_{k,1} & \cdots & K_{k,k} \end{bmatrix};$$

$$\bar{G}_1 := \left[\gamma^{-1} G_1 \quad G_{2,1} \quad \cdots \quad G_{2,k}\right]. \tag{7.50}$$

We also let $\bar{\mathcal{P}} : L_2[0,\infty) \to L_2[0,\infty)$ be the time-invariant bounded operator corresponding to the transfer function matrix $\bar{H}(sI-F)^{-1}\bar{G}_1 + \bar{J}$. That is, $\bar{\mathcal{P}}$ represents a classical linear time-invariant system

$$\begin{aligned}\dot{x}(t) &= Fx(t) + \bar{G}_1\bar{w}(t);\\ \bar{z}(t) &= \bar{H}x(t) + \bar{J}\bar{w}(t)\end{aligned} \tag{7.51}$$

where

$$\bar{w}(t) = \begin{bmatrix}\gamma\, w(t)\\ \xi_1(t)\\ \xi_2(t)\\ \vdots\\ \xi_k(t)\end{bmatrix}; \bar{z}(t) = \begin{bmatrix}z(t)\\ \zeta_1(t)\\ \zeta_2(t)\\ \vdots\\ \zeta_k(t)\end{bmatrix}; \bar{H} = \begin{bmatrix}H_1\\ H_{2,1}\\ H_{2,2}\\ \vdots\\ H_{2,k}\end{bmatrix}; \bar{J} = \begin{bmatrix}\gamma^{-1}J_1 & J_{2,1} & J_{2,2} & \dots & J_{2,k}\\ \gamma^{-1}K_1 & L_{1,1} & L_{1,2} & \dots & L_{1,k}\\ \gamma^{-1}K_2 & L_{2,1} & L_{2,2} & \dots & L_{2,k}\\ \vdots & \vdots & \vdots & & \vdots\\ \gamma^{-1}K_k & L_{k,1} & L_{k,2} & \dots & L_{k,k}\end{bmatrix}$$
$$\bar{G}_1 = \begin{bmatrix}\gamma^{-1}G_1 & G_{2,1} & G_{2,2} & \dots & G_{2,k}\end{bmatrix}. \tag{7.52}$$

For the system (7.51), the relationship between the uncertainty input $\xi_j(t)$ and output $\zeta_j(t)$ are represented by an unknown linear system as follows:

$$\begin{aligned}\dot{\mathfrak{p}}_j(t) &= \mathfrak{A}_j\mathfrak{p}_j(t) + \mathfrak{B}_j\zeta_j(t);\\ \xi_j(t) &= \mathfrak{C}_j\mathfrak{p}_j(t) + \mathfrak{D}_j\zeta_j(t)\end{aligned} \tag{7.53}$$

with $\mathfrak{A}_j$ Hurwitz and transfer function matrix

$$\mathfrak{U}_j(s) = \mathfrak{C}_j\left(sI - \mathfrak{A}_j\right)^{-1}\mathfrak{B}_j + \mathfrak{D}_j \tag{7.54}$$

which is required to satisfy

$$\|\mathfrak{U}_j(s)\|_\infty \leq 1 \tag{7.55}$$

for all $j = 1, 2, \dots, k$.

It then follows from (7.49) that

$$\left\|\bar{\mathcal{T}}\bar{\mathcal{P}}\bar{\mathcal{T}}^{-1}\phi(t)\right\|_2^2 \leq (1-\delta)\left\|\phi(t)\right\|_2^2 \tag{7.56}$$

for any $\phi(t) \in L_2[0,\infty)$ and some $\delta > 0$. Furthermore, we can rewite (7.56) as

$$\left\|\bar{\mathcal{T}}\bar{\mathcal{P}}\psi(t)\right\|_2^2 \le (1-\delta)\left\|\bar{\mathcal{T}}\psi(t)\right\|_2^2 \tag{7.57}$$

or equivalently

$$\left\|\bar{\mathcal{T}}\bar{\mathcal{P}}\psi(t)\right\|_2^2 - \left\|\bar{\mathcal{T}}\psi(t)\right\|_2^2 \le -\delta\left\|\bar{\mathcal{T}}\psi(t)\right\|_2^2 \tag{7.58}$$

where $\psi(t) := \bar{\mathcal{T}}^{-1}\phi(t)$, $\psi(t) \in L_2[0,\infty)$. From (7.58), we obtain that

$$\left\langle \psi(t), \left(\bar{\mathcal{P}}^\dagger\bar{\mathcal{T}}^\dagger\bar{\mathcal{T}}\bar{\mathcal{P}} - \bar{\mathcal{T}}^\dagger\bar{\mathcal{T}}\right)\psi(t)\right\rangle < 0 \tag{7.59}$$

where $\langle f, g\rangle$ denotes the inner product $\int_0^\infty f(t)^\dagger g(t)\,dt$ for $f(\cdot), g(\cdot) \in L_2[0,\infty)$; and thus,

$$\bar{\mathcal{P}}^\dagger\bar{\mathcal{T}}^\dagger\bar{\mathcal{T}}\bar{\mathcal{P}} - \bar{\mathcal{T}}^\dagger\bar{\mathcal{T}} < 0. \tag{7.60}$$

Now, we define

$$\bar{\mathcal{T}}^\dagger\bar{\mathcal{T}} := \begin{bmatrix} I & 0 \\ 0 & \mathcal{T}\end{bmatrix}; \quad \mathcal{T} := \begin{bmatrix} \tau_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_k \end{bmatrix} \tag{7.61}$$

and for some $\epsilon > 0$, (7.60) leads to

$$0 \ge \bar{\mathcal{P}}^\dagger \begin{bmatrix} I & 0 \\ 0 & \mathcal{T}\end{bmatrix} \bar{\mathcal{P}} - \begin{bmatrix} (1-\epsilon)I & 0 \\ 0 & \mathcal{T}\end{bmatrix} \tag{7.62}$$

which implies that

$$\begin{aligned}
0 \ge & \left\langle \begin{bmatrix} \gamma\, w(t) \\ \xi(t)\end{bmatrix}, \bar{\mathcal{P}}^\dagger \begin{bmatrix} I & 0 \\ 0 & \mathcal{T}\end{bmatrix} \bar{\mathcal{P}} \begin{bmatrix} \gamma\, w(t) \\ \xi(t)\end{bmatrix}\right\rangle \\
& - \left\langle \begin{bmatrix} \gamma\, w(t) \\ \xi(t)\end{bmatrix}, \begin{bmatrix} (1-\epsilon)I & 0 \\ 0 & \mathcal{T}\end{bmatrix}\begin{bmatrix} \gamma\, w(t) \\ \xi(t)\end{bmatrix}\right\rangle \\
= & \left\langle \begin{bmatrix} z(t) \\ \zeta(t)\end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & \mathcal{T}\end{bmatrix} \begin{bmatrix} z(t) \\ \zeta(t)\end{bmatrix}\right\rangle - \left\langle \begin{bmatrix} \gamma\, w(t) \\ \xi(t)\end{bmatrix}, \begin{bmatrix} (1-\epsilon)I & 0 \\ 0 & \mathcal{T}\end{bmatrix} \begin{bmatrix} \gamma\, w(t) \\ \xi(t)\end{bmatrix}\right\rangle \\
= & \|z(t)\|_2^2 + \langle \zeta(t), \mathcal{T}\zeta(t)\rangle - (1-\epsilon)\,\|\gamma\, w(t)\|_2^2 - \langle \xi(t), \mathcal{T}\xi(t)\rangle \\
= & \|z(t)\|_2^2 - \gamma^2(1-\epsilon)\,\|w(t)\|_2^2 + \sum_{j=1}^{k} \tau_j\left(\|\zeta(t)\|_2^2 - \|\xi(t)\|_2^2\right)
\end{aligned} \tag{7.63}$$

where

$$\xi(t) = \begin{bmatrix} \xi_1(t) \\ \vdots \\ \xi_k(t) \end{bmatrix}; \quad \zeta(t) = \begin{bmatrix} \zeta_1(t) \\ \vdots \\ \zeta_k(t) \end{bmatrix}. \tag{7.64}$$

Since (7.55) is satisfied, (7.63) results in

$$\begin{aligned} 0 &\geq \|z(t)\|_2^2 - \gamma^2(1-\epsilon)\|w(t)\|_2^2 \\ &> \|z(t)\|_2^2 - \gamma^2 \|w(t)\|_2^2 \end{aligned} \tag{7.65}$$

which implies that

$$\left\| H_1 (sI - F)^{-1} G_1 + J_1 \right\|_\infty < 1. \tag{7.66}$$

Using Lemma 7.6, we conclude that the uncertain quantum system (7.42), (7.43), (7.44), (7.45) is strict bounded real with disturbance attenuation $\gamma > 0$. □

7.4.1 Artificial uncertain quantum system

Prior to forming an artificial uncertain quantum system based on the original uncertain quantum system (7.25), (7.30), (7.31), (7.32), we need to construct a matrix K such that $(F + G_2K)$ is Hurwitz and the uncertain quantum system

$$\begin{aligned} da(t) &= (F + G_2K)\, a(t)\, dt + G_0\, dv(t) + G_1\, dw(t) + \sum_{j=1}^{k} G_{3,j}\, d\xi_j(t);\ a(0) = a_0; \\ dz(t) &= (H_1 + J_{12}K)\, a(t)\, dt; \\ d\zeta_1(t) &= (L_1 + M_1K)\, a(t)\, dt; \\ &\vdots \\ d\zeta_k(t) &= (L_k + M_kK)\, a(t)\, dt; \\ dy(t) &= H_2\, a(t)\, dt + J_{20}\, dv(t) + J_{21}\, dw(t) \end{aligned} \tag{7.67}$$

with (7.30), (7.31), (7.32) is strict bounded real with disturbance attenuation $\gamma > 0$. This requirement is satisfied under a condition, which is dependent on the existence of a solution to a parameterized algebraic Riccati equation defined as follows: Let $\kappa_1 > 0, \ldots, \kappa_k > 0$ be given constants and consider a complex algebraic Riccati equation

$$\left(F - G_2 E_1^{-1} \bar{J}_{12}^\dagger \bar{H}_1\right)^\dagger X + X \left(F - G_2 E_1^{-1} \bar{J}_{12}^\dagger \bar{H}_1\right) + X \left(\bar{G}_1 \bar{G}_1^\dagger - G_2 E_1^{-1} G_2^\dagger\right) X + \bar{H}_1^\dagger \left(I - \bar{J}_{12} E_1^{-1} \bar{J}_{12}^\dagger\right) \bar{H}_1 = 0 \tag{7.68}$$

where

$$\bar{G}_1 = \left[\gamma^{-1} G_1 \quad \sqrt{\kappa_1}^{-1} G_{3,1} \quad \cdots \quad \sqrt{\kappa_k}^{-1} G_{3,k}\right];$$
$$\bar{H}_1 = \begin{bmatrix} H_1 \\ \sqrt{\kappa_1}\, L_1 \\ \vdots \\ \sqrt{\kappa_k}\, L_1 \end{bmatrix}; \quad \bar{J}_{12} = \begin{bmatrix} J_{12} \\ \sqrt{\kappa_1}\, M_1 \\ \vdots \\ \sqrt{\kappa_k}\, M_k \end{bmatrix}; \quad E_1 = \bar{J}_{12}^\dagger \bar{J}_{12}. \tag{7.69}$$

Assumption 7.1. Given constants $\kappa_1 > 0, \ldots, \kappa_k > 0$, the uncertain quantum system (7.25), (7.30), (7.31), (7.32) is assumed to be such that $E_1 > 0$.

Lemma 7.9. *Let $\kappa_1 > 0, \ldots, \kappa_k > 0$ be given constants. Suppose that the uncertain quantum system* (7.25), (7.30), (7.31), (7.32) *is such that Assumption 7.1 is satisfied and the complex algebraic Riccati equation* (7.68) *has a stabilizing solution $X \geq 0$. Then, there exists a matrix K such that the uncertain quantum system* (7.67), (7.30), (7.31), (7.32) *is strict bounded real with disturbance attenuation $\gamma > 0$. That is, $(F + G_2 K)$ is Hurwitz and*

$$\left\| (H_1 + J_{12} K) \left(sI - (F + G_2 K)\right)^{-1} G_1 \right\|_\infty < \gamma \tag{7.70}$$

where

$$K = -E_1^{-1} \left(G_2^\dagger X + \bar{J}_{12}^\dagger \bar{H}_1\right). \tag{7.71}$$

Proof. Consider the uncertain quantum system (7.25), (7.30), (7.31), (7.32) and let $\kappa_1 > 0, \ldots, \kappa_k > 0$ be given constants. Also, suppose that the scaled quantum system

$$da(t) = F\, a(t) dt + G_0\, dv(t) + \bar{G}_1\, d\bar{w}(t) + G_2\, du(t);$$
$$d\bar{z}(t) = \bar{H}_1\, a(t) dt + \bar{J}_{12}\, du(t) \tag{7.72}$$

is such that Assumption 7.1 is satisfied and the complex algebraic Riccati equation

(7.68) has a stabilizing solution $X \geq 0$. Then, we let $K = -E_1^{-1}(G_2^\dagger X + \bar{J}_{12}^\dagger \bar{H}_1)$ be such that

$$\begin{aligned} da(t) &= (F + G_2 K)\, a(t)dt + G_0\, dv(t) + \bar{G}_1\, d\bar{w}(t); \\ d\bar{z}(t) &= (\bar{H}_1 + \bar{J}_{12} K)\, a(t)dt \end{aligned} \tag{7.73}$$

is strict bounded real with $(F + G_2 K)$ Hurwitz and

$$\|(\bar{H}_1 + \bar{J}_{12}K)(sI - (F + G_2 K))^{-1}\bar{G}_1\|_\infty < 1. \tag{7.74}$$

Moreover, $X \geq 0$ is also a stabilizing solution to the following complex algebraic Riccati equation

$$(F + G_2 K)^\dagger X + X(F + G_2 K) + X G_1 G_1^\dagger X + (H_1 + J_{12}K)^\dagger (H_1 + J_{12}K) = 0. \tag{7.75}$$

It then follows from Lemma 7.6 that there exists a matrix $P > 0$ such that

$$(F + G_2 K)^\dagger P + P(F + G_2 K) + P G_1 G_1^\dagger P + (H_1 + J_{12}K)^\dagger (H_1 + J_{12}K) < 0 \tag{7.76}$$

where $X < P$. Thus, together with Lemma 7.8, we conclude that the uncertain quantum system (7.67), (7.30), (7.31), (7.32) is strict bounded real with disturbance attenuation $\gamma > 0$. □

Using the matrix K as described in (7.71) and introducing additional uncertainty input $d\xi_{k+1}(t)$ and uncertainty output $d\zeta_{k+1}(t)$, we form an artificial uncertain quantum system as follows: (see [8])

$$\begin{aligned} da(t) &= \tilde{F}\, a(t)\, dt + G_0\, dv(t) + \tilde{G}_1\, d\tilde{w}(t) + \tilde{G}_2\, du(t) + \sum_{j=1}^{k+1} G_{3,j}\, d\xi_j(t); \\ d\tilde{z}(t) &= \tilde{H}_1\, a(t)\, dt + \tilde{J}_{12}\, du(t) + N_0\, d\xi_{k+1}(t); \\ d\zeta_1(t) &= \tilde{L}_1\, a(t)\, dt + \tilde{M}_1\, du(t) + N_1\, d\xi_{k+1}(t); \\ &\ \vdots \\ d\zeta_k(t) &= \tilde{L}_k\, a(t)\, dt + \tilde{M}_k\, du(t) + N_k\, d\xi_{k+1}(t); \\ d\zeta_{k+1}(t) &= \tilde{L}_{k+1}\, a(t)\, dt + \tilde{M}_{k+1}\, du(t) + P\, d\tilde{w}(t); \\ dy(t) &= \tilde{H}_2\, a(t)\, dt + J_{20}\, dv(t) + \tilde{J}_{21}\, d\tilde{w}(t) + N_{k+1}\, d\xi_{k+1}(t) \end{aligned} \tag{7.77}$$

where $a(0) = a_0$; $d\tilde{w}(t) = \tilde{\beta}_w(t)\,dt + d\tilde{\nu}(t)$;

$$
\begin{aligned}
d\tilde{w}(t) &= \begin{bmatrix} dw_1(t) \\ dw_2(t) \end{bmatrix}; \quad d\tilde{z}(t) = \begin{bmatrix} dz_1(t) \\ dz_2(t) \end{bmatrix}; \quad \tilde{F} = F + \tfrac{1}{2}G_2K; \\
\tilde{G}_1 &= \begin{bmatrix} G_1 & 0 \end{bmatrix}; \quad \tilde{G}_2 = \tfrac{1}{2}G_2; \quad G_{3,k+1} = \begin{bmatrix} G_2 & 0 & 0 & 0 \end{bmatrix} R^{-1}; \\
\tilde{H}_1 &= \tfrac{1}{2}\begin{bmatrix} H_1 \\ 0 \end{bmatrix}; \quad \tilde{J}_{12} = \tfrac{1}{2}\begin{bmatrix} J_{12} \\ \gamma I \end{bmatrix}; \quad N_0 = \begin{bmatrix} 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} R^{-1}; \\
\tilde{L}_1 &= L_1 + \tfrac{1}{2}M_1K; \quad \tilde{M}_1 = \tfrac{1}{2}M_1; \quad N_1 = \begin{bmatrix} M_1 & 0 & 0 & 0 \end{bmatrix} R^{-1}; \\
&\vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\
\tilde{L}_k &= L_k + \tfrac{1}{2}M_kK; \quad \tilde{M}_k = \tfrac{1}{2}M_k; \quad N_k = \begin{bmatrix} M_k & 0 & 0 & 0 \end{bmatrix} R^{-1}; \\
\tilde{L}_{k+1} &= \tfrac{1}{2}R\begin{bmatrix} K \\ H_1 \\ 0 \\ H_2 \end{bmatrix}; \quad \tilde{M}_{k+1} = \tfrac{1}{2}R\begin{bmatrix} -I \\ J_{12} \\ \gamma I \\ 0 \end{bmatrix}; \quad P = \tfrac{1}{2}R\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ J_{21} & -I \end{bmatrix}; \\
\tilde{H}_2 &= \tfrac{1}{2}H_2; \quad \tilde{J}_{21} = \tfrac{1}{2}\begin{bmatrix} J_{21} & I \end{bmatrix}; \quad N_{k+1} = \begin{bmatrix} 0 & 0 & 0 & -I \end{bmatrix} R^{-1}.
\end{aligned}
\tag{7.78}
$$

Note that R is any $n_r \times n_r$ non-singular scaling matrix, where $n_r = 2n_u + n_z + n_y$; w_2 and z_2 have the same dimensions as those of y and u, respectively.

In (7.77), the uncertainty input $d\xi_j(t)$ is related to the uncertainty output $d\zeta_j(t)$ according to (7.30) for $j = 1, 2, \ldots, k$. Also, the additional uncertainty input $d\xi_{k+1}(t)$ is related to the additional uncertainty output $d\zeta_{k+1}(t)$ according to

$$
d\xi_{k+1}(t) = \Delta_{k+1}\, d\zeta_{k+1}(t) \tag{7.79}
$$

where $\Delta_{k+1} \in \mathbf{R}$ is a real unknown scalar uncertain parameter satisfying $|\Delta_{k+1}| \leq 1$. Moreover, the H^∞ control objective for the artificial uncertain quantum system (7.77), (7.30), (7.31), (7.32), (7.79) is as follows:

$$
\begin{aligned}
&\int_0^t \left\langle \tilde{z}(s)^\dagger \tilde{z}(s) + \tilde{z}(s)^T \tilde{z}(s)^* + \varepsilon\left(\eta(s)^\dagger \eta(s) + \eta(s)^T \eta(s)^*\right) \right\rangle ds \\
&\leq (1 - \varepsilon^2) \int_0^t \left\langle \tilde{\beta}_w(s)^\dagger \tilde{\beta}_w(s) + \tilde{\beta}_w(s)^T \tilde{\beta}_w(s)^* \right\rangle ds + \pi_1 + \pi_2 t
\end{aligned}
\tag{7.80}
$$

where $\varepsilon, \pi_1, \pi_2 > 0$ are real constants.

Now, we consider two special cases for Δ_{k+1} to verify that any suitable coherent quantum controller of the form (7.33) for the artificial uncertain quantum system (7.77), (7.30), (7.31), (7.32), (7.79) is indeed stable and strict bounded real, and solves the original quantum control problem.

Special case I: $\Delta_{k+1} = 1$. With this value of Δ_{k+1}, it immediately follows that the QSDEs (7.77) become

$$\begin{aligned}
da(t) &= (F + G_2 K)\, a(t)\, dt + G_0\, dv(t) + G_1\, dw_1(t) + \sum_{j=1}^{k} G_{3,j}\, d\xi_j(t);\; a(0) = a_0;\\
dz_1(t) &= 0;\\
dz_2(t) &= \gamma\, du(t);\\
d\zeta_1(t) &= (L_1 + M_1 K)\, a(t)\, dt;\\
&\vdots\\
d\zeta_k(t) &= (L_k + M_k K)\, a(t)\, dt;\\
dy(t) &= J_{20}\, dv(t) + dw_2(t)
\end{aligned} \tag{7.81}$$

with (7.30), (7.31), (7.32). We notice that the uncertain quantum system (7.81), (7.30), (7.31), (7.32) is the same as the uncertain quantum system (7.67), (7.30), (7.31), (7.32). Hence, the uncertain quantum system (7.81), (7.30), (7.31), (7.32) is strict bounded real with disturbance attenuation $\gamma > 0$ according to the construction of the matrix K in (7.71) and Lemma 7.9. It is apparent from the QSDEs (7.81) that the control input $u(t)$ does not affect the quantum plant, but only affects the controlled output $z_2(t)$. Also, the measurement output $y(t)$ is not affected by the quantum plant but is only affected by the disturbance input $w_2(t)$ and the quantum noise $v(t)$. This situation is shown in Figure 7.1(a) where the coherent quantum controller Σ_c of the form (7.33) is detached from the uncertain quantum system $(\widetilde{\Sigma}_a, \Delta(\cdot))$ defined by (7.81), (7.30), (7.31), (7.32). It thus follows from the block diagram in Figure 7.1(a) and the closed loop H^∞ control objective (7.80) that the coherent quantum controller Σ_c must be stable and strict bounded real.

Special case II: $\Delta_{k+1} = -1$. It is straightforward to show that with this value of Δ_{k+1}, the QSDEs (7.77) reduce to the original QSDEs (7.25) with (7.30), (7.31), (7.32). Thus, if the coherent quantum controller Σ_c of the form (7.33) is

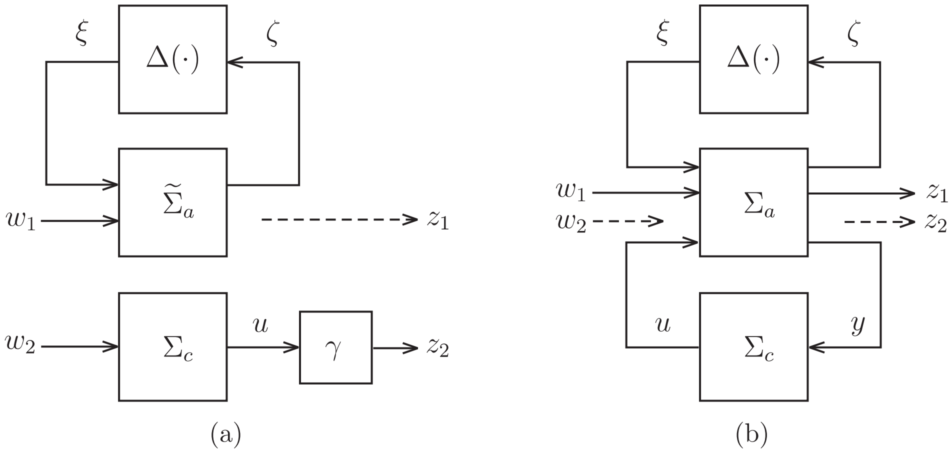


Figure 7.1: Block diagrams corresponding to special cases I and II; see [8].

applied to the artificial uncertain quantum system (7.77), (7.30), (7.31), (7.32), (7.79), we will obtain a closed loop uncertain quantum system as shown in Figure 7.1(b) where $(\Sigma_a, \Delta(\cdot))$ corresponds to the original uncertain quantum system (7.25), (7.30), (7.31), (7.32). This implies that the coherent quantum controller of the form (7.33) indeed solves the original quantum control problem where the closed loop uncertain quantum system (7.38), (7.30), (7.31), (7.32) is required to be strict bounded real with disturbance attenuation $\gamma > 0$.

From both cases, we conclude that if there exists a suitable coherent quantum controller of the form (7.33), which stabilizes the artificial uncertain quantum system (7.77), (7.30), (7.31), (7.32), (7.79) such that the resulting closed loop uncertain quantum system is strict bounded real with disturbance attenuation $\gamma > 0$, then this quantum controller also provides the same closed loop properties when it is applied to the original uncertain quantum system (7.25), (7.30), (7.31), (7.32). Also, the quantum controller itself must be stable and strict bounded real.

7.4.2 Strict bounded real quantum H^∞ controller

Along with the results in [73], we use the approach of robust H^∞ control theory presented in [187] and Lemma 7.8 to synthesize a coherent quantum controller of the form (7.33) for the artificial uncertain quantum system (7.77), (7.30), (7.31), (7.32), (7.79). To proceed with this approach, we need to introduce scaling constants $\tau_1 > 0, \ldots, \tau_{k+1} > 0$ so that we can rewrite the QSDEs (7.77) of the artificial uncertain quantum system as follows:

$$
\begin{aligned}
da(t) &= \tilde{F}\, a(t)\, dt + G_0\, dv(t) + \check{G}_1\, d\check{w}(t) + \tilde{G}_2\, du(t); \quad a(0) = a_0;\\
d\check{z}(t) &= \check{H}_1\, a(t)\, dt + \check{N}\, d\check{w}(t) + \check{J}_{12}\, du(t);\\
dy(t) &= \tilde{H}_2\, a(t)\, dt + J_{20}\, dv(t) + \check{J}_{21}\, d\check{w}(t)
\end{aligned}
\tag{7.82}
$$

where $d\check{w}(t) = \check{\beta}_w(t)\, dt + d\check{\nu}(t)$;

$$
\begin{aligned}
\check{G}_1 &= \begin{bmatrix} \gamma^{-1}\,\tilde{G}_1 & \sqrt{\tau_1}^{-1} G_{3,1} & \cdots & \sqrt{\tau_{k+1}}^{-1} G_{3,k+1} \end{bmatrix};\\
\check{J}_{21} &= \begin{bmatrix} \gamma^{-1}\,\tilde{J}_{21} & 0 & \cdots & 0 & \sqrt{\tau_{k+1}}^{-1} N_{k+1} \end{bmatrix};
\end{aligned}
$$

$$
d\check{w}(t) = \begin{bmatrix} \gamma\, d\tilde{w}(t) \\ \sqrt{\tau_1}\, d\xi_1(t) \\ \vdots \\ \sqrt{\tau_{k+1}}\, d\xi_{k+1}(t) \end{bmatrix}; \; d\check{z}(t) = \begin{bmatrix} d\tilde{z}(t) \\ \sqrt{\tau_1}\, d\zeta_1(t) \\ \vdots \\ \sqrt{\tau_{k+1}}\, d\zeta_{k+1}(t) \end{bmatrix}; \; \check{H}_1(t) = \begin{bmatrix} \tilde{H}_1 \\ \sqrt{\tau_1}\, \tilde{L}_1 \\ \vdots \\ \sqrt{\tau_{k+1}}\, \tilde{L}_{k+1} \end{bmatrix};
$$

$$
\check{J}_{12}(t) = \begin{bmatrix} \tilde{J}_{12} \\ \sqrt{\tau_1}\, \tilde{M}_1 \\ \vdots \\ \sqrt{\tau_{k+1}}\, \tilde{M}_{k+1} \end{bmatrix}; \; \check{N} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \sqrt{\tau_{k+1}}^{-1} N_0 \\ 0 & 0 & \cdots & 0 & \sqrt{\frac{\tau_1}{\tau_{k+1}}}\, N_1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \sqrt{\frac{\tau_k}{\tau_{k+1}}}\, N_k \\ \gamma^{-1}\sqrt{\tau_{k+1}}\, P & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{7.83}
$$

The H^∞ control objective corresponding to the quantum system (7.82) is

$$
\begin{aligned}
&\int_0^t \left\langle \check{z}(s)^\dagger \check{z}(s) + \check{z}(s)^T \check{z}(s)^* + \varepsilon \left(\eta(s)^\dagger \eta(s) + \eta(s)^T \eta(s)^* \right) \right\rangle ds\\
&\le (1-\varepsilon^2) \int_0^t \left\langle \check{\beta}_w(s)^\dagger \check{\beta}_w(s) + \check{\beta}_w(s)^T \check{\beta}_w(s)^* \right\rangle ds + \pi_1 + \pi_2 t
\end{aligned}
\tag{7.84}
$$

where $\varepsilon, \pi_1, \pi_2 > 0$ are real constants. Since an $\check{N}$-term in (7.82) leads to a non-standard H^∞ control problem, we apply a loop shifting transformation to eliminate this term; e.g., see [16, Sections 4.5.1 and 5.5.1] and [18, Section 17.2]. This can be done by first imposing the following assumption:

Assumption 7.2. Given constants $\tau_1 > 0, \ldots, \tau_{k+1} > 0$ and any non-singular scaling matrix R, the uncertain quantum system (7.77), (7.30), (7.31), (7.32), (7.79) is assumed to be such that $\check{N}\check{N}^\dagger < I$.

Then, we can define

$$\Phi := I - \check{N}^\dagger \check{N} > 0; \quad \check{\Phi} := I - \check{N}\check{N}^\dagger > 0 \tag{7.85}$$

and also

$$\begin{aligned} d\hat{w}(t) &:= \Phi^{\frac{1}{2}} d\check{w}(t) - \Phi^{-\frac{1}{2}} \check{N}^\dagger \left[\check{H}_1\, a(t)\, dt + \check{J}_{12}\, du(t) \right]; \\ d\hat{z}(t) &:= \check{\Phi}^{-\frac{1}{2}} \left[\check{H}_1\, a(t)\, dt + \check{J}_{12}\, du(t) \right]. \end{aligned} \tag{7.86}$$

From (7.86), it is straightforward to verify that

$$\begin{aligned} &d\check{w}(t) = \Phi^{-\frac{1}{2}} d\hat{w}(t) + \Phi^{-1} \check{N}^\dagger \left[\check{H}_1\, a(t)\, dt + \check{J}_{12}\, du(t) \right]; \\ &\|\check{w}(t)\|_2^2 - \|\check{z}(t)\|_2^2 \equiv \|\hat{w}(t)\|_2^2 - \|\hat{z}(t)\|_2^2. \end{aligned} \tag{7.87}$$

Now, we can rewrite the QSDEs (7.82) as

$$\begin{aligned} da(t) &= \hat{F}\, a(t)dt + G_0\, dv(t) + \hat{G}_1\, d\hat{w}(t) + \hat{G}_2\, du(t); \quad a(0) = a_0; \\ d\hat{z}(t) &= \hat{H}_1\, a(t)dt + \hat{J}_{12}\, du(t); \\ dy(t) &= \hat{H}_2\, a(t)dt + J_{20}\, dv(t) + \hat{J}_{21}\, d\hat{w}(t) + \hat{J}_{22}\, du(t) \end{aligned} \tag{7.88}$$

where $d\hat{w}(t) = \hat{\beta}_w(t)dt + d\hat{\nu}(t)$;

$$\begin{array}{rclcrcl} \hat{F} &=& \tilde{F} + \check{G}_1 \check{N}^\dagger \check{\Phi}^{-1} \check{H}_1; & & \hat{G}_1 &=& \check{G}_1 \Phi^{-\frac{1}{2}}; \\ \hat{G}_2 &=& \tilde{G}_2 + \check{G}_1 \check{N}^\dagger \check{\Phi}^{-1} \check{J}_{12}; & & \hat{H}_1 &=& \check{\Phi}^{-\frac{1}{2}} \check{H}_1; \\ \hat{H}_2 &=& \tilde{H}_2 + \check{J}_{21} \check{N}^\dagger \check{\Phi}^{-1} \check{H}_1; & & \hat{J}_{12} &=& \check{\Phi}^{-\frac{1}{2}} \check{J}_{12}; \\ \hat{J}_{22} &=& \check{J}_{21} \check{N}^\dagger \check{\Phi}^{-1} \check{J}_{12}; & & \hat{J}_{21} &=& \check{J}_{21} \Phi^{-\frac{1}{2}}. \end{array} \tag{7.89}$$

Furthermore, we also define

$$d\hat{y}(t) := dy(t) - \hat{J}_{22}\, du(t) \tag{7.90}$$

and substituting (7.90) into (7.88), we obtain

$$\begin{aligned} da(t) &= \hat{F}\, a(t)dt + G_0\, dv(t) + \hat{G}_1\, d\hat{w}(t) + \hat{G}_2\, du(t); \quad a(0) = a_0; \\ d\hat{z}(t) &= \hat{H}_1\, a(t)dt + \hat{J}_{12}\, du(t); \\ d\hat{y}(t) &= \hat{H}_2\, a(t)dt + J_{20}\, dv(t) + \hat{J}_{21}\, d\hat{w}(t). \end{aligned} \tag{7.91}$$

The H^∞ control objective corresponding to the quantum system (7.91) is

$$\begin{aligned}&\int_0^t \left\langle \hat{z}(s)^\dagger \hat{z}(s) + \hat{z}(s)^T \hat{z}(s)^* + \varepsilon \left(\eta(s)^\dagger \eta(s) + \eta(s)^T \eta(s)^*\right)\right\rangle ds \\ &\leq (1-\varepsilon^2) \int_0^t \left\langle \hat{\beta}_w(s)^\dagger \hat{\beta}_w(s) + \hat{\beta}_w(s)^T \hat{\beta}_w(s)^* \right\rangle ds + \pi_1 + \pi_2 t \end{aligned} \tag{7.92}$$

where $\varepsilon, \pi_1, \pi_2 > 0$ are real constants.

The solution to the coherent quantum H^∞ control problem for the linear quantum system (7.91) is given in terms of solutions to the parameterized complex algebraic Riccati equations:

$$\begin{aligned}&\left(\hat{F} - \hat{G}_2 \hat{E}_1^{-1} \hat{J}_{12}^\dagger \hat{H}_1\right)^\dagger \hat{X} + \hat{X}\left(\hat{F} - \hat{G}_2 \hat{E}_1^{-1} \hat{J}_{12}^\dagger \hat{H}_1\right) \\ &\qquad + \hat{X}\left(\hat{G}_1 \hat{G}_1^\dagger - \hat{G}_2 \hat{E}_1^{-1} \hat{G}_2^\dagger\right) \hat{X} + \hat{H}_1^\dagger \left(I - \hat{J}_{12} \hat{E}_1^{-1} \hat{J}_{12}^\dagger\right) \hat{H}_1 = 0\,; \\ &\left(\hat{F} - \hat{G}_1 \hat{J}_{21}^\dagger \hat{E}_2^{-1} \hat{H}_2\right) \hat{Y} + \hat{Y}\left(\hat{F} - \hat{G}_1 \hat{J}_{21}^\dagger \hat{E}_2^{-1} \hat{H}_2\right)^\dagger \\ &\qquad + \hat{Y}\left(\hat{H}_1^\dagger \hat{H}_1 - \hat{H}_2^\dagger \hat{E}_2^{-1} \hat{H}_2\right) \hat{Y} + \hat{G}_1 \left(I - \hat{J}_{21}^\dagger \hat{E}_2^{-1} \hat{J}_{21}\right) \hat{G}_1^\dagger = 0 \end{aligned} \tag{7.93}$$

such that the following conditions hold:

1. $\hat{F} - \hat{G}_2 \hat{E}_1^{-1} \hat{J}_{12}^\dagger \hat{H}_1 + \left(\hat{G}_1 \hat{G}_1^\dagger - \hat{G}_2 \hat{E}_1^{-1} \hat{G}_2^\dagger\right) \hat{X}$ is Hurwitz;
2. $\hat{F} - \hat{G}_1 \hat{J}_{21}^\dagger \hat{E}_2^{-1} \hat{H}_2 + \hat{Y}\left(\hat{H}_1^\dagger \hat{H}_1 - \hat{H}_2^\dagger \hat{E}_2^{-1} \hat{H}_2\right)$ is Hurwitz;
3. The spectral radius $\rho(\hat{X}\hat{Y})$ of matrix $\hat{X}\hat{Y}$ is strictly less than one.

To obtain the solutions to the Riccati equations (7.93), we require the following assumption to be satisfied.

Assumption 7.3. Given constants $\tau_1 > 0, \ldots, \tau_{k+1} > 0$ and any non-singular scaling matrix R, the uncertain quantum system (7.77), (7.30), (7.31), (7.32), (7.79) is assumed to be such that

1. $\hat{E}_1 = \hat{J}_{12}^\dagger \hat{J}_{12} > 0$;
2. $\hat{E}_2 = \hat{J}_{21} \hat{J}_{21}^\dagger > 0$.

Theorem 7.1. *Suppose that there exist constants $\kappa_1 > 0, \ldots, \kappa_k > 0$ satisfying Assumption 7.1 such that the complex algebraic Riccati equation (7.68) has a* *stabilizing solution $X \geq 0$ and let*

$$K = -E_1^{-1}\left(G_2^\dagger X + \bar{J}_{12}^\dagger \bar{H}_1\right).$$

Also, suppose that there exist a non-singular scaling matrix R and constants $\tau_1 > 0, \ldots, \tau_{k+1} > 0$ satisfying Assumption 7.2 and Assumption 7.3 such that the complex algebraic Riccati equations (7.93) *have stabilizing solutions $\hat{X} \geq 0$ and $\hat{Y} \geq 0$ such that the spectral radius $\rho(\hat{X}\hat{Y}) < 1$. Then the closed loop uncertain quantum system obtained by applying the coherent quantum controller* (7.33) *with*

$$\begin{aligned}
F_c &= \hat{F}_c - G_c \hat{J}_{22} H_c; \\
\hat{F}_c &= \hat{F} + \hat{G}_2 H_c - G_c \hat{H}_2 + \left(\hat{G}_1 - G_c \hat{J}_{21}\right)\hat{G}_1^\dagger \hat{X}; \\
G_c &= \left(I - \hat{Y}\hat{X}\right)^{-1}\left(\hat{Y}\hat{H}_2^\dagger + \hat{G}_1 \hat{J}_{21}^\dagger\right)\hat{E}_2^{-1}; \\
H_c &= -\hat{E}_1^{-1}\left(\hat{G}_2^\dagger \hat{X} + \hat{J}_{12}^\dagger \hat{H}_1\right)
\end{aligned} \tag{7.94}$$

to the uncertain quantum system (7.25), (7.30), (7.31), (7.32) *is strict bounded real with disturbance attenuation $\gamma > 0$.*

Proof. It follows from loop shifting arguments in the classical H^∞ control theory (e.g., see [16, Sections 4.5.1 and 5.5.1] and [18, Section 17.2]) that the H^∞ quantum control problem (7.82), (7.84) has a solution if and only if the complex Riccati equations in (7.93) have stabilizing solutions $\hat{X} \geq 0$ and $\hat{Y} \geq 0$ such that $\rho(\hat{X}\hat{Y}) < 1$. Moreover, a coherent quantum controller of the form (7.33) (but not necessarily stable and strict bounded real), which solves the H^∞ quantum control problem (7.82), (7.84) is defined by (7.94).

Therefore, if the conditions of the theorem are satisfied, it follows from the arguments in the proofs of Theorem 4.1 in [187] and of Theorem 7.1 in [73] that the closed loop uncertain quantum system obtained by applying the coherent quantum controller (7.33), (7.94) to the artificial uncertain quantum system (7.77), (7.30), (7.31), (7.32), (7.79) is strict bounded real with disturbance attenuation $\gamma > 0$. Moreover, it follows from the construction of the artificial uncertain quantum system (7.77), (7.32), (7.79) that the coherent quantum controller (7.33), (7.94) must be stable and satisfy condition (7.37). Thus, if this controller is applied to the original uncertain quantum system (7.25), (7.30), (7.31), (7.32), the resulting closed loop uncertain quantum system is also strict bounded real with disturbance attenuation $\gamma > 0$. □

Remark 7.2. Although the coherent quantum controller (7.33), (7.94) is guaranteed to be physically realizable, the additional uncertainty in the artificial uncertain quantum system (7.77), (7.30), (7.31), (7.32), (7.79) introduces some conservatism to the design process.

7.5 A Differential Evolution Approach

We recognize from Section 7.4 that the problem of designing a strict bounded real quantum controller (7.33) involves several parameter dependent nonlinear constraints. To find a solution to this problem, we propose to apply an evolutionary optimization method, namely the differential evolution (DE) algorithm, as given in Chapter 2. Thus, we reformulate the quantum controller design problem into an optimization problem, which is subject to nonconvex nonlinear constraints. The required parameters form a vector of decision variables defined as

$$\vartheta := \begin{bmatrix} \gamma & \kappa_1 & \cdots & \kappa_k & \tau_1 & \cdots & \tau_{k+1} \end{bmatrix}^T \tag{7.95}$$

where the dimension of ϑ is $2(k+1)$ and each element of ϑ is a positive real number. Then, the optimization problem can be stated as follows: Find an optimal solution $\vartheta^\star$ to solve

$$\min_{\vartheta} \mathsf{f}(\vartheta) \tag{7.96}$$

subject to

$$\mathsf{g_j}(\vartheta) = 0; \quad \mathsf{h_k}(\vartheta) \leq 0 \tag{7.97}$$

for $\mathsf{j} = 1, 2, \ldots, \mathsf{a}$ and $\mathsf{k} = 1, 2, \ldots, \mathsf{b}$. Here, a and b are the total number of equality and inequality constraints, respectively; and $\mathsf{f}(\vartheta)$ is an objective function to be minimized.

Since we deal with an H^∞ control problem, a suitable objective function is

$$\mathsf{f}(\vartheta) = \gamma^{\mathsf{n}} \tag{7.98}$$

where $\mathsf{n} \geq 1$ is a power constant. Referring to Section 7.4, we determine the equality constraints as

$$
\begin{aligned}
\mathsf{g}_1(\vartheta) &= \left(F - G_2 E_1^{-1} \bar{J}_{12}^{\dagger} \bar{H}_1\right)^{\dagger} X + X \left(F - G_2 E_1^{-1} \bar{J}_{12}^{\dagger} \bar{H}_1\right) \\
&\quad + X \left(\bar{G}_1 \bar{G}_1^{\dagger} - G_2 E_1^{-1} G_2^{\dagger}\right) X + \bar{H}_1^{\dagger} \left(I - \bar{J}_{12} E_1^{-1} \bar{J}_{12}^{\dagger}\right) \bar{H}_1 = 0; \\
\mathsf{g}_2(\vartheta) &= \left(\hat{F} - \hat{G}_2 \hat{E}_1^{-1} \hat{J}_{12}^{\dagger} \hat{H}_1\right)^{\dagger} \hat{X} + \hat{X} \left(\hat{F} - \hat{G}_2 \hat{E}_1^{-1} \hat{J}_{12}^{\dagger} \hat{H}_1\right) \\
&\quad + \hat{X} \left(\hat{G}_1 \hat{G}_1^{\dagger} - \hat{G}_2 \hat{E}_1^{-1} \hat{G}_2^{\dagger}\right) \hat{X} + \hat{H}_1^{\dagger} \left(I - \hat{J}_{12} \hat{E}_1^{-1} \hat{J}_{12}^{\dagger}\right) \hat{H}_1 = 0; \\
\mathsf{g}_3(\vartheta) &= \left(\hat{F} - \hat{G}_1 \hat{J}_{21}^{\dagger} \hat{E}_2^{-1} \hat{H}_2\right) \hat{Y} + \hat{Y} \left(\hat{F} - \hat{G}_1 \hat{J}_{21}^{\dagger} \hat{E}_2^{-1} \hat{H}_2\right)^{\dagger} \\
&\quad + \hat{Y} \left(\hat{H}_1^{\dagger} \hat{H}_1 - \hat{H}_2^{\dagger} \hat{E}_2^{-1} \hat{H}_2\right) \hat{Y} + \hat{G}_1 \left(I - \hat{J}_{21}^{\dagger} \hat{E}_2^{-1} \hat{J}_{21}\right) \hat{G}_1^{\dagger} = 0
\end{aligned} \tag{7.99}
$$

and the inequality constraints as

$$
\begin{array}{rclrcl}
\mathsf{h}_1(\vartheta) &=& -E_1 < 0; & \mathsf{h}_2(\vartheta) &=& -X < 0; \\
\mathsf{h}_3(\vartheta) &=& \check{N}\check{N}^{\dagger} - I < 0; & \mathsf{h}_4(\vartheta) &=& -\hat{E}_1 < 0; \\
\mathsf{h}_5(\vartheta) &=& -\hat{E}_2 < 0; & \mathsf{h}_6(\vartheta) &=& -\hat{X} < 0; \\
\mathsf{h}_7(\vartheta) &=& -\hat{Y} < 0; & \mathsf{h}_8(\vartheta) &=& \rho\left(\hat{X}\hat{Y}\right) - 1 < 0; \\
\mathsf{h}_9(\vartheta) &=& e_{\max,r}\left(\mathcal{A}_X\right) < 0; & \mathsf{h}_{10}(\vartheta) &=& e_{\max,r}\left(\mathcal{A}_{\hat{X}}\right) < 0; \\
\mathsf{h}_{11}(\vartheta) &=& e_{\max,r}\left(\mathcal{A}_{\hat{Y}}\right) < 0. & & &
\end{array} \tag{7.100}
$$

Note that $\rho(\mathcal{M})$ and $e_{\max,r}(\mathcal{M})$ denote the spectral radius and the largest real part of the eigenvalues of the matrix $\mathcal{M}$, respectively. Moreover, we define $\mathcal{A}_X$, $\mathcal{A}_{\hat{X}}$ and $\mathcal{A}_{\hat{Y}}$ as follows:

$$
\begin{aligned}
\mathcal{A}_X &:= F - G_2 E_1^{-1} \bar{J}_{12}^{\dagger} \bar{H}_1 + \left(\bar{G}_1 \bar{G}_1^{\dagger} - G_2 E_1^{-1} G_2^{\dagger}\right) X; \\
\mathcal{A}_{\hat{X}} &:= \hat{F} - \hat{G}_2 \hat{E}_1^{-1} \hat{J}_{12}^{\dagger} \hat{H}_1 + \left(\hat{G}_1 \hat{G}_1^{\dagger} - \hat{G}_2 \hat{E}_1^{-1} \hat{G}_2^{\dagger}\right) \hat{X}; \\
\mathcal{A}_{\hat{Y}} &:= \hat{F} - \hat{G}_1 \hat{J}_{21}^{\dagger} \hat{E}_2^{-1} \hat{H}_2 + \hat{Y} \left(\hat{H}_1^{\dagger} \hat{H}_1 - \hat{H}_2^{\dagger} \hat{E}_2^{-1} \hat{H}_2\right).
\end{aligned} \tag{7.101}
$$

We employ the equality and inequality constraints defined in (7.99) and (7.100) to examine the fitness of each candidate solution ϑ. Thus, the fitness evaluation proceeds as follows:

1. Compute the eigenvalues of E_1 to verify if $\mathsf{h}_1(\vartheta)$ is satisfied.

2. Evaluate $\mathsf{g}_1(\vartheta)$ to obtain a solution X to the Riccati equation in (7.68).

3. Verify if X is a stabilizing positive definite solution through the evaluation of $\mathsf{h}_2(\vartheta)$ and $\mathsf{h}_9(\vartheta)$.

4. Compute the eigenvalues of $(\check{N}\check{N}^\dagger - I)$, $\hat{E}_1$ and $\hat{E}_2$ to verify if $\mathsf{h}_3(\vartheta)$, $\mathsf{h}_4(\vartheta)$ and $\mathsf{h}_5(\vartheta)$ are satisfied.

5. Evaluate $\mathsf{g}_2(\vartheta)$ and $\mathsf{g}_3(\vartheta)$ to obtain solutions $\hat{X}$ and $\hat{Y}$ to the Riccati equations in (7.93).

6. Verify if $\hat{X}$ and $\hat{Y}$ are stabilizing positive definite solutions through the evaluation of $\mathsf{h}_6(\vartheta)$, $\mathsf{h}_7(\vartheta)$, $\mathsf{h}_{10}(\vartheta)$ and $\mathsf{h}_{11}(\vartheta)$.

7. Compute the spectral radius of the product $\hat{X}\hat{Y}$ to verify if $\mathsf{h}_8(\vartheta)$ is satisfied.

8. Evaluate the objective function $\mathsf{f}(\vartheta)$ in (7.98).

Having the fitness test routine above, we also define a set of penalty functions corresponding to the violation of each constraint in (7.99) and (7.100). That is,

$$
\begin{aligned}
\mathsf{p}_1(\vartheta) &= |e_{\min}(E_1)|^{\mathsf{s}_1}; & \mathsf{p}_2(\vartheta) &= \rho(\mathcal{C}_X)^{\mathsf{s}_2};\\
\mathsf{p}_3(\vartheta) &= |e_{\min}(X)|^{\mathsf{s}_3}; & \mathsf{p}_4(\vartheta) &= e_{\max,r}\left(\mathcal{A}_X\right)^{\mathsf{s}_4};\\
\mathsf{p}_5(\vartheta) &= e_{\max}(\check{N}\check{N}^\dagger - I)^{\mathsf{s}_5}; & \mathsf{p}_6(\vartheta) &= |e_{\min}(\hat{E}_1)|^{\mathsf{s}_6};\\
\mathsf{p}_7(\vartheta) &= |e_{\min}(\hat{E}_2)|^{\mathsf{s}_7}; & \mathsf{p}_8(\vartheta) &= \rho(\mathcal{C}_{\hat{X}})^{\mathsf{s}_8};\\
\mathsf{p}_9(\vartheta) &= \rho(\mathcal{C}_{\hat{Y}})^{\mathsf{s}_9}; & \mathsf{p}_{10}(\vartheta) &= |e_{\min}(\hat{X})|^{\mathsf{s}_{10}};\\
\mathsf{p}_{11}(\vartheta) &= |e_{\min}(\hat{Y})|^{\mathsf{s}_{11}}; & \mathsf{p}_{12}(\vartheta) &= e_{\max,r}\left(\mathcal{A}_{\hat{X}}\right)^{\mathsf{s}_{12}};\\
\mathsf{p}_{13}(\vartheta) &= e_{\max,r}\left(\mathcal{A}_{\hat{Y}}\right)^{\mathsf{s}_{13}}; & \mathsf{p}_{14}(\vartheta) &= (\rho(\hat{X}\hat{Y}) - 1)^{\mathsf{s}_{14}};\\
\mathsf{p}_{15}(\vartheta) &= \mathsf{f}(\vartheta)
\end{aligned}
\tag{7.102}
$$

where $\mathsf{s}_\mathsf{r} \geq 1$ for $\mathsf{r} = 1, 2, \ldots, 14$. Here, $e_{\min}(\mathcal{M})$ and $e_{\max}(\mathcal{M})$ denote the smallest and the largest eigenvalue of a Hermitian matrix $\mathcal{M}$, respectively. If the matrix $\mathcal{M}$ is required to be positive definite, we assign $|e_{\min}(\mathcal{M})|^{\mathsf{s}_\mathsf{r}}$ as a penalty. This is because when this requirement is violated, the matrix $\mathcal{M}$ can be either negative (semi)definite or indefinite. Moreover, we also define $\mathcal{C}_X$, $\mathcal{C}_{\hat{X}}$ and $\mathcal{C}_{\hat{Y}}$ as follows:

$$
\begin{aligned}
\mathcal{C}_X &:= \bar{H}_1^\dagger \left(I - \bar{J}_{12}E_1^{-1}\bar{J}_{12}^\dagger\right)\bar{H}_1;\\
\mathcal{C}_{\hat{X}} &:= \hat{H}_1^\dagger \left(I - \hat{J}_{12}\hat{E}_1^{-1}\hat{J}_{12}^\dagger\right)\hat{H}_1;\\
\mathcal{C}_{\hat{Y}} &:= \hat{G}_1 \left(I - \hat{J}_{21}^\dagger\hat{E}_2^{-1}\hat{J}_{21}\right)\hat{G}_1^\dagger.
\end{aligned}
\tag{7.103}
$$

7.6 An Illustrative Example

To demonstrate the coherent quantum controller design method presented in Section 7.4, we consider an example of designing a strict bounded real coherent quantum controller for a first order optical cavity; see [70] and [73]. That is,

$$\begin{aligned} da(t) &= -\frac{g}{2}\,a(t)\,dt - \sqrt{k_1}\,dv(t) - \sqrt{k_2}\,dw(t) - \sqrt{k_3}\,du(t); \\ dz(t) &= \sqrt{k_3}\,a(t)\,dt + du(t); \\ dy(t) &= \sqrt{k_2}\,a(t)\,dt + dw(t) \end{aligned} \tag{7.104}$$

where $k_1 = 2.25$, $k_2 = 1.00$, $k_3 = 1.00$ and $g = k_1 + k_2 + k_3$. We assume that the optical cavity (7.104) does not have an uncertainty term. If we follow the quantum controller design algorithms proposed in [70] and [73], we do not necessarily obtain a stable and strict bounded real coherent quantum controller. In particular, for this example, we have that $k_1 > k_2 + k_3$, but $\sqrt{k_1} < \sqrt{k_2} + \sqrt{k_3}$. This implies that the standard quantum H^∞ controller will not be physically realizable as has been pointed out in the example of [73].

Applying the DE approach to solve this control problem, we obtain $\gamma = 0.9132$ and $\tau_1 = 1.6641$ (corresponding to an additional artificial uncertainty). Then, the coherent quantum controller is obtained as

$$F_c = -11.0014; \quad G_c = -0.0118; \quad H_c = -0.4566 \tag{7.105}$$

with the corresponding H^∞ norm

$$\|H_c(sI - F_c)^{-1}G_c\|_\infty = 0.0005. \tag{7.106}$$

Thus, it is clear that the coherent quantum controller (7.105) is physically realizable because it is stable and strict bounded real.

From Lemma 7.4, we know that there exists $X_c > 0$ such that

$$F_c^\dagger X_c + X_c F_c + X_c G_c G_c^\dagger X_c + H_c^\dagger H_c = 0. \tag{7.107}$$

That is, $X_c = 0.0095$, which can be used to determine G_{c_0} and H_{c_2} as in (7.35).

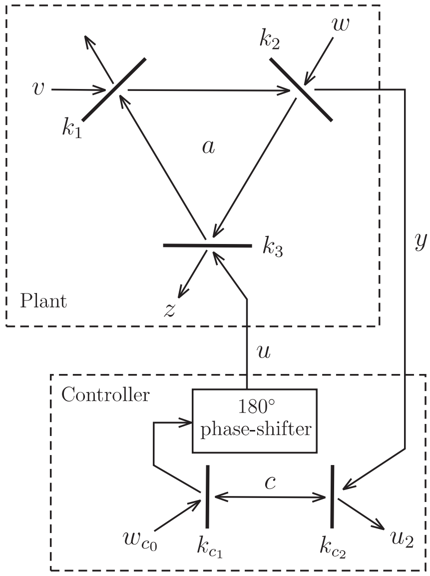


Figure 7.2: Closed loop quantum system.

Thus, for the quantum controller (7.105), we have that

$$G_{c0} := -X_c^{-1}H_c^\dagger = 48.1908; \quad H_{c_2} := -G_c^\dagger X_c = 0.0001 \tag{7.108}$$

such that the quantum controller is physically realizable (see Definition 7.6) and the conditions in (7.19) hold for $J = I$. Meanwhile, we set $G_{c_1} = 0$ and $H_{c_1} = 0$ as they are not required in the realization of a coherent quantum controller; see Remark 7.1.

Now, using (7.105) and (7.108), we can apply the algorithm presented in [77] to physically construct our coherent quantum controller as a generalized 2-mirror cavity because it has two inputs and two outputs. That is,

$$\bar{F}_c = -11.0014; \quad \bar{G}_c = \begin{bmatrix} 4.6907 & -0.0011 \end{bmatrix}; \quad \bar{H}_c = \begin{bmatrix} -4.6907 \\ 0.0011 \end{bmatrix} \tag{7.109}$$

which can be constructed using passive optical devices such as optical cavities, beam-splitters and phase shifters; see [77]. Interconnecting the coherent quantum controller (7.109) with the quantum system (7.104), we obtain a closed loop quantum system as shown in Figure 7.2 with $k_{c_1} = (4.6907)^2$, $k_{c_2} = (0.0011)^2$ and $\bar{F}_c = -\frac{1}{2}(k_{c_1} + k_{c_2})$.

7.7 Conclusions

We have presented a systematic method to synthesize a physically realizable coherent quantum robust H^∞ controller for a class of linear complex quantum stochastic systems with norm-bounded structured uncertainties. The quantum controller is required to be stable and strict bounded real. The main idea of our approach is to introduce an additional uncertainty, which is a real unknown scalar uncertain parameter, in order to form an artificial uncertain quantum system, based on which the desired quantum controller is designed. However, the additional uncertainty introduces some additional conservatism to the controller design process.

As our method involves a particular additional uncertainty, we only provide a sufficient condition, which guarantees the resulting quantum controller to be physically realizable and solves the original quantum control problem. The aim of applying such a quantum controller to the open loop uncertain quantum system is to achieve a strict bounded real closed loop uncertain quantum system with a specified disturbance attenuation level. The solution to this quantum control problem is then given in terms of the stabilizing solutions to the parameterized complex algebraic Riccati equations. Also, through an example involving the control of a first order quantum optical cavity, we have shown that our method effectively leads to a strict bounded real coherent quantum controller as required.

Chapter 8

Decentralized Coherent Quantum Robust H^∞ Control

8.1 Introduction

Extending the ideas in Chapter 4 and Chapter 5 to quantum feedback control systems, we present two systematic methods to synthesize decentralized quantum robust H^∞ controllers for a large-scale uncertain quantum system. The decentralized quantum controllers are defined in terms of the stabilizing solutions to complex algebraic Riccati equations. In these methods, the structure of the quantum plant uncertainties is exploited by assigning a scaling constant for each one of them. We assume that each structured uncertainty is an unknown linear time-invariant complex quantum system, which satisfies a norm-bound condition. Moreover, the H^∞ control objective is to achieve a robustly stable closed loop uncertain quantum system with a specified disturbance attenuation level.

A large-scale system in real-world applications is naturally comprised of interconnected subsystems. To construct a decentralized feedback controller for this system, we often find that the interconnections between subsystems are simply considered as uncertainties in addition to the plant uncertainties; e.g., see [195, 212, 322]. However, in practice, we may have partial or full knowledge on the interconnections, and hence, in our approach, we do not treat them as uncertainties. This will allow us to exploit the interconnections in order to enhance the performance of the decentralized controller; e.g., see [166].

The main idea of our approach is to treat as additional uncertainties, the neglected off-diagonal blocks of the transfer function matrix of a non-decentralized linear coherent quantum H^∞ controller; see [166]. To proceed with this idea, the non-decentralized quantum controller is required to be stable, which is immediately satisfied if the quantum controller is physically realizable; e.g., see [70, 73]. Thus, the proposed methods lead to a physically realizable decentralized quantum controller, which is robust against both quantum plant uncertainties and the additional uncertainties.

Physical realizability is an essential concern when synthesizing a quantum controller as it has to exhibit meaningful dynamics according to quantum mechanical principles; e.g., see [70, 72, 73]. In the first method, we do not immediately obtain a physically realizable decentralized quantum controller and the physical realizability of the controller must be checked before it can be implemented. This is because a physical realizability condition is not directly included in the quantum controller design algorithm. On the other hand, the second method always leads to a physically realizable decentralized quantum controller as we force the controller to be stable and strict bounded real using the approach in Chapter 7. However, the latter method inevitably leads to a more conservative decentralized quantum controller due to the use of an artificial uncertainty to ensure the physical realizability of the controller. This method then involves more constraints and design parameters than the first method, and therefore, may need more computational time to solve the quantum control problem being considered.

Since scaling constants are introduced for all uncertainties, the decentralized quantum control problem we consider then involves nonconvex nonlinear constraints. It is often difficult to find an optimal solution to this problem in the presence of such constraints. Thus, to determine the required design parameters, we apply an evolutionary optimization method, namely the differential evolution (DE) algorithm, as presented in Chapter 2. This approach has also been used in the previous chapters. Two examples are presented to show that the DE approach is applicable to synthesize a decentralized quantum H^∞ controller for a quantum optical system. In these examples, we also apply an algorithm in [77] to show that an n-th order decentralized quantum controller with m inputs can be physically constructed as a cascade of n generalized first order m-mirror optical cavities. This is an m-input-m-output interconnection, which consists only of passive optical devices such as optical cavities, beam splitters and phase shifters.

8.2 Problem Statement

In this section, we describe the decentralized quantum control problem under consideration along with the quantum H^∞ control objective. We also define the notion of physical realizability for a decentralized quantum H^∞ controller and necessary notation for the controller synthesis algorithm presented in subsequent sections. As both complex and operator matrices are involved in our derivations, we then use the following notation: $M = [m_{jk}]$, $M^* = [m^*_{jk}]$, $M^T = [m_{kj}]$ and $M^\dagger = [m^*_{kj}] = (M^*)^T$ to denote the operations as explained in Section 6.1.

8.2.1 Uncertain linear complex quantum system

We are concerned with a class of large-scale linear complex quantum stochastic systems with structured uncertainties, which are described in terms of linear quantum stochastic differential equations (QSDEs) as follows:

$$
\begin{aligned}
da(t) &= F\,a(t)dt + G_0\,dv(t) + G_1\,dw(t) + G_2\,du(t) + \sum_{l=1}^{k} G_{3,l}\,d\xi_l(t);\ a(0) = a_0;\\
dz(t) &= H_1\,a(t)dt + J_{12}\,du(t);\\
d\zeta_1(t) &= P_1\,a(t)dt + Q_1\,du(t);\\
&\ \vdots\\
d\zeta_k(t) &= P_k\,a(t)dt + Q_k\,du(t);\\
dy(t) &= H_2\,a(t)dt + J_{20}\,dv(t) + J_{21}\,dw(t)
\end{aligned}
\tag{8.1}
$$

where a is an $n\times 1$ vector of the plant annihilation operators; v is an $n_v\times 1$ vector of quantum noise; w is an $n_w\times 1$ vector of disturbance inputs; u is an $n_u\times 1$ vector of control inputs; ξ_l is an $n_{q_l}\times 1$ vector of uncertainty inputs (for $l = 1, 2, \ldots, k$); ζ_l is an $n_{s_l}\times 1$ vector of uncertainty outputs (for $l = 1, 2, \ldots, k$); z is an $n_z\times 1$ vector of controlled outputs; and y is an $n_y\times 1$ vector of 'measurement' outputs. All the coefficient matrices in (8.1) are complex matrices, which have compatible dimensions corresponding to the dimensions of the operators and signals in (8.1); see [70, 72, 73].

The disturbance input $w(t)$ and the control input $u(t)$ in (8.1) are represented respectively as follows:

$$dw(t) = \beta_w(t)\,dt + d\nu(t); \tag{8.2}$$
$$du(t) = \beta_u(t)\,dt + d\mu(t) \tag{8.3}$$

where $\beta_w(t)$ and $\beta_u(t)$ are adapted processes; and $d\nu(t)$ and $d\mu(t)$ are the noise parts of (8.2) and (8.3). Meanwhile, $dv(t)$ represents an additional quantum noise in the plant. The quantum noises $dv(t)$, $d\nu(t)$ and $d\mu(t)$ have corresponding Hermitian Ito matrices F_v, F_ν and F_μ, and Hermitian commutation matrices T_v, T_ν and T_μ, which are assumed to be

$$F_v = F_\nu = F_\mu = I; \tag{8.4}$$
$$T_v = T_\nu = T_\mu = I. \tag{8.5}$$

The l-th structured uncertainty in (8.1) is modeled as an additional unknown linear time-invariant complex quantum stochastic system:

$$\begin{aligned} d\tilde{a}_l(t) &= A_l\,\tilde{a}_l(t)\,dt + B_l\,d\zeta_l(t); \quad \tilde{a}_l(0) = \tilde{a}_{0,l}; \\ d\xi_l(t) &= C_l\,\tilde{a}_l(t)\,dt + D_l\,d\zeta_l(t) \end{aligned} \tag{8.6}$$

with A_l Hurwitz and transfer function matrix

$$\Delta_l(s) = C_l(sI - A_l)^{-1}B_l + D_l \tag{8.7}$$

which is required to satisfy

$$\|\Delta_l(s)\|_\infty \leq 1 \tag{8.8}$$

for all $l = 1, 2, \ldots, k$.

We assume that the large-scale quantum system (8.1) consists of p interconnected linear quantum subsystems. Thus, the output $dy(t)$ can also be decomposed into p components as follows:

$$dy(t) = \begin{bmatrix} dy_1(t) \\ dy_2(t) \\ \vdots \\ dy_p(t) \end{bmatrix}. \tag{8.9}$$

If we wish to control the large-scale quantum system (8.1) with a decentralized quantum controller, the decentralized control input $d\bar{u}(t)$ also has p components. That is,

$$d\bar{u}(t) = \begin{bmatrix} d\bar{u}_1(t) \\ d\bar{u}_2(t) \\ \vdots \\ d\bar{u}_p(t) \end{bmatrix}. \tag{8.10}$$

Here, $d\bar{u}_j(t)$ is an $n_{u_j} \times 1$ vector of control inputs, which is only dependent on the corresponding $dy_j(t)$ for $j = 1, 2, \ldots, p$. In this case, we do not make any assumption on the structure of the quantum system matrices F, G_2 and H_2 in (8.1). Thus, a decentralized quantum controller can be written as

$$\begin{aligned} d\bar{c}_j(t) &= F_{c_j}\ \bar{c}_j(t)dt + G_{w_{c_{0,j}}}\ dw_{c_{0,j}}(t) + G_{w_{c_{1,j}}}\ dw_{c_{1,j}}(t) + G_{c_j}\ dy_j(t);\ \bar{c}_j(0) = \bar{c}_{0,j}; \\ d\bar{u}_j(t) &= H_{c_j}\ \bar{c}_j(t)dt + dw_{c_{0,j}}(t) \end{aligned} \tag{8.11}$$

where $\bar{c}_j$ is an $n \times 1$ vector of the annihilation operators, and $w_{c_{0,j}}$ and $w_{c_{1,j}}$ are non-commutative quantum Wiener processes. The Ito matrices and commutation matrices of $w_{c_{0,j}}$ and $w_{c_{1,j}}$ are respectively assumed to be

$$\begin{aligned} F_{w_{c_{0,j}}} &= F_{w_{c_{1,j}}} = I; \\ T_{w_{c_{0,j}}} &= T_{w_{c_{1,j}}} = I. \end{aligned} \tag{8.12}$$

At time $t = 0$, it is also assumed that $a(0)$ and $\tilde{a}(0)$ commute with $\bar{c}_j(0)$. Moreover, F_{c_j} is Hurwitz and the decentralized quantum controller (8.11) has a transfer function matrix

$$T_{jj}(s) = H_{c_j}(sI - F_{c_j})^{-1}G_{c_j}. \tag{8.13}$$

8.2.2 Physical realizability and the H^∞ control objective

As mentioned in Chapter 7, the realization $\{F_{c_j}, G_{c_j}, H_{c_j}\}$ of the decentralized quantum controller (8.11) cannot be arbitrarily chosen because it does not necessarily represent a physically realizable quantum dynamical system; see [70,72,73]. Thus, a physical realizability condition for the decentralized quantum controller (8.11) is presented in the following definition and lemma.

Definition 8.1. (*Physical realizability of a decentralized quantum controller*; see [73, Definition 7.1]) The matrices F_{c_j}, G_{c_j} and H_{c_j} are said to define a physically realizable controller of the form (8.11) if there exist matrices $G_{w_{c_{0,j}}}$, $G_{w_{c_{1,j}}}$, $H_{c_{1,j}}$ and $H_{c_{2,j}}$ such that

$$\begin{aligned} d\bar{c}_j(t) &= F_{c_j}\,\bar{c}_j(t)dt + G_{w_{c_{0,j}}}\,dw_{c_{0,j}}(t) + G_{w_{c_{1,j}}}\,dw_{c_{1,j}}(t) + G_{c_j}\,dy_j(t); \\ \begin{bmatrix} d\bar{u}_j(t) \\ d\bar{u}_{1,j}(t) \\ d\bar{u}_{2,j}(t) \end{bmatrix} &= \begin{bmatrix} H_{c_j} \\ H_{c_{1,j}} \\ H_{c_{2,j}} \end{bmatrix} \bar{c}_j(t)dt + \begin{bmatrix} dw_{c_{0,j}}(t) \\ dw_{c_{1,j}}(t) \\ dy_j(t) \end{bmatrix}; \quad \bar{c}_j(0) = \bar{c}_{0,j} \end{aligned} \tag{8.14}$$

is physically realizable according to Definition 7.1 when

$$T_{y_j} := J_{20,j} T_{v_j} J_{20,j}^\dagger + J_{21,j} T_{\nu_j} J_{21,j}^\dagger = I \tag{8.15}$$

for all $j = 1, 2, \ldots, p$. Note that $J_{20,j}$, $J_{21,j}$, T_{v_j} and T_{ν_j} are the j-th partition of J_{20}, J_{21}, T_v and T_ν, which follow the partition of $dy(t)$ in (8.9).

Lemma 8.1. *(see [73, Theorem 7.2]) Suppose that* $\{F_{c_j}, G_{c_j}, H_{c_j}\}$ *is a minimal realization of the decentralized quantum controller* (8.11)*. Then, it is physically realizable if and only if* F_{c_j} *is Hurwitz and* $\|T_{jj}(s)\|_\infty \leq 1$*. This implies that the decentralized quantum controller* (8.11) *is bounded real.*

Remark 8.1. (see [73, Theorem 7.2]) The matrices $G_{w_{c_{1,j}}}$ and $H_{c_{1,j}}$ can be set to zero as the exogenous quantum noise $dw_{c_{1,j}}$ is not needed in the realization of a decentralized quantum controller of the form (8.11).

Applying the decentralized quantum controller (8.11) to the large-scale uncertain quantum system (8.1), (8.8), we obtain a closed loop uncertain quantum system such that the H^∞ control objective:

$$\begin{aligned} &\int_0^t \left\langle z(s)^\dagger z(s) + z(s)^T z(s)^* + \varepsilon\left(\bar{\eta}(s)^\dagger \bar{\eta}(s) + \bar{\eta}(s)^T \bar{\eta}(s)^*\right)\right\rangle ds \\ &\leq (\gamma^2 - \varepsilon^2) \int_0^t \left\langle \beta_w(s)^\dagger \beta_w(s) + \beta_w(s)^T \beta_w(s)^* \right\rangle ds + \pi_1 + \pi_2 t \end{aligned} \tag{8.16}$$

is satisfied for some real constants $\varepsilon, \pi_1, \pi_2 > 0$ with

$$d\bar{\eta}(t) = \begin{bmatrix} da(t)^T & d\bar{c}_1(t)^T & \cdots & d\bar{c}_p(t)^T & d\tilde{a}_1(t)^T & \cdots & d\tilde{a}_k(t)^T \end{bmatrix}^T. \tag{8.17}$$

8.2.3 A special case of quantum robust H^∞ control

The decentralized quantum controller (8.11) can also be considered as a special case of a non-decentralized quantum controller of the form

$$\begin{aligned} dc(t) &= F_c\, c(t)dt + G_{w_{c_0}}\, dw_{c_0}(t) + G_{w_{c_1}}\, dw_{c_1}(t) + G_c\, dy(t); \quad c(0) = c_0; \\ du(t) &= H_c\, c(t)dt + dw_{c_0}(t) \end{aligned} \tag{8.18}$$

where F_c is Hurwitz and which has a transfer function matrix

$$T(s) = H_c(sI - F_c)^{-1}G_c \tag{8.19}$$

with a block-diagonal structure. This special case motivates us to construct the decentralized quantum controller (8.11) based on the general non-decentralized quantum controller (8.18) designed using the algorithm in [73].

Suppose that the general non-decentralized quantum controller (8.18) has a transfer function matrix $T(s)$ as in (8.19) and $T(s)$ is partitioned according to the partition of $d\bar{u}(t)$ and $dy(t)$ in (8.10) and (8.9):

$$T(s) = \begin{bmatrix} T_{11}(s) & T_{12}(s) & \dots & T_{1p}(s) \\ T_{21}(s) & T_{22}(s) & \dots & T_{2p}(s) \\ \vdots & \vdots & \ddots & \vdots \\ T_{p1}(s) & T_{p2}(s) & \dots & T_{pp}(s) \end{bmatrix}. \tag{8.20}$$

Then, the transfer function matrix of the decentralized quantum controller (8.11) can be formed by taking only the block-diagonal parts of $T(s)$: (see [166])

$$\bar{T}(s) = \begin{bmatrix} T_{11}(s) & 0 & \dots & 0 \\ 0 & T_{22}(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T_{pp}(s) \end{bmatrix}. \tag{8.21}$$

As the off-diagonal blocks of $T(s)$ are not included in $\bar{T}(s)$, they are considered as additional uncertainties in the large-scale uncertain quantum system (8.1), (8.8). This is precisely the main idea of our approach where we do not treat the interconnections between quantum subsystems as uncertainties, but rather the neglected off-diagonal blocks of $T(s)$; e.g., see [166]. Thus, a sequence of additional uncertainty transfer function matrices can be defined as follows:

$$\begin{aligned}
\bar{\Delta}_1(s) &:= \begin{bmatrix} T_{12}(s) & T_{13}(s) & \dots & T_{1p}(s) \end{bmatrix};\\
\bar{\Delta}_2(s) &:= \begin{bmatrix} T_{21}(s) & T_{23}(s) & \dots & T_{2p}(s) \end{bmatrix};\\
&\vdots\\
\bar{\Delta}_p(s) &:= \begin{bmatrix} T_{p1}(s) & T_{p2}(s) & \dots & T_{p(p-1)}(s) \end{bmatrix}.
\end{aligned} \tag{8.22}$$

Note that $\bar{\Delta}_j(s)$ $(j = 1, 2, \dots, p)$ is stable because F_c is Hurwitz. The j-th additional uncertainty input $d\bar{\xi}_j(t)$ are then defined as

$$\begin{aligned}
d\bar{a}_j(t) &= \bar{A}_j\, \bar{a}_j(t)dt + \bar{B}_j\, d\bar{\zeta}_j(t);\\
d\bar{\xi}_j(t) &= -\bar{C}_j\, \bar{a}_j(t)dt
\end{aligned} \tag{8.23}$$

where $\bar{A}_j = F_c$, $\bar{B}_j$ and $\bar{C}_j$ are sub-matrices of G_c and H_c in (8.18), respectively, for $j = 1, 2, \dots, p$. Also, the additional uncertainty output $d\bar{\zeta}_j(t)$ are defined as

$$\begin{aligned}
d\bar{\zeta}_1(t) &= \begin{bmatrix} dy_2(t)^T & dy_3(t)^T & \cdots & dy_p(t)^T \end{bmatrix}^T\\
&= M_1 a(t)dt + N_{20,1}dv(t) + N_{21,1}dw(t);\\
d\bar{\zeta}_2(t) &= \begin{bmatrix} dy_1(t)^T & dy_3(t)^T & \cdots & dy_p(t)^T \end{bmatrix}^T\\
&= M_2 a(t)dt + N_{20,2}dv(t) + N_{21,2}dw(t);\\
&\vdots\\
d\bar{\zeta}_p(t) &= \begin{bmatrix} dy_1(t)^T & dy_2(t)^T & \cdots & dy_{(p-1)}(t)^T \end{bmatrix}^T\\
&= M_p a(t)dt + N_{20,p}dv(t) + N_{21,p}dw(t).
\end{aligned} \tag{8.24}$$

Here, $M_1, M_2, \dots, M_p$ are sub-matrices of matrix H_2; $N_{20,1}, N_{20,2}, \dots, N_{20,p}$ are sub-matrices of matrix J_{20}; and $N_{21,1}, N_{21,2}, \dots, N_{21,p}$ are sub-matrices of matrix J_{21}. Then, we can rewrite the decentralized control input $d\bar{u}(t)$ in (8.10) as

$$d\bar{u}(t) = du(t) + \sum_{j=1}^{p} L_j\, d\bar{\xi}_j(t) \tag{8.25}$$

where

$$L_1 = \begin{bmatrix} I_{n_{u_1} \times n_{u_1}} \\ 0_{\tilde{n}_{u_1} \times n_{u_1}} \end{bmatrix}; \quad L_j = \begin{bmatrix} 0_{\bar{n}_{u_{(j-1)}} \times n_{u_j}} \\ I_{n_{u_j} \times n_{u_j}} \\ 0_{\tilde{n}_{u_j} \times n_{u_j}} \end{bmatrix}; \quad L_p = \begin{bmatrix} 0_{\bar{n}_{u_{(p-1)}} \times n_{u_p}} \\ I_{n_{u_p} \times n_{u_p}} \end{bmatrix} \tag{8.26}$$

for $j = 2, 3, \ldots, p-1$. Note that $n_u = \sum_{j=1}^{p} n_{u_j}$; $\bar{n}_{u_j} = \sum_{d=1}^{j} n_{u_d}$; and $\tilde{n}_{u_j} = n_u - \bar{n}_{u_j}$ for $j = 1, 2, \ldots, p$.

8.2.4 An equivalent uncertain linear quantum system

If we apply the decentralized control input $d\bar{u}(t)$ to the large-scale uncertain quantum system (8.1), (8.8), we will obtain the same closed loop system as if we apply the non-decentralized control input $du(t)$ to the following equivalent uncertain linear quantum system:

$$\begin{aligned}
da(t) &= Fa(t)dt + G_0 dv(t) + G_1 dw(t) + G_2 du(t) + \sum_{l=1}^{k} G_{3,l} d\xi_l(t) + \sum_{j=1}^{p} G_2 L_j d\bar{\xi}_j(t); \\
dz(t) &= H_1\, a(t)dt + J_{12}\, du(t) + \sum_{j=1}^{p} J_{12} L_j\, d\bar{\xi}_j(t); \\
d\zeta_1(t) &= P_1\, a(t)dt + Q_1\, du(t) + \sum_{j=1}^{p} Q_1 L_j\, d\bar{\xi}_j(t); \\
&\ \ \vdots \\
d\zeta_k(t) &= P_k\, a(t)dt + Q_k\, du(t) + \sum_{j=1}^{p} Q_k L_j\, d\bar{\xi}_j(t); \\
dy(t) &= H_2\, a(t)dt + J_{20}\, dv(t) + J_{21}\, dw(t)
\end{aligned} \tag{8.27}$$

together with $d\bar{\zeta}_1(t), \ldots, d\bar{\zeta}_p(t)$ as defined in (8.24) and an initial condition $a(0) = a_0$. Moreover, for the j-th additional uncertainty $\bar{\Delta}_j(s)$ as given in (8.22), we define a constant $\beta_j > 0$ so that

$$\|\bar{\Delta}_j(s)\|_\infty^2 \le \beta_j, \quad \forall j = 1, 2, \ldots, p. \tag{8.28}$$

If we apply a coherent quantum robust H^∞ controller of the form (8.18) to the equivalent uncertain quantum system (8.27), (8.24), (8.8), (8.28), the corresponding H^∞ control objective is then given as follows:

$$\begin{aligned}&\int_0^t \left\langle z(s)^\dagger z(s) + z(s)^T z(s)^* + \varepsilon\left(\tilde{\eta}(s)^\dagger \tilde{\eta}(s) + \tilde{\eta}(s)^T \tilde{\eta}(s)^*\right)\right\rangle ds \\ &\leq (\gamma^2 - \varepsilon^2)\int_0^t \left\langle \beta_w(s)^\dagger \beta_w(s) + \beta_w(s)^T \beta_w(s)^*\right\rangle ds + \pi_1 + \pi_2 t \end{aligned} \tag{8.29}$$

where $\varepsilon, \pi_1, \pi_2 > 0$ are real constants and

$$d\tilde{\eta}(t) = \begin{bmatrix} da(t)^T & dc(t)^T & d\tilde{a}_1(t)^T & \cdots & d\tilde{a}_k(t)^T & d\bar{a}_1(t)^T & \cdots & d\bar{a}_p(t)^T \end{bmatrix}^T. \tag{8.30}$$

Here, $\bar{a}_j(t)$ is a vector of the annihilation operators corresponding to the additional uncertainties defined in (8.23) for $j = 1, 2, \ldots, p$.

8.3 Ordinary Decentralized Quantum H^∞ Controller

In this section, we present an algorithm to synthesize a decentralized quantum H^∞ controller as formulated in Section 8.2. Here, we only provide sufficient conditions because particular additional uncertainties in (8.28) are involved in our approach. To compute required design parameters, we apply an evolutionary optimization method, namely the differential evolution (DE) algorithm. Also, an example is considered to demonstrate the proposed algorithm.

8.3.1 Synthesis algorithm

Referring to Lemma 7.8, we then introduce two sets of scaling constants $\tau_1 > 0, \ldots, \tau_k > 0$ and $\delta_1 > 0, \ldots, \delta_p > 0$ corresponding to the structured uncertainties (8.8) and the additional uncertainties (8.28), respectively. Thus, we can rewrite the uncertain quantum system (8.27), (8.24), (8.8), (8.28) as

$$\begin{aligned} da(t) &= F\,a(t)dt + G_0\,dv(t) + \tilde{G}_1\,d\tilde{w}(t) + G_2\,du(t); \quad a(0) = a_0; \\ d\tilde{z}(t) &= \tilde{H}_1\,a(t)dt + \tilde{J}_{10}\,dv(t) + \tilde{J}_{11}\,d\tilde{w}(t) + \tilde{J}_{12}\,du(t); \\ dy(t) &= H_2\,a(t)dt + J_{20}\,dv(t) + \tilde{J}_{21}\,d\tilde{w}(t) \end{aligned} \tag{8.31}$$

where $d\tilde{w}(t) = \tilde{\beta}_w(t)dt + d\tilde{\nu}(t)$;

$$d\tilde{w}(t) = \begin{bmatrix} \gamma\, dw(t) \\ \sqrt{\tau_1}\, d\xi_1(t) \\ \vdots \\ \sqrt{\tau_k}\, d\xi_k(t) \\ \sqrt{\delta_1}\, d\bar{\xi}_1(t) \\ \vdots \\ \sqrt{\delta_p}\, d\bar{\xi}_p(t) \end{bmatrix}; \quad d\tilde{z}(t) = \begin{bmatrix} dz(t) \\ \sqrt{\tau_1}\, d\zeta_1(t) \\ \vdots \\ \sqrt{\tau_k}\, d\zeta_k(t) \\ \sqrt{\delta_1\beta_1}\, d\bar{\zeta}_1(t) \\ \vdots \\ \sqrt{\delta_p\beta_p}\, d\bar{\zeta}_p(t) \end{bmatrix};$$

$$\tilde{G}_1 = \begin{bmatrix} \gamma^{-1}G_1 & \sqrt{\tau_1}^{-1}\, G_{3,1} & \cdots & \sqrt{\tau_k}^{-1}\, G_{3,k} & \sqrt{\delta_1}^{-1}\, G_2L_1 & \cdots & \sqrt{\delta_p}^{-1}\, G_2L_p \end{bmatrix};$$

$$\tilde{H}_1 = \begin{bmatrix} H_1 \\ \sqrt{\tau_1}\, P_1 \\ \vdots \\ \sqrt{\tau_k}\, P_k \\ \sqrt{\delta_1\beta_1}\, M_1 \\ \vdots \\ \sqrt{\delta_p\beta_p}\, M_p \end{bmatrix}; \quad \tilde{J}_{10} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \sqrt{\delta_1\beta_1}\, N_{20,1} \\ \vdots \\ \sqrt{\delta_p\beta_p}\, N_{20,p} \end{bmatrix}; \quad \tilde{J}_{12} = \begin{bmatrix} J_{12} \\ \sqrt{\tau_1}\, Q_1 \\ \vdots \\ \sqrt{\tau_k}\, Q_k \\ 0 \\ \vdots \\ 0 \end{bmatrix};$$

$$\tilde{J}_{11} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \sqrt{\delta_1}^{-1}\, J_{12}L_1 & \cdots & \sqrt{\delta_p}^{-1}\, J_{12}L_p \\ 0 & 0 & \cdots & 0 & \sqrt{\tau_1/\delta_1}\, Q_1L_1 & \cdots & \sqrt{\tau_1/\delta_p}\, Q_1L_p \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \sqrt{\tau_k/\delta_1}\, Q_kL_1 & \cdots & \sqrt{\tau_k/\delta_p}\, Q_kL_p \\ \gamma^{-1}\sqrt{\delta_1\beta_1}\, N_{21,1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \gamma^{-1}\sqrt{\delta_p\beta_p}\, N_{21,p} & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix};$$

$$\tilde{J}_{21} = \begin{bmatrix} \gamma^{-1}J_{21} & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{8.32}$$

If we apply a coherent quantum robust H^∞ controller of the form (8.18) to the quantum system (8.31), the corresponding H^∞ control objective is

$$\begin{aligned} &\int_0^t \left\langle \tilde{z}(s)^\dagger \tilde{z}(s) + \tilde{z}(s)^T \tilde{z}(s)^* + \varepsilon\left(\tilde{\eta}(s)^\dagger \tilde{\eta}(s) + \tilde{\eta}(s)^T \tilde{\eta}(s)^*\right)\right\rangle ds \\ &\leq (1-\varepsilon^2)\int_0^t \left\langle \tilde{\beta}_w(s)^\dagger \tilde{\beta}_w(s) + \tilde{\beta}_w(s)^T \tilde{\beta}_w(s)^* \right\rangle ds + \pi_1 + \pi_2 t \end{aligned} \tag{8.33}$$

where $\varepsilon, \pi_1, \pi_2 > 0$ are real constants. Moreover, since the QSDEs (8.31) are not in a standard H^∞ form, we apply a loop shifting transformation to eliminate the $\tilde{J}_{11}$ term from (8.31); e.g., see [16, Sections 4.5.1 and 5.5.1] and [18, Section 17.2]. Thus, it is necessary to impose the following assumption:

Assumption 8.1. Given constants $\tau_1 > 0, \ldots, \tau_k > 0$, $\delta_1 > 0, \ldots, \delta_p > 0$, $\beta_1 > 0, \ldots, \beta_p > 0$, the uncertain quantum system (8.27), (8.24), (8.8), (8.28) is assumed to be such that $\tilde{J}_{11}\tilde{J}_{11}^\dagger < I$.

Now, we can define

$$\Phi := I - \tilde{J}_{11}^\dagger \tilde{J}_{11} > 0; \quad \tilde{\Phi} := I - \tilde{J}_{11}\tilde{J}_{11}^\dagger > 0 \tag{8.34}$$

and also

$$\begin{aligned} d\hat{w}(t) &:= \Phi^{\frac{1}{2}} d\tilde{w}(t) - \Phi^{-\frac{1}{2}} \tilde{J}_{11}^\dagger \left[\tilde{H}_1\, a(t)\, dt + \tilde{J}_{12}\, du(t)\right]; \\ d\hat{z}(t) &:= \tilde{\Phi}^{-\frac{1}{2}} \left[\tilde{H}_1\, a(t)\, dt + \tilde{J}_{12}\, du(t)\right]. \end{aligned} \tag{8.35}$$

Using (8.35), it is straightforward to verify that

$$\begin{aligned} d\tilde{w}(t) &= \Phi^{-\frac{1}{2}} d\hat{w}(t) + \Phi^{-1} \tilde{J}_{11}^\dagger \left[\tilde{H}_1\, a(t)\, dt + \tilde{J}_{12}\, du(t)\right]; \\ \|\tilde{w}(t)\|_2^2 - \|\tilde{z}(t)\|_2^2 &\equiv \|\hat{w}(t)\|_2^2 - \|\hat{z}(t)\|_2^2 . \end{aligned} \tag{8.36}$$

Therefore, we can rewrite the QSDEs (8.31) as

$$\begin{aligned} da(t) &= \hat{F}\, a(t)dt + G_0\, dv(t) + \hat{G}_1\, d\hat{w}(t) + \hat{G}_2\, du(t); \quad a(0) = a_0; \\ d\hat{z}(t) &= \hat{H}_1\, a(t)dt + \hat{J}_{12}\, du(t); \\ dy(t) &= \hat{H}_2\, a(t)dt + J_{20}\, dv(t) + \hat{J}_{21}\, d\hat{w}(t) + \hat{J}_{22}\, du(t) \end{aligned} \tag{8.37}$$

where $d\hat{w}(t) = \hat{\beta}_w(t)dt + d\hat{\nu}(t)$;

$$\begin{array}{rclrcl} \hat{F} &=& F + \tilde{G}_1 \tilde{J}_{11}^\dagger \tilde{\Phi}^{-1} \tilde{H}_1; & \hat{G}_1 &=& \tilde{G}_1 \Phi^{-\frac{1}{2}}; \\ \hat{G}_2 &=& G_2 + \tilde{G}_1 \tilde{J}_{11}^\dagger \tilde{\Phi}^{-1} \tilde{J}_{12}; & \hat{H}_1 &=& \tilde{\Phi}^{-\frac{1}{2}} \tilde{H}_1; \\ \hat{H}_2 &=& H_2 + \tilde{J}_{21} \tilde{J}_{11}^\dagger \tilde{\Phi}^{-1} \tilde{H}_1; & \hat{J}_{12} &=& \tilde{\Phi}^{-\frac{1}{2}} \tilde{J}_{12}; \\ \hat{J}_{22} &=& \tilde{J}_{21} \tilde{J}_{11}^\dagger \tilde{\Phi}^{-1} \tilde{J}_{12} & \hat{J}_{21} &=& \tilde{J}_{21} \Phi^{-\frac{1}{2}}. \end{array} \tag{8.38}$$

Note that the $\tilde{J}_{10}$ term is automatically eliminated when we apply the loop shifting transformation to the quantum system (8.31). Furthermore, defining

$$d\hat{y}(t) := dy(t) - \hat{J}_{22}\, du(t), \tag{8.39}$$

we can rewrite (8.37) as

$$\begin{aligned} da(t) &= \hat{F}\, a(t)dt + G_0\, dv(t) + \hat{G}_1\, d\hat{w}(t) + \hat{G}_2\, du(t); \quad a(0) = a_0; \\ d\hat{z}(t) &= \hat{H}_1\, a(t)dt + \hat{J}_{12}\, du(t); \\ d\hat{y}(t) &= \hat{H}_2\, a(t)dt + J_{20}\, dv(t) + \hat{J}_{21}\, d\hat{w}(t). \end{aligned} \tag{8.40}$$

The H^∞ control objective corresponding to the quantum system (8.40) is as follows:

$$\begin{aligned} &\int_0^t \left\langle \hat{z}(s)^\dagger \hat{z}(s) + \hat{z}(s)^T \hat{z}(s)^* + \varepsilon \left(\tilde{\eta}(s)^\dagger \tilde{\eta}(s) + \tilde{\eta}(s)^T \tilde{\eta}(s)^* \right) \right\rangle ds \\ &\leq (1 - \varepsilon^2) \int_0^t \left\langle \hat{\beta}_w(s)^\dagger \hat{\beta}_w(s) + \hat{\beta}_w(s)^T \hat{\beta}_w(s)^* \right\rangle ds + \pi_1 + \pi_2 t \end{aligned} \tag{8.41}$$

where $\varepsilon, \pi_1, \pi_2 > 0$ are real constants.

The solution to the coherent quantum H^∞ control problem for the quantum system (8.40) involves the solutions to the parameterized complex algebraic Riccati equations:

$$\begin{aligned} &\left(\hat{F} - \hat{G}_2 \hat{E}_1^{-1} \hat{J}_{12}^\dagger \hat{H}_1\right)^\dagger X + X \left(\hat{F} - \hat{G}_2 \hat{E}_1^{-1} \hat{J}_{12}^\dagger \hat{H}_1\right) \\ &\qquad + X \left(\hat{G}_1 \hat{G}_1^\dagger - \hat{G}_2 \hat{E}_1^{-1} \hat{G}_2^\dagger\right) X + \hat{H}_1^\dagger \left(I - \hat{J}_{12} \hat{E}_1^{-1} \hat{J}_{12}^\dagger\right) \hat{H}_1 = 0\,; \\ &\left(\hat{F} - \hat{G}_1 \hat{J}_{21}^\dagger \hat{E}_2^{-1} \hat{H}_2\right) Y + Y \left(\hat{F} - \hat{G}_1 \hat{J}_{21}^\dagger \hat{E}_2^{-1} \hat{H}_2\right)^\dagger \\ &\qquad + Y \left(\hat{H}_1^\dagger \hat{H}_1 - \hat{H}_2^\dagger \hat{E}_2^{-1} \hat{H}_2\right) Y + \hat{G}_1 \left(I - \hat{J}_{21}^\dagger \hat{E}_2^{-1} \hat{J}_{21}\right) \hat{G}_1^\dagger = 0 \end{aligned} \tag{8.42}$$

such that the following conditions hold:

1. $\hat{F} - \hat{G}_2 \hat{E}_1^{-1} \hat{J}_{12}^\dagger \hat{H}_1 + \left(\hat{G}_1 \hat{G}_1^\dagger - \hat{G}_2 \hat{E}_1^{-1} \hat{G}_2^\dagger\right) X$ is Hurwitz;
2. $\hat{F} - \hat{G}_1 \hat{J}_{21}^\dagger \hat{E}_2^{-1} \hat{H}_2 + Y \left(\hat{H}_1^\dagger \hat{H}_1 - \hat{H}_2^\dagger \hat{E}_2^{-1} \hat{H}_2\right)$ is Hurwitz;
3. The spectral radius $\rho(XY)$ of matrix XY is strictly less than one.

To solve the Riccati equations (8.42), we impose the following assumption:

Assumption 8.2. Given constants $\tau_1 > 0, \ldots, \tau_k > 0$, $\delta_1 > 0, \ldots, \delta_p > 0$, $\beta_1 > 0, \ldots, \beta_p > 0$, the equivalent uncertain quantum system (8.27), (8.24), (8.8), (8.28) is assumed to be such that $\hat{E}_1 = \hat{J}_{12}^\dagger \hat{J}_{12} > 0$ and $\hat{E}_2 = \hat{J}_{21}\hat{J}_{21}^\dagger > 0$.

Theorem 8.1. *Let $\beta_1 > 0, \ldots, \beta_p > 0$ be given constants and suppose that there exist $\tau_1 > 0, \ldots, \tau_k > 0$, $\delta_1 > 0, \ldots, \delta_p > 0$ such that Assumption 8.1 and Assumption 8.2 hold, and the complex algebraic Riccati equations* (8.42) *have stabilizing solutions $X \geq 0$ and $Y \geq 0$ such that the spectral radius $\rho(XY) < 1$. Then, the closed loop uncertain quantum system obtained by applying the quantum controller* (8.18) *with*

$$\begin{aligned}
F_c &= \hat{F}_c - G_c \hat{J}_{22} H_c; \\
\hat{F}_c &= \hat{F} + \hat{G}_2 H_c - G_c \hat{H}_2 + \left(\hat{G}_1 - G_c \hat{J}_{21}\right) \hat{G}_1^\dagger X; \\
G_c &= (I - YX)^{-1} \left(Y \hat{H}_2^\dagger + \hat{G}_1 \hat{J}_{21}^\dagger\right) E_2^{-1}; \\
H_c &= -\hat{E}_1^{-1} \left(\hat{G}_2^\dagger X + \hat{J}_{12}^\dagger \hat{H}_1\right)
\end{aligned} \tag{8.43}$$

to the equivalent uncertain quantum system (8.27), (8.24), (8.8), (8.28) *is strictly bounded real with disturbance attenuation $\gamma > 0$.*

Proof. It follows from a loop shifting arguments in the classical H^∞ control theory (e.g., see [16, Sections 4.5.1 and 5.5.1] and [18, Section 17.2]) that the H^∞ quantum control problem (8.31), (8.33) has a solution if and only if the complex Riccati equations (8.42) have stabilizing solutions $X \geq 0$ and $Y \geq 0$ such that $\rho(XY) < 1$. Moreover, a coherent quantum controller of the form (8.18) solving the H^∞ quantum control problem (8.31), (8.33) is defined by (8.43). Thus, if the conditions of the theorem are satisfied, it follows from the arguments in the proofs of Theorem 4.1 in [187] and of Theorem 7.1 [73] (Lemma 7.6) that the closed loop uncertain quantum system obtained by applying the quantum controller (8.18), (8.43) to the equivalent uncertain quantum system (8.27), (8.24), (8.8), (8.28) is strictly bounded real with disturbance attenuation $\gamma > 0$. □

Theorem 8.2. *Let $\beta_1 > 0, \ldots, \beta_p > 0$ be given constants and suppose that there exist $\tau_1 > 0, \ldots, \tau_k > 0$, $\delta_1 > 0, \ldots, \delta_p > 0$ such that Assumption 8.1 and Assumption 8.2 hold, and the complex algebraic Riccati equations* (8.42) *have stabilizing solutions $X \geq 0$ and $Y \geq 0$ such that the spectral radius $\rho(XY) < 1$. Furthermore, suppose that the coherent quantum H^∞ controller* (8.18), (8.43) *is such* *that the transfer function matrices in* (8.22) *satisfy the norm bounds in* (8.28) *and each corresponding decentralized quantum controller* (8.11) *is physically realizable. Then, the closed loop uncertain quantum system obtain by applying the decentralized coherent quantum controller* (8.11) *to the uncertain quantum system* (8.1), (8.8) *is strictly bounded real with disturbance attenuation* $\gamma > 0$.

Proof. If all conditions of the theorem are satisfied, it then follows from Theorem 8.1 that a closed loop uncertain quantum system obtained by applying the quantum controller (8.18), (8.43) to the equivalent uncertain quantum system (8.27), (8.24), (8.8), (8.28) is strictly bounded real with disturbance attenuation $\gamma > 0$. We also assume that the quantum controller (8.18), (8.43) is such that the transfer function matrices in (8.22) satisfy the norm bounds in (8.28). Furthermore, as mentioned in the construction of the equivalent uncertain quantum system (8.27), (8.24), (8.8), (8.28), a closed loop uncertain quantum system obtained by applying the decentralized quantum controller (8.11) (for all $j = 1, 2, \ldots, p$) to the uncertain quantum system (8.1), (8.6) is identical to a closed loop uncertain quantum system obtained by applying the quantum controller (8.18), (8.43) to the equivalent uncertain quantum system (8.27), (8.24), (8.8), (8.28) when the additional uncertainty inputs defined in (8.23) are applied. Hence, it follows that the decentralized quantum controller (8.11) defined by (8.18), (8.43) is such that the resulting closed loop uncertain quantum system is strictly bounded real with disturbance attenuation $\gamma > 0$. □

8.3.2 A differential evolution approach

The decentralized quantum H^∞ controller synthesis algorithm described in sub-Section 8.3.1 involves nonconvex nonlinear constraints, which are often difficult to satisfy. Thus, we reformulate the decentralized controller design problem into a constrained optimization problem, which is then solved using the differential evolution (DE) algorithm as presented in Chapter 2.

The required design parameters form a vector of decision variables defined as

$$\vartheta := \begin{bmatrix} \gamma & \tau_1 & \cdots & \tau_k & \delta_1 & \cdots & \delta_p & \beta_1 & \cdots & \beta_p \end{bmatrix}^T \tag{8.44}$$

where the dimension of ϑ is $k + 2p + 1$ and all element of ϑ are positive real numbers. Then, we define the optimization problem in terms of ϑ as follows:

Find an optimal solution $\vartheta^\star$ to solve

$$\min_{\vartheta} \mathsf{f}(\vartheta) \tag{8.45}$$

subject to

$$\mathsf{g_j}(\vartheta) = 0; \quad \mathsf{h_k}(\vartheta) \le 0 \tag{8.46}$$

for $\mathsf{j} = 1, 2, \ldots, \mathsf{a}$ and $\mathsf{k} = 1, 2, \ldots, \mathsf{b}$. Here, a and b are the total number of equality and inequality constraints, respectively; and $\mathsf{f}(\vartheta)$ is an objective function to be minimized.

A suitable objective function in relation to the decentralized quantum H^∞ controller design problem is

$$\mathsf{f}(\vartheta) = \mathsf{m}_0\gamma^{\mathsf{n}_0} + \sum_{j=1}^{p} \mathsf{m}_j \beta_j^{\mathsf{n}_j} \tag{8.47}$$

where $\mathsf{m}_0, \mathsf{m}_j \ge 1$ are weighting factors and $\mathsf{n}_0, \mathsf{n}_j \ge 1$ are power constants. Moreover, the equality constraints are

$$\begin{aligned}
\mathsf{g}_1(\vartheta) &= \left(\hat{F} - \hat{G}_2\hat{E}_1^{-1}\hat{J}_{12}^\dagger\hat{H}_1\right)^\dagger X + X\left(\hat{F} - \hat{G}_2\hat{E}_1^{-1}\hat{J}_{12}^\dagger\hat{H}_1\right) \\
&\quad + X\left(\hat{G}_1\hat{G}_1^\dagger - \hat{G}_2\hat{E}_1^{-1}\hat{G}_2^\dagger\right)X + \hat{H}_1^\dagger\left(I - \hat{J}_{12}\hat{E}_1^{-1}\hat{J}_{12}^\dagger\right)\hat{H}_1 = 0; \\
\mathsf{g}_2(\vartheta) &= \left(\hat{F} - \hat{G}_1\hat{J}_{21}^\dagger\hat{E}_2^{-1}\hat{H}_2\right)Y + Y\left(\hat{F} - \hat{G}_1\hat{J}_{21}^\dagger\hat{E}_2^{-1}\hat{H}_2\right)^\dagger \\
&\quad + Y\left(\hat{H}_1^\dagger\hat{H}_1 - \hat{H}_2^\dagger\hat{E}_2^{-1}\hat{H}_2\right)Y + \hat{G}_1\left(I - \hat{J}_{21}^\dagger\hat{E}_2^{-1}\hat{J}_{21}\right)\hat{G}_1^\dagger = 0
\end{aligned} \tag{8.48}$$

and the inequaltiy constraints are

$$\begin{array}{rclrcl}
\mathsf{h}_1(\vartheta) &=& \tilde{J}_{11}\tilde{J}_{11}^\dagger - I < 0; & \mathsf{h}_2(\vartheta) &=& -\hat{E}_1 < 0; \\
\mathsf{h}_3(\vartheta) &=& -\hat{E}_2 < 0; & \mathsf{h}_4(\vartheta) &=& -X < 0; \\
\mathsf{h}_5(\vartheta) &=& -Y < 0; & \mathsf{h}_6(\vartheta) &=& \rho\left(XY\right) - 1 < 0; \\
\mathsf{h}_7(\vartheta) &=& e_{\max,r}\left(\mathcal{A}_X\right) < 0; & \mathsf{h}_8(\vartheta) &=& e_{\max,r}\left(\mathcal{A}_Y\right) < 0; \\
\mathsf{h}_9(\vartheta) &=& e_{\max,r}\left(F_c\right) < 0; & \mathsf{h}_{10,j}(\vartheta) &=& \|\bar{\Delta}_j(s)\|_\infty^2 - \beta_j \le 0; \\
\mathsf{h}_{11,j}(\vartheta) &=& \|T_{jj}(s)\|_\infty - 1 \le 0 & & &
\end{array} \tag{8.49}$$

for $j = 1, 2, \ldots, p$. Note that $\rho(\mathcal{M})$ and $e_{\max,r}(\mathcal{M})$ denote the spectral radius and the largest real part of the eigenvalues of the matrix $\mathcal{M}$, respectively. Moreover, we define $\mathcal{A}_X$ and $\mathcal{A}_Y$ as follows:

$$\begin{aligned}\mathcal{A}_X &:= \hat{F} - \hat{G}_2\hat{E}_1^{-1}\hat{J}_{12}^\dagger\hat{H}_1 + \left(\hat{G}_1\hat{G}_1^\dagger - \hat{G}_2\hat{E}_1^{-1}\hat{G}_2^\dagger\right)X;\\ \mathcal{A}_Y &:= \hat{F} - \hat{G}_1\hat{J}_{21}^\dagger\hat{E}_2^{-1}\hat{H}_2 + Y\left(\hat{H}_1^\dagger\hat{H}_1 - \hat{H}_2^\dagger\hat{E}_2^{-1}\hat{H}_2\right).\end{aligned} \tag{8.50}$$

Having the equality and inequality constraints in (8.48) and (8.49), we now develop a fitness test routine to rate the fitness of each candidate solution with respect to those constraints. This routine proceeds as follows:

1. Compute the eigenvalues of $(\tilde{J}_{11}\tilde{J}_{11}^\dagger - I)$, $\hat{E}_1$ and $\hat{E}_2$ to verify if $\mathsf{h}_1(\vartheta)$, $\mathsf{h}_2(\vartheta)$ and $\mathsf{h}_3(\vartheta)$ hold.

2. Evaluate $\mathsf{g}_1(\vartheta)$ and $\mathsf{g}_2(\vartheta)$ to obtain solutions X and Y to the Riccati equations in (8.42).

3. Verify if X and Y are stabilizing positive definite solutions through the evaluation of $\mathsf{h}_4(\vartheta)$, $\mathsf{h}_5(\vartheta)$, $\mathsf{h}_7(\vartheta)$ and $\mathsf{h}_8(\vartheta)$.

4. Compute the spectral radius of the product XY to verify if $\mathsf{h}_6(\vartheta)$ holds.

5. Evaluate $\mathsf{h}_9(\vartheta)$ to check if F_c is Hurwitz.

6. Evaluate $\mathsf{h}_{10,j}(\vartheta)$ (for $j = 1, 2, \ldots, p$) to check if the j-th off-diagonal block of $T(s)$ in (8.20) satisfies the norm bound condition in (8.28).

7. Verify if the H^∞ norm $\|T_{jj}(s)\|_\infty$ of the j-th diagonal block of $T(s)$ is less than or equal to one by evaluating $\mathsf{h}_{11,j}(\vartheta)$ (for $j = 1, 2, \ldots, p$).

8. Evaluate the objective function $\mathsf{f}(\vartheta)$ in (8.47).

Through the fitness test, we acquire information for each candidate solution about how many constraint violations have occurred and how much penalty has been incurred. Penalty functions corresponding to the violation of the equality and inequality constraints in (8.48) and (8.49) are then defined as follows:

$$\begin{aligned}\mathsf{p}_1(\vartheta) &= e_{\max}(\tilde{J}_{11}\tilde{J}_{11}^\dagger - I)^{\mathsf{s}_1}; & \mathsf{p}_2(\vartheta) &= |e_{\min}(\hat{E}_1)|^{\mathsf{s}_2};\\ \mathsf{p}_3(\vartheta) &= |e_{\min}(\hat{E}_2)|^{\mathsf{s}_3}; & \mathsf{p}_4(\vartheta) &= \rho(\mathcal{C}_X)^{\mathsf{s}_4};\\ \mathsf{p}_5(\vartheta) &= \rho(\mathcal{C}_Y)^{\mathsf{s}_5}; & \mathsf{p}_6(\vartheta) &= |e_{\min}(X)|^{\mathsf{s}_6};\\ \mathsf{p}_7(\vartheta) &= |e_{\min}(Y)|^{\mathsf{s}_7}; & \mathsf{p}_8(\vartheta) &= e_{\max,r}\left(\mathcal{A}_X\right)^{\mathsf{s}_8};\end{aligned} \tag{8.51}$$

$$
\begin{array}{rclrcl}
\mathsf{p}_9(\vartheta) & = & e_{\max,r}\left(\mathcal{A}_Y\right)^{\mathsf{s}_9}; & \mathsf{p}_{10}(\vartheta) & = & (\rho(XY)-1)^{\mathsf{s}_{10}}; \\
\mathsf{p}_{11}(\vartheta) & = & e_{\max,r}\left(F_c\right)^{\mathsf{s}_{11}}; & \mathsf{p}_{12}(\vartheta) & = & \sum_{j=1}^{p}\mathcal{D}_j^{\mathsf{c}_j}; \\
\mathsf{p}_{13}(\vartheta) & = & \sum_{j=1}^{p}\mathcal{S}_j^{\mathsf{d}_j}; & \mathsf{p}_{14}(\vartheta) & = & \mathsf{f}(\vartheta)
\end{array}
\tag{8.52}
$$

where $\mathsf{s}_\mathsf{r} \geq 1$ for $\mathsf{r} = 1, 2, \ldots, 11$ and $\mathsf{c}_j, \mathsf{d}_j \geq 1$ for $j = 1, 2, \ldots, p$. Note that $e_{\min}(\mathcal{M})$ and $e_{\max}(\mathcal{M})$ denote the smallest and the largest eigenvalue of a Hermitian matrix $\mathcal{M}$, respectively. If the matrix $\mathcal{M}$ is required to be positive definite, we assign $|e_{\min}(\mathcal{M})|^{\mathsf{s}_\mathsf{r}}$ as a penalty. This is because when this requirement is violated, the matrix $\mathcal{M}$ can be either negative (semi)definite or indefinite. Moreover, we define $\mathcal{C}_X$, $\mathcal{C}_Y$, $\mathcal{D}_j$ and $\mathcal{S}_j$ as follows:

$$
\begin{aligned}
\mathcal{C}_X &:= \hat{H}_1^\dagger\left(I - \hat{J}_{12}\hat{E}_1^{-1}\hat{J}_{12}^\dagger\right)\hat{H}_1; \\
\mathcal{C}_Y &:= \hat{G}_1\left(I - \hat{J}_{21}^\dagger\hat{E}_2^{-1}\hat{J}_{21}\right)\hat{G}_1^\dagger; \\
\mathcal{D}_j &:= \begin{cases} \|\bar{\Delta}_j(s)\|_\infty^2, & \text{if } \mathsf{h}_{10,j}(\vartheta) \text{ is violated;} \\ 0, & \text{otherwise;} \end{cases} \\
\mathcal{S}_j &:= \begin{cases} \|T_{jj}(s)\|_\infty, & \text{if } \mathsf{h}_{11,j}(\vartheta) \text{ is violated;} \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
\tag{8.53}
$$

8.3.3 An illustrative example

We present an example to demonstrate the decentralized quantum H^∞ controller design method presented in sub-Section 8.3.1. This example belongs to a class of quantum optical systems, which only consist of passive elements; e.g., see [70,73]. In this case, we consider a decentralized quantum control problem for a cascaded linear quantum system of two identical optical cavities as shown in Figure 8.1.

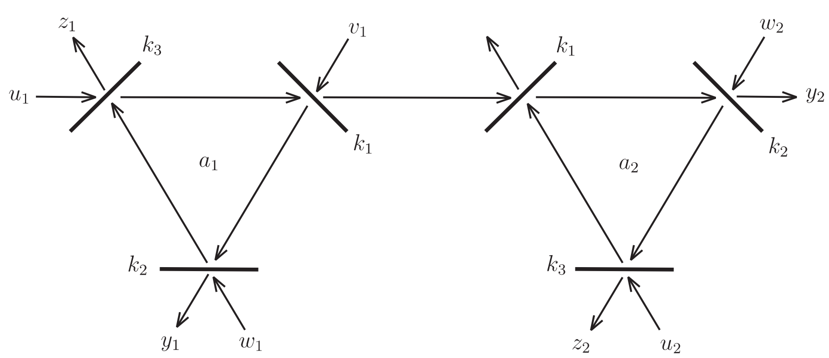


Figure 8.1: A cascaded linear quantum system of two identical optical cavities.

This quantum system can be represented as

$$\begin{aligned}
\begin{bmatrix} da_1(t) \\ da_2(t) \end{bmatrix} &= \begin{bmatrix} -\frac{g}{2} & 0 \\ -k_1 & -\frac{g}{2} \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} dt + \begin{bmatrix} -\sqrt{k_1} \\ -\sqrt{k_1} \end{bmatrix} dv_1(t) + \begin{bmatrix} -\sqrt{k_2} & 0 \\ 0 & -\sqrt{k_2} \end{bmatrix} \begin{bmatrix} dw_1(t) \\ dw_2(t) \end{bmatrix} \\
&\quad + \begin{bmatrix} -\sqrt{k_3} & 0 \\ 0 & -\sqrt{k_3} \end{bmatrix} \begin{bmatrix} du_1(t) \\ du_2(t) \end{bmatrix}; \\
\begin{bmatrix} dz_1(t) \\ dz_2(t) \end{bmatrix} &= \begin{bmatrix} \sqrt{k_3} & 0 \\ 0 & \sqrt{k_3} \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} du_1(t) \\ du_2(t) \end{bmatrix}; \\
\begin{bmatrix} dy_1(t) \\ dy_2(t) \end{bmatrix} &= \begin{bmatrix} \sqrt{k_2} & 0 \\ 0 & \sqrt{k_2} \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dw_1(t) \\ dw_2(t) \end{bmatrix}
\end{aligned} \tag{8.54}$$

where $k_1 = 2.6$, $k_2 = 0.2$, $k_3 = 0.2$ and $g = k_1 + k_2 + k_3$. We assume that the quantum plant (8.54) is known and hence, does not have uncertainty.

Applying the DE approach presented in sub-Section 8.3.2, we obtain

$$\begin{aligned}
\gamma &= 0.3381; \quad \delta_1 = 862.7180; \quad \delta_2 = 1.0011; \\
\beta_1 &= 0.0001; \quad \beta_2 = 0.1073
\end{aligned} \tag{8.55}$$

which are used to construct a non-decentralized coherent quantum robust H^∞ controller defined by the following matrices:

$$\begin{aligned}
F_c &= \begin{bmatrix} -1.1287 & 0.0004 \\ -2.8850 & -1.3929 \end{bmatrix}; \quad G_c = \begin{bmatrix} -0.4472 & 0.0000 \\ 0.0009 & -0.4464 \end{bmatrix}; \\
H_c &= \begin{bmatrix} -0.3830 & -0.0009 \\ -0.0009 & -0.3969 \end{bmatrix}.
\end{aligned} \tag{8.56}$$

The matrices G_c and H_c in (8.56) are such that the squared H^∞ norm of the additional uncertainties $\bar{\Delta}_1(s)$ and $\bar{\Delta}_2(s)$ (as defined in (8.22)):

$$\|\bar{\Delta}_1(s)\|_\infty^2 = 1.0501 \times 10^{-7}; \quad \|\bar{\Delta}_2(s)\|_\infty^2 = 0.1059 \tag{8.57}$$

are indeed less than β_1 and β_2 in (8.55), respectively. Thus, the decentralized quantum controllers are then defined by

$$F_{c_1} = F_c; \quad G_{c_1} = \begin{bmatrix} -0.4472 \\ 0.0009 \end{bmatrix}; \quad H_{c_1} = \begin{bmatrix} -0.3830 & -0.0009 \end{bmatrix}; \tag{8.58}$$

$$F_{c_2} = F_c; \quad G_{c_2} = \begin{bmatrix} 0.0000 \\ -0.4464 \end{bmatrix}; \quad H_{c_2} = \begin{bmatrix} -0.0009 & -0.3969 \end{bmatrix}. \tag{8.59}$$

The decentralized quantum controllers (8.58) and (8.59) are physically realizable because F_c is Hurwitz with eigenvalues: (see Lemma 8.1)

$$e_1 = -1.1329; \quad e_2 = -1.3888 \tag{8.60}$$

and they also have the H^∞ norms as follows:

$$\|T_{11}(s)\|_\infty = 0.1509; \quad \|T_{22}(s)\|_\infty = 0.1271. \tag{8.61}$$

Following the same steps as in (7.107) and (7.108), we are able to compute $G_{w_{c0,1}}$, $G_{w_{c0,2}}$, $H_{c2,1}$ and $H_{c2,2}$ in order to physically realize the decentralized quantum controllers in (8.58) and (8.59) (see Definition 8.1). That is,

$$\begin{aligned} G_{w_{c0,1}} &= \begin{bmatrix} -176.0655 \\ 81858.5874 \end{bmatrix}; \quad H_{c2,1} = \begin{bmatrix} 0.0291 & 0.0001 \end{bmatrix}; \\ G_{w_{c0,2}} &= \begin{bmatrix} 4.9498 \\ 12.6664 \end{bmatrix}; \quad H_{c2,2} = \begin{bmatrix} -0.0290 & 0.0253 \end{bmatrix}. \end{aligned} \tag{8.62}$$

Meanwhile, $G_{w_{c1,1}}$, $H_{c1,1}$ and $G_{w_{c1,2}}$, $H_{c1,2}$ are set to zero because we do not need exogenous quantum noise $dw_{c1,1}$ and $dw_{c1,2}$ terms in the realization of the decentralized quantum controllers in (8.58) and (8.59).

A particular physical realization of a class of linear quantum optical systems has been discussed in [77]. It presents an algorithm to construct a physically realizable coherent linear quantum feedback controller using purely passive optical devices such as optical cavities, beam-splitters and phase shifters. Applying the algorithm in [77], we then obtain the first decentralized quantum controller as

$$\begin{aligned} \bar{F}_{c_1} &= \begin{bmatrix} -1.1329 & 0 \\ 2.5015 & -1.3888 \end{bmatrix}; \\ \begin{bmatrix} \bar{G}_{w_{c0,1}} & \bar{G}_{c_1} \end{bmatrix} &= \begin{bmatrix} -1.5052 & -0.0115 \\ 1.6627 & -0.1134 \end{bmatrix}; \\ \begin{bmatrix} \bar{H}_{c_1} \\ \bar{H}_{c_{2,1}} \end{bmatrix} &= \begin{bmatrix} 1.5052 & -1.6627 \\ 0.0115 & 0.1134 \end{bmatrix} \end{aligned} \tag{8.63}$$

and the second decentralized quantum controller as

$$\bar{F}_{c_2} = \begin{bmatrix} -1.1329 & 0 \\ -2.5036 & -1.3888 \end{bmatrix};$$
$$\begin{bmatrix} \bar{G}_{w_{c0,2}} & \bar{G}_{c_2} \end{bmatrix} = \begin{bmatrix} 1.5052 & -0.0002 \\ 1.6632 & -0.1064 \end{bmatrix};$$
$$\begin{bmatrix} \bar{H}_{c_2} \\ \bar{H}_{c_{2,2}} \end{bmatrix} = \begin{bmatrix} -1.5052 & -1.6632 \\ 0.0002 & 0.1064 \end{bmatrix}. \tag{8.64}$$

The realizations in (8.63) and (8.64) imply that each decentralized quantum controller can be physically constructed as a cascade of two first order generalized 2-mirror optical cavities because it has two inputs and two outputs; see [77].

From (8.63), we obtain the parameters of the optical cavities corresponding to the first decentralized quantum controller as follows:

$$\begin{aligned} k_{c_{11}} &= (1.5052)^2; \quad k_{c_{12}} = (0.0115)^2; \\ k_{c_{13}} &= (1.6627)^2; \quad k_{c_{14}} = (0.1134)^2 \end{aligned} \tag{8.65}$$

which are shown in Figure 8.2.

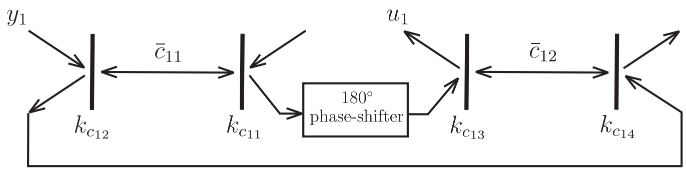


Figure 8.2: The first decentralized quantum H^∞ controller.

Moreover, from (8.64), the parameters of the optical cavities corresponding to the second decentralized quantum controller are as follows:

$$\begin{aligned} k_{c_{21}} &= (1.5052)^2; \quad k_{c_{22}} = (0.0002)^2; \\ k_{c_{23}} &= (1.6632)^2; \quad k_{c_{24}} = (0.1064)^2 \end{aligned} \tag{8.66}$$

which are also shown in Figure 8.3.

Remark 8.2. The physical realizability of the decentralized quantum controllers (8.58) and (8.59) does not immediately follow from the controller design algorithm in sub-Section 8.3.1. Thus, we always have to verify whether the resulting

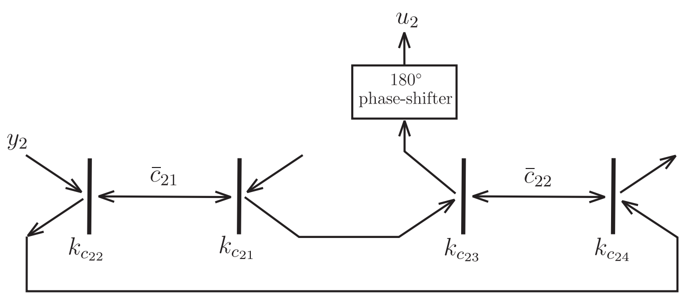


Figure 8.3: The second decentralized quantum H^∞ controller.

controllers are physically realizable before they are implemented. In order to show that this controller design algorithm indeed does not always guarantee the physical realizability of the decentralized quantum controllers, we then consider the following example:

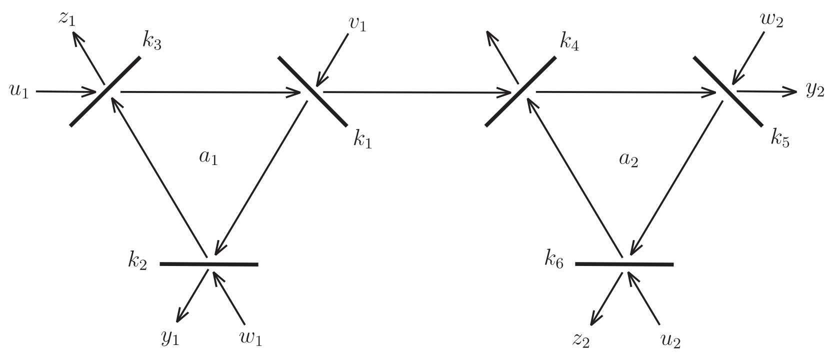


Figure 8.4: A cascaded linear quantum system of two optical cavities.

This quantum system can be represented as

$$\begin{aligned}
\begin{bmatrix} da_1(t) \\ da_2(t) \end{bmatrix} &= \begin{bmatrix} -\frac{g_1}{2} & 0 \\ -\sqrt{k_1 k_4} & -\frac{g_2}{2} \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} dt + \begin{bmatrix} -\sqrt{k_1} \\ -\sqrt{k_4} \end{bmatrix} dv_1(t) \\
&\quad + \begin{bmatrix} -\sqrt{k_2} & 0 \\ 0 & -\sqrt{k_5} \end{bmatrix} \begin{bmatrix} dw_1(t) \\ dw_2(t) \end{bmatrix} + \begin{bmatrix} -\sqrt{k_3} & 0 \\ 0 & -\sqrt{k_6} \end{bmatrix} \begin{bmatrix} du_1(t) \\ du_2(t) \end{bmatrix}; \\
\begin{bmatrix} dz_1(t) \\ dz_2(t) \end{bmatrix} &= \begin{bmatrix} \sqrt{k_3} & 0 \\ 0 & \sqrt{k_6} \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} du_1(t) \\ du_2(t) \end{bmatrix}; \\
\begin{bmatrix} dy_1(t) \\ dy_2(t) \end{bmatrix} &= \begin{bmatrix} \sqrt{k_2} & 0 \\ 0 & \sqrt{k_5} \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dw_1(t) \\ dw_2(t) \end{bmatrix} \qquad (8.67)
\end{aligned}$$

where $g_1 = k_1 + k_2 + k_3$, $g_2 = k_4 + k_5 + k_6$ and the parameter values are

$$\begin{aligned} k_1 &= 2.25; \quad k_2 = 2.00; \quad k_3 = 4.50; \\ k_4 &= 5.50; \quad k_5 = 6.00; \quad k_6 = 7.50. \end{aligned} \tag{8.68}$$

Applying the DE approach presented in sub-Section 8.3.2, we then obtain

$$\begin{aligned} \gamma &= 0.7496; \quad \delta_1 = 9.1311; \quad \delta_2 = 1.0795; \\ \beta_1 &= 0.0613; \quad \beta_2 = 0.1123. \end{aligned} \tag{8.69}$$

Given these parameters, we can construct a non-decentralized coherent quantum robust H^∞ controller as follows:

$$\begin{aligned} F_c &= \begin{bmatrix} -51.1302 & -0.0794 \\ 5.8901 & -1.8379 \end{bmatrix}; \quad G_c = \begin{bmatrix} -3.2578 & 0.0147 \\ 0.7632 & -2.3851 \end{bmatrix}; \\ H_c &= \begin{bmatrix} 9.2789 & -0.0625 \\ -0.0807 & -1.2525 \end{bmatrix}. \end{aligned} \tag{8.70}$$

The matrices G_c and H_c in (8.70) are such that the squared H^∞ norm of the additional uncertainties $\bar{\Delta}_1(s)$ and $\bar{\Delta}_2(s)$ (as defined in (8.22)):

$$\|\bar{\Delta}_1(s)\|_\infty^2 = 0.0104; \quad \|\bar{\Delta}_2(s)\|_\infty^2 = 0.0665 \tag{8.71}$$

are less than β_1 and β_2 in (8.69), respectively. Hence, the decentralized quantum controllers are as follows:

$$\begin{aligned} F_{c_1} &= F_c; \quad G_{c_1} = \begin{bmatrix} -3.2578 \\ 0.7632 \end{bmatrix}; \quad H_{c_1} = \begin{bmatrix} 9.2789 & -0.0625 \end{bmatrix}; \\ F_{c_2} &= F_c; \quad G_{c_2} = \begin{bmatrix} 0.0147 \\ -2.3851 \end{bmatrix}; \quad H_{c_2} = \begin{bmatrix} -0.0807 & -1.2525 \end{bmatrix} \end{aligned} \tag{8.72}$$

which are stable because F_c is Hurwitz with eigenvalues:

$$e_1 = -51.1207; \quad e_2 = -1.8474. \tag{8.73}$$

Also, their H^∞ norms are

$$\|T_{11}(s)\|_\infty = 0.6074; \quad \|T_{22}(s)\|_\infty = 1.6161. \tag{8.74}$$

However, $\{F_{c_2}, G_{c_2}, H_{c_2}\}$ is not physically realizable as it has the H^∞ norm: $\|T_{22}(s)\|_\infty = 1.6161$, which implies that it does not satisfy the physical realizability condition given in Lemma 8.1.

This observation motivates us to propose another method to synthesize the decentralized quantum controllers, which are ensured to be physically realizable. That is, the resulting quantum controllers are always stable and strict bounded real. This method is then used to solve the same example to show that it is indeed effective to secure the physical realizability of the quantum controllers.

8.4 Strict Bounded Real Decentralized Quantum H^∞ Controller

In Section 8.3, we have presented an algorithm to synthesize a decentralized quantum H^∞ controller. However, this algorithm may not result in a physically realizable decentralized quantum controller because the physical realizability condition given in Lemma 8.1 is not explicitly imposed on the controller design algorithm. This concern has motivated us to apply the approach of Chapter 7 to construct a strict bounded real decentralized quantum H^∞ controller, which must always be physically realizable. In this case, we employ an artificial uncertainty to ensure the physical realizability of the resulting controller. This approach, however, may result in a more conservative decentralized quantum controller than the one obtained using the controller design method in Section 8.3 due to the fact that we introduce the additional artificial uncertainty.

8.4.1 Synthesis algorithm

We refer to Section 7.4 to synthesize a strict bounded real decentralized quantum H^∞ controller based on the equivalent uncertain quantum system (8.27), (8.24), (8.8), (8.28). For this purpose, we follow the same steps as those in Section 7.4 where an artificial uncertain quantum system is used to obtain a strict bounded real quantum controller. Thus, using Lemma 7.8, we introduce a set of scaling constants $\kappa_1 > 0, \ldots, \kappa_k > 0$ and $\alpha_1 > 0, \ldots, \alpha_p > 0$ corresponding to the structured uncertainties (8.8) and the additional uncertainties (8.28), respectively.

Then, we can rewrite the QSDEs (8.27), (8.24) as follows:

$$
\begin{aligned}
da(t) &= F\,a(t)dt + G_0\,dv(t) + \bar{G}_1\,d\bar{w}(t) + G_2\,du(t); \quad a(0) = a_0;\\
d\bar{z}(t) &= \bar{H}_1\,a(t)dt + \bar{J}_{10}\,dv(t) + \bar{J}_{11}\,d\bar{w}(t) + \bar{J}_{12}\,du(t);\\
dy(t) &= H_2\,a(t)dt + J_{20}\,dv(t) + \bar{J}_{21}\,d\bar{w}(t)
\end{aligned}
\tag{8.75}
$$

where $d\bar{w}(t) = \bar{\beta}_w(t)dt + d\bar{\nu}(t)$;

$$
\begin{aligned}
d\bar{w} &= \begin{bmatrix} \gamma\,dw(t) \\ \sqrt{\kappa_1}\,d\xi_1(t) \\ \vdots \\ \sqrt{\kappa_k}\,d\xi_k(t) \\ \sqrt{\alpha_1}\,d\bar{\xi}_1(t) \\ \vdots \\ \sqrt{\alpha_p}\,d\bar{\xi}_p(t) \end{bmatrix}; \quad d\bar{z} = \begin{bmatrix} dz(t) \\ \sqrt{\kappa_1}\,d\zeta_1(t) \\ \vdots \\ \sqrt{\kappa_k}\,d\zeta_k(t) \\ \sqrt{\alpha_1\beta_1}\,d\bar{\zeta}_1(t) \\ \vdots \\ \sqrt{\alpha_p\beta_p}\,d\bar{\zeta}_p(t) \end{bmatrix};\\
\bar{G}_1 &= \begin{bmatrix} \gamma^{-1}G_1 & \sqrt{\kappa_1}^{-1}\,G_{3,1} & \cdots & \sqrt{\kappa_k}^{-1}\,G_{3,k} & \sqrt{\alpha_1}^{-1}\,G_2L_1 & \cdots & \sqrt{\alpha_p}^{-1}\,G_2L_p \end{bmatrix};\\
\bar{H}_1 &= \begin{bmatrix} H_1 \\ \sqrt{\kappa_1}\,P_1 \\ \vdots \\ \sqrt{\kappa_k}\,P_k \\ \sqrt{\alpha_1\beta_1}\,M_1 \\ \vdots \\ \sqrt{\alpha_p\beta_p}\,M_p \end{bmatrix}; \quad \bar{J}_{10} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \sqrt{\alpha_1\beta_1}\,N_{20,1} \\ \vdots \\ \sqrt{\alpha_p\beta_p}\,N_{20,p} \end{bmatrix}; \quad \bar{J}_{12} = \begin{bmatrix} J_{12} \\ \sqrt{\kappa_1}\,Q_1 \\ \vdots \\ \sqrt{\kappa_k}\,Q_k \\ 0 \\ \vdots \\ 0 \end{bmatrix};\\
\bar{J}_{11} &= \begin{bmatrix} 0 & 0 & \cdots & 0 & \sqrt{\alpha_1}^{-1}\,J_{12}L_1 & \cdots & \sqrt{\alpha_p}^{-1}\,J_{12}L_p \\ 0 & 0 & \cdots & 0 & \sqrt{\kappa_1/\alpha_1}\,Q_1L_1 & \cdots & \sqrt{\kappa_1/\alpha_p}\,Q_1L_p \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \sqrt{\kappa_k/\alpha_1}\,Q_kL_1 & \cdots & \sqrt{\kappa_k/\alpha_p}\,Q_kL_p \\ \gamma^{-1}\sqrt{\alpha_1\beta_1}\,N_{21,1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \gamma^{-1}\sqrt{\alpha_p\beta_p}\,N_{21,p} & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix};\\
\bar{J}_{21} &= \begin{bmatrix} \gamma^{-1}J_{21} & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.
\end{aligned}
\tag{8.76}
$$

Since the $\bar{J}_{11}$ term in the quantum system (8.75) leads to a non-standard H^∞ control problem, we need to apply a loop shifting transformation to eliminate that term in (8.75); e.g., see [16, Sections 4.5.1 and 5.5.1] and [18, Section 17.2]. To carry out this transformation, we impose the following assumption:

Assumption 8.3. Given constants $\kappa_1 > 0, \ldots, \kappa_k > 0$, $\alpha_1 > 0, \ldots, \alpha_p > 0$, $\beta_1 > 0, \ldots, \beta_p > 0$, the equivalent uncertain quantum system (8.27), (8.24), (8.8), (8.28) is assumed to be such that $\bar{J}_{11}\bar{J}_{11}^\dagger < I$.

Now, we can define

$$\Theta := I - \bar{J}_{11}^\dagger \bar{J}_{11} > 0; \quad \bar{\Theta} := I - \bar{J}_{11}\bar{J}_{11}^\dagger > 0 \tag{8.77}$$

and also

$$\begin{aligned} d\breve{w}(t) &:= \Theta^{\frac{1}{2}} d\bar{w}(t) - \Theta^{-\frac{1}{2}} \bar{J}_{11}^\dagger \left[\bar{H}_1\, a(t)\, dt + \bar{J}_{12}\, du(t)\right]; \\ d\breve{z}(t) &:= \bar{\Theta}^{-\frac{1}{2}} \left[\bar{H}_1\, a(t)\, dt + \bar{J}_{12}\, du(t)\right]. \end{aligned} \tag{8.78}$$

It then follows from (8.78) that

$$\begin{aligned} d\bar{w}(t) &= \Theta^{-\frac{1}{2}} d\breve{w}(t) + \Theta^{-1} \bar{J}_{11}^\dagger \left[\bar{H}_1\, a(t)\, dt + \bar{J}_{12}\, du(t)\right]; \\ \|\bar{w}(t)\|_2^2 &- \|\bar{z}(t)\|_2^2 \equiv \|\breve{w}(t)\|_2^2 - \|\breve{z}(t)\|_2^2. \end{aligned} \tag{8.79}$$

Thus, using (8.78), we can rewrite the QSDEs (8.75) as

$$\begin{aligned} da(t) &= \breve{F}\, a(t)dt + G_0\, dv(t) + \breve{G}_1\, d\breve{w}(t) + \breve{G}_2\, du(t); \quad a(0) = a_0; \\ d\breve{z}(t) &= \breve{H}_1\, a(t)dt + \breve{J}_{12}\, du(t); \\ dy(t) &= \breve{H}_2\, a(t)dt + J_{20}\, dv(t) + \breve{J}_{21}\, d\breve{w}(t) + \breve{J}_{22}\, du(t) \end{aligned} \tag{8.80}$$

where $d\breve{w}(t) = \breve{\beta}_w(t)dt + d\breve{\nu}(t)$;

$$\begin{array}{rclrcl} \breve{F} &=& F + \bar{G}_1 \bar{J}_{11}^\dagger \bar{\Theta}^{-1} \bar{H}_1; & \breve{G}_1 &=& \bar{G}_1 \Theta^{-\frac{1}{2}}; \\ \breve{G}_2 &=& G_2 + \bar{G}_1 \bar{J}_{11}^\dagger \bar{\Theta}^{-1} \bar{J}_{12}; & \breve{H}_1 &=& \bar{\Theta}^{-\frac{1}{2}} \bar{H}_1; \\ \breve{H}_2 &=& H_2 + \bar{J}_{21} \bar{J}_{11}^\dagger \bar{\Theta}^{-1} \bar{H}_1; & \breve{J}_{12} &=& \bar{\Theta}^{-\frac{1}{2}} \bar{J}_{12}; \\ \breve{J}_{22} &=& \bar{J}_{21} \bar{J}_{11}^\dagger \bar{\Theta}^{-1} \bar{J}_{12}; & \breve{J}_{21} &=& \bar{J}_{21} \Theta^{-\frac{1}{2}}. \end{array} \tag{8.81}$$

Note that the $\bar{J}_{10}$ term is automatically eliminated when we apply the loop shifting transformation to the quantum system (8.75).

We now construct a matrix K based on the quantum system (8.80) such that $(F + G_2 K)$ is Hurwitz and the following uncertain quantum system

$$
\begin{aligned}
da(t) &= (F + G_2K)\, a(t)\, dt + G_0\, dv(t) + G_1\, dw(t) + \sum_{l=1}^{k} G_{3,l}\, d\xi_l(t) + \sum_{j=1}^{p} G_2 L_j\, d\bar{\xi}_j;\\
dz(t) &= (H_1 + J_{12}K)\, a(t)\, dt + \sum_{j=1}^{p} J_{12} L_j\, d\bar{\xi}_j;\\
d\zeta_1(t) &= (P_1 + Q_1K)\, a(t)\, dt + \sum_{j=1}^{p} Q_1 L_j\, d\bar{\xi}_j;\\
&\vdots\\
d\zeta_k(t) &= (P_k + Q_kK)\, a(t)\, dt + \sum_{j=1}^{p} Q_k L_j\, d\bar{\xi}_j;\\
dy(t) &= H_2\, a(t)\, dt + J_{20}\, dv(t) + J_{21}\, dw(t)
\end{aligned}
\tag{8.82}
$$

together with $d\bar{\zeta}_1(t), \ldots, d\bar{\zeta}_p(t)$ as defined in (8.24) and an initial condition $a(0) = a_0$, is strict bounded real with disturbance attenuation $\gamma > 0$ while satisfying (8.8) and (8.28). The satisfaction of this requirement is dependent on the existence of a solution to a parameterized complex algebraic Riccati equation defined as follows: Let $\kappa_1 > 0, \ldots, \kappa_k > 0$, $\alpha_1 > 0, \ldots, \alpha_p > 0$, $\beta_1 > 0, \ldots, \beta_p > 0$ be given constants and consider a complex algebraic Riccati equation

$$
\begin{aligned}
&\left(\breve{F} - \breve{G}_2 \breve{E}_1^{-1} \breve{J}_{12}^\dagger \breve{H}_1\right)^\dagger \breve{X} + \breve{X}\left(\breve{F} - \breve{G}_2 \breve{E}_1^{-1} \breve{J}_{12}^\dagger \breve{H}_1\right)\\
&\quad + \breve{X}\left(\breve{G}_1 \breve{G}_1^\dagger - \breve{G}_2 \breve{E}_1^{-1} \breve{G}_2^\dagger\right)\breve{X} + \breve{H}_1^\dagger\left(I - \breve{J}_{12} \breve{E}_1^{-1} \breve{J}_{12}^\dagger\right)\breve{H}_1 = 0
\end{aligned}
\tag{8.83}
$$

where

$$\breve{E}_1 = \breve{J}_{12}^\dagger \breve{J}_{12}. \tag{8.84}$$

Assumption 8.4. Given constants $\kappa_1 > 0, \ldots, \kappa_k > 0$, $\alpha_1 > 0, \ldots, \alpha_p > 0$, $\beta_1 > 0, \ldots, \beta_p > 0$, the equivalent uncertain quantum system (8.27), (8.24), (8.8), (8.28) is assumed to be such that $\breve{E}_1 > 0$.

Lemma 8.2. *Let $\kappa_1 > 0, \ldots, \kappa_k > 0$, $\alpha_1 > 0, \ldots, \alpha_p > 0$, $\beta_1 > 0, \ldots, \beta_p > 0$ be given constants. Suppose that the equivalent uncertain quantum system* (8.27), (8.24), (8.8), (8.28) *is such that Assumption 8.3 and Assumption 8.4 are satisfied and the complex algebraic Riccati equation* (8.83) *has a stabilizing solution* $X \geq 0$. *Then, there exists a matrix* K *such that the uncertain quantum system* (8.82), (8.8), (8.28) *is strict bounded real with disturbance attenuation* $\gamma > 0$. *That is,* $(F + G_2 K)$ *is Hurwitz and*

$$\left\| (H_1 + J_{12} K) \left(sI - (F + G_2 K) \right)^{-1} G_1 \right\|_\infty < \gamma \tag{8.85}$$

where

$$K = -\breve{E}_1^{-1} \left(\breve{G}_2^\dagger \breve{X} + \breve{J}_{12}^\dagger \breve{H}_1 \right). \tag{8.86}$$

Proof. The proof of this lemma follows the same arguments as those in the proof of Lemma 7.9. □

Using the matrix K in (8.86) and introducing an additional uncertainty input $d\xi_{k+1}(t)$ and output $d\zeta_{k+1}(t)$, we form an artificial uncertain quantum system as follows: (see [8])

$$\begin{aligned}
da(t) &= \tilde{F}\, a(t)\, dt + G_0\, dv(t) + \tilde{G}_1\, d\tilde{w}(t) + \tilde{G}_2\, du(t) + \sum_{j=1}^{p} G_2 L_j\, d\bar{\xi}_j(t) \\
&\quad + \sum_{l=1}^{k+1} G_{3,l}\, d\xi_l(t); \quad a(0) = a_0; \\
d\tilde{z}(t) &= \tilde{H}_1\, a(t)\, dt + \tilde{J}_{12}\, du(t) + \sum_{j=1}^{p} \tilde{J}_{12,j}^L\, d\bar{\xi}_j(t) + S_0\, d\xi_{k+1}(t); \\
d\zeta_1(t) &= \tilde{P}_1\, a(t)\, dt + \tilde{Q}_1\, du(t) + \sum_{j=1}^{p} Q_1 L_j\, d\bar{\xi}_j(t) + S_1\, d\xi_{k+1}(t); \\
&\ \ \vdots \\
d\zeta_k(t) &= \tilde{P}_k\, a(t)\, dt + \tilde{Q}_k\, du(t) + \sum_{j=1}^{p} Q_k L_j\, d\bar{\xi}_j(t) + S_k\, d\xi_{k+1}(t); \\
d\zeta_{k+1}(t) &= \tilde{P}_{k+1}\, a(t)\, dt + \tilde{Q}_{k+1}\, du(t) + V_1\, d\tilde{w}(t) + \sum_{j=1}^{p} V_{2,j}\, d\bar{\xi}_j(t); \\
d\bar{\zeta}_1(t) &= M_1\, a(t)\, dt + N_{20,1}\, dv(t) + \tilde{N}_{21,1}\, d\tilde{w}(t); \\
&\ \ \vdots \\
d\bar{\zeta}_p(t) &= M_p\, a(t)\, dt + N_{20,p}\, dv(t) + \tilde{N}_{21,p}\, d\tilde{w}(t); \\
dy(t) &= \tilde{H}_2\, a(t)\, dt + J_{20}\, dv(t) + \tilde{J}_{21}\, d\tilde{w}(t) + S_{k+1}\, d\xi_{k+1}(t)
\end{aligned} \tag{8.87}$$

where $d\tilde{w}(t) = \tilde{\beta}_w(t)dt + d\tilde{\nu}(t)$;

$$d\tilde{w}(t) = \begin{bmatrix} dw_1(t) \\ dw_2(t) \end{bmatrix}; \quad d\tilde{z}(t) = \begin{bmatrix} dz_1(t) \\ dz_2(t) \end{bmatrix}; \quad \tilde{F} = F + \tfrac{1}{2}G_2K; \quad \tilde{G}_1 = \begin{bmatrix} G_1 & 0 \end{bmatrix};$$
$$\tilde{G}_2 = \tfrac{1}{2}G_2; \quad G_{3,k+1} = \begin{bmatrix} G_2 & 0 & 0 & 0 \end{bmatrix} R^{-1}; \quad \tilde{J}^L_{12,j} = \tfrac{1}{2}\begin{bmatrix} J_{12}L_j \\ 0 \end{bmatrix};$$
$$\tilde{H}_1 = \tfrac{1}{2}\begin{bmatrix} H_1 \\ 0 \end{bmatrix}; \quad \tilde{J}_{12} = \tfrac{1}{2}\begin{bmatrix} J_{12} \\ \gamma I \end{bmatrix}; \quad S_0 = \begin{bmatrix} 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} R^{-1};$$
$$\tilde{P}_l = P_l + \tfrac{1}{2}Q_lK; \quad \tilde{Q}_l = \tfrac{1}{2}Q_l; \quad S_l = \begin{bmatrix} Q_l & 0 & 0 & 0 \end{bmatrix} R^{-1}; \quad \tilde{N}_{21,j} = \begin{bmatrix} N_{21,j} & 0 \end{bmatrix};$$
$$\tilde{P}_{k+1} = \frac{R}{2}\begin{bmatrix} K \\ H_1 \\ 0 \\ H_2 \end{bmatrix}; \ \tilde{Q}_{k+1} = \frac{R}{2}\begin{bmatrix} -I \\ J_{12} \\ \gamma I \\ 0 \end{bmatrix}; \ V_1 = \frac{R}{2}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ J_{21} & -I \end{bmatrix}; \ V_{2,j} = \frac{R}{2}\begin{bmatrix} 0 \\ J_{12}L_j \\ 0 \\ 0 \end{bmatrix};$$
$$\tilde{H}_2 = \tfrac{1}{2}H_2; \quad \tilde{J}_{21} = \tfrac{1}{2}\begin{bmatrix} J_{21} & I \end{bmatrix}; \quad S_{k+1} = \begin{bmatrix} 0 & 0 & 0 & -I \end{bmatrix} R^{-1} \tag{8.88}$$

for $l = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, p$. Note that R is any $n_r \times n_r$ non-singular scaling matrix, where $n_r = 2n_u + n_z + n_y$; w_2 and z_2 have the same dimensions as those of y and u, respectively.

In (8.87), the additional uncertainty input $d\xi_{k+1}(t)$ is related to the additional uncertainty output $d\zeta_{k+1}(t)$ as follows:

$$d\xi_{k+1}(t) = \Delta_{k+1}\, d\zeta_{k+1}(t) \tag{8.89}$$

where $\Delta_{k+1} \in \mathbf{R}$ is a real unknown scalar uncertain parameter satisfying $|\Delta_{k+1}| \leq 1$. The H^∞ control objective for the artificial uncertain quantum system (8.87), (8.8), (8.28), (8.89) is as follows:

$$\begin{aligned} &\int_0^t \left\langle \tilde{z}(s)^\dagger \tilde{z}(s) + \tilde{z}(s)^T \tilde{z}(s)^* + \varepsilon\left(\tilde{\eta}(s)^\dagger \tilde{\eta}(s) + \tilde{\eta}(s)^T \tilde{\eta}(s)^*\right)\right\rangle ds \\ &\leq (1-\varepsilon^2)\int_0^t \left\langle \tilde{\beta}_w(s)^\dagger \tilde{\beta}_w(s) + \tilde{\beta}_w(s)^T \tilde{\beta}_w(s)^*\right\rangle ds + \pi_1 + \pi_2 t \end{aligned} \tag{8.90}$$

where $\varepsilon, \pi_1, \pi_2 > 0$ are real constants. Moreover, we consider two special cases for Δ_{k+1} as in Section 7.4 to verify that any suitable coherent quantum H^∞ controller of the form (8.18) for the artificial uncertain quantum system (8.87), (8.8), (8.28), (8.89) is indeed stable and strict bounded real, and solves the original decentralized quantum control problem.

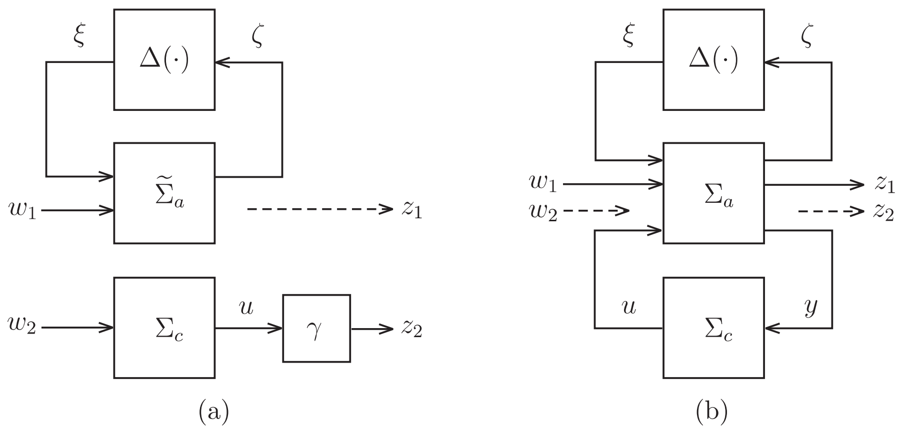


Figure 8.5: Block diagrams corresponding to special cases I and II; see [8].

Special cese I: $\Delta_{k+1} = 1$. Using this Δ_{k+1}, it follows from (8.87) that

$$
\begin{aligned}
da(t) &= (F + G_2K)\, a(t)\, dt + G_0\, dv(t) + G_1\, dw_1(t) + \sum_{j=1}^{k} G_2 L_j\, d\bar{\xi}_j(t) + G_3\, d\xi_1(t);\\
dz_1(t) &= 0;\\
dz_2(t) &= \gamma\, du(t);\\
d\zeta_1(t) &= (P_1 + Q_1K)\, a(t)\, dt + \sum_{j=1}^{k} Q_1 L_j\, d\bar{\xi}_j(t);\\
d\bar{\zeta}_1(t) &= M_1\, a(t)\, dt + N_{20,1}\, dv(t) + N_{21,1}\, dw(t);\\
&\vdots\\
d\bar{\zeta}_k(t) &= M_k\, a(t)\, dt + N_{20,k}\, dv(t) + N_{21,k}\, dw(t);\\
dy(t) &= J_{20}\, dv(t) + dw_2(t)
\end{aligned}
\tag{8.91}
$$

where $a(0) = a_0$, and conditions (8.8) and (8.28) are satisfied. Here, we recognize that the uncertain quantum system (8.91), (8.8), (8.28) is the same as the uncertain quantum system (8.82), (8.8), (8.28). Thus, the uncertain quantum system (8.91), (8.8), (8.28) is strict bounded real with disturbance attenuation $\gamma > 0$ according to the construction of the matrix K in (8.86) and Lemma 8.2. It is also apparent from the QSDEs (8.91) that the control input $u(t)$ does not affect the quantum plant, but only affects the controlled output $z_2(t)$. Moreover, the measurement output $y(t)$ is not affected by the quantum plant but is only affected by the disturbance input $w_2(t)$ and the quantum noise $v(t)$. This situation is shown in Figure 8.5(a) where the coherent quantum controller Σ_c of the form (8.18) is detached from the uncertain quantum system $(\widetilde{\Sigma}_a, \Delta(\cdot))$ defined by (8.91), (8.8), (8.28). It thus follows from the block diagram in Figure 8.5(a) and the closed loop H^∞ control objective (8.90) that the coherent quantum controller Σ_c must be stable and strict bounded real.

Special cese II: $\Delta_{k+1} = -1$. It is straightforward to show that using this value of Δ_{k+1}, we again obtain the equivalent uncertain quantum system (8.27), (8.24), (8.8), (8.28). Thus, if we can find a suitable coherent quantum H^∞ controller Σ_c of the form (8.18) for the artificial uncertain quantum system (8.87), (8.8), (8.28), (8.89), then the same quantum controller Σ_c is also suitable for the equivalent uncertain quantum system (8.27), (8.24), (8.8), (8.28) (corresponds to $(\Sigma_a, \Delta(\cdot))$ in Figure 8.5(b)) such that the H^∞ control objective (8.29) is satisfied. This implies that the resulting closed loop uncertain quantum system as shown in Figure 8.5(b) is strict bounded real with disturbance attenuation $\gamma > 0$.

From both special cases above, we conclude that if there exists a suitable quantum controller of the form (8.18) for the artificial uncertain quantum system (8.87), (8.8), (8.28), (8.89), then this quantum controller also achieves a strict bounded real closed loop system with disturbance attenuation $\gamma > 0$ when it is applied to the equivalent uncertain quantum system (8.27), (8.24), (8.8), (8.28). Moreover, this quantum controller itself must be stable and strict bounded real.

We now apply the approach in sub-Section 7.4.2 or sub-Section 8.3.1 to synthesize a strict bounded real quantum controller of the form (8.18). Thus, we introduce scaling constants $\tau_1 > 0, \ldots, \tau_{k+1} > 0$, $\delta_1 > 0, \ldots, \delta_p > 0$ so that we can rewrite the QSDEs (8.87) of the artificial uncertain quantum system as follows:

$$\begin{aligned}
da(t) &= \tilde{F}\, a(t)\, dt + G_0\, dv(t) + \check{G}_1\, d\breve{w}(t) + \tilde{G}_2\, du(t); \quad a(0) = a_0; \\
d\check{z}(t) &= \check{H}_1\, a(t)\, dt + \check{J}_{10}\, dv(t) + \check{J}_{11}\, d\breve{w}(t) + \check{J}_{12}\, du(t); \\
dy(t) &= \tilde{H}_2\, a(t)\, dt + J_{20}\, dv(t) + \check{J}_{21}\, d\breve{w}(t)
\end{aligned} \tag{8.92}$$

where $d\breve{w}(t) = \check{\beta}_w(t)\, dt + d\check{\nu}(t)$;

$$\check{J}_{21} = \begin{bmatrix} \gamma^{-1}\tilde{J}_{21} & 0 & \cdots & 0 & \sqrt{\tau_{k+1}}^{-1} S_{k+1} & 0 & \cdots & 0 \end{bmatrix}; \tag{8.93}$$

$$
d\breve{w}(t) = \begin{bmatrix} \gamma\, d\tilde{w}(t) \\ \sqrt{\tau_1}\, d\xi_1(t) \\ \vdots \\ \sqrt{\tau_{k+1}}\, d\xi_{k+1}(t) \\ \sqrt{\delta_1}\, d\bar{\xi}_1(t) \\ \vdots \\ \sqrt{\delta_p}\, d\bar{\xi}_p(t) \end{bmatrix}; \quad d\breve{z}(t) = \begin{bmatrix} d\tilde{z}(t) \\ \sqrt{\tau_1}\, d\zeta_1(t) \\ \vdots \\ \sqrt{\tau_{k+1}}\, d\zeta_{k+1}(t) \\ \sqrt{\delta_1\beta_1}\, d\bar{\zeta}_1(t) \\ \vdots \\ \sqrt{\delta_p\beta_p}\, d\bar{\zeta}_p(t) \end{bmatrix};
$$

$$
\breve{G}_1 = \left[\gamma^{-1}\tilde{G}_1 \quad \sqrt{\tau_1}^{-1}\, G_{3,1} \ \cdots \ \sqrt{\tau_{k+1}}^{-1}\, G_{3,k+1} \quad \sqrt{\delta_1}^{-1}\, G_2L_1 \ \cdots \ \sqrt{\delta_p}^{-1}\, G_2L_p\right];
$$

$$
\breve{H}_1 = \begin{bmatrix} \tilde{H}_1 \\ \sqrt{\tau_1}\, \tilde{P}_1 \\ \vdots \\ \sqrt{\tau_{k+1}}\, \tilde{P}_{k+1} \\ \sqrt{\delta_1\beta_1}\, M_1 \\ \vdots \\ \sqrt{\delta_p\beta_p}\, M_p \end{bmatrix}; \quad \bar{J}_{10} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \sqrt{\delta_1\beta_1}\, N_{20,1} \\ \vdots \\ \sqrt{\delta_p\beta_p}\, N_{20,p} \end{bmatrix}; \quad \bar{J}_{12} = \begin{bmatrix} J_{12} \\ \sqrt{\tau_1}\, \tilde{Q}_1 \\ \vdots \\ \sqrt{\tau_{k+1}}\, \tilde{Q}_{k+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix};
$$

$$
\breve{J}_{11} = \begin{bmatrix}
0 & 0 \ \cdots \ 0 & \sqrt{\frac{1}{\tau_{k+1}}}\, S_0 & \sqrt{\frac{1}{\delta_1}}\, \tilde{J}^L_{12,1} & \cdots & \sqrt{\frac{1}{\delta_p}}\, \tilde{J}^L_{12,p} \\
0 & 0 \ \cdots \ 0 & \sqrt{\frac{\tau_1}{\tau_{k+1}}}\, S_1 & \sqrt{\frac{\tau_1}{\delta_1}}\, Q_1L_1 & \cdots & \sqrt{\frac{\tau_1}{\delta_p}}\, Q_1L_p \\
\vdots & \vdots \quad\ \vdots & \vdots & \vdots & & \vdots \\
0 & 0 \ \cdots \ 0 & \sqrt{\frac{\tau_k}{\tau_{k+1}}}\, S_k & \sqrt{\frac{\tau_k}{\delta_1}}\, Q_kL_1 & \cdots & \sqrt{\frac{\tau_k}{\delta_p}}\, Q_kL_p \\
\gamma^{-1}\sqrt{\tau_{k+1}}\, V_1 & 0 \ \cdots \ 0 & 0 & \sqrt{\frac{\tau_{k+1}}{\delta_1}}\, V_{2,1} & \cdots & \sqrt{\frac{\tau_{k+1}}{\delta_p}}\, V_{2,p} \\
\gamma^{-1}\sqrt{\delta_1\beta_1}\, N_{21,1} & 0 \ \cdots \ 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots \quad\ \vdots & \vdots & \vdots & & \vdots \\
\gamma^{-1}\sqrt{\delta_p\beta_p}\, N_{21,p} & 0 \ \cdots \ 0 & 0 & 0 & \cdots & 0
\end{bmatrix}. \tag{8.94}
$$

The H^∞ control objective for the quantum system (8.92) is as follows:

$$
\begin{aligned}
&\int_0^t \left\langle \breve{z}(s)^\dagger \breve{z}(s) + \breve{z}(s)^T \breve{z}(s)^* + \varepsilon \left(\tilde{\eta}(s)^\dagger \tilde{\eta}(s) + \tilde{\eta}(s)^T \tilde{\eta}(s)^*\right)\right\rangle ds \\
&\leq (1-\varepsilon^2) \int_0^t \left\langle \breve{\beta}_w(s)^\dagger \breve{\beta}_w(s) + \breve{\beta}_w(s)^T \breve{\beta}_w(s)^*\right\rangle ds + \pi_1 + \pi_2 t
\end{aligned} \tag{8.95}
$$

Moreover, we also have a $\breve{J}_{11}$ term in (8.92), which leads to a non-standard H^∞ control problem. Thus, to eliminate this term, we impose the following assumption and then apply a loop shifting transformation with respect to (8.92); e.g., see [16, Sections 4.5.1 and 5.5.1] and [18, Section 17.2].

Assumption 8.5. Given constants $\tau_1 > 0, \ldots, \tau_{k+1} > 0$, $\delta_1 > 0, \ldots, \delta_p > 0$, $\beta_1 > 0, \ldots, \beta_p > 0$ and any non-singular scaling matrix R, the artificial uncertain quantum system (8.87), (8.8), (8.28), (8.89) is assumed to be such that $\check{J}_{11}\check{J}_{11}^\dagger < I$.

Satisfying Assumption 8.5, we now define

$$\Phi := I - \check{J}_{11}^\dagger \check{J}_{11} > 0; \quad \check{\Phi} := I - \check{J}_{11}\check{J}_{11}^\dagger > 0 \tag{8.96}$$

and also

$$\begin{aligned} d\hat{w}(t) &:= \Phi^{\frac{1}{2}} d\check{w}(t) - \Phi^{-\frac{1}{2}} \check{J}_{11}^\dagger \left[\check{H}_1\, a(t)\, dt + \check{J}_{12}\, du(t)\right]; \\ d\hat{z}(t) &:= \check{\Phi}^{-\frac{1}{2}} \left[\check{H}_1\, a(t)\, dt + \check{J}_{12}\, du(t)\right]. \end{aligned} \tag{8.97}$$

It then follows from (8.97) that

$$\begin{aligned} &d\check{w}(t) = \Phi^{-\frac{1}{2}} d\hat{w}(t) + \Phi^{-1} \check{J}_{11}^\dagger \left[\check{H}_1\, a(t)\, dt + \check{J}_{12}\, du(t)\right]; \\ &\|\check{w}(t)\|_2^2 - \|\check{z}(t)\|_2^2 \equiv \|\hat{w}(t)\|_2^2 - \|\hat{z}(t)\|_2^2. \end{aligned} \tag{8.98}$$

Now, the QSDEs (8.92) can be written as

$$\begin{aligned} da(t) &= \hat{F}\, a(t)dt + G_0\, dv(t) + \hat{G}_1\, d\hat{w}(t) + \hat{G}_2\, du(t); \quad a(0) = a_0; \\ d\hat{z}(t) &= \hat{H}_1\, a(t)dt + \hat{J}_{12}\, du(t); \\ dy(t) &= \hat{H}_2\, a(t)dt + J_{20}\, dv(t) + \hat{J}_{21}\, d\hat{w}(t) + \hat{J}_{22}\, du(t) \end{aligned} \tag{8.99}$$

where $d\hat{w}(t) = \hat{\beta}_w(t)dt + d\hat{\nu}(t)$;

$$\begin{array}{rclrcl} \hat{F} &=& \tilde{F} + \check{G}_1 \check{J}_{11}^\dagger \check{\Phi}^{-1} \check{H}_1; & \hat{G}_1 &=& \check{G}_1 \Phi^{-\frac{1}{2}}; \\ \hat{G}_2 &=& \tilde{G}_2 + \check{G}_1 \check{J}_{11}^\dagger \check{\Phi}^{-1} \check{J}_{12}; & \hat{H}_1 &=& \check{\Phi}^{-\frac{1}{2}} \check{H}_1; \\ \hat{H}_2 &=& \tilde{H}_2 + \check{J}_{21} \check{J}_{11}^\dagger \check{\Phi}^{-1} \check{H}_1; & \hat{J}_{12} &=& \check{\Phi}^{-\frac{1}{2}} \check{J}_{12}; \\ \hat{J}_{22} &=& \check{J}_{21} \check{J}_{11}^\dagger \check{\Phi}^{-1} \check{J}_{12}; & \hat{J}_{21} &=& \check{J}_{21} \Phi^{-\frac{1}{2}}. \end{array} \tag{8.100}$$

Note that the $\check{J}_{10}$ term in (8.92) is automatically eliminated when the loop shifting transformation is carried out. Furthermore, we also define

$$d\hat{y}(t) := dy(t) - \hat{J}_{22}\, du(t). \tag{8.101}$$

Substituting (8.101) to (8.99), we obtain

$$\begin{aligned} da(t) &= \hat{F}\, a(t)dt + G_0\, dv(t) + \hat{G}_1\, d\hat{w}(t) + \hat{G}_2\, du(t); \quad a(0) = a_0; \\ d\hat{z}(t) &= \hat{H}_1\, a(t)dt + \hat{J}_{12}\, du(t); \\ d\hat{y}(t) &= \hat{H}_2\, a(t)dt + J_{20}\, dv(t) + \hat{J}_{21}\, d\hat{w}(t). \end{aligned} \tag{8.102}$$

The H^∞ control objective corresponding to the quantum system (8.102) is

$$\begin{aligned} &\int_0^t \left\langle \hat{z}(s)^\dagger \hat{z}(s) + \hat{z}(s)^T \hat{z}(s)^* + \varepsilon \left(\tilde{\eta}(s)^\dagger \tilde{\eta}(s) + \tilde{\eta}(s)^T \tilde{\eta}(s)^*\right)\right\rangle ds \\ &\leq (1-\varepsilon^2) \int_0^t \left\langle \hat{\beta}_w(s)^\dagger \hat{\beta}_w(s) + \hat{\beta}_w(s)^T \hat{\beta}_w(s)^* \right\rangle ds + \pi_1 + \pi_2 t \end{aligned} \tag{8.103}$$

where $\varepsilon, \pi_1, \pi_2 > 0$ are real constants.

The solution to the coherent quantum H^∞ control problem for the quantum system (8.102) is given in terms of the solutions to the complex algebraic Riccati equations

$$\begin{aligned} &\left(\hat{F} - \hat{G}_2 \hat{E}_1^{-1} \hat{J}_{12}^\dagger \hat{H}_1\right)^\dagger \hat{X} + \hat{X}\left(\hat{F} - \hat{G}_2 \hat{E}_1^{-1} \hat{J}_{12}^\dagger \hat{H}_1\right) \\ &\qquad + \hat{X}\left(\hat{G}_1 \hat{G}_1^\dagger - \hat{G}_2 \hat{E}_1^{-1} \hat{G}_2^\dagger\right)\hat{X} + \hat{H}_1^\dagger \left(I - \hat{J}_{12} \hat{E}_1^{-1} \hat{J}_{12}^\dagger\right)\hat{H}_1 = 0\,; \\ &\left(\hat{F} - \hat{G}_1 \hat{J}_{21}^\dagger \hat{E}_2^{-1} \hat{H}_2\right)\hat{Y} + \hat{Y}\left(\hat{F} - \hat{G}_1 \hat{J}_{21}^\dagger \hat{E}_2^{-1} \hat{H}_2\right)^\dagger \\ &\qquad + \hat{Y}\left(\hat{H}_1^\dagger \hat{H}_1 - \hat{H}_2^\dagger \hat{E}_2^{-1} \hat{H}_2\right)\hat{Y} + \hat{G}_1\left(I - \hat{J}_{21}^\dagger \hat{E}_2^{-1} \hat{J}_{21}\right)\hat{G}_1^\dagger = 0 \end{aligned} \tag{8.104}$$

such that the following conditions hold:

1. $\hat{F} - \hat{G}_2 \hat{E}_1^{-1} \hat{J}_{12}^\dagger \hat{H}_1 + \left(\hat{G}_1 \hat{G}_1^\dagger - \hat{G}_2 \hat{E}_1^{-1} \hat{G}_2^\dagger\right)\hat{X}$ is Hurwitz;
2. $\hat{F} - \hat{G}_1 \hat{J}_{21}^\dagger \hat{E}_2^{-1} \hat{H}_2 + \hat{Y}\left(\hat{H}_1^\dagger \hat{H}_1 - \hat{H}_2^\dagger \hat{E}_2^{-1} \hat{H}_2\right)$ is Hurwitz;
3. The spectral radius $\rho(\hat{X}\hat{Y})$ of matrix $\hat{X}\hat{Y}$ is strictly less than one.

To compute the solutions to the Riccati equations (8.104), we need to satisfy the following assumption:

Assumption 8.6. Given constants $\tau_1 > 0, \ldots, \tau_{k+1} > 0$, $\delta_1 > 0, \ldots, \delta_p > 0$, $\beta_1 > 0, \ldots, \beta_p > 0$ and any non-singular scaling matrix R, the artificial uncertain quantum system (8.87), (8.8), (8.28), (8.89) is assumed to be such that $\hat{E}_1 = \hat{J}_{12}^\dagger \hat{J}_{12} > 0$ and $\hat{E}_2 = \hat{J}_{21}\hat{J}_{21}^\dagger > 0$.

Theorem 8.3. *Let $\beta_1 > 0, \ldots, \beta_p > 0$ be given constants and suppose that there exist constants $\kappa_1 > 0, \ldots, \kappa_k > 0$, $\alpha_1 > 0, \ldots, \alpha_p > 0$ satisfying Assumption 8.3 and Assumption 8.4 such that the complex algebraic Riccati equation* (8.83) *has a stabilizing solution $X \geq 0$; and let*

$$K = -\breve{E}_1^{-1}\left(\breve{G}_2^\dagger \breve{X} + \breve{J}_{12}^\dagger \breve{H}_1\right).$$

Also, suppose that there exist a non-singular scaling matrix R and constants $\tau_1 > 0, \ldots, \tau_{k+1} > 0$, $\delta_1 > 0, \ldots, \delta_p > 0$ satisfying Assumption 8.5 and Assumption 8.6 such that the complex algebraic Riccati equations (8.104) *have stabilizing solutions $\hat{X} \geq 0$ and $\hat{Y} \geq 0$ such that the spectral radius $\rho(\hat{X}\hat{Y}) < 1$. Then the closed loop uncertain quantum system obtained by applying the coherent quantum controller of the form* (8.18) *with*

$$\begin{aligned}
F_c &= \hat{F}_c - G_c \hat{J}_{22} H_c;\\
\hat{F}_c &= \hat{F} + \hat{G}_2 H_c - G_c \hat{H}_2 + \left(\hat{G}_1 - G_c \hat{J}_{21}\right)\hat{G}_1^\dagger \hat{X};\\
G_c &= \left(I - \hat{Y}\hat{X}\right)^{-1}\left(\hat{Y}\hat{H}_2^\dagger + \hat{G}_1 \hat{J}_{21}^\dagger\right) E_2^{-1};\\
H_c &= -\hat{E}_1^{-1}\left(\hat{G}_2^\dagger \hat{X} + \hat{J}_{12}^\dagger \hat{H}_1\right)
\end{aligned} \tag{8.105}$$

to the equivalent uncertain quantum system (8.27), (8.24), (8.8), (8.28) *is strict bounded real with disturbance attenuation $\gamma > 0$.*

Proof. It follows from a loop shifting arguments in the classical H^∞ control theory (e.g., see [16, Sections 4.5.1 and 5.5.1] and [18, Section 17.2]) that the H^∞ quantum control problem (8.92), (8.95) has a solution if and only if the complex Riccati equations in (8.104) have stabilizing solutions $\hat{X} \geq 0$ and $\hat{Y} \geq 0$ such that $\rho(\hat{X}\hat{Y}) < 1$. Moreover, a coherent quantum controller of the form (8.18) (but not necessarily stable and strict bounded real), which solves the H^∞ quantum control problem (8.92), (8.95) is defined by (8.105).

Therefore, if the conditions of the theorem are satisfied, it follows from the arguments in the proofs of Theorem 4.1 in [187] and of Theorem 7.1 in [73] that the closed loop uncertain quantum system obtained by applying the coherent quantum controller (8.18), (8.105) to the artificial uncertain quantum system (8.87), (8.8), (8.28), (8.89) is strict bounded real with disturbance attenuation $\gamma > 0$. Moreover, it follows from the construction of the artificial uncertain quantum system (8.87), (8.8), (8.28), (8.89) that the coherent quantum controller (8.18), (8.105) must be stable, strict bounded real and satisfy the condition in Lemma 8.1. Thus, if this controller is applied to the equivalent uncertain quantum system (8.27), (8.24), (8.8), (8.28), the resulting closed loop uncertain quantum system is strict bounded real with disturbance attenuation $\gamma > 0$. $\square$

Theorem 8.4. *Let $\beta_1 > 0, \ldots, \beta_p > 0$ be given constants and suppose that there exist constants $\kappa_1 > 0, \ldots, \kappa_k > 0$, $\alpha_1 > 0, \ldots, \alpha_p > 0$ satisfying Assumption 8.3 and Assumption 8.4 such that the complex algebraic Riccati equation* (8.83) *has a stabilizing solution $X \geq 0$. Also, suppose that there exist a non-singular scaling matrix R and constants $\tau_1 > 0, \ldots, \tau_{k+1} > 0$, $\delta_1 > 0, \ldots, \delta_p > 0$ such that Assumption 8.5 and Assumption 8.6 hold, and the complex algebraic Riccati equations* (8.104) *have stabilizing solutions $\hat{X} \geq 0$ and $\hat{Y} \geq 0$ such that the spectral radius $\rho(\hat{X}\hat{Y}) < 1$. Furthermore, suppose that the coherent quantum H^∞ controller* (8.18), (8.105) *is such that the transfer function matrices in* (8.22) *satisfy the norm bounds in* (8.28) *and each corresponding decentralized quantum controller* (8.11) *is physically realizable. Then, the uncertain closed loop quantum system obtain by applying the decentralized coherent quantum controller* (8.11) *to the uncertain quantum system* (8.1), (8.8) *is strict bounded real with disturbance attenuation $\gamma > 0$.*

Proof. If all conditions of the theorem are satisfied, it then follows from Theorem 8.3 that a closed loop uncertain quantum system obtained by applying the quantum controller (8.18), (8.105) to the equivalent uncertain quantum system (8.27), (8.24), (8.8), (8.28) is strictly bounded real with disturbance attenuation $\gamma > 0$. We also assume that the quantum controller (8.18), (8.105) is such that the transfer function matrices in (8.22) satisfy the norm bounds in (8.28). Furthermore, as mentioned in the construction of the equivalent uncertain quantum system (8.27), (8.24), (8.8), (8.28), a closed loop uncertain quantum system obtained by applying the decentralized quantum controller (8.11) (for all $j = 1, 2, \ldots, p$) to the uncertain quantum system (8.1), (8.6) is identical to a closed loop uncertain quantum system obtained by applying the quantum controller (8.18), (8.105) to the equivalent uncertain quantum system (8.27), (8.24), (8.8), (8.28) when the additional uncertainty inputs defined in (8.23) are applied. Hence, it follows that the decentralized quantum controller (8.11) defined by (8.18), (8.105) is such that the resulting closed loop uncertain quantum system is strict bounded real with disturbance attenuation $\gamma > 0$. □

8.4.2 A differential evolution approach

Likewise in sub-Section 8.3.2, we also provide a numerical algorithm so that the strict bounded real decentralized quantum H^∞ controller synthesis algorithm presented in sub-Section 8.4.1 is applicable. Since nonconvex nonlinear constraints are involved in the synthesis algorithm, we also apply the DE algorithm to find an optimal solution to the decentralized quantum control problem being considered. In this case, we need to reformulate the given control problem as a constrained optimization problem as stated in (8.45) and (8.46).

We now define a vector of decision variables as follows:

$$\vartheta := \begin{bmatrix} \gamma & \kappa & \alpha & \tau & \delta & \beta \end{bmatrix}^T \tag{8.106}$$

where

$$\begin{aligned} &\kappa := \begin{bmatrix} \kappa_1 & \cdots & \kappa_k \end{bmatrix}; \quad \alpha := \begin{bmatrix} \alpha_1 & \cdots & \alpha_p \end{bmatrix}; \quad \tau := \begin{bmatrix} \tau_1 & \cdots & \tau_{k+1} \end{bmatrix}; \\ &\delta := \begin{bmatrix} \delta_1 & \cdots & \delta_p \end{bmatrix}; \quad \beta := \begin{bmatrix} \beta_1 & \cdots & \beta_p \end{bmatrix}. \end{aligned} \tag{8.107}$$

The dimension of ϑ is $2k+3p+2$ and all elements of ϑ are positive real numbers. An objective function and all constraints are then defined in terms of ϑ. Since we are solving a decentralized quantum H^∞ control problem, a suitable objective function $\mathsf{f}(\vartheta)$ to be minimized is as follows:

$$\mathsf{f}(\vartheta) = \mathsf{m}_0\gamma^{\mathsf{n}_0} + \sum_{j=1}^{p} \mathsf{m}_j\beta_j^{\mathsf{n}_j} \tag{8.108}$$

where $\mathsf{m}_0, \mathsf{m}_j \geq 1$ are weighting factors and $\mathsf{n}_0, \mathsf{n}_j \geq 1$ are power constants.

Moreover, the equality constraints are

$$\begin{aligned}
\mathsf{g}_1(\vartheta) &= \left(\breve{F} - \breve{G}_2\breve{E}_1^{-1}\breve{J}_{12}^\dagger\breve{H}_1\right)^\dagger \breve{X} + \breve{X}\left(\breve{F} - \breve{G}_2\breve{E}_1^{-1}\breve{J}_{12}^\dagger\breve{H}_1\right)\\
&\quad + \breve{X}\left(\breve{G}_1\breve{G}_1^\dagger - \breve{G}_2\breve{E}_1^{-1}\breve{G}_2^\dagger\right)\breve{X} + \breve{H}_1^\dagger\left(I - \breve{J}_{12}\breve{E}_1^{-1}\breve{J}_{12}^\dagger\right)\breve{H}_1 = 0;\\
\mathsf{g}_2(\vartheta) &= \left(\hat{F} - \hat{G}_2\hat{E}_1^{-1}\hat{J}_{12}^\dagger\hat{H}_1\right)^\dagger \hat{X} + \hat{X}\left(\hat{F} - \hat{G}_2\hat{E}_1^{-1}\hat{J}_{12}^\dagger\hat{H}_1\right)\\
&\quad + \hat{X}\left(\hat{G}_1\hat{G}_1^\dagger - \hat{G}_2\hat{E}_1^{-1}\hat{G}_2^\dagger\right)\hat{X} + \hat{H}_1^\dagger\left(I - \hat{J}_{12}\hat{E}_1^{-1}\hat{J}_{12}^\dagger\right)\hat{H}_1 = 0;\\
\mathsf{g}_3(\vartheta) &= \left(\hat{F} - \hat{G}_1\hat{J}_{21}^\dagger\hat{E}_2^{-1}\hat{H}_2\right)\hat{Y} + \hat{Y}\left(\hat{F} - \hat{G}_1\hat{J}_{21}^\dagger\hat{E}_2^{-1}\hat{H}_2\right)^\dagger\\
&\quad + \hat{Y}\left(\hat{H}_1^\dagger\hat{H}_1 - \hat{H}_2^\dagger\hat{E}_2^{-1}\hat{H}_2\right)\hat{Y} + \hat{G}_1\left(I - \hat{J}_{21}^\dagger\hat{E}_2^{-1}\hat{J}_{21}\right)\hat{G}_1^\dagger = 0
\end{aligned}\tag{8.109}$$

and the inequality constraints are

$$\begin{array}{rclrcl}
\mathsf{h}_1(\vartheta) &=& \bar{J}_{11}\bar{J}_{11}^\dagger - I < 0; & \mathsf{h}_2(\vartheta) &=& -\breve{E}_1 < 0;\\
\mathsf{h}_3(\vartheta) &=& -\breve{X} < 0; & \mathsf{h}_4(\vartheta) &=& \breve{J}_{11}\breve{J}_{11}^\dagger - I < 0;\\
\mathsf{h}_5(\vartheta) &=& -\hat{E}_1 < 0; & \mathsf{h}_6(\vartheta) &=& -\hat{E}_2 < 0;\\
\mathsf{h}_7(\vartheta) &=& -\hat{X} < 0; & \mathsf{h}_8(\vartheta) &=& -\hat{Y} < 0;\\
\mathsf{h}_9(\vartheta) &=& \rho\left(\hat{X}\hat{Y}\right) - 1 < 0; & \mathsf{h}_{10}(\vartheta) &=& e_{\max,r}\left(\mathcal{A}_{\breve{X}}\right) < 0;\\
\mathsf{h}_{11}(\vartheta) &=& e_{\max,r}\left(\mathcal{A}_{\hat{X}}\right) < 0; & \mathsf{h}_{12}(\vartheta) &=& e_{\max,r}\left(\mathcal{A}_{\hat{Y}}\right) < 0;\\
\mathsf{h}_{13,j}(\vartheta) &=& \|\bar{\Delta}_j(s)\|_\infty^2 - \beta_j \le 0; & \mathsf{h}_{14,j}(\vartheta) &=& \|T_{jj}(s)\|_\infty - 1 \le 0
\end{array}\tag{8.110}$$

for $j = 1, 2, \ldots, p$. Also, we define $\mathcal{A}_{\breve{X}}$, $\mathcal{A}_{\hat{X}}$ and $\mathcal{A}_{\hat{Y}}$ as follows:

$$\begin{aligned}
\mathcal{A}_X &:= \breve{F} - \breve{G}_2\breve{E}_1^{-1}\breve{J}_{12}^\dagger\breve{H}_1 + \left(\breve{G}_1\breve{G}_1^\dagger - \breve{G}_2\breve{E}_1^{-1}\breve{G}_2^\dagger\right)\breve{X};\\
\mathcal{A}_{\hat{X}} &:= \hat{F} - \hat{G}_2\hat{E}_1^{-1}\hat{J}_{12}^\dagger\hat{H}_1 + \left(\hat{G}_1\hat{G}_1^\dagger - \hat{G}_2\hat{E}_1^{-1}\hat{G}_2^\dagger\right)\hat{X};\\
\mathcal{A}_{\hat{Y}} &:= \hat{F} - \hat{G}_1\hat{J}_{21}^\dagger\hat{E}_2^{-1}\hat{H}_2 + \hat{Y}\left(\hat{H}_1^\dagger\hat{H}_1 - \hat{H}_2^\dagger\hat{E}_2^{-1}\hat{H}_2\right).
\end{aligned}\tag{8.111}$$

To examine the fitness of a candidate solution with respect to all equality and inequality constraints in (8.109) and (8.110), we form a fitness test routine, which proceeds through the following steps:

1. Compute the eigenvalues of $(\bar{J}_{11}\bar{J}_{11}^\dagger - I)$ and $\breve{E}_1$ to verify if $\mathsf{h}_1(\vartheta)$ and $\mathsf{h}_2(\vartheta)$ are satisfied.

2. Evaluate $\mathsf{g}_1(\vartheta)$ to obtain a solution $\breve{X}$ to the Riccati equation in (8.83).

3. Verify if $\breve{X}$ is a stabilizing positive definite solution by evaluating $\mathsf{h}_3(\vartheta)$ and $\mathsf{h}_{10}(\vartheta)$.

4. Compute the eigenvalues of $(\breve{J}_{11}\breve{J}_{11}^\dagger - I)$, $\hat{E}_1$ and $\hat{E}_2$ to verify if $\mathsf{h}_4(\vartheta)$, $\mathsf{h}_5(\vartheta)$ and $\mathsf{h}_6(\vartheta)$ hold.

5. Evaluate $\mathsf{g}_2(\vartheta)$ and $\mathsf{g}_3(\vartheta)$ to obtain solutions $\hat{X}$ and $\hat{Y}$ to the Riccati equations in (8.104).

6. Verify if $\hat{X}$ and $\hat{Y}$ are stabilizing positive definite solutions by evaluating $\mathsf{h}_7(\vartheta)$, $\mathsf{h}_8(\vartheta)$, $\mathsf{h}_{11}(\vartheta)$ and $\mathsf{h}_{12}(\vartheta)$.

7. Compute the spectral radius of the product $\hat{X}\hat{Y}$ to verify if $\mathsf{h}_9(\vartheta)$ holds.

8. Evaluate $\mathsf{h}_{13,j}(\vartheta)$ (for $j = 1, 2, \ldots, p$) to check if the j-th off-diagonal block of $T(s)$ in (8.20) satisfies the norm bound condition in (8.28).

9. Verify if the H^∞ norm $\|T_{jj}(s)\|_\infty$ of the j-th diagonal block of $T(s)$ is less than or equal to one by evaluating $\mathsf{h}_{14,j}(\vartheta)$ (for $j = 1, 2, \ldots, p$).

10. Evaluate the objective function $\mathsf{f}(\vartheta)$ in (8.108).

A violation of each constraint in (8.109) and (8.110) is penalized. Thus, we need to define penalty functions, which are then accommodated in the fitness test routine. That is,

$$\begin{array}{rclrcl}
\mathsf{p}_1(\vartheta) &=& e_{\max}(\bar{J}_{11}\bar{J}_{11}^\dagger - I)^{\mathsf{s}_1}; & \mathsf{p}_2(\vartheta) &=& |e_{\min}(\breve{E}_1)|^{\mathsf{s}_2};\\
\mathsf{p}_3(\vartheta) &=& \rho(\mathcal{C}_{\breve{X}})^{\mathsf{s}_3}; & \mathsf{p}_4(\vartheta) &=& |e_{\min}(\breve{X})|^{\mathsf{s}_4};\\
\mathsf{p}_5(\vartheta) &=& e_{\max,r}\left(\mathcal{A}_{\breve{X}}\right)^{\mathsf{s}_5}; & \mathsf{p}_6(\vartheta) &=& e_{\max}(\breve{J}_{11}\breve{J}_{11}^\dagger - I)^{\mathsf{s}_6};\\
\mathsf{p}_7(\vartheta) &=& |e_{\min}(\hat{E}_1)|^{\mathsf{s}_7}; & \mathsf{p}_8(\vartheta) &=& |e_{\min}(\hat{E}_2)|^{\mathsf{s}_8};\\
\mathsf{p}_9(\vartheta) &=& \rho(\mathcal{C}_{\hat{X}})^{\mathsf{s}_9}; & \mathsf{p}_{10}(\vartheta) &=& \rho(\mathcal{C}_{\hat{Y}})^{\mathsf{s}_{10}};\\
\mathsf{p}_{11}(\vartheta) &=& |e_{\min}(\hat{X})|^{\mathsf{s}_{11}}; & \mathsf{p}_{12}(\vartheta) &=& |e_{\min}(\hat{Y})|^{\mathsf{s}_{12}};\\
\mathsf{p}_{13}(\vartheta) &=& e_{\max,r}\left(\mathcal{A}_{\hat{X}}\right)^{\mathsf{s}_{13}}; & \mathsf{p}_{14}(\vartheta) &=& e_{\max,r}\left(\mathcal{A}_{\hat{Y}}\right)^{\mathsf{s}_{14}};\\
\mathsf{p}_{15}(\vartheta) &=& (\rho(\hat{X}\hat{Y}) - 1)^{\mathsf{s}_{15}}; & \mathsf{p}_{16}(\vartheta) &=& \sum_{j=1}^p \mathcal{D}_j^{\mathsf{c}_j};\\
\mathsf{p}_{17}(\vartheta) &=& \sum_{j=1}^p \mathcal{S}_j^{\mathsf{d}_j}; & \mathsf{p}_{18}(\vartheta) &=& \mathsf{f}(\vartheta)
\end{array} \quad (8.112)$$

where $\mathsf{s}_\mathsf{r} \geq 1$ for $\mathsf{r} = 1, 2, \ldots, 15$ and $\mathsf{c}_j, \mathsf{d}_j \geq 1$ for $j = 1, 2, \ldots, p$. Moreover, we also define

$$
\begin{aligned}
\mathcal{C}_{\breve{X}} &:= \breve{H}_1^\dagger \left(I - \breve{J}_{12} \breve{E}_1^{-1} \breve{J}_{12}^\dagger \right) \breve{H}_1; \\
\mathcal{C}_{\hat{X}} &:= \hat{H}_1^\dagger \left(I - \hat{J}_{12} \hat{E}_1^{-1} \hat{J}_{12}^\dagger \right) \hat{H}_1; \\
\mathcal{C}_{\hat{Y}} &:= \hat{G}_1 \left(I - \hat{J}_{21}^\dagger \hat{E}_2^{-1} \hat{J}_{21} \right) \hat{G}_1^\dagger; \\
\mathcal{D}_j &:= \begin{cases} \|\bar{\Delta}_j(s)\|_\infty^2, & \text{if } \mathsf{h}_{13,j}(\vartheta) \text{ is violated;} \\ 0, & \text{otherwise;} \end{cases} \\
\mathcal{S}_j &:= \begin{cases} \|T_{jj}(s)\|_\infty, & \text{if } \mathsf{h}_{14,j}(\vartheta) \text{ is violated;} \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{8.113}
$$

8.4.3 Illustrative examples

We now present two examples of designing strict bound real decentralized quantum H^∞ controllers using the synthesis algorithm described in sub-Section 8.4.1. The quantum systems under consideration belong to a class of quantum optical systems with only passive components; e.g., see [70, 73]. Moreover, we can use the same approach as in sub-Section 8.3.3 to build the decentralized quantum controllers using passive components such as optical cavities, beam splitters and phase shifters; see [77].

Example 1: Cascaded two optical cavities

As the first example, we aim to construct strict bounded real decentralized quantum H^∞ controllers for the quantum system (8.67) as shown in Figure 8.4. Using the DE-based algorithm in sub-Section 8.4.2, we obtain

$$
\begin{aligned}
\gamma &= 1.1652; & \alpha_1 &= 1110.4129; & \alpha_2 &= 1051.0836; & \tau_1 &= 2.5962; \\
\delta_1 &= 15.3329; & \delta_2 &= 18.1159; & \beta_1 &= 0.0012; & \beta_2 &= 0.0013.
\end{aligned} \tag{8.114}
$$

Note that τ_1 corresponds to the artificial uncertainty, which is added to guarantee the stability and strict bounded real property of the decentralized quantum controller while achieving the closed loop H^∞ control objective.

We then use the parameters in (8.114) to construct a non-decentralized quantum H^∞ controller, which is given as follows:

$$\begin{aligned} F_c &= \begin{bmatrix} -3.5313 & -3.2273 \\ -13.0894 & -21.1233 \end{bmatrix}; \quad G_c = \begin{bmatrix} -0.5870 & -0.5541 \\ 0.0305 & -0.3741 \end{bmatrix}; \\ H_c &= \begin{bmatrix} -0.1320 & -0.0564 \\ -0.0715 & -0.2619 \end{bmatrix}. \end{aligned} \tag{8.115}$$

For this controller, the additional uncertainties $\bar{\Delta}_1(s)$ and $\bar{\Delta}_2(s)$ as defined in (8.22) have the squared H^∞ norm:

$$\|\bar{\Delta}_1(s)\|_\infty^2 = 0.0011; \quad \|\bar{\Delta}_2(s)\|_\infty^2 = 0.0013. \tag{8.116}$$

It is clear that $\|\bar{\Delta}_1(s)\|_\infty^2 \leq \beta_1$ and $\|\bar{\Delta}_2(s)\|_\infty^2 \leq \beta_2$. Thus, we can form the decentralized quantum H^∞ controllers for the quantum system (8.67) as follows:

$$\begin{aligned} F_{c_1} &= F_c; \quad G_{c_1} = \begin{bmatrix} -0.5870 \\ 0.0305 \end{bmatrix}; \quad H_{c_1} = \begin{bmatrix} -0.1320 & -0.0564 \end{bmatrix}; \\ F_{c_2} &= F_c; \quad G_{c_2} = \begin{bmatrix} -0.5541 \\ -0.3741 \end{bmatrix}; \quad H_{c_2} = \begin{bmatrix} -0.0715 & -0.2619 \end{bmatrix}. \end{aligned} \tag{8.117}$$

The eigenvalues of F_c are $e_1 = -1.3905$ and $e_2 = -23.2640$ and hence, F_c is Hurwitz. The H^∞ norms of the controllers in (8.117) are

$$\|T_{11}(s)\|_\infty = 0.0374; \quad \|T_{22}(s)\|_\infty = 0.0248. \tag{8.118}$$

Thus, according to Lemma 8.1, $\{F_{c_1}, G_{c_1}, H_{c_1}\}$ and $\{F_{c_2}, G_{c_2}, H_{c_2}\}$ in (8.117) are strict bounded real and physically realizable. Moreover, we proceed along the same steps as in (7.107) and (7.108) to compute $G_{w_{c0,1}}$, $G_{w_{c0,2}}$, $H_{c2,1}$ and $H_{c2,2}$:

$$\begin{aligned} G_{w_{c0,1}} &= \begin{bmatrix} 76.8941 \\ 694.7391 \end{bmatrix}; \quad H_{c2,1} = \begin{bmatrix} 0.0018 & -0.0001 \end{bmatrix}; \\ G_{w_{c0,2}} &= \begin{bmatrix} 62.0070 \\ 171.3056 \end{bmatrix}; \quad H_{c2,2} = \begin{bmatrix} 0.0012 & 0.0004 \end{bmatrix}. \end{aligned} \tag{8.119}$$

so that the decentralized quantum controllers (8.117) are physically realizable according to Definition 8.1.

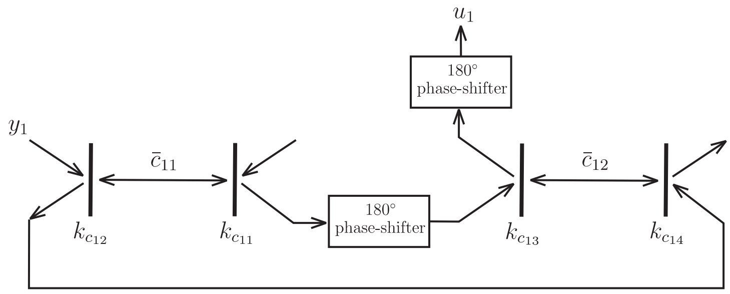


Figure 8.6: The first decentralized quantum H^∞ controller.

Using the algorithm in [77], we apply similarity transformation to the decentralized quantum controllers (8.117), (8.119). This results in equivalent realizations, which can be physically constructed using only passive optical elements such as optical cavities, beam splitters and phase shifters. Thus, the equivalent realization for the first decentralized quantum controller is

$$\begin{aligned} \bar{F}_{c_1} &= \begin{bmatrix} -1.3905 & 0 \\ 11.3733 & -23.2640 \end{bmatrix}; \\ \begin{bmatrix} \bar{G}_{w_{c_{0,1}}} & \bar{G}_{c_1} \end{bmatrix} &= \begin{bmatrix} -1.6674 & -0.0269 \\ 6.8211 & -0.0177 \end{bmatrix}; \\ \begin{bmatrix} \bar{H}_{c_1} \\ \bar{H}_{c_{2,1}} \end{bmatrix} &= \begin{bmatrix} 1.6674 & -6.8211 \\ 0.0269 & 0.0177 \end{bmatrix} \end{aligned} \tag{8.120}$$

and that for the second decentralized quantum controller is

$$\begin{aligned} \bar{F}_{c_2} &= \begin{bmatrix} -1.3905 & 0 \\ -11.3744 & -23.2640 \end{bmatrix}; \\ \begin{bmatrix} \bar{G}_{w_{c_{0,2}}} & \bar{G}_{c_2} \end{bmatrix} &= \begin{bmatrix} 1.6675 & -0.0242 \\ 6.8211 & -0.0143 \end{bmatrix}; \\ \begin{bmatrix} \bar{H}_{c_2} \\ \bar{H}_{c_{2,2}} \end{bmatrix} &= \begin{bmatrix} -1.6675 & -6.8211 \\ 0.0242 & 0.0143 \end{bmatrix}. \end{aligned} \tag{8.121}$$

The realizations (8.120) and (8.121) imply that each decentralized quantum controller can be built as a cascade of two first order generalized 2-mirror optical cavities because it has two inputs and two outputs; see [77].

The first decentralized quantum controller (8.120) is depicted in Figure 8.6

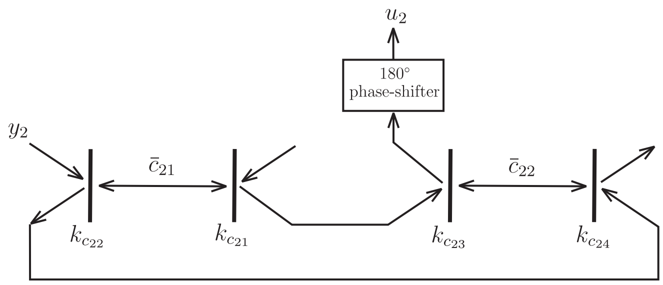


Figure 8.7: The second decentralized quantum H^∞ controller.

with the parameters:

$$\begin{aligned} k_{c_{11}} &= (1.6674)^2; \quad k_{c_{12}} = (0.0269)^2; \\ k_{c_{13}} &= (6.8211)^2; \quad k_{c_{14}} = (0.0177)^2. \end{aligned} \tag{8.122}$$

Also, the second decentralized quantum controller (8.121) is depicted in Figure 8.7 with the parameters:

$$\begin{aligned} k_{c_{21}} &= (1.6675)^2; \quad k_{c_{22}} = (0.0242)^2; \\ k_{c_{23}} &= (6.8211)^2; \quad k_{c_{24}} = (0.0143)^2. \end{aligned} \tag{8.123}$$

These results show that for the quantum system (8.67), we are indeed able to obtain the decentralized quantum H^∞ controllers, which are physically realizable, using the controller design algorithm in sub-Section 8.4.1.

Remark 8.3. We notice that the value of γ in (8.114) is larger than that in (8.69). This is reasonable because we force the decentralized quantum controllers in (8.117) to be physical realizable when applying the synthesis algorithm presented in sub-Section 8.4.1. Conversely, although it yields a smaller γ, the synthesis algorithm in sub-Section 8.3.1 results in a second decentralized quantum controller in (8.72), which is not physically realizable.

Example 2: Cascaded three optical cavities

We now consider a quantum system consisting of three interconnected subsystems, but which only have two control inputs as shown in Figure 8.8. Using this example, we show that our method can also be used to design a decentralized control system for a large-scale quantum system where the number of subsystems is more than the number of decentralized controllers. This is possible because

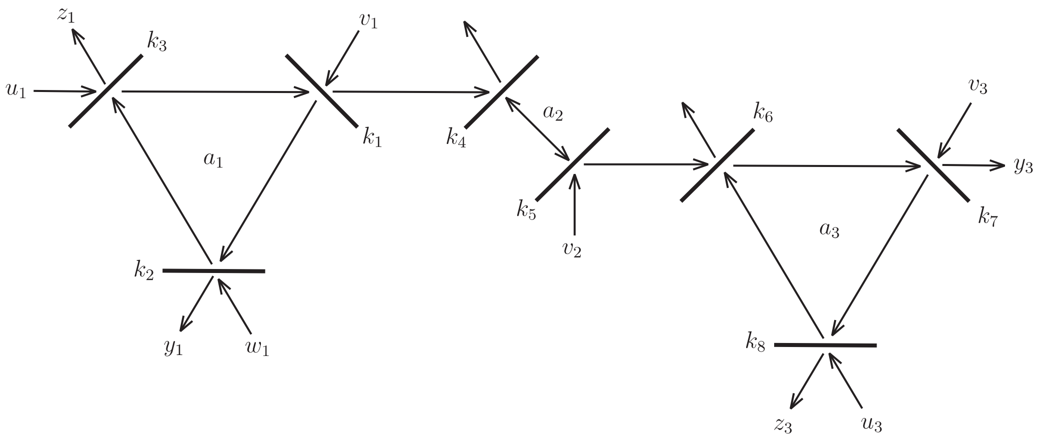


Figure 8.8: A cascaded linear quantum system of three optical cavities.

the interconnections between subsystems are known and thus, not treated as uncertainties; e.g., see [197]. In our example, although we only have direct control action on the first and the third subsystems, the second subsystem can also be stabilized as its linear interconnection with the first subsystem is known.

The quantum optical network shown in Figure 8.8 can be represented as

$$\begin{bmatrix} da_1(t) \\ da_2(t) \\ da_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{g_1}{2} & 0 & 0 \\ -\sqrt{k_1 k_4} & -\frac{g_2}{2} & 0 \\ 0 & -\sqrt{k_5 k_6} & -\frac{g_3}{2} \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \end{bmatrix} dt + \begin{bmatrix} -\sqrt{k_3} & 0 \\ 0 & 0 \\ 0 & -\sqrt{k_8} \end{bmatrix} \begin{bmatrix} du_1(t) \\ du_3(t) \end{bmatrix}$$
$$+ \begin{bmatrix} -\sqrt{k_1} & 0 & 0 \\ -\sqrt{k_4} & -\sqrt{k_5} & 0 \\ 0 & -\sqrt{k_6} & -\sqrt{k_7} \end{bmatrix} \begin{bmatrix} dv_1(t) \\ dv_2(t) \\ dv_3(t) \end{bmatrix} + \begin{bmatrix} -\sqrt{k_2} \\ 0 \\ 0 \end{bmatrix} dw_1(t);$$
$$\begin{bmatrix} dz_1(t) \\ dz_3(t) \end{bmatrix} = \begin{bmatrix} \sqrt{k_3} & 0 & 0 \\ 0 & 0 & \sqrt{k_8} \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} du_1(t) \\ du_3(t) \end{bmatrix};$$
$$\begin{bmatrix} dy_1(t) \\ dy_3(t) \end{bmatrix} = \begin{bmatrix} \sqrt{k_2} & 0 & 0 \\ 0 & 0 & \sqrt{k_7} \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \end{bmatrix} dt + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} dv_1(t) \\ dv_2(t) \\ dv_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} dw_1(t) \quad (8.124)$$

where $g_1 = k_1 + k_2 + k_3$, $g_2 = k_4 + k_5$ and $g_3 = k_6 + k_7 + k_8$. The parameters in (8.124) have the values

$$k_1 = 2.25; \quad k_2 = 1.00; \quad k_3 = 1.00; \quad k_4 = 1.00; \qquad (8.125)$$

$$k_5 = 0.50; \quad k_6 = 1.21; \quad k_7 = 0.50; \quad k_8 = 0.50. \tag{8.126}$$

Here, we also assume that the quantum system (8.124) is known and therefore, does not have any uncertainty terms in its model.

Using the DE approach in sub-Section 8.4.2, we obtain the values of all parameters required for the synthesis of the decentralized quantum controllers as

$$\begin{aligned} \gamma &= 0.9294; & \alpha_1 &= 4966.6123; & \alpha_2 &= 4.6037; & \tau_1 &= 1.5146; \\ \delta_1 &= 5.1056; & \delta_2 &= 3.3892; & \beta_1 &= 0.0005; & \beta_2 &= 0.0063. \end{aligned} \tag{8.127}$$

The parameters in (8.127) are then used to construct a non-decentralized strict bounded real quantum H^∞ controller, which are defined by

$$\begin{aligned} F_c &= \begin{bmatrix} -2.1957 & -0.4944 & 0.2830 \\ -1.5075 & -0.7530 & 0.0135 \\ -0.0625 & -0.6933 & -1.0340 \end{bmatrix}; \quad G_c = \begin{bmatrix} -0.3574 & 0.0582 \\ 0.0370 & -0.1492 \\ -0.0729 & 0.1097 \end{bmatrix}; \\ H_c &= \begin{bmatrix} -0.4021 & -0.1050 & 0.0688 \\ 0.0487 & -0.0972 & -0.2760 \end{bmatrix}. \end{aligned} \tag{8.128}$$

The matrices G_c and H_c in (8.128) are such that the additional uncertainties $\bar{\Delta}_1(s)$ and $\bar{\Delta}_2(s)$ as defined in (8.22) have the following squared H^∞ norm:

$$\|\bar{\Delta}_1(s)\|_\infty^2 = 8.1725 \times 10^{-5}; \quad \|\bar{\Delta}_2(s)\|_\infty^2 = 0.0057. \tag{8.129}$$

Indeed, $\|\bar{\Delta}_1(s)\|_\infty^2$ and $\|\bar{\Delta}_2(s)\|_\infty^2$ are less than β_1 and β_2, respectively. Thus, we can form the decentralized quantum H^∞ controllers as follows:

$$\begin{aligned} F_{c_1} &= F_c; \quad G_{c_1} = \begin{bmatrix} -0.3574 \\ 0.0370 \\ -0.0729 \end{bmatrix}; \quad H_{c_1} = \begin{bmatrix} -0.4021 & -0.1050 & 0.0688 \end{bmatrix}; \\ F_{c_2} &= F_c; \quad G_{c_2} = \begin{bmatrix} 0.0582 \\ -0.1492 \\ 0.1097 \end{bmatrix}; \quad H_{c_2} = \begin{bmatrix} 0.0487 & -0.0972 & -0.2760 \end{bmatrix} \end{aligned} \tag{8.130}$$

The eigenvalues of the matrix F_c are

$$e_1 = -2.4940; \quad e_2 = -0.2126; \quad e_3 = -1.2761 \tag{8.131}$$

and thus, F_c is Hurwitz. Moreover, the H^∞ norms of the decentralized quantum controllers are

$$\|T_{11}(s)\|_\infty = 0.0500; \quad \|T_{22}(s)\|_\infty = 0.0742. \tag{8.132}$$

Thus, according to Lemma 8.1, $\{F_{c_1}, G_{c_1}, H_{c_1}\}$ and $\{F_{c_2}, G_{c_2}, H_{c_2}\}$ are strict bounded real and physically realizable.

Following (7.107) and (7.108), we can compute $G_{w_{c0,1}}$, $G_{w_{c0,2}}$, $H_{c2,1}$ and $H_{c2,2}$ (see Definition 8.1):

$$\begin{aligned}
G_{w_{c0,1}} &= \begin{bmatrix} -58.5256 \\ -129.4753 \\ -655.8940 \end{bmatrix}; \quad H_{c_{2,1}} = \begin{bmatrix} 0.0103 & 0.0030 & -0.0017 \end{bmatrix}; \\
G_{w_{c0,2}} &= \begin{bmatrix} 112.2091 \\ 79.9897 \\ 20.4744 \end{bmatrix}; \quad H_{c_{2,2}} = \begin{bmatrix} -0.0004 & 0.0012 & -0.0032 \end{bmatrix}
\end{aligned} \tag{8.133}$$

to physically realize the decentralized quantum controllers in (8.130). Furthermore, given the controller matrices in (8.128), (8.130) and (8.133), we can now apply the algorithm in [77] so that both decentralized quantum controllers can be constructed using passive optical components such as optical cavities, beam splitters and phase shifters. Thus, the first decentralized quantum controller is represented by

$$\begin{aligned}
\bar{F}_{c_1} &= \begin{bmatrix} -2.4940 & 0 & 0 \\ -1.4561 & -0.2126 & 0 \\ -3.5668 & -1.0416 & -1.2761 \end{bmatrix}; \\
\begin{bmatrix} \bar{G}_{w_{c0,1}} & \bar{G}_{c_1} \end{bmatrix} &= \begin{bmatrix} 2.2333 & -0.0173 \\ 0.6519 & -0.0145 \\ 1.5967 & -0.0543 \end{bmatrix}; \\
\begin{bmatrix} \bar{H}_{c_1} \\ \bar{H}_{c_{2,1}} \end{bmatrix} &= \begin{bmatrix} -2.2333 & -0.6519 & -1.5967 \\ 0.0173 & 0.0145 & 0.0543 \end{bmatrix}
\end{aligned} \tag{8.134}$$

and the second decentralized quantum controller is represented by

$$\bar{F}_{c_2} = \begin{bmatrix} -2.4940 & 0 & 0 \\ 1.4557 & -0.2126 & 0 \\ 3.5679 & -1.0409 & -1.2761 \end{bmatrix};$$
$$\begin{bmatrix} \bar{G}_{w_{c0,2}} & \bar{G}_{c_2} \end{bmatrix} = \begin{bmatrix} -2.2334 & -0.0001 \\ 0.6518 & -0.0180 \\ 1.5975 & 0.0153 \end{bmatrix};$$
$$\begin{bmatrix} \bar{H}_{c_2} \\ \bar{H}_{c_{2,2}} \end{bmatrix} = \begin{bmatrix} 2.2334 & -0.6518 & -1.5975 \\ 0.0001 & 0.0180 & -0.0153 \end{bmatrix} \tag{8.135}$$

The realizations in (8.134) and (8.135) indicate that each controller is of third order and has two inputs and two outputs. Hence, it can be built as a cascade of three first order generalized 2-mirror optical cavities; see [77].

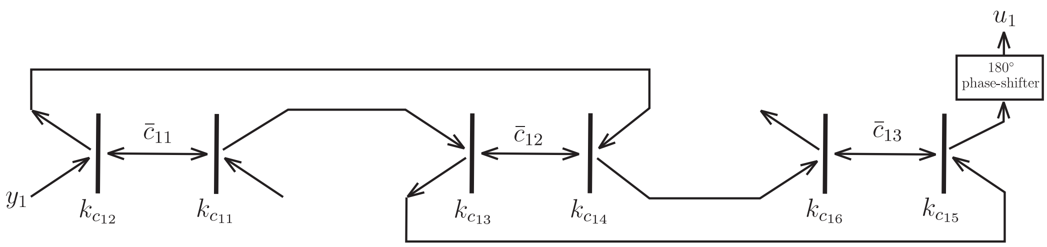


Figure 8.9: The first strict bounded real decentralized quantum H^∞ controller.

The first decentralized quantum controller (8.134) is illustrated in Figure 8.9 with the parameters:

$$\begin{aligned} k_{c_{11}} &= (2.2333)^2; & k_{c_{12}} &= (0.0173)^2; & k_{c_{13}} &= (0.6519)^2; \\ k_{c_{14}} &= (0.0145)^2; & k_{c_{15}} &= (1.5967)^2; & k_{c_{16}} &= (0.0543)^2. \end{aligned} \tag{8.136}$$

Also, the second decentralized quantum controller (8.135) has the following parameters:

$$\begin{aligned} k_{c_{21}} &= (2.2334)^2; & k_{c_{22}} &= (0.0001)^2; & k_{c_{23}} &= (0.6518)^2; \\ k_{c_{24}} &= (0.0180)^2; & k_{c_{25}} &= (1.5975)^2; & k_{c_{26}} &= (0.0153)^2. \end{aligned} \tag{8.137}$$

and is shown in Figure 8.10.

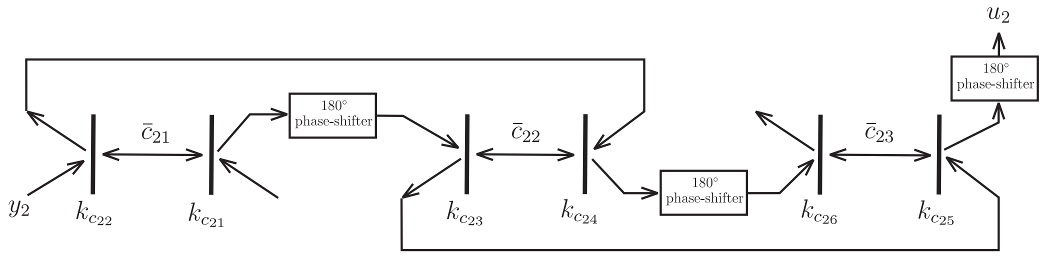


Figure 8.10: The second strict bounded real decentralized quantum H^∞ controller.

8.5 Conclusions

We have presented two systematic methods to design a decentralized coherent robust H^∞ quantum controller for a class of large-scale uncertain linear complex quantum stochastic systems with norm-bounded structured uncertainties. The large-scale quantum system is comprised of interconnected quantum subsystems where the interconnections are assumed to be partly or fully known and hence, not treated as uncertainties. Instead, we consider the neglected off-diagonal blocks of the transfer function matrix of the non-decentralized quantum controller as additional uncertainties. Thus, the resulting decentralized coherent quantum controller is robust against both the structured uncertainties in the quantum plant model and those additional uncertainties. Moreover, the quantum controller is also required to be physical realizable, which implies that the quantum controller has to be stable and bounded real.

Applying the first method, we do not immediately obtain a physically realizable decentralized quantum controller. This is because the physical realizability condition is not directly included in the controller synthesis algorithm. Meanwhile, if we apply the second method, the resulting quantum controller must be stable and strict bounded real, and hence, it is indeed physically realizable. This is achieved by introducing an artificial uncertainty to the equivalent uncertain quantum system for which a strict bounded real non-decentralized quantum controller is designed. However, this approach may result in a more conservative decentralized quantum controller than the one obtained using the first method due to the use of the additional artificial uncertainty. Also, the second method has more constraints and design parameters than the first method, and therefore, may take longer computational time to solve a quantum control problem under consideration.

To deal with the nonconvex nonlinear constraints involved in both methods, we apply the DE algorithm to compute all design parameters required for controller synthesis. Thus, we reformulate the decentralized quantum control problem as a constrained nonlinear optimization problem. Moreover, two examples of quantum optical systems are considered to demonstrate the proposed decentralized quantum controller design methods using the DE apparoach. In these examples, we have also applied the algorithm in [77] to show that an n-th order decentralized quantum optical control system with m inputs can be physically constructed using purely passive optical devices such as optical cavities, beam splitters and phase shifters. This approach results in an m-input-m-output quantum optical controller with a cascade interconnection of n first order generalized m-mirror optical cavities.

Chapter 9

Conclusions and Future Research

This chapter presents final conclusions of our research which develops new methods to synthesize non-decentralized and decentralized robust feedback control systems for classical and quantum dynamical systems. In particular, the main concerns and contributions of Chapter 3 – Chapter 8 are summarized in Section 9.1. Potential future research areas are also described in Section 9.2 in order to suggest possible extensions to what we have done in our research.

9.1 Conclusions

The main results of our research presented in this thesis consist of two parts: Chapter 3 – Chapter 5 and Chapter 6 – Chapter 8, which are designated for classical and quantum dynamical systems, respectively. A common feature in both parts is that we employ the DE algorithm given in Chapter 2 to solve nonconvex nonlinear constrained optimization problems arising in feedback control syntheses for those systems. A particular variant of the DE algorithm applied in this thesis is drift-free $DE/rand/1/either-or$. As a class of evolutionary algorithms, the DE algorithm is equipped with variation operators: mutation and recombination, and a selection operator. These operators are used to explore and exploit a numerical search space in which the optimization takes place. In addition, we also apply a static penalty-based fitness test procedure, which serves as a link to the particular controller design algorithm being considered. Each penalty function is formed based on a constraint to satisfy during the fitness test and is only applied when the constraint is violated by a candidate solution.

Specific control problems and the main contributions of Chapter 3 – Chapter 8 are described as follows:

1. In Chapter 3, we present a systematic method to construct a stable nonlinear robust H^∞ output feedback controller for a class of nonlinear uncertain systems. The admissible uncertainties and nonlinearities in the system being controlled are required to satisfy IQCs and GLCs, respectively. The H^∞ control objective is to achieve closed loop absolute stability with a specified disturbance attenuation level. Our approach to construct the nonlinear controller is to add a copy of each nonlinearity to the linear part of the controller. This is to enable the controller to exploit the system nonlinearities. Applying a standard robust H^∞ control method (see [187]) to derive the controller design algorithm, we include all copies of the nonlinearities back into the system and also characterize them with IQCs derived from GLCs.

   To ensure stability of the controller, we first solve a corresponding state feedback control problem and then introduce an additional uncertainty to form an artificial uncertain system based on the original uncertain system. Any suitable controller for the artificial uncertain system is guaranteed to be stable and solves the original control problem. The controller matrices are written in terms of stabilizing solutions to algebraic Riccati equations, which are parameterized by scaling constants associated with all IQCs. Note that the use of additional artificial uncertainty gives rise to extra conservatism in the controller design method.

2. In Chapter 4, we are concerned with decentralized state feedback robust H^∞ control for a class of large-scale linear uncertain systems. The uncertainties in the system are required to satisfy IQCs in order to be admissible. The closed loop control objective is to achieve absolute stability with a specified disturbance attenuation level. Here, the large-scale system is composed of interconnected subsystems. We assume that the interconnections between subsystems are known and therefore, we do not treat them as uncertainties, which may degrade the system performance. Instead, we neglect off-diagonal blocks of a corresponding non-decentralized state feedback gain matrix and consider them as additional uncertainties. The decentralized controllers are then capable of exploiting the interconnections and are ro-

bust against uncertainties in the systems and in themselves. This approach is useful when the number of controllers is less than the number of subsystems. The decentralized state feedback controllers are constructed using a stabilizing solution to an algebraic Riccati equation, which is parameterized by scaling constants associated with all IQCs.

3. We combine the ideas in Chapter 3 and Chapter 4 in order to derive the controller design algorithm presented in Chapter 5. In this case, we are interested in decentralized nonlinear robust H^∞ output feedback control for a class of large-scale nonlinear uncertain systems. The uncertainties and nonlinearities in the large-scale system have to satisfy IQCs and GLCs, respectively, in order to be admissible. The interconnections between subsystems are also not treated as uncertainties, but rather as useful structural information on the entire system although we do not assume how the subsystems are interconnected. However, nonlinear error systems arising from discrepancies between non-decentralized and decentralized controllers are then considered as additional uncertainties. This implies that the decentralized nonlinear controllers must be stable and are able to exploit the known interconnections while absolutely stabilizing the resulting closed loop system with a prescribed disturbance attenuation level.

   Stable decentralized controllers are obtained using the same approach as in Chapter 3, which involves solving a state feedback control problem and forming an artificial uncertain system. The use of artificial uncertainty thus introduces some additional conservatism to the controller design process. Moreover, the decentralized controllers are constructed using stabilizing solutions to algebraic Riccati equations parameterized by scaling constants corresponding to all IQCs.

4. The main contribution of Chapter 6 is a DE-based algorithm to synthesize a coherent quantum feedback controller for a class of linear quantum systems represented in terms of linear QSDEs with real and complex quadratures. The quantum controller must be physically realizable so that it exhibits meaningful dynamics according to the laws of quantum mechanics. The physical realizability condition for the quantum controller is converted into a complex algebraic Riccati equation whose coefficients are the controller

matrices. The Riccati equation is required to have an imaginary Hermitian solution, which may not be unique, in order that the quantum controller is physically realizable. We thus always include the Riccati equation as an equality constraint when we solve the quantum controller synthesis problem as a nonconvex nonlinear optimization problem using the DE-based algorithm. In this case, we are quite flexible to determine an objective function to optimize according to the particular control problem under consideration. Therefore, our approach can also be adapted to solve the coherent quantum H^∞ and LQG control problems considered in [70, 72].

The proposed DE-based algorithm is effectively used to solve an entanglement enhancement problem for an ideal quantum optical network. Here, we aim to construct a physically realizable quantum controller, which not only stabilizes the quantum network, but also enhances an initial entanglement level of the quantum network. This example indicates potential future application of a dynamic coherent quantum controller to preserve and enhance the entanglement level of a real quantum network.

5. We present a new method in Chapter 7 to construct a coherent quantum robust H^∞ controller for a class of linear complex quantum systems with norm-bounded structured uncertainties. The dynamics of an uncertain quantum system in this class is determined only by annihilation operators and is represented in terms of linear QSDEs with complex coefficients as described in [73]. Moreover, the quantum controller is required to be stable and strict bounded real, and therefore, is guaranteed to be physically realizable. The purpose of applying this quantum controller is to obtain a strict bounded real closed loop quantum system with a specified disturbance attenuation level.

   The desired quantum controller can be synthesized using the method in [8], which involves forming an artificial quantum uncertain system. Any suitable quantum controller for the artificial uncertain quantum system must be stable and strict bounded real, and also solves the original quantum robust H^∞ control problem. The quantum controller matrices are then constructed using stabilizing solutions to complex algebraic Riccati equations, which are parameterized by scaling constants associated with all uncertainties. The use of artificial uncertainty in our approach, however, introduces extra conservatism to the controller design method.

To demonstrate the proposed controller design algorithm, we consider an example of designing a strict bounded real quantum controller for a quantum optical system. This example cannot be solved using the method in [73]. We also show that the resulting quantum controller can be physically constructed using an algorithm in [77] with only passive optical elements such as optical cavities, beam splitters and phase shifters.

6. In Chapter 8, we propose two systematic methods to design a decentralized quantum robust H^∞ control system for a class of large-scale linear complex quantum systems with norm-bounded structured uncertainties. This class of uncertain quantum systems is the same as that in Chapter 7. We assume that the large-scale uncertain quantum system is composed of quantum subsystems with known interconnections between them. Nevertheless, we do not assume any structure on how the quantum subsystems are interconnected. Applying the decentralized quantum controller, we want to achieve a strict bounded real closed loop quantum system with a specified disturbance attenuation level.

   We follow the approach in [166] to derive both of the decentralized quantum controller design methods. In this case, since the interconnections are assumed to be known, they are not treated as uncertainties. Instead, we neglect off-diagonal parts of the transfer function matrix of a corresponding non-decentralized quantum controller. Those neglected parts are then considered as additional uncertainties in the large-scale uncertain quantum system. This approach then provides robustness to the decentralized quantum controller against uncertainties in the quantum system and the additional uncertainties.

   Applying the first method, we do not immediately obtain a physically realizable decentralized quantum control system. This is because we employ the results in [73] to construct the decentralized quantum controller matrices and the physically realizability condition is not directly imposed. Thus, we must always check if the decentralized quantum control system is physically realizable before it is implemented. This in turn motivates us to propose the second method, which is based on the results in Chapter 7. Using the second method, we are then guaranteed to obtain a strict bounded real decentralized quantum control system, which must be physically realizable. However, the use of an artificial uncertainty in the second method may introduce some extra conservatism in the controller design process. Despite the differences, both methods involve finding stabilizing solutions to complex algebraic Riccati equations and those solutions are used to form the controller matrices. The Riccati equations are parameterized by scaling constants corresponding to all uncertainties in the quantum system.

The efficacy of both methods is demonstrated through examples of controlling quantum optical networks consisting of passive optical elements. The resulting decentralized quantum controllers can be physically constructed using the algorithm in [77] with only passive optical elements such as optical cavities, beam splitters and phase shifters. Moreover, we also show that there is a case when the first method fails to provide a physically realizable decentralized quantum controller. The same case, however, can be solved using the second method.

9.2 Future Research

To extend the results of our research presented in this thesis, we suggest some possible future research directions as follows:

1. Regarding the results in Chapter 3 and Chapter 5, it is possible to relax the constraint on the nonlinearity from a local Lipschitz condition to consider other types of nonlinearity satisfying conditions such as monotonicity and restricted slope conditions; e.g., see [6, 185, 323]. Moreover, we can employ the ideas in those chapters to construct a stable nonlinear guaranteed cost controller for handling a worst-case-performance control problem; e.g., see [2]. Also, we can apply a reduced order controller and take into account time delays in the system, especially when dealing with a large-scale system as presented in Chapter 4 and Chapter 5; e.g., see [18, 324–326].

2. The quantum controller design algorithm presented in Chapter 6 is applicable to solve entanglement control problems for more realistic quantum

networks; e.g., see [61, 233]. Also, we may extend the ideas in Chapter 7 and Chapter 8 to solve quantum robust H^∞ and guaranteed cost control problems for a larger class of uncertain quantum systems driven by both annihilation and creation operators; e.g., see [34]. In addition, ideas of constructing a reduced order quantum controller and considering time delays in the quantum system are also of interest; e.g., see [327,328]. All these extensions are potentially useful for investigating future applications of quantum information flow control within quantum communication networks based on network flow control theory for classical systems; e.g., see [329]. Moreover, to achieve better understanding about coherent linear quantum control systems, we may approximately simulate the examples in Chapter 6 – Chapter 8 within MATLAB environment using, for example, quantum optics toolbox as described in [330, 331]. However, in general, simulating quantum systems on classical computers remains a great challenge because it requires enormous memory resources for quantum information storage; e.g., see [332–334].

3. When applying DE-based algorithms to solve our control problems in Chapter 3 – Chapter 8, it often takes a large amount of computation time to obtain the desired solutions. Thus, to improve the performance of our IQC-based algorithms, we may use parallel computation and/or combine the DE approach with reliable problem-specific algorithms for (non)-convex and/or nonsmooth optimization; e.g., see [102, 145, 159, 271, 273, 335–339]. Also, we can apply a self-adaptive strategy to determine the DE-parameter settings and dynamic penalty functions to handle constraints involved in particular control problems; e.g., see [279,281,307,340]. Moreover, it is also important to investigate numerical issues corresponding to solution characteristics (e.g., optimality, accuracy and sensitivity), convergence rate and optimization landscapes when using the DE method for designing robust control systems; e.g., see [341–343].

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