# On aspects of Ramsey theory

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**Publication Date:**
2018

**DOI:**
https://doi.org/10.26190/unsworks/20521

# ON ASPECTS OF RAMSEY THEORY

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May 2018

A thesis submitted in fulfilment of the requirements of the degree of Master of Mathematics

**PLEASE TYPE**

**THE UNIVERSITY OF NEW SOUTH WALES**
**Thesis/Dissertation Sheet**


Surname or Family name: Chng

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Other name/s:

Abbreviation for degree as given in the University calendar: MSc

School: School of Mathematics and Statistics

Faculty: Faculty of Science


Title: On aspects of Ramsey Theory

**Abstract 350 words maximum: (PLEASE TYPE)**


**This thesis presents various types of results from Ramsey Theory, most particularly, Ramsey-type theorems concerning graphs and families of sets.**

**This thesis consists of 8 chapters. In Chapter 1, we give a brief historical introduction to Ramsey Theory. Then, we introduce some necessary notation and definitions that will be consistently used throughout the thesis, including some basic knowledge of Graph Theory which is particularly useful in Chapters 2 and 3.**

**We present Ramsey-type results about graphs in Chapters 2 and 3. In Chapter 2, we introduce the classical Ramsey's Theorem which is the Ramsey-type theorem on the edge-colouring of the complete graph. We also introduce Ramsey numbers and present some results on these, especially some upper and lower bounds. In Chapter 3, we look at Ramsey-type results for monochromatic tree graphs, cycle graphs and bipartite graphs, respectively, occurring in arbitrary edge colourings of the complete graph. Then, we present the bipartite version of Ramsey's Theorem.**

**Chapters 4, 5 and 6 present other famous Ramsey-type theorems, for arithmetic progressions and other, more general, structures. In Chapter 4, we introduce and prove Van der Waerden's Theorem and we also present some results on the bounds of the Van der Waerden numbers. In Chapter 5, we present Schur's Theorem and some results relating to the Schur numbers. Then, we look into some generalisations of Schur's Theorem, including Rado's Theorem and Folkman's Theorem. In Chapter 6, we prove the Hales-Jewett Theorem. We also construct a proof of Van der Waerden's Theorem by using the Hales-Jewett Theorem.**

**Before we end our studies, in Chapter 7, we include some application of the Ramsey Theory. We look into the application of the Ramsey Theory in various fields, including graph theory, geometry and number theory. In Chapter 8, we conclude our studies. We give some overall comment on Ramsey Theory and include some possible future work on the field.**

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# Acknowledgements

First of all, I would like to deliver my highest gratitude to my supervisor, Dr Thomas Britz for his valuable guidance throughout this research project. Without his advice and encouragement, I might not be able to complete this thesis smoothly. Besides that, he has also exposed me to a broader field of mathematical research.

Next, I would also like to express my gratitude to the Public Service Department of Malaysia for the financial assistances given during my master studies. I would also like to thank all the members of the School of Mathematics and Statistics, UNSW Sydney, who have given their helping hands throughout the completion of this thesis.

Last but not least, I also wish to thank my family and my friends, especially my parents, who gave me support throughout the completion of this thesis. The days of my master studies would not be any easier without their love and encouragement.

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# Abstract

This thesis presents a representative spread of results from Ramsey Theory, most particularly, Ramsey-type theorems concerning graphs, families of sets and the integers.

This thesis consists of 8 chapters. In Chapter 1, we give a brief historical introduction to Ramsey Theory. Then, we introduce some necessary notation and definitions that will be consistently used throughout the thesis, including some basic knowledge of Graph Theory which is particularly useful in Chapters 2 and 3.

We present Ramsey-type results about graphs Chapters 2 and 3. In Chapter 2, we introduce the classical Ramsey's Theorem which is the Ramsey-type theorem on the colouring of the complete graph. We also introduce Ramsey numbers and present some results on these, especially some upper and lower bounds. In Chapter 3, we look at Ramsey-type results for monochromatic tree graphs, cycle graphs and bipartite graphs, respectively, occurring in arbitrary edge colourings of the complete graph. Then, we present the bipartite version of Ramsey's Theorem.

Chapters 4, 5 and 6 present other famous Ramsey-type theorems, for arithmetic progressions and other, more general, structures. In Chapter 4, we introduce and prove Van der Waerden's Theorem and we also present some results on the bounds of the Van der Waerden numbers. In Chapter 5, we present Schur's Theorem and some results relating to the Schur numbers. Then, we look into some generalisations of Schur's Theorem, including Rado's Theorem and Folkman's Theorem. In Chapter 6, we prove the Hales-Jewett Theorem. We also construct a proof of Van der Waerden's Theorem by using the Hales-Jewett Theorem.

Before we end our studies, in Chapter 7, we include some applications of Ramsey Theory to Graph Theory, Geometry and Number Theory. In Chapter 8, we conclude our studies with overall comments on Ramsey Theory and possible future work in this field.

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# List of Our Results



Listed above are the results that we contributed independently to.

The following theorems were discovered and proved independently and could not be found in previous literature:

Theorems 2.16–2.20, 3.5, 3.19–3.20, 4.14–4.16, 5.3–5.4, and 5.7.

The following theorems were discovered and proved independently but were later found in previous literature:

Theorems 3.7, 3.9, and 5.8–5.12.

The following theorems were found in the literature but we independently filled in missing details of their proofs and adapted and modified the proofs in order to provide better clarity of argument and notation:

Theorems 3.13–3.15.

# Chapter 1
# Introduction

## 1.1 Historical Background and Introduction

Ramsey Theory is a beautiful but difficult subject that, generally speaking, shows how, in certain orderly structures, patterns and order can never be completely eradicated by randomness or disarray. A typical result in Ramsey Theory states that *if some mathematical structure is cut into pieces, then at least one of the parts must have a given interesting property.* There are many interesting applications of Ramsey Theory, including the results in number theory, algebra, geometry, topology, set theory, logic and set theory [86].

Ramsey Theory is named after the British mathematician and philosopher Frank Plumpton Ramsey, who did seminal work in this area before his death at the age of 26 in 1930. However, the theory was brought to public attention by Paul Erdős, a Hungarian mathematician who made enormous contributions to the fields of combinatorics and graph theory. He contributed much to Ramsey Theory, especially on Ramsey's Theorem for complete graphs which states that *in any sufficiently large finitely coloured complete graph, one can find some large monochromatic substructure.* In the language of graph theory, Ramsey number $R(m, n)$ is the minimum number of vertices to ensure that a simple undirected graph with that number of vertices contains either a complete graph of order $m$ or an independent set of size $n$. The first lower bound on Ramsey numbers were obtained by Paul Erdős using probabilistic methods [22]. Together with George Szekeres, Paul Erdős also found some upper bounds on these numbers [25].

One of the key theorems of Ramsey Theory is a result on arithmetic progressions, Van der Waerden's Theorem from 1927. This theorem is named after the Dutch mathematician Bartel Leendert Van der Waerden. Van der Waerden's Theorem states that *for every positive integer $k$, there exists a positive integer $n$ such that if the set $\{1, 2, \ldots, n\}$ is partitioned into two subsets, then at least one of the subsets must contain an arithmetic progression of length $k$* [102]. This theorem is further proven by Ron Graham and B. L. Rothschild [44]. Terence Tao also constructed a topological proof of Van der Waerden's Theorem in 2008 [101].

Another result that is similar to Van der Waerden's Theorem is Schur's Theorem from 1916. This is a Ramsey-type result on integer solutions to equations and was proved by Issai Schur. The theorem states that *in any finite colouring of the natural numbers, there must be a pair of integers $x$ and $y$, such that $x$, $y$ and $x+y$ are all the same colour* [92]. This basic result was generalised by Richard Rado in 1933 to give a characterisation of the homogeneous system in which a monochromatic solution can be found in any finite colouring of the natural numbers [81]. The theorems of

Schur, Rado, Ramsey and Van der Waerden are considered to be central results of Ramsey Theory.

Another key theorem of Ramsey Theory is geometrical, namely the Hales-Jewett Theorem. It is a fundamental combinatorial result of Ramsey Theory named after Alfred W. Hales and Robert I. Jewett which states that *for* $k, r \in \mathbb{N}$*, if* $n$ *is sufficiently large, then for any* $r$*-colouring of a cube* $C_k^n = \{(x_1, \ldots, x_n) : x_i \in [0, k-1]\}$*, there is a monochromatic line* [50]. Informally speaking, Hales-Jewett Theorem states that *for any positive integers* $n$ *and* $c$*, there is a number* $H$ *such that if the cells of an* $H$*-dimensional* $n \times n \times \cdots \times n$ *cube are coloured with* $c$ *colours, there must be one row, column, or certain diagonal of length* $n$*, all of whose cells are the same colour.* Hales and Jewett showed that if the dimension is large enough, then one can always find an $n$-in-a-row tic-tac-toe that never ends in a tie [50].

## 1.2 Preliminaries and Definitions

In this section, we will introduce some definitions and theorems which will be frequently referred to throughout this thesis.

### *1.2.1 Notation*

$$\begin{aligned}
\mathbb{N} &: \text{ The set of natural numbers } \{1, 2, \ldots\}. \\
[n] &: \text{ The set } \{1, 2, \ldots, n\}. \\
[m, n] &: \text{ The set } \{m, m+1, \ldots, n\}. \\
\binom{X}{k} &: \text{ The family } \{Y \subseteq X : |Y| = k\}. \\
|X| &: \text{ The cardinality of } X.
\end{aligned}$$

### *1.2.2 The Pigeonhole Principle*

One of the basic tools used in Ramsey Theory is the *Pigeonhole Principle.* It was first formulated in 1834, by the German mathematician Peter Gustav Lejeune Dirichlet.

**Theorem 1.1** (The Pigeonhole Principle)**.** [108] *If* $n+1$ *objects are put into* $n$ *boxes, then at least one box contains two or more of the objects.*

**Theorem 1.2** (Stronger Form of the Pigeonhole Principle)**.** [108]
*Let* $q_1, q_2, \ldots, q_n$ *be positive integers. If* $q_1 + q_2 + \cdots + q_n - n + 1$ *objects are put into* $n$ *boxes, then, for some* $i \in [n]$*, there are at least* $q_i$ *objects in the* $i^{th}$ *box.*

*Proof.* Suppose the contrary, namely that the $i^{th}$ box has at most $i-1$ objects, for each $i = 1, 2, \ldots, n$. Then the total number of objects contained in the $n$ boxes is $(q_1 - 1) + (q_2 - 1) + \cdots + (q_n - 1) = q_1 + q_2 + \cdots + q_n - n$, which is less than the number of objects allocated. This is a contradiction. □

**Corollary 1.3.** [108] *If* $n(r-1)+1$ *objects are put into* $n$ *boxes, then at least one of the boxes will contains* $r$ *or more objects.*

*Proof.* It follows from the stronger form of the Pigeonhole Principle for the special case $q_1 = q_2 = \cdots = q_n = r$. □

*1.2.3 Graph Theory*

In Chapters 2 and 3, we will present the graph theory results from Ramsey Theory. Here, we introduce some graph theory definitions that will be used in those chapters.

**Definition 1.4** (Graph)**.** *A* graph *$G$ is a pair of sets $(V(G), E(G))$ where $V(G)$ is a finite non-empty set of elements called* vertices *and $E(G)$ is a set of unordered pairs of elements of $V(G)$ called* edges*.*

Figure 1.1 shows a graph $G$ with the vertex set $\{s, t, u, v, w\}$ and the edge set $\big\{\{s, t\}, \{t, u\}, \{t, w\}, \{u, v\}, \{v, w\}\big\}$.

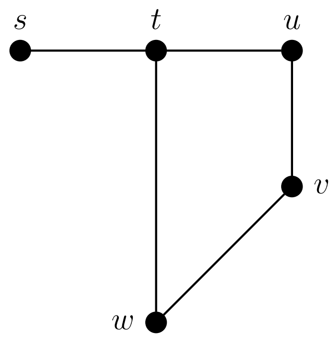
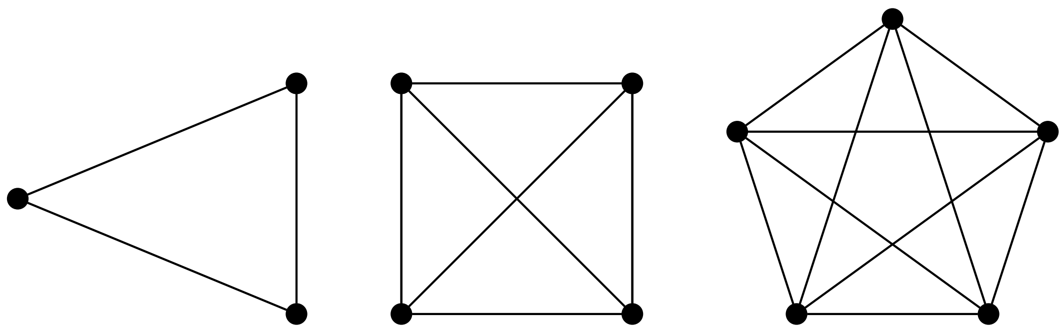



Figure 1.1: A graph $G$

**Definition 1.5** (Complete graph)**.** *Two vertices $u$ and $v$ of the graph are said to be* adjacent *if they are joined by an edge $e$. In this case, $e$ is* incident *to $u$ and $v$. A graph in which every two vertices are adjacent to each other is called a* complete graph*. A complete graph with $n$ vertices is denoted by $K_n$.*

Figure 1.2 shows some examples of complete graphs, namely $K_3$, $K_4$, and $K_5$.



(a) $K_3$ (b) $K_4$ (c) $K_5$

Figure 1.2: Complete graphs

**Definition 1.6** (Degree of a vertex)**.** *The* degree *of a vertex in a graph is the number of edges incident to it. A graph where all its vertices have the same degree is known as a* regular graph*.*

Figure 1.3 shows a graph with each of the vertices labelled by their degree.

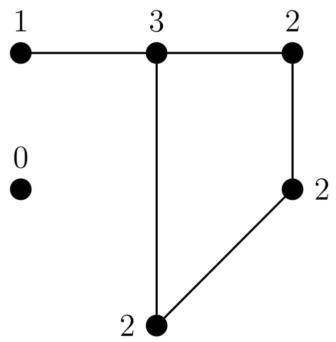
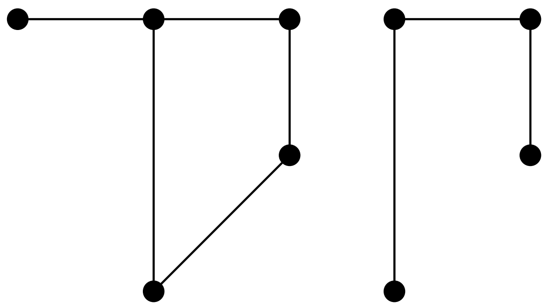



Figure 1.3: Graph $G$ with vertices labelled by their degree

**Definition 1.7** (Subgraph)**.** *A graph $H$ is a* subgraph *of $G$ if $V(S) \subseteq V(G)$ and $E(S) \subseteq E(G)$.*

Figure 1.4 shows an example of a subgraph $H$ of a graph $G$.

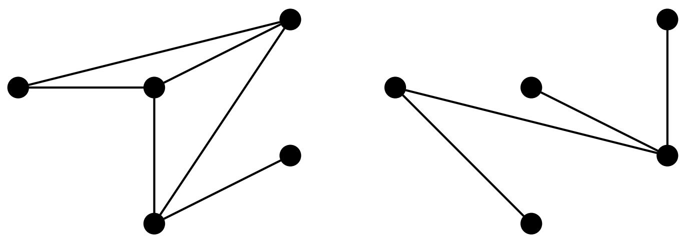


(a) $G$ (b) $H$

Figure 1.4: $H$ is a subgraph of $G$

**Definition 1.8** (Complement of a graph)**.** *Let $G$ be a graph with $n$ vertices. The* complement *of $G$, denoted by $\overline{G}$, is the graph with vertices $V(\overline{G}) = V(G)$ and edges $E(\overline{G}) = E(K_n) - E(G)$.*

Figure 1.5 shows a graph $G$ and its complement $\overline{G}$.



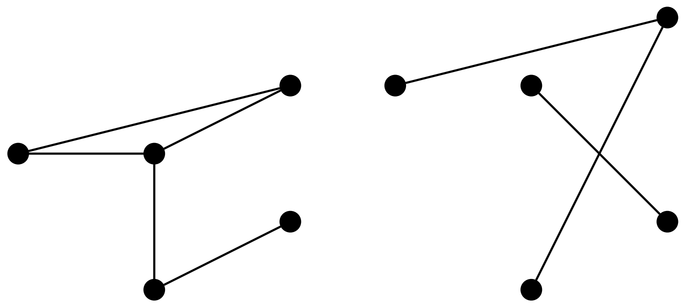
(a) $G$ (b) $\overline{G}$

Figure 1.5: A graph $G$ and its complement $\overline{G}$

**Definition 1.9** (Walks, paths and cycles)**.** *A* walk *in a graph $G$ is an alternating sequence of vertices and edges $v_0e_1v_1e_2v_2 \dots e_kv_k$ in which the ends of each edge $e_i$ are $v_{i-1}$ and $v_i$ for $i \in [k]$. It is* closed *if $v_0 = v_k$ and is* open *otherwise. A walk in which all vertices $v_0, v_1, \dots, v_k$ are distinct is called a* path*. A closed path is called a* cycle*.*

**Definition 1.10** (Connected graph)**.** *A graph $G$ is* connected *if there exists a walk between each pair of vertices in $G$. If $G$ is not connected, then it is* disconnected*.*

Figure 1.6 shows a connected graph $G$ and a disconnected graph $H$.

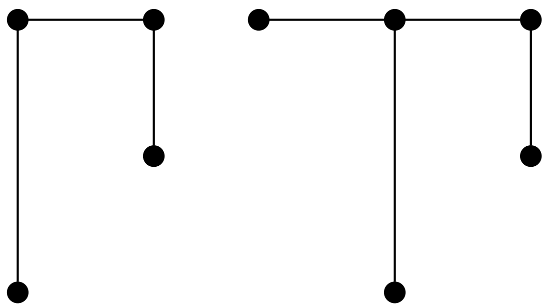


(a) $G$ (b) $H$

Figure 1.6: A connected graph $G$ and a disconnected graph $H$

**Definition 1.11** (Trees)**.** *A* tree *is a connected graph which has no cycle subgraph.*

Figure 1.7 shows some examples of trees.

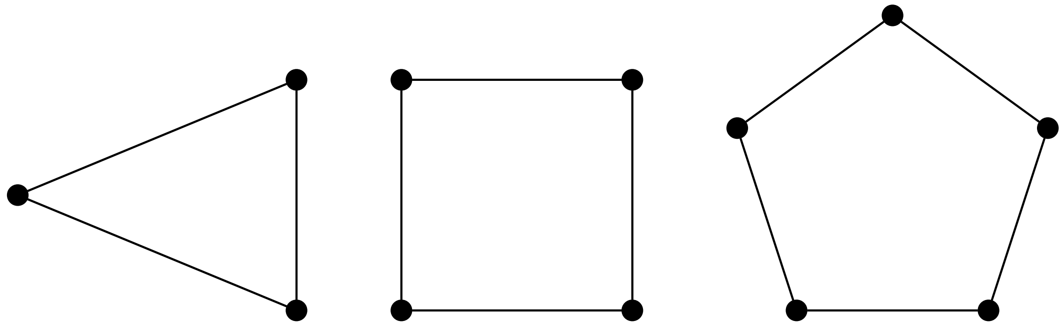


(a) $T_4$ (b) $T_5$

Figure 1.7: Trees $T_4$ and $T_5$

**Definition 1.12** (Cycle graphs)**.** *A* cycle [graph] *is a graph that consists of a single cycle. A cycle with $n$ vertices is denoted by $C_n$.*

Figure 1.8 shows cycle graphs $C_3$, $C_4$, and $C_5$.

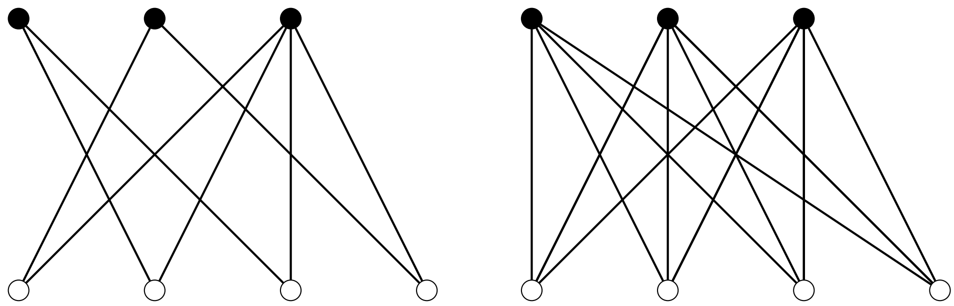


(a) $C_3$ (b) $C_4$ (c) $C_5$

Figure 1.8: Cycle graphs.

**Definition 1.13** (Bipartite graph)**.** *A* bipartite graph *is a connected graph whose the vertex set can be partition into two disjoint subsets so that each edge joins a vertex from one subset to a vertex from the other subset. The vertices of a bipartite graph can be coloured black and white according to the subset in which they belong. A bipartite graph is* complete *if each vertex from one subset is adjacent to every vertex from another subset. A complete bipartite graph is denoted by $K_{n_1,n_2}$ where $n_1$ and $n_2$ are the numbers of vertices in each subset, respectively.*

Figure 1.9 shows examples of bipartite graphs.



(a) A bipartite graph (b) A complete bipartite graph, $K_{3,4}$

Figure 1.9: Bipartite graphs

### *1.2.4 Main Theorems and Definitions*

In this subsection, we list some of the main theorems and definitions that will be discussed further in this thesis.

**Definition 1.14** (Colouring)**.** *A* colouring *is a type of labelling, assigning "colours" as the labels to elements of a mathematical structure such as a graph or some set under certain constraints.*

In this thesis, the mathematical structures that we colour are graphs (in Chapters 2 and 3) and various families defined via natural numbers (in Chapters 4, 5 and 6).

**Example 1.15.** *As mentioned in Definition 1.14, there are various type of colouring. Here, we give an example of the edge-colouring of a graph, which will be largely used in Chapters 2 and 3.*

Figure 1.10 shows an example of edge-colouring of a graph $G$. The edges $\{1,5\}$, $\{1,4\}$ and $\{2,3\}$ are blue(———), whereas the edges $\{1,2\}$ and $\{3,4\}$ are red($---$). *Note: The dashed lines are used to differentiate two colours for the black and white printed version.*

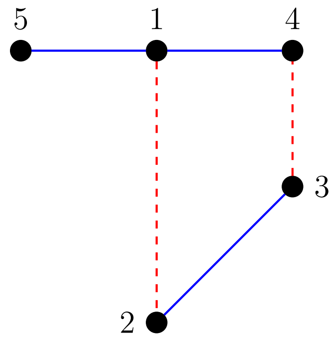



Figure 1.10: Graph $G$ with blue-colour(———) and red-coloured($---$) edges.

**Definition 1.16** (Monochromatic)**.** *A mathematical structure is* monochromatic *if all of its elements are of the same single colour.*

**Theorem 1.17** (Ramsey's Theorem for 2-colouring of the Edges of the Graph)**.** [82] *Let $m_1$ and $m_2 \in \mathbb{N}$. There exists an integer $N \in \mathbb{N}$ such that in every edge-colouring of $K_N$ with the colours $c_1$ and $c_2$, there is either a $c_1$-monochromatic $K_{m_1}$ subgraph or a $c_2$-monochromatic $K_{m_2}$ subgraph. The least such $N$ is known as the* Ramsey number *$R(m_1, m_2)$.*

**Example 1.18** (Party Problem)**.** One of the typical results in Ramsey's Theorem is the Party Problem. In the Party Problem, we are asked to find the minimum number of guests to be invited to ensure that at least $m$ of them know each other or at least $n$ of them do not know each other. This problem is equivalent to Ramsey's Theorem for two colours.

**Theorem 1.19** (Ramsey's Theorem for $r$-colouring of the Edges of the Graph)**.** [82] *If $r, m \in \mathbb{N}$ and $n$ is sufficiently large, then each $r$-colouring of the edges of $K_n$ gives a complete subgraph $K_m$ with monochromatic edges.*

**Theorem 1.20** (Ramsey's Theorem)**.** [82] *If $m_1, m_2, \ldots, m_r, k \in \mathbb{N}$ and $n$ is sufficiently large, then for each colouring of $\binom{[n]}{k}$ with colours $c_1, c_2, \ldots, c_r$, there is an $m_i$-subset $S \subseteq [n]$ such that the subfamily $\binom{S}{k}$ is coloured $c_i$ for some $i \in [r]$. The least such $n$ is denoted by $R_k(m_1, m_2, \ldots, m_r)$.*

In Chapter 4, we will present *Van der Waerden's Theorem.* Here, we introduce some definitions that will be used.

**Definition 1.21** (Arithmetic Progression)**.** *An* arithmetic progression *is a sequence of numbers such that the differences between consecutive terms is constant.*

**Theorem 1.22** (Van der Waerden's Theorem)**.** [102] *If $k, r \in (N)$, and $N$ is sufficiently large, then each $r$-colouring of $[N]$ gives a monochromatic arithmetic progression of length $k$. The least such $N$ is known as the* Van der Waerden number *$W(k, r)$.*

In Chapter 5, we will introduce *Schur's Theorem.* Here, we give some main theorems related to it that will be used in that chapter.

**Theorem 1.23** (Schur's Theorem)**.** [92] *Let $r \in \mathbb{N}$. If $\mathbb{N}$ is $r$-coloured, then there are some same-coloured $a, b, c \in \mathbb{N}$ such that $a + b = c$.*

**Theorem 1.24** (Schur's Theorem (finite))**.** [92] *Let $r \in \mathbb{N}$ and $N$ is sufficiently large, then for any $r$-colouring of $[N]$, there are some same-coloured $a, b, c \in [N]$ such that $a + b = c$. The least such $N$ is known as the* Schur number *$S(r)$.*

In Chapter 6, we will discuss the Hales-Jewett Theorem. Now, we introduce the main definition and theorem that will be used in the chapter.

**Definition 1.25** ($n$-cube over $t$ elements)**.** *We define the $n$-cube over $t$ elements by*

$$C_k^n = \{(x_1, \ldots, x_n) : x_i \in [0, t-1]\}.$$

**Definition 1.26** (Line)**.** *A line in $C_k^n$ is a set of points $x_0, \ldots, x_{k-1}$, where $x_i = (x_{i1}, \ldots, x_{in})$ so that in each coordinate $j \in [n]$, either*

$$x_{0j} = \cdots = x_{k-1,j}$$

*or*

$$x_{sj} = s\,, \text{ where } s \in [0, k-1]\,, \text{ for some } j\,.$$

**Theorem 1.27** (Hales-Jewett Theorem)**.** [50] *For all $r, t \in \mathbb{N}$ and $N$ is sufficiently large, if the vertices of $C_t^N$ are $r$-coloured, then there exists a monochromatic line.*

# Chapter 2

## Ramsey's Theorem

In this chapter, we present the main theorem in Ramsey Theory, which is Ramsey's Theorem, first proved by Flank Plumpton Ramsey in 1930 [82]. In Section 2.1, we will first present and prove a special case of it, namely that for the edge colouring of complete graphs. In doing so, we introduce the Ramsey number terminology. In Section 2.2, we then introduce and prove Ramsey's Theorem in full. In Section 2.3, we will also present some results and theorems on Ramsey numbers, including some known Ramsey numbers and bounds on them in general.

## 2.1 Ramsey's Theorem for Edge-Colouring a Graph

In this section, we state Ramsey's Theorem on the edge-colouring of the complete graph and we construct a proof of the theorem by induction. We also introduce the Ramsey Number terminology which is particularly useful in the studies of Ramsey Theory.

**Theorem 2.1** (Ramsey's Theorem for 2-colouring the Complete Graph)**.** [82]
*Let $m_1$ and $m_2 \in \mathbb{N}$. There exists an integer $N \in \mathbb{N}$ such that in every edge-colouring of $K_N$ with the colours $c_1$ and $c_2$, there is either a $c_1$-monochromatic $K_{m_1}$ subgraph or a $c_2$-monochromatic $K_{m_2}$ subgraph.*
*The least such $N$ is known as the* Ramsey number $R(m_1, m_2)$.

To prove Theorem 2.1, we first prove some auxiliary lemmas.

**Lemma 2.2.** [82]
(1) $R(m, 1) = 1 = R(1, m)$
(2) $R(m, 2) = m = R(2, m)$

*Proof.*
(1) The graph $K_1$ which only has a single vertex is trivially monochromatic.
(2) Suppose we colour all of the edges of $K_m$ with the colours $c_1$ and $c_2$. Then either there is a $c_1$-coloured $K_2$ (just a single edge) or else all the edges are $c_2$-coloured, forming $K_m$, or vice versa. □

Note that Theorem 2.1 is proven if we can show that the Ramsey Number $R(m_1, m_2)$ exists for all $m_1, m_2 \in \mathbb{N}$. Such a result has been proven by P. Erdős and G. Szekeres in 1935, as follows.

**Lemma 2.3.** [25] *For all $m_1, m_2 \geq 2$, $R(m_1, m_2) \leq R(m_1 - 1, m_2) + R(m_1, m_2 - 1)$.*

*Proof.* Let $v$ be any vertex of $K_{R(m_1-1,m_2)+R(m_1,m_2-1)}$. Partition the remaining $R(m_1 - 1, m_2) + R(m_1, m_2 - 1) - 1$ vertices into two sets $M_1$ and $M_2$, in such the way that for every vertex $w$, $w$ is in $M_1$ if the edge $\{v, w\}$ is coloured with $c_1$ and $M_2$

otherwise. Note that either $|M_1| \geq R(m_1-1, m_2)$ or $|M_2| \geq R(m_1, m_2-1)$ because otherwise $|M_1|+|M_2| \leq R(m_1-1, m_2)-1+R(m_1, m_2-1)-1$ which is impossible.

If $|M_1| \geq R(m_1-1, m_2)$, then we either have a $c_1$-coloured subgraph $K_{m_1-1}$ or a $c_2$-coloured $K_{m_2}$. For the latter case, we are done. Suppose we have a $c_1$-coloured subgraph $K_{m_1-1}$. Then take the subgraph with the vertex $v$ and all the $c_1$-coloured edges between them, we will get a $c_1$-coloured subgraph $K_{m_1}$.

Similarly, if $|M_2| \geq R(m_1, m_2-1)$, then we either have a $c_1$-coloured subgraph $K_{m_1}$, in which a case the theorem is proven, or else we have a $c_2$-coloured subgraph $K_{m_2-1}$. This subgraph together with the vertex $v$ and all the $c_2$-coloured edges between them will form a $c_2$-coloured subgraph $K_{m_2}$. In all cases, we either have a $c_1$-coloured subgraph $K_{m_1}$ or a $c_2$-coloured subgraph $K_{m_2}$. □

**Example 2.4.** Any 2-colouring of the complete graph $K_6$ will give us a monochromatic $K_3$ subgraph. Furthermore, if we colour the complete graph $K_5$ as in Figure 2.2 (Section 2.3), then no monochromatic $K_3$ subgraph can be found. Hence, we can conclude that $R(3,3)=6$. A detailed proof of $R(3,3)=6$ will be given in Section 2.3. Figure 2.1 shows an example of monochromatic $K_3$ subgraph in a $c_1$(———) and $c_2$($---$) colouring of $K_6$.

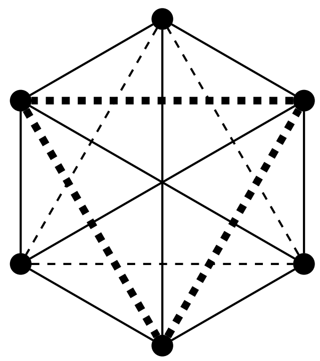


Figure 2.1: Monochromatic $c_2$-coloured $K_3$ in 2-colouring of $K_6$

**Theorem 2.5** (Ramsey's Theorem for $r$-colouring of the Complete Graph). [82] *If $r, m \in \mathbb{N}$, and $n$ is sufficiently large, then each $r$-colouring of the edges of $K_n$ gives a complete subgraph $K_m$ with monochromatic edges.*

*Proof.* We prove by induction on $r$.

For $r=1$, it is clear that we can always take any $n \geq m$ and we can find a complete subgraph $K_m$ with monochromatic edges. Suppose that the theorem is valid for $r-1$ colours. Now, we consider the $r$-colouring case. Colour each edge of $K_n$ in colours $c_1, c_2, \ldots, c_r$. Recolour each $c_{r-1}$-coloured and $c_r$-coloured edges with a new colour $c_{r-1'}$. From the induction hypothesis, for a big enough $n$, we can get a subgraph $K_{R(m,m)}$ with monochromatic edges. If the edges of this subgraph $K_{R(m,m)}$ is coloured with $c_i$ for some $i \in [r-2]$, then we can always get a $c_i$-coloured $K_m$ subgraph, and we are done. Suppose that $K_{R(m,m)}$ is coloured with $c_{r-1'}$. Since the edges of this subgraph is originally coloured with colour $c_{r-1}$ and $c_r$, thus, by the definition of $R(m,m)$, we can always get a $c_{r-1}$-coloured or $c_r$-coloured $K_m$, in which we are done in either case.

Hence by induction, the theorem is proven. □

**Definition 2.6.** *The* Ramsey Number $R(m_1, m_2, \ldots, m_r)$ *is the least integer* $N$ *such that for all* $n \geq N$*, if all the edges of* $K_n$ *are* $r$*-coloured, then there is always a monochromatic* $K_{m_i}$*, for some* $i \in [r]$*.*

## 2.2 Ramsey's Theorem

In this section, we give the full version of Ramsey's Theorem and construct a proof for it.

**Theorem 2.7** (Ramsey's Theorem)**.** [82]
*If* $m_1, m_2, \ldots, m_r, k \in \mathbb{N}$ *and* $n$ *is sufficiently large, then for each colouring of* $\binom{[n]}{k}$ *with colours* $c_1, c_2, \ldots, c_r$*, there is an* $m_i$*-subset* $S \subseteq [n]$ *such that the subfamily* $\binom{S}{k}$ *is coloured* $c_i$ *for some* $i \in [r]$*. The least such* $n$ *is denoted by* $R_k(m_1, m_2, \ldots, m_r)$*.*

*Proof.* To prove the theorem, we first show that the theorem holds for 2 colours. Then, by induction on $r$, we prove that the theorem is valid for any $r$-colouring. For 2-colouring, we need to show the existence of $R_k(m_1, m_2)$, for all $k, m_1, m_2 \in \mathbb{N}$. Note that the case $k = 2$ is none other than Theorem 2.1. Further notice that, if $m_i = k$, for some $i = 1, 2$, $R_k(m_1, m_2) = R_k(m_2, m_1) = k$ is trivial because in any 2-colouring of $\binom{k}{k}$, we will have a $k$-subset $S$ such that the subfamily $\binom{S}{k}$ is monochromatic. Now, suppose that $R_{k-1}(m_1, m_2)$,$R_k(m_1 - 1, m_2)$ and $R_k(m_1, m_2 - 1)$ exist. We want to show the existence of $R_k(m_1, m_2)$.
Take $N = R_{k-1}(R_k(m_1 - 1, m_2), R_k(m_1, m_2 - 1)) + 1$ and consider the set $\binom{[N]}{k}$. Colour all the $k$-subsets with colours $c_1$ and $c_2$ and we denote this colouring as $\chi$.

Now choose an element $x$ and consider all $(k-1)$-subsets not containing the element $x$. We call this family $S$. Note that $S$ is equivalent to the family $\binom{[N-1]}{k-1}$. Let $S$ be 2-coloured by a $c_1$ and $c_2$, by a colouring $\chi^*$ induced in such a way that $\chi^*(T) = \chi(T \cup x)$, for all $T \in S$. By the induction hypothesis, we are guaranteed one of the following cases:

(1) $S$ has a subset $M$, where $|M| = R_k(m_1 - 1, m_2)$ and all the $(k-1)$-subset of $M$ is $c_1$-coloured.

(2) $S$ has a subset $N$, where $|N| = R_k(m_1, m_2 - 1)$ and all the $(k-1)$-subset of $N$ is $c_2$-coloured.

Suppose that Case 1 holds. Then by induction hypothesis, we assume that $R_k(m_1 - 1, m_2)$ exists. Therefore, $M$ has either a subset $M_1$ with $m_1 - 1$ elements where all $k$-subset of $M_1$ are $c_1$-coloured or a subset $M_2$ with $m_2$ elements where all $k$-subset of $M_2$ are $c_2$-coloured. For the latter, we are done. Suppose that there is such a subset $M_1$, and consider $M^* = M_1 \cup x$, $|M^*| = m_1$. Take $k$-subset of $M^*$ if the $k$-subset contains element $x$. Then it is $c_1$-coloured by the induced colour of $(k-1)$-subset of $M$. On the other hand, if the $k$-subset does not contain the element $x$, then it is actually a $k$-subset of $M$ which is then $c_1$-coloured. Either way, we are done.

On the other hand, suppose that Case 2 holds. From the induction hypothesis, we assume that $R_k(m_1, m_2 - 1)$ exists. Thus $N$ has either subset $N_1$ with $m_1$ elements where all $k$-subset of $N_1$ are $c_1$-coloured, where we are done; or else, subset $N_2$ with $m_2 - 1$ elements which all $k$-subset of $N_2$ are $c_2$-coloured. Suppose the latter, and consider $N^* = N_1 \cup x$, $|N^*| = m_2$. Take $k$-subset of $M^*$. If the $k$-subset contains the element $x$, then it is $c_2$-coloured by the induced colouring of $(k-1)$-subset of $N$. If the $k$-subset does not contain the element $x$, then it is actually the $c_2$-coloured $k$-subset of $N$. Then, we are done.

Now, we have shown $R_k(m_1, m_2)$ exists for all $m_1, m_2 \in \mathbb{N}$. By induction on $r$, we want to show the theorem holds for any $r$-colouring. Assume that $R_k(m_1, m_2, \ldots, m_{r-1})$ exists. Since the theorem holds for 2 colours, $R_k(m_{r-1}, m_r)$ exists. We take $N = R_k(m_1, m_2, \ldots, m_{r-2}, R_k(m_{r-1}, m_r))$. By the induction hypothesis, we either have a $m_i$-subset $S$ of $\binom{[N]}{k}$ in which all $k$-subsets of $S$ are $c_i$-coloured, for some $i \in [r-2]$, in which we are done. Otherwise, we have a $(R_k(m_{r-1}, m_r))$-subset $S$ of $\binom{[N]}{k}$ in which all $k$-subset of $S$ are $c_{r-1}$-coloured or $c_r$-coloured. By the definition of $R_k(m_{r-1}, m_r)$, we have a set $S_1 \subseteq S$ in which all the $k$-subset of $S_1$ are $c_{r-1}$-coloured or a set $S_2 \subseteq S$ where all the $k$-subset of $S_2$ are $c_r$-coloured. In either case, we are done.

Hence by induction, Ramsey's Theorem holds for all $m_1, m_2, \ldots, m_r, k, r \in \mathbb{N}$. □

## 2.3 Ramsey Numbers

In this section, we present some known results, from old to recent, of the Ramsey numbers.

**Example 2.8.** In any group of 6 people, there are either 3 mutual friends or 3 mutual strangers.

This example is equivalent to the following statements:

(1) In any 2-colouring of $K_6$, there is a monochromatic $K_3$ subgraph.
(2) $R(3,3) \leq 6$.

*Proof.* Let $A$ be one of the groups of six. The remaining 5 people fall into one of the two classes: $F$, a set of friends of $A$ and $S$, a set of strangers to $A$. Now by the Pigeonhole Principle, one of the classes must have at least 3 people.

Case (i) : $|F| \geq 3$.
If $F$ has 3 mutual strangers, then we are done. Otherwise, $F$ has a pair of friends. This pair of friends together with $A$ will form a group of 3 mutual friends.

Case (ii): $|S| \geq 3$.
If $S$ has 3 mutual friends, then we are done. Otherwise, $S$ has a pair of strangers. This pair of strangers together with $A$ will form a group of 3 mutual strangers.

In all cases, we either have 3 mutual friends or else 3 mutual strangers. □

From Example 2.8, we have shown that $R(3,3) \leq 6$. We need to show that $R(3,3) \geq 6$ to prove $R(3,3) = 6$. In doing so, we give a counterexample. Suppose it is possible to colour the edges of the graph $K_n$ with the colour $c_1$ and $c_2$ so that $K_n$ contains neither $c_1$-coloured $K_{m_1}$ nor $c_2$-coloured $K_{m_2}$. Then we can conclude that $R(m_1, m_2) > n$.

**Example 2.9.** The figure below shows an example of $K_5$ coloured with colours $c_1$(———) and $c_2$(− − −) in such a way that there is no monochromatic $K_3$ subgraph.

With the construction of $K_5$ as shown in Figure 2.2, we have shown that $R(3,3) > 5$. Since $R(3,3)$ must be an integer, we can deduce that $R(3,3) \geq 6$.

Thus, we have shown that $R(3,3) = 6$.

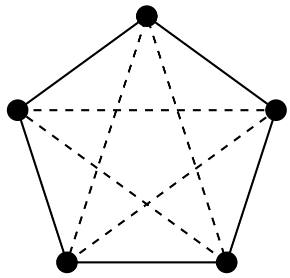


Figure 2.2: $R(3,3) > 5$

**Theorem 2.10.** [25] $R(m_1, m_2) \leq \binom{m_1+m_2-2}{m_1-1}$.

*Proof.* We prove the theorem by induction. Note that $R(1,1) = 1 \leq \binom{1+1-2}{1-1}$. For the induction hypothesis, assume that the theorem holds for $R(m_1 - 1, m_2)$ and $R(m_1, m_2 - 1)$. Now, we have $R(m_1 - 1, m_2) = \binom{m_1+m_2-3}{m_1-2}$ and $R(m_1, m_2 - 1) = \binom{m_1+m_2-3}{m_1-1}$. By Lemma 2.3, we have $R(m_1, m_2) \leq R(m_1 - 1, m_2) + R(m_1, m_2 - 1) \leq \binom{m_1+m_2-3}{m_1-2} + \binom{m_1+m_2-3}{m_1-1}$. By the recursive formula of the binomial coefficient, we get $R(m_1, m_2) \leq \binom{m_1+m_2-2}{m_1-1}$. Then, by induction, we are done. $\square$

Under certain circumstances, Lemma 2.3 has been improved by R.E. Greenwood and A.M. Gleason in 1955, as follows.

**Theorem 2.11.** [48] *If both* $R(m_1 - 1, m_2)$ *and* $R(m_1, m_2 - 1)$ *are even, then* $R(m_1, m_2) \leq R(m_1 - 1, m_2) + R(m_1, m_2 - 1) - 1$.

*Proof.* Set $N := R(m_1 - 1, m_2) + R(m_1, m_2 - 1) - 1$ and colour the edges of $K_N$ with colours $c_1$ and $c_2$. Select a vertex $v$ and partition the remaining $N - 1$ vertices into two sets $M_1$ and $M_2$ in such the way that, for every vertex $w$, $w$ is in $M_1$ if $\{v, w\}$ is coloured with $c_1$ and $w$ is in $M_2$ otherwise. Then, one of the following cases will hold:

(1) $|M_1| = R(m_1 - 1, m_2) - 1$ and $|M_2| = R(m_1, m_2 - 1) - 1$
(2) $|M_1| \geq R(m_1 - 1, m_2)$
(3) $|M_2| \geq R(m_1, m_2 - 1)$

Assume that (1) is true for all vertices in $K_N$. Then, the $c_1$-coloured subgraph will contain $c := \frac{1}{2}N(R(m_1 - 1, m_2) - 1)$ $c_1$-coloured edges, a contradiction since since $c$ is not an integer. Thus, (1) is not always true and we can therefore always choose a vertex $v$, so that either (2) or (3) holds.

Now, suppose that (2) holds. We have either a $c_1$-coloured subgraph $K_{m_1-1}$ or a $c_2$-coloured $K_{m_2}$. For the latter, we are done. Suppose there is a $c_1$-coloured subgraph $K_{m_1-1}$. Then the subgraph $K_{m_1-1}$ together with the vertex $v$, and all the $c_1$-coloured edges incident to them will form a $c_1$-coloured $K_{m_1}$.

Suppose that (3) holds. We have either a $c_1$-coloured subgraph $K_{m_1}$ or a $c_2$-coloured $K_{m_2-1}$. If there is $c_1$-coloured subgraph $K_{m_1}$, we are done. Suppose the latter. Then the subgraph $K_{m_2-1}$ together with the vertex $v$ and all the $c_2$-coloured edges incident to them will form a $c_2$-coloured $K_{m_2}$. $\square$

**Example 2.12.** Note that $R(2,4) = 4$ and $R(3,3) = 6$ are both even; therefore, Theorem 2.11 implies that $R(3,4) \leq R(2,4) + R(3,3) = 9$. Figure 2.3 shows a $(c_1, c_2)$-colouring of $K_8$ without a $c_1$-coloured (———) $K_3$ or a $c_2$-coloured (− − −) $K_4$ as subgraph. We thus have $R(3,4) \geq 9$. Hence, $R(3,4) = 9$.

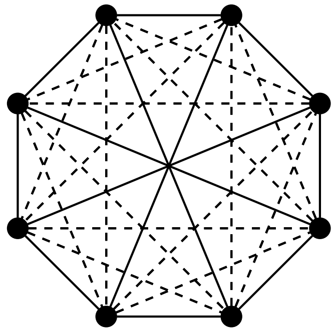


Figure 2.3: $R(3,4) > 9$.

**Theorem 2.13.** $R(m_1, m_2) = R(m_2, m_1)$.

*Proof.* By the definition of $R(m_2, m_1)$, $R(m_2, m_1)$ is the minimum number of vertices in the complete graph, such that in any edge-colouring of the complete graph $K_{R(m_2,m_1)}$ with colours $c'_1$ and $c'_2$, there is either a $c'_1$-coloured $K_{m_2}$ or $c'_2$-coloured $K_{m_1}$. Now, consider that we recoloured every edges of the graph in such a way that colour $c'_1$ will be replaced by colour $c_2$ and the colour $c'_2$ will be replaced by colour $c_1$. Then, $R(m_2, m_1)$ will be indeed the minimum number of vertices in the complete graph, such that in any edge-colouring of the complete graph $K_{R(m_2,m_1)}$ with colours $c_1$ and $c_2$, there is either a $c_1$-coloured $K_{m_1}$ or $c_2$-coloured $K_{m_2}$. Hence, $R(m_1, m_2) = R(m_2, m_1)$. □

**Example 2.14.** In Example 2.12, we have shown $R(3,4) = 9$. By Theorem 2.13, we get that $R(4,3) = 9$.

**Theorem 2.15.** [48]

(1) $R(3,5) = 14$.
(2) $R(4,4) = 18$.

*Proof.*

(1) We have $R(3,4) = 9$ by Example 2.12, and $R(2,5) = 5$ by Lemma 2.2. Hence by Lemma 2.3, we have $R(3,5) \leq 14$. Now, by the colouring of the complete graph $K_{13}$ as shown in Figure 2.4, we get $R(3,5) > 13$, and hence, we can conclude that $R(3,5) = 14$.
(2) We have $R(3,4) = 9$ by Example 2.12. By Lemma 2.3, we have $R(4,4) \leq 18$. Now, by the colouring of the complete graph $K_{17}$ shown in Figure 2.5, we get $R(4,4) > 17$, and hence, we can conclude that $R(4,4) = 18$.

□

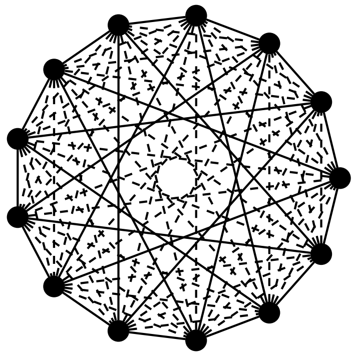


Figure 2.4: $R(3,5) > 13$.

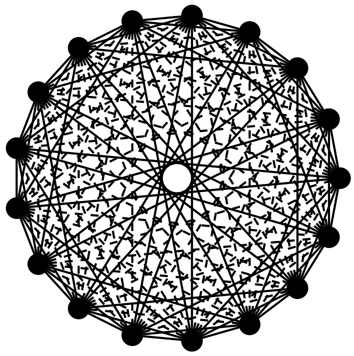


Figure 2.5: $R(4,4) > 18$.

Table 2.1 shows some of the known Ramsey numbers, with the references as cited in the table and previous discussion.

| $(m_1, m_2)$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 3 | 1 | 3 | 6 | 9 | 14 | 18 [47] | 23 [59] | 28 [73] | 36 [49] |
| 4 | 1 | 4 | 9 | 18 | 25 [74] | | | | |
| 5 | 1 | 5 | 14 | 25 | | | | | |
| 6 | 1 | 6 | 18 | | | | | | |
| 7 | 1 | 7 | 23 | | | | | | |
| 8 | 1 | 8 | 28 | | | | | | |
| 9 | 1 | 9 | 36 | | | | | | |

Table 2.1: Known Ramsey numbers

These are some of the presently known Ramsey numbers. Finding bounds on Ramsey-type numbers is a major area of research in Ramsey Theory. We now prove some bounds on Ramsey numbers that we were able to find independently, and could not be found in previous literature, namely the bounds given in Theorems 2.16–2.20 below.

**Theorem 2.16.** $R(3,m) \leq \frac{(m)(m+1)}{2}$.

*Proof.* By Lemma 2.3, we have

$$\begin{aligned}
R(3,m) &\le R(2,m) + R(3,m-1) \\
&\le R(2,m) + R(2,m-1) + R(3,m-2) \\
&\quad \vdots \\
&\le R(2,m) + R(2,m-1) + R(2,m-2) + \cdots + R(2,2) + R(3,1) \\
&= m + (m-1) + (m-2) + \cdots + 4 + 3 + 2 + 1 \\
&= \frac{m(m+1)}{2}. \qquad \square
\end{aligned}$$

In fact, since $\frac{m(m+1)}{2} = \frac{m(m+1)(m-1)!}{2(m-1)!} = \frac{(m+1)!}{2![(m+1)-2]!} = \binom{m+1}{2}$, Theorem 2.16 coincides with Theorem 2.10. Further notice that,

$$\begin{aligned}
\frac{m(m+1)}{2} &= \frac{m^2+m}{2} \\
&= \frac{2m+m^2-m}{2} \\
&= m + \frac{m^2-m}{2} \\
&= m + \frac{m(m-1)}{2} \\
&= |V(K_m)| + |E(K_m)|
\end{aligned}$$

where $|V(K_m)|$ and $|E(K_m)|$ are the number of vertices and edges in $K_m$, respectively. This makes Theorem 2.16 a special case of a conjecture by Sidorenko [93]. Note also that $R(3,3), R(2,4), R(3,5), R(2,6), R(3,9)$ and $R(2,10)$ are all even, so by Theorem 2.11, we can further improve Theorem 2.16.

**Theorem 2.17.**

(1) *For* $m \ge 4, R(3,m) \le \frac{(m)(m+1)}{2} - 1$.
(2) *For* $m \ge 6, R(3,m) \le \frac{(m)(m+1)}{2} - 2$.
(3) *For* $m \ge 10, R(3,m) \le \frac{(m)(m+1)}{2} - 3$.

*Proof.*

(1)
$$\begin{aligned}
R(3,m) &\le R(2,m) + R(3,m-1) \\
&\le R(2,m) + R(2,m-1) + R(3,m-2) \\
&\quad \vdots \\
&\le R(2,m) + R(2,m-1) + R(2,m-2) + \cdots + R(2,4) \\
&\quad + R(3,3) - 1 \\
&\le m + (m-1) + (m-2) + \cdots + R(2,4) + R(2,3) + R(2,3) - 1 \\
&= m + (m-1) + (m-2) + \cdots + 4 + 3 + 2 + 1 - 1 \\
&= \frac{m(m+1)}{2} - 1.
\end{aligned}$$

(2)
$$
\begin{aligned}
R(3,m) &\le R(2,m) + R(3,m-1) \\
&\le R(2,m) + R(2,m-1) + R(3,m-2) \\
&\quad\vdots \\
&\le R(2,m) + R(2,m-1) + R(2,m-2) + \cdots + R(2,7) + R(3,6) \\
&\le R(2,m) + R(2,m-1) + R(2,m-2) + \cdots + R(2,7) + R(2,6) \\
&\quad + R(3,5) - 1 \\
&\le R(2,m) + R(2,m-1) + R(2,m-2) + \cdots + R(2,5) + R(3,4) - 1 \\
&\le R(2,m) + R(2,m-1) + R(2,m-2) + \cdots + R(2,5) + R(2,4) \\
&\quad + R(3,3) - 1 - 1 \\
&\le m + (m-1) + (m-2) + \cdots + R(2,4) + R(2,3) + R(2,3) - 2 \\
&= m + (m-1) + (m-2) + \cdots + 4 + 3 + 3 - 2 \\
&= m + (m-1) + (m-2) + \cdots + 4 + 3 + 2 + 1 - 2 \\
&= \frac{m(m+1)}{2} - 2\,.
\end{aligned}
$$

(3)

$$
\begin{aligned}
R(3,m) &\le R(2,m) + R(3,m-1) \\
&\le R(2,m) + R(2,m-1) + R(3,m-2) \\
&\quad\vdots \\
&\le R(2,m) + R(2,m-1) + R(2,m-2) + \cdots + R(2,11) + R(3,10) \\
&\le R(2,m) + R(2,m-1) + R(2,m-2) + \cdots + R(2,11) + R(2,10) \\
&\quad + R(3,9) - 1 \\
&\le R(2,m) + R(2,m-1) + R(2,m-2) + \cdots + R(2,7) + R(3,6) - 1 \\
&\le R(2,m) + R(2,m-1) + R(2,m-2) + \cdots + R(2,7) + R(2,6) \\
&\quad + R(3,5) - 1 - 1 \\
&\le R(2,m) + R(2,m-1) + R(2,m-2) + \cdots + R(2,5) + R(3,4) - 2 \\
&\le R(2,m) + R(2,m-1) + R(2,m-2) + \cdots + R(2,5) + R(2,4) \\
&\quad + R(3,3) - 1 - 2 \\
&\le m + (m-1) + (m-2) + \cdots + R(2,4) + R(2,3) + R(2,3) - 3 \\
&= m + (m-1) + (m-2) + \cdots + 4 + 3 + 3 - 3 \\
&= m + (m-1) + (m-2) + \cdots + 4 + 3 + 2 + 1 - 3 \\
&= \frac{m(m+1)}{2} - 3\,.
\end{aligned}
$$

$\square$

**Theorem 2.18.** $R(4,m) \le \frac{m(m+1)(m+2)}{6}$.
*Furthermore, for $m \ge 5$, $R(4,m) \le \frac{m(m+1)(m+2)}{6} - 1$.*

*Proof.* By Lemma 2.3, we have

$$\begin{aligned}
R(4,m) &\le R(3,m) + R(4,m-1) \\
&\le R(3,m) + R(3,m-1) + R(4,m-2) \\
&\quad\vdots \\
&\le R(3,m) + R(3,m-1) + R(3,m-2) + \cdots + R(3,2) + R(4,1) \\
&\le \frac{(m)(m+1)}{2} + \frac{(m-1)(m)}{2} + \cdots + \frac{(4)(5)}{2} + \frac{(3)(4)}{2} + \frac{(2)(3)}{2} + 1 \\
&\le \frac{(m)(m+1)}{2} + \frac{(m-1)(m)}{2} + \cdots + \frac{(4)(5)}{2} + \frac{(3)(4)}{2} + \frac{(2)(3)}{2} + \frac{(1)(2)}{2} \\
&= \frac{m(m+1)(m+2)}{6} \,.
\end{aligned}$$

Now, note that $R(3,5)$ and $R(4,4)$ are both even. Then, for $m \ge 5$, by Theorem 2.11, we have

$$\begin{aligned}
R(4,m) &\le R(3,m) + R(4,m-1) \\
&\le R(3,m) + R(3,m-1) + R(4,m-2) \\
&\quad\vdots \\
&\le R(3,m) + R(3,m-1) + R(3,m-2) + \cdots + R(4,5) \\
&\le R(3,m) + R(3,m-1) + \cdots + R(3,5) + R(4,4) - 1 \\
&\le R(3,m) + R(3,m-1) + \cdots + R(3,5) + R(3,4) + R(4,3) - 1 \\
&\le \frac{(m)(m+1)}{2} + \frac{(m-1)(m)}{2} + \cdots + \frac{(4)(5)}{2} + \frac{(4)(5)}{2} - 1 \\
&= \frac{(m)(m+1)}{2} + \frac{(m-1)(m)}{2} + \cdots + \frac{(4)(5)}{2} \\
&\quad + \frac{(3)(4)}{2} + \frac{(2)(3)}{2} + \frac{(1)(2)}{2} - 1 \\
&= \frac{m(m+1)(m+2)}{6} - 1 \,.
\end{aligned}$$

□

**Theorem 2.19.** *For $m \ge 3$,*

$$\begin{aligned}
R(4,m) \le \frac{m(m+1)(m+2)}{6} &- 3\max\{0, m-9\} \\
&- 2\max\{0, \min\{m,9\} - 5\} - \max\{0, \min\{m,5\} - 3\} - 1 \,.
\end{aligned}$$

*Proof.* We use Theorem 2.17. First, divide the value of $m$ into 3 cases.
Case (i) : $3 \le m \le 5$.

Note that both $\max\{0, m-9\}$ and $\max\{0, \min\{m,9\}-5\}$ are equal to 0 and $\max\{0, \min\{m,5\}-3\} = m-3$. Hence, Theorem 2.19 will become

$$R(4,m) \leq \frac{m(m+1)(m+2)}{6} - (m-3) - 1\,.$$

$$\begin{aligned}
m=3: \quad R(4,3) &\leq R(3,3) + R(4,2) - 1 \\
&\leq R(3,3) + R(3,2) + R(4,1) - 1 \\
&\leq \frac{3(4)}{2} + \frac{2(3)}{2} + \frac{1(2)}{2} - 1 \\
&= \frac{3(4)(5)}{6} - 1. \\
m=4: \quad R(4,4) &\leq R(3,4) + R(4,3) \\
&= 2R(4,3) \\
&\leq 2[\frac{3(4)(5)}{6} - 1] \\
&= \frac{2(3)(4)(5)}{6} - 2 \\
&= \frac{4(5)(6)}{6} - 1 - 1. \\
m=5: \quad R(4,5) &\leq R(3,5) + R(4,4) \\
&\leq \frac{(5)(6)}{2} - 1 + \frac{(4)(5)(6)}{6} - 1 - 1 \\
&= \frac{(3)(5)(6)}{6} - 1 + \frac{(4)(5)(6)}{6} - 1 - 1 \\
&= \frac{5(6)(7)}{6} - 2 - 1
\end{aligned}$$

Case (ii) : $6 \leq m \leq 9$.
Note that $\max\{0, m-9\} = 0$, that $\max\{0, \min\{m,9\}-5\} = m-5$ and that $\max\{0, \min\{m,5\}-3\} = 2$. Hence, Theorem 2.19 will reduce to

$$R(4,m) \leq \frac{m(m+1)(m+2)}{6} - 2(m-5) - 3\,.$$

Hence,

$$\begin{aligned}
m=6: \quad R(4,6) &\leq R(3,6)+R(4,5) \\
&\leq \frac{6(7)}{2}-2+\frac{5(6)(7)}{6}-2-1 \\
&= \frac{3(6)(7)}{6}-2+\frac{5(6)(7)}{6}-2-1 \\
&= \frac{6(7)(8)}{6}-2-3\,. \\
m=7: \quad R(4,7) &\leq R(3,7)+R(4,6) \\
&\leq \frac{7(8)}{2}-2+\frac{6(7)(8)}{6}-2-3 \\
&= \frac{3(7)(8)}{6}-2+\frac{6(7)(8)}{6}-2-3 \\
&= \frac{7(8)(9)}{6}-2(2)-3\,. \\
m=8: \quad R(4,8) &\leq R(3,8)+R(4,7) \\
&\leq \frac{8(9)}{2}-2+\frac{7(8)(9)}{6}-2(2)-3 \\
&= \frac{3(8)(9)}{6}-2+\frac{7(8)(9)}{6}-2(2)-1 \\
&= \frac{8(9)(10)}{6}-2(3)-3. \\
m=9: \quad R(4,9) &\leq R(3,9)+R(4,8) \\
&\leq \frac{9(10)}{2}-2+\frac{8(9)(10)}{6}-2(3)-3 \\
&= \frac{3(9)(10)}{6}-2+\frac{8(9)(10)}{6}-2(3)-3 \\
&= \frac{9(10)(11)}{6}-2(4)-3\,.
\end{aligned}$$

Case (iii) : $m \geq 10$.
Note that $\max\{0, m-9\} = m-9$, that $\max\{0, \min\{m, 9\} - 5\} = 4$ and that $\max\{0, \min\{m, 5\} - 3\} = 2$. Hence, Theorem 2.19 reduces to

$$R(4,m) \leq \frac{m(m+1)(m+2)}{6} - 3(m-9) - 11\,.$$

We prove this case by induction.

$$
\begin{aligned}
m = 10: \quad R(4,10) &\le R(3,10) + R(4,9) \\
&\le \frac{10(11)}{2} - 3 + \frac{9(10)(11)}{6} - 2(4) - 3 \\
&= \frac{3(10)(11)}{6} - 3 + \frac{9(10)(11)}{6} - 11 \\
&= \frac{10(11)(12)}{6} - 3 - 11\,.
\end{aligned}
$$

Now, assume that the theorem valid for $R(4, m_1 - 1)$ for some $m_1 - 1 \ge 10$. We want to show that the theorem also valid for $R(4, m_1)$. Notice that, by the assumption, we have $R(4, m_1 - 1) \le \frac{m_1(m_1-1)(m_1+1)}{6} - 3[(m_1 - 1) - 9] - 11$.

$$
\begin{aligned}
R(4,m_1) &\le R(3,m_1) + R(4,m_1-1) \\
&\le \frac{m_1(m_1+1)}{2} - 3 + \frac{m_1(m_1-1)(m_1+1)}{6} - 3(m_1 - 10) - 11 \\
&= \frac{3(m_1)(m_1+1)}{6} - 3 + \frac{m_1(m_1-1)(m_1+1)}{6} - 3(m_1 - 10) - 11 \\
&= \frac{m_1(m_1+1)(m_1+2)}{6} - 3(m_1 - 9) - 11\,.
\end{aligned}
$$

Hence by induction, the theorem is valid for any $m \ge 10$. □

**Theorem 2.20.** *For $m \ge 4, R(5,m) \le \frac{m(m+1)(m+2)(m+3)}{24} - 1$.*

*Proof.* By Lemma 2.3, we have

$$
\begin{aligned}
R(5,m) &\le R(4,m) + R(5,m-1) \\
&\le R(4,m) + R(4,m-1) + R(5,m-2) \\
&\quad\vdots \\
&\le R(4,m) + R(4,m-1) + \cdots + R(4,5) + R(5,4) \\
&\le R(4,m) + R(4,m-1) + \cdots + R(4,5) + R(4,4) + R(5,3) - 1 \\
&\quad \text{since both } R(4,4) \text{ and } R(5,3) \text{ are even} \\
&\le R(4,m) + R(4,m-1) + \cdots + R(4,3) + R(4,2) + R(5,1) - 1 \\
&\le \frac{(m)(m+1)(m+2)}{6} + \frac{(m-1)(m)(m+1)}{6} + \cdots + \frac{(3)(4)(5)}{6} \\
&\quad + \frac{(2)(3)(4)}{6} + 1 - 1 \\
&\le \frac{(m)(m+1)(m+2)}{6} + \frac{(m-1)(m)(m+1)}{6} + \cdots + \frac{(3)(4)(5)}{6} \\
&\quad + \frac{(2)(3)(4)}{6} + \frac{(1)(2)(3)}{6} - 1 \\
&= \frac{m(m+1)(m+2)(m+3)}{24} - 1\,. \qquad \square
\end{aligned}
$$

A more general result on the upper bound of the Ramsey numbers was established by Ajtai, Komlós and Szemerédi [2], who showed that

$$R(m_1, m_2) \leq \frac{c_{m_1} m_2^{m_1-1}}{(\ln m_2)^(m_1-2)}\ , \text{ for some constant } c_k > 0.$$

Interested reader is referred to [2] for more details and proofs.

Next, we will discuss on the lower bound of the Ramsey numbers.

**Theorem 2.21.** [10] $R(m_1, m_2) \geq R(m_1, m_2 - 1) + 2m_1 - 3$.

*Proof.* Consider the graph $G = K_{R(m_1,m_2-1)-1}$. By the definition of $R(m_1, m_2 - 1)$, there is a 2-colouring of the edges of the graph $G$ which neither has a $c_1$-coloured $K_{m_1}$ subgraph nor a $c_2$-coloured $K_{m_2-1}$ subgraph. Consider this colouring. Note that $G$ must contain a $c_1$-coloured $K_{m_1-1}$ subgraph, for otherwise, if we add a new vertex and join it to all the edges in $G$ with $c_1$-coloured edges, we get a colouring of $K_{R(m_1,m_2-1)}$ without any monochromatic $K_{m_1}$ or $K_{m_2-1}$ subgraphs, violating the definition of $R(m_1, m_2 - 1)$. In fact, we only need to consider some $c_1$-coloured $K_{m_1-2}$ subgraph of $G$. Denote the vertices in this $K_{m_1-2}$ by $u_1, u_2, \ldots, u_{m_1-2}$.

Now, add $m_1 - 2$ more vertices to $G$ and denote them by $v_1, v_2, \ldots, v_{m_1-2}$. For each $i \in [m_1 - 2]$, join the vertices $u_i$ and $v_i$ with a $c_2$-coloured edge. Join the vertex $v_i$ to each of the other vertices $x$ in $G$ with the edges in the same colour as the edges joining $u_i$ to $x$. Let $H = K_{R(m_1,m_2-1)+m_1-3}$ be the resulting graph.

Note that in graph $H$, there is no $c_1$-coloured $K_{m_1}$. For suppose that there is one; then $u_i$ and $v_i$ cannot both be in that $K_{m_1}$. Since $v_i$'s were added to the graph by duplicating the $u_i$'s, any $v_i$ involved in the $K_{m_1}$ is isomorphic to the $K_{m_1}$ obtained by replacing $v_i$ with $u_i$. However, this contradicts with the original colouring of graph $G$. On the other hand, in the initial colouring of graph $G$, there is no $c_2$-coloured $K_{m_2-1}$. For graph $H$, the biggest degree of $c_2$-coloured complete graph is $m_2 - 1$ but any $c_2$-coloured $K_{m_2-1}$ must involve a pair of $u_i$ and $v_i$ and no other $u$ and $v$.

We then adjoin $m_1$ more vertices, denoting them by $w_1, w_2, \ldots, w_{m_1}$. Colour the edges $\{w_i, w_j\}$ with colour $c_1$ for all $i \neq j$ and the edges $\{w_i, y\}$ with colour $c_2$ for all $y \in \{u_1, u_2, \ldots, u_{m_1-2}, v_1, v_2, \ldots, v_{m_1-2}, w_1, w_2, \ldots, w_{m_1}\}$. For the edges $\{u_i, w_j\}$, colour with $c_1$ if $i \geq j$ and $c_2$ otherwise. For the edges $\{v_i, w_j\}$, we colour the other way round: colour $c_2$ if $i \geq j$ and $c_1$ otherwise.

We first prove that $K_{R(m_1,m_2-1)+2m_1-4}$ contains no $c_1$-coloured $K_{m_1}$. Assume the contrary. Since there is no such $K_{m_1}$ in $H$, the subgraph $K_{m_1}$ must involve some $w_i$. Hence, the subgraph $K_{m_1}$ must contain only vertices from the set $\{u_1, \ldots, u_{m_1-2}, v_1, \ldots, v_{m_1-2}, w_1, \ldots, w_{m_1}\}$. There are two cases to consider.
Case 1: Only one of the $w_i$'s is involved, say $w_a$. The vertices connected to $w_a$ with $c_1$-coloured edges are $\{v_1, \ldots, v_{a-1}, u_a, \ldots, u_{m_1-2}\}$. Thus, the maximum degree of the monochromatic $c_1$-coloured complete subgraph is $m_1 - 1$, which is a contradiction.
Case 2: There are two or more of the $w_i$'s involved. Without loss of generality, we may assume there are $k$ of them, $\{w_{a_1}, \ldots, w_{a_k}\}$. Note that $k \leq a_k - a_1$. Further note that the only vertices that are connected to all of them are $\{u_{a_k}, \ldots, u_{m_1-2}, v_1, v_{a_1-1}\}$. Thus, the maximum degree of the monochromatic $c_1$-coloured complete subgraph is $k + (m_1 - 2 - a_k + 1) + (a_1 - 1 + 1) \leq m_1 - 1$, which then is a contradiction.

Now, we want to prove that the graph $K_{R(m_1,m_2-1)+2m_1-4}$ does not contain a $c_2$-coloured $K_{m_2}$ subgraph. Assume the contrary. Since there is no such $K_{m_2}$ in graph $H$, again, the subgraph $K_{m_2}$ must involve some $w_i$. Because the edges $\{w_i, w_j\}$ are coloured with $c_1$ for all $i \neq j$, there is exactly one of the $w_i$'s involved, say $w_b$. Hence, the $K_{m_2}$ must include $K_{m_2-1}$ in $H$, which must use exactly one pair $u_i, v_i$. However, one of the edges $\{u_i, w_b\}$ and $\{v_i, w_b\}$ must be $c_1$-coloured, depending on the value of $i$ and $b$, which is then a contradiction.

This colouring of $K_{R(m_1,m_2-1)+2m_1-4}$ does not contain a $c_1$-coloured $K_{m_1}$ subgraph or a $c_2$-coloured $K_{m_2}$ subgraph, so the proof is complete. □

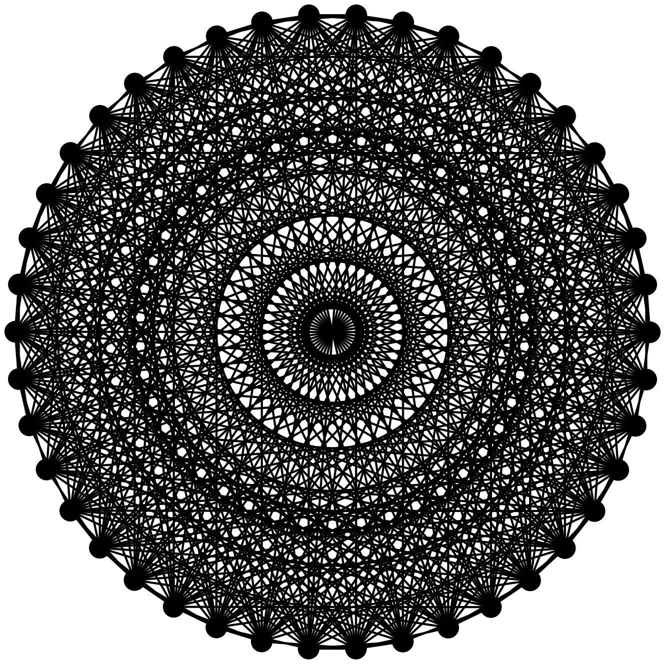
**Theorem 2.22.**

(1) $R(5,5) \leq 50$.
(2) $R(5,5) \geq 43$. [27]

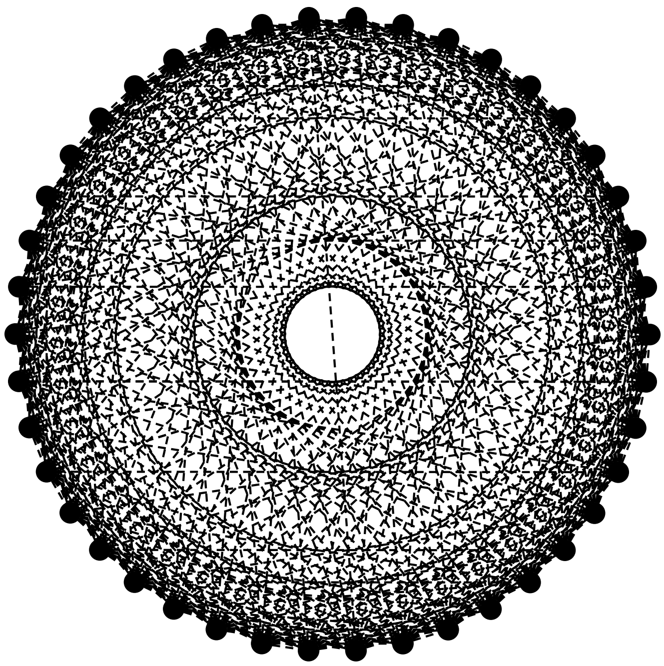
*Proof.*

(1) In [74], it is proven that $R(4,5) = 25$. By Theorem 2.11, we have $R(5,5) \leq R(4,5) + R(5,4) = 25 + 25 = 50$.
(2) Figure 2.6 shows $K_{42}$ with the edge-colouring with 2 colours $c_1$(———) and $c_2$(− − −) as demonstrated by Exoo in [27] which contains no monochromatic copies of $K_5$ in the colouring. The interested reader is referred to [27] for more details. □

Hence, we have now proven that $43 \leq R(5,5) \leq 50$. However, in [3], by using a technique of gluing induced subgraphs and verification by checking approximately two trillion separate cases by computer, Angeltveit and McKay proved that $R(5,5) \leq 48$. In fact, this is the best upper bound known for $R(5,5)$ today.



(a) No $c_1$-coloured $K_5$



(b) No $c_2$-coloured $K_5$

Figure 2.6: $R(5,5) > 42$

Table 2.2 shows some known bounds of the Ramsey numbers, after compiling the known results in the bounds of Ramsey numbers, with the references as cited, and referring to the table compiled by Radziszowski in [83].

| $m_1 \backslash m_2$ | 3 | 4 | 5 | 6 |
|---|---|---|---|---|
| 4 | 9 | 18 | 25 | $[36, 41]$[31, 75] |
| 5 | 14 | 25 | $[43, 48]$[27, 3] | $[58, 87]$[29, 58] |
| 6 | 18 | [36,41] | [58,87] | $[102, 165]$[60, 71] |
| 7 | 23 | $[49, 61]$[28, 71] | $[80, 143]$[11, 95] | $[115, 298]$[34, 71] |
| 8 | 28 | $[59, 84]$[34, 71] | $[101, 216]$[52, 95] | $[134, 495]$[34, 71] |
| 9 | 36 | $[73, 115]$[83, 71] | $[133, 316]$[68, 71] | $[183, 780]$[68, 71] |
| 10 | $[40, 42]$[32, 39] | $[92, 149]$[52, 71] | $[149, 442]$[34, 71] | $[204, 1171]$[68, 71] |
| 11 | $[47, 50]$[30, 39] | $[102, 191]$[34, 95] | $[183, 633]$[68, 56] | $[256, 1804]$[68, 56] |
| 12 | $[53, 59]$[64, 70] | $[128, 238]$[97, 95] | $[203, 848]$[68, 56] | $[294, 2566]$[68, 56] |
| 13 | $[60, 68]$[64, 39] | $[138, 291]$[34, 95] | $[233, 1138]$[68, 56] | $[347, 3703]$[68, 56] |
| 14 | $[67, 77]$[64, 39] | $[147, 349]$[34, 95] | $[267, 1461]$[68, 56] | $[326, 5033]$[Thm 2.21,[56]] |
| 15 | $[74, 87]$[64, 39] | $[155, 417]$[34, 95] | $[269, 1878]$[34, 56] | $[401, 6911]$[83, 56] |

Table 2.2: Bounds for Ramsey number $R(m_1, m_2)$ for $m_1 \leq 6$ and $m_2 \leq 15$

Let us introduce one very interesting special type of Ramsey numbers, known as diagonal Ramsey number, $R(m, m)$, or also denoted by $R(m)$. Despite much research on these numbers, $R(3) = 6$ and $R(4) = 18$ are the only known exact diagonal Ramsey numbers. The first upper bound on the $R(m)$ is a consequence of the Erdős and Szekeres proof of Ramsey theorem in 1935 [25]:

$$R(m) \leq \binom{2m-2}{m-1} \leq 4^m.$$

In 1968, Walker had established a recurrence result on $R(m, m)$ [104], proved that

$$R(m, m) \leq 4R(m, m-2) + 2.$$

The best until now upper bound was provided by Conlon in 2009 [17]:

$$R(m+1) \leq \binom{2m}{m} m^{-C\frac{\log m}{\log \log m}} \text{ , for some constant C .}$$

On the other hand, the first lower bound of the diagonal Ramsey numbers was provided by Erdős in 1947 [22], who gave a simple probabilistic proof of $R(m) \geq cm2^{\frac{m}{2}}$, for some constant $c$. In 1975, Spencer [94] improved the result to

$$R(m) \geq m2^{\frac{m}{2}}[\frac{\sqrt{2}}{e} + o(1)].$$

Before we end this chapter, we look at some results on the colouring of graph edges with more than 2 colours.

**Theorem 2.23.** $R(3,3,3) = 17$

*Proof.* We first prove that $R(3,3,3) \leq 17$. Let $A$ be any vertex in $K_{17}$. Then there are 16 edges incident with it. By the Pigeonhole Principle, at least 6 of them must be of the same colour, say $c_1$. Among these 6 vertices, if there are two vertices whose edge connecting them is $c_1$-coloured, then we have a monochromatically $c_1$-coloured $K_3$. Otherwise, all the edges in this $K_6$ are coloured with the other 2 colours, $c_2$ and $c_3$. By Example 2.8, $R(3,3) = 6$. Hence, we will get a monochromatic $K_3$. Therefore, $R(3,3,3) \leq 17$.

Now, we need to prove that $R(3,3,3) \geq 17 > 16$. To do so, we need to construct a 3-colouring of edges of $K_{16}$ which does not contain monochromatic $K_3$. We first label the vertices in $K_{16}$ as $u_1, u_2, \ldots, u_{16}$. Define the following adjacency matrix:

$$M = \{a_{ij}\} = \begin{cases} 0 & , \{u_i, u_j\} \notin E(K_{16}); \\ 1 & , \{u_i, u_j\} \text{ is } c_1\text{-coloured}; \\ 2 & , \{u_i, u_j\} \text{ is } c_2\text{-coloured}; \\ 3 & , \{u_i, u_j\} \text{ is } c_3\text{-coloured}. \end{cases}$$

Now, consider a colouring of $K_{16}$ defined by the following adjacency matrix [99].

$$M = \begin{bmatrix}
0 & 3 & 3 & 1 & 2 & 2 & 2 & 3 & 1 & 1 & 2 & 3 & 1 & 2 & 1 & 3 \\
3 & 0 & 1 & 3 & 2 & 2 & 3 & 2 & 1 & 1 & 3 & 2 & 2 & 1 & 3 & 1 \\
3 & 1 & 0 & 3 & 2 & 3 & 2 & 2 & 2 & 3 & 1 & 1 & 1 & 3 & 1 & 2 \\
1 & 3 & 3 & 0 & 3 & 2 & 2 & 2 & 3 & 2 & 1 & 1 & 3 & 1 & 2 & 1 \\
2 & 2 & 2 & 3 & 0 & 3 & 3 & 1 & 1 & 2 & 1 & 3 & 1 & 1 & 2 & 3 \\
2 & 2 & 3 & 2 & 3 & 0 & 1 & 3 & 2 & 1 & 3 & 1 & 1 & 1 & 3 & 2 \\
2 & 3 & 2 & 2 & 3 & 1 & 0 & 3 & 1 & 3 & 1 & 2 & 2 & 3 & 1 & 1 \\
3 & 2 & 2 & 2 & 1 & 3 & 3 & 0 & 3 & 1 & 2 & 1 & 3 & 2 & 1 & 1 \\
1 & 1 & 2 & 3 & 1 & 2 & 1 & 3 & 0 & 3 & 3 & 1 & 2 & 2 & 2 & 3 \\
1 & 1 & 3 & 2 & 2 & 1 & 3 & 1 & 3 & 0 & 1 & 3 & 2 & 2 & 3 & 2 \\
2 & 3 & 1 & 1 & 1 & 3 & 1 & 2 & 3 & 1 & 0 & 3 & 2 & 3 & 2 & 2 \\
3 & 2 & 1 & 1 & 3 & 1 & 2 & 1 & 1 & 3 & 3 & 0 & 3 & 2 & 2 & 2 \\
1 & 2 & 1 & 3 & 1 & 1 & 2 & 3 & 2 & 2 & 2 & 3 & 0 & 3 & 3 & 1 \\
2 & 1 & 3 & 1 & 1 & 1 & 3 & 2 & 2 & 2 & 3 & 2 & 3 & 0 & 1 & 3 \\
1 & 3 & 1 & 2 & 2 & 3 & 1 & 1 & 2 & 3 & 2 & 2 & 3 & 1 & 0 & 3 \\
3 & 1 & 2 & 1 & 3 & 2 & 1 & 1 & 3 & 2 & 2 & 2 & 1 & 3 & 3 & 0
\end{bmatrix}$$

There is no monochromatic $K_3$ in this colouring of $K_{16}$. Thus, $R(3,3,3) > 16 \geq 17$. Therefore, we have $R(3,3,3) = 17$. $\square$

Let $R(k;r) = R(k_1, k_2, \ldots, k_r)$, where $k_1 = k_2 = \cdots = k_r = k$. There is a result regarding $R(3;r)$ which has been proven by Wan in 1997 [106]. Here we state the theorem; the interested reader is referred to [106] for the detailed proof.

**Theorem 2.24.** [106] *For $r \geq 4$, $R(3;r) \leq r!(\frac{e-e^{-1}+3}{2}) + 1 \approx 2.68r! + 1$.*

Suppose that $G_1, G_2, \ldots, G_r$ are graphs. We can further extend the concept of Ramsey numbers $R(m_1, m_2, \ldots, m_r)$ by defining $R(G_1, G_2, \ldots, G_r)$ to be the least $n$ such that if the edges of a complete graph $K_n$ is $r$-coloured, then there is always a monochromatic subgraph $G_i$, for some $i \in [r]$. In next chapter, we will present some Ramsey results on certain types of graphs.

# Chapter 3
## Ramsey-type Theorems for Graphs

In this chapter, we will introduce some graph analogues of Ramsey's Theorem. In Section 3.1, we will present some Ramsey-type results for general graphs. In Section 3.2, we will present some Ramsey-type results for tree graphs. In Section 3.3, we will introduce some Ramsey-type results for cycle graphs. In Section 3.4, we will look into Ramsey's Theorem for bipartite graphs. In Section 3.5, we will present more on the result of bipartite Ramsey Number.

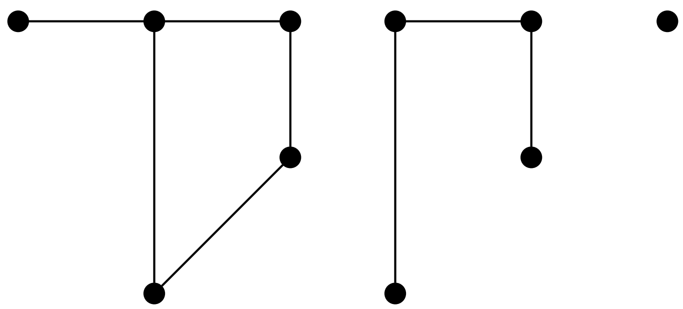
## 3.1 Ramsey-type Results for General Graphs

Before presenting Ramsey-type results for more specific types of graphs in the later subsections, we present some results that are valid for graphs in general. We first introduce some definitions that will be useful in our discussion.

**Definition 3.1** (Chromatic index)**.** *The* chromatic index *is the smallest number of colours needed to colour the edges of a graph in such a way that no two edges incident to the same vertex share the same colour. The chromatic index of the graph $G$ is denoted by $\chi(G)$.*

**Definition 3.2** (Connected component)**.** *A* connected component *of a graph is a subgraph in which any two vertices of the subgraph are connected to each other. In our study, $c(G)$ denotes the largest size of a connected component of the graph $G$.*

**Example 3.3.** Let $G$ and $H$ be graphs shown below.



(a) $G$ (b) $H$

Figure 3.1: Graphs $G$ and $H$.

Then $\chi(G) = 3, \chi(H) = 2$ and $c(G) = 5, c(H) = 4$.

**Theorem 3.4.** [16] *Let $G$ and $H$ be any graphs.* $R(G, H) \geq (c(G)-1)(\chi(H)-1)+1$.

*Proof.* Consider the complete graph $K_{(c(G)-1)(\chi(H)-1)}$. Note that we can find $\chi(H)-1$ disjointed copies of $K_{c(G)-1}$ as subgraphs of $K_{(c(G)-1)(\chi(H)-1)}$. Colour the edges of these subgraphs with colour $c_1$ and all the rest of the edges with colour $c_2$. In this way, we have no $c_1$-coloured $G$ as the largest size of the $c_1$-coloured component is $c(G)-1$. On the other hand, the $c_2$-coloured subgraph has the chromatic index of $\chi(H)-1$ and hence it is impossible to have a $c_2$-coloured $H$. Hence, $R(G,H) \geq (c(G)-1)(\chi(H)-1)+1$. □

We discovered and proved the following improvement of Theorem 3.4.

**Theorem 3.5.** *Let $k \geq 2$ and $R(G;k)$ be the least $n$ such that $k$-colouring of the complete graph $K_n$ will give us a monochromatic subgraph $G$.*
*Then $R(G;k) \geq (\chi(G)-1)(R(G;k-1)-1)+1$.*

*Proof.* Consider the complete graph $K_{(\chi(G)-1)(R(G;k-1)-1)}$. Note that we can find $(\chi(G)-1)$ disjointed copies of $K_{R(G;k-1)-1}$ as subgraphs of $K_{(\chi(G)-1)(R(G;k-1)-1)}$. Colour the edges of these subgraphs with colour $c_i$ for $1 \leq i \leq k-1$. By the definition of $R(G;k-1)$, there is a colouring of these edges such that there is no monochromatic $G$. On the other hand, we colour all the remaining edges with colour $c_k$. Since the $c_k$-coloured subgraph has the chromatic index of $\chi(G)-1$ and hence it is impossible to have a $c_k$-coloured $G$. Therefore, we have $R(G;k) \geq (\chi(G)-1)(R(G;k-1)-1)+1$. □

## 3.2 Ramsey-type Results for Trees

In this section, we will present Ramsey-type results on tree graphs.

**Theorem 3.6.** [15] $R(T_m, K_n) = (m-1)(n-1)+1$.

*Proof.* Note that $c(T_m) = m$ and $\chi(K_n) = n-1$. By Theorem 3.4, we have $R(T_m, K_n) \geq (c(T_m)-1)(\chi(K_n)-1)+1 = (m-1)(n-1)+1$. We now wish to prove that $R(T_m, K_n) \leq (m-1)(n-1)+1$. Note that $R(T_1, K_1) = R(K_1, K_1) = R(1,1) = 1$ by Lemma 2.2. Assume that $R(T'_m, K'_n) \leq (m'-1)(n'-1)+1$ for all values of $m'$ and $n'$ such that $m'+n' < m+n$. Consider any colouring of $K_{(m-1)(n-1)+1}$ with colour $c_1$ and $c_2$. By the induction assumption, we have $R(T_{m-1}, K_n) \leq (m-2)(n-1)+1 < (m-1)(n-1)+1$. Hence, in any colouring of $K_{(m-1)(n-1)+1}$, we have either $c_1$-coloured $T_{m-1}$ or $c_2$-coloured $K_n$. In latter case, we are done. Suppose that we have a $c_1$-coloured subgraph $T_{m-1}$ Remove this $c_1$-coloured $T_{m-1}$ and all the edges incident to it from the graph. Then, we will get a complete graph $K_{(m-1)(n-2)+1}$. By the induction assumption, we have $R(T_m, K_{n-1}) \leq (m-1)(n-2)+1$. Therefore, in this resulting $K_{(m-1)(n-2)+1}$, either we have some $c_1$-coloured $T_m$, in which case we are done by adding back the $c_1$-coloured $T_{m-1}$ and the other removed edges, or we have a $c_2$-coloured subgraph $K_{n-1}$. Suppose the latter case and add back $T_{m-1}$ to $K_{(m-1)(n-2)+1}$. Let $u$ be any end vertex of $T_{m-1}$. Consider all the edges joining $u$ to $K_{(m-1)(n-2)+1}$. If one of the edges is $c_1$-coloured, then a $c_1$-coloured $T_m$ is formed and we are done. Otherwise, all the edges will be $c_2$-coloured including those joining vertex $u$ to the $c_2$-coloured $K_{n-1}$ in $K_{(m-1)(n-2)+1}$, which will give us a $c_2$-coloured $K_n$. Hence, in either way, we can find either a $c_1$-coloured $T_m$ subgraph or a $c_2$-coloured $K_n$ subgraph in the colouring of $K_{(m-1)(n-1)+1}$. Thus, $R(T_m, K_n) \leq (m-1)(n-1)+1$. Therefore, we have $R(T_m, K_n) = (m-1)(n-1)+1$. □

**Theorem 3.7.** *Let $T_m$ be a tree graph with $m$ vertices. Then $m \leq R(T_3, T_m) \leq m+1$.*

*Proof.* First, note that $\chi(T_3) = 2$ and $c(T_m) = m$. Hence by Theorem 3.4, $R(T_3, T_m) = R(T_m, T_3) \geq (m-1)(2-1)+1 = m$. Now, consider any 2-colouring of $K_{m+1}$ and assume that there is no $c_1$-coloured $T_3$ in the colouring. Note that there is a $T_m$ as a subgraph in $K_{m+1}$. If all edges in this subgraph $T_m$ are $c_2$-coloured, then we are done. Suppose that is not the case. Then there is at least an edge, say $\{u, v\}$, is $c_1$-coloured. Now, let $w$ be the vertex that not included in the $T_m$ and let $a_1, \ldots, a_i$ and $b_1, \ldots, b_j$ be the vertices adjacent to $u$ and $v$ respectively. Note that the vertices $u$ and $v$ must be adjacent to all these vertices with $c_2$-coloured edges, or else we will have a $c_1$-coloured $T_3$, which is a contradiction. Consider the edges connecting the vertex $w$ and all the vertices $a_1, \ldots, a_i$. If all are $c_2$-coloured, then replace the vertex $u$ with the vertex $w$. If any of the edges is $c_1$-coloured, then all the edges connecting the vertex $w$ and the vertices $b_1, \ldots, b_j$ must be $c_2$-coloured, so replace the vertex $v$ with the vertex $w$. Now, we will get a new $T_m$. If all the edges in this $T_m$ are $c_2$-coloured, then we are done, or else, repeat the process and we can get one eventually. Hence, we have $R(T_3, T_m) \leq m+1$. Thus, we have $m \leq R(T_3, T_m) \leq m+1$. ☐

After finding and proving Theorem 3.7, we discovered that Chartrand, Gould and Polimeni [12] had proved a more complete result on $R(T_3, T_m)$, below. Interested readers are referred to [12] for a proof.

**Theorem 3.8.** [12] *If $T_m$ is any tree of order $m \geq 3$, then*

$$R(T_3, T_m) = \begin{cases} m+1, & \text{if } T_m \text{ is a complete bipartite graph } K_{1,m-1} \text{ and } m \text{ is even.} \\ m, & \text{otherwise.} \end{cases}$$

## 3.3 Ramsey-type Results for Cycles

In this section, we will present some Ramsey-type results for cycle graphs. We first derive Ramsey number for small cycles.

**Theorem 3.9.** *Let $C_3$ and $C_4$ be the cycle graph with 3 and 4 vertices, respectively. Then $R(C_3, C_3) = R(C_4, C_4) = 6$.*

*Proof.* Note that $C_3$ is isomorphic to $K_3$, so $R(C_3, C_3) = R(K_3, K_3) = R(3, 3) = 6$. Now, we need to prove that $R(C_4, C_4) = 6$. We first prove that $R(C_4, C_4) \leq 6$. Suppose the contrary, that there is no monochromatic $C_4$ in any 2-colouring of the edges of the complete graph $K_6$. As we have previously shown, there exists a monochromatic $K_3$ in any colouring of $K_6$. Without loss of generality, let the edges of the subgraph $K_3$ be $c_1$-coloured and denote the vertices of $K_3$ by $u_1, u_2$ and $u_3$. Let the remaining vertices be denoted by $v_1, v_2$ and $v_3$, respectively. For each $v_i$, there is at most one $c_1$-coloured edge connecting $v_i$ to the $c_1$-coloured $K_3$ or else some $c_1$-coloured $C_4$ is formed. This means, for each $v_i$, that there are at least 2 $c_2$-coloured edges to the subgraph $K_3$.

Now suppose that one of the vertices $v_i$, say $v_1$, has no $c_1$-coloured edge to the subgraph $K_3$. Then, there can be at most one $c_2$-coloured edge from each $v_2$ and

$v_3$ to the $c_1$-coloured $K_3$, or else a $c_2$-coloured $C_4$ will be formed, a contradiction. Therefore, there is exactly one $c_1$-coloured edge from each $v_i$ to the $K_3$.

Now, suppose there is more than one $c_1$-coloured edges from one of the $u_i$'s, say $u_1$, to the some $v_j$. Without loss of generality, we let the vertex $u_1$ be joined to $v_1$ and $v_2$ by $c_1$-coloured edges. Then, both $v_1$ and $v_2$ are adjacent to $u_2$ and $u_3$ by $c_2$-coloured edges, and this gives us a $c_2$-coloured $C_4$, a contradiction. Hence, each $v_i$ is adjacent to a different vertex of the $c_1$-coloured $K_3$ by a $c_1$-coloured edge. Without loss of generality, let the vertex $u_i$ be joined to $v_i$ by a $c_1$-coloured edge. If any of the edges $v_iv_j$ is $c_1$-coloured, then we will have a $c_1$-coloured $C_4$. If all the edges $v_iv_j$ are $c_2$-coloured, then we will get a $c_2$-coloured $C_4$. Either way, it will lead us to a contradiction. Therefore, we have $R(C_4, C_4) \leq 6$.

Now by Figure 3.2, $R(C_4, C_4) > 5$. Thus, we have $R(C_4, C_4) = 6$. □

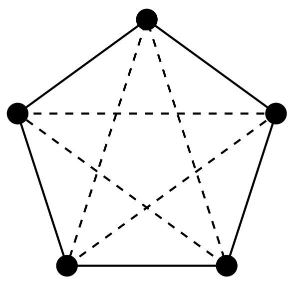


Figure 3.2: $R(C_4, C_4) > 5$.

We have looked at some Ramsey numbers on small cycles. Now, we present some Ramsey-type results on cycles more generally. In order to do prove those results, we will need the following useful proposition.

**Proposition 3.10.** [23] *Suppose that $G$ is a graph with $n$ vertices and at least $\frac{1}{2}[(c-1)(n-1)+1]$ edges. Then $G$ contains a cycle of length at least $c$.*

**Theorem 3.11.** [8] *Let $m \geq 5$ be an odd integer. Then $R(C_m, C_m) = 2m - 1$.*

*Proof.* Note that $\chi(C_m) = 3$ for odd $m$ and $c(C_m) = m$. By Theorem 3.4, $R(C_m, C_m) \geq (\chi(C_m) - 1)(c(C_m) - 1) + 1 = 2m - 1$. Now, we need to prove that $R(C_m, C_m) \leq 2m - 1$. Note that $K_{2m-1}$ has $\binom{2m-1}{2}$ edges. By the Pigeonhole Principle, at least $\frac{1}{2}\binom{2m-1}{2} = \frac{1}{4}(2m-1)(2m-2) > \frac{1}{2}[(m-1)((2m-1)-1)+1]$ of the edges are of the same colour in any 2-colouring of the edges of $K_{2m-1}$. By Proposition 3.10, there is a monochromatic cycle of the length at least $m$. Now, if we can show that the existence of a monochromatic $C_k$ in the colouring will imply the existence of a monochromatic $C_{k-1}$, then the theorem is proven.

Let $(v_0, \ldots, v_{k-1}, v_0)$ be a monochromatic $C_k$ in the edge colouring of $K_{2m-1}$. Without loss of generality, let $C_k$ be $c_1$-coloured. Suppose the contrary, that there is no monochromatic $C_{k-1}$. Now, consider the indices modulo $k$. Note that the edges of $C_k$, $\{v_i, v_{i+1}\}, 0 \leq i \leq k-1$ are $c_1$-coloured. Since there is no $c_1$-coloured $C_{k-1} = (v_0, \ldots, v_i, v_{i+2}, \ldots, v_k, v_0)$, we have the edges $\{v_i, v_{i+2}\}, 0 \leq i \leq k-1$ must be $c_2$-coloured. Since there is no $c_2$-coloured $C_{k-1} = (v_i, v_{i+4}, v_{i+6}, \ldots, v_{i-2}, v_i)$, we have the edges $\{v_i, v_{i+4}\}, 0 \leq i \leq k-1$ must be $c_1$-coloured. Now, since there is no $c_1$-coloured $C_{k-1} = (v_i, v_{i+3}, v_{i+4}, \ldots, v_{i-2}, v_{i+2}, v_{i+1}, v_i)$, we have the edges $\{v_i, v_{i+3}\}, 0 \leq i \leq k-1$ must be $c_2$-coloured. But this will give us a $c_2$-coloured $C_{k-1} = (v_1, v_3, \ldots, v_{k-6}, v_{k-3}, v_{k-5}, \ldots, v_2, v_{k-1}, v_{k-4}, v_{k-2}, v_1)$ if $k$ is odd or $c_2$-coloured $C_{k-1}$, $(v_1, v_3, \ldots, v_{k-5}, v_{k-2}, v_{k-4}, \ldots, v_2, v_k, v_{k-3}, v_{k-1}, v_1)$ if $k$ is even. In either case, we are done.

Hence by induction, we have a monochromatic cycle $C_m$ of length $m$ in any edge-colouring of $K_{2m-1}$ with 2 colours. Thus, $R(C_m, C_m) \leq 2m-1$.

Therefore, we have $R(C_m, C_m) = 2m-1$. □

**Theorem 3.12.** [85] *Let $m \geq 6$ be an even integer. Then $R(C_m, C_m) = \frac{3m}{2} - 1$.*

*Proof.* Consider a $c_1$-coloured complete graph $K_{m-1}$. Note that the complete graph does not contain $C_m$. Now, consider a $c_2$-coloured complete graph $K_{\frac{m}{2}-1}$. Join both complete graphs with $c_2$-coloured edges and form the complete graph $K_{\frac{3m}{2}-2}$. Clearly, there is no monochromatic $C_m$. Thus, we have $R(C_m, C_m) \geq \frac{3m}{2} - 1 > \frac{3m}{2} - 2$.

Now, we wish to show that $R(C_m, C_m) \leq \frac{3m}{2} - 1$. Let $D$ be the largest monochromatic cycle with $s$ vertices in the 2-colouring of $K_{\frac{3m}{2}-1}$. Let $G$ be $c_1$-coloured subgraph of $K_{\frac{3m}{2}-1}$ and $\overline{G}$ be $c_2$-coloured subgraph of $K_{\frac{3m}{2}-1}$. Without loss of generality, we let $D$ be the subgraph of $G$.

If $s < m$, by Proposition 3.10, then the number of edges of $G$, $|E(G)|$ is less than $\frac{1}{2}((m-1)(\frac{3m}{2} - 1 - 1) + 1)$. Then, we have the number of edges in $\overline{G}$,

$$\begin{aligned}
|E(\overline{G})| &= |E(K_{\frac{3m}{2}-1}| - |E(G)| \\
&> \frac{(\frac{3m}{2}-1)(\frac{3m}{2}-2)}{2} - \frac{1}{2}\Big((m-1)\Big(\frac{3m}{2} - 1 - 1\Big) + 1\Big) \\
&= \frac{6m^2 - 7m - 1}{2} \\
&> \frac{\frac{3m^2}{2} - \frac{7m}{2} + 3}{2} \\
&= \frac{1}{2}\Big((m-1)\Big(\frac{3m}{2} - 1 - 1\Big) + 1\Big).
\end{aligned}$$

By Proposition 3.10, $\overline{G}$ contains a cycle of length at least $m > s$, which is then a contradiction since $s$ is the largest size of monochromatic cycle in the colouring. Therefore, we have $s \geq m$.

Now, if $s = m$, then we are done. If $s > m$, then by the similar method of construction in Theorem 3.11, it can be shown that either $G$ or $\overline{G}$ will contains $C_m$ as the subgraph and we are done. Hence, we have $R(C_m, C_m) \leq \frac{3m}{2} - 1$. Therefore, $R(C_m, C_m) = \frac{3m}{2} - 1$, for $m \geq 6$ is an even integer. □

**Theorem 3.13.** [13] *Let $m \geq 3$,*

$$R(C_3, C_m) = \begin{cases} 6 & , \textit{if } m = 3; \\ 2m-1 & , \textit{otherwise.} \end{cases}$$

*Proof.* The outline of this proof is found in [13]; we have filled in the explicit details. From Theorem 3.9, it is known that $R(C_3, C_3) = 6$. Now for $m \geq 4$, we first need to prove that $R(C_3, C_m) \leq 2m-1$. We use induction on $m$.

Let $m = 4$. We have to show that $R(C_3, C_4) \leq 2(4) - 1 = 7$. Suppose to the contrary that there is neither $c_1$-coloured $C_3$ nor $c_2$-coloured $C_4$ in any 2-colouring of $K_7$. Note that $R(C_3, C_3) = 6 < 7$; hence, there is a monochromatic $C_3$, say $(u_1, u_2, u_3, u_1)$ in the colouring of $K_7$. Since there is no $c_1$-coloured $C_3$, the monochromatic $C_3$ must be $c_2$-coloured. Let the remaining vertices in $K_7$ be $v_1, v_2, v_3$ and $v_4$. Consider the set of vertices $\{u_2, u_3, v_1, v_2, v_3, v_4\}$. Note that these 6 vertices form $K_6$ as a subgraph of $K_7$. Hence, there is a monochromatic $C_3$ in the subgraph $K_6$ and the $C_3$ must be $c_2$-coloured. If both $u_2$ and $u_3$ are vertices in the monochromatic $C_3$, say $(u_2, u_3, v_i, u_2)$ for some $1 \leq i \leq 4$, then the vertices $\{u_2, u_1, u_3, v_i\}$ will form a $c_2$-coloured $C_4$, hence a contradiction. Now, suppose that both $u_2$ and $u_3$ are not vertices in the monochromatic $C_3$. Without the loss of generality, assume that the $c_2$-coloured $C_3$ is $(v_1, v_2, v_3, v_1)$. Consider the vertex $v_4$. Note that if there are two $c_2$-coloured edges connecting $v_4$ to the vertices $\{u_1, u_2, u_3\}$, then there will be $c_2$-coloured $C_4$, a contradiction. Hence, at least two of the vertices $\{u_1, u_2, u_3\}$, say $u_1$ and $u_2$, are connected to $v_4$ with $c_1$-coloured edges. Similarly, there at least two of the vertices $\{v_1, v_2, v_3\}$, say $v_1$ and $v_2$, are connected to $v_4$ with $c_1$-coloured edges. Now, look at the edges $\{u_1, v_1\}$ and $\{u_2, v_2\}$. If either of the edges is $c_1$-coloured, then we will have a $c_1$-coloured $C_3$. If both of the edges are $c_2$-coloured, then we will have a $c_2$-coloured $C_4$. On the other hand, suppose that one of the vertices $u_2$ and $u_3$, say $u_2$, is a vertex in the monochromatic $C_3$. Without loss of generality, let the $c_2$-coloured $C_3$ be $(u_2, v_1, v_2, u_2)$. Note that the edges $\{u_1, v_1\}$, $\{u_1, v_2\}$, $\{u_3, v_1\}$ and $\{u_3, v_2\}$ must be $c_1$-coloured, or else we will have a $c_2$-coloured $C_4$. Now, consider another subgraph $K_6$, with the vertices $\{u_1, u_3, v_1, v_2, v_3, v_4\}$. Note that there is a monochromatic $C_3$ in this subgraph and must be $c_2$-coloured since there is no $c_1$-coloured $C_3$. Further note that only one of the vertices $u_1$, $u_3$, $v_1$ and $v_2$ will be involved in the $c_2$-coloured $C_3$, or else we will have a $c_2$-coloured $C_4$. Without loss of generality, assume that the $c_2$-coloured $C_3$ is $(u_1, v_3, v_4, u_1)$. Consider these edges, $\{u_3, v_3\}$ and $\{v_2, v_3\}$. If any of these edges is $c_2$-coloured, then we will have a $c_2$-coloured $C_4$. If both of the edges are $c_1$-coloured, then we will have a $c_1$-coloured $C_3$. Either way, it will lead us to a contradiction. Hence, $R(C_3, C_4) \leq 7$.

Now, assume that $R(C_3, C_m) \leq 2m - 1$. We wish to show that $R(C_3, C_{m+1}) \leq 2(m + 1) - 1 = 2m + 1$. Consider any 2-colouring of $K_{2m+1}$ with colour $c_1$ and $c_2$. Let $G$ be a $c_1$-coloured subgraph of $K_{2m+1}$ and $\overline{G}$ be a $c_2$-coloured subgraph of $K_{2m+1}$. Suppose there is no $C_3$ in the subgraph $G$. We need to show there is a $C_{m+1}$ in the subgraph $\overline{G}$. Since $R(C_3, C_m) \leq 2m - 1$, therefore there must be a $C_m$, say $(u_1, \ldots, u_m, u_1)$, in the subgraph $\overline{G}$. Denote the remaining vertices of $K_{2m+1}$ by $v_1, v_2, \ldots, v_m, v_{m+1}$. Note that if any of the $v_i$'s is connected to two consecutive vertices in $C_m$ in $\overline{G}$, then we will have a $C_{m+1}$ in $\overline{G}$. Suppose there is no such $v_i$. There are then two cases to be considered.

First, assume that there exist two alternate vertices of $C_m$, say $u_j$ and $u_{j+2}$, which are respectively adjacent in $\overline{G}$ to two distinct $v_i$. Without loss of generality, let $u_j$ be adjacent to $v_1$ in $\overline{G}$ and $u_{j+2}$ be adjacent to $v_2$ in $\overline{G}$. If $v_1$ is adjacent to $v_2$ in $\overline{G}$, then $(u_1, \ldots, u_j, v_1, v_2, u_{j+2}, \ldots, u_m, u_1)$ will form a $C_{m+1}$ in $\overline{G}$. If $v_1$ and $v_2$ are adjacent in $G$, then consider the edges $\{v_1, u_{j+1}\}$ and $\{v_2, u_{j+1}\}$. If either of these two edges lies in $\overline{G}$, then we will have $C_{m+1}$ in $\overline{G}$. Otherwise, we will have a $C_3$ in $G$, which is a contradiction.

On the other hand, suppose that there are no two alternate vertices of $C_m$ adjacent in $\overline{G}$ to distinct $v_i$. Before we proceed to the case, note that since there is no $C_3$ in $G$, if there is a vertex in the $C_m$ that is not adjacent to any $v_i$ in $\overline{G}$ and hence adjacent to every single $v_i$'s in $G$, then every $v_i$'s must connect with each other in $\overline{G}$. However, this will form a $C_{m+1}$ (in fact a $K_{m+1}$) in $\overline{G}$. Hence, every vertices in $C_m$ must be adjacent to some $v_i$ in $\overline{G}$. Without loss of the generality, suppose that $u_j$ is adjacent to $v_1$ in $\overline{G}$. Then, $u_{j+2}$ must also adjacent to $v_1$ in $\overline{G}$ as we have assumed that there are no two alternate vertices of $C_m$ adjacent in $\overline{G}$ to distinct $v_i$. Since there is no $v_i$ connected to two consecutive vertices in $C_m$ in $\overline{G}$, $v_1$ must be adjacent to $u_{j-1}$ and $u_{j+1}$ in $G$. This will then force $u_{j-1}$ and $u_{j+1}$ to be adjacent to each other in $\overline{G}$. Then, $(u_1, \ldots, u_{j-1}, u_{j+1}, u_j, v_1, u_{j+2}, u_{j+3}, \ldots, u_m, u_1)$ will form a $C_{m+1}$ in $\overline{G}$. In either case, we will have a $C_{m+1}$ in $\overline{G}$. Thus, $R(C_3, C_{m+1}) \leq 2m+1 = 2(m+1)-1$. By induction, we have $R(C_3, C_m) \leq 2m-1$ for $m \geq 4$.

Now, we need to prove that $R(C_3, C_m) \geq 2m-1$ for $m \geq 4$. Note that there is a complete bipartite graph $K_{m-1,m-1}$ as a subgraph of $K_{2m-2}$. Colour the edges of this subgraph in colour $c_1$ and the remaining edges in colour $c_2$. In this way, we will get a $c_1$-coloured $K_{m-1,m-1}$ and $c_2$-coloured $K_{m-1} \cup K_{m-1}$. Hence, there is no $c_1$-coloured $C_3$ and $c_2$-coloured $C_m$ in this colouring. Thus, we have $R(C_3, C_m) \geq 2m-1 > 2m-2$ for $m \geq 4$.

Therefore, $R(C_3, C_m) = 2m-1$ for $m \geq 4$. Thus, the theorem is valid. □

**Theorem 3.14.** [13] *Let $m \geq 4$,*

$$R(C_4, C_m) = \begin{cases} 6, & \text{if } m = 4\,; \\ 7, & \text{if } m = 5\,; \\ m+1, & \text{otherwise}\,. \end{cases}$$

*Proof.* The outline of this proof is found in [13]; we have filled in the explicit details.

By Theorem 3.9, it is known that $R(C_4, C_4) = 6$.

For $m = 5$, consider any 2-colouring of $K_7$. Let $s$ be the largest size of the monochromatic cycle in the colouring. Let $G$ be a $c_1$-coloured subgraph of $K_7$ and $\overline{G}$ be a $c_2$-coloured subgraph of $K_7$. Suppose that $s \leq 3$. Then the largest cycle size in both $G$ and $\overline{G}$ is at most 3. By Proposition 3.10, the number of edges of $G$ and $\overline{G}$, $|E(G)|$ and $|E(\overline{G})|$, are both less than $\frac{1}{2}((3-1)(7-1)+1) = 6.5 < 7$. Therefore, $21 = |E(K_7|) = |E(G)| + |E(\overline{G})| < 14$, a contradiction. Hence, $s \geq 4$. If $4 \leq s \leq 5$, then we are done. If $s \geq 6$, then by the similar method of construction to that in Theorem 3.11, it can be shown that either $G$ or $\overline{G}$ will contains $C_5$ as the subgraph and we are done since $R(C_4, C_5) = R(C_5, C_4)$. Hence, we have $R(C_4, C_5) \leq 7$. Furthermore, Figure 3.3 shows that $R(C_4, C_5) \geq 7 > 6$. Hence, we have $R(C_4, C_5) = 7$.

For $m \geq 6$, note that there is a complete bipartite graph $K_{1,m-1}$ as a subgraph of $K_m$. Colour the edges of this subgraph in colour $c_1$ and the remaining edges in colour $c_2$. In this way, we will get a $c_1$-coloured $K_{1,m-1}$ and $c_2$-coloured $K_1 \cup K_{m-1}$. Hence, there is no $c_1$-coloured $C_4$ and $c_2$-coloured $C_m$ in this colouring. Thus for $m \geq 6$, we have $R(C_4, C_m) > m$ and so $R(C_4, C_m) \geq m+1$.

Now, we want to show that $R(C_4, C_m) \leq m+1$ for $m \geq 6$. We will proceed with induction on $m$. Let $m = 6$, consider any 2-colouring of $K_7$. Since $R(C_4, C_5) = 7$, we will have either a $c_1$-coloured $C_4$ or a $c_2$-coloured $C_5$. If there is a $c_1$-coloured $C_4$, then we are done. Suppose that is not the case. Then we have a $c_2$-coloured $C_5$; denote its vertices by $u_1, u_2, u_3, u_4$ and $u_5$, respectively. Let the remaining vertices be $v_1$ and $v_2$. If there are any consecutive vertices in $C_5$ that are connected to a same vertex, $v_1$ or $v_2$, by $c_2$-coloured edges, then we are done. Suppose there is no such vertex in $C_5$. Then each of the $v_1$ and $v_2$ must be adjacent to at least three of the vertices in $C_5$ via $c_1$-coloured edges. By the Pigeonhole Principle, one of the vertices in $C_5$, say $u_1$, must be adjacent to both $v_1$ and $v_2$ via $c_1$-coloured edges. Note that if any other vertex of $C_5$ join to both $v_1$ and $v_2$ by $c_1$-coloured edges, then we will have a $c_1$-coloured $C_4$. Assume that no such vertex exists. Every other vertex in $C_5$ must be adjacent to at least one of the $v_1$ and $v_2$ via a $c_2$-coloured edge. Without loss of generality, suppose that $u_2$ is adjacent to $v_1$ via a $c_2$-coloured edge. Then, $u_3$ must be adjacent to $v_1$ via a $c_1$-coloured edge and must be adjacent to $v_2$ via a $c_2$-coloured edge. Then, $u_2$ and $u_4$ must be both adjacent to $v_2$ via $c_1$-coloured edges. This will force $u_4$ to be adjacent to $v_1$ via a $c_2$-coloured edge. Then, again, $u_5$ must be connected to $v_1$ by a $c_1$-coloured edge and connected to $v_2$ by a $c_2$-coloured edge. Then, the edge $\{v_1, v_2\}$ must be $c_1$-coloured, or else we will have a $c_2$-coloured $C_6$. Now, look at the edge $\{u_2, u_5\}$. If the edge $\{u_2, u_5\}$ is $c_1$-coloured, then $(u_2, u_5, v_1, v_2, u_2)$ will form a $c_1$-coloured $C_4$. If the edge $\{u_2, u_5\}$ is $c_2$-coloured, then $(u_2, u_5, v_2, u_3, u_4, v_1, u_2)$ will form a $c_2$-coloured $C_6$. In either case, we can conclude that $R(C_4, C_6) \leq 7$.

Now, assume that $R(C_4, C_m) \leq m + 1$ for some $m \geq 6$. We need to show that $R(C_4, C_{m+1}) \leq (m+1)+1 = m+2$. Consider any 2-colouring of $K_{m+2}$. Let $H$ be a $c_1$-coloured subgraph of $K_{m+1}$ and $\overline{H}$ be a $c_2$-coloured subgraph of $K_{m+2}$. Suppose there is no $C_4$ in $H$. Since $R(C_4, C_m) \leq m + 1 < m + 2$, there must be a $C_m$ in $\overline{H}$. Label the vertices in $C_m$ by $u_1, \ldots, u_m$. Let $v_1$ and $v_2$ be the two vertices that are not in $C_m$. Note that if any of $v_1$ and $v_2$ is joined to two consecutive vertices of $C_m$ in $\overline{H}$, then we will have a $C_m$ in $\overline{H}$ and we are done. Suppose there is no such vertex.

Then each of $v_1$ and $v_2$ are adjacent in $H$ to at least $\frac{m}{2}$ vertices of $C_m$. If $v_1$ and $v_2$ are mutually adjacent in $H$ to two or more vertices in $C_m$, then $H$ will contain $C_4$. Hence, there are only two cases to be considered. First, $v_1$ and $v_2$ are mutually adjacent to no vertex of $C_m$ in $H$. Without loss of generality, assume $v_1$ is adjacent to $u_1$ in $H$. Then $v_2$ must be adjacent to $u_1$ in $\overline{H}$, and hence adjacent to $u_2$ and $u_m$ in $H$. In this case, $v_1$ must be adjacent to $u_2$ in $\overline{H}$ and $u_3$ in $H$. Continuing this process of reasoning, we notice that $v_1$ will be adjacent to $u_i$, where $i$ is odd, in $H$. If $m$ is odd, then both $v_1$ and $v_2$ are adjacent to $u_m$ in $H$, a contradiction. Hence, this case can only occur when $m$ is even, where $v_1$ is adjacent to $u_i$ in $H$ for odd $i$ and $v_2$ is adjacent to $u_i$ in $H$ for even $i$. Now, suppose $m$ is even. Consider these edges $\{u_2, u_m\}$ and $\{u_2, u_4\}$. If both edges are in $H$, then $(u_2, u_4, v_1, u_m, u_2)$ will form a $C_4$ in $H$. Therefore, at least one of these two edges must be in $\overline{H}$. Then, we will have a $C_m$ in $\overline{H}$. For example, let the edges $\{u_2, u_4\}$ in $\overline{H}$. Then, $(u_2, u_4, u_3, v_1, u_5, \ldots, u_m, u_1, u_2)$ forms a $C_{m+1}$. On the other hand, $v_1$ and $v_2$ are mutually adjacent to one vertex of $C_m$, say $u_1$, in $H$. Then $u_2$ must be adjacent to one of the vertices $v_1$ and $v_2$ in $\overline{H}$, without loss of generality, say $v_1$. Then $v_1$ must be adjacent to $u_3$ in $H$ and $v_2$ must be adjacent to $u_2$ in $H$ and $u_3$ in $\overline{H}$. Continuing in this way, we see that $v_1$ must be adjacent to $u_i$ for odd $i$ in $H$ and $v_2$ must be adjacent to $u_i$ for even $i$ and $u_1$ in $H$. If $u_m$ is odd, then $(u_m, v_2, u_3, u_2, v_1, u_4, \ldots, u_m)$ will form a $C_{m+1}$ in $\overline{H}$. If $u_m$ is even, then consider the edges $\{u_1, u_{m-1}\}$ and $\{u_1, u_3\}$. Note that one of these two edges must exist in $\overline{H}$, or else $(u_1, u_3, v_1, u_{m-1}, u_1)$ will form a $C_4$ in $H$. Then, we will have a $C_m$ in $\overline{H}$. For example, if the edge $\{u_1, u_{m-1}\}$ lies in $\overline{H}$, then $(v_1, u_m, u_{m-1}, u_1, u_2, \ldots, u_{m-2}, v_1)$ will form a $C_{m+1}$ in $\overline{H}$. In either way, we will have a $C_m$ in $\overline{H}$. Hence, we have $R(C_4, C_{m+1}) \leq m + 2 = (m+1) + 1$. By induction, we have $R(C_4, C_m) \leq m + 1$ for $m \geq 6$.

Therefore, the theorem is valid. $\square$

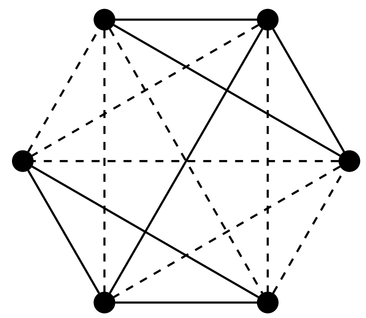


Figure 3.3: $R(C_4, C_5) > 6$.

**Theorem 3.15.** [13] *Let $m \geq 5, R(C_5, C_m) = 2m - 1$.*

*Proof.* The outline of this proof is found in [13]; we have filled in the details.

First, we will show by induction on $m$ that $R(C_5, C_m) \leq 2m - 1$ for $m \geq 5$. For $m = 5$, Theorem 3.11 implies that $R(C_5, C_5) \leq 2(5) + 1 = 10$. Now, assume that $R(C_m, C_m) \leq 2m - 1$. We wish to show that $R(C_5, C_{m+1}) \leq 2(m+1) - 1 = 2m + 1$. Consider any colouring of $K_{2m+1}$. Let $L$ be the $c_1$-coloured subgraph of $K_{2m+1}$ and $\overline{L}$ be the $c_2$-coloured subgraph of $K_{2m+1}$. Suppose there is no $C_5$ in $L$. Since $R(5, m) \leq 2m - 1 < 2m + 1$, there must be a $C_m$ in $\overline{L}$. Denote the vertices in $C_m$ by $u_1, \ldots, u_m$ and the remaining vertices by $v_1, \ldots, v_m, v_{m+1}$. Note that if any of the $v_i$'s is connected to two consecutive vertices in $C_m$ in $\overline{L}$, then we will have a $C_{m+1}$ in $\overline{L}$ and we are done. Suppose there is no such $v_i$. If all vertices $v_1, \ldots, v_{n+1}$ are adjacent to each other in $\overline{L}$ and form $K_{n+1}$, then there is a $C_{m+1}$ in $\overline{L}$ and we are done. Suppose that is not the case. Then two distinct $v_i$, say $v_1$ and $v_2$, are adjacent in $L$. Now, there are three cases to be considered.

*Case 1.* First, assume that there is a vertex from $v_i$s other than $v_1$ and $v_2$, say $v_3$, such that $v_1$ and $v_3$ are joined to a vertex $u_i$ of $C_m$ in $L$, and $v_2$ and $v_3$ are joined to a vertex $u_j$ of $C_m$ in $L$. If $i \neq j$, then $(v_1, u_i, v_3, u_j, v_2, v_1)$ forms a $C_5$ in $L$. If $i = j$, then $v_1$, $v_2$ and $v_3$ are adjacent to a vertex $u_i$, without loss of generality, say $u_1$ in $L$. Note that at least one of $u_2$ and $u_3$, say $u_2$, must be adjacent to $v_3$ in $L$. Similarly, at least one of $u_m$ and $u_{m-1}$, say $u_m$, must be adjacent to $v_3$ in $L$. Then, both $u_2$ and $u_m$ must be adjacent to $v_1$ and $v_2$ in $\overline{L}$ or else $L$ will contain $C_5$. This will force both $u_3$ and $u_{m-1}$ to be adjacent to $v_1$ and $v_2$ in $L$. Now, consider the edge $\{u_1, u_3\}$. If it is in $L$, then $(u_1, u_3, v_2, v_1, u_1)$ will form $C_5$ in $L$. If it is in $\overline{L}$, then $(u_m, v_1, u_2, u_1, u_3, \ldots, u_m)$ will form $C_m$ in $\overline{L}$.

*Case 2.* Next, assume that the first case does not hold, and there is some vertex from $v_i$s, say $v_3$, that is not adjacent in $L$ to the vertex of $C_m$ whichever is joined to

$v_1$ or $v_2$ in $L$. Note that one of the $u_1$ and $u_2$, say $u_1$, must be adjacent to $v_1$ in $L$. Then, $u_1$ must be adjacent to $v_3$ in $\overline{L}$ and, hence, both $u_m$ and $u_2$ must be adjacent to $v_3$ in $L$. Based on our assumption, both $u_m$ and $u_2$ must be adjacent to $v_1$ and $v_2$ in $\overline{L}$. Then, $u_1$ must be adjacent to $v_2$ in $L$ and both $u_{m-1}$ and $u_3$ must be adjacent to $v_1$ and $v_2$ in $L$. By our assumption, both $u_{m-1}$ and $u_3$ must be adjacent to $v_3$ in $\overline{L}$. Similarly, consider the edge $\{u_1, u_3\}$. If it is in $L$, then $(u_1, u_3, v_2, v_1, u_1)$ will form $C_5$ in $L$. If it is in $\overline{L}$, then $(u_m, v_1, u_2, u_1, u_3, \ldots, u_m)$ will form $C_m$ in $\overline{L}$.

*Case 3.* Lastly, assume that the previous two cases do not hold. Then, for each vertex from $v_i$, $i \neq 1, 2$, whenever the edges $\{v_1, u_j\}$ and $\{v_i, u_j\}$ are in $L$, the edge $\{v_2, u_j\}$ is in $\overline{L}$, or whenever the edges $\{v_2, u_j\}$ and $\{v_i, u_j\}$ are in $L$, the edge $\{v_1, u_j\}$ is in $\overline{L}$. For simplicity, we look at the vertex $v_3$. Since *Case 2* does not hold, there is at least one vertex from $C_m$, say $u_1$, that is adjacent to both $v_3$ and one of $v_1$ and $v_2$, say $v_1$, in $L$. Then, the edge $\{u_1, v_2\}$ must be in $\overline{L}$, based on our assumption. This will force both $u_m$ and $u_2$ to be adjacent to $v_2$ in $L$. Now, since *Case 1* does not hold, both $u_2$ and $u_m$ must be adjacent to $v_3$ in $\overline{L}$. Then, both $u_3$ and $u_{m-1}$ must be adjacent to $v_3$ in $L$ and $v_2$ in $\overline{L}$. Continuing this argument, $v_2$ will be adjacent to $u_i$ in $L$ for all even $i$ and $v_3$ will be adjacent to $u_i$ in $L$ for all odd $i$. Therefore, $m$ must be even for this case. Now, consider the vertices $v_4, \ldots, v_{m+1}$. If any of them are adjacent to $u_i$ in $L$ for even $i$, then $v_1$ must be adjacent to $u_i$ in $L$ for odd $i$ and in $\overline{L}$ for even $i$. Then, the edge $\{v_1, v_3\}$ must be in $L$, or else we will get a $C_{m+1}$ in $\overline{L}$. Suppose that all of them are adjacent to $u_i$ in $L$ for odd $i$. If all the edges $\{v_1, v_k\}$ are in $\overline{L}$, where $k \neq 1, 2, 3$, then we have $(u_m, u_1, v_4, v_1, v_5, u_2, \ldots, u_m)$ as a $C_{m+1}$ in $\overline{L}$. However, if there is an edge $\{v_1, v_k\}$, $k \neq 1, 2, 3$, that is in $L$, then we will get $(v_3, u_1, v_1, v_k, u_3, v_3)$ as a $C_5$ in $L$. Therefore, this subcase cannot happen, and we must have the edge $\{v_1, v_3\}$ in $L$. Relabel the vertex $v_2$ as $v_3$ and $v_3$ as $v_2$. Then, we will reach the condition in the second case where the result holds.

In each of these cases, the result holds. By induction, we have $R(C_5, C_m) \leq 2m-1$ for $m \geq 5$.

Now, note that there is complete bipartite graph $K_{m-1,m-1}$ as a subgraph of $K_{2m-2}$. Colour the edges of this subgraph in colour $c_1$ and the remaining edges in colour $c_2$. In this way, we will get a $c_1$-coloured $K_{m-1,m-1}$ and $c_2$-coloured $K_{m-1} \cup K_{m-1}$. Hence, there is neither a $c_1$-coloured $C_5$ nor a $c_2$-coloured $C_m$ in this colouring. Thus, we have $R(C_5, C_m) \geq 2m-1 > 2m-2$ for $m \geq 5$.

Therefore, $R(C_5, C_m) = 2m-1$ for $m \geq 5$. Thus, the theorem is valid. □

The complete theorem on $R(C_{m_1}, C_{m_2})$ was given by Faudree and Schelp [35] and Rosta [85] independently. However, these proofs are complicated. In 2001, a simpler proof was provided by Károlyi and Rosta [61]. The theorem is mentioned below but the interested reader is referred to the publications cited above for detailed proofs.

**Theorem 3.16.** [35, 61, 85]

$$R(C_{m_1}, C_{m_2}) = \begin{cases} 6 & \textit{if } m_1 = m_2 = 3 \textit{ or } 4; \\ 2m_2 - 1 & \textit{if } 3 \le m_1 \le m_2,\ m_1 \textit{ is odd} \\ & \textit{and } (m_1, m_2) \ne (3,3); \\ m_2 - 1 + \frac{m_1}{2} & \textit{if } 4 \le m_1 \le m_2, (m_1, m_2) \ne (4,4) \\ & \textit{and } m_1 \textit{ and } m_2 \textit{ are both even}; \\ \max\{m_2 - 1 + \frac{m_1}{2}, 2m_2 - 1\} & \textit{if } 4 \le m_1 < m_2 \\ & \textit{and } m_1 \textit{ is even and } m_2 \textit{ is odd.} \end{cases}$$

## 3.4 Ramsey-type Results for Bipartite Graphs

In this section, we present some Ramsey results on the bipartite graphs. We first study the existence of the monochromatic bipartite graph in the edge-colouring of a complete graph. Then, we will introduce the bipartite Ramsey theorem.

**Definition 3.17.** *We define $R(K_{p_1,p_2}, K_{q_1,q_2})$ as the least $N$ such that, for every edge-colouring of a complete graph $K_N$ with the colours $c_1$ and $c_2$, we can get either a monochromatic $c_1$-coloured $K_{p_1,p_2}$ or a monochromatic $c_2$-coloured $K_{q_1,q_2}$.*

Now, we will present some results on this type of Ramsey number.

**Theorem 3.18.** [51]

$$R(K_{1,m_1}, K_{1,m_2}) = \begin{cases} m_1 + m_2, & \textit{if } m_1 \textit{ or } m_2 \textit{ is odd.} \\ m_1 + m_2 - 1, & \textit{if both } m_1 \textit{ and } m_2 \textit{ are even.} \end{cases}$$

*Proof.* First, suppose that $m_1$ or $m_2$ is odd. Consider any edge-colouring of $K_{m_1+m_2}$ with the colours $c_1$ and $c_2$. For each vertex, there are $m_1 + m_2 - 1$ edges incident to it. If at least $m_1$ of them are $c_1$-coloured, then we have a $c_1$-coloured $K_{1,m_1}$. Suppose that this is not the case. Then there are at least $m_2$ $c_2$-coloured edges, forming a $c_2$-coloured $K_{1,m_2}$. Hence, we have $R(K_{1,m_1}, K_{1,m_2}) \le m_1 + m_2$. Now, we need to show that $R(K_{1,m_1}, K_{1,m_2}) \ge m_1 + m_2$. Without loss of generality, suppose $m_1$ is odd. Then $m_1 - 1$ must be even. Hence, in the complete graph $K_{m_1+m_2-1}$, there must exist a regular subgraph of degree $m_1 - 1$. We call it subgraph $G$. Note that the complement of $G$, $\overline{G}$ is a regular graph of degree $m_2 - 1$. We colour the complete graph $K_{m_1+m_2-1}$ in such a way that the edges in $G$ has colour $c_1$ and the edges in $\overline{G}$ have colour $c_2$. In this way, we have neither a $c_1$-coloured $K_{1,m_1}$ nor a $c_2$-coloured $K_{1,m_2}$. Hence, we have $R(K_{1,m_1}, K_{1,m_2}) \ge m_1 + m_2$. Therefore, $R(K_{1,m_1}, K_{1,m_2}) = m_1 + m_2$ if $m_1$ or $m_2$ is odd.

Now, suppose that $m_1$ and $m_2$ are both even. Consider any edge-colouring of $K_{m_1+m_2-1}$ with the colours $c_1$ and $c_2$. For each vertex, there are $m_1 + m_2 - 2$ edges incident to it. Note that we cannot have exactly $m_1 - 1$ of these edges coloured $c_1$ for all vertices, because if that were the case, then we would have an odd number of $c_1$-coloured edges in a graph with an odd number of vertices, which is impossible. Hence, there is at least one vertex $v$ in $K_{m_1+m_2-1}$ for which one of the following conditions holds.

(1) There are at least $m_1$ $c_1$-coloured edges incident to $v$. In this case, we have a $c_1$-coloured $K_{1,m_1}$.

(2) There are at most $m_1 - 2$ $c_1$-coloured edges incident to $v$. Then, we have more than $m_2$ $c_2$-coloured edges incident to $v$, and so we have a $c_2$-coloured $K_{1,m_2}$.

Thus, we have $R(K_{1,m_1}, K_{1,m_2}) \leq m_1 + m_2 - 1$. However in the complete graph $K_{m_1+m_2-2}$, we can find a regular subgraph of degree $m_1 - 1$ and its complementary graph is a regular graph of degree $m_2 - 2$. Thus, we have $R(K_{1,m_1}, K_{1,m_2}) \geq m_1 + m_2 - 1$. Therefore, $R(K_{1,m_1}, K_{1,m_2}) = m_1 + m_2 - 1$ if both $m_1$ and $m_2$ are even. $\square$

We have generalised the theorem above as follows.

**Theorem 3.19.**

$$R(K_{1,m_1}, \ldots, K_{1,m_k}) \leq \begin{cases} m_1 + \cdots + m_k - (k-1) + 1 & \textit{if any of } m_i \textit{ or } k \textit{ is odd;} \\ m_1 + \cdots + m_k - (k-1) & \textit{if all } m_i \textit{ and } k \textit{ are even.} \end{cases}$$

*Proof.* Suppose that any of the integers $m_1, \ldots, m_k, k$ is odd. Consider any edge-colouring of $K_{m_1+m_2+\cdots+m_k-(k-1)+1}$ with colours $c_1, c_2, \ldots, c_k$. For each vertex, there are $m_1 + m_2 + \cdots + m_k - (k-1)$ edges incident to it. By the Pigeonhole Principle, at least $m_i$ of these edges are $c_i$-coloured. Then we have a $c_i$-coloured $K_{1,m_i}$ and we are done. Hence, we have $R(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_k}) \leq m_1 + m_2 + \cdots + m_k - (k-1) + 1$ if $m_i$ or $k$ is odd.

Now, suppose that all of the integers $m_1, \ldots, m_k, k$ are even. Consider any edge-colouring of $K_{m_1+m_2+\cdots+m_k-(k-1)}$ with colours $c_1, c_2, \ldots, c_k$. For each vertex, there are $m_1 + m_2 + \cdots + m_k - k$ edges incident to it. Note that we cannot have exactly $m_1 - 1$ of these edges coloured $c_1$ for all vertices, because if that is the case, then we have an odd number of $c_1$-coloured edges in a graph with an odd number of vertices, which is impossible. Hence, for there is at least one vertex $v$ in $K_{m_1+m_2+\cdots+m_k-(k-1)}$ for which one of the following conditions holds.

(1) There are at least $m_1$ $c_1$-coloured edges incident to $v$. In this case, we have a $c_1$-coloured $K_{1,m_1}$.

(2) There are at most $m_1 - 2$ $c_1$-coloured edges incident to $v$. Then we have at least $m_2 + m_3 + \cdots + m_k - (k-2)$ remaining edges incident to $v$. By the Pigeonhole Principle, $m_i$ of them must be $c_i$-coloured, for $i = 2, 3, \ldots, k$. Hence, we have a $c_i$-coloured $K_{1,m_i}$.

Thus, we have $R(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_k}) \leq m_1 + m_2 + \cdots + m_k - (k-1)$ if all $m_i$ and $k$ are even. $\square$

**Theorem 3.20.** $R(K_{1,t}, K_{m_1,m_2}) \leq m_1 + m_2 + t - 1$.

*Proof.* Consider the complete graph $K_{m_1+m_2+t-1}$. Suppose that we colour all of the edges so that there is no $c_1$-coloured $K_{1,t}$. Then, for every vertex in $K_{m_1+m_2+t-1}$, there are at most $t-1$ $c_1$-coloured edges incident to it. Hence, we can find $m_1$ vertices of $K_{m_1+m_2+t-1}$ such that there are at most $t-1$ of the remaining vertices that are adjacent via a $c_1$-coloured edge to any one of the $m_1$ vertices. These $m_1$ vertices, together with the remaining $m_2$ vertices, form a $c_2$-coloured $K_{m_1,m_2}$. Therefore, $R(K_{1,t}, K_{m_1,m_2}) \leq m_1 + m_2 + t - 1$. $\square$

**Theorem 3.21.** [53] $R(K_{1,3}, K_{m_1,m_2}) = m_1 + m_2 + 2$.

*Proof.* From Theorem 3.20, we get $R(K_{1,3}, K_{m_1,m_2}) \leq m_1+m_2+3-1 = m_1+m_2+2$.

On the other hand, in the complete graph $K_{m_1+m_2+1}$ there exists a cycle graph $C_{m_1+m_2+1}$ as the subgraph. Suppose that we colour the complete graph $K_{m_1+m_2+1}$ in such a way that the edges in $C_{m_1+m_2+1}$ are $c_1$-coloured and the remaining edges are $c_2$-coloured. In this way, we have neither a $c_1$-coloured $K_{1,3}$ nor a $c_2$-coloured $K_{m_1,m_2}$. Therefore, we have $R(K_{1,3}, K_{m_1,m_2}) \geq m_1 + m_2 + 2$.

Hence, $R(K_{1,3}, K_{m_1,m_2}) = m_1 + m_2 + 2$. □

Previously, we were looking at the results on the existence of monochromatic, complete bipartite graph in the edge-colouring of a complete graph. Now, we will present a Ramsey-type result for edge-colourings of a complete bipartite graph. However, we first introduce the theorem below.

**Theorem 3.22.** [45] *Let $m \in \mathbb{N}$ and $0 < \varepsilon \leq 1$. There exists a sufficiently large $n$ such that if $G$ is a subgraph of $K_{n,n}$ with at least $\varepsilon n^2$ edges, then $G$ has $K_{m,m}$ as a subgraph.*

*Proof.* The idea of the proof is from [45]; we have added missing proof details here.

We can take any $n$ satisfying $n\binom{\varepsilon n}{m} \geq m\binom{n}{m}$. Let $U$ and $V$ be disjoint sets of $n$ vertices in $K_{n,n}$. Let $G$ be any subgraph of $K_{n,n}$ with at least $\varepsilon n^2$ edges. For each $i \in U$, we set $D_i = \{j \in V : \{i,j\} \in G\}$ and $d_i = |D_i|$. Thus, $\sum\limits_{i\in U} d_i \geq \varepsilon n^2$. Set $S = \{(i,X) : X \subset V, |X| = m, X \subset D_i\}$. For each $i \in U$, there are precisely $\binom{d_i}{m}$ $X$'s such that $(i,X) \in S$. Therefore, we have

$$|S| = \sum_{i\in U}\binom{d_i}{m} \geq n\binom{\frac{1}{n}\sum\limits_{i\in U} d_i}{m} \geq n\binom{\frac{\varepsilon n^2}{n}}{m} = n\binom{\varepsilon n}{m} \geq m\binom{n}{m}.$$

For $X \subseteq V, |X| = m$, we set $T_X = \{i \in U : (i,X) \in S\}$. Then, we have $|S| = \sum |T_X|$. Hence, there are $\binom{n}{m}$ summands $X$ such that $|T_X| \geq \frac{|S|}{\binom{n}{m}} \geq \frac{m\binom{n}{m}}{\binom{n}{m}} \geq m$. Let $T_X^* \subseteq T_X$ with $|T_X^*| = m$. We have that $T_X^* \cup X$, which is a subgraph of $G$, is a complete bipartite graph $K_{m,m}$. □

**Theorem 3.23** (Ramsey's Theorem for Bipartite Graphs)**.** *Let $r, m \in \mathbb{N}$. If $n$ is sufficiently large, then each $r$-colouring of the edges of $K_{n,n}$ gives a complete monochromatic subgraph $K_{m,m}$. The least of such $n$ is known as the bipartite Ramsey Number, $BR(K_{m,m}; r)$.*

*Proof.* Note that there are $n^2$ edges in the complete bipartite graph $K_{n,n}$. By the Pigeonhole Principle, in any $r$-colouring of the edges of $K_{n,n}$, there is at least one colour, say $c_i$ $(1 \leq i \leq r)$, such that at least $\frac{n^2}{r}$ of the edges are $c_i$-coloured. Now, consider Theorem 3.22. Let $\varepsilon = \frac{1}{r}$ and $G$ be the subgraph of $K_{n,n}$ which consists of all the $c_i$-coloured edges. Then, there are at least $\frac{n^2}{r} = \varepsilon n^2$ edges in the subgraph $G$. By Theorem 3.22, $G$ has some $c_i$-coloured $K_{m,m}$ - which is monochromatic. □

**Theorem 3.24.** [87] $BR(K_{m_1,m_2};r) \geq (m_1!m_2!r^{m_1m_2-1})^{\frac{1}{m_1+m_2}}$.

*Proof.* The outline of this proof is given in [87]; we have added some missing details.

Consider a $r$-colourings of the edges of $K_{n,n}$. First, note that there are $r^{n^2}$ ways to colour the edges of $K_{n,n}$. Further note that there are $r^{n^2-(m_1m_2-1)}$ ways to colour $K_{m_1,m_2}$ in order to obtain a monochromatic $K_{m_1,m_2}$. It is clear that there are $\binom{n}{m_1}\binom{n}{m_2}$ copies of $K_{m_1,m_2}$ in a complete graph $K_{n,n}$. Hence, there are at most $\binom{n}{m_1}\binom{n}{m_2}r^{n^2-(m_1m_2-1)}$ colourings of $K_{n,n}$ containing some monochromatic $K_{m_1,m_2}$. Therefore, if we have $\binom{n}{m_1}\binom{n}{m_2}r^{n^2-(m_1m_2-1)} < r^{n^2}$, then there is some $r$-colouring of the edges of $K_{n,n}$ which has no monochromatic $K_{m_1,m_2}$.

Now, suppose that we choose our $n$ such that $r^{m_1m_2-1} > (\frac{n^{m_1}}{m_1!})(\frac{n^{m_2}}{m_2!})$ or, equivalently, $n < (m_1!m_2!r^{m_1m_2-1})^{\frac{1}{m_1+m_2}}$. Then we have

$$r^{m_1m_2-1} > \left(\frac{n^{m_1}}{m_1!}\right)\left(\frac{n^{m_2}}{m_2!}\right) > \binom{n}{m_1}\binom{n}{m_2}$$

and thus

$$r^{n^2} > \binom{n}{m_1}\binom{n}{m_2}r^{n^2-(m_1m_2-1)}.$$

Therefore, $BR(K_{m_1,m_2};r) \geq (m_1!m_2!r^{m_1m_2-1})^{\frac{1}{m_1+m_2}}$. □

# Chapter 4
## Van der Waerden's Theorem

In the previous chapters, we were mainly discussing Ramsey-type results on colourings of the edges of graphs. Starting from this chapter, we will look into the Ramsey-type results on colourings of the set of integers.

In this chapter, we focus on Ramsey-type results guaranteeing the existence of monochromatic arithmetic progressions in colourings of integers. In Section 4.1, we introduce Van der Waerden's Theorem. In Section 4.2, we construct a proof of the theorem. In Section 4.3, we also present the polynomial version of Van der Waerden's Theorem. Then, in Section 4.4, we discuss some bounds on the Van der Waerden numbers.

### 4.1 Van der Waerden's Theorem

In this section, we present Van der Waerden's Theorem. Before doing so, we first introduce some terminology.

**Definition 4.1** (Arithmetic Progression)**.** *An* arithmetic progression *is a sequence of numbers such that the differences between the consecutive terms is constant. An arithmetic progression* $\{a, a+d, \ldots, a+(k-1)d\}$ *is said to be projected from term* $a$ *with common difference* $d$ *and length* $k$*.*

**Example 4.2.** $\{3, 7, 11, 15\}$ is an arithmetic progression projected from $a = 3$ and with the common difference $d = 4$ and length $k = 4$. $\{1, 4, 7, 9\}$ is not an arithmetic progression since the difference is not constant as $7 - 4 = 3$ but $9 - 7 = 2$.

**Theorem 4.3** (Van der Waerden's Theorem)**.** [102] *Let* $k, r \in \mathbb{N}$*. For sufficiently large* $n$*, each* $r$*-colouring of* $[n]$ *gives a monochromatic arithmetic progression of length* $k$*. The least of such* $n$ *is called the Van der Waerden number, denoted by* $W(k, r)$*.*

**Example 4.4.** Consider a 2-colouring of [9] in the following way: 1, 4, 5 and 8 are $c_1$-coloured and 2, 3, 6, 7 and 9 are $c_2$-coloured. Note that the $c_2$-coloured 3, 6 and 9 form a monochromatic arithmetic progression of length 3 with the common difference of 3. In fact, $W(3, 2) = 9$; a detailed proof will be given in Section 4.4.

Before we proceed to the proof of Van der Waerden's Theorem, we want to introduce the density version of the theorem, which is also known as Szemerédi's Theorem.

**Theorem 4.5** (Szemerédi's Theorem). [100] *Let $k \in \mathbb{N}$ and $S \subset \mathbb{N}$. Suppose $S$ has positive upper density, which means*

$$\limsup_{n\to\infty} \frac{|S \cup [n]|}{n} > 0,$$

*then $S$ contains infinitely many arithmetic progression of length $k$.*

Szemerédi's Theorem was first conjectured by Erdős and Turán in 1936 [26], and proven by Szemerédi in 1975 [100]. We are not going to discuss the proof here, interested reader is referred to [100].

## 4.2 Proof of Van der Waerden's Theorem

In this section, we construct a proof of Van der Waerden's Theorem. We first introduce a lemma to help us.

**Lemma 4.6.** [37] *Let $k, r \in \mathbb{N}$. Suppose that the Van der Waerden number $W(k-1, r)$ exists for all $r$. Then for all $c$, there exists a number $U(k-1, r, c)$ such that if $[U(k-1, r, c)]$ is $r$-coloured, then there exists $a \in \mathbb{N}$ for which one of the following conditions hold.*

(1) *There are $c$ monochromatic arithmetic progressions of length $k-1$, all with projected first term $a$, all of different colours, and different from $a$.*
(2) *There is a monochromatic arithmetic progression of length $k$.*

*Proof.* This proof is from [37]. We prove it by induction on $c$. For $c = 1$, we can take $U(k-1, r, 1) = 2W(k-1, r)$. Let $\chi$ be any $r$-colouring of $[U(k-1, r, 1)]$. Consider the colouring of the last half of $[U(k-1, r, 1)]$, which is of the size of $[W(k-1, r)]$. By the definition of $W(k-1, r)$, there exists a monochromatic progression of length $k-1$ in the $r$-colouring of $[W(k-1, r) + 1, 2W(k-1, r)]$, say $\{a, a+d, \ldots, a+(k-2)d\}$. Now, let $a' = a - d$ and $d' = d$. Then we will get $\{a'+d', a'+2d', \ldots, a'+(k-1)d' \in [W(k-1, r)+1, 2W(k-1, r)]\}$ such that $\chi(a'+d') = \chi(a'+2d') = \cdots = \chi(a'+(k-1)d')$. Now, note that $a' \in [W(k-1, r] \in [U(k, r, 1)]$. If $\chi(a') \neq \chi(a'+d')$, then the first condition in Lemma 4.6 holds; otherwise, we have the second condition.

Now, assume that $U(k-1, r, c)$ exists, we want to show the existence of $U(k-1, r, c+1)$. Take $U(k-1, r, c+1) = 2U(k-1, r, c)W(k-1, r^{U(k-1,r,c)})$. Let $\chi$ be any $r$-colouring of $[U(k-1, r, c+1)]$. Now, we divide $[U(k-1, r, c+1)]$ into $U(k-1, r, c)W(k-1, r^{U(k-1,r,c)})$ numbers, followed by $W(k-1, r^{U(k-1,r,c)})$ blocks of size $U(k-1, r, c)$. We denote these blocks by $B_1, B_2, \ldots, B_{W(k-1, r^{U(k-1,r,c)})}$. By the definition of $W(k-1, r^{U(k-1,r,c)})$, there exists a monochromatic arithmetic progression of length $k-1$ of blocks, say $B_A, B_{A+D}, \ldots, B_{A+(k-2)D}$, which means that these blocks are identically coloured in the $r$-colouring of $[U(k-1, r, c+1)]$. Consider the block $B_A$. Note that there are $U(k-1, r, c)$ terms in the block. If the second condition of Lemma 4.6 holds, then we are done. Otherwise, we will have $c$ arithmetic progressions of length $k-1$, all with projected first term $a$, all different colours, and different from $a$. We need to find one more monochromatic set of an arithmetic progression. Since $B_A, B_{A+D}, \ldots, B_{A+(k-2)D}$ are identically coloured, the terms $a+D, a+2D, \ldots, a+(k-2)D$ must be monochromatic. Note that this monochromatic progression of length $k-1$ is different from the previous $c$ arithmetic progressions as all these arithmetic progressions are coloured differently with $a$. Hence, we have $c+1$ arithmetic progressions of length $k-1$, all with projected first term $a$, and all of different colours.

Thus, by induction, the lemma holds. □

Now we prove the Van der Waerden's Theorem (Theorem 4.3).

*Proof.* Note that to prove the theorem, we only need to show the existence of $W(k,r)$ for all $k \geq 1$. We prove it by induction on $k$. First, notice that the case $k=1$ is trivial and that $W(1,r)=r$ since we can take any term from $r$-coloured $[r]$ and we can get a monochromatic arithmetic progression of length 1. Now, suppose that $W(k-1,r)$ exists. We wish to show that $W(k,r)$ exists. Since Lemma 4.6 is valid for all $c$, we consider the case when $c=r$. Hence, there exists $n=U(k-1,r,r)$ such that if $[n]$ is $r$-coloured, then there is a monochromatic arithmetic progression of length $k$ or $r$ arithmetic progressions of length $k-1$, all of different colours, with projected term $a$ whose colour differs from all of them. Now, note that there are only $r$ colours, so the latter case cannot happen. Therefore, we must have a monochromatic arithmetic progression of length $k$. Hence, $W(k,r)$ exists. Thus, by induction, $W(k,r)$ exists for all $k \geq 1$ and the theorem is proven. □

## 4.3 Polynomial Van der Waerden's Theorem

In this section, we are going to introduce the polynomial Van der Waerden's Theorem. First, we introduce the following definition.

**Definition 4.7** (Polynomial)**.** *A polynomial $P(x)$ is defined as an expression built from constants and variables by the means of addition, multiplication and exponential to a non-negative power.*

$$P(x)=\sum_{k=0}^{n} a_k x^n = a_n x_n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \,,$$

*where $a_i$ are constants and $x$ is the variable.*

**Theorem 4.8** (Polynomial Van der Waerden's Theorem)**.** [6]
*Let $k,r \in \mathbb{N}$ and $p_1,\ldots,p_k \in \mathbb{Z}[x]$ with $p_i(0)=0$. If $n$ is sufficiently large, then any $r$-colouring of $[n]$ will give a monochromatic $a, a+p_1(d),\ldots,a+p_k(d)$.*

We will not prove the theorem here. The interested reader is referred to [6, 105] for combinatorial proofs.

## 4.4 Van der Waerden Numbers

In this section, we will present some results on the Van der Waerden numbers.

**Theorem 4.9.**

(1) $W(k,1)=k$
(2) $W(2,r)=r+1$.

*Proof.*

(1) Note that if we colour the set $[k]$ with a single colour, then the set $[k]$ itself will form a monochromatic arithmetic progression of length $k$ with common difference 1.

(2) Consider any $r$-colouring of $[r+1]$. By the Pigeonhole Principle, at least two of them are coloured with the same colour. These two same coloured terms will form a monochromatic arithmetic progression of length 2.

□

**Theorem 4.10.** $W(3, 2) = 9$.

*Proof.* First, we need to show that $W(3, 2) \leq 9$. We divide the 2-colouring of $[9]$ into two cases.

Case 1: Both 1 and 2 are coloured with the same colour.
Without loss of generality, let both 1 and 2 be $c_1$-coloured. To avoid having monochromatic arithmetic progression of length 3, 3 must be $c_2$-coloured. Suppose that 4 is $c_1$-coloured. Then, both 6 and 7 must be $c_2$-coloured, or else $\{1, 4, 7\}$ and $\{2, 4, 6\}$ will form $c_1$-coloured arithmetic progressions of length 3. Then 5 must be $c_1$-coloured, or $\{5, 6, 7\}$ will form a $c_2$-coloured arithmetic progression. Now, if 8 is $c_1$-coloured, then $\{2, 5, 8\}$ will form a $c_1$-coloured arithmetic progression. If 8 is $c_2$-coloured, then $\{6, 7, 8\}$ will form a $c_2$-coloured arithmetic progression. On the other hand, suppose that 4 is $c_2$-coloured. Then 5 must be $c_1$-coloured or $\{3, 4, 5\}$ will form a $c_2$-coloured arithmetic progression. Then both 8 and 9 must be $c_2$-coloured, or else $\{1, 5, 9\}$ and $\{2, 5, 8\}$ will form a $c_1$-coloured arithmetic progressions of length 3. Then 7 will then be forced to be $c_1$-coloured to avoid having monochromatic progression of length 3. Now if 6 is $c_1$-coloured, then $\{5, 6, 7\}$ will form a $c_1$-coloured arithmetic progression. If 6 is $c_2$-coloured, then $\{4, 6, 8\}$ will form a $c_2$-coloured arithmetic progression. Hence, in this case, we will get a monochromatic arithmetic progression of length 3 no matter how we colour the set $[9]$.

Case 2: 1 and 2 are coloured with different colours.
Without loss of generality, we let 1 be $c_1$-coloured and 2 be $c_2$-coloured. Suppose that 3 is $c_1$-coloured. Since 1 and 3 are both $c_1$-coloured, 5 must be $c_2$-coloured. Then 8 will be forced to be $c_1$-coloured or else $\{2, 5, 8\}$ will form a $c_2$-coloured arithmetic progression. If 4 is $c_1$-coloured, then 7 must be $c_2$-coloured or else $\{1, 4, 7\}$ will form a $c_1$-coloured arithmetic progression. In this way, if 6 is $c_1$-coloured, we will get $\{4, 6, 8\}$ as a $c_1$-coloured arithmetic progression and if 6 is $c_2$-coloured, then we will get $\{5, 6, 7\}$ as a $c_2$-coloured arithmetic progression. Now, if 4 is $c_2$-coloured, then 6 must be $c_1$-coloured or else $\{2, 4, 6\}$ will form a $c_2$-coloured arithmetic progression. This will force 7 to be $c_2$-coloured or $\{6, 7, 8\}$ will form a $c_1$-coloured arithmetic progression. In this way, if 9 is $c_1$-coloured, then we will get $\{3, 6, 9\}$ as a $c_1$-coloured arithmetic progression. If 9 is $c_2$-coloured, then we will get $\{5, 7, 9\}$ as a $c_2$-coloured arithmetic progression. On the other hand, suppose that 3 be $c_2$-coloured. This will force 4 to be $c_1$-coloured or $\{2, 3, 4\}$ will form a $c_2$-coloured arithmetic progression. Since 1 and 4 are both $c_1$-coloured, 7 must be $c_2$-coloured. Then, 5 must be $c_1$-coloured, or else $\{3, 5, 7\}$ will form a $c_2$-coloured arithmetic progression. This will

force 6 to be $c_2$-coloured to avoid getting $c_1$-coloured arithmetic progression $\{4, 5, 6\}$. Since both 6 and 7 are $c_2$-coloured, we must colour 8 with $c_1$. In this way, if 9 is $c_1$-coloured, $\{1, 5, 9\}$ will form a $c_1$-coloured arithmetic progression. If 9 is $c_2$-coloured, $\{3, 6, 9\}$ will form a $c_2$-coloured arithmetic progression. Hence, in this case, we will get a monochromatic arithmetic progression of length 3 no matter how we colour the set $[9]$.

Thus, in both cases, we will get a monochromatic arithmetic progression of length 3. Hence, $W(3, 2) \leq 9$.

Now, we need to show that $W(3, 2) \geq 9$. Consider the colouring of $[8]$ in following way: 1,2,5,6 are $c_1$-coloured and 3,4,7,8 are $c_2$-coloured. In this way, we have no monochromatic arithmetic progression of length 3. Hence, $W(3, 2) \geq 9 > 8$.

Therefore, we have $W(3, 2) = 9$. $\square$

Here, we state some famous results on the upper bound of the Van der Waerden number by Gowers [41] with the proofs omitted.

**Theorem 4.11.** [41] *For $k \geq 2$, $W(k, 2) \leq 2^{2^{2^{2^{2^{k+9}}}}}$.*

**Theorem 4.12.** [41] *Let $f(k, r) = r^{2^{2^{k+9}}}$. Then $W(k, r) \leq 2^{2^{f(k,r)}}$.*

In [57], Huang and Yang had claimed that the following theorem holds. The interested reader is referred to [57] for the proof.

**Theorem 4.13.** [57] *Let $r > 5$. Then $W(3, r) < (\frac{r}{4})^{3^r}$.*

We now extend the concept of Van der Waerden Number $W(k, r)$ and define $W(k_1, \ldots, k_r; r)$ to be the least $n$ such that in each $r$-colouring of $[n]$, there is always a $c_i$-coloured arithmetic progression of length $k_i$, for some $1 \leq i \leq r$. Note that the existence of $W(k_1, \ldots, k_r; r)$ is guaranteed as $W(k_1, \ldots, k_r; r) \leq W(\max(k_1, \ldots, k_r), r)$ because a monochromatic arithmetic progression of length $\max(k_1, \ldots, k_r)$ contains an arithmetic progression of length $k_i$, for $1 \leq i \leq r$.

**Theorem 4.14.**

(1) *$W(1, k; 2) = k$.*
(2) *$W(2, k; 2) = 2k$, if $k$ is odd.*
(3) *$W(2, k; 2) = 2k - 1$, if $k$ is even.*

*Proof.*

(1) In any 2-colouring of $[k]$, we have that either all terms are $c_2$-coloured, forming an arithmetic progression of length $k$ with common difference of 1, or at least one $c_1$-coloured term, forming an arithmetic progression of length 1.
(2) Consider any 2-colouring of $[2k]$. If there is none or at least 2 $c_1$-coloured terms, then we are done. Suppose there is only one $c_1$-coloured term, say $a$. Now we partition the remaining $2k-1$ $c_2$-coloured terms into two classes: those less than $a$ and those more than $a$. By the Pigeonhole Principle, one of the partitions will have at least $k$ terms, forming a $c_2$-coloured arithmetic progression. Hence, $W(2, k; 2) \leq 2k$. Now, if we colour the $[2k - 1]$ in such a way that $k$ is $c_1$-coloured and the rest of the elements in $[2k - 1]$ are $c_2$-coloured, we will have neither a $c_1$-coloured arithmetic progression of length 2 nor a $c_2$-coloured

arithmetic progression of length $k$. Therefore, $W(2,k;2) \geq 2k > 2k-1$. Thus, we have $W(2,k;2) = 2k$.

(3) Consider any 2-colouring of $[2k-1]$. If there is none or at least 2 $c_1$-coloured terms, then we are done. Suppose that there is only one $c_1$-coloured term, say $a$. Now we partition the remaining $2k-2$ $c_2$-coloured terms into two classes, those less than $a$ and those more than $a$. If either one of the partitions have at least $k$ terms, then we will get a $c_2$-coloured arithmetic progression of length $k$. Or else, $a$ must be even and equal to $k$ and both of the partitions must contain $k-1$ terms. In this way, $1,3,5,\ldots,2k-1$ will form a $c_2$-coloured arithmetic progression of length $k$. Hence, $W(2,k;2) \leq 2k$. Now, if we colour the $[2k-2]$ in such a way that $k$ is $c_1$-coloured and the rest are $c_2$-coloured, then we will have neither a $c_1$-coloured arithmetic progression of length 2 nor a $c_2$-coloured arithmetic progression of length $k$. Therefore, $W(2,k;2) \geq 2k-1 > 2k-2$. Thus, we have $W(2,k;2) = 2k-1$.

□

**Theorem 4.15.**

$$W(k_1,k_2;2) \leq k_2 W(k_1,k_1,\ldots,k_1;2^{k_2}) = k_2 W(k_1,2^{k_2}) \leq k_2 2^{2^{2^{k_2}{2^{2^{k_1+9}}}}}.$$

*Proof.* Consider any 2-colouring of $[k_2 W(k_1,2^{k_2})]$. Now partition these $k_2 W(k_1,2^{k_2})$ terms into $W(k_1,2^{k_2})$ blocks $B_1,B_2,\ldots,B_{W(k_1,2^{k_2})}$ where

$$B_i = \{(i-1)k_2+1,(i-1)k_2+2,\ldots,(i-1)k_2+k_2\}.$$

By the definition of $W(k_1,2^{k_2})$, we have that $B_j,B_{j+d},B_{j+2d},\ldots,B_{j+(k_1-1)d}$ have the same colour. If any term in the block $B_j$ is $c_1$-coloured, say $a$, then $a+k_2d \in B_{j+d}, a+2k_2d \in B_{j+2d},\ldots,a+(k_1-1)k_2d \in B_{j+(k_1-1)d}$ will also be $c_1$-coloured and these will form a $c_1$-coloured arithmetic progression of length $k_1$. Or else, all the $k_2$ terms in the block $B_j$ are $c_2$-coloured and these will form a $c_2$-coloured arithmetic progression of length $k_2$. Thus, we have $W(k_1,k_2;2) \leq k_2 W(k_1,k_1,\ldots,k_1;2^{k_2}) = k_2 W(k_1,2^{k_2})$. By Theorem 4.12, we have $W(k_1,2^{k_2}) \leq 2^{2^{2^{k_2}{2^{2^{k_1+9}}}}}$. Therefore, $W(k_1,k_2;2) \leq k_2 2^{2^{2^{k_2}{2^{2^{k_1+9}}}}}$. □

**Theorem 4.16.** *Let $k \geq 3$. Then $W(3,k;2) < k(2^{3^{2^k}(k-2)})$.*

*Proof.* By Theorem 4.15, we have $W(3,k;2) \leq kW(3,2^k)$. Now for $k \geq 3$, we have $2^k > 5$. Hence, by Theorem 4.13, we have $W(3,2^k) < (\frac{2^k}{4})^{3^{2^k}} = 2^{3^{2^k}(k-2)}$. Thus, we have $W(3,k;2) \leq kW(3,2^k) < k(2^{3^{2^k}(k-2)})$. □

To end this chapter, we tabulate some known exact Van der Waerden numbers in the following tables, after compiling from various reference as cited respectively.

| $(k,r)$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 |
| 2 | 2 | 3 | 4 | 5 |
| 3 | 3 | 9 | 27[14] | 76[5] |
| 4 | 4 | 35[14] | 293[66] | |
| 5 | 5 | 178[96] | | |
| 6 | 6 | 1132[67] | | |

Table 4.1: Van der Waerden number $W(k,r)$.

| $(k_1,k_2)$ | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|
| 3 | 9 | 18[14] | 22[14] | 32[14] | 46[14] |
| 4 | 18 | 35 | 55[14] | 73[5] | 109[4] |
| 5 | 22 | 55 | 178 | 206[65] | 260[1] |
| 6 | 32 | 73 | 206 | 1132 | |
| 7 | 46 | 109 | 260 | | |

Table 4.2: Van der Waerden number $W(k_1,k_2,2)$.

# Chapter 5

# Schur's Theorem

In this chapter, we introduce Schur's Theorem, one of the Ramsey-type results concerning equations. In Section 5.1, we present and prove Schur's Theorem. Next, in Section 5.2, we look at some results on Schur numbers. In Section 5.3, we discuss generalisations of Schur's Theorem, specifically looking at Rado's Theorem and Folkman's Theorem.

## 5.1 Schur's Theorem

In this section, we present and prove Schur's Theorem. Schur's Theorem was given and proven by Issai Schur in his publication in 1916 [92].

**Theorem 5.1** (Schur's Theorem)**.** [92] *Let $N, r \in \mathbb{N}$. If $[N]$ is $r$-coloured, then there is some same-coloured $a, b, c \in [N]$, such that $a + b = c$, where $a$ and $b$ are not necessarily distinct. The least of such $N$ is called the* Schur number *and is denoted by $S(r)$; furthermore, such $a$, $b$ and $c$ are called a* Schur triple*.*

*Proof.* By Ramsey's Theorem (Theorem 2.7), there exists $N+1 = R(k_1, k_2, \ldots, k_r) = R(3; r)$, where $k_1 = k_2 = \cdots = k_r = 3$ such that for any $r$-colouring $K_{N+1}$, there exists a monochromatic subgraph $K_3$. Now, consider any $r$-colouring of $[N]$ and let $K_{N+1}$ be a complete graph with $N + 1$ vertices. Label each vertex of $K_N$ from 1 to $N + 1$. Colour each edge with the colour corresponding to the positive difference of the connecting vertices in the $r$-colouring of $[N]$. For instance, colour the edge connecting the vertices labelled 2 and 3 with colour 1 in the $r$-colouring of $[N]$. By the definition of $N + 1$, there is a monochromatic triangle in $K_{N+1}$, with three labelled vertices, say $i$, $j$ and $k$, for $i < j < k$. Since the edges $\{i, j\}$, $\{j, k\}$ and $\{i, k\}$ are of the same colour, $j - i$, $k - j$, and $k - i$ are of the same colour in the $r$-colouring of $[N]$. Let $a = j - i$, $b = k - j$ and $c = k - i$, and note that $a$, $b$ and $c$ are same-coloured and that $a + b = (j - i) + (k - j) = k - i = c$. Then, we have proven that the theorem is valid. $\square$

**Example 5.2.** Let $r = 2$. Consider a 2-colouring of [5] in the following way: 1, 4 and 5 are $c_1$-coloured and 2 and 3 are $c_2$-coloured. Note that 1, 4 and $1 + 4 = 5$ are all $c_1$-coloured. In fact, $S(2) = 5$; a detailed proof will be given in Section 5.2.

## 5.2 Schur's Numbers

In this section, we present some results on Schur numbers, the first of which is, as far as we can tell, new. We also independently prove that $S(1) = 2$, $S(2) = 5$ and $S(3) = 14$ (see Theorems 5.4 and 5.7). These results are without doubt not new; however, we could not find neither proofs nor original references to them in the literature.

**Theorem 5.3.** *For $r \geq 1$, $S(r) \leq R(3;r) - 1 \leq r!(\frac{e-e^{-1}+3}{2}) \approx 2.68r!$.*

*Proof.* It follows from the proof of Theorem 5.1 that there is a monochromatic Schur triple in the $r$-colouring of $[N]$, where $N+1 = R(3;r)$. Hence, $S(r) \leq R(3;r) - 1$. Now, by Theorem 2.24, we have $S(r) \leq R(3;r) - 1 \leq r!(\frac{e-e^{-1}+3}{2}) \approx 2.68r!$. □

**Theorem 5.4.**

(1) $S(1) = 2$
(2) $S(2) = 5$.

*Proof.*

(1) It is clear that there is no monochromatic Schur triple in any colouring of [1], since there is only one element in [1] and $1+1 \neq 1$. Therefore, $S(1) \geq 2 > 1$. Note that in any single-coloured [2], we can find a monochromatic Schur triple, $a+b=c$ in the colouring where $a=b=1$ and $c=2$. Hence, $S(1) \leq 2$. Thus, we have $S(1) = 2$.
(2) Consider a colouring of [4] in such a way that 1 and 4 are $c_1$-coloured and 2 and 3 are $c_2$-coloured. Note that there is no monochromatic Schur triple in this colouring. Therefore, $S(2) \geq 5 > 4$. Now, by Theorem 5.3, $S(2) \leq R(3;2) - 1 = 6 - 1 = 5$. Thus, we have $S(2) = 5$.

□

Now, we will look at some lower bounds on Schur numbers.

**Theorem 5.5.** [92] $S(r+1) \geq 3S(r) - 1 > 3S(r) - 2$.

*Proof.* The main idea of this proof is given in [92]; here, we have added some details.

Let $S(r) = n$. Then there is a $r$-colouring, say $\chi$, of $[n-1]$ that does not contain the monochromatic Schur triple. Now, let $\chi'$ be a $(r+1)$-colouring of $[3n-2]$ in such a way that

$$\chi'(x) = \begin{cases} \chi(x) & \text{for } x \in [1, n-1]\,; \\ c_{r+1} & \text{for } x \in [n, 2n-1]\,; \\ \chi(x-(2n-1)) & \text{otherwise}\,. \end{cases}$$

We claim that there is no monochromatic Schur triple in the $\chi'$-colouring of $[3n-2]$. Suppose to the contrary that there is one, say $a, b, c$, with $a \leq b < c$. Consider the colour $c_{r+1}$. Since $c = a+b \geq 2a \geq 2n \notin [n, 2n-1]$, the Schur triple cannot be $c_{r+1}$-coloured. Now, consider the other colours. Since there is no monochromatic triple in $[1, n-1]$ as $\chi' = \chi$ for $[1, n-1]$ and $[n-1+1, 2(n-1)] = [n, 2n-2]$ is $c_{r+1}$-coloured, we see that $a$ and $b$ of the Schur triple cannot both be from the interval $[1, n-1]$. Similarly, it is also impossible for both $a$ and $b$ to be from the interval $[2n, 3n-2]$ because if that is the case, then $c = a+b \geq 2a \geq 2(2n) > 3n-2 \notin [2n, 3n-2]$. Hence, we must have $a \in [1, n-1]$, $b \in [2n, 3n-2]$ and $a+b = c \in [2n, 3n-2]$ with $\chi'(a) = \chi'(b) = \chi'(c) = \chi(a)$. Now, let $b' = b - (2n-1) \in [1, n-1]$ and $c' = c - (2n-1) \in [1, n-1]$. Note that $\chi'(b) = \chi(b') = \chi(a), \chi'(c) = \chi(c') = \chi(a)$ and $a + b' = a + b - (2n-1) = c - (2n-1) = c'$: we have a monochromatic Schur triple in $\chi$, a contradiction. Hence, there is no monochromatic Schur triple in $\chi'$. Thus, $S(r+1) > 3n-2 = 3S(r) - 2$. Therefore, we have $S(r+1) \geq 3S(r) - 1$. □

**Theorem 5.6.** [92] *For $r \geq 1$, $S(r) \geq \frac{3^r+1}{2}$.*

*Proof.* We prove the theorem by induction on $r$. First, note that, by Theorem 5.4, $S(1) = 2 \geq \frac{3^1+1}{2}$. Now, assume that $S(r) \geq \frac{3^r+1}{2}$. We need to show that $S(r+1) \geq \frac{3^{r+1}+1}{2}$. By Theorem 5.5, $S(r+1) \geq 3S(r) - 1 \geq 3(\frac{3^r+1}{2}) - 1 = \frac{3^{r+1}+1}{2}$. Hence by induction, we have $S(r) \geq \frac{3^r+1}{2}$ for $r \geq 1$. $\square$

**Theorem 5.7.** $S(3) = 14$.

*Proof.* By Theorem 5.6, $S(3) \geq \frac{3^3+1}{2} = 14$. Now, we need to show that $S(3) \leq 14$. Suppose to the contrary that there is a 3-colouring of [14] that does not contain any monochromatic Schur triple. Without loss of generality, we assume that 1 is $c_1$-coloured. Since $1 + 1 = 2$, 2 cannot be $c_1$-coloured. Again, without loss of generality, we let 2 be $c_2$-coloured. Now, there are several cases to be considered.

**Case A: 3 is $c_1$-coloured.**

Note that in this case, 4 must be $c_3$-coloured. Now consider the colour of 5.

Case A1: 5 is $c_1$-coloured.

Then, 8 must be $c_2$-coloured. This will force 6 to be $c_3$-coloured. Now, note that in this case, no matter how 10 be coloured, we will get a monochromatic Schur triple, hence a contradiction.

Case A2: 5 is $c_2$-coloured.

Note that in this case, 6 can only be either $c_2$-coloured or $c_3$-coloured. We first consider the case that 6 is $c_2$-coloured. In this case, 8 must be $c_1$-coloured. This will force 7 to be $c_3$-coloured. Now, no matter how 11 be coloured, we will obtain a monochromatic Schur triple, thus a contradiction. On the other hand, if 6 is $c_3$-coloured, 10 must be $c_1$-coloured. To avoid having monochromatic Schur triple, 7 must be $c_3$-coloured, and hence 11 must be $c_2$-coloured. Now, note that no matter how 13 is coloured, we will get a monochromatic Schur triple, a contradiction.

Case A3: 5 is $c_3$-coloured.

Similar to Case A2, 6 can only be either $c_2$-coloured or $c_3$-coloured. Suppose that 6 is $c_2$-coloured. 8 must be $c_1$-coloured. Then, 9 must be $c_2$-coloured and this will force 11 to be $c_3$-coloured. Now, consider the colour of 7. We will get a monochromatic Schur triple no matter what colour is used, hence a contradiction. Now, suppose that 6 is $c_3$-coloured. Since 8 cannot be $c_3$-coloured, there are two subcases to be considered. First, 8 is $c_1$-coloured. This will force 11 to be $c_2$-coloured. Then, no matter how 9 be coloured, we will get a monochromatic Schur triple, hence a contradiction. Now, if 8 is $c_2$-coloured, then 10 need to be $c_1$-coloured. This will cause 11 to be $c_2$-coloured, and hence 13 must be $c_3$-coloured. In this case, any colouring of 9 will grant us a monochromatic Schur triple, thus a contradiction.

**Case B: 3 is $c_2$-coloured.**

In this case, 4 cannot be $c_2$-coloured. We consider the colour of 4.

Case B1: 4 is $c_1$-coloured.

In this case, 5 must be $c_3$-coloured. Since 6 cannot be $c_2$-coloured, there are two subcases to be considered. Suppose that 6 is $c_1$-coloured. Then, 10 must be $c_2$-coloured. This will force 7 to be $c_3$-coloured. In this case, no matter how we colour 12, we will get a monochromatic Schur triple, hence a contradiction. Now, suppose that 6 is $c_3$-coloured. Note that 8 cannot be $c_1$-coloured. If 8 is $c_2$-coloured, then both 10 and 11 cannot be coloured with $c_2$ and $c_3$, but this will yield a $c_1$-coloured Schur triple, thus a contradiction. On the other hand, suppose that 8 is $c_3$-coloured. In this subcase, 10, 11, 12, 13 and 14 cannot be $c_3$-coloured. If 12 is $c_1$-coloured, then 11 and 13 must be $c_2$-coloured and we will get a $c_2$-coloured Schur triple. If 12 is $c_2$-coloured, then 10 and 14 must be $c_1$-coloured. This will give us a $c_1$-coloured Schur triple, thus a contradiction.

Case B2: 4 is $c_3$-coloured.

In this case, 5 cannot be $c_2$-coloured. There are two subcases to be considered for the colouring of 5. Suppose that 5 is $c_1$-coloured. Then, 6 must be $c_3$-coloured. This forces 10 to be $c_2$-coloured; hence, 8 must be $c_1$-coloured. Then, 13 must be $c_3$-coloured. Now, any colouring of 7 will give us a monochromatic Schur triple, hence leads us to a contradiction. Now, suppose 5 is $c_3$-coloured. Note that both 8 and 10 cannot be $c_3$-coloured. Since $2+8=10$, 8 and 10 cannot be both $c_2$-coloured and hence one of them must be $c_1$-coloured. Then, 9 must be $c_2$-coloured. Consider the colouring of 6. Since 3 is $c_2$-coloured, 6 cannot be $c_2$-coloured. If 6 is $c_1$-coloured, then 7 must be $c_3$-coloured. Then, no matter how 12 be coloured, we will obtain a monochromatic Schur triple, thus gives us a contradiction. If 6 is $c_3$-coloured, then 11 must be $c_1$-coloured and hence 10 must be $c_2$-coloured. Then, no matter how we colour 12, we will have a monochromatic Schur triple, hence a contradiction.

**Case C: 3 is $c_3$-coloured.**

Consider the colour of 4; it can only be $c_1$ or $c_3$.

Case C1: 4 is $c_1$-coloured.

In this case, 8 cannot be $c_1$-coloured. Therefore, there are two subcases to be considered on the colouring of 8. First, suppose 8 is $c_2$-coloured. Then, 6 must be $c_1$-coloured and hence 10 must be $c_3$-coloured. This will force 5 to be $c_2$-coloured and 13 to be $c_1$-coloured. Now, no matter how we colour 7, we will get a monochromatic Schur triple, hence leads us to a contradiction. Next, suppose 8 is $c_3$-coloured. 5 must be $c_2$-coloured. Note that 11 cannot be $c_3$-coloured. If 11 is $c_1$-coloured, then 7 must be $c_3$-coloured. Whichever colour we use to colour 10, we will obtain a monochromatic Schur triple, thus a contradiction. If 11 is $c_3$-coloured, then 6 must be $c_1$-coloured and 10 must be $c_3$-coloured. Then, any colouring of 7 will give us a monochromatic Schur triple, hence again a contradiction.

Case C2: 4 is $c_3$-coloured.

In this case, 6, 7 and 8 cannot be $c_3$-coloured. If 7 is $c_1$-coloured, then 6 and 8 must be $c_2$-coloured. However, 2, 6 and 8 would form a $c_2$-coloured Schur triple. Hence, 7 must be $c_2$-coloured. Now, consider the colour of 6. Suppose that 6 is $c_1$-coloured; then 5 must be $c_3$-coloured and so 9 must be $c_1$-coloured. This forces both 8 and 10 to be $c_2$-coloured but 2, 8 and 10 then form a $c_2$-coloured Schur triple, a contradiction. On the other hand, suppose that 6 is $c_2$-coloured. Then 8 must be $c_1$-coloured and 9 must be $c_3$-coloured. This causes 5 to be $c_1$-coloured. However we colour 13, we will have a monochromatic Schur triple, a contradiction.

All possible cases lead us to a contradiction. Therefore, in any colouring of [14], we will have a monochromatic Schur triple. Hence, $S(3) \leq 14$.

Thus, we have $S(3) = 14$. $\square$

To end this section, we list some known Schur numbers in the table below.

| $r$ | S(r) |
|---|---|
| 1 | 2 |
| 2 | 5 |
| 3 | 14 |
| 4 | 45[40] |

Table 5.1: Known Schur numbers $S(r)$

## 5.3 Generalisations of Schur's Theorem

In this section, we present some generalisations of Schur's Theorem. Previously, we have discussed the existence of monochromatic Schur triples $a + b = c$ in colourings of $[N]$ where $N \in \mathbb{N}$. Now, we will look into the case with four terms, which is $a + b + c = d$ (Theorem 5.8), and then the case with $k$ terms, namely $x_1 + x_2 + \cdots + x_{k-1} = x_k$ (Theorem 5.9). We conceived of these results by looking at the proof of Schur's Theorem and wondering if Schur's Theorem could be generalised with respect to more terms, and it was possible to prove that it indeed was. Later, however, we found that these results already existed in the literature; see [69].

Similarly, we independently discovered and proved a bound on $k$-term Schur numbers $S(k; r)$ (Theorem 5.10), proved that $S(4; 2) = 11$ (Theorem 5.11) and proved an exact expression for $S(k; 2)$ (Theorem 5.12). These and more general results were also later found to already exist in the literature; see [69].

**Theorem 5.8.** *Let $N, r \in \mathbb{N}$. If $[N]$ is $r$-coloured, then there are same-coloured $a, b, c, d \in [N]$ such that $a + b + c = d$. We denote the least of such $N$ by $S(4; r)$.*

*Proof.* By Ramsey's Theorem (Theorem 2.7), there exists $N+1 = R(k_1, k_2, \ldots, k_r) = R(4; r)$, where $k_1 = k_2 = \cdots = k_r = 4$ such that there is a monochromatic $K_4$ in any $r$-colouring of $K_{N+1}$, . Now, consider any $r$-colouring of $[N]$. Let $K_{N+1}$ be a complete graph with $N + 1$ vertices. Label each of the vertices of $K_N$ from 1 to $N + 1$. Colour each edge with the colour corresponding to the positive difference of the end vertices in the $r$-colouring of $[N]$. By the definition of $N + 1$, there is a monochromatic $K_4$ in $K_{N+1}$, with four labelled vertices, say $i, j, k$ and $l$, for $i < j < k < l$. Since the edges $\{i, j\}$, $\{j, k\}$, $\{k, l\}$ and $\{i, l\}$ are of the same colour, it follows that $j - i$, $k - j$, $l - k$ and $l - i$ are of the same colour in the $r$-colouring of $[N]$. Let $a = j - i$, $b = k - j$, $c = l - k$ and $d = l - i$, and note that $a$, $b$, $c$ and $d$ are of the same colour and that $a + b + c = (j - i) + (k - j) + (l - k) = l - i = d$. Then, we have proven that the theorem is valid. □

**Theorem 5.9.** *Let $N, r \in \mathbb{N}$ and $k \geq 3 \in \mathbb{N}$. If $[N]$ is $r$-coloured, then there are some same-coloured $x_1, x_2, \ldots, x_k \in [N]$ such that $x_1 + x_2 + \cdots + x_{k-1} = x_k$. We denote the least of such $N$ by $S(k; r)$ where $S(3; r)$ is the Schur number $S(r)$, as defined in Theorem 5.1.*

*Proof.* By Ramsey's Theorem (Theorem 2.7), there exists $N+1 = R(k_1, k_2, \ldots, k_r) = R(k; r)$, where $k_1 = k_2 = \cdots = k_r = k$ such that for any $r$-colouring $K_{N+1}$, there exists a monochromatic $K_k$, where $k \geq 3 \in \mathbb{N}$. Now, consider any $r$-colouring of $[N]$. Let $K_{N+1}$ be a complete graph with $N+1$ vertices and label these vertices from 1 to $N+1$. Colour each edge with the colour corresponding to the positive difference of the end vertices in the $r$-colouring of $[N]$. By the definition of $N+1$, there is a monochromatic $K_k$ in $K_{N+1}$, with $k$ labelled vertices, say $v_1, v_2, \ldots, v_k$, where $v_1 < v_2 < \ldots < v_k$. Since the edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}$ and $\{v_1, v_k\}$ are of the same colour, it follows that $v_2 - v_1, v_3 - v_2, \ldots, v_k - v_{k-1}$ and $v_k - v_1$ are of the same colour in the $r$-colouring of $[N]$. Let $x_1 = v_2 - v_1, x_2 = v_3 - v_2, \ldots, x_{k-1} = v_k - v_{k-1}$ and $x_k = v_k - v_1$, and note that $x_1, x_2, \ldots, x_k$ are same-coloured and that we have $x_1 + x_2 + \cdots + x_{k-1} = (v_2 - v_1) + (v_3 - v_2) + \cdots + (v_k - v_{k-1}) = v_k - v_1 = x_k$. Thus, the theorem is valid. □

**Theorem 5.10.** *For $r \geq 1$ and $k \geq 3 \in \mathbb{N}$, $S(k; r) \leq R(k; r) - 1$.*

*Proof.* It follows from the proof of Theorem 5.9 that there are monochromatic $x_1, x_2, \ldots, x_k \in [N]$, where $x_1 + x_2 + \cdots + x_{k-1} = x_k$ in the $r$-colouring of $[N]$, in which $N + 1 = R(k; r)$. Hence, $S(k; r) \leq R(k; r) - 1$. □

**Theorem 5.11.** $S(4; 2) = 11$.

*Proof.* First, we need to show that $S(4; 2) \geq 11 > 10$. Consider the following colouring of [10]. Colour 1, 2, 9 and 10 with colour $c_1$ and 3, 4, 5, 6, 7 and 8 with colour $c_2$. In this colouring, we have no monochromatic set of four terms that satisfy $a + b + c = d$. Hence, $S(4; 2) \geq 11 > 10$. Next, we have to show that $S(4; 2) \leq 11$. Suppose to the contrary that there is no monochromatic set $\{a, b, c, d\}$ in the colouring of [11] with 2 colours. Without loss of generality, we assume that 1 is $c_1$-coloured. Since $1 + 1 + 1 = 3$, 3 must be $c_2$-coloured. Then, 9 must be $c_1$-coloured as $3 + 3 + 3 = 9$. This will force 7 and 11 to be $c_2$-coloured because $1+1+7 = 9$ and $1+1+9 = 11$. Then, 5 must be $c_1$-coloured because $3+3+5 = 11$. Now, no matter how 2 is coloured, we will have either $c_1$-coloured $1 + 2 + 2 = 5$ or $c_2$-coloured $7 + 2 + 2 = 11$. Hence, $S(4; 2) \leq 11$. Thus, we have $S(4; 2) = 11$. □

**Theorem 5.12.** *For $k \geq 3$, $S(k; 2) = (k - 1)^2 + (k - 2)$.*

*Proof.* First, we need to show that $S(k; 2) \geq (k - 1)^2 + (k - 2)$. Let $N = (k - 1)^2 + (k - 2) - 1 = (k - 1)^2 + (k - 3)$. Let $\chi$ be the colouring of $[N]$ with two colours in the following way:

$$\chi(x) = \begin{cases} c_1, & \text{for } x \in [k-1, (k-1)^2 - 1]; \\ c_2, & \text{otherwise.} \end{cases}$$

Consider all $c_1$-coloured elements. Note that the smallest $c_1$-coloured element is $k-1$. Hence, the smallest possible sum of $k$ $c_1$-coloured numbers is $(k-1)+\cdots+(k-1) = (k-1)(k-1) = (k-1)^2$ which is strictly greater than $(k-1)^2-1$ which is the largest $c_1$-coloured element. Thus, it is impossible to have monochromatically $c_1$-coloured $x_1, x_2, \ldots, x_k \in [N]$, where $x_1 + x_2 + \cdots + x_{k-1} = x_k$. Now, consider all $c_2$-coloured elements. Divide them into two partitions, $[1, k-2]$ and $[(k-1)^2, (k-1)^2+(k-3)]$. For the numbers in the first partition, all the possible sums are from $1+1+\cdots+1 = k-1$ to $(k-2)+\cdots+(k-2) = (k-1)(k-2)$, which are all $c_1$-coloured. Therefore, there is no monochromatic solution for $x_1+x_2+\cdots+x_{k-1} = x_k$ within the elements from the first partition. Similarly, in the second partition, the minimum possible sum is $(k-1)^2+\cdots+(k-1)^2 = (k-1)^3 > (k-1)^2+(k-3) = N$. Therefore, there is no monochromatic solution for $x_1+x_2+\cdots+x_{k-1} = x_k$ within the elements from the second partition. Now, consider summations involving both partitions. The minimum possible sum is

$$1+\cdots+1+(k-1)^2 = (k-1)^2+(k-2) > (k-1)^2+(k-3) = N\,.$$

Thus, there is impossible to have a monochromatically $c_2$-coloured solution to $x_1+x_2+\cdots+x_{k-1} = x_k$. Hence, there is no monochromatic solution to $x_1+x_2+\cdots+x_{k-1} = x_k$ in this 2-colouring of $[N]$. Therefore, we get $S(k;2) \geq (k-1)^2+(k-2)$.

Next, we want to show that $S(k;2) \leq (k-1)^2+(k-2)$. Let $M = (k-1)^2+(k-2)$. Suppose to the contrary that there is no monochromatic solution to $x_1+x_2+\cdots+x_{k-1} = x_k$ in some 2-colouring of $[M]$. Without loss of generality, we assume that 1 is $c_1$-coloured. Since $1+1+\cdots+1 = k-1$, $k-1$ must be $c_2$-coloured. Then $(k-1)^2$ must be $c_1$-coloured as $(k-1)+(k-1)+\cdots+(k-1) = (k-1)(k-1) = (k-1)^2$. Since $1+1+\cdots+1+(k-1)^2-(k-2) = (k-1)^2$ and $1+1+\cdots+1+(k-1)^2 = (k-1)^2+(k-2)$, both $(k-1)^2-(k-2)$ and $(k-1)^2+(k-2)$ must be $c_2$-coloured. This will force $(k-2)+(k-1)$ to be $c_1$-coloured because $(k-1)+(k-1)+\cdots+(k-1)+[(k-2)+(k-1)] = (k-1)^2+k-2$. Now, consider the colouring of $k-2$. If $k-2$ is $c_1$-coloured, then we have a monochromatic solution: $(k-2)+(k-2)+\cdots+(k-2)+[(k-2)+(k-1)] = (k-1)^2$. If $k-2$ is $c_2$-coloured, then we also have a monochromatic solution: $(k-2)+(k-2)+\cdots+(k-2)+(k-1) = (k-1)^2-(k-2)$. Either way, we have a contradiction. Hence, in any 2-colouring of $[(k-1)^2+(k-2)]$, there is a monochromatic solution to the equation $x_1+x_2+\cdots+x_{k-1} = x_k$. Therefore, $S(k;2) \leq (k-1)^2+(k-2)$. Thus, we have $S(k;2) = (k-1)^2+(k-2)$. $\square$

Next, we look at the Rado's Theorem which was proved by a student of Schur's, Richard Rado, in the 1933 paper [81]. While Schur's Theorem concerns the equation $a+b-c = 0$, Rado's Theorem addresses the equation $a_1x_1+\cdots+a_kx_k = 0$.

**Theorem 5.13** (Rado's Theorem)**.** [81] *Let $k \geq 2$ and $c_i \in \mathbb{Z}$ for $1 \leq i \leq k$. Then, for any finite colouring of $\mathbb{N}$, $c_1x_1+\cdots+c_kx_k = 0$ has a monochromatic solution $x_1, \ldots, x_k \in \mathbb{N}$ if and only*

$$\sum_{i\in I} a_i = 0 \qquad \textit{for some nonempty subset } I \subseteq [k].$$

*Proof.* The following proof follows the general outline of the proof in [63] but we have adapted it so as to provide better clarity of argument and notation.

We first prove that if there is any finite colouring of $\mathbb{N}$ such that $a_1x_1+\cdots+a_kx_k = 0$ has a monochromatic solution $x_1, \ldots, x_k \in \mathbb{N}$, then there exists a nonempty subset $I \subseteq [k]$ such that

$$\sum_{i\in I} a_i = 0\,.$$

We prove the contrapositive. Assume that there exist $a_1, \ldots, a_k$ such that no nonempty subset $I \subseteq [k]$ satisfies $\sum_{i\in I} a_i = 0$. We need to show that for some $r$, there is an $r$-colouring of $\mathbb{N}$ without monochromatic solutions to $a_1x_1+\cdots+a_kx_k = 0$. First, we choose a prime number $p$ that does not divide $\sum_{j\in J} a_j$ for any $J \subseteq [k]$. Since there are only finitely many choices for $J$, we can always do so. Now, for $n \in \mathbb{N}$, let $s$ be the largest integer such that $p^s|n$, so $n = p^s m$ where $m \not\equiv 0 \pmod p$. We define $\chi$ as a $(p-1)$-colouring of $\mathbb{N}$ in such a way that $\chi(n) = m \pmod p$ with the colours $c_1, \ldots, c_{p-1}$. We wish to show that in $\chi$, there is no monochromatic solution to $a_1x_1 + \cdots + a_kx_k = 0$. Suppose to the contrary that we have one. Let $\{y_1, \ldots, y_k\}$ be a monochromatic solution under $\chi$ with the colour $c_b$. Then, we have $1 \leq b \leq p-1$, and since $y_i \in \mathbb{N}$, for each $y_i$, there are numbers $s_i$ and $k_i$ such that $y_i = p^{s_i}(pk_i + b)$. Let $s = \min\{s_1, \ldots, s_k\}$. We have

$$0 = \sum_{i=1}^{k} a_i y_i = \sum_{i=1}^{k} a_i p^{s_i}(pk_i + b) = p^s \sum_{i=1}^{k} a_i p^{s_i - s}(pk_i + b)\,.$$

Note that $s = s_i$ and $s - s_i = 0$, for some $i$. Hence, modulo $p$, we will get

$$0 \equiv b \sum_{i=1}^{k} p^{s_i - s} a_i \pmod p\,.$$

Since $p$ is prime and $p$ does not divide $b$, we have that $p$ divides $\sum_{i\in\{1,k\};s_i=s} a_i$. This gives a contradiction since $p$ is chosen such that $p$ does not divide $\sum_{j\in J} a_j$. Hence, the result holds.

Now we need to show that if there is a nonempty subset $I \subseteq [k]$ such that

$$\sum_{i\in I} a_i = 0\,,$$

then, for any $r$-colouring of $\mathbb{N}$, $a_1x_1 + \cdots + a_kx_k = 0$ has a monochromatic solution $x_1, \ldots, x_k \in \mathbb{N}$. Suppose that $\sum_{i\in I} a_i = 0$. If $I = [k]$, then we have

$$\sum_{i=1}^{k} a_i = 0\,.$$

Then, choosing $x_1$ to be any integer and setting $x_i = x_1$ for all $2 \leq i \leq k$, we will have a monochromatic solution for $a_1x_1 + \cdots + a_kx_k = 0$. Now, suppose that $I \subset [k]$ and assume without the loss of generality that $a_1 > 0$ and $I = [m]$ where $m < k$. Let $s = a_{m+1} + a_{m+2} + \cdots + a_k$. We take $x_2 = x_3 = \cdots = x_m$ and $x_{m+1} = x_{m+2} = \cdots = x_k$. Then, the equation $a_1x_1 + \cdots + a_kx_k = 0$ will become

$a_1x_1+x_2(a_2+a_3+\cdots+a_m)+x_{m+1}(a_{m+1}+a_{m+2})+\cdots+a_k=0$. Since $a_1+\cdots+a_m=\sum_{i\in I}a_i=0$ and $s=a_{m+1}+a_{m+2}+\cdots+a_k$, we have $a_1(x_1-x_2)+sx_{m+1}=0$.

Now, we use induction on $r$. Suppose that $r=1$. Then we can choose $x_1$ and $x_2$ so that $x_2-x_1=s$ and $x_{m+1}=a_1$. Suppose that $r\geq 2$ and assume that the result holds for $r-1$. Let $b=\sum_{i=1}^k|a_i|$. Now, consider $\chi$ to be any $r$-colouring of $[W(n+1,r)b]$ where $W(n+1,r)$ is the Van der Waerden number defined in Theorem 4.3 and $n$ is the least positive integer such that in any $(r-1)$-colouring of $[n]$, there is a monochromatic solution for $a_1(x_1-x_2)+sx_{m+1}=0$. Such $n$ exists from the induction hypothesis. We want to show that, in $\chi$, we will get a monochromatic solution for $a_1(x_1-x_2)+sx_{m+1}=0$. Since $0\neq s=a_{m+1}+a_{m+2}+\cdots+a_k$ and $b=\sum_{i=1}^k|a_i|$, we have $1\leq|s|<b$. For $1\leq l\leq b$, we define $\chi_l$ to be the colouring of [W(n+1,r)] so that $\chi_l(i)=\chi(li)$. Then for each $l$, we will have a $\chi_l$-monochromatic set $\{a,a+d,\dots,a+nd\}\subseteq[W(n+1,r)]$. Hence, under the colouring of $\chi$, we have a monochromatic set $\{la,la+ld,\dots,la+lnd\}\subseteq[W(n+1,r)l]\subseteq[W(n+1,r)b]$.

Now, we let $l=|s|$ and $a'=la$. Then we will have a monochromatic set $\{a',a'+|s|d,\dots,a'+n|s|d\}\subseteq[W(n+1,r)b]$ for some $d\geq 1$ under the colouring of $\chi$. Consider this subset $\{a_1d,2a_1d,\dots,na_1d\}\subseteq[W(n+1,r)b]\subset\mathbb{N}$ under the colouring of $\chi$.

If $\chi(ja_1d)=\chi(a')$ for some $j\in[1,n]$, then consider the following cases. First, if $s<0$, then we can take $x_2=a'$, $x_1=a'+jd|s|$ and $x_{m+1}=ja_1d$: we then get $a_1(x_1-x_2)+sx_{m+1}=a_1(a'+jd|s|-a')+sja_1d=0$. If $s\geq 0$, then we take $x_2=a'+jd|s|$, $x_1=a'$ and $x_{m+1}=ja_1d$: we thereby get $a_1(x_1-x_2)+sx_{m+1}=a_1(a'-a'jd|s|)+sja_1d=0$. In both cases, we have a monochromatic solution to $a_1(x_1-x_2)+sx_{m+1}=0$.

On the other hand, if $\chi(ja_1d)\neq\chi(a')$ for all $j\in[1,n]$, then the elements of the set $\{a_1d,2a_1d,\dots,na_1d\}\subseteq[W(n+1,r)b]\subset\mathbb{N}$ are coloured with $r-1$ colours. By the induction hypothesis, we have a monochromatic solution for $a_1(x_1-x_2)+sx_{m+1}=0$.

Thus, in all cases, we have a monochromatic solution for $a_1(x_1-x_2)+sx_{m+1}=0$. Hence, by the induction, the result holds.

Therefore, the theorem is valid. □

**Example 5.14.** Consider the equation

$$x_1+x_2-x_3=0\,.$$

In this equation, $a_1=1$, $a_2=1$ and $a_3=-1$. Note that $\{a_1,a_3\}\subset\{a_1,a_2,a_3\}$ and $a_1+a_3=0$. By Theorem 5.13, $x_1+x_2-x_3=0$ has monochromatic solution. This is affirmed by Schur's Theorem (Theorem 5.1) as $x_1+x_2-x_3=0$ can be rewritten as $x_1+x_2=x_3$.

We now turn out attention to others generalisation of Schur's Theorem, now involving sum sets and product sets. We first introduce these notions, as well as a useful lemma.

**Definition 5.15** (Sum set)**.** *For any set $S\subseteq\mathbb{N}$, the* sum set*, denoted by $\sum(S)$, is the set of all finite sums of the elements of $S$.*

**Definition 5.16** (Product set)**.** *For any set $S \subseteq \mathbb{N}$, a* product set*, denoted by $\prod(S)$, is the set of all finite products of the elements of $S$.*

**Example 5.17.** Let $S = \{1, 2, 5, 8\}$. Then

$$\sum(S) = \{1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 15, 16\}$$
$$\text{and} \quad \prod(S) = \{1, 2, 5, 8, 10, 16, 40, 80\}.$$

**Lemma 5.18.** *For all $k, r \geq 1$, there is an integer $N = N(k; r)$ such that for any $r$-colouring of $[N]$, there exists $x_1 < x_2 < \cdots < x_k \in [N]$ with $\sum_{i=1}^{k} x_i < N$ where*

$$S_t = \{\sum_{r \in R} x_r : R \subseteq [k], \max_{r \in R} r = t\}$$

*is monochromatic for $t = 1, 2, \ldots, k$.*

*Proof.* The following proof is provided in [69]. We prove it by induction on $k$. For $k = 1$, the result is immediate. Now, let $r$ be arbitrary and assume that $N(k; r)$ exists. We wish to show that $N(k + 1; r)$ exists and that $N(k + 1; r) \leq 2W(N(k; r) + 2, r)$, where $W(N(k; r)+2, r)$ is the Van der Waerden number defined in Theorem 4.3. Let $m = 2W(N(k; r) + 2, r)$. Consider an arbitrary $r$-colouring of $[m]$. By the definition of $W(N(k; r) + 2, r)$, there is a monochromatic arithmetic progression

$$A = \{a, a + d, \ldots, a + (N(k; r) + 1)d\} \subseteq [\frac{m}{2}, m].$$

Now, consider the set $D = \{d, 2d, \ldots, N(k; r)d\}$. By the induction hypothesis, there exist $x_1 < x_2 < \cdots < x_k \in D \subseteq [m]$ such that the associated sets $S_1, S_2, \ldots S_k$ are each monochromatic. Now, we wish to find an $x_{k+1}$ so that $S_{k+1}$ is also monochromatic. Take $x_{k+1} = a + d$. Note that $a > \frac{m}{2}$ and $a + N(k; r)d < m$; hence $N(k; r)d < \frac{m}{2}$. Therefore, we have $x_{k+1} = a + d > N(k; r)d \geq x_k$. For $S_{k+1}$, note that $S_{k+1} \subseteq (a + d) + D \subseteq A$. Hence, $S_{k+1}$ is monochromatic. Thus, $N(k + 1; r)$ exists.

By induction, the result holds. □

**Theorem 5.19** (Folkman's theorem)**.** [90] *If $r, k \in \mathbb{N}$ and $M \in \mathbb{N}$ is sufficiently large, then for any $r$-colouring of $\mathbb{N}$, there is a $k$-subset $S \subseteq [M]$ with monochromatic $\sum(S)$.*

*Proof.* The proof generally follows that of [69]. We will prove that $M = N((k - 1)r + 1; r)$ will satisfy the conditions of the theorem, where $N((k - 1)r + 1; r)$ is defined in Lemma 5.18. Let $x_1 < x_2 < \cdots < x_{(k-1)r+1}$ satisfy Lemma 5.18 and consider the associated sets $S_1, S_2, \ldots, S_{(k-1)r+1}$. By the Pigeonhole Principle, $k$ of them must be same-coloured, say $S_{i_1}, \ldots, S_{i_k}$. Let $S = \{i_1, \ldots, i_k\} \subseteq [M]$. Then, by Lemma 5.18, $\sum(S)$ is monochromatic. Hence, the theorem is valid. □

**Example 5.20.** Consider the case $k = 2$ in Folkman's Theorem (Theorem 5.19). We can take $M = S(r)$ where $S(r)$ is the Schur number as defined in Theorem 5.1. Then, for each $r$-colouring of $[M]$, there are integers $a$, $b$ and $c$ such that the set $\{a, b, c = a + b\}$ is monochromatic.

**Theorem 5.21.** *If $r, k \in \mathbb{N}$ and $M \in \mathbb{N}$ is sufficiently large, then for any $r$-colouring of $\mathbb{N}$, there is a $k$-subset $S \subseteq [M]$ with monochromatic $\prod(S)$.*

*Proof.* Note that by Theorem 5.19, there exists an $M'$ such that for any $r$-colouring of $\mathbb{N}$, there is a $k$-subset $S' \subseteq [M']$ with monochromatic $\sum(S')$. Now, we take $M = 2^{M'}$. Let $\chi$ be any $r$-colouring of $[M]$ and $\chi'$ be the $r$-colouring of $[M']$ defined by $\chi'(i) = \chi(2^i)$ for $1 \leq i \leq M'$. By Theorem 5.19, there is a $k$-subset $S' \subseteq [M']$, say $\{s_1, \ldots, s_k\}$ with monochromatic $\sum(S')$. By the definition of $\chi'$, there is a set $S = \{2^{s_1}, \ldots, 2^{s_k}\}$ which is monochromatic under $\chi$ colouring. Note that

$$\prod_{r \in R} 2^{s_r} = 2^{\sum_{r \in R} s_r}$$

for any $R \subseteq S'$. Hence, $\prod(S)$ is monochromatic. $\square$

# Chapter 6
# The Hales-Jewett Theorem

In this chapter, we present another key theorem in Ramsey Theory, the Hales-Jewett Theorem. In Section 6.1, we present this theorem and prove it. We will also mention the density version of the theorem. In Section 6.2, we present the another proof of Van der Waerden's Theorem by using the Hales-Jewett Theorem.

## 6.1 The Hales-Jewett Theorem

In this section, we will present the Hales-Jewett Theorem and prove it. Hales-Jewett Theorem is a fundamental theorem in Ramsey Theory with a geometric focus, and was proven by Alfred W. Hales and Robert I. Jewett in 1963 [50]. Before looking into the theorem, we introduce some notation and definitions required.

**Definition 6.1** ($n$-cube over $k$ elements)**.** *We define the $n$-cube over $k$ elements by*

$$C_k^n = \{(x_1, \ldots, x_n) : x_i \in [0, k-1]\}\,.$$

**Definition 6.2** (Line)**.** *A line in $C_k^n$ is a set of points $x_0, \ldots, x_{k-1}$, where $x_i = (x_{i1}, \ldots, x_{in})$ so that in each coordinate $j \in [n]$, either*

$$x_{0j} = \cdots = x_{k-1,j}$$

*or*

$$x_{sj} = s, \text{ where } s \in [0, k-1], \text{ for some } j\,.$$

**Example 6.3.** For $k = 5, n = 4$, $\{0130, 1131, 2132, 3133, 4134\}$ forms a line in $C_k^n$. For clarity purpose, the parentheses and commas may be omitted when $k$ is small.

**Definition 6.4** (Equivalence class)**.** *There is a collection of $n+1$ equivalence classes on $C_k^n = [0, k-1]^n$ in such a way that $i$-th equivalence class is the set of all points where $k-1$ appears in the $i$ rightmost positions, for $0 \leq i \leq n$.*

**Example 6.5.** Consider $C_4^2 = [0, 3]^2$. There are 3 equivalence classes of $C_4^2$: the 0th equivalence class is $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\}$; the first equivalence class is $\{(0, 3), (1, 3), (2, 3)\}$ and the second equivalence class is $\{(3, 3)\}$.

**Definition 6.6** (Layered $c$-dimensional subspace)**.** *A $c$-dimensional subspace of $C_k^n$ is a $c$-dimensional cube. A $c$-dimensional subspace of $C_k^n$ is said to be layered if there is a line where the first $k-1$ points are monochromatic. The mentioned line is also known as a layered line.*

**Theorem 6.7** (Hales-Jewett Theorem). [50] *Let $k, r \in \mathbb{N}$. If $n$ is sufficiently large, then for any $r$-colouring of the cube $C_k^n = \{(x_1, \ldots, x_n) : x_i \in [0, k-1]\}$, there is a monochromatic line. The least of such $n$ is known as the Hales-Jewett Number and is denoted by $HJ(r, k)$.*

Before we prove the theorem, we introduce the following lemma to help us.

**Lemma 6.8.** [45] *Let $k, r \in \mathbb{N}$. Suppose that $HJ(r, k)$ exists for all $r$. Then for all $c \in \mathbb{N}$, there exists a number $LHJ(r, k, c)$ so that for $n \geq LHJ(r, k, c)$, if $C_{k+1}^n$ is $r$-coloured, then there exists a layered $c$-dimensional subspace.*

*Proof.* The proof is mainly from [45]. We use induction on $c$. Letting $c = 1$, we can take $LHJ(r, k, 1) = HJ(r, k)$. Consider any $r$-colouring of $C_{k+1}^n$ for $n \geq LHJ(r, k, 1) = HJ(r, k)$. Note that there is $C_k^n$ in $C_{k+1}^n$. By definition of $HJ(r, k)$, there is a monochromatic line in $C_k^n$ and this line is also a layered line in the 1-dimensional subspace.

Now, suppose that $LHJ(r, k, c)$ exists. We need to show that $LHJ(r, k, c+1)$ also exists. Let $m = LHJ(r, k, c)$ and $s = r^{(k+1)^m}$. Since $LHJ(r, k, 1)$ exists for all $r$, $m' = LHJ(s, k, 1) = HJ(s, k)$ must exist. We intend to show that we can take $LHJ(r, k, c+1) = m' + m$.

Let $\chi$ be an $r$-colouring on $C_{k+1}^{m'+m}$. Now, consider $x \in C_{k+1}^{m'}$ and $y \in C_{k+1}^m$. We let $xy \in C_{k+1}^{m'+m} = C_{k+1}^{m'} \times C_{k+1}^m$ denote their concatenation. Consider $\chi'$ be a $s$-colouring of $C_{k+1}^{m'}$ in such a way that

$$\chi'(x) = \chi'(x') \text{ if and only if } \chi(xy) = \chi(x'y) \text{ for all } y \in C_{k+1}^m .$$

Since there are only $s$ colours, there exists a layered line $x_0, x_1, \ldots, x_t \in C_{k+1}^{m'}$ under $\chi'$. Now, we colour $C_{k+1}^m$ by $\chi''$, where

$$\chi''(y) = \chi(x_i y), \text{ for } 0 \leq i \leq k-1 ,$$

in which the $x_i$'s are the points in the layered line. Note that there are $r$ colours in $\chi''$ and that $m = LHJ(r, k, c)$, so there is a layered $c$-dimensional subspace, say $S \subseteq C_{k+1}^m$ under the colouring of $\chi''$.

Now, let $T = \{x_i s : 0 \leq i \leq k, s \in S\} \subseteq C_{k+1}^{m'+m}$. Suppose that $S$ has equivalence classes $S_0, \ldots, S_c$. Then $T$ has equivalence classes $T_j = \{x_{is} : 0 \leq i \leq k, s \in S_j\}$, for $0 \leq j \leq c$, and $T_{c+1}$ which consists of a single point beginning with $x_k$. Note that for $x_{is}, x_{is'} \in T_j$, where $0 \leq j \leq c$, we have

$$\chi(x_{is}) = \chi''(s) = \chi''(s') = \chi(x_{is'}) .$$

Hence, $T$ is our layered $(c+1)$-dimensional subspace. Thus, $LHJ(r, k, c+1)$ exists. By induction, the result holds. □

Now, we proceed to the proof of Hales-Jewett Theorem (Theorem 6.7).

*Proof.* The proof mainly follows that given in [45]. We use induction on $k$. If $k = 1$, we can just take $n = 1$ and the result is trivial. Suppose that the theorem holds for $k$. We need to show that the result also holds for the case $k+1$. Since $HJ(r, k)$ exists by the induction hypothesis, Lemma 6.8 that $LHJ(r,k,c)$ exists for all $c \in \mathbb{N}$. Take $c = r$ and $n = LHJ(r,k,r)$. By the definition of $LHJ(r,k,r)$, if $C_{k+1}^n$ is $r$-coloured, then there exists a layered $r$-dimensional subspace. Now, let $C_{k+1}^r$ be the layered subspace and consider these $r+1$ points $x_i$ for $0 \le i \le r$:

$$x_i = (x_{i1}, \dots, x_{ir}),\ x_{ij} = \begin{cases} k, & \text{if } j \le i\,; \\ 0, & \text{if } j > i\,. \end{cases}$$

Since there are only $r$ colours, the Pigeonhole Principle implies that $x_u$ and $x_v$ are of the same colour, say $c_1$, for some $u < v$. Then, the points $y_0, \dots, y_k$ in which

$$y_s = (y_{s1}, \dots, y_{sr}),\ y_{si} = \begin{cases} k, & \text{if } i \le u\,; \\ s, & \text{if } u < i \le v\,; \\ 0, & \text{if } v < i. \end{cases}$$

will also be $c_1$-coloured and form the monochromatic line. Hence by induction, the theorem holds. □

Now, we will introduce the density version of Hales-Jewett Theorem. In this strengthened version, instead of colouring the entire $C_k^n$ with $r$ colours, we colour an arbitrary subset, say $A \subset C_k^n$, with density $0 < \delta < 1$, where $\delta = \frac{|A|}{k^n}$.

**Theorem 6.9** (Density version of Hales-Jewett Theorem)**.** [36]
*Let $k, r \in \mathbb{N}$ and $0 < \delta < 1 \in \mathbb{R}$. If $n$ is sufficiently large, then for any $r$-colouring of $A \subset C_k^n$ with density $\delta$, there is a monochromatic line.*

The proof of this theorem is rather technical so we are not going to prove it here. The interested reader is referred to [19, 36, 80].

## 6.2 Proof of Van der Waerden's Theorem by Hales-Jewett Theorem

In this section, we present another proof of Van der Waerden's Theorem (Theorem 4.3) by using the Hales-Jewett Theorem (Theorem 6.7). Van der Waerden's Theorem may indeed be proven as a corollary of the Hales-Jewett Theorem.

Let $k, r \in \mathbb{N}$. Recall that Van der Waerden's Theorem states that, for sufficiently large $n$, each $r$-colouring of $[n]$ gives a monochromatic arithmetic progression of length $k$. Recall that the Van der Waerden number $W(k,r)$ is the least such $n$.

*Proof.* This proof mainly follows that of [45]. We want to show that $W(k,r) \le k^{HJ(r,k)}$ where $HJ(r,k)$ is defined as in the Hales-Jewett Theorem (Theorem 6.7). We represent each number $a \in [k^{HJ(r,k)}]$ as $HJ(r,k)$-tuples $(a_1, \dots, a_{HJ(r,k)})$ by translating $a$ into base-$k$ expression $a = \sum_{HJ(r,k)}^{i=1} a_i k^{i-1}$, where $0 \le a_i < k$. Now, note that the $r$-colouring of $[k^{HJ(r,k)}]$ induces an $r$-colouring of $C_k^{HJ(r,k)}$. By Theorem 6.7, there is a monochromatic line of length $k$. Note that, in this monochromatic line, the coordinate of each point is either constant or increasing by one each time. Hence, by translating back every point of the monochromatic line, we will get a monochromatic progression of length $k$, with the common difference in the form $k^\alpha$ where $\alpha \in \mathbb{N}$. Then, we are done. □

# Chapter 7
## Applications of Ramsey Theory

To highlight the significance of Ramsey Theory, we include some of the applications of Ramsey Theory in this chapter. In Section 7.1, we consider the application of Ramsey Theory to Graph Theory. Next, in Section 7.2, we will show a geometric application of Ramsey Theory. Finally, in Section 7.3, we will apply Ramsey Theory to Number Theory.

## 7.1 Applications to Graph Theory

In this section, we will present some applications of Ramsey Theory to Graph Theory. The first application is to prove Mantel's Theorem.

**Theorem 7.1** (Mantel's Theorem)**.** [72] *Let $G$ be a simple graph with $n \geq 3$ vertices. If the number of edges of $G$ is $|E(G)| > \frac{n^2}{4}$, then $G$ has at least one triangle.*

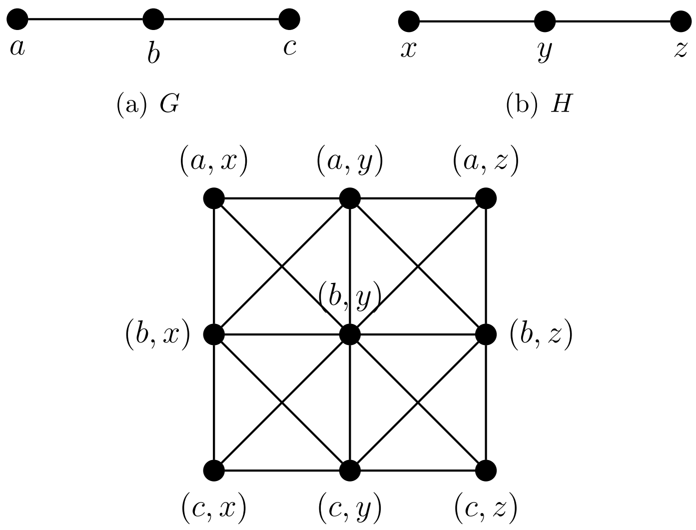
*Proof.* The proof mainly follows that of [103]. We use induction on $n$. Suppose that $n = 3$. If $|E(G)| > \frac{3^2}{4} = \frac{9}{4} \geq 3$, then $|E(G)| = 3$ and $G$ is itself a triangle $K_3$. Now, suppose that the theorem holds for the case $n-1$. We wish to show that the theorem is also true for $n$. Let $G$ be a simple graph with $|E(G)| > \frac{n^2}{4}$ and let $\{u, v\}$ be an edge of $G$. Let $H$ be the subgraph of $G$ obtained by deleting both vertices $u$ and $v$ and each edge incident to them. If $|E(H)| > \frac{(n-2)^2}{4}$, then the subgraph $H$ has at least one triangle and, hence, $G$ must also contain a triangle. If $|E(H)| \leq \frac{(n-2)^2}{4}$, then the number of edges between $H$ and the vertices $u$ and $v$ will be $|E(G) - \{u, v\}| - |E(H)| > \frac{n^2}{4} - 1 - \frac{(n-2)^2}{4} = n - 2$. Hence, there are at least $n-1$ edges between $H$ and the vertices $u$ and $v$. Note that there are only $n-2$ vertices in $H$. By the Pigeonhole Principle, some vertex in $H$ must be joined to both $u$ and $v$. Thus, $G$ has a triangle. By induction, the theorem holds for all $n \geq 3$. □

Before we go to the next application, there is a definition that we here introduce.

**Definition 7.2** (Strong product of two graphs)**.** *The* strong product *of two graphs $G$ and $H$, denoted by $G \boxtimes H$, is the graph whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edges are given as follows: $\{(a, b), (c, d)\}$ is an edge in $G \boxtimes H$ if and only if one of the following conditions holds:*

(1) *$\{a, c\} \in E(G)$ and $\{b, d\} \in E(H)$,*
(2) *$a = c$ and $\{b, d\} \in E(H)$,*
(3) *$b = d$ and $\{a, c\} \in E(G)$.*

**Example 7.3.** Figure 7.1 shows the graphs $G$ and $H$ and their normal product $G \boxtimes H$.




(a) $G$ (b) $H$

(c) $G \boxtimes H$

Figure 7.1: The normal product of graph $G$ and $H$, $G \boxtimes H$.

**Definition 7.4.** *A set of vertices in a graph is* independent *if no two of these vertices are adjacent. For each graph $G$, let $\alpha(G)$ be the largest size of an independent set in $G$.*

**Theorem 7.5.** [54] *If $G$ and $H$ are graphs, then $\alpha(G \boxtimes H) \leq R(\alpha(G)+1, \alpha(H)+1)-1$, where $R(\alpha(G)+1, \alpha(H)+1)$ is the Ramsey number as defined in Theorem 2.1.*

*Proof.* This proof is provided in [84]. Let $N = R(\alpha(G)+1, \alpha(H)+1)$. Suppose to the contrary that $\alpha(G \boxtimes H) \geq N$. Let $I$ be an independent set of $G \boxtimes H$ with $N$ vertices. Let $(a, b)$ and $(c, d)$ are two distinct vertices in $I$. Since $I$ is independent, then one of the following conditions holds:

(1) $a \neq c$ and $\{a, c\} \notin E(G)$,

(2) $b \neq d$ and $\{b, d\} \notin E(H)$.

Now, consider a 2-colouring of the complete graph $K_N$. Label each of the vertex of $K_N$ as in $I$. Colour the edge $\{(a, b), (c, d)\}$ with $c_1$ if (1) holds and $c_2$ otherwise. By the definition of $N$, there is either a $c_1$-coloured $K_{\alpha(G)+1}$ or a $c_2$-coloured $K_{\alpha(H)+1}$. Suppose that there is a $c_1$-coloured $K_{\alpha(G)+1}$. Since (1) holds for this subgraph, $\{a : a \in V(G)$ and $\{a, b\} \in K_{\alpha(G)+1}$ for some $b\}$ is an independent set of $G$ with $\alpha(G)+1$ vertices, which is a contradiction since $\alpha(G)$ is the largest size of an independent set in $G$. On the other hand, suppose that there is some $c_2$-coloured $K_{\alpha(H)+1}$. Since (2) holds for this subgraph, $\{b : b \in V(H)$ and $\{a, b\} \in K_{\alpha(H)+1}$ for some $a\}$ is an independent set of $H$ with $\alpha(H)+1$ vertices, which is a contradiction. Thus, we have $\alpha(G \boxtimes H) \leq N-1 = R(\alpha(G)+1, \alpha(H)+1)-1$. $\square$

## 7.2 Application to Geometry

In this section, we will present a geometric application of Ramsey Theory.

**Theorem 7.6.** [25] *Let $k \in \mathbb{N}$. If $n$ is sufficiently large, then among any $n$ points in the plane with no three points collinear, there are $k$ of the points that form a convex polygon.*

*Proof.* The proof mainly follows that which is outlined in [25]. We will show that we can take $n = R_3(k,k)$, where $R_3(k,k)$ is the Ramsey number as defined in Theorem 2.7. Let $K_n$ be a complete graph with $n$ vertices. Label each of the $n$ points with $1, 2, \ldots, n$ respectively in any order. Colour every triple $\{i, j, l\}$ with colour $c_1$ if $i < j < l$ is clockwise orientated and $c_2$ otherwise. By the definition of $R_3(k,k)$, there are $k$ points whose triples are monochromatic. Then, each triangle among these $k$ points is same-orientated, and hence these $k$ points will form a convex polygon. $\square$

Many study on such a minimum integer $n$ in Theorem 7.6, denoted by $ES(k)$, has been conducted over the past several decades. The most recent result is contributed by Suk, who proved that $SE(k) \leq 2^{k+6k^{\frac{2}{3}}\log_2 n}$ [98].

## 7.3 Applications to Number Theory

In this section, we present applications of Ramsey Theory to Number Theory. The first application to be presented is on the multiplicative representation of integers, proposed by Erdős in 1964.

**Theorem 7.7.** [21] *Let $A \subseteq \mathbb{N}$ where, for each $n \in \mathbb{N}$, there are $a, b \in A$ such that $n = ab$. For each $k \in \mathbb{N}$, there is some integer $n \in \mathbb{N}$ such that the equation $n = ab$ has at least $k$ solutions with $a, b \in A$.*

*Proof.* The proof mainly follows that in [77] in the form in which it was reproduced and modified in [76]. Note that $A$ must contain all prime numbers; hence it is sufficient for us to consider the integers $n$ that are products of distinct primes only. Let $M(n)$ be the set of prime factors of $n$. Whenever we have any partition of $M(n) = M_1 \cup M_2$ for which $a = \prod M_1$, $b = \prod M_2$ and $n = ab$, the definition of $A$ implies that $a, b \in A$. Now, by Ramsey's Theorem (Theorem 2.7), for a sufficiently large $n$ and $|M|$, we can have one of the partitions, without loss of generality, say $M_1$, have at least $k$ elements of $M$. Hence, there are at least $k$ ways to partition $M$. Thus, $n = ab$ have at least $k$ solutions in $A$. $\square$

Next, we are going to present an application of the Pigeonhole Principle to Number Theory, namely to prove Proizvolov's Identity, proposed by Vyacheslav Proizvolov after the 1985 All-Union Olympiad [91].

**Theorem 7.8** (Proizvolov's Identity)**.** [91]
*If $[2n]$ is biparted into sets $A = \{a_1 > \cdots > a_n\}$ and $B = \{b_1 < \cdots < b_n\}$, then*

$$\sum_{i=1}^{n} |a_i - b_i| = n^2 .$$

*Proof.* This proof follows that provided in [7]. Consider the pairs $a_i$ and $b_i$. We wish to show that one of them must be in $[n]$ and the other one must be in $[n+1, 2n]$. First, assume the contrary that both $a_i$ and $b_i$ are in $[n]$ for some $i$. Then, at least $n-i+1$ $a_j$s and $i$ $b_j$s are in $[n]$. Therefore, at least $n-i+1+i = n+1$ of the $a_j$'s and $b_j$'s lie in $[n]$. By the Pigeonhole Principle, at least two of these are identical, which is a contradiction since $A \cap B = \emptyset$. Hence, $a_i$ and $b_i$ cannot be both in $[n]$. Next, we assume that both $a_i$ and $b_i$ are in $[n+1, 2n]$. Then, at least $i$ $a_j$s and $n-i+1$ $b_j$s lie in $[n+1, 2n]$. This means that there are in total at least $i+n-i+1 = n+1$ $a_j$s and $b_j$s in $[n+1, 2n]$. Again, by the Pigeonhole Principle, at least two of them are identical, which is a contradiction. Hence, $a_i$ and $b_i$ cannot be both in $[n+1, 2n]$. Then, we have shown that one of them must be in $[n]$ and the other one must be in $[n+1, 2n]$. Thus, we have

$$\sum_{i=1}^{n} |a_i - b_i| = [(n+1) + \cdots + 2n] - (1 + \cdots + n) = n^2 .$$

$\square$

There are many other applications of Ramsey Theory, particularly in Information Theory, information retrieval, design of packet switched networks, Games Theory and many other applications. The interested reader is referred to the overviews given by F.S. Roberts [84, 86] and V. Rosta [86].

# Chapter 8
# Conclusion

In conclusion, Ramsey Theory is a rapidly developing field of mathematics. In the thesis, we have studied several different types of Ramsey-type results, including results pertaining to edge-colourings of the complete graph, monochromatic arithmetic progressions and Schur triple $a + b = c$ in colourings of integers, monochromatic lines in cube colourings. We have also presented bounds on the various types of Ramsey numbers. However, there are still many Ramsey-type topics that have not been included in this thesis. We hope that through this thesis, the reader might have found interest and appreciation for Ramsey Theory, as Ramsey Theory has become an important and active area of research. More importantly, there is still room for advancement in this field of knowledge, such as the study of the various bounds and the relationships between Ramsey Theory and the other fields of study. We look forward to conducting more such research. Before ending our thesis, we list some of the interesting open problems and conjectures in the field.

**Conjecture 8.1.** [22] *Let $R(k,k)$ denote the $k^{th}$ diagonal Ramsey number. Then,*

$$\lim_{k\to\infty} R(k,k)^{\frac{1}{k}}$$

*exists.*

**Question 8.2.** [22] *What is the limit in the Conjecture 8.1 if such a limit exists?*

**Conjecture 8.3.** [20] *Let $W(k;2)$ be the Van der Waerden number. Then,*

$$\lim_{k\to\infty} \frac{W(k;2)}{2^k} = \infty$$

*and*

$$\lim_{k\to\infty} W(k;2)^{\frac{1}{k}} = \infty.$$

**Conjecture 8.4.** [62] *Let $k \geq l > 2$. Then,*

$$W(k,l;2) \geq W(k+1,l-1) \geq W(k+2,l-2) \geq \cdots .$$

**Question 8.5.** [55] *Suppose that $r \in \mathbb{N}$ and $\mathbb{N}$ is $r$-coloured. Must there be a arbitrarily large finite set $S \subseteq \mathbb{N}$ such that $\sum(S) \cup \prod(S)$ is monochromatic?*

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