Approximation of random/stochastic partial differential equations

Author:
Kazashi, Yoshihito

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APPROXIMATION OF RANDOM/STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

A thesis submitted for the degree of
Doctor of Philosophy

By
Yoshihito Kazashi

School of Mathematics and Statistics,
UNSW Australia

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Abstract

The overarching interest of this thesis lies in approximations of partial differential equations (PDEs) with randomness or stochasticity. We focus on three rather different problems: a study of random fields on spherical shells, and its applications to PDE problems; quasi-Monte Carlo (QMC) methods for a class of PDEs with random coefficients; and a discretisation for the solution of stochastic PDEs.

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Contents

List of Symbols

Chapter 2

$\mathcal{A}$	Log-normal random field, page 37
$\mathcal{A}_{KL}$	$\exp(\mathscr{V}_{KL}T)$, page 40
$A_{k\ell}$	Expansion coefficients of $\mathfrak{c}(\cdot,\cdot)$, pages 11, 14
$\mathfrak{c}(\cdot,\cdot)$	Covariance function of T, pages 11, 14
h, h^{ang}, h^{rad}	Filter functions, page 33
J_k	Degree k Chebyshev polynomial of the first kind, page 32
μ	A finite Borel measure, pages 10, 14
P_ℓ	Legendre polynomial of degree $\ell \in \mathbb{N} \cup \{0\}$, page 10
(φ_k)	An orthonormal system for $L^2([r_{\text{in}}, r_{\text{out}}], \mu)$, pages 10, 14
r_{in}, and r_{out}	Inner, and outer radius of $\mathbb{S}_\varepsilon$, page 10
S^2	Unit sphere, page 10
$\mathbb{S}_\varepsilon$	Spherical shell, page 10
T	Gaussian random field indexed by $\mathbb{S}_\varepsilon$, page 11
$\mathscr{V}_{KL}T$	Filtered approximation of T, page 35
$\mathcal{Y}_{\ell m}$	Spherical harmonics, page 10

Chapter 3

$\bigtriangledown$, $\bigtriangledown_\ell$	Index sets, page 68
a	Log-normal random field indexed by D, page 46
$\hat{a}(\boldsymbol{y})$	$\operatorname{ess\,sup}_{x\in D} a(x,\boldsymbol{y})$, page 59
$\check{a}(\boldsymbol{y})$	$\operatorname{ess\,inf}_{x\in D} a(x,\boldsymbol{y})$, page 59
D	Bounded domain in $\mathbb{R}^d$, page 46
$\boldsymbol{\Delta} \in [0,1]^s$	Random shift, page 51
$F(\boldsymbol{y})$	$\mathcal{G}(u^s(\cdot,\boldsymbol{y}))$, page 50
$\mathcal{G}$	Continuous linear functional, page 47
n	Number of quadrature points, page 51
Φ_s^{-1}	Inverse of the cumulative normal distribution function, page 50
$(\varphi_\xi)_{\xi\in\bigtriangledown}$	Wavelet Riesz basis, page 68
$(\varphi_j)_j$	Lexicographically reordered wavelet basis, page 70

ψ_j	Spatial function, pages 46, 49, 52
$\mathcal{Q}_{s,n}(\boldsymbol{\Delta};\cdot)$	Randomly shifted lattice rule, page 51
(ρ_j)	Positive sequence, page 52
ς_j	Page 60
T	Gaussian random field indexed by D, page 46
$\mathfrak{u}$	Index set, page 53
V	$H_0^1(D)$, page 46
$\mathcal{W}^s$	Weighted unanchored Sobolev space, page 53

Chapter 4

Λ_j	Index set, page 88
$\beta_{\ell m}$	One-dimensional standard Brownian motions, page 89
H	Separable $\mathbb{R}$-Hilbert space, page 88
h_{jk}	Eigenfunction of $-A$ and Q, page 88
$\mathcal{K}_\eta$	Page 92
$\mathcal{L}_2$	Space of Hilbert–Schmidt operators, page 86
λ_j	Eigenvalue of $-A$, page 88
N	Number of temporal grids, page 92
Q	Covariance operator, page 89
q_ℓ	Eigenfunction of Q, page 89
$\left[R^J \diamond \ \cdot\ \right]^L$	Discretised stochastic convolution, page 106
$s_{\eta,\ell}$	Page 92
$(S(t))_{t\geq 0}$	C_0-semigroup generated by A, page 88
$X_{jk}(t)$	$\langle X(t), h_{jk}\rangle$, page 90
$\widehat{X}^{J,L}$	Fully discretised scheme, page 93
$X^{J,L}$	Semi-discrete scheme, page 94

Chapter 1

Introduction

The overarching interest of this thesis lies in approximations of the solution of partial differential differential equations (PDEs) with randomness or stochasticity. Theoretical analysis of the algorithms for these problems is arguably still in its growth stage: while there are already countless books on approximations for deterministic PDEs, there are many fewer books on PDEs with randomness (but also see the books [69, 59, 48, 103], all of which were published within the last decade).

We focus on three rather different problems: a study of random fields on spherical shells, and its applications to PDE problems; quasi-Monte Carlo (QMC) methods for the expected value of output functionals of solutions of a class of PDEs with random coefficients; and a discretisation for the solution of stochastic partial differential equations.

In this introductory chapter, we briefly outline the problem settings. These problem settings are technically involved, and even though all of these share an underlying theme—PDEs with randomness—results obtained in different chapters are independent. Therefore, we consider these three problems in the following separate three chapters. More detailed discussion of the problem setting will be given there. The contents of the next three chapters are based on four papers that have been completed during this PhD study. See Section 1.4 for more details.

1.1 Gaussian random fields on spherical shells

Chapter 2 is concerned with random fields on spherical shells. This chapter is primarily motivated by applications in geophysics, with other potential applications that we are currently unaware of. Motivated by such problems, we consider random fields associated with a covariance function that has rotational invariance. This invariance corresponds to a layered structure, for example that of the Earth's interior. Although these random fields are used in applications, they do not seem to have attracted much theoretical interest. This chapter aims at contributing to building a theoretical foundation. The analysis is technically difficult, since there are two variables with different nature—radial and angular variables. Even though there are numerous results on random fields on subsets of $\mathbb{R}^d$ for $d \in \mathbb{N}$, these results are not directly applicable due to the special structure—radially anisotropic but rotationally isotropic—of the covariance function we consider.

First, we analyse the path smoothness of the random field. The covariance function is assumed to be given as a series representation. We start with relating the decay rate of the expansion coefficient to the smoothness of the realisation in the radial and angular directions. We further use these results in two directions: path smoothness of the exact random fields on the spherical shells and random field approximations.

Spherical shells are a subset of $\mathbb{R}^3$. A natural question to ask is the smoothness of the realisations as functions on a subset of $\mathbb{R}^3$. Establishing this type of result will be also useful for applying available results for the usual $\mathbb{R}^d$ setting: countless recent results on partial differential equations with random coefficients are available, with the spatial domains being bounded domains in $\mathbb{R}^d$. Many of these results assume path smoothness as a function on the domain to obtain error estimates, for example for algorithms to approximate the solution of PDEs. To establish path smoothness as a function on the domain in $\mathbb{R}^3$—spherical shell—we invoke the regularity theory for elliptic PDEs.

Further, we consider an approximation of the random field, and establish an error estimate. It is of natural theoretical and practical interest how accurately we can approximate the field with a finite sum: in applications, we can use only a finite number of expansion coefficients of the assumed covariance function, except in the special occasion where a closed formula is available.

Motivated by applications, the covariance function we consider is assumed to have a series representation with each term a product of radial and spherical polynomials; thus so is the corresponding random field. Simple truncation in our setting corresponds to L^2-projection. However, L^2-projection is not the best choice when one wants a small uniform error.

We consider a so-called filtered approximation of the random field. An underlying idea is to smoothly truncate the series by multiplying the Fourier coefficients of higher order by a suitably decaying factor. It turns out that filtering can make the error equal to the best polynomial approximation error, up to a constant factor. This is particularly important when our interest is in random fields with non-smooth realisations: we will get a slower rate than the best polynomial approximation rate if we consider the L^2-projection without the filtering, and for non-smooth realisations this loss would be significant.

As an application of the approximation problem, we consider a diffusion problem with a random field as input. The coefficient of the differential operator is assumed to be given as an exponentiated random field given by a series representation. We approximate the series by a finite number of terms, and analyse how this affects the solution of the equation. This type of problem has been extensively considered in recent years, but these results, which typically make rather general assumptions, are not directly applicable to our setting. Instead of the plain vanilla truncation, we consider filtered approximation. This is advantageous in terms of the error decay rate: the error in the solution of the PDE will be bounded in terms of the supremum norm error estimate of the random coefficient, for which the filtered approximation gives a better convergence rate than simple truncation.

The content of Chapter 2 described above is a continuation of my projects on function approximation on spherical shells during my PhD study: this chapter uses a part of the results in Kazashi [53], which follows Kazashi [52]. Both of the papers [53] and [52] were completed and published by the author during this PhD study, but for coherence of this thesis they are not included here.

1.2 Quasi-Monte Carlo integration for elliptic PDEs with log-normal coefficients

Chapter 3 considers the same type of elliptic PDE, but is concerned with the approximation of the expected value of output functionals of solutions of the equation. We approximate the expectation—the integral with respect to the probability measure—by a sampling method: an equal weight quadrature rule called a quasi-Monte Carlo method (QMC). Among the QMC methods, the algorithm we consider is the randomly shifted lattice rule.

The application of QMC methods to this type of problem was initiated by computational results by Graham et al. [35], followed by error analysis by Kuo et al. [65] and Graham et al. [37]. See a recent survey paper [64] and references therein for more details.

Similarly to [37], we consider the case where the random coefficients are given by the log-normal random fields. The difference lies in the spatial functions used to represent the random field: we consider the random fields that admit a series representation in terms of spatial basis functions with *local* supports.

The motivation to consider this setting is to show that we can construct a faster approximation algorithm. To approximate the expectation well, we need to find good quadrature points. For randomly shifted lattice rules, this corresponds to finding an integer vector called the generating vector—lattice rules use a point set with a group structure, and this vector works as a generator of the group.

One way to find a good generating vector is to use an algorithm called component-by-component (CBC) construction. The CBC construction was considered in [37] as well, but under our setting we show that the CBC construction can be used with a smaller cost than the cost considered in [37].

Further, we discuss smoothness of realisations allowed by the presented theory—our theory as well as the existing theory implicitly impose a constraint on the spatial smoothness of the random fields. This implicit constraint on the path smoothness allowed is imposed to obtain an error estimate independent of the dimension of the integration. We consider a suitable class of wavelet basis as the aforementioned spatial basis, and analyse the path smoothness. To to this, we invoke a characterisation of the path smoothness in the Besov scale as well as Sobolev embedding theorems.

1.3 Non-uniform implicit Euler–Maruyama scheme for a class of stochastic evolution equations

In Chapter 4, we turn our attention to an approximation of a class of stochastic parabolic differential equations. We consider three discretisations: temporal, spatial, and the truncation of the infinite-dimensional space-valued Wiener process.

Temporally, we consider the implicit Euler–Maruyama method. For the spatial discretisation, we consider the spectral method. The Wiener process, which is assumed to admit a series representation, takes its value in an infinite-dimensional space. The Euler–Maruyama method introduces the increments of such a process, but they need to be further approximated: each increment corresponds to infinitely many random variables, but in practice we can simulate only finitely many of them. For the approximation, we truncate the Wiener process, i.e., we use a type of truncated Karhunen–Loève approximation.

We further consider a non-uniform time step. This was originally considered by Müller-Gronbach and Ritter [77, 76] for the stochastic heat equation over the hyper-cube. There, it was shown that their algorithm achieves optimality in a certain sense, and that optimality cannot be achieved by uniform time discretisations. Their use of a non-uniform time step was motivated by the following observation: upon the truncation, the Wiener increment is that of the sum of finitely many one-dimensional Wiener increments; these Wiener increments, which have the same law as normal random variables, have different variances given by the eigenvalue of the covariance operator. One should arguably change the time steps for accordingly to the variances.

In [54], which I wrote during this PhD study together with Quoc T. Le Gia, we considered essentially the same algorithm as in [76], but applied it to the stochastic heat equation on the sphere, and further analysed the error. This was not trivial, because the proofs in [76] that validate the non-uniform time step do not seem to be easily generalisable to the spherical case: the argument in [76] uses repeatedly the fact that the eigenfunctions of the Laplace operator on the cube with the Dirichlet condition are uniformly bounded. Further, the fact that these eigenfunctions are again those of the classical first order derivatives is crucial. On the sphere, we have neither of the properties. See [54] for more details.

In Chapter 4, we present the results in [54] in a more general setting. The exposition in this manner reveals that, under the multiplicative noise setting, the use of different time steps for different eigenspaces is more to do with how the operator in the stochastic forcing term acts on the eigenspaces of the covariance operator, as opposed to just the eigenvalues.

For the error estimate, we consider the temporally discrete estimate. In [54], we considered the $L^2([0,1] \times \Omega; H)$-norm estimate by considering a continuous interpolation of the approximate solution, where H denotes the Hilbert space in which the solution of the equation takes value. In this thesis, we replace the temporal integration that appears in the $L^2([0,1] \times \Omega; H)$-norm with a sum weighted by step lengths, using the values of approximated solution only on temporal grids without continuous interpolations.

Further, we study the above algorithm more deeply: we show that it satisfies a discrete analogue of an estimate called *maximal regularity*. Roughly speaking, this regularity estimate says that the spatial smoothness of the solution is controlled by that of the operator in the forcing term. In recent years, the study of discrete analogues of the maximal regularity has been attracting attention for deterministic partial differential equations. Here we consider a stochastic equation. To the best of the author's knowledge, discrete analogues of maximal regularity of numerical methods for SPDEs have not been addressed in the literature.

Maximal regularity of stochastic and deterministic equations are different in nature. Given a suitable smoothness of the initial data, the solution is "one-half spatially smoother", than the range of the diffusion operator—the regularity one can obtain in the stochastic setting is the half of the corresponding regularity in the deterministic setting [22, Chapter 6]. This estimate is optimal, in that the solution cannot be spatially smoother in general [60, Example 5.3].

1.4 Works during this PhD study

The following works have been studied and completed during this PhD study. Chapters 2, and 3 are based on [II], and [III], respectively. Chapter 4 is based on [I] and [VI].

[I] Y. Kazashi. Discrete maximal regularity of an implicit Euler–Maruyama scheme for a class of stochastic evolution equations. *Electron. Commun. Probab.* 23 (2018), no. 29, 1–14

[II] Y. Kazashi. Gaussian random fields on spherical shells: Regularity, approximation, and elliptic partial differential equations. (submitted)

[III] Y. Kazashi. Quasi-Monte Carlo integration with product weights for elliptic PDEs with log-normal coefficients. *IMA J. Numer. Anal.* (2018) https://doi.org/10.1093/imanum/dry028

[IV] Y. Kazashi. A fully discretised filtered polynomial approximation on spherical shells. *J. Comput. Appl. Math.* 333 (2018), pp. 428–441

[V] Y. Kazashi. A fully discretised polynomial approximation on spherical shells. *GEM - Int. J. Geomathematics* 7 (2016), pp. 299–323

[VI] Y. Kazashi and Q. T. Le Gia. A non-uniform discretization of stochastic heat equations with multiplicative noise on the unit sphere. 2017. arXiv: 1706.02838 (submitted)

Chapter 2

Gaussian random fields on spherical shells: regularity, approximation, and elliptic partial differential equations

Abstract of this chapter

Gaussian random fields on spherical shells that are radially anisotropic and rotationally isotropic are considered. The smoothness of the covariance function is connected to the sample continuity, partial differentiability, and the Sobolev smoothness. Based on the regularity results, convergence rates of filtered approximations are established: Gaussian and log-normal random fields approximated with filtering, and a class of elliptic partial differential equations with approximated log-normal random coefficients, are considered.

2.1 Introduction

In this chapter, we develop a theory of Gaussian random fields on spherical shells. We first study the smoothness of the realisations. Based on the regularity results we establish, we study convergence rates of several approximations.

Random fields on spherical shells are often used to study planetary structures of the earth—for example, to model the velocity perturbations that characterise correlated random media [11, 12, 25, 38, 46, 73, 84, 87]. Nevertheless, theoretical analysis does not seem to have attracted attention.

Motivated by such practical problems, we consider Gaussian random fields. Gaussian random fields on any index set are characterised by the mean function, which without loss of generality we take as zero, and the covariance function defined on the index set. Fixing a covariance function guarantees the unique (up to the law) existence of the corresponding random field, and thus governs the various properties such as radial anisotropy, angular isotropy, and the smoothness of the realisations.

For random fields on the sphere, invariance of the covariance structure under rotation, also called *isotropy*, is often demanded by applications [71]. In contrast, on spherical shells we have isotropy in the angular direction, but not in the radial direction; for example, we have the layered, and hence radially anisotropic structure of the interior of the Earth [24, 25, 29, 38, 49, 72, 73, 84, 86]. We consider a class of covariance function that gives random fields with a transversely isotropic but radially anisotropic structure.

Often in applications, these random fields are approximated by a suitable method such as truncated series, and used in simulations without mathematical justification. Regardless of the method users choose, the smoothness of the realisations is a key to understanding how accurate the simulated random fields are. On the sphere, Lang and Schwab [66] analysed the regularity of realisations of isotropic Gaussian random fields. There, the keys to the analysis were the properties of the spherical harmonics. Covariance functions that define strongly isotropic Gaussian random fields are characterised by the Legendre polynomial expansion [71, Section 5.2], which are, in view of the addition theorem, nothing but kernels diagonalised by the spherical harmonics. The use of spherical harmonics makes the regularity analysis simple, since the spherical harmonics form eigenfunctions of the Laplace–Beltrami operator, whose domain can be characterised as the Sobolev space on the sphere [68, Chapter 7]. In contrast, in the shell setting the necessity to respect both the angular isotropy and radial anisotropy introduces difficulties in treating the realisations as a function on the spherical shell as a subset in $\mathbb{R}^3$.

On spherical shells, to achieve the angular isotropy and radial anisotropy in practice, covariance functions represented by a series expansion in terms of a product basis—product of spherical functions and radial functions—are often considered. As for the analysis of the angular direction, we essentially follow the spherical case [66] to show the partial differentiability in the angular direction. For the radial direction, we do not have structures such as isotropy or periodicity, let alone eigenfunctions of a suitable differential operator which we can easily relate to function spaces of practical use. We show, when the covariance function in the radial direction is represented by a suitable complete orthonormal system of an L^2 space on the interval, that the realisations have continuity, and further, partial differentiability in the radial and angular direction.

Moreover, since spherical shells are naturally embedded into $\mathbb{R}^3$, the behaviour of the realisations as a function on the subset of $\mathbb{R}^3$ is of interest. A difficulty arising is that the covariance function does not possess a natural description in terms of the Cartesian coordinates in $\mathbb{R}^3$. We utilise the partial differentiability results we establish, and invoke the regularity theory of the elliptic partial differential equations. We show that, under a suitable properties of the covariance function (see Assumption 2.1), the realisations are in a Sobolev space on spherical shells.

As applications of the regularity theory to be developed, we consider approximations of Gaussian random fields, and log-normal random fields, applied to a class of elliptic partial differential equations, and provide error analysis. Since realisations of random fields can be highly oscillatory, we wish to consider approximation methods with small point-wise error. For this reason, we consider so-called *filtered approximations*. An underlying idea is to smoothly truncate the series by multiplying the Fourier coefficients of higher order by a suitable factor decaying smoothly to zero. Filtered approximations have also been considered for other regions, including the sphere (see [82, 91] and references therein). Further, we consider log-normal random fields, and their approximation with filtering. For both of these approximation methods, the regularity theory developed is a key to deriving approximation errors; the expected squared uniform error, and the expected L^p error with suitable p, are shown to decay algebraically with respect to the degree of polynomials used to approximate the random fields, given a suitable smoothness of the covariance function in the radial and the angular directions. As a further application, we consider a class of elliptic partial differential equations (PDEs) that have the log-normal random field as a coefficient. We assess the effect to the solution u of the PDE caused by approximating the coefficient. The expected error $u - \widetilde{u}$ with a suitable norm, where $\widetilde{u}$ is the solution of the PDE with the approximated coefficient, turns out to be bounded by the approximation error of the coefficient in terms of the expected squared uniform norm. A similar problem on the sphere was considered by Herrmann et al. [40]. In [40], the numerical treatment of partial differential equations was also considered; we will investigate this application for the spherical shell in future work.

This chapter is organised as follows. In Section 2.2 we give some necessary background information on Gaussian random fields on spherical shells, and set up some notation. In Section 2.3, we establish the continuity of realisations of random fields. In Section 2.4 we develop further smoothness, namely, partial differentiability and the Sobolev smoothness of the realisations. Based on the theory developed in Section 2.4, in Section 2.5 we consider a filtered approximation of the Gaussian and log-normal random fields, and analyse the error.

Further, a class of elliptic partial equations with log-normal random coefficients are considered, and error analysis is provided. Section 2.6 concludes the chapter.

2.2 Random fields on spherical shells

We denote a point on a spherical shell $\mathbb{S}_\varepsilon := \{x \in \mathbb{R}^3 \mid r_{\text{in}} \le \|x\|_2 \le r_{\text{out}} = r_{\text{in}} + \varepsilon\}$ by x, where $\|\cdot\|_2$ is the Euclidean norm, $\varepsilon > 0$, $0 < r_{\text{in}} \le 1 \le r_{\text{out}} < \infty$, and $r_{\text{in}} \ne r_{\text{out}}$. We denote the unit sphere by S^2. Given $x, x_1 \in \mathbb{S}_\varepsilon$, etc., we often write $r_x := \|x\|_2$, $\sigma_x := \frac{x}{r_x}$, and $(r_x, \sigma_x) := r_x\sigma_x = x$ or $r_1 := \|x_1\|_2$, $\sigma_1 := \frac{x_1}{r_1}$, and $(r_1, \sigma_1) := r_1\sigma_1 = x_1$. For $f\colon \mathbb{S}_\varepsilon = [r_{\text{in}}, r_{\text{out}}] \times S^2 \to \mathbb{R}$, we often write $f(x)$, $f(r_x\sigma_x)$, $f(x_1)$, or $f(r_1\sigma_1)$ as $f(r_x, \sigma_x)$, or $f(r_1, \sigma_1)$. We use the spherical coordinate system

$$(r \sin\theta \cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) \quad (r \in [0,\infty),\ \theta \in [0,\pi],\ \varphi \in [0, 2\pi)),$$

where for $\theta \in \{0, \pi\}$ we let $\varphi = 0$. Further, we use the notation $SO(3) := \{O \in GL(3,\mathbb{R}) \mid O^\top O = 1, \det O = 1\}$ by identifying rotations and their matrix representations. For a topological space X, we denote the σ-algebra of Borel sets in X by $\mathcal{B}(X)$.

Definition 1. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $\mathcal{B}(\mathbb{S}_\varepsilon)$ be the Borel σ-algebra associated with the metric space $(\mathbb{S}_\varepsilon, \|\cdot\|_2)$. We call a function $T\colon \mathbb{S}_\varepsilon \times \Omega \to \mathbb{R}$ a (jointly measurable) *random field* on a spherical shell $\mathbb{S}_\varepsilon$ if T is $\mathcal{B}(\mathbb{S}_\varepsilon) \otimes \mathscr{F}$-measurable. Further, we say a random field T has a rotationally invariant covariance, if for every $O \in SO(3)$ we have

$$\mathbb{E}[T(r_1,\sigma_1)T(r_2,\sigma_2)] = \mathbb{E}[T(r_1, O\sigma_1)T(r_2, O\sigma_2)] \tag{2.2.1}$$

∎

For a random field $T\colon \mathbb{S}_\varepsilon \times \Omega \ni (x,\omega) \mapsto T(x;\omega) \in \mathbb{R}$, we abuse the notation slightly by writing $T(r,\sigma;\omega) := T(x;\omega)$ for $x = r\sigma$, $r \in [r_{\text{in}}, r_{\text{out}}]$, $\sigma = S^2$; and similarly, $T(x) := T(r,\sigma) := T(x;\omega)$, when the dependence on ω is unimportant.

Let $\int_{S^2} \mathrm{d}S$ be the integration with respect to the usual spherical measure normalised so that $\int_{S^2} \mathrm{d}S = 4\pi$, and let $\{\mathcal{Y}_{\ell m}\}$ be the real spherical harmonics normalised so that they form an orthonormal system of $L^2(S^2)$, where the inner product is defined with $\int_{S^2} \mathrm{d}S$. Further, let P_ℓ be the Legendre polynomial of degree $\ell \in \mathbb{N} \cup \{0\}$ normalised so that $P_\ell(1) = 1$. See, e.g., [6] for more details. Furthermore, let (φ_k) be an orthonormal system of $L^2([r_{\text{in}}, r_{\text{out}}], \mu)$ with respect to a finite Borel measure μ on $[r_{\text{in}}, r_{\text{out}}]$ such that the Lebesgue measure restricted to $\mathcal{B}([r_{\text{in}}, r_{\text{out}}])$ is absolutely continuous with respect to μ, for example, the Chebyshev measure on Borel sets. More details of μ will be discussed later.

We consider a zero mean Gaussian random field $\{T(x;\cdot)\}_{x\in\mathbb{S}_\varepsilon}$ with the rotationally invariant covariance function $\mathbb{E}[T(x)T(x')] =: \mathfrak{c}(x,x')\colon \mathbb{S}_\varepsilon \times \mathbb{S}_\varepsilon \to \mathbb{R}$ given by

$$\mathfrak{c}(x,x') := \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} A_{k\ell}\varphi_k(r_x)\varphi_k(r_{x'})\frac{2\ell+1}{4\pi}P_\ell(\sigma_x\cdot\sigma_{x'}), \quad \text{for any } x,x'\in\mathbb{S}_\varepsilon, \tag{2.2.2}$$

where $A_{k\ell}$ are non-negative coefficients such that the series is uniformly and absolutely convergent. By fixing r_x, $r_{x'}$ (resp. σ_x, $\sigma_{x'}$) we see $\mathfrak{c}(x,x')$ as a function on S^2 (resp. $[r_{\text{in}}, r_{\text{out}}]$). Correspondingly we use the notations

$$\mathfrak{c}(x,x') = \mathfrak{c}_{r_x,r_{x'}}(\sigma_x,\sigma_{x'}) = \mathfrak{c}^{\sigma_x\sigma_{x'}}(r_x,r_{x'}) = \mathfrak{c}((r_x,\sigma_x),(r_{x'},\sigma_{x'})).$$

Letting $\mathcal{C}_\ell(r_x,r_{x'}) := \sum_{k=0}^{\infty} A_{k\ell}\varphi_k(r_x)\varphi_k(r_{x'})$, we have

$$\mathfrak{c}(x,x') = \sum_{\ell=0}^{\infty}\mathcal{C}_\ell(r_x,r_{x'})\frac{2\ell+1}{4\pi}P_\ell(\sigma_x\cdot\sigma_{x'}).$$

A covariance function of this form is used in the field of geophysics. The quantity $\mathcal{C}_\ell(r_x,r_{x'})$ is often called (after a suitable normalisation) the *radial correlation function* in seismic and convection modelling, and describes the radially anisotropic structure of the interior of the Earth [49, 84, 86]. If $r_x = r^* = r_{x'}$, then $\mathcal{C}_\ell(r^*,r^*)$ defines the *depth-dependent spherical harmonics spectrum* (per degree ℓ) on the sphere $\{x\in\mathbb{R}^3 \mid \|x\|_2 = r^*\}$, which is also employed for modelling in geosciences [24, 25, 29, 38, 72, 73]. Since the functions such as $\mathfrak{c}(\cdot,\cdot)$ or $\mathcal{C}_\ell(\cdot,\cdot)$ are often estimated by suitable data [14, 24, 25, 38, 72, 84], we assume that (2.2.2) is given.

Remark 1. If $r := r_1 = r_2$, then, the notion of rotationally invariant covariance coincides with the 2-weakly isotropicity in the context of random fields on the sphere [66, 71]. ∎

It is easy to check that the kernel $\mathfrak{c}(x,x')$ is positive definite on $\mathbb{S}_\varepsilon\times\mathbb{S}_\varepsilon$. Recall that the unique (up to the law) existence of a zero mean Gaussian random field is guaranteed once we specify such a function. Indeed, we can check that $\mathfrak{c}(x,x')$ is the reproducing kernel in a reproducing kernel Hilbert space $H_{\mathfrak{c}}$ equipped with a suitable inner product that has $\{\sqrt{A_{k\ell}}\varphi_k\mathcal{Y}_{\ell m}\}_{k\ell m}$ as a complete orthonormal system. Let $\{X_{k\ell m}\}$ be a collection of independent random variables on a suitable probability space $(\Omega, \mathscr{F}, \mathbb{P})$ such that $X_{k\ell m} \sim \mathcal{N}(0,1)$, where $\mathcal{N}(a, b^2)$ denotes the normal distribution with mean a and variance b^2. Let $\widehat{T}_{k\ell m}(\omega) := \sqrt{A_{k\ell}} X_{k\ell m}(\omega)$, and

$$T(x;\omega) := \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \widehat{T}_{k\ell m}(\omega) \varphi_k(r_x) \mathcal{Y}_{\ell m}(\sigma_x), \quad x \in \mathbb{S}_\varepsilon. \tag{2.2.3}$$

Then, for each $x \in \mathbb{S}_\varepsilon$ the series is $L^2_{\mathbb{P}}(\Omega)$-norm convergent. Note that changing the order of the sum above does not change the law on $\mathbb{R}^{\mathbb{S}_\varepsilon}$ of the resulting random field. Thus, T is uniquely determined up to the law independently of the order of the sum. Further, from the $L^2_{\mathbb{P}}(\Omega)$-norm convergence, noting the independence of $\{X_{k\ell m}\}$ we have the almost sure convergence [57, Theorem 5.2]. From the construction, T is $\mathcal{B}(\mathbb{S}_\varepsilon) \otimes \mathscr{F}$-measurable.

We noted that given $\mathfrak{c}$ we can construct the corresponding Gaussian random field. The following proposition states a partial converse: assuming the joint measurability of the random field, the expansion takes the same form with some (φ_k).

Proposition 2.2.1. *(i) For a zero-mean random field T' on a spherical shell to have rotationally invariant covariance, it is sufficient that the covariance function is given by a uniform and absolutely convergent series of the form (2.2.2) with a complete orthonormal system (φ_k) for some $L^2([r_{\text{in}}, r_{\text{out}}], \mu)$ with a finite Borel measure μ on $\mathcal{B}([r_{\text{in}}, r_{\text{out}}])$.*

(ii) Assume a zero-mean random field T' has a rotationally invariant covariance, and the covariance function $\mathfrak{c}' \colon \mathbb{S}_\varepsilon \times \mathbb{S}_\varepsilon \to \mathbb{R}$ is continuous. Fix a finite Borel measure μ on $\mathcal{B}([r_{\text{in}}, r_{\text{out}}])$. Then, $\mathfrak{c}'$ can be expanded in the form (2.2.2) with a complete orthonormal system (φ_k) for $L^2([r_{\text{in}}, r_{\text{out}}], \mu)$.

Proof. The statement (i) follows from $P_\ell(\sigma_1 \cdot \sigma_2) = P_\ell(O\sigma_1 \cdot O\sigma_2)$. To show (ii), first note that by virtue of the rotational invariance, for arbitrarily fixed $r_1, r_2 \in [r_{\text{in}}, r_{\text{out}}]$ the function $\mathfrak{c}'_{r_1 r_2}$ defined by

$$\mathfrak{c}'_{r_1 r_2}(\sigma_1 \cdot \sigma_2) \colon = \mathbb{E}[T'(r_1, \sigma_1) T'(r_2, \sigma_2)], \quad \sigma_1, \sigma_2 \in S^2,$$

is well-defined as a univariate function $[-1,1] \to \mathbb{R}$. To see this, suppose $\sigma_1 \cdot \sigma_2 = \widetilde{\sigma}_1 \cdot \widetilde{\sigma}_2$. For some $O, \widetilde{O} \in SO(3)$ we have $O\sigma_2 = \sigma_{\text{north}} = \widetilde{O}\widetilde{\sigma}_2$, where

$\sigma_{\rm north} := (0,0,1)^\top$. Then, from $O\sigma_1 \cdot \sigma_{\rm north} = \sigma_1 \cdot \sigma_2 = \widetilde{O}\widetilde{\sigma}_1 \cdot \sigma_{\rm north}$, for some rotation $O^* \in SO(3)$ that fixes $\sigma_{\rm north}$ we have $O^*O\sigma_1 = \widetilde{O}\widetilde{\sigma}_1$. Thus, we have

$$\mathfrak{c}'_{r_1r_2}(\sigma_1 \cdot \sigma_2) = \mathbb{E}[T'(r_1, O^*O\sigma_1)T'(r_2, \sigma_{\rm north})] = \mathfrak{c}'_{r_1r_2}(\widetilde{\sigma}_1 \cdot \widetilde{\sigma}_2),$$

and thus $\mathfrak{c}'_{r_1r_2}(\cdot)$ depends only on the inner product $t = \sigma_1 \cdot \sigma_2$.

We show that $\mathfrak{c}'_{r_1r_2}$ is square integrable on $[-1,1]$. In this regard, we note that for $\theta \in [0,\pi]$ we can take $\sigma \in S^2$ such that $\cos\theta = \sigma \cdot \sigma_{\rm north}$. Then, noting $2\pi \int_{-1}^1 |\mathfrak{c}'_{r_1r_2}(t)|^2 \,\mathrm{d}t = \int_0^{2\pi}\!\int_0^{\pi} |\mathfrak{c}'_{r_1r_2}(\cos\theta)|^2 \sin\theta \,\mathrm{d}\theta\mathrm{d}\xi$ we have

$$\begin{aligned}
2\pi \int_{-1}^1 |\mathfrak{c}'_{r_1r_2}(t)|^2 \,\mathrm{d}t = &\int_{S^2} |\mathbb{E}[T'(r_1,\sigma)T'(r_2,\sigma_{\rm north})]|^2 \,\mathrm{d}S \\
\leq &\mathbb{E}\big[T'(r_2,\sigma_{\rm north})^2\big] \int_{S^2} \mathbb{E}\big[T'(r_1,\sigma)^2\big] \,\mathrm{d}S \\
= &4\pi\mathbb{E}\big[T'(r_2,\sigma_{\rm north})^2\big]\mathbb{E}\big[T'(r_1,\sigma_{\rm north})^2\big] < \infty,
\end{aligned}$$

where in the last line the rotational invariance is used. Thus, $\mathfrak{c}'_{r_1r_2}$ is square integrable on $[-1,1]$. Therefore, with some $\mathcal{C}_\ell(r_1,r_2)$ we have the Legendre polynomial expansion

$$\mathfrak{c}'_{r_1r_2}(\sigma_1 \cdot \sigma_2) = \sum_{\ell=0}^\infty \mathcal{C}_\ell(r_1,r_2) \frac{2\ell+1}{4\pi} P_\ell(\sigma_1 \cdot \sigma_2). \tag{2.2.4}$$

Now, consider the kernel $\mathfrak{c}'^{\sigma_1\sigma_2}(r_1,r_2) := \mathfrak{c}'((r_1,\sigma_1),(r_2,\sigma_2))$ on $[r_{\rm in}, r_{\rm out}] \times [r_{\rm in}, r_{\rm out}]$. Since $\mathfrak{c}'^{\sigma_1\sigma_2}(r_1,r_2)$ is continuous and symmetric positive definite, from Mercer's theorem (for example, [61]) there is an orthonormal system (φ_k) of $L^2_\mu([r_{\rm in}, r_{\rm out}])$ consisting of eigenfunctions of the integral operator associated with $\mathfrak{c}'^{\sigma_1\sigma_2}$ such that the corresponding eigenvalues $\lambda_k^{(\sigma_1,\sigma_2)}$ are non-negative and that

$$\mathfrak{c}'^{\sigma_1\sigma_2}(r_1,r_2) = \sum_{k=0}^\infty \lambda_k^{(\sigma_1,\sigma_2)} \varphi_k(r_1)\varphi_k(r_2), \tag{2.2.5}$$

where the series is uniformly and absolutely convergent. We observe $\lambda_k^{(\sigma_1,\sigma_2)}$ is given by

$$\lambda_k^{(\sigma_1,\sigma_2)} = \int_{r_{\rm in}}^{r_{\rm out}}\!\int_{r_{\rm in}}^{r_{\rm out}} \sum_{\ell=0}^\infty \frac{2\ell+1}{4\pi} P_\ell(\sigma_1\cdot\sigma_2)\mathcal{C}_\ell(s_1,s_2)\varphi_k(s_1)\varphi_k(s_2)\mathrm{d}\mu(s_1)\mathrm{d}\mu(s_2),$$

and that we can rewrite $\mathfrak{c}'^{\sigma_1\sigma_2}(r_1, r_2)$ as

$$\mathfrak{c}'^{\sigma_1\sigma_2}(r_1, r_2) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\left(\int_{r_{\text{in}}}^{r_{\text{out}}}\int_{r_{\text{in}}}^{r_{\text{out}}}\sum_{\ell=0}^{\infty}\mathcal{C}_\ell(s_1, s_2)\varphi_k(s_2)\varphi_k(s_1)\,\mathrm{d}\mu(s_2)\mathrm{d}\mu(s_1)\right) \times \frac{2\ell+1}{4\pi}P_\ell(\sigma_1\cdot\sigma_2)\varphi_k(r_1)\varphi_k(r_2),$$

which is the desired form. □

We use the following assumption throughout this chapter.

Assumption 2.1. We consider the Gaussian random field T, given by (2.2.3), associated with the covariance function given by (2.2.2), with a complete orthonormal system (φ_k) of $L^2_\mu([r_{\text{in}}, r_{\text{out}}])$, where μ is a finite Borel measure such that the Lebesgue measure restricted to $\mathcal{B}([r_{\text{in}}, r_{\text{out}}])$ is absolutely continuous with respect to μ. Further, $A_{k\ell}$ as in (2.2.2) are all non-negative and satisfy,

$$\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}(k+1)^{4\eta+1}(\ell+1)^{2s+1}A_{k\ell} < \infty, \tag{2.2.6}$$

for some $\eta, s \geq 0$. Furthermore, we assume (φ_k) satisfies $(\varphi_k) \subset C^{\lfloor\eta\rfloor}([r_{\text{in}}, r_{\text{out}}])$, and $\sup_{r\in[r_{\text{in}}, r_{\text{out}}]}|\varphi_k(r)| \leq c_0$ for some finite constant $c_0 > 0$. When $\eta > 1$, we assume for any $k \in \mathbb{N}$

$$\sup_{r\in[r_{\text{in}}, r_{\text{out}}]}\left|\frac{\mathrm{d}}{\mathrm{d}r}\varphi_k(r)\right| \leq c_1 k^2, \tag{2.2.7}$$

and for sufficiently large k,

$$\sup_{r\in[r_{\text{in}}, r_{\text{out}}]}\left|\frac{\mathrm{d}^n}{\mathrm{d}r^n}\varphi_k(r)\right| \leq c_n k^{2n}, \tag{2.2.8}$$

where $n \in \{0, \ldots, \lfloor\eta\rfloor\}$, and $c_1, c_n > 0$ are some fixed constants. ▮

Here, $\eta, s \geq 0$ are parameters that correspond to the smoothness of the covariance functions, and in turn, realisations of T; the parameter η is related to the smoothness of the covariance function (2.2.2), and in turn, the realisations of T as a function in r on the interval $[r_{\text{in}}, r_{\text{out}}]$, and s is related to the smoothness as a function on S^2. In more detail, we will see that solely the decay of $A_{k\ell}$ determines the smoothness of T in the radial direction and the angular direction, and further the Sobolev smoothness.

The conditions on (φ_k) in Assumption 2.1 is motivated by geophysics. To be more precise, the following settings have been considered in applications. To treat the radial direction in the Earth modelling, the Chebyshev polynomial of the first kind $\{J_k\}$ mapped affinely to $[r_{\text{in}}, r_{\text{out}}]$ from $[-1,1]$ is often considered [14, 43, 49, 86, 92, 93], in particular, in the random field setting [84], and $\{J_k\}$ satisfies the above assumption. See, for example [34, A.7]. In the Chebyshev case μ is the Chebyshev measure $\int_\cdot \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$ on $[-1,1]$ mapped affinely to $[r_{\text{in}}, r_{\text{out}}]$. We remark that the assumed uniform boundedness of (φ_k) in k and the decay rate of the derivative above is merely for simplicity, and not essential. For functions (φ_k) with different bounds, it suffices to adjust the condition (2.2.6) above accordingly, together with modifying proofs that utilise the boundedness of (φ_k) independently of k, for example, Proposition 2.3.2, Proposition 2.3.3 or Proposition 2.4.6.

2.3 Continuity of realisations

We start with establishing the continuity of the realisations. We need the following lemma.

Lemma 2.3.1. *For any $x_1, x_2 \in \mathbb{S}_\varepsilon$, we have*

$$\arccos(\sigma_1 \cdot \sigma_2) + |r_2 - r_1| \leq c_{\text{met}} \|x_1 - x_2\|_2 \,,$$

where $c_{\text{met}} = \frac{\pi}{\sqrt{2}r_{\text{in}}}$.

Proof. Note that we have $1 - \cos\theta_0 \geq \frac{2}{\pi^2}\theta_0^2$, for any $\theta_0 \in [0,\pi]$. Thus, letting $\theta := \arccos(\sigma_1 \cdot \sigma_2) \in [0,\pi]$, we have

$$\|x_1 - x_2\|_2^2 = (r_1 - r_2)^2 + 2r_1 r_2(1 - \cos\theta) \geq {r_{\text{in}}}^2 \frac{4}{\pi^2}\frac{1}{2}[|r_1 - r_2| + \theta]^2, \tag{2.3.1}$$

where we used $1 \geq {r_{\text{in}}}^2 \frac{4}{\pi^2}$. Therefore, $|r_1 - r_2| + \theta \leq \frac{\pi}{\sqrt{2}r_{\text{in}}} \|x_1 - x_2\|_2$. □

Proposition 2.3.2. *Suppose Assumption 2.1 holds with $s \geq \frac{1}{2}$ and $\eta > 1$. Further, let T be the corresponding random field as in (2.2.3). Then, there exist constants $\delta_1, \delta_2 > 0$ and $c = c_{s,\eta} > 0$ such that for any $x_1, x_2 \in \mathbb{S}_\varepsilon$ we have*

$$\mathbb{E}\big[|T(x_1) - T(x_2)|^{\delta_1}\big] \leq c \|x_1 - x_2\|_2^{3+\delta_2}. \tag{2.3.2}$$

Further, T has a continuous modification.

Remark 2. In the following, taking a continuous modification if necessary, we see each realisation $T(\cdot\,;\omega)$ as a continuous function on $\mathbb{S}_\varepsilon$. ▮

Proof of Proposition 2.3.2. Clearly, we have

$$\begin{aligned}&\mathbb{E}\big[|T(r_1,\sigma_1)-T(r_2,\sigma_2)|^2\big]\\&\quad\le 2\mathbb{E}\big[|T(r_1,\sigma_1)-T(r_1,\sigma_2)|^2\big]+2\mathbb{E}\big[|T(r_1,\sigma_2)-T(r_2,\sigma_2)|^2\big].\end{aligned}\tag{2.3.3}$$

First, since

$$\mathbb{E}\big[|T(r_1,\sigma_1)-T(r_1,\sigma_2)|^2\big]=\mathbb{E}[T(r_1,\sigma_1)^2]+\mathbb{E}[T(r_1,\sigma_2)^2]-2\mathbb{E}[T(r_1,\sigma_1)T(r_1,\sigma_2)],$$

from (2.2.2) we have

$$\mathbb{E}\big[|T(r_1,\sigma_1)-T(r_1,\sigma_2)|^2\big]=\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}A_{k\ell}[\varphi_k(r_1)]^2\frac{2\ell+1}{2\pi}\Big[1-P_\ell(\sigma_1\cdot\sigma_2)\Big],$$

noting that the above series is uniformly and absolutely convergent Assumption 2.1. From the proof of [66, Lemma 4.2], for an arbitrary $\gamma\in[0,1]$ we have $|1-P_\ell(\sigma_1\cdot\sigma_2)|\le 2\arccos(\sigma_1\cdot\sigma_2)^{2\gamma}(2\ell+1)^{2\gamma}$. Hence, for $\gamma_1\in[0,\min\{1,s\}]$ we have

$$\begin{aligned}&2\mathbb{E}\big[|T(r_1,\sigma_1)-T(r_1,\sigma_2)|^2\big]\\&\qquad\le c_0\left[\frac{2}{\pi}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}A_{k\ell}(2\ell+1)^{2\gamma_1+1}\right]\arccos(\sigma_1\cdot\sigma_2)^{2\gamma_1}<\infty.\end{aligned}\tag{2.3.4}$$

Next, similarly to the above we have

$$\mathbb{E}\big[|T(r_1,\sigma_2)-T(r_2,\sigma_2)|^2\big]\le\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}A_{k\ell}\frac{2\ell+1}{4\pi}\big[\varphi_k(r_1)-\varphi_k(r_2)\big]^2.\tag{2.3.5}$$

Assumption 2.1 implies

$$|\varphi_k(r_1)-\varphi_k(r_2)|\le 2c_0,\quad\text{and}\quad|\varphi_k(r_1)-\varphi_k(r_2)|\le c_1k^2|r_1-r_2|.\tag{2.3.6}$$

Thus, for an arbitrary $\gamma\in[0,1]$ using these two bounds we have

$$|\varphi_k(r_1)-\varphi_k(r_2)|\le(2c_0)^{1-\gamma}c_1^{\gamma}k^{2\gamma}|r_1-r_2|^{\gamma}.\tag{2.3.7}$$

Therefore, from (2.3.5) and (2.3.7) for $\gamma_2 \in [0, \min\{1, \eta + 1/4\}]$ we have

$$\begin{aligned} &2\mathbb{E}\left[|T(r_1,\sigma_2) - T(r_2,\sigma_2)|^2\right] \\ &\quad\le 2^{2-2\gamma_2} c_0^{2-2\gamma_2} c_1^{2\gamma_2} \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} A_{k\ell}(2\ell+1)k^{4\gamma_2}|r_1 - r_2|^{2\gamma_2} < \infty. \end{aligned} \tag{2.3.8}$$

Now, let $\gamma_1 = \gamma_2 := \frac{\beta}{2} := \min\{1, s, \eta + \frac{1}{4}\}$. Then, with a finite constant

$$c_{\text{pol}} := \max\Big\{\frac{2c_0}{\pi}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} A_{k\ell}(2\ell+1)^{\beta+1}, 4c_0^{2-\beta}c_1^{\beta}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} A_{k\ell}(2\ell+1)k^{2\beta}\Big\},$$

from (2.3.3), (2.3.4), and (2.3.8) together with Lemma 2.3.1 we have

$$\mathbb{E}\left[|T(x_1) - T(x_2)|^2\right] \le c_{\text{pol}} c_{\text{met}}^{\beta} \left\|x_1 - x_2\right\|_2^{\beta}. \tag{2.3.9}$$

Since $T(x_1) - T(x_2) \sim \mathcal{N}(0, \varsigma_*^2)$ with $\varsigma_*^2 := \mathbb{E}\left[|T(x_1) - T(x_2)|^2\right]$, we observe that, with $X_{\text{std}} \sim \mathcal{N}(0,1)$ we have for any $m \in \mathbb{N}$

$$\mathbb{E}[|T(x_1) - T(x_2)|^{2m}] = \varsigma_*^{2m}\mathbb{E}[|X_{\text{std}}|^{2m}] = C_{m,\beta}\left\|x_1 - x_2\right\|_2^{2m\beta}, \tag{2.3.10}$$

where $C_{m,\beta} := (c_{\text{pol}}c_{\text{met}}^{\beta})^{2m}\mathbb{E}[|X_{\text{std}}|^{2m}]$. Taking $m > \frac{3}{2\beta}$, we have (2.3.2), with $\delta_1 := 2m$, and $\delta_2 := 2\beta m - 3$.

As a consequence of the estimate (2.3.2), the existence of a continuous modification follows from a variant of Kolmogorov–Totoki's theorem. See, for example, [62, Theorem 4.1]. □

We introduce the space of equivalence classes of Borel measurable functions $L^2_\mu(\mathbb{S}^\circ_\varepsilon) := \Big\{f\colon \mathbb{S}^\circ_\varepsilon \to \mathbb{R} \,\Big|\, \|f\|_{L^2_\mu(\mathbb{S}^\circ_\varepsilon)} < \infty\Big\} / \sim$, where we let $\mathbb{S}^\circ_\varepsilon$ denote the interior of $\mathbb{S}_\varepsilon$. We let $\int_{\mathbb{S}_\varepsilon} \mathrm{d}\mathbb{S}_\varepsilon$ denote the integral with respect to the product measure defined by μ on $[r_{\text{in}}, r_{\text{out}}]$ and the surface measure on S^2. Then, we define

$$\|f\|^2_{L^2_\mu(\mathbb{S}^\circ_\varepsilon)} := \int_{\mathbb{S}_\varepsilon} |f|^2 \,\mathrm{d}\mathbb{S}_\varepsilon = \int_{S^2}\int_{r_{\text{in}}}^{r_{\text{out}}} |f(r_x,\sigma_x)|^2 \mathrm{d}\mu(r_x)\mathrm{d}S(\sigma_x),$$

and $f \sim g \iff \|f - g\|_{L^2_\mu(\mathbb{S}^\circ_\varepsilon)} = 0$. In this chapter, because of the product structure of the domain—the radial interval and the sphere—unless we need to see $L^2_\mu(\mathbb{S}^\circ_\varepsilon)$ as a normed space, it is often convenient to see $L^2_\mu(\mathbb{S}^\circ_\varepsilon)$ as a collection of product measurable functions that are square integrable, without quotienting. Since our interest is often in a specific representative T, for simplicity we abuse the notation by using the same notation $L^2_\mu(\mathbb{S}^\circ_\varepsilon)$ for this set. Then, given $f \in$

$L^2_\mu(\mathbb{S}^\circ_\varepsilon)$ for $r \in [r_{\text{in}}, r_{\text{out}}]$ the r-section $f(r, \cdot)$, and for $\sigma \in S^2$ the σ-section $f(\cdot, \sigma)$ of f are well defined.

We need the following result for the later analysis.

Proposition 2.3.3. *Under Assumption 2.1, we have* $\mathbb{E}[\|T\|^2_{L^2_\mu(\mathbb{S}^\circ_\varepsilon)}] < \infty$.

Proof. Note that T is $\mathcal{B}(\mathbb{S}_\varepsilon) \otimes \mathscr{F}$-measurable. From

$$\int_{\mathbb{S}_\varepsilon} \mathbb{E}[|T(x)|^2] \, \mathrm{d}\mathbb{S}_\varepsilon \le c_0^2 \int_{S^2} \int_{r_{\text{in}}}^{r_{\text{out}}} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} A_{k\ell} \frac{(2\ell+1)}{4\pi} \mathrm{d}\mu(r) \, \mathrm{d}S < \infty,$$

we have $\mathbb{E}[\|T\|^2_{L^2_\mu(\mathbb{S}^\circ_\varepsilon)}] = \int_{\mathbb{S}_\varepsilon} \mathbb{E}[|T(x)|^2] \, \mathrm{d}\mathbb{S}_\varepsilon < \infty$. □

Remark 3. We saw in the discussion after (2.2.3) that the series (2.2.3) is convergent for each $x \in \mathbb{S}_\varepsilon$, almost surely. As a consequence of the previous proposition, almost surely we have $\|T\|^2_{L^2_\mu(\mathbb{S}^\circ_\varepsilon)} < \infty$, and thus from Fubini–Tonelli we have the $L^2_\mu(\mathbb{S}^\circ_\varepsilon)$-Fourier coefficient

$$\int_{\mathbb{S}_\varepsilon} T(r, \sigma; \omega) \varphi_{k'}(r) \mathcal{Y}_{\ell' m'}(\sigma) \, \mathrm{d}\mathbb{S}_\varepsilon = \widehat{T}_{k'\ell'm'}(\omega). \tag{2.3.11}$$

Thus, we see that

$$\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \widehat{T}_{k\ell m}(\omega) \varphi_k(r_x) \mathcal{Y}_{\ell,m}(\sigma_x) \tag{2.3.12}$$

is convergent in the $L^2_\mu(\mathbb{S}^\circ_\varepsilon)$-sense as well, almost surely. ∎

Proposition 2.3.4. *(i) Let* $\widehat{T}_k^{\text{rad}}(\sigma; \omega) := \int_{r_{\text{in}}}^{r_{\text{out}}} T(r, \sigma; \omega) \varphi_k(r) \mathrm{d}\mu(r)$. *Then, almost surely we have*

$$\widehat{T}_k^{\text{rad}}(\sigma; \omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \widehat{T}_{k\ell m}(\omega) \mathcal{Y}_{\ell m}(\sigma) \ \textit{in} \quad L^2(S^2), \ \textit{and } S^2\textit{-a.e.}$$

(ii) Let $\widehat{T}_{\ell m}^{\text{ang}}(r; \omega) := \int_{S^2} T(r, \sigma; \omega) \mathcal{Y}_{\ell m}(\sigma) \, \mathrm{d}S$. *Then, almost surely we have*

$$\widehat{T}_{\ell m}^{\text{ang}}(r; \omega) = \sum_{k=0}^{\infty} \widehat{T}_{k\ell m}(\omega) \varphi_k(r) \ \textit{in} \quad L^2_\mu([r_{\text{in}}, r_{\text{out}}]), \ \textit{and } [r_{\text{in}}, r_{\text{out}}]\textit{-a.e.}$$

Proof. We first claim that $\widehat{T}_k^{\mathrm{rad}}(\cdot;\omega) \in L^2(S^2)$ almost surely. Indeed, from Jensen's inequality we have almost surely

$$\int_{S^2} \left|\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega)\right|^2 \mathrm{d}S \le \mu(|r_{\mathrm{out}} - r_{\mathrm{in}}|) \int_{\mathbb{S}_\varepsilon} |T(r,\sigma;\omega)|^2 |\varphi_k(r)|^2 \,\mathrm{d}\mathbb{S}_\varepsilon < \infty.$$

Therefore, from (2.3.11) we have

$$\begin{aligned}
\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega) &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\langle \widehat{T}_k^{\mathrm{rad}}(\sigma;\omega), \mathcal{Y}_{\ell m} \right\rangle_{L^2(S^2)} \mathcal{Y}_{\ell m}(\sigma) \\
&= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\langle \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} T(r,\cdot;\omega)\varphi_k(r)\,\mathrm{d}\mu(r), \mathcal{Y}_{\ell m} \right\rangle_{L^2(S^2)} \mathcal{Y}_{\ell m}(\sigma) \\
&= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \widehat{T}_{k\ell m}(\omega)\mathcal{Y}_{\ell,m}(\sigma),
\end{aligned}$$

in the $L^2(S^2)$-sense. Further, we claim that almost surely the convergence of the series $\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \widehat{T}_{k\ell m}(\omega)\mathcal{Y}_{\ell,m}(\sigma)$ is S^2-a.e. point-wise. To see this, for any $L \in \mathbb{N}$ from the independence of $\{\widehat{T}_{k\ell m}(\omega)\}$ we have

$$\mathbb{E}\left[\left(\sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \widehat{T}_{k\ell m}(\omega)\mathcal{Y}_{\ell,m}(\sigma)\right)^2\right] = \sum_{\ell=0}^{L} \frac{2\ell+1}{4\pi} A_{k\ell}, \qquad \text{for any } \sigma \in S^2.$$

Thus, $\sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \widehat{T}_{k\ell m}(\omega)\mathcal{Y}_{\ell,m}(\sigma)$ is convergent in $L^2_{\mathbb{P}}(\Omega)$, and thus by virtue of the independence, almost surely. Since they have the same Fourier coefficients, they define the same $L^2(S^2)$ function, and thus $\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \widehat{T}_{k\ell m}(\omega)\mathcal{Y}_{\ell,m}(\sigma)$, which completes the proof of the first claim.

The second assertion can be checked similarly, noting the absolutely continuity of the Borel measure on $[r_{\mathrm{in}}, r_{\mathrm{out}}]$ with respect to μ. $\square$

Remark 4. From Remark 2, for each ω we have $\sup_{x\in\mathbb{S}_\varepsilon} |T(r_x,\sigma_x;\omega)\varphi_k(r_x)| < \infty$. Thus, the integral $\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega) = \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} T(r,\sigma;\omega)\varphi_k(r)\mathrm{d}\mu(r)$ parametrised by σ is continuous in σ for each ω. Similarly, $\widehat{T}_{\ell m}^{\mathrm{ang}}(r;\omega)$ is continuous in r for each ω.

2.4 Smoothness of the realisations

2.4.1 Partial differentiability of realisations

In this section, we discuss how the decay of $A_{k\ell}$ as in the covariance function (2.2.2) controls the partial differentiability of the realisations of T. We need the following result.

Lemma 2.4.1. *Suppose Assumption 2.1 holds. Let $\widehat{T}_k^{\mathrm{rad}}$ and $\widehat{T}_{\ell m}^{\mathrm{ang}}$ be defined as in Proposition 2.3.4. Then, for each $\sigma \in S^2$, we have*

$$\mathbb{E}\left[\sum_{k=0}^{\infty}(k+1)^{4\eta+1}\left|\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega)\right|^2\right] = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}(k+1)^{4\eta+1}A_{k\ell}\frac{2\ell+1}{4\pi}, \tag{2.4.1}$$

Further, for each $r \in [r_{\mathrm{in}}, r_{\mathrm{out}}]$ we have

$$\mathbb{E}\left[\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(\ell+1)^{2s}\left|\widehat{T}_{\ell m}^{\mathrm{ang}}(r;\omega)\right|^2\right] = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}(2\ell+1)(\ell+1)^{2s}A_{k\ell}\varphi_k(r)^2, \tag{2.4.2}$$

Proof. We first observe

$$\mathbb{E}\left[\left|\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega)\right|^2\right] = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}A_{k\ell}\mathcal{Y}_{\ell m}(\sigma)^2, \tag{2.4.3}$$

which follows from the definition of $\widehat{T}_k^{\mathrm{rad}}$, the orthogonality of (φ_k), and (2.2.2) together with Fubini–Tonelli's theorem. Further, note that from (2.4.3) the monotone convergence theorem yields

$$\mathbb{E}\left[\sum_{k=0}^{\infty}(k+1)^{4\eta+1}\left|\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega)\right|^2\right] = \sum_{k=0}^{\infty}(k+1)^{4\eta+1}\left(\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}A_{k\ell}\mathcal{Y}_{\ell m}(\sigma)^2\right),$$

and thus the addition theorem implies the first assertion.

The second claim can be shown following the same argument. □

Proposition 2.4.2. *Suppose Assumption 2.1 holds with $\eta > 1$ such that $\eta > \lfloor \eta \rfloor$, and $s \geq 0$. Then, $T(r,\sigma;\omega)$ is $\lfloor \eta \rfloor$-times continuously partial differentiable with respect to r on $[r_{\mathrm{in}}, r_{\mathrm{out}}]$, for each $\sigma \in S^2$, a.s.*

Proof. First, we show the series $\sum_{k=0}^{\infty}\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega)\,\varphi_k(r)$ is uniformly convergent as a function of r for each $\sigma \in S^2$, and that the limit T^* is $\lfloor \eta \rfloor$-times continuously differentiable in each σ, almost surely.

From Lemma 2.4.1, we have $\sum_{k=0}^{\infty}(k+1)^{4\eta+1}\left|\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega)\right|^2 < \infty$, a.s. Therefore, almost surely

$$\sum_{k=0}^{\infty}(k+1)^{2\lfloor\eta\rfloor}\left|\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega)\right| \le \sqrt{C_\eta \sum_{k=0}^{\infty}(k+1)^{4\eta+1}\left|\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega)\right|^2} < \infty, \tag{2.4.4}$$

where $C_\eta := \sum_{k=0}^{\infty}\frac{1}{(k+1)^{4\eta-4\lfloor\eta\rfloor+1}} < \infty$. From (2.2.8) for sufficiently large k, say $k \ge k_0$, we have

$$\left|\frac{\partial^{\lfloor\eta\rfloor}}{\partial r^{\lfloor\eta\rfloor}}\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega)\,\varphi_k(r)\right| \le c_{\lfloor\eta\rfloor}(k+1)^{2\lfloor\eta\rfloor}\left|\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega)\right|. \tag{2.4.5}$$

Thus, from (2.4.4), for any $r \in [r_{\mathrm{in}}, r_{\mathrm{out}}]$ we have

$$\left|\frac{\partial^{\lfloor\eta\rfloor}}{\partial r^{\lfloor\eta\rfloor}}\left(\sum_{k=0}^{\infty}\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega)\,\varphi_k(r)\right)\right| = \left|\sum_{k=0}^{\infty}\frac{\partial^{\lfloor\eta\rfloor}}{\partial r^{\lfloor\eta\rfloor}}\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega)\,\varphi_k(r)\right|.$$

Hence, for any $r \in [r_{\mathrm{in}}, r_{\mathrm{out}}]$ it follows that almost surely

$$\left|\sum_{k=0}^{\infty}\frac{\partial^{\lfloor\eta\rfloor}}{\partial r^{\lfloor\eta\rfloor}}\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega)\,\varphi_k(r)\right| \le C\sum_{k=k_0}^{\infty}(k+1)^{2\lfloor\eta\rfloor}\left|\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega)\right| < \infty. \tag{2.4.6}$$

Therefore, by the Weierstrass M-test the series $\frac{\partial^{\lfloor\eta\rfloor}}{\partial r^{\lfloor\eta\rfloor}}\sum_{k=0}^{\infty}\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega)\,\varphi_k(r)$ is continuous in r almost surely.

Since T is the $L^2_\mu([r_{\mathrm{in}}, r_{\mathrm{out}}])$-limit of the series with the same Fourier coefficients, we have $T = \sum_{k=0}^{\infty}\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega)\,\varphi_k(r)$, $[r_{\mathrm{in}}, r_{\mathrm{out}}]$-a.e., where the right hand side is the uniformly convergent limit. In view of Proposition 2.3.2, from the continuity of T the equality holds everywhere on $[r_{\mathrm{in}}, r_{\mathrm{out}}]$, which is the desired result. □

Fix $r \in [r_{\mathrm{in}}, r_{\mathrm{out}}]$, and let $S^2(r)$ be the sphere with radius r, and $L^2(S^2(r))$ be the corresponding L^2 space with the spherical measure normalised as $\int_{S^2(r)} \mathrm{d}S(r) = 4\pi r^2$. By letting $\tilde{f}(x/\|x\|_2) := f(rx/\|x\|_2)$, any square integrable function $f\colon S^2(r) \to \mathbb{R}$ can be seen as a function on S^2.

Now, we note $\{\frac{1}{r}\mathcal{Y}_{\ell m}\}$ is a complete orthonormal system for $L^2(S^2(r))$. Further we note $\|f\|_{L^2(S^2(r))} = r\|\tilde{f}\|_{L^2(S^2)}$ and $\langle f, \frac{1}{r}\mathcal{Y}_{\ell m}\rangle_{L^2(S^2(r))} = r\langle \tilde{f}, \mathcal{Y}_{\ell m}\rangle_{L^2(S^2)}$. We abuse the notation slightly by writing these as $\|f\|_{L^2(S^2(r))} = r\|f\|_{L^2(S^2)}$, and $\langle f, \frac{1}{r}\mathcal{Y}_{\ell m}\rangle_{L^2(S^2(r))} = r\langle f, \mathcal{Y}_{\ell m}\rangle_{L^2(S^2)}$. Furthermore, we note that the Laplace–Beltrami operator $\Delta_{S^2(r)}$ on $S^2(r)$ with the round metric with radius r is given by $\Delta_{S^2(r)} = \frac{1}{r^2}\Delta_{S^2}$, so that $-\Delta_{S^2(r)}\frac{1}{r}\mathcal{Y}_{\ell m} = \frac{1}{r^2}\ell(\ell+1)\frac{1}{r}\mathcal{Y}_{\ell m}$.

We define $H^s(S^2(r))$ as the domain of $(-4\Delta_{S^2(r)}+I)^{\frac{s}{2}}\colon L^2(S^2(r)) \to L^2(S^2(r))$ $(s \geq 0)$: let $H^s(S^2(r)) := \{f \in L^2(S^2(r)) \mid \|f\|_{H^s(S^2(r))} < \infty\}$ with

$$\|f\|^2_{H^s(S^2(r))} := \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Big(\frac{4\ell^2}{r^2} + \frac{4\ell}{r^2} + 1\Big)^s \langle f, 1/r\mathcal{Y}_{\ell m}\rangle^2_{L^2(S^2(r))} \tag{2.4.7}$$

$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(4\ell^2 + 4\ell + r^2)^s \frac{1}{r^{2s}}\langle f, 1/r\mathcal{Y}_{\ell m}\rangle^2_{L^2(S^2(r))} \tag{2.4.8}$$

$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(2\ell + r)^{2s}\frac{1}{r^{2s-2}}\langle f, \mathcal{Y}_{\ell m}\rangle^2_{L^2(S^2)}. \tag{2.4.9}$$

Note that the norms $\|f\|_{H^s(S^2(r_1))}$ and $\|f\|_{H^s(S^2(r_2))}$ for $r_1, r_2 \in [r_{\text{in}}, r_{\text{out}}]$ are equivalent, where the functions are seen to be defined on the same sphere. In view of this equivalence, in the following we often make use of the Sobolev space $H^s(S^2)$ by seeing $f \in L^2(S^2(r))$ as a function on S^2. For more details, see for example [68] together with Appendix A.

Recall from the discussion below (2.2.3) that for any $r\sigma \in \mathbb{S}_\varepsilon$, we have $T(r,\sigma;\cdot) \in L^2_{\mathbb{P}}(\Omega)$. We record the following result regarding the path smoothness of the r-section of T.

Lemma 2.4.3. ([66, Proof of Theorem 4.6]) *Let Assumption 2.1 hold. Then, for each $r \in [r_{\text{in}}, r_{\text{out}}]$ we have $T(r,\cdot\,;\omega) \in H^s(S^2)$ almost surely.*

The following results are later used to show Sobolev smoothness and approximation results.

Proposition 2.4.4. *Suppose Assumption 2.1 holds with $\eta > 1$ such that $\eta > \lfloor\eta\rfloor$. Then,*

$$\mathbb{E}\big[\|T^{(\lfloor\eta\rfloor,0)}\|^2_{\infty,2}\big] := \mathbb{E}\bigg[\int_{S^2}\sup_{r\in[r_{\text{in}},r_{\text{out}}]}\bigg|\frac{\partial^{\lfloor\eta\rfloor}}{\partial r^{\lfloor\eta\rfloor}}T(r,\sigma)\bigg|^2 \mathrm{d}S\bigg] < \infty. \tag{2.4.10}$$

Proof. From Proposition 2.4.2 and (2.4.6), almost surely we have

$$\bigg|\frac{\partial^{\lfloor\eta\rfloor}}{\partial r^{\lfloor\eta\rfloor}}T\bigg|^2 \leq c\bigg(\sum_{k=0}^{\infty}(k+1)^{4\eta+1}\Big|\widehat{T}_k^{\text{rad}}(\sigma;\omega)\Big|^2\bigg)\sum_{k=0}^{\infty}\frac{1}{(k+1)^{4\eta-4\lfloor\eta\rfloor+1}} < \infty, \tag{2.4.11}$$

the right hand side of which does not depend on r. From Proposition 2.4.2, and Lemma 2.4.1, for each $\sigma \in S^2$ it follows that

$$\mathbb{E}\left[\sup_{r\in[r_{\text{in}},r_{\text{out}}]}\left|\frac{\partial^{\lfloor\eta\rfloor}}{\partial r^{\lfloor\eta\rfloor}}T\right|^2\right] \leq cC_\eta\left(\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}(k+1)^{4\eta+1}A_{k\ell}\frac{2\ell+1}{4\pi}\right), \tag{2.4.12}$$

where $C_\eta := \sum_{k=0}^{\infty}\frac{1}{(k+1)^{4\eta-4\lfloor\eta\rfloor+1}}$. We note that $\sup_{r\in[r_{\text{in}},r_{\text{out}}]}\left|\frac{\partial^{\lfloor\eta\rfloor}}{\partial r^{\lfloor\eta\rfloor}}T(r,\sigma;\omega)\right|^2$ is $\mathcal{B}(S^2)\otimes\mathscr{F}$-measurable. Therefore, we obtain

$$\begin{aligned}\mathbb{E}&\left[\int_{S^2}\sup_{r\in[r_{\text{in}},r_{\text{out}}]}\left|\frac{\partial^{\lfloor\eta\rfloor}}{\partial r^{\lfloor\eta\rfloor}}T(r,\sigma)\right|^2\mathrm{d}S\right]\\ &\leq cC_\eta\left(\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}(k+1)^{4\eta+1}A_{k\ell}(2\ell+1)\right)<\infty.\end{aligned}$$

□

With an extra condition on the parameter s, we can show a stronger result.

Proposition 2.4.5. *Suppose Assumption 2.1 holds with $\eta > 1$ such that $\eta > \lfloor\eta\rfloor$, and $s > \frac{1}{2}$. Then, we have*

$$\mathbb{E}\left[\|T^{(\lfloor\eta\rfloor,0)}\|^2_{\infty,\operatorname{ess\,sup}}\right] := \mathbb{E}\left[\operatorname*{ess\,sup}_{\sigma\in S^2}\sup_{r\in[r_{\text{in}},r_{\text{out}}]}\left|\frac{\partial^{\lfloor\eta\rfloor}}{\partial r^{\lfloor\eta\rfloor}}T\right|^2\right]<\infty. \tag{2.4.13}$$

Proof. For any $s_0 \in (\frac{1}{2}, s)$ we claim

$$\mathbb{E}\left[\operatorname*{ess\,sup}_{\sigma\in S^2}|\widehat{T}_k^{\text{rad}}(\sigma;\omega)|^2\right] \leq C_{s_0}\sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell'}(2\ell'+1)^{1+2s_0}A_{k\ell'}, \qquad \sigma\in S^2,$$

where $C_{s_0} := \frac{1}{4\pi}\sum\limits_{\ell'=0}^{\infty}\frac{1}{(2\ell'+1)^{2s_0}}$. Indeed, applying the Cauchy–Schwarz inequality twice yields

$$\begin{aligned}\left|\widehat{T}_k^{\text{rad}}(\sigma;\omega)\right| &\leq \sum_{\ell'=0}^{\infty}\left(\sum_{m'=-\ell'}^{\ell'}\frac{2\ell'+1}{4\pi}|\widehat{T}_{k\ell'm'}(\omega)|^2\right)^{\frac{1}{2}}\\ &\leq\left[\sum_{\ell'=0}^{\infty}\frac{1}{(2\ell'+1)^{2s_0}}\right]^{\frac{1}{2}}\left[\sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell'}\frac{(2\ell'+1)^{1+2s_0}}{4\pi}|\widehat{T}_{k\ell'm'}(\omega)|^2\right]^{\frac{1}{2}},\end{aligned}$$

thus $\operatorname{ess\,sup}_{\sigma_{S^2}} \left|\widehat{T}_k^{\mathrm{rad}}(\sigma;\omega)\right|^2 \le C_{s_0}\sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell'}(2\ell'+1)^{1+2s_0}|\widehat{T}_{k\ell'm'}(\omega)|^2$. From $\mathbb{E}[|\widehat{T}_{k\ell'm'}(\omega)|^2] = A_{k\ell'}$, the claim follows. From this fact and (2.4.11), we have

$$\begin{aligned}
&\mathbb{E}\left[\operatorname*{ess\,sup}_{\sigma\in S^2}\sup_{r\in[r_{\mathrm{in}},r_{\mathrm{out}}]}\left|\frac{\partial^{\lfloor\eta\rfloor}}{\partial r^{\lfloor\eta\rfloor}}T\right|^2\right] \\
&\le \left(C_{s_0}\sum_{k=0}^{\infty}(k+1)^{4\eta+1}\sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell'}(2\ell'+1)^{1+2s_0}A_{k\ell'}\right)\sum_{k=0}^{\infty}\frac{1}{(k+1)^{4\eta-4\lfloor\eta\rfloor+1}} < \infty.
\end{aligned}$$

□

The following result on the second moment of the supremum of the classical angular derivative will be used later to show the filtered approximation results.

Proposition 2.4.6. *Suppose Assumption 2.1 holds with $\eta > 1$ such that $\eta > \lfloor\eta\rfloor$, and $s > 1$. Then, for non-negative integers $0 \le t < \frac{s-1}{2}$, we have*

$$\mathbb{E}\big[\|T^{(0,2t)}\|^2_{\mathrm{ess\,sup},\infty}\big] := \mathbb{E}\left[\operatorname*{ess\,sup}_{r\in(r_{\mathrm{in}},r_{\mathrm{out}})}\sup_{\sigma\in S^2}|\Delta^t_{S^2}T(r,\sigma)|^2\right] < \infty. \tag{2.4.14}$$

Proof. It suffices to show $\mathbb{E}\big[\operatorname{ess\,sup}_{r\in(r_{\mathrm{in}},r_{\mathrm{out}})}\|T(r,\cdot;\cdot)\|^2_{H^s(S^2)}\big] < \infty$. Indeed, using the Sobolev's embedding theorem on the sphere (for example, [41]) $\Delta^t_{S^2}T$ can be shown to be continuous in σ and

$$\sup_{\sigma\in S^2}|\Delta^t_{S^2}T|^2 \le c\|T(r,\cdot;\cdot)\|^2_{H^s(S^2)}, \quad (s > 2t+1),$$

where c is independent of T. We show

$$\mathbb{E}\left[\operatorname*{ess\,sup}_{r\in(r_{\mathrm{in}},r_{\mathrm{out}})}\sup_{\sigma\in S^2}|\Delta^t_{S^2}T(r,\sigma)|^2\right] \le c\mathbb{E}\left[\operatorname*{ess\,sup}_{r\in(r_{\mathrm{in}},r_{\mathrm{out}})}\|T(r,\cdot;\cdot)\|^2_{H^s(S^2)}\right] < \infty.$$

Observe the $\mathscr{F}/\mathcal{B}(\mathbb{R})$-measurability of the integrands above.

From the definition of Sobolev norm and Parseval's identity, for each $r \in [r_{\mathrm{in}}, r_{\mathrm{out}}]$ almost surely we have

$$\begin{aligned}
\|T\|^2_{H^s(S^2)} &= \|(-4\Delta_{S^2}+1)^{\frac{s}{2}}T\|^2_{L^s(S^2)} \\
&= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left|\left\langle\sum_{\ell'=0}^{\infty}(2\ell'+1)^s\sum_{m'=-\ell'}^{\ell'}\widehat{T}^{\mathrm{ang}}_{\ell'm'}(r;\omega)\mathcal{Y}_{\ell'm'},\mathcal{Y}_{\ell m}\right\rangle_{L^2(S^2)}\right|^2 \\
&= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(2\ell+1)^{2s}\left|\widehat{T}^{\mathrm{ang}}_{\ell m}(r;\omega)\right|^2,
\end{aligned}$$

where the continuity of $\langle \cdot, \mathcal{Y}_{\ell m}\rangle_{L^2(S^2)} : L^2(S^2) \to \mathbb{R}$, and $\langle \mathcal{Y}_{\ell' m'}, \mathcal{Y}_{\ell m}\rangle_{L^2(S^2)} = \delta_{\ell\ell'}\delta_{mm'}$ are used. Hence, we have

$$\mathbb{E}\Bigg[\operatorname*{ess\,sup}_{r\in(r_{\text{in}},r_{\text{out}})} \|T\|^2_{H^s(S^2)}\Bigg] \leq \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(2\ell+1)^{2s}\mathbb{E}\Bigg[\operatorname*{ess\,sup}_{r\in(r_{\text{in}},r_{\text{out}})} \left|\widehat{T}^{\text{ang}}_{\ell m}(r;\omega)\right|^2\Bigg]. \tag{2.4.15}$$

We now obtain a bound for $\mathbb{E}\big[\operatorname{ess\,sup}_{r\in(r_{\text{in}},r_{\text{out}})} \big|\widehat{T}^{\text{ang}}_{\ell m}(r;\omega)\big|^2\big]$. From Proposition 2.3.4 almost surely we have $\widehat{T}^{\text{ang}}_{\ell m}(r;\omega) = \sum_{k'=0}^{\infty}\widehat{T}_{k'\ell m}(\omega)\varphi_{k'}(r)$. Assumption 2.1 we have $|\widehat{T}^{\text{ang}}_{\ell m}(r;\omega)|^2 \leq \big(c_0\sum_{k=0}^{\infty}|\widehat{T}_{k\ell m}(\omega)|\big)^2$, and hence

$$\mathbb{E}\Bigg[\operatorname*{ess\,sup}_{r\in(r_{\text{in}},r_{\text{out}})} |\widehat{T}^{\text{ang}}_{\ell m}(r;\omega)|^2\Bigg] \leq c_0^2\Big(\sum_{k'=0}^{\infty}\sqrt{A_{k'\ell}}\Big)^2, \tag{2.4.16}$$

where $\mathbb{E}[|\widehat{T}_{k'\ell m}|^2] = A_{k'\ell}$ is used. Clearly,

$$\sum_{k=0}^{\infty}\sqrt{A_{k\ell}} \leq \left(\sum_{k=0}^{\infty}A_{k\ell}(k+1)^{\eta-\lfloor\eta\rfloor+1}\right)^{\frac{1}{2}}\left(\sum_{k=0}^{\infty}\frac{1}{(k+1)^{\eta-\lfloor\eta\rfloor+1}}\right)^{\frac{1}{2}} < \infty.$$

This bound, together with (2.4.15) and (2.4.16), yields

$$\begin{aligned}&\mathbb{E}\Bigg[\operatorname*{ess\,sup}_{r\in(r_{\text{in}},r_{\text{out}})} \|T\|^2_{H^s(S^2)}\Bigg]\\ \leq& c_0^2\left(\sum_{k=0}^{\infty}\frac{1}{(k+1)^{\eta-\lfloor\eta\rfloor+1}}\right)\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}(2\ell+1)^{2s+1}(k+1)^{\eta-\lfloor\eta\rfloor+1}A_{k\ell} < \infty,\end{aligned}$$

which completes the proof. □

2.4.2 Sobolev smoothness of realisations

In this section we show that the realisations of T are in a suitable Sobolev space consisting of (equivalence classes of) functions on $\mathbb{S}^{\circ}_{\varepsilon}$. We often use results regarding derivatives on spherical shells in the deterministic setting, which are provided in the appendix. We start with the following partial differentiability of the realisations of T.

Lemma 2.4.7. *Suppose Assumption 2.1 holds with $\eta > 1$ such that $\eta > \lfloor\eta\rfloor$, $s > 0$, and let $\alpha \in \{0,\ldots,\lfloor\eta\rfloor\}$, $\beta \in \{0,\ldots,\lfloor\frac{s}{2}\rfloor\}$. Then, $\int_{S^2}\big|\Delta^{\beta}_{S^2}\frac{\partial^\alpha}{\partial r^\alpha}T\big|^2\,\mathrm{d}S$*

is continuous in r on $[r_{\text{in}}, r_{\text{out}}]$ for all $\omega \in \Omega$, redefining the probability space if necessary, and we have

$$\mathbb{E}\Big[\sup_{r\in[r_{\text{in}},r_{\text{out}}]} \int_{S^2} \Big|\Delta_{S^2}^{\beta} \frac{\partial^\alpha}{\partial r^\alpha} T\Big|^2 \mathrm{d}S\Big] < \infty.$$

In particular, $\mathbb{E}\Big[\Big\|\Delta_{S^2}^{\beta}\frac{\partial^\alpha}{\partial r^\alpha} T\Big\|^2_{L^2(\mathbb{S}^{\circ}_{\varepsilon})}\Big] < \infty$.

Proof. Fix $\alpha \in \{0, \dots, \lfloor \eta \rfloor\}$ and $\beta \in \{0, \dots, \lfloor \frac{s}{2} \rfloor\}$. For any fixed $r \in [r_{\text{in}}, r_{\text{out}}]$, we have

$$\int_{S^2} \Big|\Delta_{S^2}^{\beta}\frac{\partial^\alpha}{\partial r^\alpha} T(r,\sigma;\omega)\Big|^2 \mathrm{d}S = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} (\ell^2+\ell)^{2\beta} \Big|\Big\langle \frac{\partial^\alpha}{\partial r^\alpha} T(r,\cdot;\omega), \mathcal{Y}_{\ell m}\Big\rangle_{L^2(S^2)}\Big|^2.$$

Thus, in view of the discussion from the equations (2.4.5) to (2.4.6), as a uniformly convergent series we have $\frac{\partial^\alpha}{\partial r^\alpha} T(r,\sigma;\omega) = \sum_{k=0}^{\infty} \widehat{T}_k^{\text{rad}}(\sigma;\omega) \frac{\partial^\alpha}{\partial r^\alpha}\varphi_k(r)$, a.s. Now, we note that $\sum\limits_{k=0}^{\infty} \widehat{T}_k^{\text{rad}}(\sigma;\omega) \frac{\partial^\alpha}{\partial r^\alpha}\varphi_k(r)$ is also the $L^2(S^2)$-limit of the sequence

$$\{\sum_{k=0}^{K} \widehat{T}_k^{\text{rad}}(\sigma;\omega) \frac{\partial^\alpha}{\partial r^\alpha}\varphi_k(r)\}_K,$$

for each $r \in [r_{\text{in}}, r_{\text{out}}]$, a.s. Therefore,

$$\begin{aligned}
&\int_{S^2} \Big|\Delta_{S^2}^{\beta}\frac{\partial^\alpha}{\partial r^\alpha} T(r,\sigma;\omega)\Big|^2 \mathrm{d}S \\
&= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} (\ell^2+\ell)^{2\beta} \Big|\sum_{k=0}^{\infty} \Big\langle \widehat{T}_k^{\text{rad}}(\cdot;\omega) \frac{\partial^\alpha}{\partial r^\alpha}\varphi_k(r), \mathcal{Y}_{\ell m}\Big\rangle_{L^2(S^2)}\Big|^2 \\
&= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} (\ell^2+\ell)^{2\beta} \Big|\sum_{k=0}^{\infty} \Big\langle \sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell'} \widehat{T}_{k\ell'm'}(\omega)\mathcal{Y}_{\ell'm'} \frac{\partial^\alpha}{\partial r^\alpha}\varphi_k(r), \mathcal{Y}_{\ell m}\Big\rangle_{L^2(S^2)}\Big|^2 \\
&= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} (\ell^2+\ell)^{2\beta} \Big|\sum_{k=0}^{\infty}\sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell'} \Big\langle \widehat{T}_{k'\ell'm'}(\omega)\mathcal{Y}_{\ell'm'} \frac{\partial^\alpha}{\partial r^\alpha}\varphi_k(r), \mathcal{Y}_{\ell m}\Big\rangle_{L^2(S^2)}\Big|^2 \\
&= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} (\ell^2+\ell)^{2\beta} \Big|\sum_{k=0}^{\infty} \widehat{T}_{k\ell m}(\omega) \frac{\partial^\alpha}{\partial r^\alpha}\varphi_k(r)\Big|^2,
\end{aligned}$$

where in the second and third equality, from Proposition 2.3.4, we use the fact that $\widehat{T}_k^{\text{rad}}(\sigma;\omega) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \widehat{T}_{k\ell m}(\omega)\mathcal{Y}_{\ell m}(\sigma)$ is S^2-a.e. convergent, and respectively $L^2(S^2)$-convergent.

Next, from $\alpha \le \lfloor \eta \rfloor$, we observe that $h_{\ell m}(r;\omega) := \sum\limits_{k=0}^{\infty} \widehat{T}_{k\ell m}(\omega) \frac{\partial^\alpha}{\partial r^\alpha} \varphi_k(r)$ ($\ell \in \mathbb{N} \cup \{0\}$) is continuous in r on $[r_{\text{in}}, r_{\text{out}}]$ a.s. Then, we claim

$$\int_{S^2} \left| \Delta_{S^2}^{\beta} \frac{\partial^\alpha}{\partial r^\alpha} T(r,\sigma;\omega) \right|^2 \mathrm{d}S = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell^2+\ell)^{2\beta} |h_{\ell m}(r;\omega)|^2 \tag{2.4.17}$$

is continuous in r on $[r_{\text{in}}, r_{\text{out}}]$ almost surely. To see this, note that for any $r \in [r_{\text{in}}, r_{\text{out}}]$ we have

$$\begin{aligned} &\mathbb{E}\Big[\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell^2+\ell)^{2\beta} |h_{\ell m}(r;\omega)|^2 \Big] = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell^2+\ell)^{2\beta} \mathbb{E}[|h_{\ell m}(r;\omega)|^2] \\ &\le \Big(\sum_{k=0}^{\infty} \frac{1}{(k+1)^{4\lfloor \eta \rfloor - 4\eta + 1}} \Big) \Big(\sum_{\ell=0}^{\infty} (2\ell+1)(\ell^2+\ell)^{2\beta} \Big(\sum_{k=0}^{\infty} (k+1)^{4\eta+1} A_{k\ell} \Big) \Big) < \infty. \end{aligned}$$

Thus, in view of the Weierstrass M-test $\sum_{\ell=0}^{\infty} (2\ell+1)(\ell^2+\ell)^{2\beta} |h_{\ell m}(r;\omega)|^2$ is uniformly convergent, a.s. Hence, from the continuity of $h_\ell(\cdot;\omega)$ on $[r_{\text{in}}, r_{\text{out}}]$, the a.s. continuity of $\int_{S^2} \left| \Delta_{S^2}^{\beta} \frac{\partial^\alpha}{\partial r^\alpha} T(r,\sigma;\omega) \right|^2 \mathrm{d}S$ follows as claimed. Therefore,

$$\mathbb{E}\Big[\sup_{r \in [r_{\text{in}}, r_{\text{out}}]} \Big(\int_{S^2} \left| \Delta_{S^2}^{\beta} \frac{\partial^\alpha}{\partial r^\alpha} T(r,\sigma;\omega) \right|^2 \mathrm{d}S \Big) \Big]$$

is well defined, and we have

$$\begin{aligned} &\frac{1}{r_{\text{out}} - r_{\text{in}}} \mathbb{E}\Big[\int_{r_{\text{in}}}^{r_{\text{out}}} \int_{S^2} \left| \Delta_{S^2}^{\beta} \frac{\partial^\alpha}{\partial r^\alpha} T(r,\sigma;\omega) \right|^2 r^2 \mathrm{d}S \mathrm{d}r \Big] \\ &\qquad \le \mathbb{E}\Bigg[\sup_{r \in [r_{\text{in}}, r_{\text{out}}]} \Bigg(\int_{S^2} \left| \Delta_{S^2}^{\beta} \frac{\partial^\alpha}{\partial r^\alpha} T(r,\sigma;\omega) \right|^2 r^2 \mathrm{d}S \Bigg) \Bigg] < \infty. \end{aligned} \tag{2.4.18}$$

□

We now define the trace space $H^s(\partial \mathbb{S}_\varepsilon^{\mathrm{o}})$. Observe that for any $x^* \in \partial \mathbb{S}_\varepsilon^{\mathrm{o}}$, there exists $\delta > 0$ and $G \colon \mathbb{R}^2 \to \mathbb{R}$ such that—upon relabelling and reorienting the coordinates axes if necessary—

$$\mathbb{S}_\varepsilon^{\mathrm{o}} \cap \mathfrak{B}(x^*, \delta) = \{ x \in \mathfrak{B}(x^*, \delta) \mid x_3 > G(x_1, x_2) \},$$

where $\mathfrak{B}(x^*, \delta) = \{ x \in \mathbb{R}^3 \mid \|x - x^*\| \le \delta \}$. Indeed, without loss of generality suppose $x^* = (x_1^*, x_2^*, x_3^*) \in \partial \mathbb{S}_\varepsilon^{\mathrm{o}}$ is on the lower outer hemisphere, i.e., $x^* \in \{ x \in \mathbb{R} \mid x_1^2 + x_2^2 + x_3^2 = r_{\text{out}}, x_3 < 0 \}$, in which $r_{\text{out}} - x_1^2 - x_2^2 > 0$ is always satisfied. Then, for any $\delta \in (0, \min\{r_{\text{out}} - r_{\text{in}}, |x_3^*|\})$ we have $\mathbb{S}_\varepsilon^{\text{o}} \cap \mathfrak{B}(x^*, \delta) = \left\{x \in \mathfrak{B}(x^*, \delta) \,\middle|\, x_3 > -\sqrt{r_{\text{out}} - x_1^2 - x_2^2}\right\}$. Let $U_x \subset \mathfrak{B}(x, \delta_x) \subset \mathbb{R}^3$ be an open set. Then, $\bigcup_{x \in \partial\mathbb{S}_\varepsilon^{\text{o}}} U_x (\supset \partial\mathbb{S}_\varepsilon^{\text{o}})$ is an open covering of $\partial\mathbb{S}_\varepsilon^{\text{o}}$. Since $\partial\mathbb{S}_\varepsilon^{\text{o}} = S^2(r_{\text{in}}) \cup S^2(r_{\text{out}})$, where $S^2(r_{\text{in}})$ and respectively $S^2(r_{\text{out}})$ are the inner and outer spheres, is compact, we can take finite $x_1, \ldots, x_{p_{\text{in}}} \in S^2(r_{\text{in}})$ and $x_{p_{\text{in}}+1}, \ldots, x_{p_{\text{in}}+p_{\text{out}}} \in S^2(r_{\text{out}})$ such that $\partial\mathbb{S}_\varepsilon^{\text{o}} \subset \bigcup_{j=1}^{p_{\text{in}}+p_{\text{out}}} U_j$, where $U_j := U_{x_j}$. Reshaping U_x in the radial direction, if necessary, so that if $x \in S^2(r_{\text{in}})$ then $\operatorname{dist}(U_x, S^2(r_{\text{out}})) > \frac{\varepsilon}{2}$, and if $x \in S^2(r_{\text{out}})$ then $\operatorname{dist}(U_x, S^2(r_{\text{in}})) > \frac{\varepsilon}{2}$, we can take U_j so that $U_j \cap U_k = \emptyset$ if $j \in \{1, \ldots, p_{\text{in}}\}$ and $k \in \{p_{\text{in}} + 1, \ldots, p_{\text{in}} + p_{\text{out}}\}$.

With a partition of unity $(\varsigma_j)_{j=1,\ldots,p_{\text{in}}+p_{\text{out}}}$ that is subordinate to the open cover $(U_j)_{j=1,\ldots,p_{\text{in}}+p_{\text{out}}}$, we can decompose a function f on $\partial\mathbb{S}_\varepsilon^{\text{o}}$ as

$$f(x) = \sum_{j=1}^{p_{\text{in}}+p_{\text{out}}} f(x)\varsigma_j(x) \qquad (x \in \partial\mathbb{S}_\varepsilon^{\text{o}}). \tag{2.4.19}$$

Let $Q := \left\{y = (y_1, y_2, y_3) \in \mathbb{R}^3 \mid \sqrt{y_1^2 + y_2^2} < 1,\ -1 < y_3 < 1\right\}$. Since $\partial\mathbb{S}_\varepsilon^{\text{o}}$ is an C^∞-boundary, for $j = 1, \ldots, p_{\text{in}} + p_{\text{out}}$ we have a C^∞-mapping $\Psi_j \colon \mathbb{R}^3 \supset U_j \to Q \subset \mathbb{R}^3$ that straightens out the boundary, with the inverse $\Psi_j^{-1} \colon Q \to U_j$, which is also a C^∞-mapping. Define the function

$$\begin{aligned} \Psi_j^*(f\varsigma_j) \colon Q \cap \{y \in \mathbb{R}^3 \mid y_3 = 0\} \to{}& \mathbb{R} \\ : \qquad\qquad (y_1, y_2, 0) \mapsto{}& f(\Psi_j^{-1}(y_1, y_2, 0))\, \varsigma_j(\Psi_j^{-1}(y_1, y_2, 0)), \end{aligned}$$

then $\Psi_j^*(f\varsigma_j)$ has a compact support in $Q \cap \{y \in \mathbb{R}^3 \mid y_3 = 0\}$. Let

$$H^s(\partial\mathbb{S}_\varepsilon^{\text{o}}) := \{f \in L^2(\partial\mathbb{S}_\varepsilon^{\text{o}}) \mid \Psi_j^*(f\varsigma_j) \in H^s(\mathbb{R}^2),\ j = 1, \ldots, p_{\text{in}} + p_{\text{out}}\}.$$

Further, define the norm by $\|f\|_{H^s(\partial\mathbb{S}_\varepsilon^{\text{o}})} := \left(\sum_{j=1}^{p_{\text{in}}+p_{\text{out}}} \left\|\Psi_j^*(f\varsigma_j)\right\|_{H^s(\mathbb{R}^2)}^2\right)^{\frac{1}{2}}$. We have $\|f\|_{H^s(\partial\mathbb{S}_\varepsilon^{\text{o}})}^2 = \sum_{j=1}^{p_{\text{in}}} \|\Psi_j^*(f\varsigma_j)\|_{H^s(\mathbb{R}^2)}^2 + \sum_{j=p_{\text{in}}+1}^{p_{\text{in}}+p_{\text{out}}} \|\Psi_j^*(f\varsigma_j)\|_{H^s(\mathbb{R}^2)}^2$, but since $U_j \cap U_k = \emptyset$ if $j \in \{1, \ldots, p_{\text{in}}\}$ and $k \in \{p_{\text{in}} + 1, \ldots, p_{\text{in}} + p_{\text{out}}\}$, each term is equivalent to $\left\|f|_{S^2(r_{\text{in}})}\right\|_{H^s(S^2(r_{\text{in}}))}^2$ and respectively $\left\|f|_{S^2(r_{\text{out}})}\right\|_{H^s(S^2(r_{\text{out}}))}^2$ as in (2.4.7). See [68, Section 1.7.3]. Therefore, we have the equivalence

$$\|f\|_{H^s(\partial\mathbb{S}_\varepsilon^{\text{o}})} \sim \left(\left\|f|_{S^2(r_{\text{in}})}\right\|_{H^s(S^2(r_{\text{in}}))}^2 + \left\|f|_{S^2(r_{\text{out}})}\right\|_{H^s(S^2(r_{\text{out}}))}^2\right)^{\frac{1}{2}}. \tag{2.4.20}$$

For further details of trace spaces, see, e.g., [30, 68].

Our first goal in this section is to show $\Delta^{\iota} T \in H^1(\mathbb{S}_\varepsilon^{\circ})$ with suitable integers ι. We start with the following lemma.

Lemma 2.4.8. *Suppose Assumption 2.1 is satisfied with $\eta > 1$ such that $\eta > \lfloor \eta \rfloor$. Let ι, ι', and ι'' be non-negative integers. If ι satisfies $\iota \le \min\{\lfloor \frac{\eta}{2} \rfloor, \frac{s}{2}\} - 1$, then we have $\mathbb{E}\big[\, \|\Delta^{\iota+1} T\|^2_{L^2(\mathbb{S}_\varepsilon^{\circ})}\big] < \infty$. Further, if $\iota' \le \min\{\lfloor \frac{\eta}{2} \rfloor, \frac{s}{2} - \frac{1}{4}\}$, we have*

$$\mathbb{E}\Big[\Big\| \Delta^{\iota'} T|_{\partial \mathbb{S}_\varepsilon^{\circ}} \Big\|^2_{H^{\frac{1}{2}}(\partial \mathbb{S}_\varepsilon^{\circ})}\Big] < \infty. \tag{2.4.21}$$

Furthermore, if $\iota'' \le \min\big\{\lfloor \frac{\eta}{2} \rfloor, \frac{s}{2} - \frac{3}{4}\big\}$, then we have

$$\mathbb{E}\Big[\, \Big\| \Delta^{\iota''-j} T|_{\partial \mathbb{S}_\varepsilon^{\circ}} \Big\|^2_{H^{2j+\frac{3}{2}}(\partial \mathbb{S}_\varepsilon^{\circ})}\Big] < \infty, \quad \textit{for } j = 0, \ldots, \iota''. \tag{2.4.22}$$

Proof. The first assertion follows from Proposition A.11 and Lemma 2.4.7. To show the second assertion, first we claim that $\Delta^{\iota'} T|_{\partial \mathbb{S}_\varepsilon^{\circ}}$ is well-defined as an $L^2(\partial \mathbb{S}_\varepsilon^{\circ})$-function. To see this, we note that in view of Proposition A.11 the operator $\Delta^{\iota} \colon L^2(\mathbb{S}_\varepsilon^{\circ}) \to L^2(\mathbb{S}_\varepsilon^{\circ})$ can be written as $\Delta^{\iota} = \sum_{0 \le \alpha, 2\beta \le 2\iota} \frac{c_{\alpha\beta}}{p_{\alpha\beta}(r)} \Delta^{\beta}_{S^2} \frac{\partial^\alpha}{\partial r^\alpha}$, as in (A.4). Thus, from Lemma 2.4.7 we see that $\Delta^{\iota} T = \sum_{\alpha,\beta} \frac{c_{\alpha\beta}}{p_{\alpha\beta}(r)} \Delta^{\beta}_{S^2} \frac{\partial^\alpha}{\partial r^\alpha} T$ on $\mathbb{S}_\varepsilon^{\circ}$, and further that for each $r \in [r_{\text{in}}, r_{\text{out}}]$ we have

$$\int_{S^2} \Big| \Delta^{\beta}_{S^2} \frac{\partial^\alpha}{\partial r^\alpha} T(r, \sigma; \omega) \Big|^2 \,\mathrm{d}S < \infty, \quad \text{a.s.} \tag{2.4.23}$$

Hence, $\Delta^{\iota'} T(r_{\text{in}}, \cdot; \omega) \in L^2(S^2(r_{\text{in}}))$, and $\Delta^{\iota'} T(r_{\text{out}}, \cdot; \omega) \in L^2(S^2(r_{\text{in}}))$, a.s. To show (2.4.21), from (2.4.20) it suffices to show

$$\mathbb{E}\big[\big\|\Delta^{\iota'} T(r_{\text{in}}, \cdot)\big\|^2_{H^{\frac{1}{2}}(S^2(r_{\text{in}}))}\big] < \infty, \quad \text{and} \quad \mathbb{E}\big[\big\|\Delta^{\iota'} T(r_{\text{out}}, \cdot)\big\|^2_{H^{\frac{1}{2}}(S^2(r_{\text{out}}))}\big] < \infty.$$

Let $g(\sigma; \omega) := \Delta^{\iota'} T(r_{\text{in}}, \cdot; \omega)$. Then, from $\iota' \le \min\{\lfloor \frac{\eta}{2} \rfloor, \frac{s}{2} - \frac{1}{4}\}$, in view of Lemma 2.4.7 we have

$$\mathbb{E}[\|g(r_{\text{in}}, \cdot; \cdot)\|^2_{L^2(S^2(r_{\text{in}}))}] < \infty, \quad \text{and} \quad \mathbb{E}\Big[\big\|\Delta^{\frac{1}{4}}_{S^2} g(r_{\text{in}}, \cdot; \cdot)\big\|^2_{L^2(S^2(r_{\text{in}}))}\Big] < \infty.$$

Thus, from (2.4.7) we have $\mathbb{E}\,\|g(r_{\text{in}}, \cdot; \cdot)\|^2_{H^{\frac{1}{2}}(S^2(r_{\text{in}}))} < \infty$. By a similar argument we see $\mathbb{E}[\|g(r_{\text{out}}, \cdot; \cdot)\|^2_{H^{\frac{1}{2}}(S^2(r_{\text{out}}))}] < \infty$.

The estimate (2.4.22) can be shown similarly. $\square$

Now, we show that $\Delta^{\iota} T \in H^1(\mathbb{S}_\varepsilon^{\circ})$.

Proposition 2.4.9. *Suppose Assumption 2.1 is satisfied with $\eta > 2$ such that $\eta > \lfloor \eta \rfloor$. If a non-negative integer ι satisfies $0 \le \iota \le \min\{\lfloor \frac{\eta}{2} \rfloor, \frac{s}{2}\} - 1$, Then, we have*

$$\mathbb{E}[\|\Delta^{\iota} T\|_{H^1(\mathbb{S}_\varepsilon^{\circ})}] \le C\mathbb{E}\Big[\, \big\|\Delta^{\iota+1} T\big\|_{L^2(\mathbb{S}_\varepsilon^{\circ})} + \big\|\Delta^{\iota} T|_{\partial \mathbb{S}_\varepsilon^{\circ}}\big\|_{H^{\frac{1}{2}}(\partial \mathbb{S}_\varepsilon^{\circ})}\Big] < \infty, \tag{2.4.24}$$

where the constant $C > 0$ is independent of T.

Proof. First note that from Lemma 2.4.8 we have $\mathbb{E}[\|\Delta^{\iota+1} T\|_{L^2(\mathbb{S}_\varepsilon^{\circ})}] < \infty$, and $\mathbb{E}[\big\|\Delta^{\iota} T|_{\partial \mathbb{S}_\varepsilon^{\circ}}\big\|_{H^{\frac{1}{2}}(\partial \mathbb{S}_\varepsilon^{\circ})}] < \infty$.

Consider the case $\iota = 0$.

First, since $T|_{\partial \mathbb{S}_\varepsilon^{\circ}} \in H^{\frac{1}{2}}(\partial \mathbb{S}_\varepsilon^{\circ})$ a.s., there exists $u_T \in H^1(\mathbb{S}_\varepsilon^{\circ})$ such that $u_T|_{\partial \mathbb{S}_\varepsilon^{\circ}} = T|_{\partial \mathbb{S}_\varepsilon^{\circ}}$ (for example, [80, Théorèm 5.7]). For $v \in H_0^1(\mathbb{S}_\varepsilon^{\circ})$, define

$$F_{-\Delta T + \Delta u_T}(v) := \int_{\mathbb{S}_\varepsilon^{\circ}} (-\Delta T v - \nabla u_T \cdot \nabla v) \, \mathrm{d}x. \tag{2.4.25}$$

Then, we have $F_{-\Delta T + \Delta u_T} \in H^{-1}(\mathbb{S}_\varepsilon^{\circ})$. Now, consider the problem:

$$\text{Find } u_0 \in H_0^1(\mathbb{S}_\varepsilon^{\circ}) : \quad a(u_0, v) = F_{-\Delta T + \Delta u_T}(v) \quad \text{for all } v \in H_0^1(\mathbb{S}_\varepsilon^{\circ}), \tag{2.4.26}$$

where the bilinear form is defined as $a(u_0, v) = \int_{\mathbb{S}_\varepsilon^{\circ}} \nabla u_0 \cdot \nabla v \, \mathrm{d}x$. From Lax–Milgram lemma, the weak solution $u_0 \in H_0^1(\mathbb{S}_\varepsilon^{\circ})$ of the above problem uniquely exists.

Let $T^* := u_0 + u_T$ so that we have $T^*|_{\partial \mathbb{S}_\varepsilon^{\circ}} = u_T|_{\partial \mathbb{S}_\varepsilon^{\circ}} = T|_{\partial \mathbb{S}_\varepsilon^{\circ}}$, and further,

$$\int_{\mathbb{S}_\varepsilon^{\circ}} \nabla T^* \cdot \nabla v \, \mathrm{d}x = -\int_{\mathbb{S}_\varepsilon^{\circ}} \Delta T v, \qquad \text{for any } v \in H_0^1(\mathbb{S}_\varepsilon^{\circ}). \tag{2.4.27}$$

This implies $\int_{\mathbb{S}_\varepsilon^{\circ}} (T^* - T) \Delta \phi \, \mathrm{d}x = 0$, for any $\phi \in C_c^{\infty}(\mathbb{S}_\varepsilon^{\circ})$, and thus there exists a unique harmonic function h such that $h = T^* - T$, $\mathbb{S}_\varepsilon^{\circ}$-a.e.

Now, from [33, Theorem 2.4, 2.14] the solution of the problem:

$$\text{Find } h \text{ such that} \qquad \Delta h^* = 0 \ \text{ in } \mathbb{S}_\varepsilon^{\circ}, \ \text{ and } \quad h^* = 0 \ \text{ on } \partial \mathbb{S}_\varepsilon^{\circ},$$

uniquely exists and $h^* \equiv 0$ on $\mathbb{S}_\varepsilon = \mathbb{S}_\varepsilon^{\circ} \cup \partial \mathbb{S}_\varepsilon^{\circ}$. But h satisfies the above and thus $0 = h^* = h = T^* - T$ almost everywhere on $\mathbb{S}_\varepsilon^{\circ}$. Together with $T|_{\partial \mathbb{S}_\varepsilon^{\circ}} = T^*|_{\partial \mathbb{S}_\varepsilon^{\circ}}$, we have $T = T^* = u_0 + u_T$ almost everywhere on $\mathbb{S}_\varepsilon$.

Now, we show $T \in H^1(\mathbb{S}_\varepsilon^\circ)$. From (2.4.26) and the Poincaré's inequality we have $\|u_0\|_{H_0^1(\mathbb{S}_\varepsilon^\circ)} \leq c\|\Delta T\|_{L^2(\mathbb{S}_\varepsilon^\circ)} + \|u_T\|_{H_0^1(\mathbb{S}_\varepsilon^\circ)}$ with a constant $c = c(\mathbb{S}_\varepsilon^\circ) > 0$. Therefore,

$$\|T\|_{H^1(\mathbb{S}_\varepsilon^\circ)} \leq \|u_0\|_{H^1(\mathbb{S}_\varepsilon^\circ)} + \|u_T\|_{H^1(\mathbb{S}_\varepsilon^\circ)} \leq C(\|\Delta T\|_{L^2(\mathbb{S}_\varepsilon^\circ)} + \|u_T\|_{H^1(\mathbb{S}_\varepsilon^\circ)}),$$

where the constant $C > 0$ is independent of T.

The above inequality holds for arbitrary $g \in H^1(\mathbb{S}_\varepsilon^\circ)$ such that $\mathcal{T}r(g) = T|_{\partial\mathbb{S}_\varepsilon^\circ}$ in place of u_T, where $\mathcal{T}r$ is the trace operator. Hence, by virtue of the continuity of the right inverse of the trace operator (see, e.g., [80]) we have

$$\|T\|_{H^1(\mathbb{S}_\varepsilon^\circ)} \leq C(\|\Delta T\|_{L^2(\mathbb{S}_\varepsilon^\circ)} + \inf_{\{g \in H^1 |\, \mathcal{T}r(g) = T|_{\partial\mathbb{S}_\varepsilon^\circ}\}} \|g\|_{H^1(\mathbb{S}_\varepsilon^\circ)}) \tag{2.4.28}$$

$$\leq C(\|\Delta T\|_{L^2(\mathbb{S}_\varepsilon^\circ)} + \big\|T|_{\partial\mathbb{S}_\varepsilon^\circ}\big\|_{H^{\frac{1}{2}}(\partial\mathbb{S}_\varepsilon^\circ)}), \tag{2.4.29}$$

where the generic constants C above are not necessarily the same, but independent of T. Thus, we have $\mathbb{E}[\|T\|_{H^1(\mathbb{S}_\varepsilon^\circ)}] \leq C(\mathbb{E}[\|\Delta T\|_{L^2(\mathbb{S}_\varepsilon^\circ)} + \big\|T|_{\partial\mathbb{S}_\varepsilon^\circ}\big\|_{H^{\frac{1}{2}}(\partial\mathbb{S}_\varepsilon^\circ)}])$.

The case $\iota \geq 1$ follows from the same argument. $\square$

Under a stronger smoothness assumption on the covariance function, we have $T \in H^2(\mathbb{S}_\varepsilon^\circ)$.

Theorem 2.4.10. *Suppose Assumption 2.1 holds with $\eta > 2$ and $s \geq 2$. Then, we have*

$$\mathbb{E}[\|T\|_{H^2(\mathbb{S}_\varepsilon^\circ)}] \leq C\mathbb{E}\big[\, \|T\|_{L^2(\mathbb{S}_\varepsilon^\circ)} + \|\Delta T\|_{L^2(\mathbb{S}_\varepsilon^\circ)} + \big\|T|_{\partial\mathbb{S}_\varepsilon^\circ}\big\|_{H^{\frac{3}{2}}(\partial\mathbb{S}_\varepsilon^\circ)}\big], \tag{2.4.30}$$

where the constant $C > 0$ is independent of T.

Proof. First, note that $T|_{\partial\mathbb{S}_\varepsilon^\circ}(\cdot;\omega) \in H^{\frac{3}{2}}(\partial\mathbb{S}_\varepsilon^\circ)$ a.s., from Lemma 2.4.8. Thus, from the smoothness of the boundary, there exists (see, e.g., [80, Théorèm 5.8]) a continuous operator $\mathcal{Z}\colon H^{\frac{3}{2}}(\partial\mathbb{S}_\varepsilon^\circ) \to H^2(\mathbb{S}_\varepsilon^\circ)$ such that $(\mathcal{T}r \circ \mathcal{Z})T|_{\partial\mathbb{S}_\varepsilon^\circ} = T|_{\partial\mathbb{S}_\varepsilon^\circ}$. Letting $g = \mathcal{Z}T|_{\partial\mathbb{S}_\varepsilon^\circ}$, from Proposition 2.4.9 together with Lemma 2.4.8 we have $T - g \in H_0^1(\mathbb{S}_\varepsilon^\circ)$.

Now, we have $T \in H^1(\mathbb{S}_\varepsilon^\circ)$ and $\Delta T \in L^2(\mathbb{S}_\varepsilon)$, but trivially,

$$\langle \nabla T, \nabla v\rangle_{L^2(\mathbb{S}_\varepsilon^\circ)} = \langle \Delta T, v\rangle_{L^2(\mathbb{S}_\varepsilon^\circ)} \text{ for any } v \in C_c^2(\mathbb{S}_\varepsilon^\circ), \quad \text{and} \quad T|_{\partial\mathbb{S}_\varepsilon^\circ} = T|_{\partial\mathbb{S}_\varepsilon^\circ} \text{ on } \partial\mathbb{S}_\varepsilon^\circ.$$

Therefore, from the regularity theory for elliptic partial differential equations (see, e.g., [33, Chapter 8]), with $C_{\mathcal{Z}} := \|\mathcal{Z}\|_{H^{\frac{3}{2}}(\partial\mathbb{S}^{\circ}_{\varepsilon}) \to H^2(\mathbb{S}^{\circ}_{\varepsilon})}$ we have

$$\mathbb{E}[\|T\|_{H^2(\mathbb{S}^{\circ}_{\varepsilon})}] \leq C\mathbb{E}\big[\,\|T\|_{L^2(\mathbb{S}^{\circ}_{\varepsilon})} + \|\Delta T\|_{L^2(\mathbb{S}^{\circ}_{\varepsilon})} + C_{\mathcal{Z}}\,\big\|T|_{\partial\mathbb{S}^{\circ}_{\varepsilon}}\big\|_{H^{\frac{3}{2}}(\partial\mathbb{S}^{\circ}_{\varepsilon})}\big]. \tag{2.4.31}$$

□

Proposition 2.4.11. *Suppose Assumption 2.1 holds with $\eta > 2$ such that $\eta > \lfloor\eta\rfloor$ and $s \geq 2$. Then, for any integer $\kappa \geq 0$ such that $2\kappa + 2 \leq \min\{\lfloor\eta\rfloor, s\}$, we have*

$$\mathbb{E}[\|T\|_{H^{2\kappa+2}(\mathbb{S}^{\circ}_{\varepsilon})}] \leq C_{2\kappa}\mathbb{E}\Big[\Big(\,\|T\|_{L^2(\mathbb{S}^{\circ}_{\varepsilon})} + \|\Delta T\|_{H^{2\kappa}(\mathbb{S}^{\circ}_{\varepsilon})} + \big\|T|_{\partial\mathbb{S}^{\circ}_{\varepsilon}}\big\|_{H^{2\kappa+\frac{3}{2}}(\partial\mathbb{S}^{\circ}_{\varepsilon})}\Big)\Big],$$

where the constant $C_{2\kappa} > 0$ depends on κ but independent of T.

Proof. Following the proof of Theorem 2.4.10, the standard bootstrap argument, namely, a repeated use of Lemma 2.4.8 and Proposition 2.4.9 yields the result. See [33, Theorem 8.12]. □

2.5 Approximations and a class of PDEs

As an application of the regularity theory developed in previous sections, we consider an approximation of random fields, and a class of elliptic partial differential equations that have a log-normal random field as a coefficient. In this section, we assume $\varphi_k = J_k$ is the degree k Chebyshev polynomial of the first kind mapped affinely to $[r_{\text{in}}, r_{\text{out}}]$ from $[-1, 1]$. As pointed out before, to treat the radial direction in the Earth modelling, the Chebyshev polynomial is often considered. In principle, the discussion in this section can be applied to any Jacobi polynomials with the parameters (α, β) both greater than -1 such that the resulting measure $\mu = \mu(\alpha, \beta)$ satisfies $\|\cdot\|_{L^2((r_{\text{in}}, r_{\text{out}}))} \leq \|\cdot\|_{L^2_\mu((r_{\text{in}}, r_{\text{out}}))}$, by adjusting the decaying factor such as (2.2.6), (2.2.7), or (2.2.8). For example, all $-1 < \alpha = \beta \leq 0$ satisfies this condition.

To motivate the discussion on the approximation, we start with the following boundary value problem of the elliptic PDE:

$$\text{Find } u\colon \quad -\nabla\cdot(\mathcal{A}\nabla u) = f \quad \text{in} \quad \mathbb{S}^{\circ}_{\varepsilon}, \quad \text{and} \quad u|_{\partial\mathbb{S}^{\circ}_{\varepsilon}} = 0 \quad \text{on} \quad \partial\mathbb{S}^{\circ}_{\varepsilon}, \tag{2.5.1}$$

where $f \in L^2(\mathbb{S}^{\circ}_{\varepsilon})$. The coefficient $\mathcal{A} = \mathcal{A}(\cdot, \omega)\colon \mathbb{S}^{\circ}_{\varepsilon} \to \mathbb{R}$ is taken as a log-normal random field, that is, we let $\mathcal{A} := \exp T$.

Often, in practice the quantity $\{A_{k\ell}\}$ are estimated with a suitable statistical procedure. Here, we suppose $\{A_{k\ell}\}_{0\le k\le K,\, 0\le \ell\le L}$ is given, and consider an approximation $\mathcal{A}_{KL}$ of $\mathcal{A}$ using $\{A_{k\ell}\}_{0\le k\le K,\, 0\le \ell\le L}$. Our goal is to estimate the error $\|u-\widetilde{u}\|_X$ with respect to a suitable Sobolev norm $\|\cdot\|_X$ clarified below, where $\widetilde{u}$ is the solution of (2.5.1) with the coefficient $\mathcal{A}_{KL}$.

For the existence and uniqueness of the solution of the problem above, we consider the weak formulation and invoke the Lax–Milgram lemma. Later we see that by a variant of Strang's lemma, a key estimate to bound $\|u-\widetilde{u}\|_X$ is the error $\sup_{x\in\mathbb{S}_\varepsilon} |\mathcal{A}(x;\omega)-\mathcal{A}_{KL}(x;\omega)|$. For the sake of a good uniform convergence, we consider a filtered approximation. We start with the approximation of the random field T.

2.5.1 Filtered approximation of Gaussian random fields

Recall that as we saw in (2.2.3) the random field T is given as

$$T(x;\omega)=\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sqrt{A_{k\ell}}X_{k\ell m}(\omega)J_k(r_x)\mathcal{Y}_{\ell m}(\sigma_x),\quad x\in\mathbb{S}_\varepsilon, \tag{2.5.2}$$

with i.i.d. standard normal random variables $\{X_{k\ell m}\}$. Thus, in view of Remark 3, knowing $A_{k\ell}$ ($0\le k\le K$, $0\le\ell\le L$) corresponds to knowing Fourier coefficients of T up to the degree K and L, given that $X_{k\ell m}$ ($0\le k\le K$, $0\le\ell\le L$) can be generated; from (2.3.11), the truncated sum $\sum_{k=0}^{K}\sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell}\widehat{T}_{k\ell m}(\omega)J_k\mathcal{Y}_{\ell m}$ is precisely the partial Fourier sum for $\mathbb{P}$-almost every ω with $\widehat{T}_{k\ell m}(\omega)=\sqrt{A_{k\ell}}X_{k\ell m}(\omega)$.

However, the naive L^2-projection is not the best choice when one wants a small point-wise error—recall how the Fourier series of a function on the torus may fail to converge on any measure zero set if f is merely continuous. This motivates the use of *filtering*: we smoothly truncate the series by multiplying $\widehat{T}_{k\ell m}(\omega)$, the Fourier coefficients, of higher order by a suitably small factor. In more details, first define a *filter* function $h\colon [0,\infty)\times[0,\infty)\to[0,\infty)$ as follows. Consider the non-increasing functions $h^{\mathrm{ang}},h^{\mathrm{rad}}\colon[0,\infty)\to[0,\infty)$ with $\operatorname{supp}(h^{\mathrm{ang}}),\operatorname{supp}(h^{\mathrm{rad}})\subset[0,2]$ with the following properties:

(a) $h^{\mathrm{rad}}(t)=1$ for $t\in[0,1]$, and $h^{\mathrm{ang}}(s)=1$ for $s\in[0,1]$.

(b) h^{rad}, respectively h^{ang}, are absolutely continuous and their derivatives are of bounded variation.

Then, we define the *filter* function $h\colon[0,\infty)\times[0,\infty)\to[0,\infty)$ by

$$h\colon(s,t)\mapsto h^{\mathrm{rad}}(s)h^{\mathrm{ang}}(t). \tag{2.5.3}$$

With this filter function, for $f \in C(\mathbb{S}_\varepsilon)$ let us define the (continuous) *filtered approximation* $\mathscr{V}_{KL} f$ of f by

$$\mathscr{V}_{KL} f = \sum_{k=0}^{2K-1} \sum_{\ell=0}^{2L-1} \sum_{m=-\ell}^{\ell} h\Big(\frac{k}{K}, \frac{\ell}{L}\Big) \hat{f}_{k\ell m} J_k \mathcal{Y}_{\ell m}, \tag{2.5.4}$$

where $\hat{f}_{k\ell m} = \int_{\mathbb{S}_\varepsilon^\circ} f J_k \mathcal{Y}_{\ell m} \,\mathrm{d}\mathbb{S}_\varepsilon$, and $K, L \geq 1$. The discretised filtered approximation, that is, the approximation of the same form as above but $\hat{f}_{k\ell m}$ being replaced by a quadrature rule with a suitable precision, is considered in [53]. There, the continuous filtered approximation is considered to derive the error for the discretised filtered approximation. Thus, the error estimate for the continuous version is readily available. See also [91] for the continuous filtered approximation in the angular direction in more detail. We have the following supremum norm estimate essentially from [53, Corollary 5.4].

Proposition 2.5.1. *Suppose that f is $\lfloor \eta \rfloor$-times continuously partially differentiable with respect to r ($\lfloor \eta \rfloor \in \{1, \dots, K\}$) and satisfies*

$$\|f^{(\lfloor \eta \rfloor, 0)}\|_{\infty, \mathrm{ess\,sup}} = \operatorname*{ess\,sup}_{\sigma \in S^2} \sup_{r \in [r_{\mathrm{in}}, r_{\mathrm{out}}]} \left| \frac{\partial^{\lfloor \eta \rfloor}}{\partial r^{\lfloor \eta \rfloor}} f(r, \sigma) \right| < \infty.$$

Suppose furthermore that $f(r, \cdot) \in C^{2t}(S^2)$ ($t \in \{1, 2, 3, \dots\}$) for each r and satisfies

$$\|f^{(0,2t)}\|_{\mathrm{ess\,sup}, \infty} = \operatorname*{ess\,sup}_{r \in [r_{\mathrm{in}}, r_{\mathrm{out}}]} \sup_{\sigma \in S^2} |\Delta_{S^2}^t f(r, \sigma)| < \infty.$$

Then, we have

$$\begin{aligned} &\sup_{(r,\sigma) \in \mathbb{S}_\varepsilon} |f(r, \sigma) - \mathscr{V}_{KL} f(r, \sigma)| \\ &\qquad \leq C \frac{(K - \lfloor \eta \rfloor)!}{K!} \|f^{(\lfloor \eta \rfloor, 0)}\|_{\infty, \mathrm{ess\,sup}} + C' L^{-2t} \|f^{(0,2t)}\|_{\mathrm{ess\,sup}, \infty}, \end{aligned}$$

where the positive constants C, C' are independent of K, L, and f.

Proof. In [53], an error estimate for a fully discretised filtered approximation is considered. From the discussion above, for the continuous filtered approximation above, we can easily check that we have the error estimate of the same form as [53, Theorem 5.3 and Corollary 5.4], now with respect to the essential supremum norm:

$$\begin{aligned}&\operatorname*{ess\,sup}_{(r,\sigma)\in\mathbb{S}_\varepsilon}|f(r,\sigma)-\mathscr{V}_{KL}f(r,\sigma)|\\&\qquad\le C\frac{(K-\lfloor\eta\rfloor)!}{K!}\|f^{(\lfloor\eta\rfloor,0)}\|_{\infty,\operatorname{ess\,sup}}+C'L^{-2t}\|f^{(0,2t)}\|_{\operatorname{ess\,sup},\infty}.\end{aligned}$$

Since f and $\mathscr{V}_{KL}f$ are continuous on $\mathbb{S}_\varepsilon$, we have

$$\sup_{(r,\sigma)\in\mathbb{S}_\varepsilon}|f(r,\sigma)-\mathscr{V}_{KL}f(r,\sigma)|=\operatorname*{ess\,sup}_{(r,\sigma)\in\mathbb{S}_\varepsilon}|f(r,\sigma)-\mathscr{V}_{KL}f(r,\sigma)|,$$

which completes the proof. $\square$

Now, we consider the approximation

$$\mathscr{V}_{KL}T(r,\sigma)=\sum_{k=0}^{2K-1}\sum_{\ell=0}^{2L-1}\sum_{m=-\ell}^{\ell}h\Big(\frac{k}{K},\frac{\ell}{L}\Big)\sqrt{A_{k\ell}}X_{k\ell m}(\omega)J_k\mathcal{Y}_{\ell m}. \tag{2.5.5}$$

Then, from the regularity theory developed in Section 2.4, we obtain the following result.

Theorem 2.5.2. *Suppose Assumption 2.1 holds with the parameters $\eta>1$ such that $\eta>\lfloor\eta\rfloor$ and $s>3$. Let $t\in(0,\frac{s-1}{2})$ be an integer. For T given by (2.5.2), let $\mathscr{V}_{KL}T$ be defined by (2.5.5). Then, we have*

$$\mathbb{E}\Big[\sup_{(r,\sigma)\in\mathbb{S}_\varepsilon}|T(r,\sigma)-\mathscr{V}_{KL}T(r,\sigma)|^2\Big]\le\mathscr{E}(\eta,t,K,L,T), \tag{2.5.6}$$

with

$$\begin{aligned}&\mathscr{E}(\eta,t,K,L,T)\\&:=C\frac{((K-\eta)!)^2}{(K!)^2}\mathbb{E}[\|T^{(\lfloor\eta\rfloor,0)}\|^2_{\infty,\operatorname{ess\,sup}}]+C'L^{-4t}\mathbb{E}[\|T^{(0,2t)}\|^2_{\operatorname{ess\,sup},\infty}],\end{aligned} \tag{2.5.7}$$

where $C,C'>0$ are constants independent of K, L, and T. Further, we have

$$\Big(\mathbb{E}[\|T-\mathscr{V}_{KL}T\|^{2n}_{L^2(\mathbb{S}^\circ_\varepsilon)}]\Big)^{\frac{1}{2n}}\le C''_n\sqrt{\mathscr{E}(\eta,t,K,L,T)}, \tag{2.5.8}$$

for any $n\in\mathbb{N}$ where $C''_n>0$ is a constant independent of K, L, and T.

Proof. Note that in view of Lemma 2.3.2 we have $T\in C(\mathbb{S}_\varepsilon)$ and thus the $\mathscr{F}$-measurability is preserved under taking supremum over $\mathbb{S}_\varepsilon$. From Propositions

2.4.5 and 2.4.6, we have

$$\mathbb{E}[\|T^{(\lfloor\eta\rfloor,0)}\|^2_{\infty,\mathrm{ess\,sup}}] < \infty, \text{ and } \mathbb{E}[\|T^{(0,2t)}\|^2_{\mathrm{ess\,sup},\infty}] < \infty.$$

Thus, (2.5.6) follows from Proposition 2.5.1.

Next, we observe that $T(r,\sigma) - \mathscr{V}_{KL}T(r,\sigma)$ is a normal random variable. To see this, note that for $K' \geq 2K-1$ and $L' \geq 2L-1$,

$$e_{K'L'}(x) := \sum_{k=0}^{K'}\sum_{\ell=0}^{L'}\sum_{m=-\ell}^{\ell} \sqrt{A_{k\ell}} X_{k\ell m}(\omega) J_k(r_x)\mathcal{Y}_{\ell m}(\sigma_x) - \mathscr{V}_{KL}T(x),$$

is a sum of independent normal random variables for each $x \in \mathbb{S}_\varepsilon$. For any fixed $x \in \mathbb{S}_\varepsilon$, $\{e_{K'L'}\}_{(K',L')\in\mathbb{N}\times\mathbb{N}}$ is a convergent net in $L^2_{\mathbb{P}}(\Omega)$ with the limit $T(x) - \mathscr{V}_{KL}T(x)$, where $\mathbb{N}\times\mathbb{N}$ is equipped with the relation $\leq$ defined by $(K_1', L_1') \leq (K_2', L_2')$ if and only if $K_1' \leq K_2'$ and $L_1' \leq L_2'$. But linear combinations of independent Gaussian random variables form a closed subspace of $L^2_{\mathbb{P}}(\Omega)$ (for example, [42, Section 1.6]). Hence, for any $x \in \mathbb{S}_\varepsilon$ the limit of $\{e_{K'L'}\}_{(K',L')\in\mathbb{N}\times\mathbb{N}}$ is again a normal random variable: $T(x) - \mathscr{V}_{KL}T(x) \sim \mathcal{N}(0, \varsigma^2(x))$ for some $\varsigma^2(x)$. Thus, with $X_{\mathrm{std}} \sim \mathcal{N}(0,1)$ for any $n \in \mathbb{N}$ we have

$$\mathbb{E}[|T(x) - \mathscr{V}_{KL}T(x)|^{2n}] = \mathbb{E}[|T(x) - \mathscr{V}_{KL}T(x)|^2]^n \mathbb{E}[|X_{\mathrm{std}}|^{2n}]. \tag{2.5.9}$$

Integrating both sides over $\mathbb{S}_\varepsilon$, by virtue of Jensen's inequality and (2.5.6) we have

$$\begin{aligned}
\mathbb{E}[\|T - \mathscr{V}_{KL}T\|^{2n}_{L^2(\mathbb{S}_\varepsilon^\circ)}] &\leq C_{r_{\mathrm{in}},r_{\mathrm{out}}}\mathbb{E}\Big[\sup_{x\in\mathbb{S}_\varepsilon}|T(x) - \mathscr{V}_{KL}T(x)|^{2n}\Big] \\
&\leq C_{r_{\mathrm{in}},r_{\mathrm{out}}}\mathbb{E}[|X_{\mathrm{std}}|^{2n}] \\
&\quad\times\left(C\frac{((K-\eta)!)^2}{(K!)^2}\mathbb{E}[\|T^{(\lfloor\eta\rfloor,0)}\|^2_{\infty,\mathrm{ess\,sup}}] + C'L^{-4t}\mathbb{E}[\|T^{(0,2t)}\|^2_{\mathrm{ess\,sup},\infty}]\right)^n,
\end{aligned}$$

and thus (2.5.8) follows. □

2.5.2 Lognormal random field on spherical shells

We consider log-normal random fields on spherical shells and estimate the approximation error. Our analysis uses Theorem 2.5.2 and bounds for moments of log-normal random fields.

Definition 2. We call a random field $\mathcal{A}$ on a spherical shell log-normal if there exists a Gaussian random field T on the shell $\mathbb{S}_\varepsilon$ such that

$$\mathcal{A} = \exp(T). \tag{2.5.10}$$

From the continuity of the exponential function, $\mathcal{A}$ is well-defined. ∎

We need the following bound for the error estimate and the well-posedness of the weak formulation of the problem (2.5.1). The following argument follows Charrier [17, Proposition 3.10].

Proposition 2.5.3. *Suppose Assumption 2.1 holds with $\eta > 1$ and $s > \frac{1}{2}$. Then, for any $p > 0$ there exists a constant $c_p > 0$ depending only on p such that*

$$\mathbb{E}\left[\exp(p \sup_{x\in\mathbb{S}_\varepsilon} |T(x)|)\right] \le c_p. \tag{2.5.11}$$

Further, there exists a constant c'_p depending only on p (independent of K and L) such that

$$\mathbb{E}\left[\exp\left(p \sup_{x\in\mathbb{S}_\varepsilon} |\mathscr{V}_{KL} T(x)|\right)\right] \le c'_p, \tag{2.5.12}$$

Proof. Without loss of generality, we assume $\eta > \lfloor \eta \rfloor$. In view of Lemma 2.3.2, we may see T as a continuous function on $\mathbb{S}_\varepsilon$, which implies $\operatorname{ess\,sup}_{x\in\mathbb{S}_\varepsilon} |T(x)|^2 = \sup_{x\in\mathbb{S}_\varepsilon} |T(x)|^2$. Thus, from the proof of Proposition 2.4.6 , we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{x\in\mathbb{S}_\varepsilon} |T(x)|^2\right] &\le \mathbb{E}\left[\operatorname*{ess\,sup}_{r\in(r_{\text{in}},r_{\text{out}})} \sup_{\sigma\in S^2} |\Delta^0_{S^2} T(x)|^2\right] \\
&\le c c_0^2 \left(\sum_{k=0}^\infty \frac{1}{(k+1)^{\eta-\lfloor\eta\rfloor+1}}\right) \sum_{\ell=0}^\infty \sum_{k=0}^\infty (2\ell+1)^{2s+1}(k+1)^{\eta-\lfloor\eta\rfloor+1} A_{k\ell} \\
&=: D < \infty.
\end{aligned}$$

Then, as a consequence of the Markov's inequality, for any $b > 0$ we have

$$\mathbb{P}(\sup_{x\in\mathbb{S}_\varepsilon} |T(x)| > b|) \le \frac{D^2}{b^2}.$$

Now, take a $s_0 \in (0,1)$ such that $\log(\frac{1-s_0}{s_0}) \le -2$ and let $b_0 := \frac{D}{\sqrt{1-s_0}}$ so that $s_0 \le 1 - \mathbb{P}(\sup_{x\in\mathbb{S}_\varepsilon} |T(x)| > b_0|)$. Further, choose $\lambda > 0$ such that $32\lambda s_0^2 < 1$

holds. Then, since $\frac{1-\mathbb{P}(\sup_{x\in\mathbb{S}_\varepsilon}|T(x)|\le b_0|)}{\mathbb{P}(\sup_{x\in\mathbb{S}_\varepsilon}|T(x)|\le b_0|)} = \frac{\mathbb{P}(\sup_{x\in\mathbb{S}_\varepsilon}|T(x)|>b_0|)}{1-\mathbb{P}(\sup_{x\in\mathbb{S}_\varepsilon}|T(x)|>b_0|)} \le \frac{1-s_0}{s_0}$, we have

$$\log\left(\frac{1-\mathbb{P}(\sup_{x\in\mathbb{S}_\varepsilon}|T(x)|\le b_0|)}{\mathbb{P}(\sup_{x\in\mathbb{S}_\varepsilon}|T(x)|\le b_0|)}\right) + 32\lambda s_0^2 \le -1,$$

which is the assumption of Fernique's theorem (see, for example [22, Theorem 2.7]). Using this theorem we have $\mathbb{E}\left[\exp(\lambda\sup_{x\in\mathbb{S}_\varepsilon}|T(x)|^2)\right] \le e^{16\lambda s_0^2} + \frac{e^2}{e^2-1}$. Noting $pt \le \lambda t^2 + \frac{p^2}{4\lambda}$ $(p,t,\lambda\in\mathbb{R})$, for any $p>0$ we have

$$\mathbb{E}\left[\exp(p\sup_{x\in\mathbb{S}_\varepsilon}|T(x)|)\right] \le \mathbb{E}\left[\exp(\lambda\sup_{x\in\mathbb{S}_\varepsilon}|T(x)|^2)\right]\mathrm{e}^{\frac{p^2}{4\lambda}} \le \mathrm{e}^{16\lambda s_0^2+\frac{p^2}{4\lambda}} + \frac{\mathrm{e}^{2+\frac{p^2}{4\lambda}}}{\mathrm{e}^2-1}.$$

For each ω, from [53, Theorem 3.2] (see also [75, Theorem 3.1]) and [99, Lemma 2.1] we observe

$$\sup_{x\in\mathbb{S}_\varepsilon}|\mathscr{V}_{KL}T(x;\omega)|^2 \le c'\sup_{x\in\mathbb{S}_\varepsilon}|T(x;\omega)|^2, \tag{2.5.13}$$

with a constant $c'>0$ independent of K, L, and T. Then, it follows that

$$\sup_{x\in\mathbb{S}_\varepsilon}|\mathscr{V}_{KL}T(x;\omega)|^2 \le c'D < \infty.$$

Taking the expectation on both sides yields

$$\mathbb{E}\left[\sup_{x\in\mathbb{S}_\varepsilon}|\mathscr{V}_{KL}T(x)|^2\right] \le c'D < \infty. \tag{2.5.14}$$

and thus repeating the same argument as above yields

$$\mathbb{E}\left[\exp\left(p\sup_{x\in\mathbb{S}_\varepsilon}|\mathscr{V}_{KL}T(x)|\right)\right] \le \mathrm{e}^{16\lambda' s_0^2+\frac{p^2}{4\lambda'}} + \frac{\mathrm{e}^{2+\frac{p^2}{4\lambda'}}}{\mathrm{e}^2-1}. \tag{2.5.15}$$

□

We are ready to state the error estimate for log-normal random fields. We need the following result to obtain the truncation error for the PDE.

Proposition 2.5.4. *Suppose Assumption 2.1 holds with* $\eta>1$ *such that* $\eta>\lfloor\eta\rfloor$ *and* $s>3$*. Let* $t\in(0,\frac{s-1}{2})$ *be an integer. For* T *given by* (2.5.2)*, let* $\mathscr{V}_{KL}T$ *be* *defined by* (2.5.5)*. Then, we have*

$$\mathbb{E}\left[\sup_{x\in\mathbb{S}_\varepsilon}|\exp\big(T(x)\big)-\exp(\mathscr{V}_{KL}T(x))|^2\right]\leq 2(c_2+c_2')\mathscr{E}(\eta,t,K,L,T), \qquad (2.5.16)$$

where c_2 *and* c_2' *are constants in Proposition* 2.5.3 *with* $p=2$*, and* $\mathscr{E}(\eta,t,K,L,T)$ *is defined in Theorem* 2.5.2*. Further, we have*

$$\left(\mathbb{E}[\|\exp(T)-\exp(\mathscr{V}_{KL}T)\|^{2n}_{L^2(\mathbb{S}^\circ_\varepsilon)}]\right)^{\frac{1}{2n}}\leq C_n\sqrt{\mathscr{E}(\eta,t,K,L,T)}, \qquad (2.5.17)$$

for any integer $n\geq 1$*, where* $C_n>0$ *is a constant independent of* K*,* L*, and* T*.*

Proof. The proof is based on the argument by Lang and Schwab [66, Lemma 6.1]. Fix ω. Note that for all $a,b\in\mathbb{R}$ we have

$$|\mathrm{e}^a-\mathrm{e}^b|=\left|\int_b^a\mathrm{e}^s\,\mathrm{d}s\right|\leq\max\{\mathrm{e}^a,\mathrm{e}^b\}|b-a|\leq(\mathrm{e}^a+\mathrm{e}^b)|b-a|.$$

Thus, we have

$$\begin{aligned}&\sup_{x\in\mathbb{S}_\varepsilon}|\exp\big(T(x)\big)-\exp(\mathscr{V}_{KL}T(x))|^2\\&\leq 2\big[\sup_{x\in\mathbb{S}_\varepsilon}\exp\big(2T(x)\big)+\sup_{x\in\mathbb{S}_\varepsilon}\exp(2\mathscr{V}_{KL}T(x))\big]\sup_{x\in\mathbb{S}_\varepsilon}|T(x)-\mathscr{V}_{KL}T(x)|^2.\end{aligned}$$

Taking the expectation of both sides, from Theorem 2.5.2 and Proposition 2.5.3 we obtain (2.5.16).

Similarly, the estimate (2.5.17) follows from Proposition 2.5.3, and the proof of Theorem 2.5.2. □

2.5.3 A class of Elliptic PDEs with a random coefficient

In this section, we consider a class of elliptic partial differential equations that have a log-normal random field $\mathcal{A}$ as a coefficient of the differential operator. Based on results in the previous sections, we assess the effect on the solution caused by approximating the coefficient $\mathcal{A}$.

We consider the weak formulation of the problem (2.5.1). Let $X:=H^1_0(\mathbb{S}^\circ_\varepsilon)$ be the zero-trace Sobolev space equipped with the norm $\|g\|_X=\left(\int_{\mathbb{S}^\circ_\varepsilon}|\nabla g|^2\,\mathrm{d}x\right)^{\frac{1}{2}}$. We consider the following weak formulation of (2.5.1):

$$\text{Find}\quad u\in X:\quad a(u,v)=F(v),\ \text{ for all } v\in X, \qquad (2.5.18)$$

where

$$a(u, v) := a_\omega(u, v) := \int_{\mathbb{S}_\varepsilon^\circ} \mathcal{A}(x; \omega)(\nabla u \cdot \nabla v)\, \mathrm{d}x \quad \text{for } u, v \in X, \tag{2.5.19}$$

and $F(v) := \int_{\mathbb{S}_\varepsilon^\circ} f v \,\mathrm{d}x$ for $v \in X$. Clearly, F is linear and continuous on X.

To show the well-posedness of the problem (2.5.18), we use the Lax–Milgram lemma. To invoke it, we show the following. Here, we use the continuity of the paths. For a similar almost-sure result using Proposition 2.5.3, see [17].

Proposition 2.5.5. *Suppose that the same assumption as Proposition 2.5.4 is satisfied. Then, the bilinear form $a_\omega(\cdot, \cdot)\colon X \times X \to \mathbb{R}$ defined by (2.5.19) is bounded and coercive for each $\omega \in \Omega$.*

Proof. In view of Proposition 2.3.2 the paths of T are continuous on $\overline{\mathbb{S}_\varepsilon^\circ} = \mathbb{S}_\varepsilon$, and thus for each ω there exists $\alpha_{\max}(\omega),\ \alpha_{\min}(\omega) > 0$ such that

$$\alpha_{\min}(\omega) \leq \min_{x \in \overline{\mathbb{S}_\varepsilon^\circ}} \mathcal{A}(x; \omega) \tag{2.5.20}$$

and that

$$\max_{x \in \overline{\mathbb{S}_\varepsilon^\circ}} |\mathcal{A}(x; \omega)| = \max_{x \in \overline{\mathbb{S}_\varepsilon^\circ}} \mathcal{A}(x; \omega) \leq \alpha_{\max}(\omega) < \infty. \tag{2.5.21}$$

Therefore, for any $v, x \in X$, we have $|a(v, w)| \leq \alpha_{\max}(\omega) \|v\|_X \|w\|_X$. Thus, the bilinear form $a(\cdot, \cdot)\colon X \times X \to \mathbb{R}$ is bounded for each fixed ω. Further,

$$a(v, v) \geq \alpha_{\min}(\omega) \int_{\mathbb{S}_\varepsilon^\circ} \nabla v \cdot \nabla v \,\mathrm{d}x = \alpha_{\min}(\omega) \|v\|_X^2,$$

and thus coercive for each ω. $\square$

From this result, together with the boundedness of F on X, we can conclude that the problem (2.5.18) is well-posed.

Let $\mathcal{A}_{KL}(x; \omega) := \exp(\mathscr{V}_{KL} T(x; \omega))$, we define

$$a_{KL}(u, v) = a_{KL,\omega}(u, v) := \int_{\mathbb{S}_\varepsilon^\circ} \mathcal{A}_{KL}(x; \omega)(\nabla u \cdot \nabla v)\, \mathrm{d}x.$$

Similarly to the above, we consider

$$\text{Find } \widetilde{u} \in X: \quad a_{KL}(\widetilde{u}, v) = F(v), \text{ for all } v \in X. \tag{2.5.22}$$

By invoking Proposition 2.5.3 we have $0 < \alpha'_{\text{min}}(\omega) \leq \min_{x\in\mathbb{S}_\varepsilon} \mathcal{A}_{KL}(x;\omega)$, and further $\max_{x\in\mathbb{S}_\varepsilon} \mathcal{A}_{KL}(x;\omega) \geq \alpha'_{\text{max}}(\omega) < \infty$ independently of K and L, and thus (2.5.22) is well-posed almost surely. If we invoke Proposition 2.5.4 which requires a stronger condition, together with (2.5.20) and (2.5.21) then we have the well-posedness of (2.5.22) for all ω.

We wish to evaluate $\|u - \widetilde{u}\|_X$, the effect on the solution u caused by approximating T with $\mathscr{V}_{KL}T$. First, we need the following variant of Strang's lemma.

Lemma 2.5.6. *For $\omega \in \Omega$, let $u(\omega)$ and $\widetilde{u}(\omega)$ be the solution of (2.5.18) and (2.5.22), respectively. Suppose the same assumption as Proposition 2.5.4 is satisfied. Then, we have*

$$\|u(\omega) - \widetilde{u}(\omega)\|_X \leq \frac{1}{\alpha'_{\text{min}}(\omega)\alpha_{\text{min}}(\omega)} \left\{ \sup_{x\in\mathbb{S}_\varepsilon} |\mathcal{A}(x;\omega) - \mathcal{A}_{KL}(x;\omega)| \|f\|_{L^2(\mathbb{S}^\circ_\varepsilon)} \right\}. \tag{2.5.23}$$

Proof. Fix ω and omit ω in the following. From $\min_{x\in\mathbb{S}_\varepsilon} \mathcal{A}_{KL}(x) \geq \alpha'_{\text{min}}$ and $a(u, u-\widetilde{u}) - a(u, u-\widetilde{u}) = 0$, we have

$$\begin{aligned}\alpha'_{\text{min}} \|u - \widetilde{u}\|_X^2 &\leq a_{KL}(u-\widetilde{u}, u-\widetilde{u}) \\ &= a_{KL}(u, u-\widetilde{u}) - a_{KL}(\widetilde{u}, u-\widetilde{u}) + [a(u, u-\widetilde{u}) - a(u, u-\widetilde{u})].\end{aligned}$$

Since u and $\widetilde{u}$ are the solution of (2.5.18) and (2.5.22), respectively, we have $a(u, u-\widetilde{u}) = F(u-\widetilde{u})$ and $a_{KL}(\widetilde{u}, u-\widetilde{u}) = F(u-\widetilde{u})$. Thus, we have $\alpha'_{\text{min}} \|u - \widetilde{u}\|_X^2 \leq a_{KL}(u, u-\widetilde{u}) - a(u, u-\widetilde{u})$, and dividing both sides by $\|u - \widetilde{u}\|_X$ $(\neq 0)$ yields

$$\alpha'_{\text{min}} \|u - \widetilde{u}\|_X \leq \frac{a_{KL}(u, u-\widetilde{u}) - a(u, u-\widetilde{u})}{\|u - \widetilde{u}\|_X}. \tag{2.5.24}$$

Now, from $|a(u, u-\widetilde{u}) - a_{KL}(u, u-\widetilde{u})| \leq \sup_{x\in\mathbb{S}_\varepsilon} |\mathcal{A}(x) - \mathcal{A}_{KL}(x)| \|u\|_X \|u - \widetilde{u}\|_X$, it follows that

$$\|u - \widetilde{u}\|_X \leq \frac{1}{\alpha'_{\text{min}}} \left\{ \sup_{x\in\mathbb{S}_\varepsilon} |\mathcal{A}(x) - \mathcal{A}_{KL}(x)| \|u\|_X \right\}. \tag{2.5.25}$$

Noting $\|u\|_X \leq \frac{1}{\alpha_{\text{min}}} \|f\|_{L^2(\mathbb{S}^\circ_\varepsilon)}$ the statement follows. □

The following remark states that taking the expectation of $\|u(\omega) - \widetilde{u}(\omega)\|_X$ is valid.

Remark 5. Under the same assumption as Proposition 2.5.4, we see that the mapping $\|u - \widetilde{u}\|_X : \Omega \to \mathbb{R}$ is a random variable. ∎

Proof. First note that the statement of Lemma 2.5.6 remains true if $\mathcal{A}_{KL}$ is replaced with arbitrary continuous functions that are close to A in $(C(\mathbb{S}_\varepsilon), \sup_{x\in\mathbb{S}_\varepsilon} |\cdot|)$, in particular, functions $\{\mathcal{A}_N\}$ $(N \in \mathbb{N})$ such that $\mathcal{A}_N \to \mathcal{A}(\omega)$ in $C(\mathbb{S}_\varepsilon)$. Thus, the above result shows the mapping $\mathcal{A} \mapsto u$ is continuous from $(C(\mathbb{S}_\varepsilon), \sup_{x\in\mathbb{S}_\varepsilon} |\cdot|)$ to X.

Further, note that T is $C(\mathbb{S}_\varepsilon)$-valued random variable, i.e., $: \Omega \to C(\mathbb{S}_\varepsilon)$ is $\mathscr{F}/\mathcal{B}(C(\mathbb{S}_\varepsilon))$-measurable. To see this, let $\sigma(\{\mathcal{C}\}) := \sigma(\{\mathcal{C}(x_1,\dots,x_n;B_n)\})$ be the Kolmogorov's σ-algebra, that is, the σ-algebra generated by the family of all cylindrical sets

$$\begin{aligned}&\{\mathcal{C}(x_1,\dots,x_n;B_n)\}\\ &\quad := \{\mathcal{C}(x_1,\dots,x_n;B_n) \mid n\in\mathbb{N}, x_1,\dots,x_n\in\mathbb{S}_\varepsilon, B_n\in\mathcal{B}(\mathbb{R}^n)\},\end{aligned} \tag{2.5.26}$$

where $\mathcal{C}(x_1,\dots,x_n;B_n) = \{w \in C(\mathbb{S}_\varepsilon) \mid (w(x_1),\dots,w(x_n)) \in B_n\}$ is a cylindrical set. Then, it is easy to check $T\colon (\Omega,\mathscr{F}) \to (C(\mathbb{S}_\varepsilon), \sigma(\{\mathcal{C}\}))$ is measurable. From $\sigma(\{\mathcal{C}\}) = \mathcal{B}(C(\mathbb{S}_\varepsilon))$ (see, e.g., [45]), we can conclude that T is $\mathscr{F}/\mathcal{B}(C(\mathbb{S}_\varepsilon))$-measurable.

From the continuity of the paths, $\mathcal{A}$ is also $\mathscr{F}/\mathcal{B}(C(\mathbb{S}_\varepsilon))$-measurable, and thus from the continuity of $\mathcal{A} \mapsto u$, we conclude that $u\colon (\Omega,\mathscr{F}) \to (X, \mathcal{B}(\|\cdot\|_X))$ is measurable. Following the same argument, so is $\widetilde{u}$. From the continuity of $\|\cdot\|_X : X \to \mathbb{R}$, we see that $\|u - \widetilde{u}\|_X$ is a random variable. $\square$

Finally, we obtain the estimate on the expected error of $\|u(\omega) - \widetilde{u}(\omega)\|_X$.

Theorem 2.5.7. *Suppose the same assumption as Proposition 2.5.4 is satisfied. Let $u(\omega)$ and $\widetilde{u}(\omega)$ be the solution of* (2.5.18) *and* (2.5.22), *respectively. Then, we have*

$$\mathbb{E}[\|u(\omega) - \widetilde{u}(\omega)\|_X] \le \sqrt{2}\|f\|_{L^2(\mathbb{S}_\varepsilon^\circ)}(c_4'c_4)^{\frac14}(c_2 + c_2')^{\frac12}\sqrt{\mathscr{E}(\eta,t,K,L,T)}, \tag{2.5.27}$$

where c_2, c_2', c_4, c_4' are constants from Proposition 2.5.3 with $p = 2, 4$. Further, $\mathscr{E}(\eta,t,K,L,T)$ is defined in Theorem 2.5.2.

Proof. We claim $\mathbb{E}\left[\frac{1}{(\alpha_{\min})^4}\right] < c_4$, $\mathbb{E}\left[\frac{1}{(\alpha'_{\min})^4}\right] < c_4'$. Indeed, we have

$$\frac{1}{\min_{x\in\mathbb{S}_\varepsilon}\exp(T(x))} = \exp(-\min_{x\in\mathbb{S}_\varepsilon} T(x)) \le \exp(\sup_{x\in\mathbb{S}_\varepsilon}|T(x)|), \tag{2.5.28}$$

and thus from Proposition 2.5.3

$$\mathbb{E}\left[\frac{1}{[\min_{x\in\mathbb{S}_\varepsilon}\exp(T(x))]^4}\right] \le \mathbb{E}\Big[\exp(4\sup_{x\in\mathbb{S}_\varepsilon}|T(x)|)\Big] \le c_4. \tag{2.5.29}$$

Similarly, $\mathbb{E}\left[\frac{1}{(\alpha'_{\min})^4}\right] < c'_4$. Therefore, in view of Lemma 2.5.6 and Remark 5, we have

$$\mathbb{E}[\,\|u(\omega)-\widetilde{u}(\omega)\|_X] \le \|f\|_{L^2(\mathbb{S}_\varepsilon^\circ)}(c'_4c_4)^{\frac{1}{4}}\mathbb{E}\left[\sup_{x\in\mathbb{S}_\varepsilon}|\mathcal{A}(x;\omega)-\mathcal{A}_{KL}(x;\omega)|^2\right]^{\frac{1}{2}},$$

and hence from Proposition 2.5.4 the statement follows. □

2.6 Conclusion

In this chapter, we considered the Gaussian random fields on the spherical shell that are isotropic in the angular, and anisotropic in the radial direction. Under a suitable assumption on the covariance function, we established the continuity and partial differentiability of the realisations, and further, utilising the regularity theory of the elliptic PDEs, the Sobolev smoothness. Based on the regularity theory developed, we provided error analyses for the filtered approximation of the Gaussian and log-normal random fields. Furthermore, we considered a class of elliptic PDEs with log-normal random coefficients, and analysed the error incurred by approximating the variable coefficient of the differential operator with the filtered approximation.

Chapter 3

Quasi-Monte Carlo integration with product weights for elliptic PDEs with log-normal coefficients

Abstract of this chapter

Quasi-Monte Carlo (QMC) integration of output functionals of solutions of the diffusion problem with a log-normal random coefficient is considered. The random coefficient is assumed to be given by an exponential of a Gaussian random field that is represented by a series expansion of some system of functions. Graham et al. [36] developed a lattice-based QMC theory for this problem and established a quadrature error decay rate ≈ 1 with respect to the number of quadrature points. The key assumption there was a suitable summability condition on the aforementioned system of functions. As a consequence, product-order-dependent (POD) weights were used to construct the lattice rule. In this chapter, a different assumption on the system is considered. This assumption, originally considered by Bachmayr et al. [10] to utilise the locality of support of basis functions in the context of polynomial approximations applied to the same type of the diffusion problem, is shown to work well in the same lattice-based QMC method considered by Graham et al.: the assumption leads us to product weights, which enables the construction of the QMC method with a smaller computational cost than Graham et al. A quadrature error decay rate ≈ 1 is established, and the theory developed here is applied to a wavelet stochastic model. By a characterisation of the Besov smoothness, it is shown that a wide class of path smoothness can be treated with this framework.

3.1 Introduction

This chapter is concerned with quasi-Monte Carlo (QMC) integration of output functionals of solutions of the diffusion problem with a random coefficient of the form

$$-\nabla \cdot (a(x,\boldsymbol{y})\nabla u(x,\boldsymbol{y})) = f(x) \quad \text{in } D \subset \mathbb{R}^d, \qquad u = 0 \quad \text{on } \partial D, \tag{3.1.1}$$

where $\boldsymbol{y} \in \Omega$ is an element of a suitable probability space $(\Omega, \mathscr{F}, \mathbb{P})$ (clarified below), and $D \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary. Our interest is in the log-normal case, that is, $a(\cdot,\cdot)\colon D \times \Omega \to \mathbb{R}$ is assumed to have the form

$$a(x,\boldsymbol{y}) = a_*(x) + a_0(x)\exp(T(x,\boldsymbol{y})) \tag{3.1.2}$$

with continuous functions $a_*(x) \geq 0$, $a_0(x) > 0$ on $\overline{D}$, and Gaussian random field $T(\cdot,\cdot)\colon D \times \Omega \to \mathbb{R}$ represented by a series expansion

$$T(x,\boldsymbol{y}) = \sum_{j=1}^{\infty} Y_j(\boldsymbol{y})\psi_j(x), \quad x \in D, \tag{3.1.3}$$

where $\{Y_j\}$ is a collection of independent standard normal random variables on $(\Omega, \mathscr{F}, \mathbb{P})$, and $(\psi_j)_{j\geq 1}$ is a suitable system of real-valued measurable functions on D.

To handle a wide class of a and f, we consider the weak formulation of the problem (3.1.1). By V we denote the zero-trace Sobolev space $H_0^1(D)$ endowed with the norm

$$\|v\|_V := \Big(\int_D |\nabla v(x)|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}}, \tag{3.1.4}$$

and by $V' := H^{-1}(D)$ the topological dual space of V. For the given random coefficient $a(x,\boldsymbol{y})$, we define the bilinear form $\mathscr{A}(\boldsymbol{y};\cdot,\cdot)\colon V \times V \to \mathbb{R}$ by

$$\Omega \ni \boldsymbol{y} \mapsto \mathscr{A}(\boldsymbol{y};v,w) := \int_D a(x,\boldsymbol{y})\nabla v(x)\cdot\nabla w(x)\,\mathrm{d}x \quad \text{for all } v,w \in V. \tag{3.1.5}$$

Then, for any $\boldsymbol{y} \in \Omega$, the weak formulation of (3.1.1) reads: find $u(\cdot,\boldsymbol{y}) \in V$ such that

$$\mathscr{A}(\boldsymbol{y};u(\cdot,\boldsymbol{y}),v) = \langle f,v\rangle \qquad \text{for all } v \in V, \tag{3.1.6}$$

where f is assumed to be in V', and $\langle\cdot,\cdot\rangle$ denotes the duality paring between V' and V. We impose further conditions to ensure the well-posedness of the problem, which we will discuss later.

The ultimate goal is to compute $\mathbb{E}[\mathcal{G}(u(\cdot))]$, the expected value of $\mathcal{G}(u(\cdot, \boldsymbol{y}))$, where $\mathcal{G}$ is a linear bounded functional on V. The problem (3.1.1), and of computing $\mathbb{E}[\mathcal{G}(u(\cdot))]$ often arises in many applications such as hydrology [23, 78, 79], and has attracted attention in computational uncertainty quantification (UQ). See, for example, Cohen and DeVore [20], Schwab and Gittelson [89], and Kuo and Nuyens [64] and references therein. Two major ways to tackle this problem are function approximation, and quadrature, in particular, quasi-Monte Carlo (QMC) methods.

Our interest is in QMC. It is now well known that the QMC methods beats the plain-vanilla Monte Carlo methods in various settings when applied to the problems of computing $\mathbb{E}[\mathcal{G}(u(\cdot))]$ [36, 64, 65]. Among the QMC methods, the algorithm we consider is *randomly shifted lattice rules*.

Graham et al. [36] showed that when the randomly shifted lattice rules are applied to the class of PDEs we consider, a QMC convergence rate, in terms of expected root square mean root, ≈ 1 is achievable, which is known to be optimal for lattice rules in the function space they consider. More precisely, they showed that quadrature points for randomly shifted lattice rules that achieve such a rate can be constructed using an algorithm called component-by-component (CBC) construction. The algorithm uses *weights*, which represents the relative importance of subsets of the variables of the integrand, as an input, and the cost of it is dependent on the type of weights. The weights considered by Graham et al. [36] are so-called product-order-dependent (POD) weights, which were determined by minimising an error bound. For POD weights, the CBC construction takes $\mathcal{O}(sn\log n + s^2 n)$ operations, where n is the number of quadrature points and s is the dimension of truncation $\sum_{j=1}^{s} Y_j(\boldsymbol{y})\psi_j(x)$.

The contributions of the current chapter are twofold: proof of a convergence rate ≈ 1 with product weights, and an application to a stochastic model with wavelets. In more detail, we show that for the currently considered problem, the CBC construction can be constructed with weights called product weights, and achieves the optimal rate ≈ 1 in the function space we consider, and further, we show that the developed theory can be applied to a stochastic model which covers a wide class of wavelet bases.

Often in practice, we want to approximate the random coefficients well, and consequently s has to be taken to be large, in which case the second term of $\mathcal{O}(sn\log n + s^2 n)$ becomes dominant. The use of the POD weights originates from the summability condition imposed on (ψ_j) by Graham et al. [36]. We consider a different condition, the one proposed by Bachmayr et al. [10] to utilise the locality of supports of (ψ_j) in the context of polynomial approximations applied to PDEs with random coefficients. We show that under this condition, the shifted lattice rule for the PDE problem can be constructed with a CBC algorithm with the computational cost $\mathcal{O}(sn\log n)$, the cost with the product weights as shown by Dick et al. [28]. Further, the stochastic model we consider broadens the range of applicability of the QMC methods to the PDEs with log-normal coefficients. One concern about the conditions, in particular the summability condition on (ψ_j), imposed in Graham et al. [36] is that it is so strong that only random coefficients with smooth realisations are in the scope of the theory. We show that at least for $d = 1, 2$, such random coefficients (e.g., realisations with just some Hölder smoothness) can be considered.

Upon finalising the paper on which this chapter is based, we learnt about the paper by Herrmann and Schwab [39]. Our works share the same spirit in that we are all inspired by the work by Bachmayr et al. [10]. We provide a different, arguably simpler, proof for the same convergence rate with the exponential weight function, and we discuss the roughness of the realisations that can be considered.

Herrmann and Schwab [39] develops a theory under the setting essentially the same as ours. In contrast to this chapter, they treat the truncation error in a general setting, and as for the QMC integration error, they consider both the exponential weight functions and the Gaussian weight function for the weighted Sobolev space. As for the exponential weight function, the current chapter and Herrmann and Schwab [39] impose essentially the same assumptions (Assumption 3.1 below), and show the same convergence rate. However, our proof strategy is different, which turns out to result in different (product) weights. This can lead to a smaller implied constant in the estimate especially when the fields fluctuation is large, as we discuss later. Further, in contrast to Herrmann and Schwab [39], we provide a discussion of the roughness of the realisations of random coefficients as mentioned above. In Section 3.5, we provide a discussion via the Besov *characterisation* of the realisations of the random coefficients and the embedding results.

We now give a remark on the uniform case setting. One of the keys in the current chapter, which deals with the log-normal case, is the estimate of the derivative given in Corollary 3.3.2. This result essentially follows from the bounds obtained by Bachmayr et al. [10]. The similar argument employed in the current chapter is applicable to the randomly shifted lattice rules applied to PDEs with uniform random coefficients considered by Kuo, Schwab, and Sloan [65], using the derivative bounds for the uniform case considered by Bachmayr, Cohen, and Migliorati [9]. For this, we defer to Gantner, Herrmann, and Schwab [32], in which not only the randomly shifted lattice rules but also higher order QMCs were considered.

The outline of the rest of the chapter is as follows. In Section 3.2, we describe the problem we consider in detail. In Section 3.3, we discuss bounds on mixed derivatives. Then, in Section 3.4 we develop the QMC theory applied to the PDE problem with log-normal coefficients using the product weights. Section 3.5 provides an application of the theory: we consider a stochastic model represented by a wavelet Riesz basis. Then, we close this chapter with concluding remarks in Section 3.6.

3.2 Setting

We assume that the Gaussian random field T admits a series representation as in (3.1.3). We fix $(\Omega, \mathscr{F}, \mathbb{P}) := (\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}), \mathbb{P}_Y)$, where $\mathbb{N} := \{1, 2, \dots, \}$, $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$ is the Borel σ-algebra generated by the product topology in $\mathbb{R}^{\mathbb{N}}$, and $\mathbb{P}_Y := \prod_{j=1}^{\infty} \mathbb{P}_{Y_j}$ is the product measure on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}))$ defined by the standard normal distributions $\{\mathbb{P}_{Y_j}\}_{j\in\mathbb{N}}$ on $\mathbb{R}$ (see, for example, Itô [44, Chapter 2] for details). Then, for each $\boldsymbol{y} \in \Omega$ we may see $Y_j(\boldsymbol{y})$ $(j \in \mathbb{N})$ as given by the projection (or the canonical coordinate function)

$$\Omega = \mathbb{R}^{\mathbb{N}} \ni \boldsymbol{y} \mapsto Y_j(\boldsymbol{y}) =: y_j \in \mathbb{R}.$$

Note in particular that from the continuity of the projection, the mapping $\boldsymbol{y} \mapsto y_j$ is $\mathcal{B}(\mathbb{R}^{\mathbb{N}})/\mathcal{B}(\mathbb{R})$-measurable.

In the following, we write T above as

$$T(x, \boldsymbol{y}) = \sum_{j=1}^{\infty} y_j \psi_j(x), \quad x \in D, \tag{3.2.1}$$

and see it as a deterministically parametrised function on D. We will impose a condition considered by Bachmayr et al. [10] on (ψ_j), see Assumption 3.1 below, that is particularly suitable for ψ_j with local support.

To ensure the law on $\mathbb{R}^D$ is well defined, we suppose

$$\sum_{j=1}^{\infty} \psi_j(x)^2 < \infty \text{ for all } x \in D, \tag{3.2.2}$$

so that the covariance function $\mathbb{E}[T(x_1)T(x_2)] = \sum_{j\geq 1} \psi_j(x_1)\psi_j(x_2)$ $(x_1, x_2 \in D)$ is well-defined.

We consider the parametrised elliptic partial differential equation (3.1.1). To prove well-posedness of the variational problem (3.1.6), we use the Lax–Milgram lemma. Conditions which ensure that the bilinear form $\mathscr{A}(\boldsymbol{y};\cdot,\cdot)$ defined by the diffusion coefficient a is coercive and bounded are discussed later.

Motivated by UQ applications, we are interested in expected values of bounded linear functionals of the solution of the above PDEs. We note that the linearity is for the sake of the theoretical interest. Theoretical treatment of non-linear functionals will require suitable smoothness and mild growth of suitable derivatives, but it is almost unexplored with an exception being an attempt by Scheichl, Stuart, and Teckentrup [88].

Given a continuous linear functional $\mathcal{G} \in V'$ we wish to compute $\mathbb{E}[\mathcal{G}(u(\cdot))] := \int_{\mathbb{R}^{\mathbb{N}}} \mathcal{G}(u(\cdot, \boldsymbol{y}))\mathrm{d}\mathbb{P}_Y(\boldsymbol{y})$, where the measurability of the integrands will be discussed later. To compute $\mathbb{E}[\mathcal{G}(u(\cdot))]$ we use a sampling method: generate realisations of $a(x, \boldsymbol{y})$, which yields the solution $u(x, \boldsymbol{y})$ via the PDE (3.1.1), and from these we compute $\mathbb{E}[\mathcal{G}(u(\cdot))]$.

In practice, these operations cannot be performed exactly, and numerical methods need to be employed. This chapter gives an analysis of the error incurred by the method outlined as follows. We truncate the series (3.2.1) to s terms for some integer $s \geq 1$, which results in the coefficient $a(x, (y_1, \dots, y_s, 0, 0, 0, \dots))$ of the PDE (3.1.1). Further, the expectation of the random variable that is the solution of the corresponding solution of the PDE applied to the linear functional $\mathcal{G}$ is approximated by a QMC method.

Let $u^s(x) = u^s(x, \boldsymbol{y})$ be the solution of (3.1.1) with $\boldsymbol{y} = (y_1, \dots, y_s, 0, 0, 0, \dots)$, that is, of the problem: Find $u^s \in V$ such that

$$-\nabla \cdot (a(x, (y_1, \dots, y_s, 0, 0, \cdots))\nabla u^s(x)) = f(x) \quad \text{in } D, \qquad u^s = 0 \text{ on } \partial D. \tag{3.2.3}$$

Here, even though the dependence of u^s on $\boldsymbol{y}$ is only on $(y_1, \dots, y_s)$, we abuse the notation slightly by writing $u^s(x, \boldsymbol{y}) := u^s(y_1, \dots, y_s, 0, 0, 0, \dots)$.

Let $\Phi_s^{-1} \colon [0,1]^s \ni \boldsymbol{v} \mapsto \Phi_s^{-1}(\boldsymbol{v}) \in \mathbb{R}^s$ be the inverse of the cumulative normal distribution function applied to each entry of $\boldsymbol{v}$. We write

$$F(\boldsymbol{y}) := F(y_1, \dots, y_s) = \mathcal{G}(u^s(\cdot, \boldsymbol{y})), \tag{3.2.4}$$

and

$$I_s(F) := \int_{\boldsymbol{v}\in(0,1)^s} F(\Phi_s^{-1}(\boldsymbol{v}))\mathrm{d}\boldsymbol{v} = \int_{\boldsymbol{y}\in\mathbb{R}^s} \mathcal{G}(u^s(\cdot,\boldsymbol{y}))\prod_{j=1}^{s}\phi(y_j)\mathrm{d}\boldsymbol{y} = \mathbb{E}[\mathcal{G}(u^s)], \tag{3.2.5}$$

where ϕ is the probability density function of the standard normal random variable. The measurability of the mapping $\mathbb{R}^s \ni \boldsymbol{y} \mapsto \mathcal{G}(u^s(\cdot,\boldsymbol{y})) \in \mathbb{R}$ will be discussed later.

In order to approximate $I_s(F)$, we employ a QMC method called a *randomly shifted lattice rule*. This is an equal-weight quadrature rule of the form

$$\mathcal{Q}_{s,n}(\boldsymbol{\Delta};F) := \frac{1}{n}\sum_{i=1}^{n} F\left(\Phi_s^{-1}\left(\mathrm{frac}\left(\frac{i\boldsymbol{z}}{n}+\boldsymbol{\Delta}\right)\right)\right),$$

where the function $\mathrm{frac}(\cdot)\colon \mathbb{R}^s \ni \boldsymbol{y} \mapsto \mathrm{frac}(\boldsymbol{y}) \in [0,1)^s$ takes the fractional part of each component in $\boldsymbol{y}$. Here, $\boldsymbol{z} \in \mathbb{N}^s$ is a carefully chosen point called the (deterministic) *generating vector* and $\boldsymbol{\Delta} \in [0,1]^s$ is the *random shift*. We assume the random shift $\boldsymbol{\Delta}$ is a $[0,1]^s$-valued uniform random variable defined on a suitable probability space different from $(\Omega,\mathscr{F},\mathbb{P})$. For further details of the randomly shifted lattice rules, we refer to the surveys Dick, Kuo, and Sloan [27] and Kuo and Nuyens [64] and references therein.

We want to evaluate the root-mean-square error

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}\left[\left(\mathbb{E}[\mathcal{G}(u)] - \mathcal{Q}_{s,n}(\boldsymbol{\Delta};F)\right)^2\right]}. \tag{3.2.6}$$

where $\mathbb{E}^{\boldsymbol{\Delta}}$ is the expectation with respect to the random shift. Note that in practice the solution u^s needs to be approximated by some numerical scheme $\widetilde{u}^s$, such as the finite element method as considered in Graham et al. [36] and Kuo and Nuyens [64], which results in computing $\widetilde{F}(\boldsymbol{y}) := \mathcal{G}(\widetilde{u}^s(\boldsymbol{y}))$. Thus, the error $e_{s,n} := \sqrt{\mathbb{E}^{\boldsymbol{\Delta}}\left[\left(\mathbb{E}[\mathcal{G}(u)] - \mathcal{Q}_{s,n}(\boldsymbol{\Delta};\widetilde{F})\right)^2\right]}$ is what we need to evaluate in practice. Via the trivial decompositions we have, using $\mathbb{E}^{\boldsymbol{\Delta}}[\mathcal{Q}_{s,n}(\boldsymbol{\Delta};\widetilde{F})] = \mathbb{E}[\mathcal{G}(\widetilde{u}^s)]$ (see, for example, Dick, Kuo, and Sloan [27]),

$$e_{s,n}^2 = (\mathbb{E}[\mathcal{G}(u) - \mathcal{G}(\widetilde{u}^s)])^2 + \mathbb{E}^{\boldsymbol{\Delta}}\left[\left(\mathbb{E}[\mathcal{G}(\widetilde{u}^s)] - \mathcal{Q}_{s,n}(\boldsymbol{\Delta};\widetilde{F})\right)^2\right]. \tag{3.2.7}$$

For the sake of simplicity, we forgo the discussion on the numerical approximation of the solution of the PDE. Instead, we discuss the smoothness of the realisations of the random coefficient. Then, given a suitable smoothness of the boundary ∂D, the convergence rate of $\mathbb{E}[\mathcal{G}(u)-\mathcal{G}(\widetilde{u}^s)]$ is typically obtained from the smoothness of the realisations of the coefficients $a(\cdot,\boldsymbol{y})$, via the regularity of the solution u. See Graham et al. [36] and Kuo and Nuyens [64]. Therefore in the following, we concentrate on the truncation error and the quadrature error, the second and the third term of the above decomposition, and the realisations of a.

In the course of the error analyses, we assume (ψ_j) satisfies the following assumption.

Assumption 3.1. The system (ψ_j) satisfies the following. There exists a positive sequence (ρ_j) such that

$$\sup_{x\in D}\sum_{j\geq 1}\rho_j|\psi_j(x)| =: \kappa < \ln 2, \tag{\textbf{b1}}$$

and further,

$$(1/\rho_j)\in \ell^q \qquad \text{for some } q\in(0,1]. \tag{\textbf{b2}}$$

We also use the following weaker assumption.

Assumption 3.1′. The same as Assumption 3.1, only with the condition (**b2**) being replaced with

$$(1/\rho_j)\in \ell^q \qquad \text{for some } q\in(0,\infty). \tag{\textbf{b2}$'$}$$

We note that (**b2′**), and thus also (**b2**), implies $\rho_j\to\infty$ as $j\to\infty$.

Some remarks on the assumptions are in order. First note that Assumption 3.1′ implies $\sum_{j\geq 1}|\psi_j(x)|<\infty$ for any $x\in D$, and hence (3.2.2). Assumption 3.1′ is used to obtain an estimate on the mixed derivative with respect to the random parameter y_j, and further, ensures the almost surely well-posedness of the problem (3.1.6) — see Corollary 3.3.2 and Remark 1. Assumption 3.1 is used to obtain a dimension-independent QMC error estimate — see Theorem 3.4.4, and Theorem 3.5.1. The stronger the condition (**b2**) the system (ψ_j) satisfies, that is, the smaller is q, the smoother the realisations of the random coefficient become. In Section 3.5.2, we discuss smoothness of realisations allowed by these conditions.

3.3 Bounds on mixed derivatives

In this section, we discuss bounds on mixed derivatives. In order to motivate the discussion in this section, first we explain how the derivative bounds come into play in the QMC analysis developed in the next section.

Application of QMC methods to elliptic PDEs with log-normal random coefficients was initiated with computational results by Graham et al. [35], and an analysis was followed by Graham et al. [36]. Following the discussion by Graham et al. [36], we assume the integrand f is in the space called the *weighted unanchored Sobolev space* $\mathcal{W}^s$, consisting of measurable functions $f\colon \mathbb{R}^s \to \mathbb{R}$ such that

$$\begin{aligned} &\|f\|^2_{\mathcal{W}^s} \\ &= \sum_{\mathfrak{u}\subseteq\{1:s\}} \frac{1}{\gamma_{\mathfrak{u}}} \int_{\mathbb{R}^{|\mathfrak{u}|}} \left(\int_{\mathbb{R}^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} f}{\partial y_{\mathfrak{u}}} (\boldsymbol{y}_{\mathfrak{u}}; \boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}}) \prod_{j\in\{1:s\}\setminus\mathfrak{u}} \phi(y_j) \, \mathrm{d}\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}} \right)^2 \prod_{j\in\mathfrak{u}} w_j^2(y_j) \, \mathrm{d}\boldsymbol{y}_{\mathfrak{u}} \\ &< \infty, \end{aligned} \tag{3.3.1}$$

where we assume, similarly to Graham et al. [36], that

$$w_j^2(y_j) = \exp(-2\alpha_j|y|) \tag{3.3.2}$$

for some $\alpha_j > 0$. Here, $\{1 : s\}$ is a shorthand notation for the set $\{1, \dots, s\}$; $|\mathfrak{u}|$ denotes the cardinality of the set $\mathfrak{u} \subseteq \{1 : s\}$; $\frac{\partial^{|\mathfrak{u}|}F}{\partial y_{\mathfrak{u}}}$ denotes the mixed first derivative with respect to each of the "active" variables y_j with $j \in \mathfrak{u} \subseteq \{1 : s\}$; and $\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}}$ denotes the "inactive" variables y_j with $j \notin \mathfrak{u}$. Further, weights $(\gamma_{\mathfrak{u}})$ describe the relative importance of the variables $\{y_j\}_{j\in\mathfrak{u}}$. Note that the measure $\int_\cdot \mathrm{d}y_{\mathfrak{u}}$ and $\int_\cdot \frac{1}{\gamma_{\mathfrak{u}}} \mathrm{d}y_{\mathfrak{u}}$ differ by at most a constant factor depending on $\mathfrak{u}$. Weights $(\gamma_{\mathfrak{u}})$ play an important role in deriving error estimates independently of the dimension s, and further, in obtaining the generating vector $\boldsymbol{z}$ for the lattice rule via the component-by-component (CBC) algorithm. We note that the norm $\|\cdot\|_{\mathcal{W}^s}$ resembles the one appearing in the Koksma-Hlawka inequality. For more details on this connection, we refer the interested readers to [81, Section 2.6.1], [27] and references therein.

Depending on the problem, different types of weights have been considered to derive error estimates. For the randomly shifted lattice rules, "POD weights" and "product weights" have been considered [27, 64]. POD weights, which stands for "product and order dependent form", are the weights of the form

$\gamma_{\mathfrak{u}} = \Gamma_{|\mathfrak{u}|} \prod_{j\in\mathfrak{u}} \Upsilon_j$ specified by suitable two sets of non-negative numbers $\{\Gamma_k\}$ and $\{\Upsilon_j\}$; and product weights are the ones of the form $\gamma_{\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_j$ with a suitable set of non-negative numbers $\{\gamma_j\}$. When applied to the PDE parametrised with log-normal coefficients, the result in Graham et al. [36] suggests the use of POD weights for the problem.

We wish to develop a theory on the applicability of product weights, which has an advantage in terms of computational cost. The computational cost of the CBC construction is $\mathcal{O}(sn\log n + ns^2)$ in the case of POD weights, compared to $\mathcal{O}(sn\log n)$ for product weights [28]. Since we often want to approximate the random field well, and so necessarily we have large s, the applicability of product weights is of clear interest.

Estimates of derivatives of the integrand $F(\boldsymbol{y})$ with respect to the parameter $\boldsymbol{y}$, that is, the variable with which $F(\boldsymbol{y})$ is integrated, are one of the keys in the error analysis of QMC. In Graham et al. [36], it was the estimates being of "POD-form" that led their theory to the POD weights. Under an assumption on the system (ψ_j), which is different from that in Graham et al. [36], we show that the derivative estimates turn out to be of "product-form", and further that, under a suitable assumption, we achieve the same error convergence rate close to 1 with product weights.

Now, we derive an estimate of the product form. Let

$$\mathcal{F} := \{\mu = (\mu_1, \mu_2, \dots) \in \mathbb{N}_0^{\mathbb{N}} \mid \text{ all but finite number of components of } \mu \text{ are zero}\}.$$

For $\mu \in \mathcal{F}$ we use the notation $|\mu| = \sum_{j\geq 1} \mu_j$, $\mu! = \prod_{j\geq 1} \mu_j!$, $\rho^{\mu} = \prod_{j\geq 1} \rho_j^{\mu_j}$ for $\rho = (\rho_j)_{j\geq 1} \in \mathbb{R}^{\mathbb{N}}$, and

$$\partial^{\mu} u = \frac{\partial^{|\mu|}}{y_{j(1)}^{\mu_{j(1)}} \cdots y_{j(k)}^{\mu_{j(k)}}} u, \tag{3.3.3}$$

where $k = \#\{j \mid \mu_j \neq 0\}$.

We have the following bound on mixed derivatives of order $r \geq 1$ (although in our application we will need only $r = 1$). The proof follows essentially the same argument as the proof by Bachmayr et al. [10, Theorem 4.1]. Here, we show a tighter bound by changing the condition from $\frac{\ln 2}{\sqrt{r}}$ to $\frac{\ln 2}{r}$ in Bachmayr et al. [10, (91)], and we have $\rho^{2\mu}$ in (3.3.4) in place of $\frac{\rho^{2\mu}}{\mu!}$ in the left hand side of Bachmayr et al. [10, (92)].

Proposition 3.3.1. *Let $r \geq 1$ be an integer. Suppose (ψ_j) satisfies the condition (**b1**) with $\ln 2$ replaced by $\frac{\ln 2}{r}$, with a positive sequence (ρ_j). Then, there exists a constant $C_0 = C_0(r)$ that depends on κ and r, such that*

$$\sum_{\substack{\mu \in \mathcal{F} \\ \|\mu\|_\infty \leq r}} \rho^{2\mu} \int_D a(\boldsymbol{y}) |\nabla(\partial^\mu u(\boldsymbol{y}))|^2 \, \mathrm{d}x \leq C_0 \int_D a(\boldsymbol{y}) |\nabla u(\boldsymbol{y})|^2 \, \mathrm{d}x. \tag{3.3.4}$$

for all $\boldsymbol{y}$ that satisfy $\left\| \sum_{j \geq 1} y_j \psi_j \right\|_{L^\infty(D)} < \infty$, where $u(\boldsymbol{y})$ is the solution of (3.1.6) for such $\boldsymbol{y}$. The same bound holds also for $u^s(\boldsymbol{y})$, the solution of (3.1.6) with $\boldsymbol{y} = (y_1, \dots, y_s, 0, 0, \dots)$.

Proof. Let

$$\Lambda_k := \{\mu \in \mathcal{F} \mid |\mu| = k \text{ and } \|\mu\|_{\ell_\infty} \leq r\},$$

and

$$S_\mu := \{\nu \in \mathcal{F} \mid \nu \leq \mu \text{ and } \nu \neq \mu\} \text{ for } \mu \in \mathcal{F},$$

with $\leq$ denoting the component-wise partial order between multi-indices. Let us introduce the notation $\|v\|_{a(\boldsymbol{y})}^2 := \int_D a(\boldsymbol{y}) |\nabla v|^2 \, \mathrm{d}x$ for all $v \in V$, and let

$$\sigma_k := \sum_{\mu \in \Lambda_k} \rho^{2\mu} \|\partial^\mu u(\boldsymbol{y})\|_{a(\boldsymbol{y})}^2 .$$

We show below that we can choose $\delta = \delta(r) < 1$ such that

$$\sigma_k \leq \sigma_0 \delta^k \qquad \text{for all } k \geq 0. \tag{3.3.5}$$

Note that if this holds then we have

$$\sum_{\|\mu\|_\infty \leq r} \rho^{2\mu} \|\partial^\mu u(\boldsymbol{y})\|_{a(\boldsymbol{y})}^2 = \sum_{k=0}^{\infty} \sum_{\mu \in \Lambda_k} \rho^{2\mu} \|\partial^\mu u(\boldsymbol{y})\|_{a(\boldsymbol{y})}^2 = \sum_{k=0}^{\infty} \sigma_k \leq \sigma_0 \sum_{k=0}^{\infty} \delta^k < \infty, \tag{3.3.6}$$

and the statement will follow with $C_0 = C_0(r) = \sum_{k=0}^{\infty} \delta(r)^k$.

We now show $\sigma_k \leq \sigma_0 \delta^k$. Note that from the assumption $\| \sum_{j \geq 1} y_j \psi_j \|_{L^\infty(D)} < \infty$, in view of Bachmayr et al. [10, Lemma 3.2] we have $\partial^\mu u \in V$ for any $\mu \in \mathcal{F}$.

Thus, by taking $v := \partial^\mu u$ ($\mu \in \Lambda_k$) in Bachmayr et al. [10, (74)], we have

$$\begin{aligned}\sigma_k &= \sum_{\mu\in\Lambda_k} \rho^{2\mu}\int_D a(\boldsymbol{y})|\nabla\partial^\mu u(\boldsymbol{y})|^2 \,\mathrm{d}x \\ &\leq \sum_{\mu\in\Lambda_k}\sum_{\nu\in S_\mu}\left(\prod_{j\geq 1}\frac{\mu_j!\rho_j^{\mu_j-\nu_j}\rho^{\mu_j}\rho^{\nu_j}}{\nu_j!(\mu_j-\nu_j)!}\right)\int_D a(\boldsymbol{y})\left(\prod_{j\geq 1}|\psi_j|^{\mu_j-\nu_j}\right)|\nabla\partial^\nu u(\boldsymbol{y})||\nabla\partial^\mu u(\boldsymbol{y})|\,\mathrm{d}x.\end{aligned} \tag{3.3.7}$$

Using the notation

$$\epsilon(\mu,\nu)(x) := \epsilon(\mu,\nu) := \frac{\mu!}{\nu!}\frac{\rho^{\mu-\nu}|\psi|^{\mu-\nu}}{(\mu-\nu)!}, \tag{3.3.8}$$

and the Cauchy–Schwarz inequality for the sum over S_μ, it follows that

$$\sigma_k \tag{3.3.9}$$

$$\leq \int_D \sum_{\mu\in\Lambda_k}\sum_{\nu\in S_\mu}\epsilon(\mu,\nu)a(\boldsymbol{y})|\rho^\nu\nabla\partial^\nu u(\boldsymbol{y})||\rho^\mu\nabla\partial^\mu u(\boldsymbol{y})|\,\mathrm{d}x \tag{3.3.10}$$

$$\leq \int_D \sum_{\mu\in\Lambda_k}\left(\sum_{\nu\in S_\mu}\epsilon(\mu,\nu)a(\boldsymbol{y})|\rho^\nu\nabla\partial^\nu u(\boldsymbol{y})|^2\right)^{\frac12}\left(\sum_{\nu\in S_\mu}\epsilon(\mu,\nu)a(\boldsymbol{y})|\rho^\mu\nabla\partial^\mu u(\boldsymbol{y})|^2\right)^{\frac12}\mathrm{d}x. \tag{3.3.11}$$

Let

$$S_{\mu,\ell} := \{\nu\in S_\mu \mid |\mu-\nu| = \ell\}.$$

Then, for $\mu \in \Lambda_k$ we have

$$S_\mu = \{\nu\in\mathcal{F}\mid \nu\leq\mu,\ \nu\neq\mu\} = \bigcup_{\ell=1}^{|\mu|}\{\nu\in\mathcal{F}\mid\nu\leq\mu,\ |\mu-\nu|=\ell\} = \bigcup_{\ell=1}^{|\mu|}S_{\mu,\ell},$$

and further, from $|\mu| = k$, we have

$$\sum_{\nu\in S_\mu}\epsilon(\mu,\nu) = \sum_{\ell=1}^{k}\sum_{\nu\in S_{\mu,\ell}}\epsilon(\mu,\nu) = \sum_{\ell=1}^{k}\sum_{\nu\in S_{\mu,\ell}}\frac{\mu!}{\nu!}\frac{\rho^{\mu-\nu}|\psi|^{\mu-\nu}}{(\mu-\nu)!}. \tag{3.3.12}$$

Since $\nu\in S_{\mu,\ell}$ implies $\sum_{j\in\operatorname{supp}\mu}(\mu_j-\nu_j) = \ell$, there are ℓ factors in

$$\frac{\mu!}{\nu!} = \prod_{j\in\operatorname{supp}\mu}\mu_j(\mu_j-1)\cdots(\nu_j+1).$$

From $\mu_j \le r$ ($j \in \operatorname{supp}\mu$), each of the factors is at most r. Thus,

$$\frac{\mu!}{\nu!} \le r^\ell \quad \text{for } \mu \in \Lambda_k,\ \nu \in S_{\mu,\ell}.$$

Therefore, from the multinomial theorem, for each $x \in D$ it follows from (3.3.12) that

$$\sum_{\nu \in S_\mu} \epsilon(\mu,\nu) \le \sum_{\ell=1}^{k} r^\ell \sum_{\nu \in S_{\mu,\ell}} \frac{\rho^{\mu-\nu}|\psi|^{\mu-\nu}}{(\mu-\nu)!} \le \sum_{\ell=1}^{k} r^\ell \sum_{|\tau|=\ell} \frac{\rho^\tau |\psi|^\tau}{\tau!} = \sum_{\ell=1}^{k} r^\ell \frac{1}{\ell!} \sum_{|\tau|=\ell} \frac{\ell!}{\tau!}\rho^\tau |\psi|^\tau \tag{3.3.13}$$

$$= \sum_{\ell=1}^{k} r^\ell \frac{1}{\ell!} (\sum_{j=1}^{\infty} \rho_j |\psi_j|)^\ell \le \sum_{\ell=1}^{k} r^\ell \frac{1}{\ell!} \kappa^\ell \le e^{r\kappa} - 1 \le e^{\ln 2} - 1 = 1. \tag{3.3.14}$$

Inserting into (3.3.11), we have

$$\begin{aligned} &\sum_{\mu \in \Lambda_k} \rho^{2\mu} \left\| \partial^\mu u(\boldsymbol{y}) \right\|_{a(\boldsymbol{y})}^2 \\ &\le \int_D \sum_{\mu \in \Lambda_k} \left(\sum_{\nu \in S_\mu} \epsilon(\mu,\nu) a(\boldsymbol{y}) |\rho^\nu \nabla \partial^\nu u(\boldsymbol{y})|^2 \right)^{\frac{1}{2}} \left(a(\boldsymbol{y}) |\rho^\mu \nabla \partial^\mu u(\boldsymbol{y})|^2 \right)^{\frac{1}{2}} \,\mathrm{d}x. \end{aligned}$$

Again applying the Cauchy–Schwarz inequality to the summation over Λ_k and then to the integral, we have

$$\begin{aligned} \sigma_k &\le \int_D \left(\sum_{\mu \in \Lambda_k} \sum_{\nu \in S_\mu} \epsilon(\mu,\nu) a(\boldsymbol{y}) |\rho^\nu \nabla \partial^\nu u(\boldsymbol{y})|^2 \right)^{\frac{1}{2}} \left(\sum_{\mu \in \Lambda_k} a(\boldsymbol{y}) |\rho^\mu \nabla \partial^\mu u(\boldsymbol{y})|^2 \right)^{\frac{1}{2}} \,\mathrm{d}x \\ &\le \left(\int_D \sum_{\mu \in \Lambda_k} \sum_{\nu \in S_\mu} \epsilon(\mu,\nu) a(\boldsymbol{y}) |\rho^\nu \nabla \partial^\nu u(\boldsymbol{y})|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \sigma_k^{\frac{1}{2}}, \end{aligned}$$

and hence

$$\sigma_k \le \int_D \sum_{\mu \in \Lambda_k} \sum_{\nu \in S_\mu} \epsilon(\mu,\nu) a(\boldsymbol{y}) |\rho^\nu \nabla \partial^\nu u(\boldsymbol{y})|^2 \,\mathrm{d}x. \tag{3.3.15}$$

Now, for any $k \geq 1$ and any $\nu \in \Lambda_\ell = \{\nu \in \mathcal{F} \mid |\nu| = \ell,\ \|\nu\|_\infty \leq r\}$ with $\ell \leq k-1$, let

$$R_{\nu,\ell,k} := \{\mu \in \Lambda_k \mid \nu \in S_\mu\} = \{\mu \in \mathcal{F} \mid |\mu| = k,\ \|\mu\|_\infty \leq r,\ \mu \geq \nu,\ \mu \neq \nu\}.$$

Then, for fixed $k \geq 1$ we can write

$$\bigcup_{\mu\in\Lambda_k} \bigcup_{\nu\in S_\mu} (\mu,\nu) = \bigcup_{\ell=0}^{k-1} \bigcup_{\nu\in\Lambda_\ell} \bigcup_{\mu\in R_{\nu,\ell,k}} (\mu,\nu). \tag{3.3.16}$$

Thus, we have

$$\sum_{\mu\in\Lambda_k} \sum_{\nu\in S_\mu} \epsilon(\mu,\nu) a(\boldsymbol{y}) |\rho^\nu \nabla \partial^\nu u(\boldsymbol{y})|^2 = \sum_{\ell=0}^{k-1} \sum_{\nu\in\Lambda_\ell} a(\boldsymbol{y}) |\rho^\nu \nabla \partial^\nu u(\boldsymbol{y})|^2 \sum_{\mu\in R_{\nu,\ell,k}} \epsilon(\mu,\nu). \tag{3.3.17}$$

Now, note that $k - \ell = \sum_{j\in\operatorname{supp}\mu} \mu_j - \sum_{j\in\operatorname{supp}\mu} \nu_j = |\mu - \nu|$. Thus, we have $\frac{\mu!}{\nu!} \leq r^{k-\ell}$. It follows that

$$\sum_{\mu\in R_{\nu,\ell,k}} \epsilon(\mu,\nu) = \sum_{\nu\in R_{\nu,\ell,k}} \frac{\mu!}{\nu!} \frac{\rho^{\mu-\nu}|\psi|^{\mu-\nu}}{(\mu-\nu)!} \leq r^{k-\ell} \sum_{\nu\in R_{\nu,\ell,k}} \frac{\rho^{\mu-\nu}|\psi|^{\mu-\nu}}{(\mu-\nu)!} \tag{3.3.18}$$

$$\leq r^{k-\ell} \sum_{|\tau|=k-\ell} \frac{\rho^\tau |\psi|^\tau}{\tau!} \leq r^{k-\ell} \frac{1}{(k-\ell)!} \kappa^{k-\ell}. \tag{3.3.19}$$

Then, substituting (3.3.19) into (3.3.17) we obtain from (3.3.15)

$$\sigma_k \leq \sum_{\ell=0}^{k-1} \frac{1}{(k-\ell)!} (r\kappa)^{k-\ell} \sigma_\ell. \tag{3.3.20}$$

From the assumption we have $\kappa < \frac{\ln 2}{r}$. Thus, we can take $\delta = \delta(r) < 1$ such that $\kappa < \delta \frac{\ln 2}{r}$.

We show $\sigma_k \leq \sigma_0 \delta^k$ for all $k \geq 0$ by induction. This is clearly true for $k = 0$. Suppose $\sigma_\ell \leq \sigma_0 \delta^\ell$ holds for $\ell = 0, \ldots, k-1$. Then, for $\ell = k$ we have

$$\sigma_k \leq \sum_{\ell=0}^{k-1} \frac{1}{(k-\ell)!} (r\kappa)^{k-\ell} \sigma_\ell \leq \sum_{\ell=0}^{k-1} \frac{1}{(k-\ell)!} (r\kappa)^{k-\ell} \sigma_0 \delta^\ell \leq \sum_{\ell=0}^{k-1} \frac{1}{(k-\ell)!} (\delta \ln 2)^{k-\ell} \sigma_0 \delta^\ell \tag{3.3.21}$$

$$= \sigma_0 \delta^k \sum_{\ell=0}^{k-1} \frac{1}{(k-\ell)!} (\ln 2)^{k-\ell} \leq \sigma_0 \delta^k (e^{\ln 2} - 1) = \sigma_0 \delta^k, \tag{3.3.22}$$

which completes the proof. $\square$

With the notation

$$\check{a}(\boldsymbol{y}) := \operatorname*{ess\,inf}_{x\in D} a(x,\boldsymbol{y}), \qquad \text{and } \hat{a}(\boldsymbol{y}) := \operatorname*{ess\,sup}_{x\in D} a(x,\boldsymbol{y}), \tag{3.3.23}$$

we have the following corollary, where here and from now on we set $r = 1$.

Corollary 3.3.2. *Suppose (ψ_j) satisfies Assumption 3.1′ with a positive sequence (ρ_j). Then, for $C_0 = C_0(1)$ as in Proposition 3.3.1 for any $\mathfrak{u} \subset \mathbb{N}$ of finite cardinality we have*

$$\left\|\frac{\partial^{|\mathfrak{u}|}u(\boldsymbol{y})}{\partial \boldsymbol{y}_{\mathfrak{u}}}\right\|_V \le \sqrt{C_0}\frac{\|f\|_{V'}}{\check{a}(\boldsymbol{y})}\prod_{j\in\mathfrak{u}}\frac{1}{\rho_j} < \infty, \quad \textit{almost surely,} \tag{3.3.24}$$

where $\|\cdot\|_{V'}$ is the norm in the dual space V'. The same bound holds also for $\left\|\frac{\partial^{|\mathfrak{u}|}u^s}{\partial \boldsymbol{y}_{\mathfrak{u}}}\right\|_V$, with $\boldsymbol{y} = (y_1,\dots,y_s,0,0,\dots)$.

Proof. First, if $\boldsymbol{y} \in \mathbb{R}^{\mathbb{N}}$ satisfies $\|\sum_{j\ge1} y_j\psi_j\|_{L^\infty(D)} < \infty$, then we have $\frac{1}{(\check{a}(\boldsymbol{y}))} < \infty$:

$$\check{a}(\boldsymbol{y}) \ge \big(\inf_{x\in D} a_0(x)\big)\exp\Big(-\operatorname*{ess\,sup}_{x\in D}\Big|\sum_{j\ge1} y_j\psi_j(x)\Big|\Big), \tag{3.3.25}$$

and thus

$$\frac{1}{\check{a}(\boldsymbol{y})} \le \frac{1}{\big(\inf_{x\in D} a_0(x)\big)}\exp\Big(\operatorname*{ess\,sup}_{x\in D}\big|\sum_{j\ge1} y_j\psi_j(x)\big|\Big). \tag{3.3.26}$$

Now, from $(1/\rho_j) \in \ell^q$ for some $q \in (0,\infty)$, in view of Bachmayr et al. [10, Remark 2.2] we have

$$\mathbb{E}\Big[\exp\Big(k\Big\|\sum_{j\ge1} y_j\psi_j\Big\|_{L^\infty(D)}\Big)\Big] < \infty,$$

for any $0 \le k<\infty$. Thus, $\|\sum_{j\ge1} y_j\psi_j\|_{L^\infty(D)} < \infty$, and the right hand side of (3.3.24) is bounded with full (Gaussian) measure. We remark that the $\mathcal{B}(\mathbb{R}^{\mathbb{N}})/\mathcal{B}(\mathbb{R})$-measurability of the mapping $\boldsymbol{y} \mapsto \big\|\sum_{j\ge1} y_j\psi_j\big\|_{L^\infty(D)}$ is not an issue. See Bachmayr et al. [10, Remark 2.2] noting the continuity of norms, together with, for example, Reed and Simon [85, Appendix to IV. 5].

Now, recalling the standard argument regarding the continuous dependence of the solution of the variational problem (3.1.6) on f, we have $\int_D a(\boldsymbol{y})|\nabla(u(\boldsymbol{y}))|^2\,\mathrm{d}x \le$

$\frac{\|f\|^2_{V'}}{\breve{a}(\boldsymbol{y})}$. Then the claim follows from Proposition 3.3.1, noting that for any $\mathfrak{u} \subset \mathbb{N}$ of finite cardinality we have

$$\breve{a}(\boldsymbol{y}) \int_D \left| \nabla \left(\frac{\partial^{|\mathfrak{u}|} u}{\partial \boldsymbol{y}_{\mathfrak{u}}} \right) \right|^2 \mathrm{d}x \leq \sum_{\substack{\mu \in \mathcal{F} \\ \|\mu\|_\infty \leq 1}} \rho^{2\mu} \int_D a(\boldsymbol{y}) |\nabla(\partial^\mu u(\boldsymbol{y}))|^2 \,\mathrm{d}x. \tag{3.3.27}$$

$\square$

Remark 1. We note that following a similar discussion to the above, $\hat{a}(\boldsymbol{y})$ can be bounded almost surely. Thus, under the Assumption 3.1′, the well-posedness of the problem (3.1.6) readily follows almost surely. Further, Assumption 3.1′ implies the measurability of the mapping $\mathbb{R}^s \ni \boldsymbol{y} \mapsto \mathcal{G}(u^s(\cdot, \boldsymbol{y})) \in \mathbb{R}$. See Bachmayr et al. [10, Corollary 2.1, Remark 2.2] noting $\mathcal{G} \in V'$, together with the fact that a strongly $\mathscr{F}$-measurable V-valued mapping is weakly $\mathscr{F}$-measurable. For more details on the measurability of vector-valued functions, see for example, Reed and Simon [85] and Yosida [102].

3.4 QMC integration error with product weights

Based on the bound on mixed derivatives obtained in the previous section, now we derive a QMC convergence rate with product weights.

We first introduce some notations. Let

$$\varsigma_j(\lambda) := 2 \left(\frac{\sqrt{2\pi} \exp(\alpha_j^2/\Lambda^*)}{\pi^{2-2\Lambda^*}(1-\Lambda^*)\Lambda^*} \right)^\lambda \zeta\left(\lambda + \frac{1}{2}\right), \tag{3.4.1}$$

where $\Lambda^* := \frac{2\lambda - 1}{4\lambda}$, and $\zeta(x) := \sum_{k=1}^\infty k^{-x}$ denotes the Riemann zeta function.

We record the following result by Graham et al. [36].

Theorem 3.4.1. [36, Theorem 15] *Let $f \in \mathcal{W}^s$. Given s, $n \in \mathbb{N}$ with $2 \leq n \leq 10^{30}$, weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subset \mathbb{N}}$, and the standard normal density function ϕ, a randomly shifted lattice rule with n points in s dimensions can be constructed by a component-by-component algorithm such that, for all $\lambda \in (1/2, 1]$,*

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}} \left| I_s(f) - \mathcal{Q}_{s,n}(\boldsymbol{\Delta}; f) \right|^2} \leq 9 \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq \{1:s\}} \gamma_{\mathfrak{u}}^\lambda \prod_{j \in \mathfrak{u}} \varsigma_j(\lambda) \right)^{\frac{1}{2\lambda}} n^{-\frac{1}{2\lambda}} \|f\|_{\mathcal{W}^s}. \tag{3.4.2}$$

For the weight function (3.3.2) we assume that the α_j satisfy for some constants $0 < \alpha_{\min} < \alpha_{\max} < \infty$,

$$\max\left\{\frac{\ln 2}{\rho_j}, \alpha_{\min}\right\} < \alpha_j \le \alpha_{\max}, \qquad j \in \mathbb{N}. \tag{3.4.3}$$

For example, under Assumption 3.1′ letting $\alpha_j := 1 + \frac{\ln 2}{\rho_j}$ satisfies (3.4.3) with $\alpha_{\min} := 1$ and $\alpha_{\max} := 1 + \sup_{j\ge 1} \frac{\ln 2}{\rho_j}$.

We have the following bound on $\|F\|^2_{\mathcal{W}^s}$. The argument is essentially by Graham et al. [36, Theorem 16].

Proposition 3.4.2. *Suppose Assumption 3.1′ is satisfied with a positive sequence (ρ_j) such that*

$$(1/\rho_j) \in \ell^1. \tag{3.4.4}$$

Then, for F as in (3.2.4) we have

$$\|F\|^2_{\mathcal{W}^s} \le (C^*)^2 \sum_{\mathfrak{u}\subseteq\{1:s\}} \frac{1}{\gamma_{\mathfrak{u}}} \left(\frac{1}{\prod_{j\in\mathfrak{u}} \rho_j}\right)^2 \prod_{j\in\mathfrak{u}} \frac{1}{\alpha_j - (\ln 2)/\rho_j}, \tag{3.4.5}$$

with a positive constant

$$C^* := \frac{\|f\|_{V'}\,\|\mathcal{G}\|_{V'}\sqrt{C_0}}{\inf_{x\in D} a_0(x)} \left[\exp\left(\frac{1}{2}\sum_{j\ge 1}\frac{(\ln 2)^2}{\rho_j^2} + \frac{2}{\sqrt{2\pi}}\sum_{j\ge 1}\frac{\ln 2}{\rho_j}\right)\right] < \infty.$$

Proof. In this proof we abuse the notation slightly and $\boldsymbol{y}$ always denotes

$$(y_1, \dots, y_s, 0, 0, \dots) \in \mathbb{R}^{\mathbb{N}}.$$

From (**b1**) and (3.4.4), in view of Corollary 3.3.2 for $\mathbb{P}_Y$-almost every $\boldsymbol{y}$ we have

$$\left|\frac{\partial^{|\mathfrak{u}|} F}{\partial \boldsymbol{y}_{\mathfrak{u}}}\right| \le \|\mathcal{G}\|_{V'} \left\|\frac{\partial^{|\mathfrak{u}|} u^s}{\partial \boldsymbol{y}_{\mathfrak{u}}}\right\|_V \le \|\mathcal{G}\|_{V'}\sqrt{C_0}\frac{1}{\prod_{j\in\mathfrak{u}}\rho_j}\frac{\|f\|_{V'}}{\check{a}(\boldsymbol{y})}. \tag{3.4.6}$$

Since

$$\sup_{x\in D}\sum_{j\ge 1}|y_j||\psi_j(x)| \le \left(\sup_{j\ge 1}\frac{|y_j|}{\rho_j}\right)\sup_{x\in D}\sum_{j\ge 1}\rho_j|\psi_j(x)| \le \left(\sum_{j\ge 1}\frac{|y_j|}{\rho_j}\right)\sup_{x\in D}\sum_{j\ge 1}\rho_j|\psi_j(x)|,$$

the condition (**b1**) and equations (3.4.6) and (3.3.26) together with $y_j = 0$ for $j > s$, imply

$$\left|\frac{\partial^{|\mathfrak{u}|}F}{\partial \boldsymbol{y}_{\mathfrak{u}}}\right| \leq \frac{K^*}{\prod_{j\in\mathfrak{u}}\rho_j} \prod_{j\in\{1:s\}} \exp\left(\frac{\ln 2}{\rho_j}|y_j|\right), \tag{3.4.7}$$

where $K^* := \frac{\|f\|_{V'}\|\mathcal{G}\|_{V'}\sqrt{C_0}}{\inf_{x\in D} a_0(x)}$. Then, it follows from (3.3.1) that

$$\begin{aligned}
&\|F\|^2_{\mathcal{W}^s}\\
&= \sum_{\mathfrak{u}\subseteq\{1:s\}} \frac{1}{\gamma_{\mathfrak{u}}} \int_{\mathbb{R}^{|\mathfrak{u}|}} \left(\int_{\mathbb{R}^{s-|\mathfrak{u}|}} \left|\frac{\partial^{|\mathfrak{u}|}F}{\partial \boldsymbol{y}_{\mathfrak{u}}}(\boldsymbol{y}_{\mathfrak{u}};\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}})\right| \prod_{j\in\{1:s\}\setminus\mathfrak{u}} \phi(y_j)\,\mathrm{d}\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}}\right)^2 \prod_{j\in\mathfrak{u}} w_j^2(y_j)\,\mathrm{d}\boldsymbol{y}_{\mathfrak{u}}\\
&\leq \sum_{\mathfrak{u}\subseteq\{1:s\}} \frac{1}{\gamma_{\mathfrak{u}}}\\
&\quad\times \int_{\mathbb{R}^{|\mathfrak{u}|}} \left(\int_{\mathbb{R}^{s-|\mathfrak{u}|}} \frac{K^*}{\prod_{j\in\mathfrak{u}}\rho_j} \prod_{j\in\{1:s\}} \exp\left(\frac{\ln 2}{\rho_j}|y_j|\right) \prod_{j\in\{1:s\}\setminus\mathfrak{u}} \phi(y_j)\,\mathrm{d}\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}}\right)^2 \prod_{j\in\mathfrak{u}} w_j^2(y_j)\,\mathrm{d}\boldsymbol{y}_{\mathfrak{u}}\\
&= (K^*)^2 \sum_{\mathfrak{u}\subseteq\{1:s\}} \frac{1}{\gamma_{\mathfrak{u}}} \left(\frac{1}{\prod_{j\in\mathfrak{u}}\rho_j}\right)^2\\
&\quad\times \left(\int_{\mathbb{R}^{s-|\mathfrak{u}|}} \prod_{j\in\{1:s\}\setminus\mathfrak{u}} \exp\left(\frac{\ln 2}{\rho_j}|y_j|\right) \prod_{j\in\{1:s\}\setminus\mathfrak{u}} \phi(y_j)\,\mathrm{d}\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}}\right)^2\\
&\quad\times \int_{\mathbb{R}^{|\mathfrak{u}|}} \prod_{j\in\mathfrak{u}} \exp\left(\frac{2\ln 2}{\rho_j}|y_j|\right) \prod_{j\in\mathfrak{u}} w_j^2(y_j)\,\mathrm{d}\boldsymbol{y}_{\mathfrak{u}}.
\end{aligned}$$

Note that this takes essentially the same form as Graham et al. [36, (4.14)]. Thus, the rest of the proof is in parallel to that of Graham et al. [36, Theorem 16].

Noting that $2\alpha_j - \frac{2\ln 2}{\rho_j} < 0$, and following the same argument as in Graham et al. [36, (4.15)–(4.17)], we have

$$\|F\|^2_{\mathcal{W}^s} \leq (K^*)^2 \sum_{\mathfrak{u}\subseteq\{1:s\}} \frac{1}{\gamma_{\mathfrak{u}}} \left(\frac{1}{\prod_{j\in\mathfrak{u}}\rho_j}\right)^2 \left(\prod_{j\in\{1:s\}\setminus\mathfrak{u}} 2\exp\left(\frac{(\ln 2)^2}{2\rho_j^2}\right)\Phi\left(\frac{\ln 2}{\rho_j}\right)\right)^2 \prod_{j\in\mathfrak{u}} \frac{1}{\alpha_j - \frac{\ln 2}{\rho_j}}, \tag{3.4.8}$$

with $\Phi(\cdot)$ denoting the cumulative standard normal distribution function. Comparing this to Graham et al. [36, Equation (4.17)], the statement follows from the rest of the proof of Graham et al. [36, Theorem 16]. $\square$

As in Graham et al. [36, Theorem 17], from Theorem 3.4.1 and Proposition 3.4.2 we have the following.

Proposition 3.4.3. *For each $j \geq 1$, let $w_j(t) = \exp(-2\alpha_j|t|)$ ($t \in \mathbb{R}$) with α_j satisfying (3.4.3). Given s, $n \in \mathbb{N}$ with $2 \leq n \leq 10^{30}$, weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u}\subset\mathbb{N}}$, and the standard normal density function ϕ, a randomly shifted lattice rule with n points in s dimensions can be constructed by a component-by-component algorithm such that, for all $\lambda \in (1/2, 1]$,*

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}} \left| I_s(F) - \mathcal{Q}_{s,n}(\boldsymbol{\Delta}; F) \right|^2} \leq 9C^* C_{\boldsymbol{\gamma},s}(\lambda) n^{-\frac{1}{2\lambda}}, \tag{3.4.9}$$

with

$$C_{\boldsymbol{\gamma},s}(\lambda) := \left(\sum_{\emptyset\neq\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{\lambda} \prod_{j\in\mathfrak{u}} \varsigma_j(\lambda) \right)^{\frac{1}{2\lambda}} \left(\sum_{\mathfrak{u}\subseteq\{1:s\}} \frac{1}{\gamma_{\mathfrak{u}}} \left(\frac{1}{\prod_{j\in\mathfrak{u}} \rho_j} \right)^2 \prod_{j\in\mathfrak{u}} \frac{1}{[\alpha_j - \ln 2/\rho_j]} \right)^{\frac{1}{2}}, \tag{3.4.10}$$

and C^* defined as in Proposition 3.4.2.

We choose weights of the product form

$$\gamma_{\mathfrak{u}} = \gamma_{\mathfrak{u}}^*(\lambda) := \left[\left(\frac{1}{\prod_{j\in\mathfrak{u}} \rho_j} \right)^2 \prod_{j\in\mathfrak{u}} \frac{1}{\varsigma_j(\lambda)[\alpha_j - \ln 2/\rho_j]} \right]^{\frac{1}{1+\lambda}} \tag{3.4.11}$$

In particular, with $\alpha_j := 1 + \ln 2/\rho_j$ we have

$$\gamma_{\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \left(\frac{1}{\rho_j^2 \varsigma_j(\lambda)} \right)^{\frac{1}{1+\lambda}}. \tag{3.4.12}$$

Then, it turns out that under a suitable value of λ the constant (3.4.10) can be bounded independently of s, and we have the QMC error bound as follows.

Theorem 3.4.4. *For each $j \geq 1$, let $w_j(t) = \exp(-2\alpha_j|t|)$ ($t \in \mathbb{R}$) with α_j satisfying (3.4.3). Let $\varsigma_{\max}(\lambda)$ be ς_j defined by (3.4.1) but α_j being replaced by*

$\alpha_{\max}$. *Suppose* (ψ_j) *satisfies Assumption 3.1. Suppose further that, we choose* λ *as*

$$\lambda = \begin{cases} \frac{1}{2-2\delta} \text{ for arbitrary } \delta \in (0, \frac{1}{2}] & \text{when } q \in (0, \frac{2}{3}] \\ \frac{q}{2-q} & \text{when } q \in (\frac{2}{3}, 1], \end{cases} \tag{3.4.13}$$

and choose the weights $\gamma_{\mathfrak{u}}$ *as in (3.4.11). Then, given* s, $n \in \mathbb{N}$ *with* $n \leq 10^{30}$, *and the standard normal density function* ϕ, *a randomly shifted lattice rule with* n *points in* s *dimensions can be constructed by a component-by-component algorithm such that*

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}} \left| I_s(F) - \mathcal{Q}_{s,n}(\boldsymbol{\Delta}; F) \right|^2} \leq \begin{cases} 9C_{\rho,q,\delta} C^* n^{-(1-\delta)} & \text{when } 0 < q \leq \frac{2}{3}, \\ 9C_{\rho,q} C^* n^{-\frac{2-q}{2q}} & \text{when } \frac{2}{3} < q \leq 1. \end{cases} \tag{3.4.14}$$

where the constants $C_{\rho,q,\delta}$, *(resp.* $C_{\rho,q}$*) are independent of* s *but depend on* $\rho := (\rho_j)$, q *and* δ *(resp.* ρ *and* q*), and* C^* *is defined as in Proposition 3.4.2. In particular, with* $\alpha_j := 1 + \ln 2/\rho_j$ *the finite constants* $C_{\rho,q,\delta}$, *and* $C_{\rho,q}$ *are both given by*

$$C_{\rho,q,\delta} = C_{\rho,q} = \left(\prod_{j=1}^{\infty} \left(1 + \left(\frac{\varsigma_j(\lambda)}{\rho_j^{2\lambda}} \right)^{\frac{1}{1+\lambda}} \right) - 1 \right)^{\frac{1}{2\lambda}} \left(\prod_{j=1}^{\infty} \left(1 + \left(\frac{\varsigma_j(\lambda)}{\rho_j^{2\lambda}} \right)^{\frac{1}{1+\lambda}} \right) \right)^{\frac{1}{2}},$$

with λ *given by (3.4.13).*

Proof. Let $\beta_j(\lambda) := \left(\frac{(\varsigma_j(\lambda))^{\frac{1}{\lambda}}}{\rho_j^2 [\alpha_j - \ln 2/\rho_j]} \right)^{\frac{\lambda}{1+\lambda}}$. Observe that with the choice of weights (3.4.11) we have

$$C_{\boldsymbol{\gamma},s}(\lambda) = \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq \{1:s\}} \prod_{j \in \mathfrak{u}} \beta_j(\lambda) \right)^{\frac{1}{2\lambda}} \left(\sum_{\mathfrak{u} \subseteq \{1:s\}} \prod_{j \in \mathfrak{u}} \beta_j(\lambda) \right)^{\frac{1}{2}} \tag{3.4.15}$$

$$= \left(\left(\prod_{j=1}^{s} (1 + \beta_j(\lambda)) \right) - 1 \right)^{\frac{1}{2\lambda}} \left(\prod_{j=1}^{s} (1 + \beta_j(\lambda)) \right)^{\frac{1}{2}}. \tag{3.4.16}$$

Now, let $\mathcal{J} := \inf_{j\geq 1}(\alpha_j - \ln 2/\rho_j)$, which is a positive value from (3.4.3). Further, note that $\varsigma_j(\lambda) \leq \varsigma_{\max}(\lambda)$ for $j \geq 1$. Then, from $\beta_j(\lambda) \geq 0$ we have

$$\prod_{j=1}^{s}(1+\beta_j(\lambda)) \leq \prod_{j=1}^{s}\exp(\beta_j(\lambda)) \leq \exp\Big(\sum_{j\geq 1}\beta_j(\lambda)\Big) \tag{3.4.17}$$

$$\leq \exp\left(\left[\frac{[\varsigma_{\max}(\lambda)]^{\frac{1}{\lambda}}}{\mathcal{J}}\right]^{\frac{\lambda}{1+\lambda}}\sum_{j\geq 1}\left[\frac{1}{\rho_j}\right]^{\frac{2\lambda}{1+\lambda}}\right). \tag{3.4.18}$$

Thus, if $\sum_{j\geq 1}\left[\frac{1}{\rho_j}\right]^{\frac{2\lambda}{1+\lambda}} < \infty$ we can conclude that $C_{\gamma,s}(\lambda)$ is bounded independently of s.

We discuss the relation between q and the exponent $\frac{2\lambda}{1+\lambda}$. First note that from $\lambda \in (\frac{1}{2}, 1]$, we have $\frac{2}{3} < \frac{2\lambda}{1+\lambda} \leq 1$. Suppose $0 < q \leq \frac{2}{3}$. in this case, we always have $q < \frac{2\lambda}{1+\lambda}$, and thus $(1/\rho_j) \in \ell^{\frac{2\lambda}{1+\lambda}}$. Thus, $\sum_{j\geq 1}\left[\frac{1}{\rho_j}\right]^{\frac{2\lambda}{1+\lambda}} < \infty$ follows. Letting $\lambda := \frac{1}{2-2\delta}$ with an arbitrary $\delta \in (0, \frac{1}{2}]$, we obtain the result for $q \in (0, \frac{2}{3}]$. Next, consider the case $\frac{2}{3} < q \leq 1$. Then, letting $\lambda := \lambda(q) = \frac{q}{2-q}$, we have $\lambda \in (1/2, 1]$ and

$$\frac{2\lambda}{1+\lambda} = \frac{2\frac{q}{2-q}}{1+\frac{q}{2-q}} = \frac{2q}{2-q+q} = q, \tag{3.4.19}$$

and thus $\sum_{j\geq 1}\left[\frac{1}{\rho_j}\right]^{\frac{2\lambda}{1+\lambda}} < \infty$. □

3.4.1 On the estimate of the constant

The estimate (3.4.14) gives the same convergence rate as the one obtained by Herrmann and Schwab [39, Theorem 13]. The weights used there are simpler than (3.4.12). See Herrmann and Schwab [39, Equation (24)]. The essential difference is that we incorporate the function $1/\varsigma_j(\lambda)$ into the weights as in (3.4.11) and (3.4.12). An advantage of this is that, roughly speaking, when the magnitude of $\{\sup_{x\in D}|\psi_j|\}$ is large, our estimate gives a smaller constant, as shown in Proposition 3.4.6 below.

To make a comparison, following Herrmann and Schwab [39] we let $a_* \equiv 0$, and $a_0 \equiv 1$. Suppose the sequence $\{\rho_j\}$ that satisfies Assumption 3.1 is given by

$\rho_j = c_b \frac{1}{b_j}$ with a constant $c_b > 0$ and a sequence $\{b_j\}$, and let

$$K_{\mathrm{HS}} := \sup_{x \in D} \sum_{j \geq 1} \frac{|\psi_j(x)|}{b_j} = \frac{\kappa}{c_b} < \frac{\ln 2}{c_b}. \tag{HS-A1}$$

This is essentially the same assumption as Herrmann and Schwab [39, Assumption (A1)]. We quote the following result.

Theorem 3.4.5. [39, Theorem 13] *Suppose (ψ_j) satisfies Assumption 3.1 with a sequence $\{\rho_j\}$ that is of the form $\rho_j = c_b \frac{1}{b_j}$ with a constant $c_b > 0$ and a sequence $\{b_j\}$. Let $w_j(t) = \exp(-2\alpha|t|)$ $(t \in \mathbb{R})$ with a parameter $\alpha > \frac{\kappa}{c_b} \sup_{j \geq 1}\{b_j\}$. Let $\varsigma_{\mathrm{HS}}(\lambda)$ be ς_j defined by (3.4.1) but with α_j being replaced by α. Suppose further that λ is chosen as in (3.4.13), and that the weights $\gamma_{\mathfrak{u}}$ are chosen as $\gamma_{\mathfrak{u}} := \prod_{j \in \mathfrak{u}} b_j^{\frac{2}{1+\lambda}}$. Then, given s, $n \in \mathbb{N}$ with $n \leq 10^{30}$, and the standard normal density function ϕ, a randomly shifted lattice rule with n points in s dimensions can be constructed by a component-by-component algorithm such that*

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}} \left| I_s(F) - \mathcal{Q}_{s,n}(\boldsymbol{\Delta}; F) \right|^2} \leq 9 \|f\|_{V'} \, \|\mathcal{G}\|_{V'} \, \sqrt{C_0} C_{\mathrm{HS},1} C_{\mathrm{HS},2} C_{\mathrm{HS},3} n^{-\frac{1}{2\lambda}}, \tag{3.4.20}$$

with

$$C_{\mathrm{HS},1} := \exp\left(\sum_{j \geq 1} \left((K_{\mathrm{HS}} b_j)^2 + \frac{2}{\sqrt{2\pi}} K_{\mathrm{HS}} b_j \right) \right), \tag{3.4.21}$$

$$C_{\mathrm{HS},2} := \exp\left(\frac{1}{2\lambda} \sum_{j \geq 1} b_j^q \varsigma_{\mathrm{HS}}(\lambda) \right), \quad \textit{and} \quad C_{\mathrm{HS},3} := \exp\left(\frac{1}{2} \sum_{j \geq 1} \frac{b_j^q / c^2}{\alpha/2 - 2K_{\mathrm{HS}} b_j} \right), \tag{3.4.22}$$

with an arbitrarily fixed constant $c \in (0, \ln 2/K_{\mathrm{HS}})$, where C_0 is defined in Proposition 3.3.1. ∎

To compare, we note that (3.4.14) can be further bounded as

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}} \left| I_s(F) - \mathcal{Q}_{s,n}(\boldsymbol{\Delta}; F) \right|^2} \leq 9 \|f\|_{V'} \, \|\mathcal{G}\|_{V'} \, \sqrt{C_0} C_1 C_2 n^{-\frac{1}{2\lambda}}, \tag{3.4.23}$$

with

$$C_1 := \exp\left(\sum_{j \geq 1} \left(\frac{1}{2} \left(\frac{\kappa}{\rho_j} \right)^2 + \frac{2}{\sqrt{2\pi}} \frac{\kappa}{\rho_j} \right) \right), \tag{3.4.24}$$

and

$$C_2 := \exp\left(\left(\frac{1}{2\lambda}+\frac{1}{2}\right)[\varsigma_{\max}(\lambda)]^{\frac{1}{1+\lambda}}\sum_{j\geq 1}\left[\frac{1}{\rho_j}\right]^{\frac{2\lambda}{1+\lambda}}\right), \tag{3.4.25}$$

with the choice $\alpha_j := 1+\frac{\kappa}{\rho_j}$. Note that the scalar $\ln 2$ in (3.4.3) and (3.4.14) can be replaced by κ, which is defined as in (**b1**).

We have the following result on the comparison of the constants.

Proposition 3.4.6. *Fix $\varepsilon_{\mathrm{HS}} > 0$ arbitrarily. Let the assumptions of Theorem 3.4.5 hold with $\alpha := \varepsilon_{\mathrm{HS}} + \kappa \sup_j\{1/\rho_j\}$. Then, we have $C_1 < C_{\mathrm{HS},1}$ and $1 < C_{\mathrm{HS},3}$. Further, for $\lambda \in (1/2, 1]$ suppose*

$$\kappa \sup_{j\geq 1}\frac{1}{\rho_j} \geq \frac{\sqrt{1+\lambda}+1}{\lambda}$$

holds. Then, we have $C_2 < C_{\mathrm{HS},2}$, and therefore $C_1C_2 < C_{\mathrm{HS},1}C_{\mathrm{HS},2}C_{\mathrm{HS},3}$.

Proof. Clearly, we have $1 < C_{\mathrm{HS},3}$. The equations (HS-A1) and $\rho_j = c_b\frac{1}{b_j}$ imply $K_{\mathrm{HS}}b_j = \frac{\kappa}{\rho_j}$, and thus $C_1 < C_{\mathrm{HS},1}$ follows.

To show $C_2 < C_{\mathrm{HS},2}$, it suffices to show

$$\left(\frac{1}{2\lambda}+\frac{1}{2}\right)\left(\frac{\sqrt{2\pi}}{\pi^{2-2\Lambda^*}(1-\Lambda^*)\Lambda^*}\right)^{\frac{\lambda}{1+\lambda}} \tag{3.4.26}$$

$$\left[2\exp(\lambda\widetilde{\alpha}_{\max}^2/\Lambda^*)\right]^{\frac{1}{1+\lambda}}\sum_{j\geq 1}\left[\frac{1}{\rho_j}\right]^{\frac{2\lambda}{1+\lambda}}\zeta(\lambda+1/2)^{\frac{1}{1+\lambda}}$$

$$< \frac{1}{\lambda}\left(\frac{\sqrt{2\pi}}{\pi^{2-2\Lambda^*}(1-\Lambda^*)\Lambda^*}\right)^{\lambda}\exp(\lambda\alpha^2/\Lambda^*)\sum_{j\geq 1}b_j^{\frac{2\lambda}{1+\lambda}}\zeta(\lambda+1/2). \tag{3.4.27}$$

For $\lambda \in (1/2, 1]$, we have $\Lambda^* = \Lambda^*(\lambda) = \frac{2\lambda-1}{4\lambda} \in (0, 1/4]$, and thus

$$1 < \frac{\sqrt{2\pi}}{\pi^2}\frac{16}{3} = \frac{\sqrt{2\pi}}{\pi^{2-2\Lambda^*(1/2)}(1-\Lambda^*(1))\Lambda^*(1)} \leq \frac{\sqrt{2\pi}}{\pi^{2-2\Lambda^*(\lambda)}(1-\Lambda^*(\lambda))\Lambda^*(\lambda)}.$$

Hence, we have

$$\left(\frac{\sqrt{2\pi}}{\pi^{2-2\Lambda^*}(1-\Lambda^*)\Lambda^*}\right)^{\frac{\lambda}{1+\lambda}} < \left(\frac{\sqrt{2\pi}}{\pi^{2-2\Lambda^*}(1-\Lambda^*)\Lambda^*}\right)^{\lambda}. \tag{3.4.28}$$

Further, from $(1/(2\lambda)+1/2)2^{\frac{1}{1+\lambda}} \le 2^{\frac{\lambda}{1+\lambda}}/\lambda$, noting $2 < \zeta(3/2) \le \zeta(\lambda+1/2)$ we have

$$\Big(\frac{1}{2\lambda}+\frac{1}{2}\Big)2^{\frac{1}{1+\lambda}}\zeta(\lambda+1/2)^{\frac{1}{1+\lambda}} \le \frac{2^{\frac{\lambda}{1+\lambda}}}{\lambda}\zeta(\lambda+1/2)^{\frac{1}{1+\lambda}} \le \frac{\zeta(\lambda+1/2)}{\lambda}. \qquad (3.4.29)$$

We now show $\frac{1}{1+\lambda}\widetilde{\alpha}_{\max} = 1+\sup_{j\ge 1}\frac{\kappa}{\rho_j} < \alpha$. The assumption $\kappa\sup_j \frac{1}{\rho_j} \ge \frac{\sqrt{1+\lambda}+1}{\lambda} = \frac{1}{\sqrt{1+\lambda}-1}$ implies $\frac{(1+\kappa\sup_j\{1/\rho_j\})^2}{(\kappa\sup_j\{1/\rho_j\})^2} \le 1+\lambda$, and thus

$$\frac{1}{1+\lambda}(1+\kappa\sup_j\{1/\rho_j\})^2 < (\varepsilon_{\mathrm{HS}}+\kappa\sup_j\{1/\rho_j\})^2. \qquad (3.4.30)$$

Hence, the above together with (3.4.28) and (3.4.29) we conclude that (3.4.27) holds, which is the desired result. □

3.5 Application to a wavelet stochastic model

Cioica et al. [18] considered a stochastic model in which users can choose the smoothness at will. In this section, we consider the Gaussian case, and show that the theory developed in Section 3.4 can be applicable for the model with a wide range of smoothness.

3.5.1 Stochastic model

For simplicity we assume $D\subset\mathbb{R}^d$ is a bounded convex polygonal domain. Consider a wavelet system $(\varphi_\xi)_{\xi\in\bigtriangledown}$ that is a Riesz basis for $L^2(D)$-space. We explain the notations and outline the standard properties we assume as follows. The indices $\xi\in\bigtriangledown$ typically encodes both the scale, often denoted by $|\xi|$, and the spatial location, and also the type of the wavelet. Since our analysis does not rely on the choice of a type of wavelet, we often use the notation $\xi=(\ell,k)$, and $\bigtriangledown=\{(\ell,k)\mid \ell\ge\ell_0, k\in\bigtriangledown_\ell\}$ where $\bigtriangledown_\ell$ is some countable index set. The scale level ℓ of φ_ξ is denoted by $|\xi|=|(\ell,k)|=\ell$. Furthermore, $(\widetilde{\varphi}_\xi)_{\xi\in\bigtriangledown}$ denotes the dual wavelet basis, i.e., $\langle\varphi_\xi,\widetilde{\varphi}_{\xi'}\rangle_{L^2(D)}=\delta_{\xi\xi'}$, $\xi,\xi'\in\bigtriangledown$.

In the following, $\alpha\lesssim\beta$ means that α can be bounded by some constant times β uniformly with respect to any parameters on which α and β may depend. Further, $\alpha\sim\beta$ means that $\alpha\lesssim\beta$ and $\beta\lesssim\alpha$.

We list the assumption on wavelets:

(W1) the wavelets $(\varphi_\xi)_{\xi\in\bigtriangledown}$ form a Riesz basis for $L^2(D)$;

(W2) the cardinality of the index set $\bigtriangledown_\ell$ satisfies $\#\bigtriangledown_\ell = C_\bigtriangledown 2^{\ell d}$ for some constant $C_\bigtriangledown>0$, where d is the spatial dimension of D;

(W3) the wavelets are local. That is, the supports of $\varphi_{\ell,k}$ are contained in balls of diameter $\sim 2^{-\ell}$, and do not overlap too much in the following sense: There exists a constant $M > 0$ independent of ℓ such that for each given ℓ for any $x \in D$,

$$\#\{k \in \nabla_\ell \mid \varphi_{\ell,k}(x) \neq 0\} \leq M; \tag{3.5.1}$$

(W4) the wavelets satisfy the cancellation property

$$|\langle v, \varphi_\xi \rangle_{L^2(D)}| \lesssim 2^{-|\xi|(\frac{d}{2}+\tilde{m})} |v|_{W^{\tilde{m},\infty}(\mathrm{supp}(\varphi_\xi))},$$

for $|\xi| \geq \ell_0$ with some parameter $\tilde{m} \in \mathbb{N}$, where $|\cdot|_{W^{\tilde{m},\infty}}$ denotes the usual Sobolev semi-norm. That is, the inner product is small when the function v is smooth on the support $\mathrm{supp}(\varphi_\xi)$;

(W5) the wavelet basis induces characterisations of Besov spaces $B^t_{\mathsf{q}}(L_{\mathsf{p}}(D))$ for $1 \leq \mathsf{p}, \mathsf{q} < \infty$ and all t with $d \max\{1/\mathsf{p} - 1, 0\} < t < t_*$ for some parameter $t_* > 0$:

$$\|v\|_{B^t_{\mathsf{q}}(L_{\mathsf{p}}(D))} := \left(\sum_{\ell=\ell_0}^{\infty} 2^{\ell\left(t+d\left(\frac{1}{2}-\frac{1}{\mathsf{p}}\right)\right)\mathsf{q}} \left(\sum_{k \in \nabla_\ell} |\langle v, \tilde{\varphi}_{\ell,k} \rangle_{L^2(D)}|^{\mathsf{p}} \right)^{\frac{\mathsf{q}}{\mathsf{p}}} \right)^{\frac{1}{\mathsf{q}}}. \tag{3.5.2}$$

The upper bound t_* depends on the choice of wavelet basis. Since t we consider is typically small, here for simplicity we *define* the Besov norm as above.

(W6) the wavelets satisfy

$$\sup_{x \in D} |\varphi_{\ell,k}(x)| = C_\varphi 2^{\frac{\beta_0 d}{2}\ell} \qquad \text{with some } \beta_0 \in \mathbb{R}_+, \tag{3.5.3}$$

for some constant $C_\varphi > 0$. Typically we have $\varphi_{\ell,k} \sim 2^{\frac{d}{2}\ell} \psi(2^\ell(x - x_{\ell,k}))$, for some bounded function ψ. In this case we have $\beta_0 = 1$.

See Cioica et al. [18, section 2.1] and references therein for further details. See also Cohen [19], DeVore [26], and Urban [95].

We now investigate a stochastic model expanded by the wavelet basis described above. Let $\{Y_{\ell,k}\}$ be a collection of independent standard normal random variables on a suitable probability space $(\Omega', \mathscr{F}', \mathbb{P}')$. We assume the random field

(3.1.2) is given with T such that

$$T(x, \boldsymbol{y}') = \sum_{\ell=\ell_0}^{\infty} \sum_{k \in \nabla_\ell} Y_{\ell,k}(\boldsymbol{y}') \sigma_\ell \varphi_{\ell,k}(x), \tag{3.5.4}$$

where

$$\sigma_\ell := 2^{-\frac{\beta_1 d}{2}\ell} \text{ with } \beta_1 > 1. \tag{3.5.5}$$

Thanks to the decaying factor σ_ℓ, in view of (W1) the series (3.5.4) converges $\mathbb{P}'$-almost surely in $L^2(D)$: $\mathbb{E}_{\mathbb{P}'}\Big(\sum_{\ell=\ell_0}^{\infty} \sum_{k \in \nabla_\ell} Y_{\ell,k}(\boldsymbol{y}')^2 \sigma_\ell^2\Big) = C_\nabla \sum_{\ell=\ell_0}^{\infty} 2^{-(\beta_1 - 1)d\ell} < \infty$. Further, σ_ℓ will be used for $\{\sigma_\ell \varphi_{\ell,k}\}$ to satisfy the condition (**b1**).

To replace (3.1.2), we consider the following log-normal stochastic model:

$$a(x, \boldsymbol{y}') = a_*(x) + a_0(x) \exp\Bigg(\sum_{\ell=\ell_0}^{\infty} \sum_{k \in \nabla_\ell} Y_{\ell,k}(\boldsymbol{y}') \sigma_\ell \varphi_{\ell,k}(x)\Bigg). \tag{3.5.6}$$

In the following, we argue that we can reorder $\sigma_\ell \varphi_{\ell,k}$ lexicographically as $\sigma_j \varphi_j$ and see it as ψ_j, while keeping the law.

Throughout this section, we assume that the parameters β_0 and β_1 satisfy

$$0 < \beta_1 - \beta_0, \tag{3.5.7}$$

and that point evaluation $\varphi_{\ell,k}(x)$ ($(\ell, k) \in \nabla$) is well-defined for any $x \in D$. Under this assumption, reordering $(Y_{\ell,k} \sigma_\ell \varphi_{\ell,k})$ lexicographically does not change the law of (3.5.4) on $\mathbb{R}^D$. To see this, from the Gaussianity it suffices to show that the covariance function $\mathbb{E}_{\mathbb{P}'}[T(\cdot)T(\cdot)] \colon D \times D \to \mathbb{R}$ is invariant under the reordering.

Fix $x \in D$ arbitrarily. For any L, L' ($L > L'$), from the independence of $\{Y_{\ell,k}\}$ we have

$$\mathbb{E}_{\mathbb{P}'}\Bigg(\sum_{\ell=\ell_0}^{L} \sum_{k \in \nabla_\ell} Y_{\ell,k}(\boldsymbol{y}') \sigma_\ell \varphi_{\ell,k}(x) - \sum_{\ell=\ell_0}^{L'} \sum_{k \in \nabla_\ell} Y_{\ell,k}(\boldsymbol{y}') \sigma_\ell \varphi_{\ell,k}(x)\Bigg)^2 = \sum_{\ell=L'+1}^{L} \sum_{k \in \nabla_\ell} \sigma_\ell^2 \varphi_{\ell,k}^2(x) \tag{3.5.8}$$

$$\leq C_\varphi^2 M \sum_{\ell=L'+1}^{L} 2^{-(\beta_1 - \beta_0)d\ell} \tag{3.5.9}$$

$$< \infty. \tag{3.5.10}$$

Hence, the sequence $\left\{\sum_{\ell=\ell_0}^{L}\sum_{k\in\nabla_\ell} Y_{\ell,k}(\boldsymbol{y}')\sigma_\ell\varphi_{\ell,k}(x)\right\}_L$ is convergent in $L^2(\Omega',\mathbb{P}')$. The continuity of the inner product $\mathbb{E}_{\mathbb{P}'}[\cdot,\cdot]$ on $L^2(\Omega')$ in each variable yields

$$\mathbb{E}_{\mathbb{P}'}[T(x_1)T(x_2)] = \sum_{\ell=\ell_0}^{\infty}\sum_{k\in\nabla_\ell}\sum_{\ell'=\ell_0'}^{\infty}\sum_{k'\in\nabla_{\ell'}} \mathbb{E}_{\mathbb{P}'}[Y_{\ell,k}(\boldsymbol{y}')\sigma_\ell\varphi_{\ell,k}(x_1)Y_{\ell',k'}(\boldsymbol{y}')\sigma_{\ell'}\varphi_{\ell',k'}(x_2)] \tag{3.5.11}$$

$$= \sum_{\ell=\ell_0}^{\infty}\sum_{k\in\nabla_\ell} \sigma_\ell^2\varphi_{\ell,k}(x_1)\varphi_{\ell,k}(x_2), \qquad \text{for any } x_1,\, x_2 \in D. \tag{3.5.12}$$

But we have $\sum_{\ell=\ell_0}^{\infty}\sum_{k\in\nabla_\ell}\sigma_\ell^2|\varphi_{\ell,k}(x_1)\varphi_{\ell,k}(x_2)| \le C_\varphi^2 M\sum_{\ell=L'+1}^{L} 2^{-(\beta_1-\beta_0)d\ell}$. Hence,

$$\mathbb{E}_{\mathbb{P}'}[T(x_1)T(x_2)] = \sum_{j\ge1}\sigma_j^2\varphi_j(x_1)\varphi_j(x_2), \qquad x_1,\, x_2 \in D.$$

Following a similar discussion, we see that the series $\sum_{j\ge1}\sigma_j^2 y_j\varphi_j(x)$ converges in $L^2(\Omega)$ for each $x \in D$, and has the covariance function

$$\sum_{\ell=\ell_0}^{\infty}\sum_{k\in\nabla_\ell}\sigma_\ell^2\varphi_{\ell,k}(x_1)\varphi_{\ell,k}(x_2).$$

Hence the law on $\mathbb{R}^D$ is the same. Thus, abusing the notation slightly we write $T(\cdot,\boldsymbol{y}) := T(\cdot,\boldsymbol{y}')$, $y_{\ell,k} := Y_{\ell,k}(\boldsymbol{y}')$, $\Omega = \mathbb{R}^{\mathbb{N}} := \Omega'$, $\mathscr{F} := \mathscr{F}'$, $\mathbb{P}_Y := \mathbb{P}'$, and $\mathbb{E}[\cdot] := \mathbb{E}_{\mathbb{P}'}[\cdot]$.

Remark 2. Our theory at present is restricted to the Gaussian random fields with the covariance functions of the form (3.5.12). Although there are attempts to represent a given Gaussian random fields with wavelet-like functions (see Bachmayr, Cohen, and Migliorati [8] and references therein), unfortunately it does not seem to be the case that arbitrary covariance functions, in particular Matérn covariance functions, are representable as in (3.5.12) with wavelets with the properties (W1–W6).

Next, we discuss the applicability of the theory developed in Section 3.4 to the wavelet stochastic model above. We need to check Assumption 3.1.

Take $\theta \in (0, \frac{d}{2}(\beta_1-\beta_0))$, and for $\xi = (\ell, k)$ let

$$\rho_\xi := c2^{\theta|\xi|} = c2^{\theta\ell}, \tag{3.5.13}$$

with some constant $0 < c < \ln 2\big(MC_\varphi\sum_{\ell=\ell_0}^{\infty} 2^{\ell(\theta-\frac{d}{2}(\beta_1-\beta_0))}\big)^{-1}$.

Then, by virtue of the locality property (3.5.1) we have (**b1**) as follows:

$$\sup_{x\in D}\sum_{\xi}\rho_\xi|\sigma_\xi\varphi_\xi(x)| \leq \sum_{\ell=\ell_0}^{\infty}\rho_\ell \sup_{x\in D}\sum_{k\in\nabla_\ell}|2^{-\frac{\beta_1 d\ell}{2}}\varphi_{\ell,k}(x)| \tag{3.5.14}$$

$$\leq cMC_\varphi \sum_{\ell=\ell_0}^{\infty} 2^{\theta\ell}2^{-\frac{\beta_1 d\ell}{2}}2^{\frac{\beta_0 d}{2}\ell} < \ln 2. \tag{3.5.15}$$

Further, we note that by reordering for sufficiently large j we have

$$\sup_{x\in D}|\sigma_j\varphi_j(x)| \sim j^{-\frac{1}{2}(\beta_1-\beta_0)}, \tag{3.5.16}$$

To see this, first recall that there are $\mathcal{O}(2^{\ell d})$ wavelets at level ℓ. Thus, for an arbitrary but sufficiently large j we have $2^{\ell_j d} \lesssim j \lesssim 2^{(\ell_j+1)d}$, for some $\ell_j \geq \ell_0$, which is equivalent to

$$2^{-(\ell_j+1)d} \lesssim j^{-1} \lesssim 2^{-\ell_j d}.$$

Now, let $\xi_j \in \nabla_{\ell_j}$ be the index corresponding to j. Since $|\xi_j| = \ell_j$, noting $\beta_1 - \beta_0 > 0$ we have

$$\sup_{x\in D}|\sigma_j\varphi_j(x)| = \sup_{x\in D}|\sigma_{\ell_j}\varphi_{\xi_j}(x)| = C_\varphi 2^{-\frac{\beta_1 d}{2}\ell_j}2^{\frac{\beta_0 d}{2}\ell_j} \lesssim C_\varphi 2^{\frac{d}{2}\beta^*} j^{-\frac{1}{2}(\beta_1-\beta_0)}, \tag{3.5.17}$$

for any $\beta^* \geq \beta_0 - \beta_1$. The opposite direction can be derived as

$$j^{-\frac{1}{2}(\beta_1-\beta_0)} \lesssim 2^{-\ell_j d(\frac{1}{2}(\beta_1-\beta_0))} = \frac{1}{C_\varphi}\sup_{x\in D}|\sigma_j\varphi_j(x)|. \tag{3.5.18}$$

Similarly, we have

$$\rho_j \sim j^{\frac{\theta}{d}}. \tag{3.5.19}$$

Thus, to have $\sum_{j\geq 1}\frac{1}{\rho_j} < \infty$, the weakest condition on the summability on $(1/\rho_j)$ for Assumption 3.1 to be satisfied, it is necessary (and sufficient) to have $\theta > d$.

The following proposition summarises the discussion above.

Theorem 3.5.1. *Suppose the random coefficient (3.1.2) is given by T as in (3.5.4) with $(\varphi_{\ell,k})$ that satisfies (3.5.3), and non-negative numbers (σ_ℓ) that satisfy (3.5.5). Let (ρ_ξ) be defined by (3.5.13). Further, assume β_0 and β_1 satisfy*

$$\frac{2}{q} < \beta_1 - \beta_0, \tag{3.5.20}$$

for some $q \in (0,1]$*. Then, the reordered system* $(\sigma_j\varphi_j)$ *with the reordered* (ρ_j) *satisfies Assumption 3.1, and under the same conditions on* $w_j(t)$*,* α_j*, and* ς_j *as in Theorem 3.4.4 we have the QMC error bound (3.4.14):*

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}\left|I_s(F)-\mathcal{Q}_{s,n}(\boldsymbol{\Delta};F)\right|^2}=\begin{cases}O(n^{-(1-\delta)}) & \text{when } 0<q\le\frac{2}{3},\\ O(n^{-\frac{2-q}{2q}}) & \text{when } \frac{2}{3}<q\le 1,\end{cases}$$

where $\delta \in (0,1/2]$ *is arbitrary, and the implied constants are as in Theorem 3.4.4.*

Proof. Take $\theta \in (\frac{d}{q}, \frac{d}{2}(\beta_1-\beta_0))$, and define (ρ_ξ) as in (3.5.13), reorder the components lexicographically, and denote the reordered (ρ_ξ) by (ρ_j). Then, we have (**b2**)

$$\sum_{j\ge 1}\left(\frac{1}{\rho_j}\right)^q \lesssim \sum_{j\ge 1}\left(\frac{1}{j}\right)^{\frac{q\theta}{d}} < \infty. \tag{3.5.21}$$

Further, from $\theta-\frac{\beta_1 d}{2}+\frac{\beta_0 d}{2}<0$ we have (3.5.15), and thus (**b1**) holds. Hence, from the discussion in this section Assumption 3.1 is satisfied, and thus in view of Theorem 3.4.4 we have (3.4.14). □

3.5.2 Smoothness of the stochastic model

Hölder smoothness of the realisations

Often, random fields T with realisations that are not smooth are regularly of interest. In this section, we see that the stochastic model we consider (3.5.6) allows reasonably rough random fields (Hölder smoothness) for $d=1,2$. The result is shown via Sobolev embedding results. We provide a necessary and sufficient condition to have specified Sobolev smoothness (Theorem 3.5.2). Recall that embedding results are in general optimal (see, for example, [1, 4.12, 4.40–4.44]), and in this sense, we have a sharp condition for our model to have Hölder smoothness. A building block is a Besov characterisation of the realisations which is essentially due to Cioica et al. [18, Theorem 6]. Here we define $s := s(L) := \sum_{\ell=\ell_0}^{L}\#(\nabla_\ell)$, that is, the truncation is considered in terms of the level L.

Theorem 3.5.2. [18, Theorem 6] *Let* $\bar{\mathtt{p}}, \bar{\mathtt{q}} \in [1, \infty)$*, and* $t \in (d \max\{1/\bar{\mathtt{p}} - 1, 0\}, t_*)$*, where* t_* *is the parameter in (W5). Then,*

$$t < d\Big(\frac{\beta_1 - 1}{2}\Big) \tag{3.5.22}$$

if and only if $T \in B^t_{\bar{\mathtt{q}}}(L_{\bar{\mathtt{p}}}(D))$ *a.s. Further, if* (3.5.22) *is satisfied, then the stochastic model* (3.5.6) *satisfies* $\mathbb{E}[\|T^{s(L)}\|^{\bar{\mathtt{q}}}_{B^t_{\bar{\mathtt{q}}}(L_{\bar{\mathtt{p}}}(D))}] \leq \mathbb{E}[\|T\|^{\bar{\mathtt{q}}}_{B^t_{\bar{\mathtt{q}}}(L_{\bar{\mathtt{p}}}(D))}] < \infty$ *for all* $L \in \mathbb{N}$*.*

Proof. First, from the proof of Cioica et al. [18, Theorem 6], we see that $T \in B^t_{\bar{\mathtt{q}}}(L_{\bar{\mathtt{p}}}(D))$ a.s., is equivalent to

$$\sum_{\ell=\ell_0}^{\infty} 2^{\ell(t+d(1/2-1/\bar{\mathtt{p}}))\bar{\mathtt{q}}} \sigma_\ell^{\bar{\mathtt{q}}} (\#\nabla_\ell)^{\bar{\mathtt{q}}/\bar{\mathtt{p}}} \sim \sum_{\ell=\ell_0}^{\infty} 2^{\ell\bar{\mathtt{q}}(t - \frac{d}{2}(\beta_1 - 1))} < \infty,$$

which holds from the assumption $t < d\big(\frac{\beta_1-1}{2}\big)$. Similarly, from the proof of Cioica et al. [18, Theorem 6] we have

$$\mathbb{E}[\|T\|^{\bar{\mathtt{q}}}_{B^t_{\bar{\mathtt{q}}}(L_{\bar{\mathtt{p}}}(D))}] \lesssim \sum_{\ell=\ell_0}^{\infty} 2^{\ell(t+d(1/2-1/\bar{\mathtt{p}}))\bar{\mathtt{q}}} \sigma_\ell^{\bar{\mathtt{q}}} (\#\nabla_\ell)^{\bar{\mathtt{q}}/\bar{\mathtt{p}}} < \infty.$$

Finally, from (W5) we have

$$\mathbb{E}[\|T^s\|^{\bar{\mathtt{q}}}_{B^t_{\bar{\mathtt{q}}}(L_{\bar{\mathtt{p}}}(D))}] = \sum_{\ell=\ell_0}^{s} 2^{\ell(t+d(1/2-1/\bar{\mathtt{p}}))\bar{\mathtt{q}}} \mathbb{E}\Big[\Big(\sum_{k\in\nabla_\ell} |Y_{\ell,k}|^{\bar{\mathtt{p}}}\Big)^{\bar{\mathtt{q}}/\bar{\mathtt{p}}}\Big] \leq \mathbb{E}[\|T\|^{\bar{\mathtt{q}}}_{B^t_{\bar{\mathtt{q}}}(L_{\bar{\mathtt{p}}}(D))}],$$

completing the proof. □

To establish the Hölder smoothness, we employ embedding results. To invoke them, we first establish that the realisations are continuous; we want the measurability, and want to keep the law of T on $\mathbb{R}^D$.

The Hölder norm involves taking the supremum over the uncountable set D, and thus whether the resulting function $\Omega \ni \boldsymbol{y} \mapsto \|T(\cdot, \boldsymbol{y})\|_{C^{t_1}(\overline{D})} \in \mathbb{R}$, where $t_1 \in (0, 1]$ is a Hölder exponent, is an $\mathbb{R}$-valued random variable is not immediately clear. We see that by the continuity the measurability is preserved.

Sobolev embeddings are achieved by finding a suitable representative by changing values of functions on measure zero sets of D. This change could affect the law on $\mathbb{R}^D$, since it is determined by the laws of arbitrary finitely many random variables $(T(x_1), \ldots, T(x_m))$ ($\{x_i\}_{i=1,\ldots,m} \subset D$) on $\mathbb{R}^m$. To avoid this, we establish the existence of continuous modification, thereby taking the continuous element of a Besov function that respects the law of T from the outset.

We want realisations of T to have continuous paths. Now, suppose that there exist positive constants ι_1, C_{KT}, and $\iota_2(> d)$ satisfying

$$\mathbb{E}[|T(x_1) - T(x_2)|^{\iota_1}] \leq C_{\mathrm{KT}} \|x_1 - x_2\|_2^{\iota_2}, \qquad \text{for any } x_1, x_2 \in D. \tag{3.5.23}$$

Then, by virtue of Kolmogorov–Totoki's theorem [62, Theorem 4.1] T has a continuous modification. Further, the continuous modification is uniformly continuous on D and it can be extended to the closure $\overline{D}$. Thus, we want T to satisfy (3.5.23).

A Hölder smoothness of $(\varphi_{k,l})$ is sufficient for (3.5.23) to hold.

Proposition 3.5.3. *Suppose that (σ_ℓ) satisfies (3.5.5). Further, suppose that for each $(\ell, k) \in \triangledown$, the function $\varphi_{\ell,k}$ is t_0-Hölder continuous on D for some $t_0 \in (0, 1]$. Then, (3.5.23) holds, in particular, T has a modification that is uniformly continuous on D and can be extended to the closure $\overline{D}$.*

Proof. It suffices to show (3.5.23) holds. Fix $x_1, x_2 \in D$ arbitrarily. First note that

$$\sigma_*^2 := \mathbb{E}[|T(x_1) - T(x_2)|^2] = \sum_{\ell=\ell_0}^{\infty} \sum_{k \in \triangledown_\ell} \sigma_\ell^2 (\varphi_{\ell,k}(x_1) - \varphi_{\ell,k}(x_2))^2 \tag{3.5.24}$$

$$\leq C \|x_1 - x_2\|_2^{2t_0} \sum_{\ell=\ell_0}^{\infty} \sum_{k \in \triangledown_\ell} \sigma_\ell^2 < \infty, \tag{3.5.25}$$

where C is the t_0-Hölder constant. Then, since $T(x_1) - T(x_2) \sim \mathcal{N}(0, \sigma_*^2)$ we observe that, with $X_{\mathrm{std}} \sim \mathcal{N}(0, 1)$ we have

$$\mathbb{E}[|T(x_1) - T(x_2)|^{2m}] = \mathbb{E}[|X_{\mathrm{std}} \sigma_*|^{2m}] = \sigma_*^{2m} \mathbb{E}[|X_{\mathrm{std}}|^{2m}] \tag{3.5.26}$$

$$\leq C^m \|x_1 - x_2\|_2^{2t_0 m} \left(\sum_{\ell=\ell_0}^{\infty} \sum_{k \in \triangledown_\ell} \sigma_\ell^2 \right)^m \mathbb{E}[|X_{\mathrm{std}}|^{2m}], \text{ for any } m \in \mathbb{N}. \tag{3.5.27}$$

Taking $m > \frac{d}{2t_0}$, we have (3.5.23) with $\iota_1 := 2m$,

$$C_{\mathrm{KT}} := C^m \big(\sum_{\ell=\ell_0}^{\infty} \sum_{k \in \triangledown_\ell} \sigma_\ell^2 \big)^m \mathbb{E}[|X_{\mathrm{std}}|^{2m}],$$

and $\iota_2 := 2t_0m(> d)$, and thus the statement follows. $\square$

In the following, we assume $\varphi_{\ell,k}$ is t_0-Hölder continuous on D for some $t_0 \in (0,1]$. Note that under this assumption, we may assume $\varphi_{\ell,k}$ is continuous on $\overline{D}$.

Using the fact that $T(\cdot,\boldsymbol{y}) \in B_2^t(L_2(D)) = H^t(D)$ a.s., now we establish the expected Hölder smoothness of the random coefficient a. This implies a spatial regularity of the solution u, given a suitable regularity of D and f. In turn, for example, the convergence rate of the finite element method using the piecewise linear functions are readily obtained, under a certain condition on the output functional $\mathcal{G}$. See [94, Lemma 3.3] or [36, Theorem 6].

First, we argue that to analyse the Hölder smoothness of the realisations of a, without loss of generality we may assume $a_* \equiv 0$ and $a_0 \equiv 1$. To see this, suppose a_*, a_0 in (3.5.6) satisfies $a_*, a_0 \in C^{t_1}(\overline{D})$ for some $t_1 \in (0,1]$. By virtue of

$$|\mathrm{e}^a - \mathrm{e}^b| = \left|\int_a^b \mathrm{e}^r \,\mathrm{d}r\right| \le \max\{\mathrm{e}^a,\mathrm{e}^b\}|b-a| \le (\mathrm{e}^a+\mathrm{e}^b)|b-a| \text{ for all } a,b\in\mathbb{R}, \tag{3.5.28}$$

for any $x_0, x_1, x_2 \in \overline{D}$ $(x_1 \neq x_2)$ we have

$$\left|\mathrm{e}^{T(x_0)}\right| + \frac{\left|\mathrm{e}^{T(x_1)} - \mathrm{e}^{T(x_2)}\right|}{\|x_1-x_2\|_2^{t_1}} \le \Big(\sup_{x\in\overline{D}}\left|\mathrm{e}^{T(x)}\right|\Big)\Big(1+2\frac{|T(x_2)-T(x_3)|}{\|x_1-x_2\|_2^{t_1}}\Big). \tag{3.5.29}$$

Noting that $\left\|a_0\mathrm{e}^T\right\|_{C^{t_1}(\overline{D})} \le C_{t_1}\left\|a_0\right\|_{C^{t_1}(\overline{D})}\left\|\mathrm{e}^T\right\|_{C^{t_1}(\overline{D})}$ (see, for example [33, p. 53]) we have

$$\|a\|_{C^{t_1}(\overline{D})} \le \|a_*\|_{C^{t_1}(\overline{D})} + C_{t_1}\|a_0\|_{C^{t_1}(\overline{D})}\Big(\sup_{x\in\overline{D}}|\mathrm{e}^{T(x)}|\Big)\Big(1+2\|T\|_{C^{t_1}(\overline{D})}\Big). \tag{3.5.30}$$

Thus, given $a_*, a_0 \in C^{t_1}(\overline{D})$, it suffices to show $(\sup_{x\in\overline{D}}|\mathrm{e}^{T(x)}|)(1+2\|T\|_{C^{t_1}(\overline{D})}) < \infty$ for the Hölder smoothness of the realisations of a. Therefore, in the rest of this subsection, for simplicity we assume $a_* \equiv 0$ and $a_0 \equiv 1$.

In order to invoke embedding results we assume t_* satisfies $\frac{d}{2} < \lfloor t_* \rfloor$, and that we can take $t \in (0, \frac{d}{2}(\beta_1 - 1))$ such that $\frac{d}{2} < \lfloor t \rfloor$. For the latter to hold, taking $\beta_1 \ge 3$, implying $\frac{d}{2} < \lfloor \frac{d}{2}(\beta_1-1)\rfloor$, is sufficient, which is always satisfied for the presented QMC theory to be applicable. See Section 3.5.2.

Now, take $t_1 \in (0,1] \cap (0, \lfloor t \rfloor - \frac{d}{2}]$. Then, from $B_2^t(L_2(D)) = H^t(D)$ and the Sobolev embedding (for example, [1, Theorem 4.12]) we have

$$\|a\|_{C^{t_1}(\overline{D})} \lesssim \Big(\sup_{x\in\overline{D}} |a(x)|\Big)\Big(1 + 2\,\|T\|_{B_2^t(L_2(D))}\Big), \tag{3.5.31}$$

Similarly, we have $\|a^s\|_{C^{t_1}(\overline{D})} \lesssim \big(\sup_{x\in\overline{D}} |a^s(x)|\big)\Big(1 + 2\,\|T^s\|_{B_2^t(L_2(D))}\Big)$.

We want to take the expectation of $\|a\|_{C^{t_1}(\overline{D})}$. To do this, we establish the $\mathscr{F}/\mathcal{B}(\mathbb{R})$-measurability of $\boldsymbol{y} \mapsto \|a(\cdot,\boldsymbol{y})\|_{C^{t_1}(\overline{D})}$. Taking continuous modifications of T if necessary, we may assume paths of a are continuous on $\overline{D}$. Then, from the continuity of the mapping

$$\{(x_1,x_2) \in \overline{D} \times \overline{D} \mid x_1 \neq x_2\} \ni (x_1,x_2) \mapsto \frac{|a(x_1) - a(x_2)|}{\|x_1 - x_2\|_2^{t_1}} \in \mathbb{R},$$

with a countable set G that is dense in $\{(x_1,x_2) \in \overline{D} \times \overline{D} \mid x_1 \neq x_2\} \subset \mathbb{R}^d \times \mathbb{R}^d$ we have

$$\sup_{x_1,x_2\in\overline{D},\, x_1\neq x_2} \frac{|a(x_1) - a(x_2)|}{\|x_1 - x_2\|_2^{t_1}} = \sup_{(x_1,x_2)\in G} \frac{|a(x_1) - a(x_2)|}{\|x_1 - x_2\|_2^{t_1}}. \tag{3.5.32}$$

Thus, $\boldsymbol{y} \mapsto \|a(\cdot,\boldsymbol{y})\|_{C^{t_1}(\overline{D})}$, and by the same argument, $\boldsymbol{y} \mapsto \|a^s(\cdot,\boldsymbol{y})\|_{C^{t_1}(\overline{D})}$, are $\mathcal{B}(\mathbb{R}^{\mathbb{N}})/\mathcal{B}(\overline{\mathbb{R}})$-measurable, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$.

From

$$\mathbb{E}[\|T^s\|_{C(\overline{D})}] \lesssim \mathbb{E}[\|T^s\|_{B_2^t(L_2(D))}] \leq \mathbb{E}[\|T\|_{B_2^t(L_2(D))}] \lesssim \Big(\sum_{\ell=\ell_0}^{\infty} 2^{\ell(2t-d(\beta_1-1))}\Big)^{1/2} < \infty$$

independently of s, and $\mathbb{E}[\|T\|_{C(\overline{D})}] \lesssim \big(\sum_{\ell=\ell_0}^{\infty} 2^{\ell(2t-d(\beta_1-1))}\big)^{1/2} < \infty$, following the discussion by Charrier [17, Proof of Proposition 3.10] utilising the Fernique's theorem there exists a constant $M_p > 0$ independent of p such that

$$\max\Big\{\mathbb{E}[\exp(p\,\|T^s(\cdot,\boldsymbol{y})\|_{C(\overline{D})})], \mathbb{E}[\exp(p\,\|T(\cdot,\boldsymbol{y})\|_{C(\overline{D})})]\Big\} < M_p, \tag{3.5.33}$$

for any $p \in (0,\infty)$. Together with, $\sup_{x\in\overline{D}} |a(x)| \leq \exp(\sup_{x\in\overline{D}} |T(x)|)$, we have

$$\max\Big\{\mathbb{E}[(\sup_{x\in\overline{D}} |a^s(x)|)^{2p}], \mathbb{E}[(\sup_{x\in\overline{D}} |a(x)|)^{2p}]\Big\} < M_{2p}, \quad \text{for any } p \in (0,\infty).$$

Hence, from (3.5.31) we conclude that

$$\mathbb{E}[\|a\|^p_{C^{t_1}(\overline{D})}] \le \max\{1, 2^{p-1/2}\}\sqrt{\mathbb{E}\Big[\Big(\sup_{x\in\overline{D}}|a(x)|\Big)^{2p}\Big]}\sqrt{1+4^p\mathbb{E}\big[\,\|T\|^{2p}_{B^t_2(L_2(D))}\,\big]} < \infty. \tag{3.5.34}$$

Similarly, we have

$$\begin{aligned}&\mathbb{E}[\|a^s\|^p_{C^{t_1}(\overline{D})}]\\ &\le \max\{1, 2^{p-1/2}\}\sqrt{\mathbb{E}\Big[\Big(\sup_{x\in\overline{D}}|a^s(x)|\Big)^{2p}\Big]}\sqrt{1+4^p\mathbb{E}\big[\,\|T^s\|^{2p}_{B^t_2(L_2(D))}\,\big]} < \infty,\end{aligned}$$

where the right hand side can be bounded independently of s.

On the smoothness of the realisations our theory can treat

We now discuss the smoothness of the realisations that the currently developed theory permits. From the conditions imposed on the basis functions, e.g., the summability conditions, random fields with smooth realisations are easily in the scope of the QMC theory applied to PDEs. Here, the capability of taking a reasonably rough random field into account is of interest. Thus, we are interested in the smallest β_1 our theory allows us to take : Given β_0, in view of Theorem 3.5.2 the smaller the decay rate β_1 of σ_ℓ is, the rougher the realisations of (3.5.4) are.

Typically, L^2 wavelet Riesz basis have growth rate $\beta_0 = 1$, where β_0 is the parameter for the growth rate as in (W6). Then, the condition $2 < \beta_1 - \beta_0$, the weakest condition on β_1 in Theorem 3.5.1, is equivalent to

$$\beta_1 = 3 + \frac{2}{d}\varepsilon, \qquad \text{for any } \varepsilon > 0, \tag{A1}$$

where the factor $\frac{2}{d}$ is introduced to simplify the notation in the following discussion. In the following, we let $\beta_0 = 1$, take β_1 as in (A1), and discuss the smoothness of (3.5.4) achieved by taking small $\varepsilon > 0$, i.e., smallest β_1 possible. Our discussion will be based on Sobolev embedding results.

We first note that from $B^t_2(L^2(D)) = H^t(D)$, in view of Theorem 3.5.2, $T(\cdot, \boldsymbol{y}) \in H^t(D)$ a.s. if and only if the condition (3.5.22), holds.

In applications, $d = 1, 2, 3$ are of interest. We recall the following embedding results. See, for example, Adams and Fournier [1, p. 85]. For $d = 1, 2,$ and 3

	$t < \frac{d}{2}(\beta_1 - 1)$	$t < \frac{d}{2}(\beta_1 - 1)$ with $\beta_1 = 3 + 2\varepsilon/d$ for some $(\varepsilon > 0)$
$d = 1$	$t < (\beta_1 - 1)/2$	$t < 1 + \varepsilon$
$d = 2$	$t < (\beta_1 - 1)$	$t < 2 + \varepsilon$
$d = 3$	$t < \frac{3}{2}(\beta_1 - 1)$	$t < 3 + \varepsilon$

Table 3.1: For $\beta_0 = 1$, the upper bound for the exponent t for realisations of T to have H^t-smoothness is $d/2(\beta_1 - 1)$. Column 2 shows this upper bound varied with the spatial dimension d. Column 3 shows the smallest bound on t allowed by the presented QMC theory: The case $\beta_1 = 3 + 2\varepsilon/d$ for small $\varepsilon > 0$

respectively, with $\beta_1 = 3 + 2\varepsilon/d$ the condition (3.5.22) reads $t < 1+\varepsilon$, $t < 2+\varepsilon$, and $t < 3+\varepsilon$. See Table 3.1, which summarises the condition (3.5.22) with (A1).

For $d = 1$ and $d = 2$, realisations that allowed by $t < 1 + \varepsilon$, and respectively $t < 2 + \varepsilon$, seem to be rough enough. For $d = 1$, $H^1(D)$ is characterised as a space of absolutely continuous functions. Since in practice we employ a suitable numerical method to solve PDEs, the validity of point evaluations demands $a(\cdot, \boldsymbol{y}) \in C(D)$. For $d = 2$, we know $H^2(D)$ can be embedded to $C^{0,t}(\overline{D})$, $(t \in (0, 1))$. This is a standard assumption to have the convergence of FEM with the hat function elements on polygonal domains.

For $d = 3$, we have $t < 3 + \varepsilon$. We know $H^3(D) = H^{1+2}(D)$ can be embedded to $C^{1,t}(\overline{D})$, $(t \in (0, 2 - \frac{3}{2}] = (0, \frac{1}{2}])$. In practice, we employ quadrature rules to compute the integrals in the bilinear form. That $a \in C^{1,t}(\overline{D})$ $(t \in (0, \frac{1}{2}])$ is a reasonable assumption to get the convergence rate for FEM with quadratures. As a matter of fact, we want $a(\cdot, \boldsymbol{y}) \in C^{2r}(\overline{D})$ to have the $\mathcal{O}(H^{2r})$ convergence of the expected $L^p(\Omega)$-moment of $L^2(D)$-error even for C^2-bounded domains. See Charrier, Scheichl, and Teckentrup [16, Remark 3.14], and Teckentrup et al. [94, Remark 3.2].

Finally, we note these embedding results are in general optimal (see, for example, [1, 4.12, 4.40–4.44]), and in this sense, together with the characterisation (Theorem 3.5.2), the condition for our model to have Hölder smoothness is sharp.

3.5.3 Dimension truncation error

In this section we estimate the truncation error $\mathbb{E}\,\|u - u^s\|_V$. As in the previous section, the truncation is considered in terms of the level L and we let $s = s(L) = \sum_{\ell=\ell_0}^{L} \#(\triangledown_\ell)$. Let a^s be $a(x, \boldsymbol{y})$ with $y_j = 0$ for $j > s$, and define $\check{a}^s(\boldsymbol{y})$, $\hat{a}^s(\boldsymbol{y})$ accordingly.

Proposition 3.5.4. *Let u be the solution of the variational problem* (3.1.6) *with the coefficient given by the stochastic model* (3.5.6) *defined with* (3.5.4) *and* (3.5.5). *Let* $u^{s(L)}$ *be the solution of the same problem but with* $y_j := 0$ *for* $j > s(L)$. *Then, we have*

$$\mathbb{E}[\|u - u^{s(L)}\|_V] \lesssim \Big(\sum_{\ell=L+1}^{\infty} 2^{\ell(-d(\beta_1-1)+\epsilon')} \Big)^{\frac{1}{2}}, \tag{3.5.35}$$

for any $\epsilon' \in (0, 2\min\{t_*, d/2(\beta_1 - 1)\})$.

Proof. By a variant of Strang's lemma, we have

$$\|u - u^s\|_V \le \|a - a^s\|_{L^\infty(D)} \frac{\|f\|_{V'}}{\check{a}(\boldsymbol{y})\check{a}^s(\boldsymbol{y})} \tag{3.5.36}$$

for $\boldsymbol{y}$ such that $\check{a}(\boldsymbol{y})$, $\check{a}^s(\boldsymbol{y}) > 0$. We first derive an estimate on $\|a - a^s\|_{L^\infty(D)}$.

Fix $t \in (0, \min\{t_*, d/2(\beta_1 - 1)\})$ arbitrarily, where t_* is the parameter in (W5). For $t \in (0, \frac{d}{2}(\beta_1 - 1))$, choose $\bar{\mathfrak{p}}_0 \in [1, \infty)$ such that $\frac{d}{\bar{\mathfrak{p}}_0} \le t$ so that we can invoke the Besov embedding results. Since $\max\{d(\frac{1}{\bar{\mathfrak{p}}_0} - 1), 0\} < t$, from Theorem 3.5.2 there exists a set $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 1$ and $T(\cdot, \boldsymbol{y}) \in B^t_{\bar{\mathfrak{q}}}(L^{\bar{\mathfrak{p}}_0}(D))$ for all $\boldsymbol{y} \in \Omega_0$ with any $\bar{\mathfrak{q}} \in [1, \infty)$. Then, letting $T^L(x, \boldsymbol{y}) := \sum_{\ell=\ell_0}^{L} \sum_{k \in \nabla_\ell} y_{\ell,k}\sigma_\ell\varphi_{\ell,k}(x)$, from the embedding result of Besov spaces [1, Chapter 7], and the characterisation by wavelets (W5) for any L, $L' \ge 1$ ($L \ge L'$) we have

$$\Big\|T^L(\cdot, \boldsymbol{y}) - T^{L'}(\cdot, \boldsymbol{y})\Big\|_{L^\infty(D)} \lesssim \Big\|T^L(\cdot, \boldsymbol{y}) - T^{L'}(\cdot, \boldsymbol{y})\Big\|_{B^t_{\bar{\mathfrak{q}}}(L^{\bar{\mathfrak{p}}_0}(D))} \tag{3.5.37}$$

$$\sim \Bigg(\sum_{\ell=L'+1}^{L} 2^{\ell(t+d(1/2-1/\bar{\mathfrak{p}}_0))\bar{\mathfrak{q}}} \Big(\sum_{k \in \nabla_\ell} |\sigma_\ell y_{\ell,k}|^{\bar{\mathfrak{p}}_0} \Big)^{\bar{\mathfrak{q}}/\bar{\mathfrak{p}}_0} \Bigg)^{1/\bar{\mathfrak{q}}} < \infty, \tag{3.5.38}$$

for all $\boldsymbol{y} \in \Omega_0$. Thus, the sequence $\{T^L(\cdot, \boldsymbol{y})\}_L$ ($\boldsymbol{y} \in \Omega_0$) is Cauchy, and thus convergent in $L^\infty(D)$. Hence, we obtain

$$\|T(\cdot, \boldsymbol{y}) - T^L(\cdot, \boldsymbol{y})\|^{\bar{\mathfrak{q}}}_{L^\infty(D)} \lesssim \sum_{\ell=L+1}^{\infty} 2^{\ell(t+d(1/2-1/\bar{\mathfrak{p}}))\bar{\mathfrak{q}}} \Big(\sum_{k \in \nabla_\ell} |\sigma_\ell y_{\ell,k}|^{\bar{\mathfrak{p}}} \Big)^{\bar{\mathfrak{q}}/\bar{\mathfrak{p}}} \quad \text{a.s.,} \tag{3.5.39}$$

for all $\bar{\mathtt{p}} \in [1, \infty)$ such that $\frac{d}{\bar{\mathtt{p}}} \le t$, and any $\bar{\mathtt{q}} \in [1, \infty)$. For such $\bar{\mathtt{p}}$ and $\bar{\mathtt{q}}$, from Cioica et al. [18, Proof of Theorem 6], we have

$$\mathbb{E}\big[\, \|T(\cdot, \boldsymbol{y}) - T^L(\cdot, \boldsymbol{y})\|^{\bar{\mathtt{q}}}_{L^\infty(D)} \,\big] \lesssim \sum_{\ell=L+1}^{\infty} 2^{\ell(t+d(1/2-1/\bar{\mathtt{p}}))\bar{\mathtt{q}}} \sigma_\ell^{\bar{\mathtt{q}}} (\#\nabla_\ell)^{\bar{\mathtt{q}}/\bar{\mathtt{p}}} \tag{3.5.40}$$

$$\sim \sum_{\ell=L+1}^{\infty} 2^{\ell\bar{\mathtt{q}}(t-\frac{d}{2}(\beta_1-1))} < \infty. \tag{3.5.41}$$

Further, from (3.5.28) we have

$$\begin{aligned} &\mathbb{E}\Big[\, \|a(x,\boldsymbol{y}) - a^{s(L)}(x,\boldsymbol{y})\|^2_{L^\infty(D)} \Big] \\ &\le (\sup_{x\in D} |a_0(x)|^2) \\ &\quad \times \mathbb{E}[\exp(2\,\|T(\cdot,\boldsymbol{y})\|_{L^\infty(D)}) + \exp(2\|T^L(\cdot,\boldsymbol{y})\|_{L^\infty(D)})] \mathbb{E}\big[\, \|T - T^L\|^2_{L^\infty(D)} \,\big]. \end{aligned} \tag{3.5.42}$$

The sequence (ρ_ξ) defined by (3.5.13), when reordered, satisfies $(1/\rho_j) \in \ell^{\frac{d}{\theta}+\varepsilon}$ for any $\varepsilon > 0$. Thus, from the proof of Corollary 3.3.2, as in Bachmayr et al. [10, Remark 2.2], we have

$$\max\Big\{\mathbb{E}[\exp(2\,\|T(\cdot,\boldsymbol{y})\|_{L^\infty(D)})], \mathbb{E}[\exp(2\|T^L(\cdot,\boldsymbol{y})\|_{L^\infty(D)})]\Big\} < M_2, \tag{3.5.43}$$

where the constant $M_2 > 0$ is independent of L.

Together with (3.5.36), we have

$$\mathbb{E}[\|u - u^s\|_V] \le \|f\|_{V'}\, \mathbb{E}\Big[\frac{1}{(\check{a}(\boldsymbol{y}))^4}\Big]^{\frac{1}{4}} \mathbb{E}\Big[\frac{1}{(\check{a}^s(\boldsymbol{y}))^4}\Big]^{\frac{1}{4}} \mathbb{E}[\|a - a^s\|^2_{L^\infty(D)}]^{\frac{1}{2}} < \infty, \tag{3.5.44}$$

where Cauchy–Schwarz inequality is employed in the right hand side of (3.5.36). To see the finiteness of the right hand side of (3.5.44), note that

$$\frac{1}{\check{a}(\boldsymbol{y})} \le \frac{1}{\inf_{x\in D} a_0(x)} \exp(\|T\|_{L^\infty(D)}),\ \ \frac{1}{\check{a}^s(\boldsymbol{y})} \le \frac{1}{\inf_{x\in D} a_0(x)} \exp(\|T^L\|_{L^\infty(D)}),$$

and further, from the same argument as above, we have

$$\max\Big\{\mathbb{E}[\exp(4\,\|T(\cdot,\boldsymbol{y})\|_{L^\infty(D)})], \mathbb{E}[\exp(4\|T^L(\cdot,\boldsymbol{y})\|_{L^\infty(D)})]\Big\} < M_4, \tag{3.5.45}$$

where the constant $M_4 > 0$ is independent of L.

Therefore, from (3.5.41), (3.5.42), and (3.5.44) we obtain

$$\mathbb{E}[\|u - u^{s(L)}\|_V] \lesssim \mathbb{E}\big[\, \|T - T^L\|^2_{L^\infty(D)} \big]^{\frac{1}{2}} \lesssim \Big(\sum_{\ell=L+1}^{\infty} 2^{\ell(2t-d(\beta_1-1))} \Big)^{\frac{1}{2}}. \qquad (3.5.46)$$

Letting $\varepsilon' := 2t$ completes the proof. □

We conclude this section with a remark on other examples to which the currently developed QMC theory is applicable. Bachmayr et al. [10] considered so-called functions (ψ_j) with finitely overlapping supports, for example, indicator functions of a partition of the domain D. It is easy to find a positive sequence (ρ_j) such that Assumption 3.1 holds, and thus Theorem 3.4.4 readily follows. However, for these examples, due to the lack of smoothness it does not seem that it is easy to obtain a meaningful analysis as given above, and thus we forgo elaborating them.

3.6 Concluding remark

We considered a QMC theory for a class of elliptic partial differential equations with a log-normal random coefficient. Using an estimate on the partial derivative with respect to the parameter $y_{\mathfrak{u}}$ that is of product form, we established a convergence rate ≈ 1 of randomly shifted lattice rules. Further, we considered a stochastic model with wavelets, and analysed the smoothness of the realisations, and truncation errors.

In closing we note that the currently developed theory works well for (ψ_j) with local supports such as wavelets as described, but does not work so well for functions with arbitrary supports. In fact, under the same summability condition

$$\sum_{j\geq 1} (\sup_{x\in D} |\psi_j(x)|)^p < \infty \text{ for some } p \in (0,1]$$

considered by Graham et al. [36], letting $\rho_j := c(\sup_{x\in D} |\psi_j(x)|)^{p-1}$ with a suitable constant $c > 0$, one can apply Theorem 3.4.4 with $q := q(p) := \frac{p}{1-p}$. Consequently, one gets the convergence rate ≈ 1 for $p \in (0, \frac{2}{5} + \varepsilon]$ for small ε. Under a weaker summability condition than this — similarly to the uniform case [65, p. 3368], for $p \in (0, \frac{1}{2}]$ — the rate ≈ 1 with product weights already follows from the results by Graham et al. [36]; we are grateful to Frances Y. Kuo for bringing this point to our attention.

Another point related to the above concerns the cost of the CBC construction. Suppose that we can represent a given random field with two representations:

spatial functions with local support and global support. Let $s(L)$ be the truncation degree as in Section 3.5.2 for the local support, and $\tilde{s}$ be the one for global support as in Graham et al. [36]. We mentioned that the generating vector for the lattice rule can be constructed with the cost $O(s(L)n \log n)$ with the CBC construction algorithm. In Graham et al. [36], the POD weights led to the cost $O(\tilde{s}n \log n + \tilde{s}^2 n)$. Given a target error, it is not clear which order is larger: we might require $s(L) \gg \tilde{s}$ to have a desired truncation error.

Chapter 4

Discrete error estimate and discrete maximal regularity of a non-uniform implicit Euler–Maruyama scheme for a class of stochastic evolution equations

Abstract of this chapter

An implicit Euler–Maruyama method with non-uniform step-size applied to a class of stochastic evolution equations is studied. A spectral method is used for the spatial discretization and the truncation of the Wiener process. A time discretised error estimate is shown. Further, a discrete analogue of maximal L^2-regularity of the scheme and the stochastic convolution is established, which has the same form as their continuous counterpart.

4.1 Introduction

Our interest in this chapter lies in an approximation of the solution of the class of stochastic partial differential equations (SPDEs) of parabolic type. We consider three discretisations: temporal, spatial, and the truncation of the infinite-dimensional space-valued Wiener process. We establish a temporally discretised error estimate, and a discrete analogue of maximal regularity estimate.

In more detail, with a positive self-adjoint generator $-A$ with compact inverse densely defined on a separable Hilbert space H, we consider the equation

$$\begin{cases} \mathrm{d}X(t) &= AX(t)\mathrm{d}t + B(t, X(t))\mathrm{d}W(t), \qquad \text{for } \ t \in (0,1] \\ \ \ X(0) &= \xi, \end{cases} \tag{4.1.1}$$

where the mild solution X takes values in H. The assumption on B and the Q-Wiener process W will be discussed later.

In practice, such SPDEs need to be discretised. Temporally, we consider the implicit Euler–Maruyama method with a non-uniform step—a generalisation of the uniform step. Spatially, we consider the spectral method. We further discretise the Wiener process.

The Wiener process, which is assumed to admit a series representation, takes its value in an infinite-dimensional space. The Euler–Maruyama method introduces the increments of such a process, but they need to be further approximated: each increment corresponds to infinitely many random variables, but in practice we can simulate only finitely many of them. For the approximation, we truncate the Wiener process, i.e., we use a type of truncated Karhunen–Loève approximation.

This chapter is concerned with two properties of the algorithm described above: approximation error, and discrete maximal regularity. We first establish an error estimate (Theorem 4.4.8) in terms of the $L^2([0,1]\times\Omega;H)$-norm, where the temporal integration in the error estimate is treated discretely. We then show that the aforementioned algorithm satisfies a discrete analogue of an estimate called maximal regularity (Corollary 4.5.6).

Maximal regularity is a fundamental concept in the theory of deterministic partial differential equations (see, for example [3, 63, 70] and references therein). Similarly, in the study of stochastic partial differential equations the maximal regularity is an important analysis tool [22, 21] as well as an active research area [97, 96, 7, 98]. In our setting, the above equation (4.1.1) can be shown to satisfy the maximal regularity estimate of the form

$$\int_0^1 \mathbb{E}[\|X(s)\|^2_{D(A^{\iota+\frac{1}{2}})}]\,\mathrm{d}s \leq \|\xi\|^2_{D(A^{\iota+\frac{1}{2}})} + \int_0^1 \mathbb{E}\big\|B(r,X(r))\big\|^2_{\mathcal{L}_2(H_0,D(A^{\iota}))}\,\mathrm{d}r, \tag{4.1.2}$$

where $\iota \geq 0$ is a suitable parameter depending on the operator B; $D(A^{\iota+\frac{1}{2}})$ is the domain of the fractional power $A^{\iota+\frac{1}{2}}$ of A in H; and $\mathcal{L}_2(H_0,D(A^{\iota}))$ is the space of Hilbert–Schmidt operator from H_0, the Cameron–Martin space associated with Q, to $D(A^{\iota})$. More details will be discussed later. In Corollary 4.5.6, we establish a discrete analogue of the estimate (4.1.2).

To motivate our study in this chapter, we now review the literature. The aforementioned discretisation with non-uniform step was originally considered by Müller-Gronbach and Ritter [77, 76] for the stochastic heat equation over the hyper-cube. In terms of the time step size, our error estimate (Theorem 4.4.8) has the same convergence rate as what Müller-Gronbach and Ritter [76] obtained, where the error was estimated in terms of the $L^2([0,1]\times\Omega;H)$-norm with the continuous time integration. In [77, 76], the resulting non-uniform scheme was shown to achieve an asymptotic optimality, which in general cannot be achieved by schemes with uniform step-size.

The error analysis presented here is based on [54], a paper completed by the author during this PhD study together with Quoc T. Le Gia. In [54], we considered essentially the same algorithm as in [76], but applied to the stochastic heat equation on the sphere, and further analysed the error. This was not trivial. The proofs in [76] that validate the non-uniform time step do not seem to be easily generalisable to the spherical case: in the argument in [76], the fact that the eigenfunctions of the Laplace operator on the cube with the Dirichlet condition are uniformly bounded is repeatedly used in the proof; further the fact that these eigenfunctions are again those of the classical first order derivatives is crucial. On the sphere, we have neither of the properties. See [54] for more details.

We have noted that the problem formulation (4.1.1) generalises the heat equation considered in [76] as well as [54]. There, the temporal step-sizes were related to the variance of the projected one-dimensional Wiener processes. Their use of non-uniform time step was motivated by the following observation: upon the truncation, the Wiener increment is that of the sum of finitely many one-dimensional Wiener increments; these Wiener increments, which have the same law as the normal random variables, have different variances—the eigenvalue of the covariance operator; and thus one should change the time steps for accordingly to the variances.

The generalisation presented here reveals what seems to be more essential—how the stochastic forcing term operator acts on eigenspaces of the covariance operator (see Assumption 4.2), as opposed to the eigenvalues of the covariance operator—to determine the step-size. Assumption 4.2 is satisfied in [76] and [54], where the stochastic heat equations over the unit cube and the unit sphere, respectively, are considered. Other contributions to the error analysis of the numerical methods for SPDEs include [83, 100, 101, 47].

In recent years, the study of discrete analogues of the maximal regularity has been attracting attention for deterministic partial differential equations [2, 5, 15, 55, 56, 58, 67]; to the best of the author's knowledge, corresponding properties of numerical methods for stochastic PDEs have not been addressed in the literature. We focus on the case where the operator A and the covariance operator Q share the same eigensystems. This prototypical setting is partly motivated by applications in environmental modelling and astrophysics, where covariance operators—of the random fields [13, 71], and of the Wiener process for the stochastic heat equations [66, 4] for example—the eigenspaces of which are the same as those of the Laplace operators play important roles.

Maximal regularity of stochastic and deterministic equations are different in nature. As we see in (4.1.2), given a suitable smoothness of the initial data the solution is "one-half spatially smoother", than the range of the diffusion operator $B(t,x)$. This estimate optimal, in that the solution cannot be spatially smoother in general (see [60, Example 5.3]). To put it another way, as described in [22, Chapter 6], the regularity one can obtain is the half of the corresponding regularity for the deterministic case.

The structure of this chapter is as follows. Section 4.2 recalls some definitions and basic results needed in this chapter. Section 4.3 introduces the discretised scheme we consider. We provide a discrete error estimate in Section 4.4. Then, in Section 4.5 we show a discrete maximal regularity. We conclude this chapter in Section 4.6.

4.2 Setting

By H we denote a separable $\mathbb{R}$-Hilbert space $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$. Let $-A : D(A) \subset H \to H$ be a self-adjoint, positive definite linear operator that is densely defined on H, with compact inverse $-A^{-1}$. Then, A is the generator of the C_0-semigroup $(S(t))_{t\geq 0} := (\mathrm{e}^{At})_{t\geq 0}$ acting on H that is analytic. Further, there exists a complete orthonormal system $\{h_{jk}\}$ for H such that $-Ah_{jk} = \lambda_j h_{jk}$, each eigenspace is of finite dimensional, and

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots ,$$

and $\lambda_j \to \infty$ as $j \to \infty$, unless $-A^{-1}$ is of finite rank. Let $\operatorname{span}\{h_{jk} \mid j \in \Lambda_j\}$ denote the j-th eigenspace with an index set Λ_j of a finite cardinality. Then, we have the spectral representation

$$S(t)x = \sum_{j=1}^{\infty} \sum_{k\in\Lambda_j} \mathrm{e}^{-\lambda_j t} \left\langle x, h_{jk}\right\rangle h_{jk} \in H.$$

For $r \in \mathbb{R}$, let us define the domain $D(A^r)$ of the fractional power A^r of A by

$$D(A^r) := \left\{ x \in H \;\middle|\; \|x\|_{D(A^r)}^2 = \sum_{j=1}^{\infty} \sum_{k\in\Lambda_j} \lambda_j^{2r} \left\langle x, h_{jk}\right\rangle^2 < \infty \right\}.$$

We obtain a separable Hilbert space $(D(A^r), \langle\cdot,\cdot\rangle_{D(A^r)}, \|\cdot\|_{D(A^r)})$ by setting $\langle\cdot,\cdot\rangle_{D(A^r)} := \langle A^r\cdot, A^r\cdot\rangle$.

For more details for the set up above, see for example [51, 70, 90, 102].

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space equipped with a filtration satisfying the usual conditions. By $W\colon [0,1]\times\Omega \to H$ we denote the Q-Wiener process with a covariance operator Q of the trace class. We assume that the Wiener process W is adapted to the filtration. Further, we assume that the eigenfunctions $h_{\ell m}$ of A is also eigenfunctions of Q with

$$Qh_{\ell m} = q_\ell h_{\ell m},$$

such that $\operatorname{Tr}(Q) = \sum_{\ell=1}^{\infty}\sum_{m\in\Lambda_\ell}\langle Qh_{\ell m}, h_{\ell m}\rangle = \sum_{\ell=1}^{\infty}\sum_{m\in\Lambda_\ell} q_\ell < \infty$. It is well-known that W taking values in H can be characterised as

$$W(t) = \sum_{\ell=0}^{\infty}\sum_{m\in\Lambda_\ell}\sqrt{q_\ell}\beta_{\ell m}(t)h_{\ell m} \quad \text{a.s.},$$

where $\beta_{\ell m}$ are independent one-dimensional standard Brownian motions with the zero initial condition realised on $(\Omega, \mathscr{F}, \mathbb{P})$ that are adapted to the underlying filtration, and that the series converges in the Bochner space $L^2(\Omega; C([0,1]; H))$.

The Q-Wiener process takes values in H by construction. Here, since A and Q are assumed to share the same eigenfunctions, we can provide finer characterisations of the regularity.

Remark 3. Let $r \geq 0$ and $t \in (0,1]$. Then, $\sum_{\ell=0}^{\infty}\sum_{m\in\Lambda_\ell}\lambda_\ell^{2r}q_\ell < \infty$ if and only if $W(t) \in D(A^r)$, a.s. Indeed, we have

$$\mathbb{E}[\|W(t)\|_{D(A^r)}^2] = t\sum_{\ell=1}^{\infty}\sum_{m\in\Lambda_\ell}\lambda_\ell^{2r}q_\ell.$$

We introduce the Hilbert space $H_0 = Q^{1/2}(H)$ equipped with the inner product

$$\langle h_1, h_2\rangle_0 = \langle Q^{-1/2}h_1, Q^{-1/2}h_2\rangle \quad \text{for } h_1, h_2 \in H,$$

where $Q^{-1/2} := (Q^{1/2}|_{(\ker(Q^{1/2}))^\perp})^{-1}\colon Q^{\frac{1}{2}}(H) \to (\ker(Q^{1/2}))^\perp$ is the pseudo-inverse of $Q^{1/2}$.

In the following, $a \preceq b$ means that a can be bounded by some constant times b uniformly with respect to any parameters on which a and b may depend. Throughout this chapter, we assume the following.

Assumption 4.1. For some $\iota \geq 0$, we assume $B\colon [0,1]\times D(A^\iota) \to \mathcal{L}_2(H_0, D(A^\iota))$ is $\mathcal{B}([0,1]) \otimes \mathcal{B}(D(A^\iota))/\mathcal{B}(\mathcal{L}_2(H_0, D(A^\iota)))$-measurable, where for a given normed space $(\mathscr{X}, \|\cdot\|_{\mathscr{X}})$ the Borel σ-algebra associated with the norm topology is denoted by $\mathcal{B}(\mathscr{X})$. Further, let B satisfy the Lipschitz condition: for $t \in [0,1]$, $u, v \in D(A^\iota)$ we have

$$\|B(t,u) - B(t,v)\|_{\mathcal{L}_2(H_0, D(A^\iota))} \preceq \|u - v\|_{D(A^\iota)},$$

and the linear growth condition: for $t \in [0,1]$, $u \in D(A^\iota)$ we have

$$\|B(t,u)\|_{\mathcal{L}_2(H_0, D(A^\iota))} \preceq 1 + \|u\|_{D(A^\iota)}.$$

We recall the following existence result, which can be found in, for example, [22, Section 7.1].

Theorem 4.2.1. *Suppose that B satisfies Assumption 4.1 with some $\iota \geq 0$. Then, for $\xi \in D(A^\iota)$ there exists a $D(A^\iota)$-valued continuous process $(X(t))_{t\in[0,1]}$ adapted to the underlying filtration satisfying the usual conditions such that*

$$X(t) = S(t)\xi + \int_0^t S(t-s)B(s, X(s))\mathrm{d}W(s), \quad t \in [0,1] \quad a.s. \tag{4.2.1}$$

Moreover, this process is uniquely determined a.s., and it is called the mild solution of the stochastic evolution equation (4.1.1). Further, for any $p \geq 2$ we have

$$\sup_{t\in[0,1]} \mathbb{E}\|X(t)\|^p_{D(A^\iota)} < \infty. \tag{4.2.2}$$

For the mild solution X, let

$$X(t) = \sum_{j=1}^{\infty} \sum_{k\in\Lambda_j} X_{jk}(t) h_{jk}, \quad X_{jk}(t) = \langle X(t), h_{jk} \rangle .$$

Then, the processes $X_{jk} = (X_{jk}(t))_{t\in[0,1]}$ satisfy the following bi-inifinite system of stochastic differential equations:

$$\begin{cases} \mathrm{d}X_{jk}(t) &= -\lambda_j X_{jk}(t)\mathrm{d}t + \sum_{\ell=1}^{\infty} \sum_{m\in\Lambda_\ell} \sqrt{q_\ell} \langle B(t, X(t))h_{\ell m}, h_{jk} \rangle \mathrm{d}\beta_{\ell m}(t) \\ X_{jk}(0) &= \langle \xi, h_{jk} \rangle , \qquad \text{for} \quad j \in \mathbb{N},\ k \in \Lambda_j. \end{cases}$$

Each process X_{jk} is given as

$$X_{jk}(t) = \mathrm{e}^{-\lambda_j t} \left\langle \xi, h_{jk} \right\rangle + \sum_{\ell=1}^{\infty} \sum_{m \in \Lambda_\ell} \sqrt{q_\ell} \int_0^t \mathrm{e}^{-\lambda_j (t-s)} \left\langle B(s, X(s)) h_{\ell m}, h_{jk} \right\rangle \mathrm{d}\beta_{\ell m}(s),$$

where the series in the second term is convergent in $L^2(\Omega)$, due to (4.2.2) and Assumption 4.1.

We have the following spatial regularity result.

Proposition 4.2.2. *Suppose Assumption 4.1 is satisfied with some $\iota \geq 0$, and the initial condition satisfies $\xi \in D(A^\iota)$. Then, we have the estimate*

$$\int_0^1 \mathbb{E}\|X(s)\|^2_{D(A^{\iota+1/2})} \,\mathrm{d}s \leq \|\xi\|^2_{D(A^\iota)} + \int_0^1 \mathbb{E}\big\|B(r, X(r))\big\|^2_{\mathcal{L}_2(H_0, D(A^\iota))} \,\mathrm{d}r. \quad (4.2.3)$$

Proof. Itô's isometry yields

$$\lambda_j^{2\iota+1} \mathbb{E}(X_{jk}(s))^2 = \exp(-2\lambda_j s) \lambda_j^{2\iota+1} \left\langle \xi, h_{jk} \right\rangle^2 + \int_0^s \exp(-2\lambda_j (s-r)) \lambda_j \mathbb{E}\|B^*(r, X(r)) \lambda_j^{-\iota} h_{jk}\|^2_{H_0} \mathrm{d}r,$$

where $B^*(r, X(r))$ denotes the adjoint operator of $B(r, X(r))$. Therefore, it holds that $\int_0^1 \mathbb{E}[\lambda_j^{2\iota+1} |X_{jk}(s)|^2] \,\mathrm{d}s \leq \lambda_j^{2\iota} \left\langle \xi, h_{jk} \right\rangle^2 + \int_0^1 \mathbb{E}\|B^*(r, X(r)) \lambda_j^{-\iota} h_{jk}\|^2_{H_0} \,\mathrm{d}r$, and thus summing over $j \geq 1$, $k \in \Lambda_j$ yields the desired result. □

Remark 4. We note that the solution is spatially one half smoother than the range of $B(t, x)$. This is in general optimal, in that the solution cannot be spatially smoother in general ([60, Example 5.3]). Note that here we need only $\xi \in D(A^\iota)$; in contrast, to obtain $\sup_{s \in [0,1]} \mathbb{E}\|X(s)\|^2_{D(A^{\iota+1/2})}$ we need $\xi \in D(A^{\iota+1/2})$. For more details, see [60, 59] and references therein. For recent developments of maximal regularity theory, see [97, 96].

4.3 Discretisation

This section introduces the scheme proposed by Müller-Gronbach and Ritter [77, 76]. In this regard, let us first discretise the interval $[0, 1]$ with a uniform partition, i.e., we partition the interval with $t_i = i/n$, for $i = 0, 1, 2, \ldots, n$. For integers $J, L \in \mathbb{N}$, an Itô–Galerkin approximation $\overline{X}(t_i)$ to (4.2.1) with the temporal discretisation being the implicit Euler–Maruyama scheme with a uniform time discretisation is given by

$$\overline{X}^{J,L}(t_i) = \sum_{j=1}^{J} \sum_{k \in \Lambda_j} \overline{X}_{jk}^{J,L}(t_i) h_{jk}, \quad \text{for} \quad i = 0, \ldots, N, \tag{4.3.1}$$

with coefficients $\langle \overline{X}(t_i), h_{jk} \rangle$ defined by $\overline{X}_{jk}^{J,L}(0) = \langle \xi, h_{jk} \rangle$, and

$$\begin{aligned} \overline{X}_{jk}^{J,L}(t_i) &= \left(1 + \frac{\lambda_j}{n}\right)^{-1} \Bigg(\overline{X}_{jk}^{J,L}(t_{i-1}) \\ &\quad + \sum_{\ell=1}^{L} \sum_{m \in \Lambda_\ell} \sqrt{q_\ell} \left\langle B(t_{i-1}, \overline{X}^{J,L}(t_{i-1})) h_{\ell m}, h_{jk} \right\rangle (\beta_{\ell m}(t_i) - \beta_{\ell m}(t_{i-1})) \Bigg). \end{aligned}$$

Müller-Gronbach and Ritter [77, 76] noted that the projected Q-Wiener processes $\sqrt{q_\ell}\beta_{\ell m} = \sqrt{\langle Q h_{\ell m}, h_{\ell m} \rangle}\beta_{\ell m} = \langle W(t), h_{\ell m} \rangle$ have varying variances depending on the index ℓ. This observation motivated them to use different stepsizes depending on ℓ. Following them, we evaluate the standard one-dimensional Wiener process $\beta_{\ell m}$ at each level $\ell = 1, \ldots, L$ at the corresponding $n_\ell \in \mathbb{N}$ nodes

$$0 < t_{1,\ell} < \cdots < t_{n_\ell,\ell} = 1, \quad \text{where} \quad t_{i,\ell} = \frac{i}{n_\ell} \quad \text{for} \quad i = 0, \ldots, n_\ell.$$

Then, the discretisation of the truncated Q-Wiener process $\sum_{\ell=1}^{L} \sum_{m \in \Lambda_\ell} \sqrt{q_\ell} \beta_{\ell m} h_{\ell m}$ in general results in a non-uniform time discretisation:

$$0 =: \tau_0 < \cdots < \tau_N := 1, \quad \text{where} \quad \{\tau_0, \ldots, \tau_N\} := \bigcup_{\ell=1}^{L} \{t_{0,\ell}, \ldots, t_{n_\ell,\ell}\},$$

and $t_{0,\ell} = \tau_0 = 0$ for all $\ell \in \mathbb{N}$. To write our scheme in the recursive form, we introduce the following notations. Let

$$\mathcal{K}_\eta := \left\{ \ell \in \{0, 1, \ldots, L\} \,\middle|\, \tau_\eta \in \{t_{0,\ell}, \ldots, t_{n_\ell,\ell}\} \right\}, \tag{4.3.2}$$

for $\eta = 0, \ldots, N$ and we define $s_{\eta,\ell}$ for $\eta = 1, \ldots, N$ and $\ell = 1, \ldots, L$ by

$$s_{\eta,\ell} := \max \left\{ \{t_{0,\ell}, \ldots, t_{n_\ell,\ell}\} \cap [0, \tau_\eta) \right\}.$$

We further introduce the following notation for the product of eigenvalues of the operator $(I - \frac{1}{\tau_\nu - \tau_{\nu-1}} A)^{-1}$, which we use for the approximation of the semigroup generated by A. For any $\tau_{\eta_1} \le \tau_{\eta_2}$, we let

$$\mathfrak{R}_j(\tau_{\eta_1}, \tau_{\eta_2}) := \prod_{\nu=\eta_1+1}^{\eta_2} \frac{1}{1 + \lambda_j(\tau_\nu - \tau_{\nu-1})}, \tag{4.3.3}$$

with the convention $\prod_\emptyset = 1$. Note that $s_{\eta,\ell}, t_{i-1,\ell} \in \{\tau_1, \dots, \tau_N\}$. Then, for $\eta = 1, \dots, N$, the drift-implicit Euler–Maruyama scheme in the recursive form is given by,

$$\begin{aligned} \widehat{X}^{J,L}_{jk}(\tau_\eta) = \mathfrak{R}_j(\tau_{\eta-1}, \tau_\eta) \Bigg(\widehat{X}^{J,L}_{jk}(\tau_{\eta-1}) + \sum_{\ell \in \mathcal{K}_\eta} \sum_{m \in \Lambda_\ell} \sqrt{q_\ell} \left\langle B(s_{\eta,\ell}, \widehat{X}^{J,L}(s_{\eta,\ell})) h_{\ell m}, h_{jk} \right\rangle \\ \times \mathfrak{R}_j(s_{\eta,\ell}, \tau_{\eta-1}) (\beta_{\ell m}(\tau_\eta) - \beta_{\ell m}(s_{\eta,\ell})) \Bigg). \end{aligned}$$

Equivalently, the above can be written in the convolution form

$$\begin{aligned} &\widehat{X}^{J,L}_{jk}(\tau_\eta) \\ &= \mathfrak{R}_j(\tau_0, \tau_\eta) \langle \xi, h_{jk} \rangle + \sum_{\ell=1}^{L} \sum_{m \in \Lambda_\ell} \sum_{\tau_1 \le t_{i,\ell} \le \tau_\eta} \sqrt{q_\ell} \left\langle B(t_{i-1,\ell}, \widehat{X}^{J,L}(t_{i-1,\ell})) h_{\ell m}, h_{jk} \right\rangle \\ &\qquad \times \mathfrak{R}_j(t_{i-1,\ell}, \tau_\eta) (\beta_{\ell m}(t_{i,\ell}) - \beta_{\ell m}(t_{i-1,\ell})). \end{aligned} \tag{4.3.4}$$

Then, we use

$$\widehat{X}^{J,L}(\tau_\eta) = \sum_{j=1}^{J} \sum_{k \in \Lambda_j} \widehat{X}^{J,L}_{jk}(\tau_\eta) h_{jk} \quad \text{for} \quad \eta = 1, \dots, N \tag{4.3.5}$$

as our approximate solution.

We note that this scheme generalises the aforementioned approximation $\overline{X}^{J,L}$ with the uniform time step as in (4.3.1): $\overline{X}^{J,L}$ is nothing but $\widehat{X}^{J,L}$ with $n_\ell = N$ for $\ell = 1, \dots, L$.

4.4 Discrete time integral error estimate

The aim of this section is to show the error estimate given in Theorem 4.4.8. We study the error under the following Lipschitz condition. This property will be used to justify the use of different step-sizes depending on ℓ. Assumption 4.2 is satisfied in [76] and [54], where the stochastic heat equations over the unit cube and the unit sphere, respectively, are considered.

Assumption 4.2. We assume that for any $\ell \geq 1$, $x, x_1, x_2 \in H$ and $t \in [0,1]$ we have a positive number γ_ℓ such that $\sum_{\ell=1}^{\infty} \gamma_\ell < \infty$ and

$$\sum_{m \in \Lambda_\ell} \|B(t,x)\sqrt{q_\ell} h_{\ell m}\|^2 \leq \gamma_\ell (1 + \|x\|^2), \tag{4.4.1}$$

and

$$\sum_{m \in \Lambda_\ell} \|(B(t,x_1) - B(t,x_2))\sqrt{q_\ell} h_{\ell m}\|^2 \leq \gamma_\ell \|x_1 - x_2\|^2. \tag{4.4.2}$$

Further, for any $s, t \in [0,1]$ and $x \in H$ we have

$$\sum_{m \in \Lambda_\ell} \|(B(s,x) - B(t,x))\sqrt{q_\ell} h_{\ell m}\|^2 \leq \gamma_\ell (1 + \|x\|^2)|t - s|. \tag{4.4.3}$$

Assumption 4.2 describes the regularity of $B \colon [0,t] \times H \to \mathcal{L}_2(H_{0,\ell}, H)$, where $H_{0,\ell} := \operatorname{span}\{\sqrt{q_\ell} h_\ell \mid m \in \Lambda_\ell\} \subset H_0$, is the ℓ-th eigenspace of Q in H_0. For example, (4.4.2) can be rewritten as

$$\|B(t,x_1) - B(t,x_2)\|_{\mathcal{L}_2(H_{0,\ell},H)} \leq \sqrt{\gamma_\ell} \|x_1 - x_2\|.$$

Note that Assumption 4.2 states the separately continuity of B from the product of separable metric spaces $[0,1] \times H$ to $\mathcal{L}_2(H_0, H)$. Thus, by a property of the Carathèodory function B is product measurable. Therefore, Assumption 4.2 implies Assumption 4.1 with $\iota = 0$.

We start our analysis by considering the following semi-discrete—temporally continuous—scheme given by

$$X^{J,L}(t) = \sum_{j=1}^{J} \sum_{k \in \Lambda_j} X^{J,L}_{jk}(t) h_{jk}, \tag{4.4.4}$$

with the real-valued process $X^{J,L}_{jk} = (X^{J,L}_{jk}(t))_{t \in [0,1]}$ that solve the finite-dimensional system

$$\mathrm{d}X^{J}_{jk}(t) = -\lambda_j X^{J,L}_{jk}(t)\mathrm{d}t + \sum_{\ell=1}^{L} \sum_{m \in \Lambda_\ell} \left\langle B(t, X^{J,L}(t))\sqrt{q_\ell} h_{\ell m}, h_{jk} \right\rangle \mathrm{d}\beta_{\ell m}(t),$$

with the initial condition $X_{jk}^{J,L}(0) = \langle \xi, h_{jk} \rangle$. Each process $X_{jk}^{J,L}$ is given as

$$
\begin{aligned}
X_{jk}^{J,L}(t) =& \mathrm{e}^{-\lambda_j t} \left\langle \xi, h_{jk} \right\rangle \\
&+ \sum_{\ell=1}^{\infty} \sum_{m \in \Lambda_\ell} \int_0^t \mathrm{e}^{-\lambda_j (t-s)} \left\langle B(s, X^{J,L}(s)) \sqrt{q_\ell} h_{\ell m}, h_{jk} \right\rangle \mathrm{d}\beta_{\ell m}(s).
\end{aligned} \tag{4.4.5}
$$

From (4.2.2) with $\iota = 0$ we have

$$
\sup_{s \in [0,1]} \mathbb{E} \|X(s)\|^2 < c. \tag{4.4.6}
$$

Further, by the same argument given in [76, (6.8)] we have

$$
\sup_{s \in [0,1]} \mathbb{E} \|X^{J,L}(s)\|^2 < c. \tag{4.4.7}
$$

We have the following truncation error estimate with a discretised temporal integral.

Proposition 4.4.1. *Let Assumption 4.2 hold. Then, we have*

$$
\mathbb{E} \Big[\sum_{\eta=1}^{N} \|X(\tau_\eta) - X^{J,L}(\tau_\eta)\|^2 (\tau_\eta - \tau_{\eta-1}) \Big] \preceq \frac{1}{\lambda_{J+1}} + \sum_{\ell=L+1}^{\infty} \gamma_\ell,
$$

where $X^{J,L}$ is defined as in (4.4.4).

Proof. We first note that Assumption 4.1 is satisfied.

For any $t \in [0,1]$ we have the trivial decomposition

$$
X(t) = Y^{J,L}(t) + Y^{J,\mathrm{c}L}(t) + \sum_{j \geq J+1} \sum_{k \in \Lambda_J} X_{jk} h_{jk}(t),
$$

where

$$
\begin{aligned}
Y^{J,L}(t) :=& \sum_{j=1}^{J} \sum_{k \in \Lambda_k} \Big(\exp(-\lambda_j t) \left\langle \xi, h_{jk} \right\rangle \\
&+ \sum_{\ell=1}^{L} \sum_{m \in \Lambda_\ell} \int_0^t \exp(-\lambda_j (t-s)) \langle B(X(s)) \sqrt{q_\ell} h_{\ell m}, h_{jk} \rangle \mathrm{d}\beta_{\ell m}(s) \Big), \\
Y^{J,\mathrm{c}L}(t) :=& \sum_{j=1}^{J} \sum_{k \in \Lambda_k} \sum_{\ell \geq L+1} \sum_{m \in \Lambda_\ell} \int_0^t \exp(-\lambda_j (t-s)) \langle B(s, X(s)) \sqrt{q_\ell} h_{\ell m}, h_{jk} \rangle \mathrm{d}\beta_{\ell m}(s).
\end{aligned}
$$

Then, we have

$$\|X(t)-X^{J,L}(t)\|^2 \preceq \|Y^{J,\mathrm{c}L}(t)\|^2 + \Big\|\sum_{j\geq J+1}\sum_{k\in\Lambda_J} X_{jk}(t)h_{jk}\Big\|^2 + \|Y^{J,L}(t)-X^{J,L}(t)\|^2. \tag{4.4.8}$$

First, from (4.4.1) in Assumption 4.2 and (4.4.6) we have

$$\mathbb{E}\|Y^{J,\mathrm{c}L}(t)\|^2 \preceq \sum_{\ell\geq L+1}\sum_{m\in\Lambda_\ell}\int_0^t \mathbb{E}\big[\|B(s,X(s))\sqrt{q_\ell}h_{\ell m}\|^2\big]\,\mathrm{d}s \preceq \sum_{\ell\geq L+1}\gamma_\ell \leq c. \tag{4.4.9}$$

Next, fixing arbitrary $j\in\mathbb{N}$ and $k\in\Lambda_j$ we have for any $\eta\in\{1,\dots,N\}$

$$\begin{aligned}\sum_{\nu=1}^{\eta}\mathbb{E}[|X_{jk}(\tau_\nu)|^2](\tau_\nu-\tau_{\nu-1}) &= \sum_{\nu=1}^{\eta}\int_{\tau_{\nu-1}}^{\tau_\nu}\mathrm{e}^{-2\lambda_j\tau_\nu}|\left\langle\xi,h_{jk}\right\rangle|^2\,\mathrm{d}t\\ &+\sum_{\ell=1}^{\infty}\sum_{m\in\Lambda_\ell}\sum_{\nu=1}^{\eta}\int_{\tau_{\nu-1}}^{\tau_\nu}\int_0^{\tau_\nu}\mathrm{e}^{-2\lambda_j(\tau_\nu-s)}\mathbb{E}|\left\langle B(s,X(s))\sqrt{q_\ell}h_{\ell m},h_{jk}\right\rangle|^2\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

Letting $\mathscr{P}_J^\perp x := \sum_{j\geq J+1}\sum_{k\in\Lambda_j}\left\langle x,h_{jk}\right\rangle h_{jk}$ for $x\in H$, we have

$$\begin{aligned}\sum_{j\geq J+1}\sum_{k\in\Lambda_j}\sum_{\ell=1}^{\infty}\sum_{m\in\Lambda_\ell}&\mathrm{e}^{-2\lambda_j(\tau_\nu-s)}\mathbb{E}|\left\langle B(s,X(s))\sqrt{q_\ell}h_{\ell m},h_{jk}\right\rangle|^2\\ &=\sum_{\ell=1}^{\infty}\sum_{m\in\Lambda_\ell}\mathbb{E}\|S(\tau_\nu-s)\mathscr{P}_J^\perp B(s,X(s))\sqrt{q_\ell}h_{\ell m}\|^2\\ &\leq \|S(\tau_\nu-s)\|_{V_J^\perp\to H}\mathbb{E}\|B(s,X(s))\|^2_{\mathcal{L}_2(H_0,H)},\end{aligned}$$

where $V_J^\perp := \{x\in H \mid \langle x,h_{jk}\rangle = 0,\ \text{ for } j\geq J+1,\ k\in\Lambda_j\}$. Thus, noting $\|S(\tau_\nu-s)\|_{V_J^\perp\to H} = \mathrm{e}^{-\lambda_{J+1}(\tau_\nu-s)}$ from the Lipschitz condition and (4.4.6) we have

$$\begin{aligned}&\sum_{\nu=1}^{\eta}\sum_{j\geq J+1}\sum_{k\in\Lambda_j}\mathbb{E}[|X_{jk}(\tau_\nu)|^2](\tau_\nu-\tau_{\nu-1})\\ &\leq \int_0^{\tau_\eta}\mathrm{e}^{-2\lambda_{J+1}t}\,\mathrm{d}t\|\xi\|^2 + \frac{1}{2\lambda_{J+1}}\Big(1+\sup_{s_0\in[0,1]}\mathbb{E}\|X(s_0)\|^2\Big)\sum_{\nu=1}^{\eta}\int_{\tau_{\nu-1}}^{\tau_\nu}\mathrm{d}t \preceq \frac{1}{\lambda_{J+1}} \leq c.\end{aligned} \tag{4.4.10}$$

Finally, we consider a bound of $\mathbb{E}\|Y^{J,L}(t) - X^{J,L}(t)\|^2$. Similarly to the above, for any $j \in \mathbb{N}$, $k \in \Lambda_j$ and any $t \in [0,1]$ we have

$$\begin{aligned}
&\int_0^t \mathbb{E}|X_{jk}(r)|^2 \, \mathrm{d}r \\
&\qquad\qquad \leq \frac{1}{2\lambda_j} | \left\langle \xi, h_{jk} \right\rangle |^2 + \frac{1}{2\lambda_j} \sum_{\ell=1}^{\infty} \sum_{m \in \Lambda_\ell} \int_0^1 \mathbb{E}| \left\langle B(s, X(s)) \sqrt{q_\ell} h_{\ell m}, h_{jk} \right\rangle |^2 \, \mathrm{d}s.
\end{aligned}$$

Letting $C_j := | \left\langle \xi, h_{jk} \right\rangle |^2 + \int_0^1 \mathbb{E}| \left\langle \sqrt{q_\ell} h_{\ell m}, B(s, X(s))^* h_{jk} \right\rangle_{H_0} |^2 \, \mathrm{d}s$, the linear growth condition (4.4.1) and $\int_0^1 \mathbb{E}\|X(t)\|^2 \, \mathrm{d}t < c$ implies

$$\sum_{j=1}^{\infty} \sum_{k \in \Lambda_j} C_j \preceq \|\xi\| + \int_0^1 (1 + \mathbb{E}\,\|X(t)\|^2) \, \mathrm{d}s \leq c.$$

and thus

$$\int_0^1 \sum_{j \geq J+1} \sum_{k \in \Lambda_j} \mathbb{E}|X_{jk}(r)|^2 \, \mathrm{d}r \preceq \frac{1}{\lambda_{J+1}} \leq c. \tag{4.4.11}$$

Now, the Lipschitz condition (4.4.2) implies

$$\begin{aligned}
&\mathbb{E}\|Y^{J,L}(t) - X^{J,L}(t)\|^2 \\
&= \sum_{\ell=1}^{L} \sum_{m \in \Lambda_\ell} \int_0^t \exp(-2\lambda_j(t-s)) \mathbb{E}[\langle (B(s, X(s)) - B(s, X^{J,L}(s))) \sqrt{q_\ell} h_{\ell m}, h_{jk} \rangle^2] \, \mathrm{d}s \\
&\preceq \int_0^t \mathbb{E}[\|X(s) - X^{J,L}(s)\|^2] \, \mathrm{d}s \\
&\preceq \sum_{\ell \geq L+1} \gamma_\ell + \int_0^1 \sum_{j \geq J+1} \sum_{k \in \Lambda_j} \mathbb{E}|X_{jk}(s)|^2 \, \mathrm{d}s + \int_0^t \mathbb{E}\|Y^{J,L}(s) - X^{J,L}(s)\|^2 \, \mathrm{d}s, \\
&\preceq 2c + \int_0^t \mathbb{E}\|Y^{J,L}(s) - X^{J,L}(s)\|^2 \, \mathrm{d}s,
\end{aligned}$$

where in the last line (4.4.9) and (4.4.11) are used. Noting $\mathbb{E}\|Y^{J,L}(s)\|^2 \leq \mathbb{E}\|X(s)\|^2$, in view of (4.4.6) and (4.4.7) we have

$$\sup_{s \in [0,1]} \mathbb{E}\|Y^{J,L}(s) - X^{J,L}(s)\|^2 < \infty,$$

and thus the Gronwall's lemma implies

$$\mathbb{E}\|Y^{J,L}(t) - X^{J,L}(t)\|^2 \leq \sum_{\ell \geq L+1} \gamma_\ell + \int_0^1 \sum_{j \geq J+1} \sum_{k \in \Lambda_j} \mathbb{E}|X_{jk}(s)|^2 \,\mathrm{d}s.$$

From this inequality, together with (4.4.8), (4.4.9), (4.4.10), and (4.4.11), the statement follows. □

We need the following estimate on the discretised solution due to [76].

Lemma 4.4.2. *Suppose Assumption 4.2 is satisfied. Then, $\widehat{X}^{J,L}$ defined as in (4.3.5) satisfies*

$$\mathbb{E}\big[\max_{\tau_\nu \in \{\tau_0, \dots, \tau_N\}} \|\widehat{X}^{J,L}(\tau_\nu)\|^2\big] \preceq 1.$$

Proof. The statement is essentially implied by [76, Lemma 6.4]. □

We now proceed to discuss properties of the spectral approximation $\mathfrak{R}_j$ of the semigroup. We use the notation

$$n^* := \max_{\ell=1,\dots,L} n_\ell.$$

Lemma 4.4.3. *For any $\ell \in \{1, \dots, L\}$ and $i \in \{1 \dots, n_\ell\}$, we have*

$$\sum_{\tau_{\eta^*(i-1,\ell)+1} \leq \tau_\eta \leq \tau_N} \big(\mathfrak{R}_j(t_{i-1,\ell}, \tau_\eta) - \exp(-\lambda_j(\tau_\eta - t_{i-1,\ell}))\big)^2 (\tau_\eta - \tau_{\eta-1}) \preceq \frac{1}{n^*}.$$

Proof. Define constant interpolations

$$g_{j,t_{i-1,\ell}}(t) := \begin{cases} 1 & t = t_{i-1,\ell} \\ \mathfrak{R}_j(t_{i-1,\ell}, \tau_\eta) & t \in (\tau_{\eta-1}, \tau_\eta], \quad \eta = \eta^*(i-1,\ell)+1, \dots, N. \end{cases}$$

and

$$h_{j,t_{i-1,\ell}}(t) := \begin{cases} 1 & t = t_{i-1,\ell} \\ \exp\big(-\lambda_j(\tau_\eta - t_{i-1,\ell})\big) & t \in (\tau_{\eta-1}, \tau_\eta], \ \eta = \eta^*(i-1,\ell)+1, \dots, N. \end{cases}$$

Let $f(t) := g_{j,t_{i-1,\ell}}(t) - h_{j,t_{i-1,\ell}}(t)$. Then, we have

$$\sum_{\tau_{\eta^*(i-1,\ell)+1} \leq \tau_\eta \leq \tau_N} \big(\mathfrak{R}_j(t_{i-1,\ell}, \tau_\eta) - \exp(-\lambda_j(\tau_\eta - t_{i-1,\ell}))\big)^2 (\tau_\eta - \tau_{\eta-1})$$
$$= \int_{t_{i-1,\ell}}^1 \big(g_{j,t_{i-1,\ell}}(t) - h_{j,t_{i-1,\ell}}(t)\big)^2 \,\mathrm{d}t = \int_{t_{i-1,\ell}}^1 (f(t))^2 \,\mathrm{d}t.$$

Put $k^* := \lceil t_{i-1,\ell} n^* \rceil$. Then, noting $0 \leq f \leq 1$, we have

$$\int_{t_{i-1,\ell}}^1 (f(t))^2 \,\mathrm{d}t \preceq \frac{1}{n^*} + \sum_{k=k^*}^{n^*-1} \int_{k/n^*}^{(k+1)/n^*} (f(t))^2 \,\mathrm{d}t.$$

Following the argument in the proof [76, Lemma 6.3] to show [76, (6.9)], we have

$$\sum_{k=k^*}^{n^*-1} \sup_{t \in [k/n^*, (k+1)/n^*]} (f(t))^2 \preceq 1.$$

Thus, the Gronwall's lemma implies the result. □

We need the mean-square continuity. The following result is a slight generalisation of [77, Lemma 1]. See [22, Theorem 9.1], [59, Theorem 2.25], and [60, Theorem 4.1] for further related results. We have the following temporal regularity result.

Proposition 4.4.4. *Suppose Assumption 4.1 is satisfied with some $\iota \geq 0$, and that the initial condition satisfies $\xi \in D(A^\iota)$. Then, the mild solution is continuous in the mean-square sense on $[0,1]$. Further, the function*

$$\psi(s) := \sum_{j=1}^{\infty} \sum_{k \in \Lambda_j} \lambda_j^{2\iota+1} \mathbb{E}(X_{jk}^2(s))$$

satisfies $\psi \in L_1([0,1])$, and we have the estimate

$$\mathbb{E}\|X(s) - X(t)\|_{D(A^\iota)}^2 \leq C|t-s|\big(1 + \psi(\min\{s,t\})\big). \tag{4.4.12}$$

Proof. First note that $\mathbb{E}\|X(s)-X(t)\|^2_{D(A^\iota)} = \sum_{j=1}^\infty \sum_{k\in\Lambda_j} \mathbb{E}\lambda_j^{2\iota}(X_{jk}(s)-X_{jk}(t))^2$. For $s<t$ we have

$$X_{jk}(t) - X_{jk}(s) = \mathrm{e}^{-\lambda_j(t-s)-1} X_{jk}(s) \\ + \sum_{\ell=1}^\infty \sum_{m\in\Lambda_\ell} \int_s^t e^{-\lambda_j(t-r)} \left\langle B(r,X(r))\sqrt{q_\ell}h_{\ell m}, h_{jk}\right\rangle \mathrm{d}\beta_{\ell m}(r).$$

Thus, by virtue of the Itô's isometry we have

$$\mathbb{E}\lambda_j^{2\iota}(X_{jk}(s) - X_{jk}(t))^2 = [\exp(-\lambda_j(t-s)) - 1]^2 \lambda_j^{2\iota}\mathbb{E}(X_{jk}^2(s)) \\ + \int_s^t \exp(-2\lambda_j(t-r))\mathbb{E}\|B^*(r,X(r))\lambda_j^{-\iota}h_{jk}\|^2_{H_0}\,\mathrm{d}r.$$

Now, put

$$\mathtt{I} := \sum_{j=1}^\infty \sum_{k\in\Lambda_j} [\exp(-\lambda_j(t-s)) - 1]^2 \mathbb{E}(\lambda_j^{2\iota} X_{jk}^2(s)), \tag{4.4.13}$$

and

$$\mathtt{II} := \sum_{j=1}^\infty \sum_{k\in\Lambda_j} \int_s^t \exp(-2\lambda_j(t-r))\mathbb{E}\|B^*(r,X(r))\lambda_j^{-\iota}h_{jk}\|^2_{H_0}\,\mathrm{d}r.$$

We use (4.2.2) and the linear growth condition to obtain

$$\mathtt{II} \le \mathbb{E}\Big(\int_s^t \big\|B^*(r,X(r))\big\|^2_{\mathcal{L}_2(D(A^\iota),H_0)}\mathrm{d}r\Big) \le c(t-s). \tag{4.4.14}$$

This inequality together with (4.4.13) yields the mean-square continuity on $[0,1]$. Further, since $1-\exp(-x)\le x$ we have

$$\mathtt{I} \le c(t-s)\psi(s),$$

and thus together with (4.4.14) we have (4.4.12).

Finally, since

$$\int_0^1 \mathbb{E}(\lambda_j^{2\iota+1}X_{jk}^2(s))ds \le \lambda_j^{2\iota}\left\langle \xi, h_{jk}\right\rangle^2 + \int_0^1 \mathbb{E}\|B^*(r,X(r))\lambda_j^{-\iota}h_{jk}\|^2_{H_0}\,\mathrm{d}r,$$

we conclude $\psi\in L_1([0,1])$. □

Remark 5. In the previous result, the estimate (4.4.12) and the mean-square continuity on $[0,1]$ are two different results: if we assume only $\xi\in D(A^\iota)$ as opposed to $\xi \in D(A^{\iota+1/2})$, then (4.4.12) itself implies the mean-square continuity only on $(0,1]$.

We need the error bound for piecewise constant interpolation of $X^{J,L}$, essentially due to Müller-Gronbach and Ritter [76, Lemma 6.2].

Lemma 4.4.5 ([76])**.** *Suppose Assumption 4.1 is satisfied with $\iota = 0$. Let $J \in \mathbb{N}$ and $L \in \mathbb{N}$. Then, for any $\ell = 1, \dots, L$ we have*

$$\sum_{i=1}^{n_\ell} \int_{t_{i-1,\ell}}^{t_{i,\ell}} \mathbb{E}\|X^{J,L}(s) - X^{J,L}(t_{i-1,\ell})\|^2 \,\mathrm{d}s \preceq \frac{1}{n_\ell},$$

where $X^{J,L}$ is defined as in (4.4.4).

Proof. In view of (4.4.7) and Proposition 4.4.4, the same argument as the proof of [76, Lemma 6.2] is applicable. □

Now, we need the interpolated scheme defined as follows. An error estimate for this scheme, a slight generalisation of [76, Theorem 4.2], will be used as an intermediate step to obtain our final result. With $\mathcal{S}_j(\cdot,\cdot)$ as in (4.5.8), we introduce the interpolated scheme $\widetilde{X}^{J,L}$ defined by

$$\begin{aligned}\widetilde{X}^{J,L}(t) :=& \mathcal{S}_j(\tau_0, t)\left\langle \xi, h_{jk}\right\rangle \\ &+ \sum_{\ell=1}^{L} \sum_{m\in\Lambda_\ell} \sum_{\tau_1 \le t_{i,\ell} \le \tau_\eta} \mathcal{S}_j(t_{i-1,\ell}, t)\left\langle B(t_{i-1,\ell}, \widehat{X}^{J,L}(t_{i-1,\ell}))\sqrt{q_\ell} h_{\ell m}, h_{jk}\right\rangle \\ &\times (\beta_{\ell m}(t_{i,\ell}) - \beta_{\ell m}(t_{i-1,\ell})), \quad \text{for } t \in (\tau_{\eta-1}, \tau_\eta].\end{aligned}$$

Lemma 4.4.6 ([76])**.** *Let Assumption 4.2 hold. Then, we have*

$$\int_0^1 \mathbb{E}\|X^{J,L}(s) - \widetilde{X}^{J,L}(s)\|^2 \,\mathrm{d}s \preceq \sum_{\ell=1}^{L} \frac{\gamma_\ell}{n_\ell},$$

where $X^{J,L}$ is defined as in (4.4.4). Further, for any $\ell \in \{1, \dots, L\}$ we have

$$\sum_{i=1}^{n_\ell} \int_{t_{i-1,\ell}}^{t_{i,\ell}} \mathbb{E}\|\widetilde{X}^{J,L}(s) - \widetilde{X}^{J,L}(t_{i-1,\ell})\|^2 \,\mathrm{d}s \preceq \frac{1}{n_\ell} + \sum_{\ell'=1}^{L} \frac{\gamma_{\ell'}}{n_{\ell'}}.$$

Proof. The first assertion follows from [76, Proof of Theorem 4.2], only that we also use (4.4.3) and (4.4.7) to treat the dependence of B on t. The second assertion follows from [76, Lemma 6.6]. See also [54, Lemma 5.7]. □

Under Assumption 4.2, we have the following estimate on the temporal discretisation error.

Proposition 4.4.7. *Suppose Assumption 4.2 is satisfied. Then, we have*

$$\mathbb{E}\Big[\sum_{\eta=1}^{N}\|X^{J,L}(\tau_\eta)-\widehat{X}^{J,L}(\tau_\eta)\|^2(\tau_\eta-\tau_{\eta-1})\Big]\preceq\sum_{\ell=1}^{L}\frac{\gamma_\ell}{n_\ell},$$

where $X^{J,L}$, and $\widehat{X}^{J,L}$ are defined as in (4.4.4), and (4.3.5), respectively.

Proof. Let

$$U^{(0)}_{j,k}(\tau_\eta)=\sum_{\ell\in\{1,\dots,L\}\setminus\mathcal{K}_\eta}\sum_{m\in\Lambda_\ell}\mathfrak{R}_j(s_{\eta,\ell},\tau_\eta)\left\langle B(s_{\eta,\ell},\widehat{X}^{J,L}(s_{\eta,\ell}))\sqrt{q_\ell}h_{\ell m},h_{jk}\right\rangle \\ \times(\beta_{\ell m}(\tau_\eta)-\beta_{\ell m}(s_{\eta,\ell})),$$

and $t_{i^*,\ell}:=t_{i^*(\eta,\ell),\ell}:=\tau_\eta$, where $\mathfrak{R}_j(\cdot,\cdot)$ is defined as in (4.3.3). Then, we have

$$\widehat{X}^{J,L}_{j,k}(\tau_\eta)=\mathfrak{R}_j(\tau_0,\tau_\eta)\left\langle\xi,h_{jk}\right\rangle-U^{(0)}_{j,k}(\tau_\eta) \\ +\sum_{\ell=1}^{L}\sum_{m\in\Lambda_\ell}\int_0^{\tau_\eta}\sum_{i=1}^{n_\ell}\mathfrak{R}_j(t_{i-1,\ell},\tau_\eta)\Big\langle B(t_{i-1,\ell},\widehat{X}^{J,L}(t_{i-1,\ell}))\sqrt{q_\ell}h_{\ell m},h_{jk}\Big\rangle\mathbb{I}_{(t_{i-1,\ell},t_{i,\ell}]}(s)\mathrm{d}\beta_{\ell m}(s). \tag{4.4.15}$$

Now, for $\kappa=1,2,3$ we define

$$U^{(\kappa)}_{j,k}(\tau_\eta)=\sum_{\ell=1}^{L}\sum_{m\in\Lambda_\ell}\int_0^{\tau_\eta}\sum_{i=1}^{n_\ell}V^{(\kappa)}_{j,k,\ell,m,i}(s,\tau_\eta)\mathbb{I}_{(t_{i-1,\ell},t_{i,\ell}]}(s)\mathrm{d}\beta_{\ell m}(s),$$

with

$$V^{(1)}_{j,k,\ell,m,i}(s,\tau_\eta)=\mathrm{e}^{-\lambda_j(\tau_\eta-s)}\left\langle\big(B(s,X^{J,L}(s))-B(t_{i-1,\ell},X^{J,L}(t_{i-1,\ell}))\big)\sqrt{q_\ell}h_{\ell m},h_{jk}\right\rangle,$$
$$V^{(2)}_{j,k,\ell,m,i}(s,\tau_\eta)=\mathrm{e}^{-\lambda_j(\tau_\eta-s)} \\ \times\left\langle\big(B(t_{i-1,\ell},X^{J,L}(t_{i-1,\ell}))-B(t_{i-1,\ell},\widehat{X}^{J,L}(t_{i-1,\ell}))\big)\sqrt{q_\ell}h_{\ell m},h_{jk}\right\rangle,$$
$$V^{(3)}_{j,k,\ell,m,i}(s,\tau_\eta)=\big(\mathrm{e}^{-\lambda_j(\tau_\eta-s)}-\mathfrak{R}_j(t_{i-1,\ell},\tau_\eta)\big)\left\langle B(t_{i-1,\ell},\widehat{X}^{J,L}(t_{i-1,\ell}))\sqrt{q_\ell}h_{\ell m},h_{jk}\right\rangle.$$

Then, from (4.4.5) and (4.4.15) we see

$$X^{J,L}_{jk}(\tau_\eta)-\widehat{X}^{J,L}_{jk}(\tau_\eta)=(\mathrm{e}^{-\lambda_j\tau_\eta}-\mathfrak{R}_j(\tau_0,\tau_\eta))\left\langle\xi,h_{jk}\right\rangle \\ +U^{(0)}_{j,k}(\tau_\eta)+U^{(1)}_{j,k}(\tau_\eta)+U^{(2)}_{j,k}(\tau_\eta)+U^{(3)}_{j,k}(\tau_\eta).$$

Now, since $\frac{1}{n^*} \preceq \frac{1}{n^*}\sum_{\ell=1}^{L}\gamma_\ell \preceq \sum_{\ell=1}^{L}\frac{\gamma_\ell}{n_\ell}$, Lemma 4.4.3 implies

$$\sum_{j=1}^{J}\sum_{k\in\Lambda_j}\langle\xi,h_{jk}\rangle\sum_{\eta=1}^{N}\big(\exp(-\lambda_j\tau_\eta)-\mathfrak{R}_j(\tau_0,\tau_\eta)\big)^2(\tau_\eta-\tau_{\eta-1}) \preceq \sum_{\ell=1}^{L}\frac{\gamma_\ell}{n_\ell}.$$

Further, Assumption 4.2 together with (4.4.7) and Lemma 4.4.5 yields

$$\begin{aligned}
&\sum_{j=0}^{J}\sum_{k\in\Lambda_j}\mathbb{E}(U_{j,k}^{(1)}(\tau_\eta))^2\\
&\preceq \sum_{\ell=1}^{L}\gamma_\ell\sum_{i=1}^{n_\ell}\int_{t_{i-1,\ell}}^{t_{i,\ell}}\Big[(1+\mathbb{E}\|X^{J,L}(s)\|^2)(s-t_{i-1,\ell})+\mathbb{E}\|X^{J,L}(s)-X^{J,L}(t_{i-1,\ell})\|^2\Big]\,\mathrm{d}s\\
&\preceq \sum_{\ell=1}^{L}\gamma_\ell\sum_{i=1}^{n_\ell}\left[\frac{(t_{i,\ell}-t_{i,\ell-1})^2}{2}+\int_{t_{i-1,\ell}}^{t_{i,\ell}}\mathbb{E}\|X^{J,L}(s)-X^{J,L}(t_{i-1,\ell})\|^2\,\mathrm{d}s\right]\preceq\sum_{\ell=1}^{L}\gamma_\ell/n_\ell.
\end{aligned}$$

From Lemma 4.4.6, $f(s) := \mathbb{E}\|X^{J,L}(s)-\widetilde{X}^{J,L}(s)\|^2$ satisfies $\int_0^{\tau_\eta} f(s)\,\mathrm{d}s \preceq \sum_{\ell=1}^{L}\frac{\gamma_\ell}{n_\ell}$. Noting that $\widetilde{X}^{J,L}(\tau_\nu)=\widehat{X}^{J,L}(\tau_\nu)$ for all $\nu=1,\dots,N$, from Assumption 4.2 in view of Lemmata 4.4.5 and 4.4.6 we have

$$\begin{aligned}
&\sum_{j=0}^{J}\sum_{k\in\Lambda_j}\mathbb{E}(U_{j,k}^{(2)}(\tau_\eta))^2\\
&\preceq\sum_{\ell=1}^{L}\gamma_\ell\sum_{i=0}^{n_\ell}\int_{\tau_\eta\wedge t_{i-1,\ell}}^{\tau_\eta\wedge t_{i,\ell}}\Big(\mathbb{E}\|X^{J,L}(t_{i-1,\ell})-X^{J,L}(s)\|^2\\
&\qquad\qquad+f(s)+\mathbb{E}\|\widetilde{X}^{J}(s)-\widetilde{X}^{J}(t_{i-1,\ell})\|^2\Big)\,\mathrm{d}s\\
&\preceq\sum_{\ell=1}^{L}\frac{\gamma_\ell}{n_\ell}.
\end{aligned}$$

We consider the discrete temporal integral $\sum_{\eta=1}^{N}\sum_{j=1}^{J}\sum_{k\in\Lambda_j}\mathbb{E}(U_{j,k}^{(3)}(\tau_\eta))^2(\tau_\eta-\tau_{\eta-1})$ for the term $\mathbb{E}(U_{j,k}^{(3)}(\tau_\eta))^2$. To utilise estimates we have derived, we estimate $\exp(-\lambda_j(\tau_\eta-s))-\mathfrak{R}_j(t_{i-1,\ell},\tau_\eta)$ and $\Big\langle B(t_{i-1,\ell},\widehat{X}^{J,L}(t_{i-1,\ell}))h_{\ell m},h_{jk}\Big\rangle$ separately. An extra care is needed to do this, due to the dependence of the time steps $(t_{i,\ell})$ on ℓ.

Let $E_j(t_{i-1,\ell}, s, \tau_\eta) := \left|\mathrm{e}^{-\lambda_j(\tau_\eta - s)} - \mathrm{e}^{-\lambda_j(\tau_\eta - t_{i-1,\ell})}\right|^2 \mathbb{I}_{(t_{i-1,\ell}, t_{i,\ell}]}(s)$. If $n_\ell \le \lambda_j$ we have

$$\begin{aligned}&\sum_{i=1}^{n_\ell} \int_0^{\tau_\eta} \exp(-2\lambda_j(\tau_\eta - t_{i-1,\ell}))\mathbb{I}_{(t_{i-1,\ell}, t_{i,\ell}]}(s)\,\mathrm{d}s \\ &\le \sum_{i=1}^{n_\ell} \int_0^{\tau_\eta} \exp(-2\lambda_j(\tau_\eta - s))\mathbb{I}_{(t_{i-1,\ell}, t_{i,\ell}]}(s)\,\mathrm{d}s = \int_0^{\tau_\eta} \exp(-2n_\ell(\tau_\eta - s))\,\mathrm{d}s \le \frac{1}{2n_\ell},\end{aligned}$$

and thus

$$\sum_{i=1}^{n_\ell} \int_0^{\tau_\eta} E_j(t_{i-1,\ell}, s, \tau_\eta)\,\mathrm{d}s \le \frac{1}{n_\ell}.$$

On the other hand, if $\lambda_j \le n_\ell$, noting that $|\mathrm{e}^{-\lambda_j(\tau_\eta - s)} - \mathrm{e}^{-\lambda_j(\tau_\eta - t_{i-1,\ell})}| \le \mathrm{e}^{-\lambda_j(\tau_\eta - s)} \frac{\lambda_j}{n_\ell}$ for $s \in (t_{i-1,\ell}, t_{i,\ell}]$ with $s \le \tau_\eta$, we have

$$\sum_{i=1}^{n_\ell} \int_0^{\tau_\eta} E_j(t_{i-1,\ell}, s, \tau_\eta)\,\mathrm{d}s \le \frac{\lambda_j^2}{n_\ell^2} \frac{1}{2\lambda_j} \le \frac{1}{2n_\ell}.$$

Thus, there exists $j^* \in \{j' \mid \lambda_{j'} \le n^*\}$ such that for any $j \ge 1$ we have

$$\begin{aligned}&\sum_{i=1}^{n_\ell} \int_0^{\tau_\eta} E_j(t_{i-1,\ell}, s, \tau_\eta)\,\mathrm{d}s \\ &\le \sum_{i=1}^{n_\ell} \max\left\{\int_0^{\tau_\eta} E_{j^*}(t_{i-1,\ell}, s, \tau_\eta)\,\mathrm{d}s, \sup_{j \in \{j' \mid \lambda_{j'} > n^*\}} \int_0^{\tau_\eta} E_j(t_{i-1,\ell}, s, \tau_\eta)\,\mathrm{d}s\right\} \le \frac{1}{n_\ell}.\end{aligned} \tag{4.4.16}$$

Further, let $F_j(t_{i-1,\ell}, s, \tau_\eta) := \left|\mathrm{e}^{-\lambda_j(\tau_\eta - t_{i-1,\ell})} - \mathfrak{R}_j(t_{i-1,\ell}, \tau_\eta)\right|^2 \mathbb{I}_{(t_{i-1,\ell}, t_{i,\ell}]}(s)$. For each $\ell \in \{1, \ldots, L\}$ and $j \ge 1$, letting $\delta\tau_\eta := \tau_\eta - \tau_{\eta-1}$ we have

$$\begin{aligned}&\sum_{i=1}^{n_\ell} \sum_{\eta=1}^{N} \int_0^{\tau_\eta} F_j(t_{i-1,\ell}, s, \tau_\eta)\,\mathrm{d}s\delta\tau_\eta = \sum_{i=1}^{n_\ell} \sum_{\eta=\eta^*(i-1,\ell)+1}^{N} \int_0^{\tau_\eta} F_j(t_{i-1,\ell}, s, \tau_\eta)\,\mathrm{d}s\delta\tau_\eta \\ &\le \frac{1}{n_\ell} \sum_{i=1}^{n_\ell} \sup_{j \ge 1} \left(\sum_{\eta=\eta^*(i-1,\ell)+1}^{N} F_j(t_{i-1,\ell}, s, \tau_\eta)\delta\tau_\eta\right) \preceq \frac{1}{n_\ell},\end{aligned}$$

where in the last inequality Lemma 4.4.3 is used. Thus, we have

$$\sum_{i=1}^{n_\ell} \sup_{j \ge 1} \sum_{\eta=1}^{N} \int_0^{\tau_\eta} F_j(t_{i-1,\ell}, s, \tau_\eta)\,\mathrm{d}s\delta\tau_\eta \preceq \frac{1}{n_\ell}. \tag{4.4.17}$$

Hence, letting $a_{jk\ell m}(t_{i-1,\ell}) := \mathbb{E}\big[\big\langle B(t_{i-1,\ell}, \widehat{X}^{J,L}(t_{i-1,\ell}))\sqrt{q_\ell}h_{\ell m}, h_{jk}\big\rangle^2\big]$, for all $\ell \in \{1, \dots, L\}$ we have

$$\begin{aligned}
&\sum_{\eta=1}^{N}\sum_{i=1}^{n_\ell}\sum_{\{j|\lambda_j\le n^*\}}\sum_{k\in\Lambda_j}\sum_{m\in\Lambda_\ell} a_{jk\ell m}(t_{i-1,\ell}) \\
&\quad\times \int_0^{\tau_\eta}\big(\exp(-\lambda_j(\tau_\eta - s)) - \mathfrak{R}_j(t_{i-1,\ell},\tau_\eta)\big)^2 \mathbb{I}_{(t_{i-1,\ell},t_{i,\ell}]}(s)\,\mathrm{d}s\,\delta\tau_\eta \\
&\le 2\sum_{i=1}^{n_\ell}\bigg(\sum_{\eta=1}^{N}\int_0^{\tau_\eta} E_{j^*}(t_{i-1,\ell}, s, \tau_\eta)\,\mathrm{d}s\,\delta\tau_\eta + \sup_{j'\ge 1}\sum_{\eta=1}^{N}\int_0^{\tau_\eta} F_{j'}(t_{i-1,\ell}, s, \tau_\eta)\,\mathrm{d}s\,\delta\tau_\eta\bigg) \\
&\quad\times \sum_{\{j|\lambda_j\le n^*\}}\sum_{k\in\Lambda_j}\sum_{m\in\Lambda_\ell} a_{jk\ell m}(t_{i-1,\ell}).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\sum_{\eta=1}^{N}\sum_{i=1}^{n_\ell}\sum_{\{j|j\le J, n^*<\lambda_j\}}\sum_{k\in\Lambda_j}\sum_{m\in\Lambda_\ell} a_{jk\ell m}(t_{i-1,\ell}) \\
&\le 2\sum_{i=1}^{n_\ell}\bigg(\sum_{\eta=1}^{N}\sup_{j'\in\{j|\lambda_j>n^*\}}\int_0^{\tau_\eta} E_{j'}(t_{i-1,\ell}, s, \tau_\eta)\,\mathrm{d}s\,\delta\tau_\eta \\
&\qquad + \sup_{j'\ge 1}\sum_{\eta=1}^{N}\int_0^{\tau_\eta} F_{j'}(t_{i-1,\ell}, s, \tau_\eta)\,\mathrm{d}s\,\delta\tau_\eta\bigg)\sum_{\{j|j\le J, n^*<\lambda_j\}}\sum_{k\in\Lambda_j}\sum_{m\in\Lambda_\ell} a_{jk\ell m}(t_{i-1,\ell}).
\end{aligned}$$

Therefore, from Assumption 4.2 and Lemma 4.4.2, together with from (4.4.16) and (4.4.17) we have

$$\begin{aligned}
&\sum_{\eta=1}^{N}\sum_{j=1}^{J}\sum_{k\in\Lambda_j}\mathbb{E}\big(U_{j,k}^{(3)}(\tau_\eta)\big)^2(\tau_\eta - \tau_{\eta-1}) \\
&= \sum_{\eta=1}^{N}\sum_{\ell=1}^{L}\sum_{i=1}^{n_\ell}\sum_{j=1}^{J}\sum_{k\in\Lambda_j}\sum_{m\in\Lambda_\ell} a_{jk\ell m}(t_{i-1,\ell}) \\
&\quad\times \int_0^{\tau_\eta}\Big(\exp(-\lambda_j(\tau_\eta - s)) - \mathfrak{R}_j(t_{i-1,\ell},\tau_\eta)\Big)^2 \mathbb{I}_{(t_{i-1,\ell},t_{i,\ell}]}(s)\,\mathrm{d}s(\tau_\eta - \tau_{\eta-1}) \\
&\le 2\sum_{\ell=1}^{L}\frac{1}{n_\ell}\sum_{j=1}^{J}\sum_{k\in\Lambda_j}\sum_{m\in\Lambda_\ell} a_{jk\ell m}(t_{i-1,\ell}) \preceq \sum_{\ell=1}^{L}\frac{\gamma_\ell}{n_\ell}.
\end{aligned}$$

Finally, from Assumption 4.2 we have

$$\sum_{j=1}^{J}\sum_{k\in\Lambda_j}\mathbb{E}(U_{j,k}^{(0)}(\tau_\eta))^2 \le \sum_{j=1}^{J}\sum_{k\in\Lambda_j}\Big(\sum_{\ell=1}^{L}\sum_{m\in\Lambda_\ell} a_{jk\ell m}(s_{\eta,\ell})\Big)\frac{1}{n_\ell} \preceq \sum_{\ell=1}^{L}\frac{\gamma_\ell}{n_\ell},$$

which completes the proof. □

Now, we are ready to state the fully discretised error estimate.

Theorem 4.4.8. *Suppose Assumption 4.2 holds. Then, we have*

$$\mathbb{E}\Big[\sum_{\eta=1}^{N}\|X(\tau_\eta)-\widehat{X}^{J,L}(\tau_\eta)\|^2(\tau_\eta-\tau_{\eta-1})\Big] \preceq \frac{1}{\lambda_{J+1}}+\sum_{\ell=L+1}^{\infty}\gamma_\ell+\sum_{\ell=1}^{L}\frac{\gamma_\ell}{n_\ell},$$

where $\widehat{X}^{J,L}$ *is defined as in* (4.3.5).

Proof. The statement follows from Propositions 4.4.1 and 4.4.7. □

4.5 Discrete regularity estimate

We now turn our attention to a discrete analogue of maximal regularity of the scheme introduced in Section 4.3.

First, let $\mathscr{P}_J x := \sum_{j=1}^{J}\sum_{k\in\Lambda_j}\langle x, h_{jk}\rangle h_{jk}$ for $x\in H$. Further, by writing $\prod_\emptyset = I$ we let

$$R(\tau_{\eta_1},\tau_{\eta_2};A) := \prod_{\nu=\eta_1+1}^{\eta_2}\Big(I-\frac{1}{\tau_\nu-\tau_{\nu-1}}A\Big)^{-1}, \tag{4.5.1}$$

where the meaning of the product symbol is unambiguous due to the commutativity of resolvents.

For $j\in\{1,\dots,J\}$, $k\in\Lambda_j$, and $\eta\in\{1,\dots,N\}$, define

$$\begin{aligned}&\big[R^J\diamond B(\cdot,\widehat{X}^{J,L}(\cdot))\big]^L(\tau_\eta)\\ &:= \sum_{\ell=1}^{L}\sum_{m\in\Lambda_\ell}\sum_{\tau_1\le t_{i,\ell}\le\tau_\eta}\mathscr{P}_J R(t_{i-1,\ell},\tau_\eta;A)B(t_{i-1,\ell},\widehat{X}^{J,L}(t_{i-1,\ell}))\sqrt{q_\ell}h_{\ell m}\\ &\quad\times(\beta_{\ell m}(t_{i,\ell})-\beta_{\ell m}(t_{i-1,\ell})).\end{aligned} \tag{4.5.2}$$

For $\xi=0$ and $B(t_{i-1,\ell},\widehat{X}^{J,L}(t_{i-1,\ell}))=B(t_{i-1,\ell})$ the equation (4.5.2) is a discrete analogue of the stochastic convolution. The Fourier coefficients of (4.5.2) are given by

$$
\begin{aligned}
&\big[R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))\big]^L_{jk}(\tau_\eta) := \Big\langle \big[R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))\big]^L(\tau_\eta), h_{jk} \Big\rangle \\
&= \sum_{\ell=1}^{L} \sum_{m\in\Lambda_\ell} \sum_{\tau_1 \le t_{i,\ell} \le \tau_\eta} \sqrt{q_\ell}\,\mathfrak{R}_j(t_{i-1,\ell}, \tau_\eta) \Big\langle B(t_{i-1,\ell}, \widehat{X}^{J,L}(t_{i-1,\ell})) h_{\ell m}, h_{jk} \Big\rangle \\
&\quad \times (\beta_{\ell m}(t_{i,\ell}) - \beta_{\ell m}(t_{i-1,\ell})), \qquad \text{for } j \in \{1,\dots,J\},\ k \in \Lambda_j, \text{ and } \eta \in \{1,\dots,N\}.
\end{aligned}
$$

Then, noting that by the assumptions on A we have $((I-\lambda A)^{-1})^* = (I-\lambda A)^{-1}$ for $\lambda \in (0,\infty)$, the Fourier coefficients of the discretised solution are given by

$$
\widehat{X}^{J,L}_{jk}(\tau_\eta) = \mathfrak{R}_j(\tau_0, \tau_\eta) \langle \xi, h_{jk} \rangle + \big[R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))\big]^L_{jk}(\tau_\eta).
$$

Our first goal is to estimate the expectation of $\|\widehat{X}^{J,L}(\tau_\eta)\|^2_{D(A^r)}$. For any $r \ge 0$ we have

$$
\begin{aligned}
\mathbb{E}\|\widehat{X}^{J,L}(\tau_\eta)\|^2_{D(A^r)} &= \sum_{j=1}^{J} \sum_{k\in\Lambda_j} \lambda_j^{2r} \big|\mathfrak{R}_j(\tau_0,\tau_\eta) \langle \xi, h_{jk}\rangle \big|^2 \\
&\quad + \sum_{j=1}^{J} \sum_{k\in\Lambda_j} \lambda_j^{2r} \mathbb{E}\big|\big[R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))\big]^L_{jk}(\tau_\eta)\big|^2. \qquad (4.5.3)
\end{aligned}
$$

We see the second term in the right hand side of (4.3.4) as the stochastic integral of a representation of an elementary process.

Let $\mathscr{P}_{\ell m} x := \langle x, h_{\ell m}\rangle h_{\ell m}$ for $\ell \ge 1$, $m \in \Lambda_\ell$, and let $\iota \ge 0$ be the index from Assumption 4.1. For $\nu \in \{1,\dots,\eta\}$, we define an $\mathcal{L}_2(H_0, D(A^\iota))$-valued random variable $(\phi^{J,(\eta)}_{\ell m})_{\nu-1}$ by

$$
(\phi^{J,(\eta)}_{\ell m})_{\nu-1} := \begin{cases} \mathscr{P}_J R(s_{\nu,\ell}, \tau_\eta; A) B(s_{\nu,\ell}, \widehat{X}^{J,L}(s_{\nu,\ell})) \mathscr{P}_{\ell m} & \text{if } \ell \in \Xi_\nu \qquad (4.5.4a) \\ 0_{H_0 \to H} & \text{if } \ell \notin \Xi_\nu, \qquad (4.5.4b) \end{cases}
$$

where

$$
\Xi_\nu := \big\{ \ell \in \{1,\dots,L\} \;\big|\; \ell \in \mathcal{K}_\mu \text{ for some } \mu \in \{\nu,\dots,\eta\} \big\}. \qquad (4.5.5)
$$

We elaborate on the notation. First, note the following: for $\ell \notin \mathcal{K}_\nu$, $\nu \in \{0,\dots,\eta\}$ if the index $i' \in \{1,\dots,n_\ell\}$ is such that $s_{\nu,\ell} = t_{i'-1,\ell}$, then we have $\tau_\nu < t_{i',\ell}$. The separate treatment (4.5.4b) corresponds to the construction of the algorithm: suppose $\ell \in \{1,\dots,L\}$ and $i^* \in \{1,\dots,n_\ell\}$ satisfy $s_{\eta,\ell} = t_{i^*-1,\ell}$

and $\tau_\eta < t_{i^*,\ell}$, then the evaluation $\beta_{\ell m}(t_{i^*,\ell})$ of the Brownian motion $\beta_{\ell m}$ at $t_{i^*,\ell}$ is not used to obtain $\widehat{X}^{J,L}_{jk}(\tau_\eta)$; only up to $\beta_{\ell m}(t_{0,\ell}), \ldots, \beta_{\ell m}(t_{i^*-1,\ell})$ are used.

Let us define the elementary process $\Phi^{J,(\eta)}_{\ell m} \colon \Omega \times [0,\tau_\eta] \to \mathcal{L}_2(H_0, D(A^\iota))$ by

$$\Phi^{J,(\eta)}_{\ell m}(\omega,t) := \sum_{\nu=1}^{\eta} (\phi^{J,(\eta)}_{\ell m})_{\nu-1}(\omega)\, \mathbb{I}_{(\tau_{\nu-1},\tau_\nu]}(t).$$

Then, we have the following.

Lemma 4.5.1. *Let $\left[R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))\right]^L_{jk}(\cdot)$ be defined by (4.5.2) and let Assumption 4.1 hold with $\iota \geq 0$. Then, for $j = 1\ldots, J$, and $k \in \Lambda_j$ we have*

$$\left[R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))\right]^L_{jk}(\tau_\eta) = \left\langle \int_0^{\tau_\eta} \sum_{\ell=1}^{L} \sum_{m\in\Lambda_\ell} \Phi^{J,(\eta)}_{\ell m}(s)\mathrm{d}W(s), h_{jk} \right\rangle.$$

Proof. Fix $\eta \in \{1,\ldots,N\}$. Let $\mathcal{S}_\mu := \mathcal{K}_{\eta-\mu} \setminus (\bigcup_{\mu' \in \{0,\ldots,\mu-1\}} \mathcal{K}_{\eta-\mu'})$ for $\mu, \eta \in \{1,\ldots,N\}$ with $\mu \leq \eta$, and let $\mathcal{S}_0 := \mathcal{K}_\eta$. Then, we have

$$\begin{aligned}
&\left[R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))\right]^L_{jk}(\tau_\eta) \\
&= \sum_{\nu=1}^{\eta} \sum_{\ell\in\mathcal{K}_\eta} \sum_{m\in\Lambda_\ell} \sqrt{q_\ell}\,\mathfrak{R}_j(s_{\nu,\ell},\tau_\eta) \left\langle B(s_{\nu,\ell}, \widehat{X}^{J,L}(s_{\nu,\ell}))h_{\ell m}, h_{jk}\right\rangle (\beta_{\ell m}(\tau_\nu) - \beta_{\ell m}(\tau_{\nu-1})) \\
&+ \sum_{\nu=1}^{\eta-1} \sum_{\ell\in\mathcal{S}_1} \sum_{m\in\Lambda_\ell} \sqrt{q_\ell}\,\mathfrak{R}_j(s_{\nu,\ell},\tau_\eta) \left\langle B(s_{\nu,\ell}, \widehat{X}^{J,L}(s_{\nu,\ell}))h_{\ell m}, h_{jk}\right\rangle (\beta_{\ell m}(\tau_\nu) - \beta_{\ell m}(\tau_{\nu-1})) \\
&\vdots \\
&+ \sum_{\nu=1}^{\eta-\mu} \sum_{\ell\in\mathcal{S}_\mu} \sum_{m\in\Lambda_\ell} \sqrt{q_\ell}\,\mathfrak{R}_j(s_{\nu,\ell},\tau_\eta) \left\langle B(s_{\nu,\ell}, \widehat{X}^{J,L}(s_{\nu,\ell}))h_{\ell m}, h_{jk}\right\rangle (\beta_{\ell m}(\tau_\nu) - \beta_{\ell m}(\tau_{\nu-1})) \\
&\vdots \\
&+ \sum_{\ell\in\mathcal{S}_{\eta-1}} \sum_{m\in\Lambda_\ell} \sqrt{q_\ell}\,\mathfrak{R}_j(s_{1,\ell},\tau_\eta) \left\langle B(s_{1,\ell}, \widehat{X}^{J,L}(s_{1,\ell}))h_{\ell m}, h_{jk}\right\rangle (\beta_{\ell m}(\tau_1) - \beta_{\ell m}(\tau_0)).
\end{aligned}$$

Further, we can rewrite the above as

$$\begin{aligned}
&\left[R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))\right]^L_{jk}(\tau_\eta) \\
&= \sum_{\mu=0}^{\eta-1} \sum_{\nu=1}^{\eta-\mu} \sum_{\ell\in\mathcal{S}_\mu} \sum_{m\in\Lambda_\ell} \left\langle B(s_{\nu,\ell}, \widehat{X}^{J,L}(s_{\nu,\ell}))\mathscr{P}_{\ell m}(W(\tau_\nu) - W(\tau_{\nu-1})), R(s_{\nu,\ell},\tau_\eta; A)\mathscr{P}_J h_{jk}\right\rangle.
\end{aligned}$$

By the assumptions on A we have $((I-\lambda A)^{-1})^* = (I-\lambda A)^{-1}$ for $\lambda \in (0,\infty)$, and thus

$$\big[R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))\big]_{jk}^L(\tau_\eta) = \Big\langle \sum_{\nu=1}^{\eta} \Big(\sum_{\ell=1}^{L} \sum_{m\in\Lambda_\ell} (\phi_{\ell m}^{J,(\eta)})_{\nu-1} \Big) \big(W(\tau_\nu) - W(\tau_{\nu-1})\big), h_{jk} \Big\rangle .$$

By definition of the stochastic integral of elementary processes the statement follows. $\square$

Using the previous result, we obtain the following estimate.

Proposition 4.5.2. *Let Assumption 4.1 hold with some $\iota \geq 0$. Let $\eta \in \{1,\dots,N\}$. For $p \geq 1$, suppose that the process defined by (4.5.4a)–(4.5.4b) satisfies*

$$\mathbb{E}\Big[\sum_{\nu=1}^{\eta} \Big\| \sum_{\ell=1}^{L} \sum_{m\in\Lambda_\ell} (\phi_{\ell m}^{J,(\eta)})_{\nu-1} \Big\|^2_{\mathcal{L}_2(H_0, D(A^\iota))} (\tau_\nu - \tau_{\nu-1}) \Big] < \infty. \tag{4.5.6}$$

Then, we have

$$\begin{aligned} &\mathbb{E}\Big[\big\| \big[R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))\big]^L(\tau_\eta) \big\|^2_{D(A^\iota)} \Big] \\ &\qquad \leq \mathbb{E}\Big[\sum_{\nu=1}^{\eta} \Big\| \sum_{\ell=1}^{L} \sum_{m\in\Lambda_\ell} (\phi_{\ell m}^{J,(\eta)})_{\nu-1} \Big\|^2_{\mathcal{L}_2(H_0, D(A^\iota))} (\tau_\nu - \tau_{\nu-1}) \Big]. \end{aligned} \tag{4.5.7}$$

Proof. For any $\eta \in \{1,\dots,N\}$, from Lemma 4.5.1 we have

$$\begin{aligned} &\mathbb{E}\big[\big\| \big[R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))\big]^L(\tau_\eta) \big\|^2_{D(A^\iota)} \big] \\ &\qquad = \mathbb{E}\Big[\sum_{j=1}^{J} \sum_{k\in\Lambda_j} \lambda_j^{2r} \Big| \Big\langle \int_0^{\tau_\eta} \sum_{\ell=1}^{L} \sum_{m\in\Lambda_\ell} \Phi_{\ell m}^{J,(\eta)}(s) \mathrm{d}W(s), h_{jk} \Big\rangle \Big|^2 \Big]. \end{aligned}$$

It follows that

$$
\begin{aligned}
&\mathbb{E}\Big[\sum_{j=1}^{J}\sum_{k\in\Lambda_j}\Big|\Big\langle\int_0^{\tau_\eta}\sum_{\ell=1}^{L}\sum_{m\in\Lambda_\ell}\Phi_{\ell m}^{J,(\eta)}(s)\mathrm{d}W(s),\lambda_j^r h_{jk}\Big\rangle\Big|^2\Big] \\
&\quad\le \mathbb{E}\Big[\Big\|\int_0^{\tau_\eta}\sum_{\ell=1}^{L}\sum_{m\in\Lambda_\ell}\Phi_{\ell m}^{J,(\eta)}(s)\mathrm{d}W(s)\Big\|_{D(A^\iota)}^2\Big] \\
&\quad= \mathbb{E}\Big[\int_0^{\tau_\eta}\Big\|\sum_{\ell=1}^{L}\sum_{m\in\Lambda_\ell}\Phi_{\ell m}^{J,(\eta)}(s)\Big\|_{\mathcal{L}_2(H_0,D(A^\iota))}^2\mathrm{d}s\Big] \\
&\quad= \mathbb{E}\Big[\sum_{\nu=1}^{\eta}\Big\|\sum_{\ell=1}^{L}\sum_{m\in\Lambda_\ell}(\phi_{\ell m}^{J,(\eta)})_{\nu-1}\Big\|_{\mathcal{L}_2(H_0,D(A^\iota))}^2(\tau_\nu-\tau_{\nu-1})\Big] < \infty,
\end{aligned}
$$

where in the first equality Itô's isometry, and in the last inequality the condition (4.5.6) is used. Thus, the statement follows. □

We need the following estimate for the process $(\phi_{\ell m}^{J,(\eta)})_{\nu-1}$ as in (4.5.4a) and (4.5.4b) in terms of the Hilbert–Schmidt norm.

Lemma 4.5.3. *Suppose that Assumption 4.1 is satisfied. Fix an arbitrary integer $\eta\in\{1,\dots,N\}$. Then, for any $\nu\in\{0,\dots,\eta\}$, we have*

$$
\begin{aligned}
&\Big\|\sum_{\ell=1}^{L}\sum_{m\in\Lambda_\ell}(\phi_{\ell m}^{J,(\eta)})_{\nu-1}\Big\|_{\mathcal{L}_2(H_0,D(A^\iota))} = \Big\|\sum_{\ell\in\Xi_\nu}\sum_{m\in\Lambda_\ell}(\phi_{\ell m}^{J,(\eta)})_{\nu-1}\Big\|_{\mathcal{L}_2(H_0,D(A^\iota))} \\
&\le \Big(\sum_{\ell\in\Xi_\nu}\sum_{m\in\Lambda_\ell}\sum_{j=1}^{J}\sum_{k\in\Lambda_j}\lambda_j^{2\iota}|\mathfrak{R}_j(s_{\nu,\ell},\tau_\eta)|^2\Big|\Big\langle B(s_{\nu,\ell},\widehat{X}^{J,L}(s_{\nu,\ell}))\sqrt{q_\ell}h_{\ell m},h_{jk}\Big\rangle\Big|^2\Big)^{\frac{1}{2}},
\end{aligned}
$$

where Ξ_ν is defined by (4.5.5).

Proof. Note that if $\ell\notin\Xi_\nu$, then $\Big\|(\phi_{\ell m}^{J,(\eta)})_{\nu-1}\sqrt{q_\ell}h_{\ell m}\Big\|_{D(A^\iota)}^2 = 0$. Thus, noting that $\mathscr{P}_{\ell m}h_{\ell' m'} = 0$ unless $\ell=\ell'$ and $m=m'$, from the definition of $(\phi_{\ell m}^{J,(\eta)})_{\nu-1}$ we have

$$
\begin{aligned}
\Big\|\sum_{\ell=1}^{L}\sum_{m\in\Lambda_\ell}(\phi_{\ell m}^{J,(\eta)})_{\nu-1}\Big\|_{\mathcal{L}_2(H_0,D(A^\iota))}^2 &= \sum_{\ell'=1}^{L}\sum_{m'\in\Lambda_{\ell'}}\Big\|(\phi_{\ell' m'}^{J,(\eta)})_{\nu-1}\sqrt{q_{\ell'}}h_{\ell' m'}\Big\|_{D(A^\iota)}^2 \\
&= \sum_{\ell'\in\Xi_\nu}\sum_{m'\in\Lambda_{\ell'}}\Big\|(\phi_{\ell' m'}^{J,(\eta)})_{\nu-1}\sqrt{q_{\ell'}}h_{\ell' m'}\Big\|_{D(A^\iota)}^2.
\end{aligned}
$$

Fix $\ell \in \Xi_\nu$. For any $\eta \in \{1,\dots,N\}$ and $\nu \in \{1,\dots,\eta\}$ we have

$$\begin{aligned}
&\left\|(\phi^{J,(\eta)}_{\ell m})_{\nu-1}\sqrt{q_\ell}h_{\ell m}\right\|^2_{D(A^\iota)} \\
&= \sum_{j=1}^{\infty}\sum_{k\in\Lambda_j}\lambda_j^{2\iota}\left|\left\langle \mathscr{P}_J R(s_{\nu,\ell},\tau_\eta;A)B(s_{\nu,\ell},\widehat{X}^{J,L}(s_{\nu,\ell}))\sqrt{q_\ell}h_{\ell m},h_{jk}\right\rangle\right|^2 \\
&\le \sum_{j=1}^{J}\sum_{k\in\Lambda_j}\lambda_j^{2\iota}|\mathfrak{R}_j(s_{\nu,\ell},\tau_\eta)|^2\left|\left\langle B(s_{\nu,\ell},\widehat{X}^{J,L}(s_{\nu,\ell}))\sqrt{q_\ell}h_{\ell m},h_{jk}\right\rangle\right|^2.
\end{aligned}$$

Hence, the statement follows. $\square$

The following lemma is important to show the maximal regularity estimate of the same form as the continuous counterpart (4.2.3), studied in [22, Proposition 6.18] and [21].

Lemma 4.5.4. *For any $j \ge 1$, $\ell \ge 1$, and $i \in \{1,\dots,n_\ell\}$, we have*

$$\sum_{t_{i,\ell}\le\tau_\eta\le\tau_N}|\mathfrak{R}_j(t_{i-1,\ell},\tau_\eta)|^2(\tau_\eta-\tau_{\eta-1}) \le \frac{2}{\lambda_j},$$

where $\mathfrak{R}_j(\cdot,\cdot)$ is defined by (4.3.3).

Proof. For $\tau_{\eta_0} \in \{\tau_0,\dots,\tau_N\}$ define a continuous interpolation $\mathcal{S}_j(\tau_{\eta_0},\cdot)\colon[0,1]\to\mathbb{R}$ of $\mathfrak{R}_j(\tau_{\eta_0},\tau_\eta)$ by

$$\mathcal{S}_j(\tau_{\eta_0},t) := \prod_{\nu=\eta_0+1}^{N}\frac{1}{1+\lambda_j(t\wedge\tau_\nu - t\wedge\tau_{\nu-1})}, \qquad t\in[0,1]. \tag{4.5.8}$$

Then, for $t \in (\tau_{\eta-1},\tau_\eta]$, $\eta \in \{1,\dots,N\}$, we have

$$\mathcal{S}_j(\tau_{\eta_0},t)\mathbb{I}_{\{(\tau_{\eta-1},\tau_\eta]\}}(t) = \mathcal{S}_j(\tau_{\eta_0},t) \ge \mathfrak{R}_j(\tau_{\eta_0},\tau_\eta)\mathbb{I}_{\{(\tau_{\eta-1},\tau_\eta]\}}(t).$$

Further, for $\ell = 1,\dots,L$ and $i = 1,\dots,n_\ell$, let $\tau_{\eta^*} := \tau_{\eta^*(i,\ell)} := t_{i,\ell}$. Then, we have

$$\begin{aligned}
\sum_{t_{i,\ell}\le\tau_\eta\le\tau_N}|\mathfrak{R}_j(t_{i-1,\ell},\tau_\eta)|^2(\tau_\eta-\tau_{\eta-1}) &\le \int_{\tau_{\eta^*-1}}^1\sum_{\eta=\eta^*(i,\ell)}^{N}|\mathcal{S}_j(t_{i-1,\ell},s)|^2\mathbb{I}_{\{(\tau_{\eta-1},\tau_\eta]\}}(s)\,\mathrm{d}s \\
&= \int_{\tau_{\eta^*-1}}^1|\mathcal{S}_j(t_{i-1,\ell},s)|^2\,\mathrm{d}s \le \int_{t_{i-1,\ell}}^1|\mathcal{S}_j(t_{i-1,\ell},s)|^2\,\mathrm{d}s.
\end{aligned}$$

For $t \in [t_{\kappa-1,\ell}, t_{\kappa-1,\ell}]$ with $\kappa \geq i$, the elementary inequality $\frac{1}{1+(b-a)}\frac{1}{1+(c-b)} \leq \frac{1}{1+(c-a)}$ $(0 \leq a \leq b \leq c)$ implies $\mathcal{S}_j(t_{i-1,\ell}, t) \leq \frac{1}{(1+\lambda_j/n_\ell)^{\kappa-i}} \cdot \frac{1}{1+\lambda_j(t-t_{\kappa-1,\ell})}$, and therefore

$$
\begin{aligned}
&\int_{t_{i-1,\ell}}^{1} |\mathcal{S}_j(t_{i-1,\ell}, s)|^2 \,\mathrm{d}s = \sum_{\kappa=i}^{n_\ell} \int_{t_{\kappa-1,\ell}}^{t_{\kappa,\ell}} |\mathcal{S}_j(t_{i-1,\ell}, s)|^2 \,\mathrm{d}s \\
&\leq \sum_{\kappa=i}^{n_\ell} \frac{1}{(1+\frac{\lambda_j}{n_\ell})^{2\kappa-2i}} \int_{t_{\kappa-1,\ell}}^{t_{\kappa,\ell}} \frac{1}{(1+\lambda_j(s-t_{\kappa-1,\ell}))^2} \,\mathrm{d}s \\
&= \sum_{\kappa=i}^{n_\ell} \frac{1}{(1+\frac{\lambda_j}{n_\ell})^{2\kappa-2i}} \frac{1}{\lambda_j + 1/(t_{\kappa,\ell}-t_{\kappa-1,\ell})} \leq \frac{1}{\lambda_j+n_\ell} \sum_{\kappa=i}^{n_\ell} \frac{1}{(1+\frac{\lambda_j}{n_\ell})^{2\kappa-2i}}.
\end{aligned}
$$

If $\frac{\lambda_j}{n_\ell} \geq 1$, then $\frac{1}{\lambda_j+n_\ell}\sum_{\kappa=i}^{n_\ell} \frac{1}{(1+\lambda_j/n_\ell)^{2\kappa-2i}} \leq \frac{2}{\lambda_j}$, and otherwise $(1+\frac{\lambda_j}{n_\ell})^2 \leq 4$ and thus

$$
\frac{1}{\lambda_j+n_\ell} \sum_{\kappa=i}^{n_\ell} \frac{1}{(1+\frac{\lambda_j}{n_\ell})^{2\kappa-2i}} \leq \frac{1}{n_\ell} \frac{1}{1-1/(1+\frac{\lambda_j}{n_\ell})^2} \leq \frac{4}{2\lambda_j + \lambda_j^2/n_\ell} \leq \frac{2}{\lambda_j}.
$$

Hence, we have $\sum_{t_{i,\ell} \leq \tau_\eta \leq \tau_N} |\mathfrak{R}_j(t_{i-1,\ell}, \tau_\eta)|^2 (\tau_\eta - \tau_{\eta-1}) \leq \frac{2}{\lambda_j}$, as claimed. □

We are ready to state our main result in this chapter.

Theorem 4.5.5. *Suppose Assumption 4.1 is satisfied with some $\iota \geq 0$. Then, we have*

$$
\begin{aligned}
&\sum_{\eta=1}^{N} \mathbb{E}\Big[\Big\| \big[R^J \diamond B(\cdot, \widehat{X}^{J,L}(\cdot))\big]^L(\tau_\eta) \Big\|^2_{D(A^{\iota+1/2})} \Big] (\tau_\eta - \tau_{\eta-1}) \\
&\leq 2\mathbb{E}\Big[\sum_{\ell=1}^{\infty} \sum_{m \in \Lambda_\ell} \sum_{i=1}^{n_\ell} \Big\| \mathscr{P}_J B(t_{i-1,\ell}, \widehat{X}^{J,L}(t_{i-1,\ell})) \mathscr{P}_L \sqrt{q_\ell} h_{\ell m} \Big\|^2_{D(A^\iota)} (t_{i,\ell} - t_{i-1,\ell}) \Big].
\end{aligned}
$$

In particular, $\overline{X}^{J,L}$ defined as in (4.3.1) satisfies

$$
\begin{aligned}
&\sum_{i=1}^{N} \mathbb{E}\Big[\Big\| \big[R^J \diamond B(\cdot, \overline{X}^{J,L}(\cdot))\big]^L(t_i) \Big\|^2_{D(A^{\iota+1/2})} \Big] (t_i - t_{i-1}) \\
&\qquad \leq 2 \sum_{i=1}^{N} \mathbb{E}\Big[\Big\| \mathscr{P}_J B(t_{i-1}, \overline{X}^{J,L}(t_{i-1})) \mathscr{P}_L \Big\|^2_{\mathcal{L}_2(H_0, D(A^\iota))} \Big] (t_i - t_{i-1}).
\end{aligned}
$$

Proof. We first show that for $\eta = 1, \dots, N$, we have

$$\mathbb{E}\Big[\sum_{\nu=1}^{\eta}\Big\|\sum_{\ell=1}^{L}\sum_{m\in\Lambda_\ell}(\phi_{\ell m}^{J,(\eta)})_{\nu-1}\Big\|^2_{\mathcal{L}_2(H_0,D(A^{\iota+1/2}))}(\tau_\nu-\tau_{\nu-1})\Big] < \infty. \tag{4.5.9}$$

In view of Lemma 4.5.3, we have

$$\begin{aligned}
&\sum_{\eta=1}^{N}\mathbb{E}\Big[\sum_{\nu=1}^{\eta}\Big\|\sum_{\ell=1}^{L}\sum_{m\in\Lambda_\ell}(\phi_{\ell m}^{J,(\eta)})_{\nu-1}\Big\|^2_{\mathcal{L}_2(H_0,D(A^{\iota+1/2}))}(\tau_\nu-\tau_{\nu-1})\Big](\tau_\eta-\tau_{\eta-1})\\
&\le \sum_{\eta=1}^{N}\mathbb{E}\Big[\sum_{j=1}^{J}\sum_{k\in\Lambda_j}\sum_{\nu=1}^{\eta}\sum_{\ell\in\Xi_\nu}\sum_{m\in\Lambda_\ell}\lambda_j^{2\iota+1}|\mathfrak{R}_j(s_{\nu,\ell},\tau_\eta)|^2\\
&\qquad\times\Big|\Big\langle B(s_{\nu,\ell},\widehat{X}^{J,L}(s_{\nu,\ell}))\sqrt{q_\ell}h_{\ell m},h_{jk}\Big\rangle\Big|^2(\tau_\nu-\tau_{\nu-1})\Big](\tau_\eta-\tau_{\eta-1})\\
&= \mathbb{E}\Big[\sum_{j=1}^{J}\sum_{k\in\Lambda_j}\sum_{\ell=1}^{L}\sum_{m\in\Lambda_\ell}\sum_{\eta=1}^{N}\sum_{\tau_1\le t_{i,\ell}\le\tau_\eta}\lambda_j^{2\iota+1}|\mathfrak{R}_j(t_{i-1,\ell},\tau_\eta)|^2\\
&\qquad\times\Big|\Big\langle B(t_{i-1,\ell},\widehat{X}^{J,L}(t_{i-1,\ell}))\sqrt{q_\ell}h_{\ell m},h_{jk}\Big\rangle\Big|^2(t_{i,\ell}-t_{i-1,\ell})(\tau_\eta-\tau_{\eta-1})\Big].
\end{aligned}\tag{4.5.10}$$

Since

$$\bigcup_{\eta=1}^{N}\bigcup_{\tau_1\le t_{i,\ell}\le\tau_\eta}\{\tau_\eta,t_{i,\ell}\}=\bigcup_{i=1}^{n_\ell}\bigcup_{t_{i,\ell}\le\tau_\eta\le\tau_N}\{\tau_\eta,t_{i,\ell}\},$$

the right hand side of (4.5.10) can be rewritten as

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{j=1}^{J}\sum_{k\in\Lambda_j}\sum_{\ell=1}^{L}\sum_{m\in\Lambda_\ell}\sum_{i=1}^{n_\ell}\sum_{t_{i,\ell}\le\tau_\eta\le\tau_N}\lambda_j^{2\iota+1}|\mathfrak{R}_j(t_{i-1,\ell},\tau_\eta)|^2\\
&\qquad\times\Big|\Big\langle B(t_{i-1,\ell},\widehat{X}^{J,L}(t_{i-1,\ell}))\sqrt{q_\ell}h_{\ell m},h_{jk}\Big\rangle\Big|^2(t_{i,\ell}-t_{i-1,\ell})(\tau_\eta-\tau_{\eta-1})\Big]\\
&= \mathbb{E}\Big[\sum_{j=1}^{J}\sum_{k\in\Lambda_j}\sum_{\ell=1}^{L}\sum_{m\in\Lambda_\ell}\sum_{i=1}^{n_\ell}\lambda_j^{2\iota+1}\Big|\Big\langle B(t_{i-1,\ell},\widehat{X}^{J,L}(t_{i-1,\ell}))\sqrt{q_\ell}h_{\ell m},h_{jk}\Big\rangle\Big|^2\\
&\qquad\times(t_{i,\ell}-t_{i-1,\ell})\sum_{t_{i,\ell}\le\tau_\eta\le\tau_N}|\mathfrak{R}_j(t_{i-1,\ell},\tau_\eta)|^2(\tau_\eta-\tau_{\eta-1})\Big].
\end{aligned}\tag{4.5.11}$$

From Lemma 4.5.4, (4.5.10) and (4.5.11), due to Assumption 4.1 we have (4.5.9).

From (4.5.9), we note that Proposition 4.5.2 implies

$$\sum_{\eta=1}^{N}\mathbb{E}\Big[\Big\|\big[R^J\diamond B(\cdot,\widehat{X}^{J,L}(\cdot))\big]^L(\tau_\eta)\Big\|^2_{D(A^{\iota+1/2})}\Big](\tau_\eta-\tau_{\eta-1})$$
$$\leq\sum_{\eta=1}^{N}\mathbb{E}\Big[\sum_{\nu=1}^{\eta}\Big\|\sum_{\ell=1}^{L}\sum_{m\in\Lambda_\ell}(\phi^{J,(\eta)}_{\ell m})_{\nu-1}\Big\|^2_{\mathcal{L}_2(H_0,D(A^{\iota+1/2}))}(\tau_\nu-\tau_{\nu-1})\Big](\tau_\eta-\tau_{\eta-1}).$$

Therefore, again from Lemma 4.5.4 together with (4.5.10) and (4.5.11) we obtain

$$\sum_{\eta=1}^{N}\mathbb{E}\Big[\Big\|\big[R^J\diamond B(\cdot,\widehat{X}^{J,L}(\cdot))\big]^L(\tau_\eta)\Big\|^2_{D(A^{\iota+1/2})}\Big](\tau_\eta-\tau_{\eta-1})$$
$$\leq 2\mathbb{E}\Big[\sum_{j=1}^{J}\sum_{k\in\Lambda_j}\sum_{\ell=1}^{L}\sum_{m\in\Lambda_\ell}\sum_{i=1}^{n_\ell}\lambda_j^{2\iota}\Big|\Big\langle B(t_{i-1,\ell},\widehat{X}^{J,L}(t_{i-1,\ell}))\sqrt{q_\ell}h_{\ell m},h_{jk}\Big\rangle\Big|^2$$
$$\times(t_{i,\ell}-t_{i-1,\ell})\Big]$$
$$=2\mathbb{E}\Big[\sum_{\ell=1}^{\infty}\sum_{m\in\Lambda_\ell}\sum_{i=1}^{n_\ell}\Big\|\mathscr{P}_JB(t_{i-1,\ell},\widehat{X}^{J,L}(t_{i-1,\ell}))\mathscr{P}_L\sqrt{q_\ell}h_{\ell m}\Big\|^2_{D(A^\iota)}(t_{i,\ell}-t_{i-1,\ell})\Big].$$

When $n_\ell=N$ for all $\ell\in\{1,\dots,L\}$, we have $t_{i,\ell}-t_{i-1,\ell}=t_i-t_{i-1}$ $(i=1,\dots,N)$. Thus, repeating the same argument as above completes the proof. □

As a consequence of the previous result, given a suitable regularity of the initial condition, the approximate solution has the spatial regularity "one-half smoother"—the same as the continuous counterpart [22]—than the range of the operator $B(t,x)$.

Corollary 4.5.6. *Let Assumption 4.1 hold and $\xi\in D(A^\iota)$. Then, we have*

$$\Big(\sum_{\eta=1}^{N}\mathbb{E}\big[\|\widehat{X}^{J,L}(\tau_\eta)\|^2_{D(A^{\iota+1/2})}\big](\tau_\eta-\tau_{\eta-1})\Big)^{\frac12}\preceq\|\mathscr{P}_J\xi\|_{D(A^\iota)}$$
$$+\Big(\mathbb{E}\Big[\sum_{\ell=1}^{\infty}\sum_{m\in\Lambda_\ell}\sum_{i=1}^{n_\ell}\Big\|\mathscr{P}_JB(t_{i-1,\ell},\widehat{X}^{J,L}(t_{i-1,\ell}))\mathscr{P}_L\sqrt{q_\ell}h_{\ell m}\Big\|^2_{D(A^\iota)}(t_{i,\ell}-t_{i-1,\ell})\Big]\Big)^{\frac12}.$$

In particular, $\overline{X}^{J,L}$ defined as in (4.3.1) satisfies

$$
\begin{aligned}
&\Big(\sum_{i=1}^{n}\mathbb{E}\big[\|\overline{X}^{J,L}(t_i)\|^2_{D(A^{\iota+1/2})}\big](t_i-t_{i-1})\Big)^{\frac{1}{2}}\\
&\preceq \big\|\mathscr{P}_J\xi\big\|_{D(A^{\iota})}+\Big(\sum_{i=1}^{n}\mathbb{E}\Big[\Big\|\mathscr{P}_J B(t_{i-1},\overline{X}^{J,L}(t_{i-1}))\mathscr{P}_L\Big\|^2_{\mathcal{L}_2(H_0,D(A^{\iota}))}\Big](t_i-t_{i-1})\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Proof. From Lemma 4.5.4 we have

$$
\begin{aligned}
&\sum_{\eta=1}^{N}\mathbb{E}\big[\,\big\|R(\tau_0,\tau_\eta;A)\mathscr{P}_J\xi\big\|^2_{D(A^{\iota+1/2})}\big](\tau_\eta-\tau_{\eta-1})\\
&=\mathbb{E}\Big[\sum_{j=1}^{J}\sum_{k\in\Lambda_j}\lambda_j^{2\iota+1}\big|\left\langle \xi,h_{jk}\right\rangle\big|^2\sum_{\eta=1}^{N}\big|\mathfrak{R}_j(\tau_0,\tau_\eta)\big|^2(\tau_\eta-\tau_{\eta-1})\Big]\\
&\le 2\mathbb{E}\Big[\sum_{j=1}^{J}\sum_{k\in\Lambda_j}\lambda_j^{2\iota}\big|\left\langle \xi,h_{jk}\right\rangle\big|^2\Big].
\end{aligned}
$$

Then, from (4.5.3) and Theorem 4.5.5 the first statement follows. Letting $n_\ell = n$ for $\ell = 1,\dots,L$ establishes the second statement. □

Remark 6. The results in this section can be generalised to non-uniform grids on each level. Let $0 < t_{1,\ell} < \cdots < t_{n_\ell,\ell} = 1$ be the temporal grids that satisfies the following: Letting $\delta_\ell^{\max} := \max_{i=1,\dots,n_\ell}\{t_{i,\ell}-t_{i-1,\ell}\}$, $\delta_\ell^{\min} := \min_{i=1,\dots,n_\ell}\{t_{i,\ell}-t_{i-1,\ell}\}$, we have a constant $c_{\mathrm{disc}} \ge 1$ such that $\delta_\ell^{\max}/\delta_\ell^{\min} \le c_{\mathrm{disc}}$ holds. Then, the statement of Lemma 4.5.4 can be replaced by

$$
\sum_{t_{i,\ell}\le\tau_\eta\le\tau_N}|\mathfrak{R}_j(t_{i-1,\ell},\tau_\eta)|^2(\tau_\eta-\tau_{\eta-1})\le\frac{2c_{\mathrm{disc}}}{\lambda_j},
$$

and that of Theorem 4.5.5 by

$$
\begin{aligned}
&\sum_{\eta=1}^{N}\mathbb{E}\Big[\Big\|\big[R^J\diamond B(\cdot,\widehat{X}^{J,L}(\cdot))\big]^L(\tau_\eta)\Big\|^2_{D(A^{\iota+1/2})}\Big](\tau_\eta-\tau_{\eta-1})\\
&\le 2c_{\mathrm{disc}}\mathbb{E}\Big[\sum_{\ell=1}^{\infty}\sum_{m\in\Lambda_\ell}\sum_{i=1}^{n_\ell}\Big\|\mathscr{P}_J B(t_{i-1,\ell},\widehat{X}^{J,L}(t_{i-1,\ell}))\mathscr{P}_L\sqrt{q_\ell}h_{\ell m}\Big\|^2_{D(A^{\iota})}(t_{i,\ell}-t_{i-1,\ell})\Big].
\end{aligned}
$$

4.6 Conclusion

In this chapter, we considered an implicit Euler–Maruyama scheme for a class of stochastic partial differential equations. We showed a discrete error estimate

(Theorem 4.4.8), and further showed that the scheme satisfies a discrete analogue of the maximal L^2-regularity (Corollary 4.5.6).

Appendix A

Weak derivatives on spherical shells

The representation (2.2.3) of T have a structure inherited from the covariance function (2.2.2); T is expanded by the product of radial and angular complete orthonormal systems. To show the Sobolev smoothness of T we wish to utilise this structure. For this reason, we consider the following sequence of smooth functions $\{(\mathcal{JP})_n f\}$ that approximates $f \in L^2(\mathbb{S}_\varepsilon^{\mathrm{o}})$. These functions plays a role equivalent to the Friedrichs mollifier ([30, 31, 33]), only they are more suited to the shell setting. We define the inner product $\langle f, g\rangle_{\frac{1}{r^2}} = \int_{\mathbb{S}_\varepsilon^{\mathrm{o}}} fg\frac{1}{r^2}\,\mathrm{d}x = \iint_{\mathbb{S}_\varepsilon^{\mathrm{o}}} fg\,\mathrm{d}r\mathrm{d}S$. Then, the induced norm $\|\cdot\|_{\frac{1}{r^2}}$ satisfies

$$r_{\mathrm{in}}\,\|f\|_{\frac{1}{r^2}} \leq \|f\|_{L^2(\mathbb{S}_\varepsilon^{\mathrm{o}})} \leq r_{\mathrm{out}}\,\|f\|_{\frac{1}{r^2}}\,. \tag{A.1}$$

Often, it is easier to work with $\|\cdot\|_{\frac{1}{r^2}}$. Let j_n be a standard mollifier on $\mathbb{R}$ with support $[\frac{1}{n}, \frac{1}{n}]$, and P_n is the Legendre polynomial of degree n. For $f \in L^2(\mathbb{S}_\varepsilon^{\mathrm{o}})$, define $(\mathcal{JP})_n f$, $\mathcal{J}_n f$, and $\mathcal{P}_n f$ as follows: For $r\sigma \in \mathbb{S}_\varepsilon^{\mathrm{o}}$, let

$$[(\mathcal{JP})_n f](r\sigma) := \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_{S^2} f(r_0,\sigma_0) j_n(r-r_0) P_n(\sigma\cdot\sigma_0)\,\mathrm{d}r_0\mathrm{d}S_0,$$

$$[\mathcal{J}_n f](r\sigma) := \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} f(r_0,\sigma) j_n(r-r_0)\,\mathrm{d}r_0,$$

and

$$[\mathcal{P}_n f](r\sigma) := \int_{S^2} f(r,\sigma_0) P_n(\sigma\cdot\sigma_0)\,\mathrm{d}S_0.$$

We remark that $(\mathcal{JP})_n f$, and $(\mathcal{J})_n f$ can be extended to $\mathbb{S}_{\varepsilon *}^{\mathrm{o}} := \{x \in \mathbb{R} \mid r_{\mathrm{in}}^* < \|x\|_2 < r_{\mathrm{out}}^*\} \supset \mathbb{S}_\varepsilon$ for arbitrary $r_{\mathrm{in}}^* \in (0, r_{\mathrm{in}})$ and $r_{\mathrm{out}}^* \in (r_{\mathrm{out}}, \infty)$ as follows: for any representative $f \in L^2(\mathbb{S}_\varepsilon^{\mathrm{o}})$, for each σ as a function on $(r_{\mathrm{in}}, r_{\mathrm{out}})$ we extend $[(\mathcal{JP})_n f](\cdot,\sigma)$, respectively $[\mathcal{J}_n f](\cdot,\sigma)$ to $C^\infty((r_{\mathrm{in}}^*, r_{\mathrm{out}}^*))$-functions by extending $f(\cdot,\sigma)$ on $\mathbb{R}$ by 0, and restricting the resulting $[(\mathcal{JP})_n f](\cdot,\sigma)$, respectively $[\mathcal{J}_n f](\cdot,\sigma)$ to $(r_{\mathrm{in}}^*, r_{\mathrm{out}}^*)$.

First, we show $(\mathcal{JP})_n f$ is a smooth function on $\mathbb{S}_\varepsilon$.

Lemma A.1. *Let $f \in L^2(\mathbb{S}^{\mathrm{o}}_\varepsilon)$. Then, for each $n \in \mathbb{N} \cup \{0\}$ we have*

$$\sup_{x \in \mathbb{S}^{\mathrm{o}}_{\varepsilon *}} \left| \Delta^\iota [(\mathcal{JP})_n f](x) \right| \le C_{n,\iota}, \tag{A.2}$$

for every $\iota \in \mathbb{N} \cup \{0\}$. Further,

$$[(\mathcal{JP})_n f](r_{\mathrm{in}}, \cdot), [(\mathcal{JP})_n f](r_{\mathrm{out}}, \cdot) \in C^\infty(S^2). \tag{A.3}$$

Proof. Take $\alpha, \beta \in \mathbb{N} \cup \{0\}$ arbitrarily. In the following, we often indicate the variable σ that Δ_{S^2} acts on by writing $\Delta_{S^2,\sigma}$. From the smoothness of j_n, and $P_n(\sigma \cdot \sigma_0)$, together with $f := f^{\mathrm{ext}}|_{\mathbb{S}^{\mathrm{o}}_{\varepsilon *}} \in L^1(\mathbb{S}^{\mathrm{o}}_{\varepsilon *})$, for any $x \in \mathbb{S}^{\mathrm{o}}_{\varepsilon *}$ we have

$$\begin{aligned}
&\left| \frac{\partial^\alpha}{\partial r^\alpha} \Delta^\beta_{S^2,\sigma} [(\mathcal{JP})_n f](x) \right| = \left| \Delta^\beta_{S^2,\sigma} \frac{\partial^\alpha}{\partial r^\alpha} [(\mathcal{JP})_n f](x) \right| \\
&\le \left(\sup_{s \in \mathbb{R}} \left| \frac{\partial^\alpha}{\partial r^\alpha} j_n(s) \right| \right) \left(\sup_{t \in S^2} (n^2+n)^\beta |P_n(t)| \right) \iint_{\mathbb{S}_\varepsilon} |f(r_0, \sigma_0)| \, \mathrm{d}r_0 \mathrm{d}S_0 < \infty.
\end{aligned}$$

Thus, the first claim follows noting that Δ^ι can be written as a finite sum of differential operators

$$\Delta^\iota = \sum_{0 \le \alpha, 2\beta \le 2\iota} \frac{c_{\alpha\beta}}{p_{\alpha\beta}(r)} \Delta^\beta_{S^2} \frac{\partial^\alpha}{\partial r^\alpha}, \tag{A.4}$$

where $c_{\alpha\beta} \in \mathbb{R}$ are constants, and $p_{\alpha\beta}(r) (\neq 0)$ are non-zero polynomials of $r \in [r_{\mathrm{in}}, r_{\mathrm{out}}]$ such that $\sup_{r \in [r_{\mathrm{in}}, r_{\mathrm{out}}]} \left| \frac{1}{p_{\alpha\beta}(r)} \right| < \infty$.

For (A.3), from $f \in L^2(\mathbb{S}^{\mathrm{o}}_\varepsilon)$ we observe $\mathcal{J}_n f(r', \cdot) \in L^2(S^2)$ for arbitrary $r' \in [r_{\mathrm{in}}, r_{\mathrm{out}}]$. Then, (A.3) readily follows. □

Corollary A.2. *Let $f \in L^2(\mathbb{S}^{\mathrm{o}}_\varepsilon)$. Then, changing the value on measure zero sets if necessary we have*

$$(\mathcal{JP})_n f \in C^\infty(\mathbb{S}_\varepsilon).$$

Proof. Let $\Gamma(x-y) := -\frac{1}{4\pi} \frac{1}{\|x-y\|_2}$ $(x, y \in \mathbb{R}^3)$, and

$$w_1(x) := \int_{\mathbb{S}^{\mathrm{o}}_{\varepsilon *}} \Gamma(x-y) \, \Delta[(\mathcal{JP})_n f](y) \, \mathrm{d}y, \quad (x \in \mathbb{R}^3),$$

where $\mathbb{S}^{\mathrm{o}}_{\varepsilon *} = \{x \in \mathbb{R} \mid r^*_{\mathrm{in}} < \|x\|_2 < r^*_{\mathrm{out}}\} \supset \mathbb{S}_\varepsilon$. From (A.2), we have $[(\mathcal{JP})_n f] \in L^2(\mathbb{S}^{\mathrm{o}}_{\varepsilon *})$. Thus, from [50, Theorem 12.1.1] we have $w_1|_{\mathbb{S}^{\mathrm{o}}_{\varepsilon *}} \in H^2(\mathbb{S}^{\mathrm{o}}_{\varepsilon *})$ and a.e. in

$\mathbb{S}_{\varepsilon*}^{o}$ we have

$$\Delta w_1 = \Delta[(\mathcal{JP})_n f] \leq C_{n,1},$$

and thus, for a.e. in $\mathbb{S}_{\varepsilon*}^{o}$ we have $\Delta\big(w_1 - (\mathcal{JP})_n f\big) = 0$. Since weakly harmonic functions are almost everywhere harmonic, there exists a harmonic function v_1 such that $v_1 = w_1 - (\mathcal{JP})_n f$ a.e. in $\mathbb{S}_{\varepsilon*}^{o}$.

Since $\Delta[(\mathcal{JP})_n f] \leq C_{n,1}$, from [33, Lemma 4.1] for $i \in \{1,2,3\}$ we have

$$\frac{\partial}{\partial x_i} w_1(x) = \int_{\mathbb{S}_{\varepsilon*}^{o}} \frac{\partial}{\partial x_i}\Gamma(x-y)\,[\Delta(\mathcal{JP})_n f](y)\,\mathrm{d}y \quad (x \in \mathbb{R}^3), \tag{A.5}$$

and $w_1|_{\mathbb{S}_{\varepsilon*}^{o}} \in C^1(\mathbb{S}_{\varepsilon*}^{o})$. From $(\mathcal{JP})_n f = v_1 + w_1$ almost everywhere in $\mathbb{S}_{\varepsilon*}^{o}$, we can conclude that there exists $((\mathcal{JP})_n f)^{\sim} \in C^1(\mathbb{S}_{\varepsilon*}^{o})$ such that $((\mathcal{JP})_n f)^{\sim} = (\mathcal{JP})_n f$ a.e. in $\mathbb{S}_{\varepsilon*}^{o}$. Note that in particular $((\mathcal{JP})_n f)^{\sim} \in C^1(\mathbb{S}_{\varepsilon})$.

Noting $\Delta^{\iota}[(\mathcal{JP})_n f] \leq C_{n,\iota}$, repeating the argument yields

$$((\mathcal{JP})_n f)^{\sim}|_{\mathbb{S}_{\varepsilon}} \in C^{\infty}(\mathbb{S}_{\varepsilon}).$$

□

We have established $(\mathcal{JP})_n f \in C^{\infty}(\mathbb{S}_{\varepsilon}^{o})$. To use $(\mathcal{JP})_n f$ as a mollifier, we next show that the sequence $\{(\mathcal{JP})_n f\}$, and derivatives of this sequence, approximates f, and respectively derivatives of f in $L^2(\mathbb{S}_{\varepsilon}^{o})$.

We first show two lemmata that claim $\mathcal{J}_n f$ and $\mathcal{P}_n f$ approximate f in the radial and angular directions.

Lemma A.3. *For $f \in L^2(\mathbb{S}_{\varepsilon}^{o})$, we have*

$$\mathcal{J}_n f \to f \quad (n \to \infty) \quad \textit{in } L^2(\mathbb{S}_{\varepsilon}^{o}).$$

Proof. For S^2-almost every σ we have $f(\cdot,\sigma) \in L^2((r_{\text{in}}, r_{\text{out}}))$. Thus, for any $\varepsilon > 0$ there exists $n_0 = n_0(\sigma)$ such that for all $n \geq n_0$ we have

$$g_n(\sigma) := \int_{r_{\text{in}}}^{r_{\text{out}}} \Big|\mathcal{J}_n f(\cdot,\sigma) - f(\cdot,\sigma)\Big|^2 \,\mathrm{d}r < \varepsilon, \qquad \text{for } S^2\text{-almost every } \sigma.$$

From the property of the mollifier j_n, for S^2-almost every $\sigma \in S^2$, for each $n \in \mathbb{N}$, we have $g_n(\sigma) \leq 4\,\|f(\cdot,\sigma)\|^2_{L^2((r_{\text{in}},r_{\text{out}}))}$. Together with $\int_{S^2} 4\,\|f(\cdot,\sigma)\|^2_{L^2((r_{\text{in}},r_{\text{out}}))}\,\mathrm{d}S = 4\,\|f(\cdot,\sigma)\|^2_{L^2(\mathbb{S}_{\varepsilon}^{o})} < \infty$, the dominated convergence theorem and (A.1) yields the desired result. □

It is easy to show that $\mathcal{P}_n f$ approximates f up to derivatives.

Lemma A.4. *Let $\beta \in \mathbb{N}$ and $\beta' \in \{0, 1, \ldots, \beta\}$. Suppose $f \in L^2(\mathbb{S}_\varepsilon^{\circ})$ satisfies $\int_{\mathbb{S}_\varepsilon^{\circ}} \left|\Delta_{S^2}^{\beta'} f(x)\right|^2 \mathrm{d}x < \infty$, where the Laplace–Beltrami operator $\Delta_{S^2}^{\beta'}$ is defined in the weak sense. Then,*

$$\Delta_{S^2}^{\beta'} \mathcal{P}_n f \to \Delta_{S^2}^{\beta'} f \quad (n \to \infty) \quad \text{in } L^2(\mathbb{S}_\varepsilon^{\circ}),$$

for each $\beta' \in \{0, 1, \ldots, \beta\}$.

Proof. Fix $\beta' \in \{0, 1, \ldots, \beta\}$. From the assumption, for $(r_{\text{in}}, r_{\text{out}})$-almost every r we have $f(r, \cdot)$, $\Delta_{S^2}^{\beta'} f(r, \cdot) \in L^2(S^2)$. From the standard differentiation under the integral sign and the integration by parts on the sphere, and the definition of the weak Laplace–Beltrami operator, for $(r_{\text{in}}, r_{\text{out}})$-almost every r we note $[\Delta_{S^2}^{\beta'}(\mathcal{P}_n f)](r, \sigma) = [\mathcal{P}_n(\Delta_{S^2}^{\beta'} f)](r, \sigma)$.

Since $\Delta_{S^2}^{\beta'} f(\cdot, \sigma) \in L^2(S^2)$ for $(r_{\text{in}}, r_{\text{out}})$-almost every r, for any $\varepsilon > 0$ there exists $n_1 = n_1(r)$ such that for all $n \geq n_1$ we have

$$\int_{S^2} \left|\mathcal{P}_n \left(\Delta_{S^2}^{\beta'} f\right)(r, \cdot) - \Delta_{S^2}^{\beta'} f(r, \cdot)\right|^2 \mathrm{d}S < \varepsilon,$$

for $(r_{\text{in}}, r_{\text{out}})$-almost every r. The result follows by the same argument as Lemma A.3, noting $\|[\mathcal{P}_n(\Delta_{S^2}^{\beta'} f)](r, \cdot)\|_{L^2(S^2)} \leq \left\|\Delta_{S^2}^{\beta'} f(r, \cdot)\right\|_{L^2(S^2)}$. □

Now, we are ready to state $(\mathcal{J}\mathcal{P})_n f$ approximates $f \in L^2(\mathbb{S}_\varepsilon^{\circ})$.

Proposition A.5. *For $f \in L^2(\mathbb{S}_\varepsilon^{\circ})$, we have*

$$\|(\mathcal{J}\mathcal{P})_n f - f\|_{L^2(\mathbb{S}_\varepsilon^{\circ})} \to 0 \quad (n \to \infty). \tag{A.6}$$

Proof. Trivially, we have $(\mathcal{J}\mathcal{P})_n f - f = \mathcal{J}_n f - f + \mathcal{J}_n(\mathcal{P}_n f - f)$, and thus noting that $\|\mathcal{J}_n f\|_{L^2(\mathbb{S}_\varepsilon^{\circ})} \leq \|f\|_{L^2(\mathbb{S}_\varepsilon^{\circ})}$ we have

$$\|(\mathcal{J}\mathcal{P})_n f - f\|_{L^2(\mathbb{S}_\varepsilon^{\circ})} \leq \|\mathcal{J}_n f - f\|_{L^2(\mathbb{S}_\varepsilon^{\circ})} + \|\mathcal{P}_n f - f\|_{L^2(\mathbb{S}_\varepsilon^{\circ})}, \tag{A.7}$$

and thus from Lemma A.3 and A.4, $\|(\mathcal{J}\mathcal{P})_n f - f\|_{L^2(\mathbb{S}_\varepsilon^{\circ})} \to 0$ $(n \to \infty)$. □

Further, we wish to show $(\mathcal{J}\mathcal{P})_n f$ approximates derivatives of f in $L^2(\mathbb{S}_\varepsilon^{\circ})$ as well. We start from the definition of the derivative in the sense of $L^2(\mathbb{S}_\varepsilon^{\circ})$-limit, and discuss its properties.

Definition 3. For $f \in L^2(\mathbb{S}^{\mathrm{o}}_\varepsilon)$, if there exist $v \in L^2(\mathbb{S}^{\mathrm{o}}_\varepsilon)$ and a sequence $\{f_n\} \subset C^{\alpha+2\beta}(\mathbb{S}^{\mathrm{o}}_\varepsilon)$ such that $f_n \to f$ $(n \to \infty)$ in $L^2(\mathbb{S}^{\mathrm{o}}_\varepsilon)$ and

$$\left\| \frac{\partial^\alpha}{\partial r^\alpha} \Delta^\beta_{S^2} f_n - v \right\|_{L^2(\mathbb{S}^{\mathrm{o}}_\varepsilon)} = \left\| \Delta^\beta_{S^2} \frac{\partial^\alpha}{\partial r^\alpha} f_n - v \right\|_{L^2(\mathbb{S}^{\mathrm{o}}_\varepsilon)} \to 0 \quad (n \to \infty), \tag{A.8}$$

we write

$$\frac{\partial^\alpha}{\partial r^\alpha} \Delta^\beta_{S^2} f = \Delta^\beta_{S^2} \frac{\partial^\alpha}{\partial r^\alpha} f := v \in L^2(\mathbb{S}^{\mathrm{o}}_\varepsilon) \quad (\alpha, \beta \in \mathbb{N}), \tag{A.9}$$

and call them $(L^2(\mathbb{S}^{\mathrm{o}}_\varepsilon)$-)*strong derivative* $\Delta^\beta_{S^2} \frac{\partial^\alpha}{\partial r^\alpha} f$ of f. ▮

The following lemma states that the strong derivative $\Delta^\beta_{S^2} \frac{\partial^\alpha}{\partial r^\alpha} f \in L^2(\mathbb{S}^{\mathrm{o}}_\varepsilon)$ is the weak derivative in the sense of Definition 4, and further that by virtue of the fundamental lemma of the variational calculus, the definition of $\Delta^\beta_{S^2} \frac{\partial^\alpha}{\partial r^\alpha} f$ is independent of the choice of $\{f_n\}$. More details on the weak derivatives will be discussed later in Proposition A.10.

Lemma A.6. *Let* $\frac{\partial^\alpha}{\partial r^\alpha} \Delta^\beta_{S^2} f = \Delta^\beta_{S^2} \frac{\partial^\alpha}{\partial r^\alpha} f \in L^2(\mathbb{S}^{\mathrm{o}}_\varepsilon)$ *be the strong derivative. Then, we have*

$$\begin{aligned}\int_{S^2}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} & \frac{\partial^\alpha}{\partial r^\alpha} \Delta^\beta_{S^2} f(r\sigma)\phi(r\sigma) r^2 \, \mathrm{d}r\mathrm{d}S \\ &= (-1)^\alpha \int_{S^2}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} f(r\sigma) \Delta^\beta_{S^2} \frac{\partial^\alpha}{\partial r^\alpha} (\phi(r\sigma) r^2) \, \mathrm{d}r\mathrm{d}S,\end{aligned}$$

for any $\phi \in C^\infty_c(\mathbb{S}^{\mathrm{o}}_\varepsilon)$.

Proof. Let $\{f_n\} \subset C^{\alpha+2\beta}(\mathbb{S}^{\mathrm{o}}_\varepsilon)$ be an approximating sequence as in Definition 3. We have

$$\int_{\mathbb{S}^{\mathrm{o}}_\varepsilon} \frac{\partial^\alpha}{\partial r^\alpha} \Delta^\beta_{S^2} f_n(x)\phi(x) \, \mathrm{d}x = (-1)^\alpha \iint_{\mathbb{S}^{\mathrm{o}}_\varepsilon} f_n(r\sigma) \Delta^\beta_{S^2} \frac{\partial^\alpha}{\partial r^\alpha} (\phi(r\sigma) r^2) \, \mathrm{d}r\mathrm{d}S. \tag{A.10}$$

From the assumption, we have $\frac{\partial^\alpha}{\partial r^\alpha} \Delta^\beta_{S^2} f_n \to \frac{\partial^\alpha}{\partial r^\alpha} \Delta^\beta_{S^2} f$ and thus the continuity of the functional $\langle \cdot, \phi \rangle_{L^2(\mathbb{S}^{\mathrm{o}}_\varepsilon)}$ implies the limit $\int_{\mathbb{S}^{\mathrm{o}}_\varepsilon} \frac{\partial^\alpha}{\partial r^\alpha} \Delta^\beta_{S^2} f(x)\phi(x) \, \mathrm{d}x$ on the left hand side. Similarly, from the assumption $f_n \to f$ in $L^2(\mathbb{S}^{\mathrm{o}}_\varepsilon)$. Thus, noting $\Delta^\beta_{S^2} \frac{\partial^\alpha}{\partial r^\alpha} (\phi(r\sigma) r^2) \in L^2(\mathbb{S}^{\mathrm{o}}_\varepsilon)$, the continuity of $\langle \cdot, g \rangle_{\frac{1}{r^2}}$ $(g \in L^2(\mathbb{S}^{\mathrm{o}}_\varepsilon))$ on $L^2(\mathbb{S}^{\mathrm{o}}_\varepsilon)$ implies that the right hand side of (A.10) has the limit $(-1)^\alpha \iint_{\mathbb{S}^{\mathrm{o}}_\varepsilon} f(r\sigma) \Delta^\beta_{S^2} \frac{\partial^\alpha}{\partial r^\alpha} (\phi(r\sigma) r^2) \, \mathrm{d}r\mathrm{d}S$. Hence, the result follows. ☐

The following proposition shows that $(\mathcal{JP})_n f$ has the same kind of local approximating property as a mollifier usually does.

If a set $U \subset \mathbb{S}_\varepsilon^{\mathrm{o}}$ has a compact closure in $\mathbb{S}_\varepsilon^{\mathrm{o}}$, we write $U \Subset \mathbb{S}_\varepsilon^{\mathrm{o}}$.

Proposition A.7. *Suppose we have the strong derivative $\frac{\partial^\alpha}{\partial r^\alpha}\Delta_{S^2}^\beta f = \Delta_{S^2}^\beta \frac{\partial^\alpha}{\partial r^\alpha} f \in L^2(\mathbb{S}_\varepsilon^{\mathrm{o}})$. Let $U \Subset \mathbb{S}_\varepsilon^{\mathrm{o}}$ be a subdomain. Then, we have*

$$\left\| \frac{\partial^\alpha}{\partial r^\alpha}\Delta_{S^2}^\beta (\mathcal{JP})_n f - \frac{\partial^\alpha}{\partial r^\alpha}\Delta_{S^2}^\beta f \right\|_{L^2(U)} \to 0 \quad (n \to \infty). \tag{A.11}$$

Proof. From the smoothness of $j_n(\cdot - r_0)$, and $P_n(\cdot \cdot \sigma_0)$, in the classical sense we have $\frac{\partial^\alpha}{\partial r^\alpha}\Delta_{S^2}^\beta[(\mathcal{JP})_n f](r\sigma) = \Delta_{S^2}^\beta \frac{\partial^\alpha}{\partial r^\alpha}[(\mathcal{JP})_n f](r\sigma)$. We note

$$\begin{aligned}
&\Delta_{S^2}^\beta \frac{\partial^\alpha}{\partial r^\alpha}[(\mathcal{JP})_n f](r,\sigma) \\
&\quad = (-1)^\alpha \int_{S^2}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} f(r_0,\sigma_0)\left(\frac{\partial^\alpha}{\partial r_0^\alpha} j_n(r-r_0)\right)\Delta_{S^2,\sigma_0}^\beta P_n(\sigma\cdot\sigma_0)\,\mathrm{d}r_0 \mathrm{d}S_0.
\end{aligned}$$

For $x = r\sigma \in U$, define $g_{r\sigma}^{(n)}(r_0\sigma_0) := j_n(r-r_0)P_n(\sigma\cdot\sigma_0)$. From $U \Subset \mathbb{S}_\varepsilon^{\mathrm{o}}$, we can take $N \in \mathbb{N}$ such that $\mathrm{dist}(U, \partial\mathbb{S}_\varepsilon^{\mathrm{o}}) > \frac{1}{N}$. Noting $\mathrm{supp}(j_n) \subset [-\frac{1}{n}, \frac{1}{n}]$, we have $g_{r\sigma}^{(n)} \in C_c^\infty(\mathbb{S}_\varepsilon^{\mathrm{o}})$ for $n \geq N$. Thus, from Lemma A.6 we see that $\Delta_{S^2}^\beta \frac{\partial^\alpha}{\partial r^\alpha}[(\mathcal{JP})_n f](r,\sigma)$ can be rewritten as

$$\begin{aligned}
&(-1)^\alpha \int_{S^2}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} f(r_0,\sigma_0)\frac{\partial^\alpha}{\partial r_0^\alpha}\Delta_{S^2,\sigma_0}^\beta g_{r\sigma}^{(n)}(r_0\sigma_0)\,\mathrm{d}r_0\mathrm{d}S_0 \\
&\qquad\qquad = \left[(\mathcal{JP})_n \Delta_{S^2}^\beta \frac{\partial^\alpha}{\partial r^\alpha} f\right](r,\sigma), \quad \text{for } x = r\sigma \in U.
\end{aligned}$$

Thus, letting

$$F(x) := \begin{cases} \frac{\partial^\alpha}{\partial r^\alpha}\Delta_{S^2}^\beta f(x) & x \in U \\ 0 & x \in \mathbb{S}_\varepsilon^{\mathrm{o}} \setminus U, \end{cases} \tag{A.12}$$

we have $\left\| \frac{\partial^\alpha}{\partial r^\alpha}\Delta_{S^2}^\beta (\mathcal{JP})_n f - \frac{\partial^\alpha}{\partial r^\alpha}\Delta_{S^2}^\beta f \right\|_{L^2(U)} = \|(\mathcal{JP})_n F - F\|_{L^2(\mathbb{S}_\varepsilon^{\mathrm{o}})}$, and thus from Proposition A.5 the result follows. □

In Definition 3, we defined the derivative as the L^2-limit of suitably smooth functions. We see that the class of functions we can consider under this definition is reasonably large—they are also weak derivatives in the following sense. We define the weak derivative as follows. Then, we show in Proposition A.10 that they are also the strong derivative under a reasonable condition.

Definition 4. Let $\alpha, \beta \in \mathbb{N} \cup \{0\}$. Suppose $f, v \in L^1(\mathbb{S}_\varepsilon^{\circ})$ satisfies

$$\iint_{\mathbb{S}_\varepsilon^{\circ}} f(r\sigma) \Delta_{S^2}^{\beta} \frac{\partial^\alpha}{\partial r^\alpha} \left(\phi(r\sigma) r^2\right) \mathrm{d}r \mathrm{d}S = (-1)^\alpha \int_{\mathbb{S}_\varepsilon^{\circ}} v(r\sigma)\phi(r\sigma) r^2 \,\mathrm{d}r\mathrm{d}S, \tag{A.13}$$

for any $\phi \in C_c^\infty(\mathbb{S}_\varepsilon^{\circ})$. Then, we call $\widetilde{\Delta_{S^2}^{\beta} \frac{\partial^\alpha}{\partial r^\alpha}} f = \widetilde{\frac{\partial^\alpha}{\partial r^\alpha} \Delta_{S^2}^{\beta}} f := v$ the *weak* derivative of f. ▮

The uniqueness follows from the fundamental lemma of variational calculus. Similarly, from Lemma A.6, if we have the strong derivative $\Delta_{S^2}^{\beta} \frac{\partial^\alpha}{\partial r^\alpha} f \in L^2(\mathbb{S}_\varepsilon^{\circ})$, then $\Delta_{S^2}^{\beta} \frac{\partial^\alpha}{\partial r^\alpha} f = \widetilde{\Delta_{S^2}^{\beta} \frac{\partial^\alpha}{\partial r^\alpha}} f$ almost everywhere in $\mathbb{S}_\varepsilon^{\circ}$.

Proposition A.10 shows we have a partially converse result, namely, if

$$\widetilde{\Delta_{S^2}^{\beta} \frac{\partial^{\alpha'}}{\partial r^{\alpha'}}} f \in L^2(\mathbb{S}_\varepsilon^{\circ}) \quad \text{for all} \quad \alpha' \in \{0, \dots, \alpha\},$$

then $\widetilde{\Delta_{S^2}^{\beta} \frac{\partial^\alpha}{\partial r^\alpha}} f = \Delta_{S^2}^{\beta} \frac{\partial^\alpha}{\partial r^\alpha} f$. We show two lemmata to prove Proposition A.10.

Lemma A.8. *Suppose* $\widetilde{\frac{\partial^\alpha}{\partial r^\alpha} \Delta_{S^2}^{\beta}} f = \widetilde{\Delta_{S^2}^{\beta} \frac{\partial^\alpha}{\partial r^\alpha}} f \in L^2(\mathbb{S}_\varepsilon^{\circ})$. *For any subdomain* $U \Subset \mathbb{S}_\varepsilon^{\circ}$, *we have*

$$\left\| \frac{\partial^\alpha}{\partial r^\alpha} \Delta_{S^2}^{\beta} (\mathcal{JP})_n f - \widetilde{\frac{\partial^\alpha}{\partial r^\alpha} \Delta_{S^2}^{\beta}} f \right\|_{L^2(U)} \to 0 \quad (n \to \infty). \tag{A.14}$$

Proof. Following the proof of Proposition A.7, with the classical derivatives we have

$$\begin{aligned} &\Delta_{S^2}^{\beta} \frac{\partial^\alpha}{\partial r^\alpha} [(\mathcal{JP})_n f](r, \sigma) \\ &\quad = (-1)^\alpha \int_{S^2} \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} f(r_0, \sigma_0) \left(\frac{\partial^\alpha}{\partial r_0^\alpha} j_n(r - r_0) \right) \Delta_{S^2, \sigma_0}^{\beta} P_n(\sigma \cdot \sigma_0) \,\mathrm{d}r_0 \mathrm{d}S_0. \end{aligned}$$

Similarly to the proof of Proposition A.7, letting $g_{r\sigma}^{(n)}(r_0\sigma_0) := j_n(r - r_0) P_n(\sigma \cdot \sigma_0)$, $(x = r\sigma \in U)$ for sufficiently large n we have $g_{r\sigma}^{(n)} \in C_c^\infty(\mathbb{S}_\varepsilon^{\circ})$. From the definition of $\widetilde{\frac{\partial^\alpha}{\partial r^\alpha} \Delta_{S^2}^{\beta}} f$ for any $x = r\sigma \in U$ we have

$$\Delta_{S^2}^{\beta} \frac{\partial^\alpha}{\partial r^\alpha} [(\mathcal{JP})_n f](r, \sigma) = \left[(\mathcal{JP})_n \widetilde{\frac{\partial^\alpha}{\partial r^\alpha} \Delta_{S^2}^{\beta}} f \right] (r, \sigma).$$

The rest follows from the proof of Proposition A.7. $\square$

Lemma A.9. *Let $\alpha, \beta \in \mathbb{N} \cup \{0\}$. Suppose $\widetilde{\Delta_{S^2}^{\beta} \frac{\partial^{\alpha'}}{\partial r^{\alpha'}}} f \in L^2(\mathbb{S}_\varepsilon^{\mathrm{o}})$ for all $\alpha' \in \{0, \ldots, \alpha\}$. Suppose further that $\zeta \in C^\infty(\mathbb{S}_\varepsilon^{\mathrm{o}})$ is constant in the angular direction, and $\sup_{x \in \mathbb{S}_\varepsilon^{\mathrm{o}}} \left| \frac{\partial^{\alpha'}}{\partial r^{\alpha'}} \zeta(x) \right| < \infty$ for each $\alpha' \in \{0, \ldots, \alpha\}$. Then, we have $\widetilde{\Delta_{S^2}^{\beta} \frac{\partial^{\alpha}}{\partial r^{\alpha}}}(f\zeta) \in L^2(\mathbb{S}_\varepsilon^{\mathrm{o}})$ and*

$$\widetilde{\Delta_{S^2}^{\beta} \frac{\partial^{\alpha}}{\partial r^{\alpha}}}(f\zeta) = \sum_{\alpha'=0}^{\alpha} \binom{\alpha}{\alpha'} \Big(\widetilde{\Delta_{S^2}^{\beta} \frac{\partial^{\alpha'}}{\partial r^{\alpha'}}} f \Big) \Big(\frac{\partial^{\alpha - \alpha'}}{\partial r^{\alpha - \alpha'}} \zeta \Big). \tag{A.15}$$

Proof. We show the claim inductively. From $\sup_{x \in \mathbb{S}_\varepsilon^{\mathrm{o}}} \left| \frac{\partial^{\alpha'}}{\partial r^{\alpha'}} \zeta(x) \right| < \infty$ for $0 \leq \alpha' \leq \alpha$, it suffices to show (A.15) holds.

For $\alpha = \beta = 0$, clearly (A.15) holds. Assume that (A.15) holds for $(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}$ such that $\alpha \leq \alpha^* - 1$, and $\beta \leq \beta^* - 1$. We show the following equalities:

$$\widetilde{\Delta_{S^2}^{\beta+1} \frac{\partial^{\alpha}}{\partial r^{\alpha}}}(f\zeta) = \sum_{\alpha'=0}^{\alpha} \binom{\alpha}{\alpha'} \Big(\widetilde{\Delta_{S^2}^{\beta+1} \frac{\partial^{\alpha'}}{\partial r^{\alpha'}}} f \Big) \Big(\frac{\partial^{\alpha - \alpha'}}{\partial r^{\alpha - \alpha'}} \zeta \Big), \tag{A.16}$$

and

$$\widetilde{\Delta_{S^2}^{\beta} \frac{\partial^{\alpha+1}}{\partial r^{\alpha+1}}}(f\zeta) = \sum_{\alpha'=0}^{\alpha+1} \binom{\alpha+1}{\alpha'} \Big(\widetilde{\Delta_{S^2}^{\beta} \frac{\partial^{\alpha'}}{\partial r^{\alpha'}}} f \Big) \Big(\frac{\partial^{\alpha+1 - \alpha'}}{\partial r^{\alpha+1 - \alpha'}} \zeta \Big). \tag{A.17}$$

From the inductive hypothesis, we have $\widetilde{\Delta_{S^2}^{\beta} \frac{\partial^{\alpha}}{\partial r^{\alpha}}}(f\zeta) \in L^2(\mathbb{S}_\varepsilon^{\mathrm{o}})$ and

$$\begin{aligned} &\iint_{\mathbb{S}_\varepsilon^{\mathrm{o}}} f\zeta \Delta_{S^2}^{\beta+1} \frac{\partial^{\alpha}}{\partial r^{\alpha}} (\phi r^2) \, \mathrm{d}r \mathrm{d}S \\ &\quad = (-1)^\alpha \int_{\mathbb{S}_\varepsilon^{\mathrm{o}}} \sum_{\alpha'=0}^{\alpha} \binom{\alpha}{\alpha'} \Big(\widetilde{\Delta_{S^2}^{\beta} \frac{\partial^{\alpha'}}{\partial r^{\alpha'}}} f \Big) \Big(\frac{\partial^{\alpha - \alpha'}}{\partial r^{\alpha - \alpha'}} \zeta \Big) \Delta_{S^2} \phi \, \mathrm{d}x. \end{aligned} \tag{A.18}$$

For each $r \in [r_{\mathrm{in}}, r_{\mathrm{out}}]$ we see the functions ϕ, ζ, and $\frac{\partial^{\alpha-\alpha'}}{\partial r^{\alpha-\alpha'}} \zeta$ as functions on S^2. We extend these functions on S^2 to $\mathbb{R}^3$, in the standard manner to define the spherical Laplacians. Since ζ is constant in the angular direction, from the product rule for the Laplacian we have

$$\Delta_{S^2} \Big(\phi \frac{\partial^{\alpha-\alpha'}}{\partial r^{\alpha-\alpha'}} \zeta \Big) = \Delta \Big[\phi \Big(r, \frac{x}{\|x\|_2} \Big) \frac{\partial^{\alpha-\alpha'}}{\partial r^{\alpha-\alpha'}} \zeta \Big(r, \frac{x}{\|x\|_2} \Big) \Big] \Big|_{S^2} \tag{A.19}$$

$$= \frac{\partial^{\alpha-\alpha'}}{\partial r^{\alpha-\alpha'}} \zeta \Big(r, \frac{x}{\|x\|_2} \Big) \Delta \phi \Big(r, \frac{x}{\|x\|_2} \Big) \Big|_{S^2} + 0 = \frac{\partial^{\alpha-\alpha'}}{\partial r^{\alpha-\alpha'}} \zeta (\Delta_{S^2} \phi), \tag{A.20}$$

where Δ here acts on the second argument and r is a fixed parameter. Thus, noting that $\frac{\partial^{\alpha-\alpha'}}{\partial r^{\alpha-\alpha'}}\zeta\cdot\phi\in C_c^\infty(\mathbb{S}_\varepsilon^{\mathrm{o}})$, for $\alpha'=0,\dots,\alpha$ we see

$$\int_{\mathbb{S}_\varepsilon^{\mathrm{o}}}\Big(\widetilde{\Delta_{S^2}^{\beta}\frac{\partial^{\alpha'}}{\partial r^{\alpha'}}}f\Big)\Big(\frac{\partial^{\alpha-\alpha'}}{\partial r^{\alpha-\alpha'}}\zeta\Big)\Delta_{S^2}\phi\,\mathrm{d}x\int_{\mathbb{S}_\varepsilon^{\mathrm{o}}}\Big(\widetilde{\Delta_{S^2}^{\beta+1}\frac{\partial^{\alpha'}}{\partial r^{\alpha'}}}f\Big)\Big(\frac{\partial^{\alpha-\alpha'}}{\partial r^{\alpha-\alpha'}}\zeta\Big)\phi\,\mathrm{d}x.$$

Hence, together with (A.18) we have (A.16).

To show (A.17), from the hypothesis for $\psi\in C_c^\infty(\mathbb{S}_\varepsilon^{\mathrm{o}})$ we have

$$\begin{aligned}&\iint_{\mathbb{S}_\varepsilon^{\mathrm{o}}}f\zeta\Delta_{S^2}^{\beta}\frac{\partial^{\alpha+1}}{\partial r^{\alpha+1}}\Big(\psi\frac{1}{r^2}r^2\Big)\,\mathrm{d}r\mathrm{d}S\\&\qquad=(-1)^{\alpha}\iint_{\mathbb{S}_\varepsilon^{\mathrm{o}}}\sum_{\alpha'=0}^{\alpha}\binom{\alpha}{\alpha'}\Big(\widetilde{\Delta_{S^2}^{\beta}\frac{\partial^{\alpha'}}{\partial r^{\alpha'}}}f\Big)\Big(\frac{\partial^{\alpha-\alpha'}}{\partial r^{\alpha-\alpha'}}\zeta\Big)\frac{\partial}{\partial r}\psi\,\mathrm{d}r\mathrm{d}S.\end{aligned}\tag{A.21}$$

Now, from $\frac{\partial^{\alpha-\alpha'}}{\partial r^{\alpha-\alpha'}}\zeta\psi\in C_c^\infty(\mathbb{S}_\varepsilon^{\mathrm{o}})$ we have

$$\begin{aligned}&\int_{\mathbb{S}_\varepsilon^{\mathrm{o}}}\Big(\widetilde{\Delta_{S^2}^{\beta}\frac{\partial^{\alpha'}}{\partial r^{\alpha'}}}f\Big)\Big(\frac{\partial^{\alpha-\alpha'}}{\partial r^{\alpha-\alpha'}}\zeta\Big)\frac{\partial}{\partial r}\psi\,\mathrm{d}r\mathrm{d}S\\&\qquad=-\int_{\mathbb{S}_\varepsilon^{\mathrm{o}}}\Big(\widetilde{\Delta_{S^2}^{\beta}\frac{\partial^{\alpha'+1}}{\partial r^{\alpha'+1}}}f\Big)\Big(\frac{\partial^{\alpha-\alpha'}}{\partial r^{\alpha-\alpha'}}\zeta\psi\Big)\,\mathrm{d}r\mathrm{d}S\\&\qquad\quad-\int_{\mathbb{S}_\varepsilon^{\mathrm{o}}}\Big(\widetilde{\Delta_{S^2}^{\beta}\frac{\partial^{\alpha'}}{\partial r^{\alpha'}}}f\Big)\Big(\frac{\partial^{\alpha-\alpha'+1}}{\partial r^{\alpha-\alpha'+1}}\zeta\Big)\psi\,\mathrm{d}r\mathrm{d}S.\end{aligned}$$

This together with (A.21), letting $F_j:=\widetilde{\Delta_{S^2}^{\beta}\frac{\partial^j}{\partial r^j}}f$ and $Z_j:=\frac{\partial^j}{\partial r^j}\zeta$ we have

$$\int_{\mathbb{S}_\varepsilon^{\mathrm{o}}}f\zeta\Delta_{S^2}^{\beta}\frac{\partial^{\alpha+1}}{\partial r^{\alpha+1}}\psi\,\mathrm{d}r\mathrm{d}S=(-1)^{\alpha+1}\int_{\mathbb{S}_\varepsilon^{\mathrm{o}}}\sum_{\alpha'=0}^{\alpha+1}\binom{\alpha+1}{\alpha'}F_{\alpha'}Z_{\alpha-\alpha'+1}\psi\,\mathrm{d}r\mathrm{d}S,\tag{A.22}$$

where we used the Pascal's rule. Thus, for any $\psi\in C_c^\infty(\mathbb{S}_\varepsilon^{\mathrm{o}})$ we have

$$\begin{aligned}&\iint_{\mathbb{S}_\varepsilon^{\mathrm{o}}}f\zeta\Delta_{S^2}^{\beta}\frac{\partial^{\alpha+1}}{\partial r^{\alpha+1}}\psi\,\mathrm{d}r\mathrm{d}S\\&\qquad=(-1)^{\alpha+1}\int_{\mathbb{S}_\varepsilon^{\mathrm{o}}}\sum_{\alpha'=0}^{\alpha+1}\binom{\alpha+1}{\alpha'}\widetilde{\Delta_{S^2}^{\beta}\frac{\partial^{\alpha'}}{\partial r^{\alpha'}}}f\frac{\partial^{\alpha-\alpha'+1}}{\partial r^{\alpha-\alpha'+1}}\zeta\psi\,\mathrm{d}r\mathrm{d}S.\end{aligned}\tag{A.23}$$

Therefore, for any $\phi\in C_c^\infty(\mathbb{S}_\varepsilon^{\mathrm{o}})$, letting $\psi:=\phi r^2\in C_c^\infty(\mathbb{S}_\varepsilon^{\mathrm{o}})$ we have (A.17). $\square$

We are ready to state the following result that claims, under a mild assumption, if we have a weak derivative defined in Definition 4, it is also a strong derivative as in Definition 3.

Proposition A.10. *Let $\alpha, \beta \in \mathbb{N} \cup \{0\}$. For $f \in L^2(\mathbb{S}_\varepsilon^{\mathrm{o}})$, suppose $\widetilde{\Delta_{S^2}^{\beta} \frac{\partial^{\alpha'}}{\partial r^{\alpha'}}} f \in L^2(\mathbb{S}_\varepsilon^{\mathrm{o}})$ for $\alpha' \in \{0, \dots, \alpha\}$. Then, we have*

$$\widetilde{\Delta_{S^2}^{\beta} \frac{\partial^{\alpha}}{\partial r^{\alpha}}} f = \Delta_{S^2}^{\beta} \frac{\partial^{\alpha}}{\partial r^{\alpha}} f \quad \textit{almost everywhere on } \mathbb{S}_\varepsilon^{\mathrm{o}}. \tag{A.24}$$

Proof. It suffices to show for any $\epsilon > 0$ there exists $w \in C^\infty(\mathbb{S}_\varepsilon^{\mathrm{o}})$ such that $\|w - f\|_{L^2(\mathbb{S}_\varepsilon^{\mathrm{o}})} < \epsilon$, and $\left\| \Delta_{S^2}^{\beta} \frac{\partial^{\alpha}}{\partial r^{\alpha}} w - \widetilde{\Delta_{S^2}^{\beta} \frac{\partial^{\alpha}}{\partial r^{\alpha}}} f \right\|_{L^2(\mathbb{S}_\varepsilon^{\mathrm{o}})} < \epsilon$.

Given Proposition A.5, Lemma A.8, and Lemma A.9, it is an easy exercise following the argument by Meyers–Serrin [74] on the global approximation of Sobolev functions by smooth functions (for example [1, Theorem 3.17]), and thus we omit the proof. □

The following result shows if we have the L^2-strong (and thus weak) derivatives $\frac{\partial^\alpha}{\partial r^\alpha} \Delta_{S^2}^{\beta} f \in L^2(\mathbb{S}_\varepsilon^{\mathrm{o}})$ with suitable orders, then f is in the domain of the weak Laplace operator.

Proposition A.11. *Let $\iota^* \geq 0$ be an integer. Suppose $f \in L^2(\mathbb{S}_\varepsilon^{\mathrm{o}})$ satisfies $\frac{\partial^\alpha}{\partial r^\alpha} \Delta_{S^2}^{\beta} f \in L^2(\mathbb{S}_\varepsilon^{\mathrm{o}})$ for all pairs (α, β) of integers such that $0 \leq \alpha + 2\beta \leq 2\iota^*$. Then, we have*

$$\Delta^\iota f \in L^2(\mathbb{S}_\varepsilon^{\mathrm{o}}), \qquad \textit{for all } \iota \in \{1, \dots, \iota^*\}, \tag{A.25}$$

where Δ is in the weak sense.

Proof. Fix $\iota \in \{1, \dots, \iota^*\}$. Formally, let $D := \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^2}$. Define D^ι inductively as

$$\begin{aligned}
&\frac{\partial^2}{\partial r^2} D^{\iota-1} + \frac{2}{r}\frac{\partial}{\partial r} D^{\iota-1} + \frac{1}{r^2}\Delta_{S^2} D^{\iota-1} \\
&= \frac{\partial^{2\iota}}{\partial r^{2\iota}} + \frac{2\iota}{r}\frac{\partial^{2\iota-1}}{\partial r^{2\iota-1}} + \frac{\iota}{r^2}\frac{\partial^{2\iota-2}}{\partial r^{2\iota-2}}\Delta_{S^2} + \frac{\iota(\iota-1)}{r^4}\frac{\partial^{2\iota-4}}{\partial r^{2\iota-4}}\left(\frac{1}{2}\Delta_{S^2}^2 + \Delta_{S^2}\right) \\
&\quad + \cdots + \frac{1}{r^{2\iota}}\Delta_{S^2}^{\iota} =: D^\iota.
\end{aligned}$$

For any $\phi \in C_c^\infty(\mathbb{S}_\varepsilon^{\mathrm{o}})$, we have $\frac{1}{r^N}\phi \in C_c^\infty(\mathbb{S}_\varepsilon^{\mathrm{o}})$ for arbitrary $N \in \mathbb{N}$. Thus, in view of Lemma A.6, the expression

$$\begin{aligned}
&\iint_{\mathbb{S}_\varepsilon^{\mathrm{o}}} D^\iota f \phi\, r^2 \mathrm{d}r\mathrm{d}S \\
&= (-1)^{2\iota} \iint_{\mathbb{S}_\varepsilon^{\mathrm{o}}} f \frac{\partial^{2\iota}}{\partial r^{2\iota}}\big(\phi r^2\big)\mathrm{d}r\mathrm{d}S + (-1)^{2\iota-1} \iint_{\mathbb{S}_\varepsilon^{\mathrm{o}}} f\Big(2\iota \frac{\partial^{2\iota-1}}{\partial r^{2\iota-1}}\big(\phi r\big)\Big)\mathrm{d}r\mathrm{d}S \\
&+ (-1)^{2\iota-2} \iint_{\mathbb{S}_\varepsilon^{\mathrm{o}}} f\Big(\iota \frac{\partial^{2\iota-2}}{\partial r^{2\iota-2}}\Delta_{S^2}\phi\Big)\mathrm{d}r\mathrm{d}S \\
&\ + \cdots + (-1)^0 \iint_{\mathbb{S}_\varepsilon^{\mathrm{o}}} f \Delta_{S^2}^\iota\Big(\frac{1}{r^{2\iota}}\phi r^2\Big)\mathrm{d}r\mathrm{d}S,
\end{aligned}$$

is well defined, and $D^\iota f \in L^2(\mathbb{S}_\varepsilon^{\mathrm{o}})$, where the left hand side is the sum of L^2-strong derivatives of f, and the right hand side is the classical derivatives.

One can check that for $\iota = 1$ we have $\iint_{\mathbb{S}_\varepsilon^{\mathrm{o}}} Df\phi\, r^2\mathrm{d}r\mathrm{d}S = \iint_{\mathbb{S}_\varepsilon^{\mathrm{o}}} f\big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^2}\big)\phi r^2\mathrm{d}r\mathrm{d}S$, and inductively that for any $\phi \in C_c^\infty(\mathbb{S}_\varepsilon^{\mathrm{o}})$ we have

$$\begin{aligned}
&\iint_{\mathbb{S}_\varepsilon^{\mathrm{o}}} D^\iota f \phi\, r^2 \mathrm{d}r\mathrm{d}S \\
&= \iint_{\mathbb{S}_\varepsilon^{\mathrm{o}}} f\Big[\underbrace{\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^2}\Big)\cdots\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^2}\Big)}_{\iota\text{-times}}\phi\Big] r^2\mathrm{d}r\mathrm{d}S,
\end{aligned}$$

where the right hand side is the classical derivative. Hence, we have

$$\int_{\mathbb{S}_\varepsilon^{\mathrm{o}}} D^\iota f \phi\, \mathrm{d}x = \int_{\mathbb{S}_\varepsilon^{\mathrm{o}}} f \Delta^\iota \phi\, \mathrm{d}x \quad \text{for any} \quad \phi \in C_c^\infty(\mathbb{S}_\varepsilon^{\mathrm{o}}). \tag{A.26}$$

The fundamental lemma of variational calculus yields $\Delta^\iota f = D^\iota f \in L^2(\mathbb{S}_\varepsilon^{\mathrm{o}})$. $\square$

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