Rational trigonometry of a tetrahedron over a general metrical framework

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Rational trigonometry of a tetrahedron over a general metrical framework

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Abstract 350 words maximum: (PLEASE TYPE)

This thesis sets up a framework for rational trigonometry in three dimensions, using a linear algebraic approach to extend the classical trigonometric framework of years past, as well as the two-dimensional rational trigonometric framework of Wildberger, beyond the usual Euclidean setting to arbitrary symmetric bilinear forms and arbitrary fields not of characteristic 2. We will use two complementary techniques to establish such a framework.

In addition to a generalised scalar product which is defined by a symmetric bilinear form, we define a generalised vector product. Furthermore, we derive analogs of classical results attributed to Lagrange, Cauchy and Binet, and use these to establish formulas for the quadrances, quadreas, quadrume, spreads, dihedral spreads, solid spreads and dual solid spreads of a general tetrahedron. While we aim to generalise and prove previously stated formulas of Wildberger, as well as classical formulas attributed to Richardson, we also establish new results such as the Three-dimensional quadrea theorem and the Quadrume theorem.

The other technique is to introduce standard co-ordinates, where affine transformations are used to transform to a particularly simple example, and all the complexity resides in the algebraic expression for the symmetric bilinear form rather than the generality of the tetrahedron itself. Using this technique, we derive the Tetrahedron cross law and the Dihedral cross relation.

Throughout this thesis, we use a simple example from Khafre's pyramid to illustrate the ideas we have formulated, and in the final chapter we examine the special cases of the regular, isosceles and trirectangular tetrahedral, as well as a general tetrahedron in a relativistic setting and a general tetrahedron over a finite field.

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Contents

List of Figures

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Preface

In 2005, Wildberger introduced the field of rational trigonometry in [59]. In it, he presents a purely algebraic approach to the subject of trigonometry, which is currently reliant on the classical notion of circular functions and square roots, whose precise definitions involve infinite processes which are computationally approximate in nature. By framing the study of trigonometry in the realm of rational numbers, calculations with regards to the triangle are computationally exact and we obtain a much more general form of geometry that extends to arbitrary fields not of characteristic 2 (to avoid zero denominators) and to other types of geometries, which are parameterised by an arbitrary symmetric bilinear form.

We will extend the rational or algebraic study of geometry and trigonometry in two dimensions introduced by Wildberger in [55] and [59] to three dimensions. We will use the tools of linear algebra to formulate two complementary approaches by which to study general tetrahedra. One approach involves a generalised vector product, so that we can define the trigonometric quantities of the tetrahedron and formulate some results pertaining to it. The other approach involves an affine map from a general tetrahedron to a specific tetrahedron, so that we may study a specific tetrahedron over a general symmetric bilinear form as opposed to a general tetrahedron over a specific bilinear form.

Introduction

To provide motivation for this thesis, we summarise briefly the standard approach to trigonometry, which involves considering a general triangle in the Euclidean 2-space $\mathbb{E}^2$ typically over the "real number field", denoted by $\mathbb{R}$. The points A_0, A_1 and A_2 of this triangle, as well as the corresponding distances and angles, are denoted and illustrated in Figure 1.

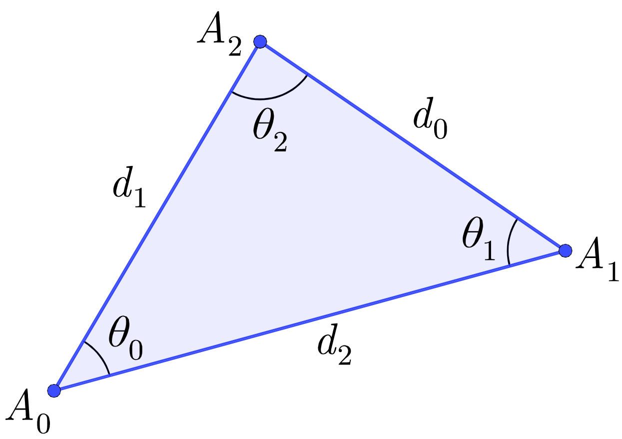

Figure 1: Triangle with distances and angles displayed

In what follows, we may suppose that the angles θ_0, θ_1 and θ_2 are measured in radians. Given these quantities, the following results are standard:

1. The sum of the angles is equal to two right angles (Proposition 13, Book I of *Elements* [26]), i.e.
$$\theta_0 + \theta_1 + \theta_2 = \pi.$$

2. *Pythagoras' theorem* (Proposition 48, Book I of *Elements*) - We have that $\theta_0 = \frac{\pi}{2}$ precisely when
$$d_0^2 = d_1^2 + d_2^2.$$

3. *Cosine law* - We have that
$$d_0^2 = d_1^2 + d_2^2 - 2d_1d_2\cos\theta_0,$$

$$d_1^2 = d_0^2 + d_2^2 - 2d_0 d_2 \cos\theta_1,$$

and

$$d_2^2 = d_0^2 + d_1^2 - 2d_0 d_1 \cos\theta_2.$$

4. *Sine law* - We have that
$$\frac{\sin\theta_0}{d_0} = \frac{\sin\theta_1}{d_1} = \frac{\sin\theta_2}{d_2}.$$

In co-ordinate geometry, the distances d_0, d_1 and d_2 are obtained by taking the square root of the sum of squares of the co-ordinate differences; here, we assume the Cartesian plane and all its associated properties. As for the angles θ_0, θ_1 and θ_2, we draw an arc of radius 1 centred at the intersection of the two sides and thus compute the arclength, which typically requires methods from calculus. This methodology presents various difficulties, so working classically in trigonometry has its limitations. Over arbitrary fields not of characteristic 2, especially in finite fields, not all numbers have square roots. As an example, in the field of 11 elements (denoted by $\mathbb{F}_{11}$ and typically consisting of integers from -5 to 5) we have that the only squares in $\mathbb{F}_{11}$ are -2, 0, 1, 3, 4 and 5; the other five elements of $\mathbb{F}_{11}$ do not have square roots. Thus, the current framework of classical trigonometry is generally restricted to the "real number field" and the formulas do not generalise easily to other fields. Additional difficulties include:

- the reliance of approximations in the calculations of square roots and arclengths;
- the reliance of extensive tables to calculate only some of the trigonometric values;
- the implicit reliance on differential calculus for the definition of angles and circular functions, as well as their inverses;
- the complexity associated with teaching this content to students; and
- inherent difficulties with angles, with particular emphasis on moving to three dimensions and possibly higher.

The story does not improve by much with the use of vectors. We introduce the Euclidean scalar product [48, p. 16] and take its square root to obtain the distance. As for angles, if $\|v\| \equiv \sqrt{v \cdot v}$ then we can use

$$v \cdot w = \|v\| \, \|w\| \cos\theta$$

to introduce angles. Setting
$$x \equiv \frac{v \cdot w}{\|v\| \, \|w\|},$$
the Maclaurin series [52] of $\arcsin x$, for $|x| \leq 1$, is

$$\arcsin x = \sum_{n \geq 0} \frac{(2n)!}{4^n \, (n!)^2 \, (2n+1)} x^{2n+1} = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots$$

and then

$$\begin{aligned}\theta &= \arccos x = \frac{\pi}{2} - \arcsin x \\ &= \frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots\right).\end{aligned}$$

We note the unending/non-terminating aspect of the Maclaurin series, with relation to our desire to obtain exact computations.

In the framework of spherical geometry, as laid out classically in Moritz [42], we have analogous formulas with similar computational limitations. Given a spherical triangle with spherical distances a, b and c, and respective opposite spherical angles A, B and C, the Spherical sine law is

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

and the Spherical cosine law is given by the set of relations

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

$$\cos b = \cos a \cos c + \sin a \sin c \cos B$$

and

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

We also have a second family of Spherical cosine laws, given by

$$\cos A = \cos B \cos C - \sin B \sin C \cos a,$$

$$\cos B = \cos A \cos C - \sin A \sin C \cos b$$

and

$$\cos C = \cos A \cos B - \sin A \sin B \cos c.$$

As a replacement for distances and angles, the notions of quadrance and spread are introduced in [55] and [59, Chap. 5 and 6], for the two-dimensional Euclidean space $\mathbb{E}^2$ over the rational number field, equipped with the usual definition of Euclidean scalar product. Given two points $A_1 \equiv [x_1, y_1]$ and $A_2 \equiv [x_2, y_2]$, the **quadrance** between them is the number

$$\begin{aligned}Q(A_1, A_2) &\equiv (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (x_2 - x_1, y_2 - y_1) \cdot (x_2 - x_1, y_2 - y_1).\end{aligned}$$

Furthermore, if we have two lines l_1 and l_2 with respective direction vectors $v_1 \equiv (x_1, y_1)$

and $v_2 \equiv (x_2, y_2)$, then the **spread** between them is the number

$$s(l_1, l_2) \equiv 1 - \frac{(x_1 x_2 + y_1 y_2)^2}{(x_1^2 + x_2^2)(y_1^2 + y_2^2)} = \frac{(x_1 y_2 - x_2 y_1)^2}{(x_1^2 + x_2^2)(y_1^2 + y_2^2)}.$$

The last equality goes back to a result of Diophantus, as well as Brahmagupta and Fibonacci; the history of this is highlighted in Stillwell [50, pp. 72-76].

We can interpret the notions of quadrance and spread classically as the squared distance and the square of the sine of an angle, respectively. Working rationally does not limit us to the trigonometric identities, which inputs angles, and will allow us to expand our current framework beyond Euclidean geometry naturally.

Motivated by the study of a triangle in two-dimensional Euclidean (affine) geometry from *Elements* and its rational reformulations in [59], this thesis extends the framework of the latter to the three-dimensional affine space, which we will denote by $\mathbb{A}^3$, over an arbitrary field $\mathbb{F}$ not of characteristic 2. We will associate to $\mathbb{A}^3$ the vector space $\mathbb{V}^3$, which will be equipped with a general symmetric bilinear form, so that the current framework is extended beyond Euclidean geometry. Here, the usual notions of points and vectors are given to us. The focus of this thesis is to understand the fundamental object in three-dimensional geometry: a general tetrahedron. We do this in a way that will prepare us for higher-dimensional generalisations. We build on Altshiller-Court [1] and Richardson [45] on the classical trigonometry of a tetrahedron to provide an introductory framework for the rational trigonometry of a general tetrahedron in the generalised affine and vector spaces.

We first introduce some elementary concepts in $\mathbb{A}^3$ before defining a symmetric bilinear form on $\mathbb{V}^3$, which generalises the Euclidean scalar product. This gives us a generalised metrical framework by which the matrix representing the symmetric bilinear form becomes an important factor in our calculations. In addition to this, we generalise the Euclidean vector product based on this symmetric bilinear form and link it with matrix adjugation. With the generalised scalar and vector products, we define the scalar and vector triple and quadruple products, and draw inspiration from Spiegel's manual [48, p. 16] to present easier, indirect methods of calculating them.

In addition to points, we also have lines and planes in $\mathbb{A}^3$, as well as triangles and tetrahedra in $\mathbb{A}^3$ and various other objects associated to them. From here, we are able to build on the notions of quadrance and spread to define the quadrea and quadrume, as well as the dihedral spread, solid spread and dual solid spread. As an example, we can start with a tetrahedron with points A_0, A_1, A_2 and A_3 and define the quadrances of the tetrahedron to be Q_{01}, Q_{02}, Q_{03}, Q_{12}, Q_{13} and Q_{23}, where for $0 \leq i < j \leq 3$

$$Q_{ij} \equiv Q(A_i, A_j).$$

Based on this, the quadreas and quadrume of this tetrahedron can be expressed in terms of the quadrances as

$$\mathcal{A}_{012} \equiv (Q_{01}+Q_{02}+Q_{12})^2 - 2\left(Q_{01}^2+Q_{02}^2+Q_{12}^2\right),$$

$$\mathcal{A}_{013} \equiv (Q_{01}+Q_{03}+Q_{13})^2 - 2\left(Q_{01}^2+Q_{03}^2+Q_{13}^2\right),$$

$$\mathcal{A}_{023} \equiv (Q_{02}+Q_{03}+Q_{23})^2 - 2\left(Q_{02}^2+Q_{03}^2+Q_{23}^2\right),$$

$$\mathcal{A}_{123} \equiv (Q_{12}+Q_{13}+Q_{23})^2 - 2\left(Q_{12}^2+Q_{13}^2+Q_{23}^2\right)$$

and

$$\mathcal{V} \equiv \frac{1}{2}\begin{vmatrix} 2Q_{01} & Q_{01}+Q_{02}-Q_{12} & Q_{01}+Q_{03}-Q_{13} \\ Q_{01}+Q_{02}-Q_{12} & 2Q_{02} & Q_{02}+Q_{03}-Q_{23} \\ Q_{01}+Q_{03}-Q_{13} & Q_{02}+Q_{03}-Q_{23} & 2Q_{03} \end{vmatrix}.$$

For $0 \le j < k \le 3$ with j and k distinct from i, let $s_{i;jk}$ be the spread between the lines through the pairs of points (A_i, A_j) and (A_i, A_l). Then, the spread can be expressed in terms of the quadrances and quadreas as

$$s_{i;jk} = \frac{\mathcal{A}_{ijk}}{4Q_{ij}Q_{ik}}.$$

Based on the results of a seminar by Wildberger [63], the dihedral spreads and solid spreads of this tetrahedron can be expressed in terms of the quadrances, quadreas and quadrume as

$$E_{01} = \frac{4Q_{01}\mathcal{V}}{\mathcal{A}_{012}\mathcal{A}_{013}}, \quad E_{02} = \frac{4Q_{02}\mathcal{V}}{\mathcal{A}_{012}\mathcal{A}_{023}}, \quad E_{03} = \frac{4Q_{03}\mathcal{V}}{\mathcal{A}_{013}\mathcal{A}_{023}},$$

$$E_{23} = \frac{4Q_{23}\mathcal{V}}{\mathcal{A}_{023}\mathcal{A}_{123}}, \quad E_{13} = \frac{4Q_{13}\mathcal{V}}{\mathcal{A}_{013}\mathcal{A}_{123}}, \quad E_{12} = \frac{4Q_{12}\mathcal{V}}{\mathcal{A}_{012}\mathcal{A}_{123}},$$

$$\mathcal{S}_0 = \frac{\mathcal{V}}{4Q_{01}Q_{02}Q_{03}}, \quad \mathcal{S}_1 = \frac{\mathcal{V}}{4Q_{01}Q_{12}Q_{13}},$$

$$\mathcal{S}_2 = \frac{\mathcal{V}}{4Q_{02}Q_{12}Q_{23}} \quad \text{and} \quad \mathcal{S}_3 = \frac{\mathcal{V}}{4Q_{03}Q_{13}Q_{23}}.$$

Based on this, we can define $C_{ij} \equiv 1 - E_{ij}$ to be the dihedral crosses of this tetrahedron, for $0 \le i < j \le 3$. On a similar note, we can express the dual solid spread in terms of the quadrances and quadrume as

$$\mathcal{D}_0 = \frac{4\mathcal{V}^2}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}}, \quad \mathcal{D}_1 = \frac{4\mathcal{V}^2}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{123}},$$

$$\mathcal{D}_2 = \frac{4\mathcal{V}^2}{\mathcal{A}_{012}\mathcal{A}_{023}\mathcal{A}_{123}} \quad \text{and} \quad \mathcal{D}_3 = \frac{4\mathcal{V}^2}{\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}}.$$

These will be important quantities that we associate to a general tetrahedron in $\mathbb{A}^3$, and a significant part of our discussion will involve justifying the expressions for the latter three quantities. We will also link them to various quantities from projective geometry and trigonometry, which Wildberger has framed in the rational sense in [55], [60], [61], [64], [65] and [66]; the general framework will be reviewed in the thesis.

Throughout the latter parts of the thesis, we refer to a specific tetrahedron and use it as a running example for these results. This tetrahedron is based on the second of three pyramids in Giza, Egypt, which houses the tomb of Khafre and is located in front of the famous Great Sphinx.

Of independent but special interest, we make the observation that a pair of lines of a tetrahedron passing through a distinct pair of two points are skew, i.e. they are non-parallel and non-intersecting lines. This motivates us to compute the quadrance between such pairs of lines and uncover a connection with a rational version of a result of Bretschneider [6], which is also illustrated in Coolidge [15]. In the classical framework, Bretschneider's result computes the area of a general quadrangle (a collection of four coplanar points) based on the distance between any two points of it. We also present two results which express the equality of ratios of certain trigonometric quantities, based on the results from Richardson [45].

As an additional technology that can be used to simplify some calculations, we can consider an affine map which sends a general tetrahedron in $\mathbb{A}^3$ to a unique tetrahedron $\overline{B_0B_1B_2B_3}$, where

$$B_0 \equiv [0,0,0], \quad B_1 \equiv [1,0,0], \quad B_2 \equiv [0,1,0] \quad \text{and} \quad B_3 \equiv [0,0,1].$$

Inspired by the contents of Nguyen Le's doctoral thesis [35] and her joint paper with Wildberger [36], we will name this tetrahedron the **Standard tetrahedron** and this will form the content of Chapter 3. We will use the trigonometric quantities of the Standard tetrahedron to prove more complex results at the cost of a brute force approach. Motivated by Richardson [45] and Lee [37], we introduce rational analogs to two substantial results, which we will call the Tetrahedron cross law and the Dihedral cross relation. In the former result, we involve the dihedral crosses to find a relationship between the quadreas of the four faces of a tetrahedron; in the latter result, we see that the six dihedral crosses will satisfy a relation involving a very large polynomial.

Based on the expressions for the trigonometric quantities above, the Tetrahedron cross law takes the form

$$\begin{aligned}
&\left[\begin{array}{c}\left[(\mathcal{A}_{012}+\mathcal{A}_{013}+\mathcal{A}_{023}-\mathcal{A}_{123})^2-4\left(\mathcal{A}_{012}\mathcal{A}_{013}C_{01}+\mathcal{A}_{012}\mathcal{A}_{023}C_{02}+\mathcal{A}_{013}\mathcal{A}_{023}C_{03}\right)^2\right] \\ -64\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\left(\mathcal{A}_{012}C_{01}C_{02}+\mathcal{A}_{013}C_{01}C_{03}+\mathcal{A}_{023}C_{02}C_{03}\right)\end{array}\right]^2 \\
= \;& 4096\mathcal{A}_{012}^2\mathcal{A}_{013}^2\mathcal{A}_{023}^2\left(\mathcal{A}_{012}+\mathcal{A}_{013}+\mathcal{A}_{023}-\mathcal{A}_{123}\right)^2 C_{01}C_{02}C_{03}.
\end{aligned}$$

Furthermore, define

$$X \equiv C_{01}C_{23}, \quad Y \equiv C_{02}C_{13}, \quad Z \equiv C_{03}C_{12},$$

$$x \equiv C_{01}+C_{23}, \quad y \equiv C_{02}+C_{13}, \quad z \equiv C_{03}+C_{12},$$

$$W \equiv C_{01}C_{02}C_{03} + C_{01}C_{12}C_{13} + C_{02}C_{12}C_{23} + C_{03}C_{13}C_{23},$$
$$P \equiv \frac{1}{2}\left(1 - x - y - z + X + Y + Z\right),$$
$$R \equiv P + z - Z, \quad S \equiv P + y - Y, \quad T \equiv P + x - X,$$
$$U \equiv \frac{1}{2}\left(P^2 - W + XY + XZ + YZ\right)$$

and

$$V \equiv \frac{1}{2}\left(U^2 - XYR^2 - XZS^2 - YZT^2\right).$$

Then, the Dihedral cross relation takes the form

$$V^2 = XYZ\left(XR^2S^2 + YR^2T^2 + ZS^2T^2 + 2RSTU\right).$$

To end the thesis, we apply the results of the previous chapters to some examples of special tetrahedra in the final chapter. We start with three particular tetrahedra:

- *regular tetrahedron* - here, the quadrances between any two points of the tetrahedron are all equal.
- *isosceles tetrahedron* or *disphenoid* - this is a tetrahedron where opposite edges (pairs of edges of a tetrahedron with no common points) have equal quadrances.
- *trirectangular tetrahedron* - a tetrahedron where the edges/vectors emanating from a point are mutually perpendicular with respect to the arbitrary symmetric bilinear form.

In addition to these three special tetrahedra, we will consider two further tetrahedra. In one case, we will examine an example tetrahedron in $\mathbb{A}^3$ over the rational number field where we equip a relativistic bilinear form based on a two-dimensional analog mentioned by Wildberger in [56], [57] and [58], which is related to the Minkowski scalar product [40]. In the other case, we will examine another tetrahedron in $\mathbb{A}^3$, but over the finite field $\mathbb{F}_{11}$ with 11 elements, which will be expressed as integers between -5 and 5.

It is a measure of the generality of the rational trigonometric formulation that we are able to extend our study even to such non-standard situations. We will end the thesis with some remarks on further directions.

Chapter 1

Rational trigonometry in three dimensions

We start by considering three-dimensional affine space, denoted by $\mathbb{A}^3$, over a general (number) field $\mathbb{F}$ not of characteristic 2. Points are algebraically expressed as a triple enclosed in square brackets, e.g. $A = [x, y, z]$. Two points $A_1 \equiv [x_1, y_1, z_1]$ and $A_2 \equiv [x_2, y_2, z_2]$ are **equal** precisely when

$$x_1 = x_2, \quad y_1 = y_2 \quad \text{and} \quad z_1 = z_2.$$

If A_1 and A_2 are not equal, then they are **distinct**. We can also talk about multiple points being distinct if any two are distinct.

Noting that points may represent (absolute) positions in space if co-ordinate axes have been specified, we will also want to consider relative displacements between points. If $A_1 \equiv [x_1, y_1, z_1]$ and $A_2 \equiv [x_2, y_2, z_2]$, then we define the associated **(displacement) vector**

$$\overrightarrow{A_1A_2} \equiv (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

Two vectors $v_1 \equiv (x_1, y_1, z_1)$ and $v_2 \equiv (x_2, y_2, z_2)$ are **equal** precisely when

$$x_1 = x_2, \quad y_1 = y_2 \quad \text{and} \quad z_1 = z_2.$$

The (displacement) vector $v = \overrightarrow{A_1A_2}$ is the zero vector $\mathbf{0} \equiv (0, 0, 0)$ precisely when A_1 and A_2 are equal.

We also follow some non-standard conventions introduced by Wildberger in his YouTube playlist [68]. Namely, we will write the following operation on two points:

$$A_2 - A_1 \equiv \overrightarrow{A_1A_2}.$$

Note that we are not introducing a general linear algebraic structure on points here; only the *difference* of points is defined, not their *sum*. In conjunction with this, if $A \equiv [x, y, z]$

is a point and $v \equiv (a,b,c)$ is a vector, then we define the **sum** $A+v$ to be the point

$$A + v = [x,y,z] + (a,b,c) \equiv [x+a, y+b, z+c].$$

Thus $A+v=B$ is equivalent to $v=\overrightarrow{AB}$. Note that we will require that the point A be written on the left, and the vector v be written on the right, so that $v+A$ has no meaning for us.

It now makes sense to consider expressions of the form $(A+v_1)+v_2$ and to define **addition** on two vectors v_1 and v_2 by the requirement that

$$(A+v_1)+v_2 = A+(v_1+v_2).$$

Clearly this is equivalent to the rule that

$$(a_1,b_1,c_1)+(a_2,b_2,c_2) = (a_1+a_2, b_1+b_2, c_1+c_2).$$

Consistent with this, we define for any non-zero $\lambda \in \mathbb{F}$ the **scalar multiple** of a vector $v \equiv (a,b,c)$ by

$$\lambda v = \lambda(a,b,c) \equiv (\lambda a, \lambda b, \lambda c).$$

This allows us to obtain and define our usual three-dimensional vector (or linear) space associated with $\mathbb{A}^3$ over $\mathbb{F}$, which we will denote by $\mathbb{V}^3$. We will identify vectors with $1 \times n$ matrices so that the usual apparatus of matrices and linear maps present in linear algebra may be applied to vectors.

Throughout the thesis, we will also apply the *Zero denominator convention* from [59, p. 28], which states that a statement involving fractions and rational functions is assumed to be empty if its denominator is zero.

We define the **signed volume** of three ordered vectors $v_1 \equiv (x_1,y_1,z_1)$, $v_2 \equiv (x_2,y_2,z_2)$ and $v_3 \equiv (x_3,y_3,z_3)$ to be the number

$$v(v_1,v_2,v_3) \equiv \frac{1}{6}\det\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = \frac{x_1y_2z_3 - x_1y_3z_2 - x_2y_1z_3 + x_2y_3z_1 + x_3y_1z_2 - x_3y_2z_1}{6}.$$

Note that this quantity is an element of $\mathbb{F}$; as 2 and 3 are factors of 6, fields of characteristics 3 will also be excluded from the discussion of signed volumes.

If $w \equiv w_1 + w_2$, for $w_1, w_2 \in \mathbb{V}^3$, then we have that

$$v(w,v_2,v_3) = v(w_1,v_2,v_3) + v(w_2,v_2,v_3)$$

and for some non-zero $\lambda \in \mathbb{F}$ we also have that

$$v(\lambda v_1, v_2, v_3) = \lambda v(v_1,v_2,v_3).$$

These properties will hold with the other inputs, as the signed volume is a trilinear operation. Moreover,

$$\begin{aligned} & v\left(v_1,v_2,v_3\right)=v\left(v_2,v_3,v_1\right)=v\left(v_3,v_1,v_2\right) \\ = \quad & -v\left(v_1,v_3,v_2\right)=-v\left(v_2,v_1,v_3\right)=-v\left(v_1,v_3,v_2\right). \end{aligned}$$

1.1 Symmetric bilinear form on $\mathbb{V}^3$

Suppose that we are given a 3×3 symmetric matrix

$$B\equiv\begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix}.$$

This matrix determines a **symmetric bilinear form** [39, p. 1-2] on $\mathbb{V}^3$ defined by

$$v\cdot_B w\equiv vBw^T.$$

We will also say that the matrix B represents the symmetric bilinear form defined above. We will call such an operation the B-**scalar product**. This is a concept used extensively in [55] and also by Le [35] to extend the framework of rational trigonometry to more general metrical situations. When B is the 3×3 identity matrix, the B-scalar product corresponds to the usual notion of the Euclidean scalar product, which we simply write as $v\cdot w$.

The symmetric bilinear form also gives us a B-**quadratic form**

$$Q_B\left(v\right)\equiv v\cdot_B v.$$

For vectors v and w in $\mathbb{V}^3$ and a non-zero λ in $\mathbb{F}$, we have the property

$$Q_B\left(\lambda v\right)=\lambda^2 Q_B\left(v\right),$$

as well as

$$Q_B\left(v+w\right)=Q_B\left(v\right)+Q_B\left(w\right)+2\left(v\cdot_B w\right)$$

and

$$Q_B\left(v-w\right)=Q_B\left(v\right)+Q_B\left(w\right)-2\left(v\cdot_B w\right).$$

Hence, we can express the B-scalar product between two vectors $v,w\in\mathbb{V}^3$ in terms of the B-quadratic form as

$$v\cdot_B w=\frac{Q_B\left(v+w\right)-Q_B\left(v\right)-Q_B\left(w\right)}{2}=\frac{Q_B\left(v\right)+Q_B\left(w\right)-Q_B\left(v-w\right)}{2}.$$

Consistent with the terminology of Havlicek and Weiß [25], the B-scalar product is then the **polar form** of the B-quadratic form, and the above relation will thus be called the **polarisation formula**.

The symmetric bilinear form on $\mathbb{V}^3$ represented by the matrix B is **non-degenerate** if, for any vector $v \in \mathbb{V}^3$, $v \cdot_B w = 0$ implies that $w = 0$. This will be true precisely when B is invertible. We will assume this throughout the thesis unless otherwise stated.

Two vectors v and w in $\mathbb{V}^3$ are **perpendicular** with respect to the B-scalar product precisely when

$$v \cdot_B w = 0,$$

in which case we say that the vectors v and w are B-**perpendicular** and use the notation "$v \perp_B w$".

Extending the concept of Euclidean vector projection in Anton and Rorres [2, p. 206], as well as Strang [51, p. 174], the B-**projection** of a vector w in the direction of v is defined as

$$(\mathrm{proj}_v\, w)_B \equiv \left(\frac{v \cdot_B w}{v \cdot_B v}\right) v.$$

This has the following unique property, which is well-known in Euclidean geometry.

Lemma 1 *For vectors v and w in $\mathbb{V}^3$, let $u \equiv (\mathrm{proj}_v\, w)_B$. Then, v is B-perpendicular to $w - u$.*

Proof. Using the properties of the symmetric bilinear form and the definition of the B-projection, calculate the B-scalar product of v and $w - u$ to get

$$\begin{aligned} v \cdot_B (w - u) &= v \cdot_B w - v \cdot_B u \\ &= v \cdot_B w - v \cdot_B \left(\frac{v \cdot_B w}{v \cdot_B v}\right) v \\ &= v \cdot_B w - \left(\frac{v \cdot_B w}{v \cdot_B v}\right)(v \cdot_B v) \\ &= 0. \end{aligned}$$

So v is B-perpendicular to $w - u$, as required. ■

1.2 The vector product in $\mathbb{V}^3$

Given two vectors $v_1 \equiv (x_1, y_1, z_1)$ and $v_2 \equiv (x_2, y_2, z_2)$ in $\mathbb{V}^3$, the usual notion of the Euclidean vector product, introduced by Lagrange [34] and formalised by Gibbs [22, p. 65], is defined and denoted as follows:

$$\begin{aligned} v_1 \times v_2 &= (x_1, y_1, z_1) \times (x_2, y_2, z_2) \\ &\equiv (y_1 z_2 - y_2 z_1, x_2 z_1 - x_1 z_2, x_1 y_2 - x_2 y_1). \end{aligned}$$

Recall that the **adjugate matrix** of a 3×3 invertible matrix M, as defined separately in Gantmacher [21, pp. 76-89] and Strang [51, p. 248], is

$$\operatorname{adj} M \equiv (\det M)\, M^{-1}.$$

This satisfies the property that for 3×3 invertible matrices M and N

$$\operatorname{adj}(MN) = (\operatorname{adj} N)\,(\operatorname{adj} M)$$

and if I is the 3×3 identity matrix then

$$M\,(\operatorname{adj} M) = (\operatorname{adj} M)\, M = (\det M)\, I.$$

This notion extends also to $n \times n$ invertible matrices, but we will not need that.

For the matrix

$$B = \begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix}$$

representing an arbitrary symmetric bilinear form on $\mathbb{V}^3$, we will use the notation

$$\operatorname{adj} B = \begin{pmatrix} a_2a_3 - b_1^2 & b_1b_2 - a_3b_3 & b_1b_3 - a_2b_2 \\ b_1b_2 - a_3b_3 & a_1a_3 - b_2^2 & b_2b_3 - a_1b_1 \\ b_1b_3 - a_2b_2 & b_2b_3 - a_1b_1 & a_1a_2 - b_3^2 \end{pmatrix} \equiv \begin{pmatrix} \alpha_1 & \beta_3 & \beta_2 \\ \beta_3 & \alpha_2 & \beta_1 \\ \beta_2 & \beta_1 & \alpha_3 \end{pmatrix}.$$

We now define a generalised version of the Euclidean vector product, called the B-**vector product**, between two vectors v_1 and v_2 to be

$$v_1 \times_B v_2 \equiv (v_1 \times v_2)\operatorname{adj} B.$$

The motivation for this definition is given in the following theorem. A similar result has been explored by Collomb [14], where a 3×3 matrix is inverted and the determinant of it is calculated using the Euclidean vector product.

Theorem 2 (Adjugate vector product theorem) *Let v_1, v_2 and v_3 be three linearly independent vectors in $\mathbb{V}^3$, and let M be the matrix with rows v_1, v_2 and v_3, i.e.*

$$M \equiv \begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{pmatrix}.$$

If the adjugate of the matrix MB is written as

$$\operatorname{adj}(MB) \equiv \begin{pmatrix} | & | & | \\ w_1^T & w_2^T & w_3^T \\ | & | & | \end{pmatrix}$$

then

$$w_1 = v_2 \times_B v_3, \quad w_2 = v_3 \times_B v_1 \quad \textit{and} \quad w_3 = v_1 \times_B v_2.$$

Proof. Suppose $v_1 \equiv (x_1, y_1, z_1)$, $v_2 \equiv (x_2, y_2, z_2)$ and $v_3 \equiv (x_3, y_3, z_3)$, so that

$$M = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}.$$

Then

$$\begin{aligned} \operatorname{adj} M &= \begin{pmatrix} y_2z_3 - y_3z_2 & y_3z_1 - y_1z_3 & y_1z_2 - y_2z_1 \\ x_3z_2 - x_2z_3 & x_1z_3 - x_3z_1 & x_2z_1 - x_1z_2 \\ x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \end{pmatrix} \\ &= \begin{pmatrix} | & | & | \\ (v_2 \times v_3)^T & (v_3 \times v_1)^T & (v_1 \times v_2)^T \\ | & | & | \end{pmatrix}. \end{aligned}$$

Since $\operatorname{adj}(MB) = \operatorname{adj} B \operatorname{adj} M$ and B is symmetric,

$$\begin{aligned} (\operatorname{adj}(MB))^T &= (\operatorname{adj} M)^T \operatorname{adj} B \\ &= \begin{pmatrix} - & v_2 \times v_3 & - \\ - & v_3 \times v_1 & - \\ - & v_1 \times v_2 & - \end{pmatrix} \operatorname{adj} B \\ &= \begin{pmatrix} - & (v_2 \times v_3) \operatorname{adj} B & - \\ - & (v_3 \times v_1) \operatorname{adj} B & - \\ - & (v_1 \times v_2) \operatorname{adj} B & - \end{pmatrix} \\ &= \begin{pmatrix} - & v_2 \times_B v_3 & - \\ - & v_3 \times_B v_1 & - \\ - & v_1 \times_B v_2 & - \end{pmatrix}. \end{aligned}$$

Take the transpose of this matrix to obtain

$$w_1 = v_2 \times_B v_3, \quad w_2 = v_3 \times_B v_1 \quad \text{and} \quad w_3 = v_1 \times_B v_2,$$

as required. ∎

The B-vector product is a *bilinear* operation, i.e. for vectors v_1, v_2 and v_3 in $\mathbb{V}^3$ and a scalar λ in $\mathbb{F}$, we have that

$$v_1 \times_B (v_2 + v_3) = v_1 \times_B v_2 + v_1 \times_B v_3,$$

$$(v_1 + v_2) \times_B v_3 = v_1 \times_B v_3 + v_2 \times_B v_3$$

and

$$(\lambda v_1) \times_B v_2 = v_1 \times_B (\lambda v_2) = \lambda (v_1 \times_B v_2).$$

The *alternating property* is also satisfied, i.e. for $v \in \mathbb{V}^3$,

$$v \times_B v = \mathbf{0}.$$

The B-vector product is *anti-symmetric*, i.e.

$$v_1 \times_B v_2 = -v_2 \times_B v_1,$$

as the bilinear and alternating properties gives

$$(v_1 + v_2) \times_B (v_1 + v_2) = v_1 \times_B v_2 + v_2 \times_B v_1 = \mathbf{0}.$$

1.3 Results from generalised vector geometry

We now present some more complicated results regarding B-vector products. In what follows, the proofs will be based on modifications of standard Euclidean arguments, as in Spiegel's manual on vector analysis [48, pp. 16-34].

We start with the **Euclidean scalar triple product** of three vectors v_1, v_2 and v_3 in $\mathbb{V}^3$, which, defined in Gibbs [22, pp. 68-71], can be expressed as

$$[v_1, v_2, v_3] \equiv v_1 \cdot (v_2 \times v_3) = \det \begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{pmatrix} = 6v(v_1, v_2, v_3).$$

This links the Euclidean scalar triple product to the signed volume of three vectors.

We can generalise this definition for an arbitrary symmetric bilinear form with matrix representation B; we will call this the B**-scalar triple product** and define it by

$$[v_1, v_2, v_3]_B \equiv v_1 \cdot_B (v_2 \times_B v_3).$$

The following result allows for the evaluation of the B-scalar triple product, which extends a result of Gibbs [22, pp. 68-71]; this is a generalised version of the Euclidean scalar triple product defined above.

Theorem 3 (Scalar triple product theorem) *Suppose v_1, v_2 and v_3 are vectors in $\mathbb{V}^3$ and let M be the matrix with rows v_1, v_2 and v_3, i.e.*

$$M \equiv \begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{pmatrix}.$$

Then,

$$\begin{aligned} & [v_1, v_2, v_3]_B = [v_2, v_3, v_1]_B = [v_3, v_1, v_2]_B \\ = & -[v_1, v_3, v_2]_B = -[v_2, v_1, v_3]_B = -[v_3, v_2, v_1]_B \\ = & (\det B)[v_1, v_2, v_3] = \det(MB). \end{aligned}$$

Proof. By the definition of the B-scalar triple product, the B-scalar product and the B-vector product,

$$\begin{aligned} [v_1, v_2, v_3]_B &= v_1 \cdot_B (v_2 \times_B v_3) \\ &= v_1 B\left((v_2 \times v_3) \operatorname{adj} B\right)^T \\ &= v_1 (B \operatorname{adj} B)(v_2 \times v_3)^T. \end{aligned}$$

Since $B \operatorname{adj} B = (\det B) I_3$ and $v_1 \cdot (v_2 \times v_3) = \det M$,

$$\begin{aligned} [v_1, v_2, v_3]_B &= (\det B)\left(v_1 (v_2 \times v_3)^T\right) \\ &= (\det B)(v_1 \cdot (v_2 \times v_3)), \\ &= \det B \det M \\ &= \det(MB). \end{aligned}$$

The other results follow by symmetry. ■

From the Scalar triple product theorem, we can make an important implication regarding vector products, which is framed in the literatures as a property of vector products, for example in Anton and Rorres [2, p. 215].

Corollary 4 *For vectors v and w in $\mathbb{V}^3$, v and w are both B-perpendicular to $v \times_B w$, i.e.*

$$v \perp_B (v \times_B w) \quad \text{and} \quad w \perp_B (v \times_B w).$$

Proof. If one row or column of a matrix is a non-zero multiple of another then its determinant is zero. Therefore, by the Scalar triple product theorem,

$$v \cdot_B (v \times_B w) = [v, v, w]_B = \det \begin{pmatrix} - & v & - \\ - & v & - \\ - & w & - \end{pmatrix} \det B = 0.$$

Similarly $[w, v, w]_B = 0$ and thus $v \perp_B (v \times_B w)$ and $w \perp_B (v \times_B w)$. ■

The B-**vector triple product** of three vectors $v_1, v_2, v_3 \in \mathbb{V}^3$ will be defined by

$$\langle v_1, v_2, v_3 \rangle_B \equiv v_1 \times_B (v_2 \times_B v_3).$$

We can evaluate this by generalising the following result of Lagrange [34] for B-vector products. A detailed proof in English is given for the Euclidean case by Chapman and

Milne [9], while [48, pp. 28-29] provides a brute-force approach for the Euclidean case which involves expressing the vectors in three-dimensional co-ordinate form and expressing the Euclidean scalar and vector products in terms of them.

Theorem 5 (Lagrange's formula) *For vectors v_1, v_2 and v_3 in $\mathbb{V}^3$, the B-vector triple product $\langle v_1, v_2, v_3\rangle_B$ can be expressed as*

$$\langle v_1, v_2, v_3\rangle_B = (\det B)\left[(v_1 \cdot_B v_3)\, v_2 - (v_1 \cdot_B v_2)\, v_3\right].$$

Proof. Let $w \equiv \langle v_1, v_2, v_3\rangle_B$. We have that $(v_2 \times_B v_3) \perp_B w$, as well as

$$v_2 \perp_B (v_2 \times_B v_3) \quad \text{and} \quad v_3 \perp_B (v_2 \times_B v_3).$$

Therefore, w is parallel to a linear combination of v_2 and v_3, i.e. for some $\alpha, \beta \in \mathbb{F}$,

$$w = \alpha v_2 + \beta v_3.$$

Furthermore, since $v_1 \perp_B w$, the definition of B-perpendicularity shows that

$$w \cdot_B v_1 = \alpha\,(v_1 \cdot_B v_2) + \beta\,(v_1 \cdot_B v_3) = 0.$$

This equality is true precisely when $\alpha = \lambda\,(v_1 \cdot_B v_3)$ and $\beta = -\lambda\,(v_1 \cdot_B v_2)$, for some non-zero $\lambda \in \mathbb{F}$. Hence,

$$w = \lambda\left[(v_1 \cdot_B v_3)\, v_2 - (v_1 \cdot_B v_2)\, v_3\right].$$

To proceed, we first want to prove that λ is independent of the choices v_1, v_2 and v_3, so that we can compute w for arbitrary v_1, v_2, v_3. First, suppose that λ is dependent on v_1, v_2, v_3; in other words, let $\lambda \equiv \lambda\,(v_1, v_2, v_3)$. Given another vector $d \in \mathbb{V}^3$, we have

$$w \cdot_B d = \lambda\,(v_1, v_2, v_3)\left[(v_1 \cdot_B v_3)\,(v_2 \cdot_B d) - (v_1 \cdot_B v_2)\,(v_3 \cdot_B d)\right].$$

Directly substituting the definition of w, we use the Scalar triple product theorem to obtain

$$w \cdot_B d = (v_1 \times_B (v_2 \times_B v_3)) \cdot_B d = v_1 \cdot_B ((v_2 \times_B v_3) \times_B d) = -v_1 \cdot_B \langle d, v_2, v_3\rangle_B.$$

Thus,

$$\begin{aligned} -v_1 \cdot_B \langle d, v_2, v_3\rangle_B &= -v_1 \cdot_B \left(\lambda\,(d, v_2, v_3)\left[(d \cdot_B v_3)\, v_2 - (d \cdot_B v_2)\, v_3\right]\right) \\ &= \lambda\,(d, v_2, v_3)\left[(v_1 \cdot_B v_3)\,(v_2 \cdot_B d) - (v_1 \cdot_B v_2)\,(v_3 \cdot_B d)\right]. \end{aligned}$$

Since this expression is equal to $w \cdot_B d$, we deduce that $\lambda\,(v_1, v_2, v_3) = \lambda\,(d, v_2, v_3)$ and hence λ must be independent of the choice of v_1. Given this observation, suppose instead that $\lambda \equiv \lambda(v_2, v_3)$, so that

$$w \cdot_B d = \lambda(v_2, v_3)\left[(v_1 \cdot_B v_3)(v_2 \cdot_B d) - (v_1 \cdot_B v_2)(v_3 \cdot_B d)\right].$$

By direct substitution of w, we use the Scalar triple product theorem to obtain

$$w \cdot_B d = (v_1 \times_B (v_2 \times_B v_3)) \cdot_B d = (v_2 \times_B v_3) \cdot_B (d \times_B v_1) = v_2 \cdot_B \langle v_3, d, v_1\rangle_B.$$

Similarly,

$$\begin{aligned} v_2 \cdot_B \langle v_3, d, v_1\rangle_B &= v_2 \cdot_B \lambda(d_2, v_3)\left((v_1 \cdot_B v_3) d - (v_3 \cdot_B d) v_1\right) \\ &= \lambda(d, v_1)\left[(v_1 \cdot_B v_3)(v_2 \cdot_B d) - (v_1 \cdot_B v_2)(v_3 \cdot_B d)\right]. \end{aligned}$$

Since this expression is also equal to $w \cdot_B d$, we deduce that $\lambda(v_2, v_3) = \lambda(d, v_1)$ and conclude that λ is indeed independent of v_2 and v_3, in addition to v_1. At this point, we can find λ by substitution of arbitrary vectors for v_1, v_2 and v_3. In that case, suppose that $v_2 \equiv (1,0,0) \equiv e_1$ and $v_1 = v_3 \equiv (0,1,0) \equiv e_2$; then, noting the definition of $\operatorname{adj} B$, we have

$$v_2 \times_B v_3 = (0,0,1)\begin{pmatrix} \alpha_1 & \beta_3 & \beta_2 \\ \beta_3 & \alpha_2 & \beta_1 \\ \beta_2 & \beta_1 & \alpha_3 \end{pmatrix} = (\beta_2, \beta_1, \alpha_3)$$

and hence

$$\begin{aligned} \langle v_1, v_2, v_3\rangle_B &= \left[(0,1,0) \times (\beta_2, \beta_1, \alpha_3)\right]\begin{pmatrix} \alpha_1 & \beta_3 & \beta_2 \\ \beta_3 & \alpha_2 & \beta_1 \\ \beta_2 & \beta_1 & \alpha_3 \end{pmatrix} \\ &= (\alpha_3, 0, -\beta_2)\begin{pmatrix} \alpha_1 & \beta_3 & \beta_2 \\ \beta_3 & \alpha_2 & \beta_1 \\ \beta_2 & \beta_1 & \alpha_3 \end{pmatrix} \\ &= \left(\alpha_1\alpha_3 - \beta_2^2, \alpha_3\beta_3 - \beta_1\beta_2, 0\right) \\ &= (\det B)(a_2, -b_3, 0). \end{aligned}$$

Since $v_1 \cdot_B v_2 = e_1 B e_2^T = b_3$ and $v_1 \cdot_B v_3 = e_2 B e_2^T = a_2$, it follows that

$$\begin{aligned} (\det B)(a_2, -b_3, 0) &= (\det B)\left[(v_1 \cdot_B v_3) e_1 - (v_1 \cdot_B v_2) e_2\right] \\ &= (\det B)\left[(v_1 \cdot_B v_3) v_2 - (v_1 \cdot_B v_2) v_3\right] \\ &= \langle v_1, v_2, v_3\rangle_B. \end{aligned}$$

From this, we deduce that $\lambda = \det B$ and hence

$$\langle v_1, v_2, v_3\rangle_B = (\det B)\left[(v_1 \cdot_B v_3) v_2 - (v_1 \cdot_B v_2) v_3\right],$$

as required. ■

The B-vector product is generally not an associative operation, i.e. the six variants of $\langle v_1, v_2, v_3\rangle_B$ generally yield different results. The following result, dating back to 1829 and attributed to Jacobi [31] in the Euclidean case, illustrates the link between three of them.

Theorem 6 (Jacobi identity) *For vectors v_1, v_2 and v_3 in $\mathbb{V}^3$,*

$$\langle v_1, v_2, v_3\rangle_B + \langle v_2, v_3, v_1\rangle_B + \langle v_3, v_1, v_2\rangle_B = 0.$$

Proof. Apply Lagrange's formula on each of the three summands to get

$$\langle v_1, v_2, v_3\rangle_B = (\det B)\left[(v_1 \cdot_B v_3)\, v_2 - (v_1 \cdot_B v_2)\, v_3\right],$$

$$\langle v_2, v_3, v_1\rangle_B = (\det B)\left[(v_1 \cdot_B v_2)\, v_3 - (v_2 \cdot_B v_3)\, v_1\right],$$

and

$$\langle v_3, v_1, v_2\rangle_B = (\det B)\left[(v_2 \cdot_B v_3)\, v_1 - (v_1 \cdot_B v_3)\, v_2\right].$$

So,

$$\begin{aligned} & \langle v_1, v_2, v_3\rangle_B + \langle v_2, v_3, v_1\rangle_B + \langle v_3, v_1, v_2\rangle_B \\ = \; & (\det B)\left[(v_1 \cdot_B v_3)\, v_2 - (v_1 \cdot_B v_2)\, v_3\right] + (\det B)\left[(v_1 \cdot_B v_2)\, v_3 - (v_2 \cdot_B v_3)\, v_1\right] \\ & + (\det B)\left[(v_2 \cdot_B v_3)\, v_1 - (v_1 \cdot_B v_3)\, v_2\right] \\ = \; & 0, \end{aligned}$$

as required. ■

Combining this result with the alternating and bilinear properties of the cross product, we deduce that $\mathbb{V}^3$ forms a Lie algebra with respect to vector addition and the B-vector product. This concept is defined in Belifante and Kolman [4, pp. 12-13], as well as Humphreys [30, p. 1], for a general vector space over an arbitrary field that is endowed with a general Lie bracket, for which the Euclidean vector product is provided as the simplest example of a Lie algebra over the "real number field". This provides motivation for a study of three-dimensional Lie algebras founded purely on generalised vector products over general fields, which we will not pursue here.

The following result allows us to compute the vector $(v_1 \times_B v_2) \times_B (v_3 \times_B v_4)$, based on the properties of Euclidean vector products. We will call such an operation a B-**quadruple vector product**, which extends the result in the Euclidean case by Gibbs [22, pp. 76-77].

Theorem 7 (Quadruple vector product theorem) *For vectors v_1, v_2, v_3 and v_4 in $\mathbb{V}^3$,*

$$\begin{aligned} (v_1 \times_B v_2) \times_B (v_3 \times_B v_4) &= (\det B)\left([v_1, v_2, v_4]_B\, v_3 - [v_1, v_2, v_3]_B\, v_4\right) \\ &= (\det B)\left([v_1, v_3, v_4]_B\, v_2 - [v_2, v_3, v_4]_B\, v_1\right). \end{aligned}$$

Proof. If $u \equiv v_1 \times_B v_2$, then use Lagrange's formula to get

$$\begin{aligned}(v_1 \times_B v_2) \times_B (v_3 \times_B v_4) &= \langle u, v_3, v_4\rangle_B \\ &= (\det B)\left[(u \cdot_B v_4)\, v_3 - (u \cdot_B v_3)\, v_4\right].\end{aligned}$$

From the Scalar triple product theorem,

$$u \cdot_B v_3 = [v_1, v_2, v_3]_B \quad \text{and} \quad u \cdot_B v_4 = [v_1, v_2, v_4]_B\,.$$

Therefore,

$$(v_1 \times_B v_2) \times_B (v_3 \times_B v_4) = (\det B)\left[[v_1, v_2, v_4]_B\, v_3 - [v_1, v_2, v_3]_B\, v_4\right].$$

Also, if $w \equiv v_3 \times_B v_4$, then use Lagrange's formula to get

$$\begin{aligned}(v_1 \times_B v_2) \times_B (v_3 \times_B v_4) &= (v_1 \times_B v_2) \times_B w = -\langle w, v_1, v_2\rangle_B = \langle w, v_2, v_1\rangle_B \\ &= (\det B)\left[(w \cdot_B v_1)\, v_2 - (w \cdot_B v_2)\, v_1\right].\end{aligned}$$

From the Scalar triple product theorem,

$$w \cdot_B v_1 = [v_1, v_3, v_4]_B \quad \text{and} \quad w \cdot_B v_2 = [v_2, v_3, v_4]_B\,.$$

Therefore,

$$(v_1 \times_B v_2) \times_B (v_3 \times_B v_4) = (\det B)\left[[v_1, v_3, v_4]_B\, v_2 - [v_2, v_3, v_4]_B\, v_1\right].$$

Thus, the desired result is obtained. ■

The following corollary will come in handy as we progress through the thesis.

Corollary 8 *For vectors v_1, v_2 and v_3 in $\mathbb{V}^3$, and*

$$M \equiv \begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{pmatrix},$$

we have

$$(v_1 \times_B v_2) \times_B (v_1 \times_B v_3) = \left[(\det B)^2 (\det M)\right] v_1.$$

Proof. By the Quadruple vector product theorem,

$$(v_1 \times_B v_2) \times_B (v_1 \times_B v_3) = (\det B)\left([v_1, v_2, v_3]_B\, v_1 - [v_1, v_2, v_1]_B\, v_3\right).$$

Since $[v_1, v_2, v_1]_B = 0$,

$$(v_1 \times_B v_2) \times_B (v_1 \times_B v_3) = (\det B)\,[v_1, v_2, v_3]_B\, v_1.$$

We then use the Scalar triple product theorem to obtain

$$(v_1 \times_B v_2) \times_B (v_1 \times_B v_3) = (\det B)^2 (\det M) v_1,$$

as required. ■

In addition to the B-quadruple vector product, we can also talk about a **B-quadruple scalar product**, which is the operation given by $(v_1 \times_B v_2) \cdot_B (v_3 \times_B v_4)$. The following result allows us to compute B-quadruple scalar products purely in terms of B-scalar products; this is a generalisation of a result of independent works by Binet [5] and Cauchy [8], as highlighted by Brualdi and Schneider [7], and is thus called the Binet-Cauchy identity. The Euclidean version of the following result is also explained and proven in English in [48, p. 29].

Theorem 9 (Binet-Cauchy identity) *For vectors v_1, v_2, v_3 and v_4 in $\mathbb{V}^3$,*

$$(v_1 \times_B v_2) \cdot_B (v_3 \times_B v_4) = (\det B) [(v_1 \cdot_B v_3)(v_2 \cdot_B v_4) - (v_1 \cdot_B v_4)(v_2 \cdot_B v_3)].$$

Proof. Let $w \equiv v_1 \times_B v_2$, so that by the Scalar triple product theorem,

$$w \cdot_B (v_3 \times_B v_4) = [v_4, w, v_3]_B.$$

By Lagrange's formula,

$$\begin{aligned} w \times_B v_3 &= (v_1 \times_B v_2) \times_B v_3 = -\langle v_3, v_1, v_2 \rangle_B \\ &= -(\det B) [(v_2 \cdot_B v_3) v_1 - (v_1 \cdot_B v_3) v_2] \\ &= (\det B) [(v_1 \cdot_B v_3) v_2 - (v_2 \cdot_B v_3) v_1]. \end{aligned}$$

Hence,

$$\begin{aligned} (v_1 \times_B v_2) \cdot_B (v_3 \times_B v_4) &= ((\det B) [(v_1 \cdot_B v_3) v_2 - (v_2 \cdot_B v_3) v_1]) \cdot v_4 \\ &= (\det B) [(v_1 \cdot_B v_3)(v_2 \cdot_B v_4) - (v_1 \cdot_B v_4)(v_2 \cdot_B v_3)]. \end{aligned}$$

Thus, the desired result is obtained. ■

The following result is a special case of the Binet-Cauchy identity, which will be important for calculations in later parts of the thesis. This is a generalisation of another result of Lagrange [34], which is also shown in English by Steele [49, pp. 37-39].

Theorem 10 (Lagrange's identity) *Given vectors v_1 and v_2 in $\mathbb{V}^3$,*

$$Q_B (v_1 \times_B v_2) = (\det B) \left[Q_B (v_1) Q_B (v_2) - (v_1 \cdot_B v_2)^2 \right].$$

Proof. We start with the Binet-Cauchy identity

$$(v_1 \times_B v_2) \cdot_B (v_3 \times_B v_4) = (\det B) [(v_1 \cdot_B v_3)(v_2 \cdot_B v_4) - (v_1 \cdot_B v_4)(v_2 \cdot_B v_3)].$$

If $v_3 = v_1$ and $v_4 = v_2$, then we substitute these quantities into our equation to obtain the desired result. ■

1.4 Geometric objects in $\mathbb{A}^3$

Besides points, the fundamental objects in the affine geometry of $\mathbb{A}^3$ include lines and planes which we now introduce.

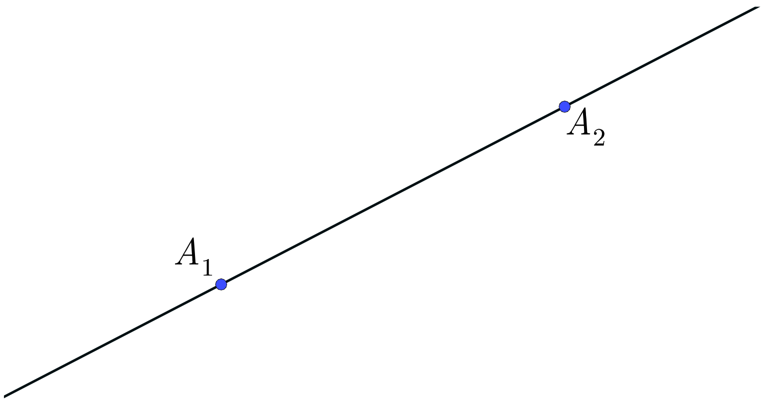

Figure 1.1: Line A_1A_2

A **line** in $\mathbb{A}^3$ is an expression involving two distinct points, say A_1 and A_2, which can be denoted by A_1A_2 (see Figure 1.1); it has the condition that two lines A_1A_2 and B_1B_2 are equal precisely when the points B_1 and B_2 can be expressed in the form $A_1 + \lambda\overrightarrow{A_1A_2}$, for some $\lambda \in \mathbb{F}$. In such a case, we say that the vector $\overrightarrow{A_1A_2}$ (or any non-zero multiple of it) is called a **direction vector** of the line A_1A_2. The line A_1A_2 can be represented by an affine combination, i.e. any point X on the line A_1A_2 can be written as

$$\begin{aligned} X &= A_1 + \lambda\overrightarrow{A_1A_2} = A_1 + \lambda\left(A_2 - A_1\right) \\ &= \left(1-\lambda\right)A_1 + \lambda A_2, \end{aligned}$$

for some $\lambda \in \mathbb{F}$.

A **plane** in $\mathbb{A}^3$ is an expression involving three non-collinear points, say A_1, A_2 and A_3, which is denoted by $A_1A_2A_3$ (see Figure 1.2); it has the condition that two planes $A_1A_2A_3$ and $B_1B_2B_3$ are equal precisely when the points B_1, B_2 and B_3 can be expressed in the form $A_1 + \lambda\overrightarrow{A_1A_2} + \mu\overrightarrow{A_1A_3}$, for some $\lambda, \mu \in \mathbb{F}$. In such a case, we say that the vectors $\overrightarrow{A_1A_2}$ and $\overrightarrow{A_1A_3}$ (or any two non-zero linearly independent vectors which are linear combinations of them) are **spanning vectors** of the plane $A_1A_2A_3$. The plane $A_1A_2A_3$ can be similarly expressed as an affine combination, i.e. for any point X lying on $A_1A_2A_3$,

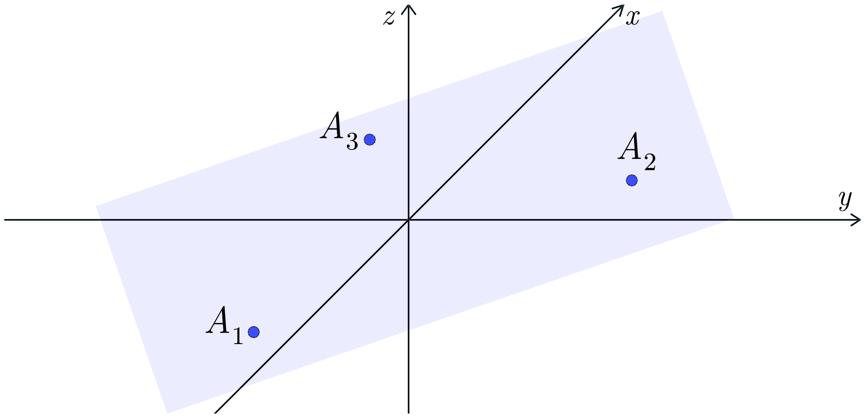

Figure 1.2: Plane $A_0A_1A_2$

we have for $\lambda, \mu \in \mathbb{F}$ that

$$\begin{aligned} X &= A_1 + \lambda \overrightarrow{A_1A_2} + \mu \overrightarrow{A_1A_3} \\ &= A_1 + \lambda (A_2 - A_1) + \mu (A_3 - A_1) \\ &= (1 - \lambda - \mu) A_1 + \lambda A_2 + \mu A_3. \end{aligned}$$

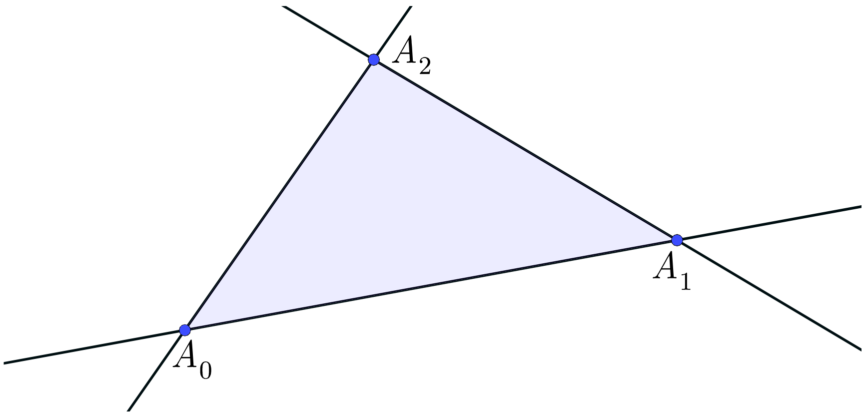

Figure 1.3: Triangle $\overline{A_0A_1A_2}$

A **triangle** in $\mathbb{A}^3$ will be defined as an unordered collection of three points in $\mathbb{A}^3$. Given three points A_0, A_1 and A_2 in $\mathbb{A}^3$, we denote a triangle by $\overline{A_0A_1A_2}$ (see Figure 1.3). In addition to the points A_0, A_1 and A_2, such a triangle also has lines

$$l_{01} \equiv A_0A_1, \quad l_{02} \equiv A_0A_2 \quad \text{and} \quad l_{12} \equiv A_1A_2.$$

We define a **side** of a triangle to be a pair of points which determine a line, and a **vertex** of a triangle to be a pair of lines which determine a point. The triangle $\overline{A_0A_1A_2}$ has sides

$$\overline{A_0A_1}, \quad \overline{A_0A_2} \quad \text{and} \quad \overline{A_1A_2},$$

as well as vertices

$$\overline{l_{01}l_{02}}, \quad \overline{l_{01}l_{12}} \quad \text{and} \quad \overline{l_{02}l_{12}}.$$

Similarly, a **tetrahedron** in $\mathbb{A}^3$ will be defined as an unordered collection of four points in $\mathbb{A}^3$. Given four points A_0, A_1, A_2 and A_3 in $\mathbb{A}^3$, we denote a tetrahedron with these points by $\overline{A_0A_1A_2A_3}$ (see Figure 1.4). In addition to points, such a tetrahedron also has lines

$$l_{01} \equiv A_0A_1, \quad l_{02} \equiv A_0A_2, \quad l_{03} \equiv A_0A_3,$$

$$l_{12} \equiv A_1A_2, \quad l_{13} \equiv A_1A_3 \quad \text{and} \quad l_{23} \equiv A_2A_3,$$

as well as planes

$$A_0A_1A_2, \quad A_0A_1A_3, \quad A_0A_2A_3 \quad \text{and} \quad A_1A_2A_3.$$

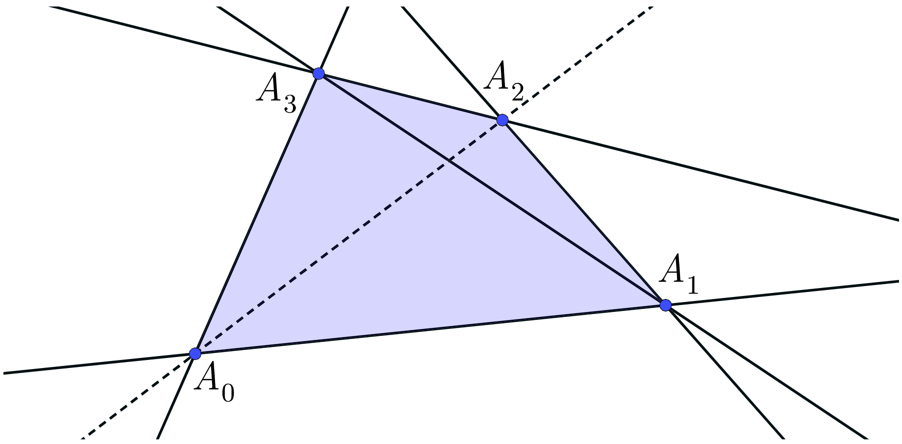

Figure 1.4: Tetrahedron $\overline{A_0A_1A_2A_3}$

The **edge** of a tetrahedron is a collection of two points of it which defines a line, and the **triangle** of a tetrahedron is a collection of three points of it which defines a plane. The six sides of the tetrahedron $\overline{A_0A_1A_2A_3}$ are

$$\overline{A_0A_1}, \quad \overline{A_0A_2}, \quad \overline{A_0A_3}, \quad \overline{A_1A_2}, \quad \overline{A_1A_3} \quad \text{and} \quad \overline{A_2A_3},$$

and the four triangles of the tetrahedron $\overline{A_0A_1A_2A_3}$ are

$$\overline{A_0A_1A_2}, \quad \overline{A_0A_1A_3}, \quad \overline{A_0A_2A_3} \quad \text{and} \quad \overline{A_1A_2A_3}.$$

A **vertex** of a tetrahedron is a collection of two concurrent lines of it. The twelve vertices of the tetrahedron $\overline{A_0A_1A_2A_3}$ are

$$\overline{l_{01}l_{02}}, \quad \overline{l_{01}l_{12}}, \quad \overline{l_{02}l_{12}}, \quad \overline{l_{01}l_{03}}, \quad \overline{l_{01}l_{13}}, \quad \overline{l_{03}l_{13}},$$

$$\overline{l_{02}l_{03}}, \quad \overline{l_{02}l_{23}}, \quad \overline{l_{03}l_{23}}, \quad \overline{l_{12}l_{13}}, \quad \overline{l_{12}l_{23}} \quad \text{and} \quad \overline{l_{13}l_{23}}.$$

A **corner** of a tetrahedron is a collection of three concurrent lines of it. The four corners of the tetrahedron $\overline{A_0A_1A_2A_3}$ are

$$\overline{l_{01}l_{02}l_{03}}, \quad \overline{l_{01}l_{12}l_{13}}, \quad \overline{l_{02}l_{12}l_{23}} \quad \text{and} \quad \overline{l_{03}l_{13}l_{23}}.$$

1.5 Projective geometry

By extending our understanding of geometry from the affine setting to the projective setting, we will introduce some new ideas and notions that will aid us in our study of projective geometry. We specifically use the methods in [55] and [60], emphasising projective geometry from the view of spherical/elliptic geometry.

We will steer away from the usual formulations involving "infinite sets", and rather frame the main definitions in terms of types of objects.

1.5.1 Projective points, lines and triangles

We will associate to $\mathbb{V}^3$, the associated vector space of $\mathbb{A}^3$, the two-dimensional projective space denoted by $\mathbb{P}^2$, where the objects of interest are projective points and projective lines.

A **projective point** p in $\mathbb{P}^2$ is a single vector enclosed in square brackets, with the condition that two projective points p_1 and p_2 are equal precisely when one of the vectors is a (non-zero) scalar multiple of the other. For a non-zero vector $v \equiv (x, y, z) \in \mathbb{V}^3$ and a non-zero scalar $\lambda \in \mathbb{F}$, we can denote a projective point p by

$$p \equiv [v] = [(x, y, z)] = [\lambda v] \,.$$

A **projective line** L in $\mathbb{P}^2$ is a list of two linearly independent vectors enclosed in double square brackets, with the condition that two projective lines L_1 and L_2 are equal precisely when the vectors in L_2 can be written as a linear combination of the vectors in L_1. By elementary linear algebra, this is a symmetric condition. For $v_1 \equiv (x_1, y_1, z_1)$ and $v_2 \equiv (x_2, y_2, z_2)$ in $\mathbb{V}^3$, we may denote such a projective line by

$$L \equiv [[v_1, v_2]] = [[\alpha v_1 + \beta v_2, \gamma v_1 + \delta v_2]] = \left[\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}\right],$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{F}$ satisfy the condition

$$\det \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \alpha\delta - \beta\gamma \neq 0.$$

We note the use of double brackets when defining projective lines. This is to indicate the invariance of the choice of two linearly independent vectors for the projective line under a linear map. Such notation is unnecessary for projective points, as invariance under a 1×1

matrix and invariance under non-zero scalar multiplication are equivalent notions. Hence, we will use single brackets to denote projective points.

We will make an important association between projective points and lines in $\mathbb{P}^2$ with one-dimensional and two-dimensional subspaces (respectively) of $\mathbb{V}^3$, which is central to calculations in projective geometry and trigonometry. We also note that lower-case lettering in projective geometry is reserved for projective points and upper-case lettering in projective geometry is reserved for projective lines. In the case of affine geometry, this notation is reversed; while subtle, this allows us to highlight the natural connection, and distinction, between affine space and projective space.

For $v_1 \equiv (x_1, y_1, z_1)$, $v_2 \equiv (x_2, y_2, z_2)$ and $v_3 \equiv (x_3, y_3, z_3)$, we say that the projective point $p = [v_3]$ is **incident with** the projective line $L = [[v_1, v_2]]$ precisely when

$$\det \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = 0.$$

This is well-defined and equivalent to saying that p **lies on** L, or that L **passes through** p.

Given two distinct projective points $p_1 \equiv [v_1]$ and $p_2 \equiv [v_2]$, we can define the projective line in $\mathbb{P}^2$ passing through p_1 and p_2 to be the **join** of p_1 and p_2, and denote this by

$$p_1 p_2 \equiv [[v_1, v_2]] \,.$$

We will also define the **meet** of two distinct projective lines L_1 and L_2 to be the projective point p that lies on both L_1 and L_2. If

$$L_1 \equiv [[v_1, w_1]] \quad \text{and} \quad L_2 \equiv [[v_2, w_2]]$$

then we denote their meet by

$$L_1 L_2 \equiv [u] \,,$$

where the vector u satisfies the equations

$$\det \begin{pmatrix} - & v_1 & - \\ - & w_1 & - \\ - & u & - \end{pmatrix} = \det \begin{pmatrix} - & v_2 & - \\ - & w_2 & - \\ - & u & - \end{pmatrix} = 0.$$

Given three linearly independent vectors v_1, v_2 and v_3 in $\mathbb{V}^3$ and

$$p_1 \equiv [v_1] \,, \quad p_2 \equiv [v_2] \quad \text{and} \quad p_3 \equiv [v_3] \,,$$

we define a **projective triangle** in $\mathbb{P}^2$ to be an unordered collection of these three projective points and denote this by $\overline{p_1 p_2 p_3}$ (see Figure 1.5). A construction of [64], we will also refer to a projective triangle as a **tripod**.

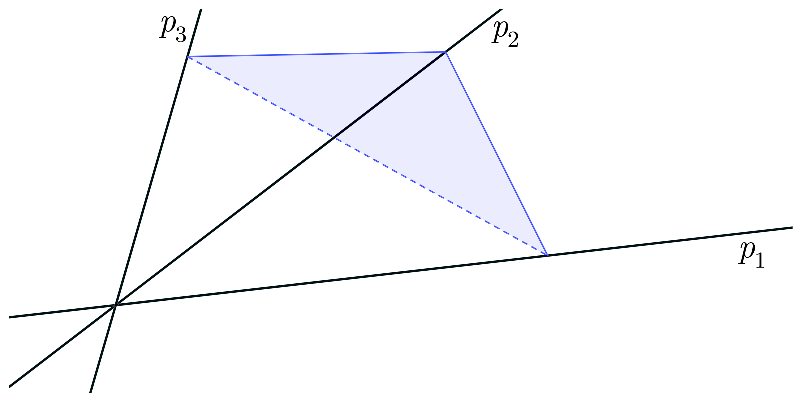

Figure 1.5: Tripod $\overline{p_1p_2p_3}$

With p_1, p_2 and p_3 as the **projective points** of the tripod $\overline{p_1p_2p_3}$, we also have the joins of any two points of it as the **projective lines** and denote them by

$$L_{12} \equiv p_1p_2, \quad L_{13} \equiv p_1p_3 \quad \text{and} \quad L_{23} \equiv p_2p_3.$$

In addition to the points and lines, we take the **projective sides** of the tripod to be the set of any two points of it which defines a line and the **projective vertices** of the tripod to be the set of any two lines of it which defines a point. We will denote the projective sides of the tripod by

$$\overline{p_1p_2}, \quad \overline{p_1p_3} \quad \text{and} \quad \overline{p_2p_3},$$

and the projective vertices by

$$\overline{L_1L_2}, \quad \overline{L_1L_3} \quad \text{and} \quad \overline{L_2L_3}.$$

1.5.2 Symmetric bilinear form, perpendicularity and duality

Since our objects in $\mathbb{P}^2$ can be viewed as familiar one-dimensional and two-dimensional subspaces of $\mathbb{V}^3$, we will perform further calculations on $\mathbb{P}^2$ in terms of $\mathbb{V}^3$.

Recall that a 3×3 symmetric matrix B determines a symmetric bilinear form, or B-scalar product, on $\mathbb{V}^3$ defined by $u \cdot_B v \equiv uBv^T$; in addition to this, we also defined the B-vector product to be $u \times_B v \equiv (u \times v) \operatorname{adj} B$.

We can then express a projective line $L \equiv [[v_1, v_2]]$ as

$$L \equiv \langle v_1 \times_B v_2 \rangle = \langle \lambda \left(v_1 \times_B v_2 \right) \rangle$$

for any non-zero $\lambda \in \mathbb{F}$. For projective points $p_1 \equiv [v_1]$ and $p_2 \equiv [v_2]$, we can define the B-**normal** of the projective line $L \equiv p_1p_2$ to be the projective point

$$p_1 \times_B p_2 \equiv [v_1 \times_B v_2] \, .$$

We now define the B-**dual** of the tripod $\overline{p_1p_2p_3}$ to be the tripod $\overline{r_1r_2r_3}$, where

$$r_1 \equiv p_2 \times_B p_3, \quad r_2 \equiv p_1 \times_B p_3 \quad \text{and} \quad r_3 \equiv p_1 \times_B p_2.$$

Such a tripod, which is another construction of [64], will be called the B-**dual projective**

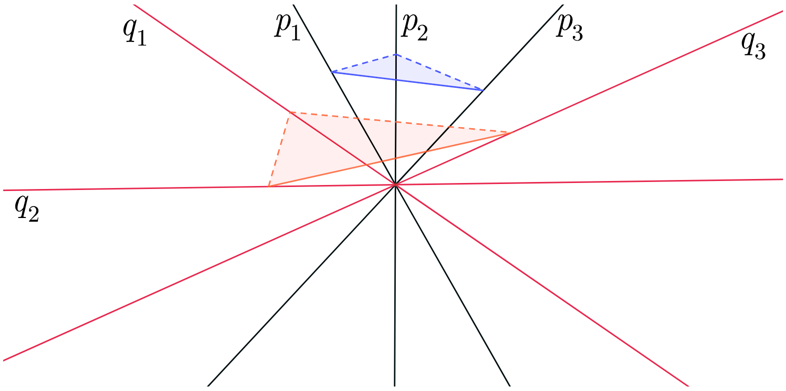

Figure 1.6: B-dual tripod $\overline{q_1q_2q_3}$ of tripod $\overline{p_1p_2p_3}$

triangle of the tripod $\overline{p_1p_2p_3}$ (see Figure 1.6 for a Euclidean example); we will also call this a B-**dual tripod** of $\overline{p_1p_2p_3}$. There is a notion of B-duality for tripods.

Proposition 11 *If the B-dual of the tripod $\overline{p_1p_2p_3}$ is $\overline{r_1r_2r_3}$, then the B-dual of the tripod $\overline{r_1r_2r_3}$ is $\overline{p_1p_2p_3}$.*

Proof. Let $p_1 \equiv [v_1]$, $p_2 \equiv [v_2]$ and $p_3 \equiv [v_3]$, so that

$$r_1 = p_2 \times_B p_3 = [v_2 \times_B v_3]\,, \quad r_2 = p_1 \times_B p_3 = [v_1 \times_B v_3]$$

and

$$r_3 = p_1 \times_B p_2 = [v_1 \times_B v_2]\,.$$

Suppose that the B-dual tripod of $\overline{r_1r_2r_3}$ is given by $\overline{t_1t_2t_3}$, where

$$t_1 \equiv r_2 \times_B r_3, \quad t_2 \equiv r_1 \times_B r_3 \quad \text{and} \quad t_3 \equiv r_1 \times_B r_2.$$

By the definition of the B-normal, we use Corollary 8 to get

$$\begin{aligned} t_1 &= [(v_1 \times_B v_3) \times_B (v_1 \times_B v_2)] \\ &= [(\det B)\,[v_1, v_3, v_2]_B\, v_1]\,. \end{aligned}$$

Since B is non-degenerate and the vectors v_1, v_2 and v_3 are linearly independent, $(\det B)\,[v_1, v_3, v_2]_B \neq 0$ and thus by the definition of a projective point

$$t_1 = [v_1] = p_1.$$

By symmetry, we must have that $t_2 = p_2$ and $t_3 = p_3$, and hence $\overline{t_1 t_2 t_3} = \overline{p_1 p_2 p_3}$. Thus, the B-dual tripod of $\overline{r_1 r_2 r_3}$ is $\overline{p_1 p_2 p_3}$. ■

1.6 Rational trigonometric quantities

We now proceed from the geometry of $\mathbb{A}^3$ and $\mathbb{P}^2$ to the affine and projective rational trigonometry in the respective spaces. We assume that the associated vector space $\mathbb{V}^3$ of $\mathbb{A}^3$ is equipped with a symmetric bilinear form with matrix representation

$$B = \begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix}.$$

1.6.1 Affine rational trigonometry

We first extend Wildberger's definition of the quadrance to an arbitrary symmetric bilinear form. The B-**quadrance** between two points A_1 and A_2 in $\mathbb{A}^3$ is

$$Q_B\left(A_1, A_2\right) \equiv Q_B\left(\overrightarrow{A_1 A_2}\right) = \overrightarrow{A_1 A_2} \cdot_B \overrightarrow{A_1 A_2}.$$

In what follows, define **Archimedes' function** [59, p. 64] as

$$A\left(a, b, c\right) \equiv \left(a+b+c\right)^2 - 2\left(a^2+b^2+c^2\right).$$

Simple algebraic rewriting gives us also the asymmetric forms

$$A\left(a, b, c\right) = 4ab - \left(a+b-c\right)^2 = 4ac - \left(a+c-b\right)^2 = 4bc - \left(b+c-a\right)^2.$$

Wildberger introduced Archimedes' function to give a rational analog to the well-known Heron's formula, as shown in English by Heath [27, pp. 321-323]. The name comes from the fact that Arab sources have attributed this formula to Archimedes, as highlighted in [58]. For a triangle $\overline{A_0 A_1 A_2}$ in the planar Euclidean setting with B-quadrances

$$Q_{01} \equiv Q_B\left(A_0, A_1\right), \quad Q_{02} \equiv Q_B\left(A_0, A_2\right) \quad \text{and} \quad Q_{12} \equiv Q_B\left(A_1, A_2\right),$$

[59, p. 68] showed that the quantity $A\left(Q_{01}, Q_{02}, Q_{12}\right)$ is equal to 16 times the squared area of $\overline{A_0 A_1 A_2}$; note that in the Euclidean setting, the matrix B is the 2×2 identity matrix. Motivated by this, we can use Archimedes' function and our vector formulation to obtain a new result for rational trigonometry in three dimensions.

Theorem 12 (Three-dimensional quadrea theorem) *Given a triangle $\overline{A_0 A_1 A_2}$ in $\mathbb{A}^3$, suppose that $v_{01} \equiv \overrightarrow{A_0 A_1}$, $v_{02} \equiv \overrightarrow{A_0 A_2}$ and $v_{12} \equiv \overrightarrow{A_1 A_2}$, and let*

$$Q_{01} \equiv Q_B\left(A_0, A_1\right), \quad Q_{02} \equiv Q_B\left(A_0, A_2\right) \quad \textit{and} \quad Q_{12} \equiv Q_B\left(A_1, A_2\right).$$

Then,

$$Q_B(v_{01} \times_B v_{02}) = Q_B(v_{01} \times_B v_{12}) = Q_B(v_{02} \times_B v_{12}) = \frac{\det B}{4} A(Q_{01}, Q_{02}, Q_{12}).$$

Proof. If $v_{01} \equiv \overrightarrow{A_0A_1}$ and $v_{02} \equiv \overrightarrow{A_0A_2}$, Lagrange's identity gives

$$Q_B(v_{01} \times_B v_{02}) = (\det B)\left[Q_B(v_{01}) Q_B(v_{02}) - (v_{01} \cdot_B v_{02})^2\right].$$

Since

$$v_{01} \cdot_B v_{02} = \frac{Q_{01} + Q_{02} - Q_{12}}{2},$$

we have

$$\begin{aligned} Q_B(v_{01} \times_B v_{02}) &= (\det B)\left[Q_{01}Q_{02} - \left(\frac{Q_{01} + Q_{02} - Q_{12}}{2}\right)^2\right] \\ &= \frac{\det B}{4}\left[4Q_{01}Q_{02} - (Q_{01} + Q_{02} - Q_{12})^2\right] \\ &= \frac{\det B}{4} A(Q_{01}, Q_{02}, Q_{12}). \end{aligned}$$

By symmetry, we will obtain the same result when computing $Q_B(v_{01} \times_B v_{12})$ and $Q_B(v_{02} \times_B v_{12})$. ∎

This result motivates us to define the B-**quadrea** of a triangle $\overline{A_0A_1A_2}$ with B-quadrances

$$Q_{01} \equiv Q_B(A_0, A_1), \quad Q_{02} \equiv Q_B(A_0, A_2) \quad \text{and} \quad Q_{12} \equiv Q_B(A_1, A_2)$$

to be the quantity

$$\mathcal{A}_B\left(\overline{A_0A_1A_2}\right) \equiv A(Q_{01}, Q_{02}, Q_{12}),$$

so that

$$\begin{aligned} \mathcal{A}_B\left(\overline{A_0A_1A_2}\right) &= \frac{4}{\det B} Q_B\left(\overrightarrow{A_0A_1} \times_B \overrightarrow{A_0A_2}\right) \\ &= \frac{4}{\det B} Q_B\left(\overrightarrow{A_1A_0} \times_B \overrightarrow{A_1A_2}\right) \\ &= \frac{4}{\det B} Q_B\left(\overrightarrow{A_2A_0} \times_B \overrightarrow{A_2A_1}\right). \end{aligned}$$

This is an extension of the usual definition of quadrea as given in [59, p. 68].

If we have a tetrahedron $\overline{A_0A_1A_2A_3}$ in $\mathbb{A}^3$ with $v_1 \equiv \overrightarrow{A_0A_1}$, $v_2 \equiv \overrightarrow{A_0A_2}$ and $v_3 \equiv \overrightarrow{A_0A_3}$, then we define its B-**quadrume** as

$$\mathcal{V}_B\left(\overline{A_0A_1A_2A_3}\right) \equiv 144(\det B)(v(v_1, v_2, v_3))^2.$$

We can express the B-quadrume of the tetrahedron $\overline{A_0A_1A_2A_3}$ in terms of its B-quadrances; this result will explain the choice of the factor 144 in the definition.

Theorem 13 (Quadrume theorem) *For a tetrahedron $\overline{A_0A_1A_2A_3}$ in $\mathbb{A}^3$, define $Q_{ij} \equiv Q_B(A_i, A_j)$, for $0 \le i < j \le 3$. The B-quadrume of the tetrahedron $\overline{A_0A_1A_2A_3}$ can be expressed as*

$$\mathcal{V}_B\left(\overline{A_0A_1A_2A_3}\right) = \frac{1}{2}\begin{vmatrix} 2Q_{01} & Q_{01}+Q_{02}-Q_{12} & Q_{01}+Q_{03}-Q_{13} \\ Q_{01}+Q_{02}-Q_{12} & 2Q_{02} & Q_{02}+Q_{03}-Q_{23} \\ Q_{01}+Q_{03}-Q_{13} & Q_{02}+Q_{03}-Q_{23} & 2Q_{03} \end{vmatrix}.$$

Proof. Let $v_1 \equiv \overrightarrow{A_0A_1}$, $v_2 \equiv \overrightarrow{A_0A_2}$ and $v_3 \equiv \overrightarrow{A_0A_3}$, and let

$$M \equiv \begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{pmatrix}.$$

Using the Scalar triple product theorem, we have from the definition of the B-quadrume that

$$\begin{aligned} \mathcal{V}_B\left(\overline{A_0A_1A_2A_3}\right) &= 144 \det B \left[\frac{\det(M)}{6}\right]^2 \\ &= 4 \det\left(MBM^T\right) \\ &= 4\begin{vmatrix} v_1 \cdot_B v_1 & v_1 \cdot_B v_2 & v_1 \cdot_B v_3 \\ v_1 \cdot_B v_2 & v_2 \cdot_B v_2 & v_2 \cdot_B v_3 \\ v_1 \cdot_B v_3 & v_2 \cdot_B v_3 & v_3 \cdot_B v_3 \end{vmatrix}. \end{aligned}$$

The diagonal entries evaluate to

$$v_1 \cdot_B v_1 = Q_B(v_1) = Q_{01}, \quad v_2 \cdot_B v_2 = Q_B(v_2) = Q_{02}$$

and

$$v_3 \cdot_B v_3 = Q_B(v_3) = Q_{03}.$$

By the polarisation formula, we also have

$$v_1 \cdot_B v_2 = \frac{Q_{01}+Q_{02}-Q_{12}}{2}, \quad v_1 \cdot_B v_3 = \frac{Q_{01}+Q_{03}-Q_{13}}{2}$$

and

$$v_2 \cdot_B v_3 = \frac{Q_{02}+Q_{03}-Q_{23}}{2}.$$

Substitute the above six quantities into the expression for $\mathcal{V}_B\left(\overline{A_0A_1A_2A_3}\right)$ and simplify to get

$$\mathcal{V}_B\left(\overline{A_0A_1A_2A_3}\right) = \frac{1}{2}\begin{vmatrix} 2Q_{01} & Q_{01}+Q_{02}-Q_{12} & Q_{01}+Q_{03}-Q_{13} \\ Q_{01}+Q_{02}-Q_{12} & 2Q_{02} & Q_{02}+Q_{03}-Q_{23} \\ Q_{01}+Q_{03}-Q_{13} & Q_{02}+Q_{03}-Q_{23} & 2Q_{03} \end{vmatrix}.$$

■

Remark 14 *The B-quadrume $\mathcal{V}_B\left(\overline{A_0A_1A_2A_3}\right)$ can be written as the polynomial*

$$\begin{aligned}&4Q_{01}Q_{02}Q_{03}+\left(Q_{01}+Q_{02}-Q_{12}\right)\left(Q_{01}+Q_{03}-Q_{13}\right)\left(Q_{02}+Q_{03}-Q_{23}\right)\\&-Q_{01}\left(Q_{02}+Q_{03}-Q_{23}\right)^2-Q_{02}\left(Q_{01}+Q_{03}-Q_{13}\right)^2-Q_{03}\left(Q_{01}+Q_{02}-Q_{12}\right)^2,\end{aligned}$$

which is a symmetric expression in the B-quadrances. This means that the choice of vectors in the definition of the B-quadrume is arbitrary, as long as they emanate from a single point of the tetrahedron $\overline{A_0A_1A_2A_3}$.

The following result is of use to us as we progress throughout the thesis.

Theorem 15 (Quadrume matrix product theorem) *For a tetrahedron $\overline{A_0A_1A_2A_3}$ in $\mathbb{A}^3$, let $v_i \equiv \overrightarrow{A_0A_i}$ for $i=1,2,3$ and let*

$$M \equiv \begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{pmatrix}.$$

Then,

$$\mathcal{V}_B\left(\overline{A_0A_1A_2A_3}\right)=4\det\left(MBM^T\right).$$

Proof. This is a direct result from the proof of the Quadrume theorem. ■

The determinant present in the definition is called the **Cayley-Menger determinant**. Appearing in Audet [3], Dörrie [19, pp. 285-289] and Sommerville [47, pp. 124-126], this determinant forms the basis for calculating higher-dimensional trigonometric quantities. While named after Cayley and Menger, this formula was known to Euler and dates back to works of Tartaglia.

Given two lines l_1 and l_2 in $\mathbb{A}^3$ with respective direction vectors v_1 and v_2, we define the B-**spread** between them to be

$$s_B\left(l_1,l_2\right)\equiv 1-\frac{\left(v_1\cdot_B v_2\right)^2}{Q_B\left(v_1\right)Q_B\left(v_2\right)},$$

which by Lagrange's identity can be rewritten as

$$s_B\left(l_1,l_2\right)=\frac{Q_B\left(v_1\times_B v_2\right)}{\left(\det B\right)Q_B\left(v_1\right)Q_B\left(v_2\right)}.$$

While the former expression is an extension of Wildberger's definition of the spread [59, p. 73], the latter expression is a reformulation in terms of the B-vector product. Complementary to the spread, we will also define the B-**cross** between the same two lines l_1 and l_2 to be

$$c_B\left(l_1,l_2\right)\equiv 1-s_B\left(l_1,l_2\right)=\frac{\left(v_1\cdot_B v_2\right)^2}{Q_B\left(v_1\right)Q_B\left(v_2\right)}.$$

This has been defined in the Euclidean case in [59, p. 74].

In what follows, we can represent a plane Π in $\mathbb{A}^3$ with spanning vectors v and w in terms of a B-**normal vector** $n \equiv v \times_B w$, so that any point X on Π which passes through a given point A satisfies the equation

$$\overrightarrow{AX} \cdot_B n = 0.$$

So, given two planes Π_1 and Π_2 in $\mathbb{A}^3$ with B-normal vectors n_1, n_2, we define the B-**dihedral spread**, or just the **dihedral spread**, between them to be

$$E_B(\Pi_1, \Pi_2) \equiv 1 - \frac{(n_1 \cdot_B n_2)^2}{Q_B(n_1)\, Q_B(n_2)}.$$

This can be rewritten using Lagrange's identity as

$$E_B(\Pi_1, \Pi_2) = \frac{Q_B(n_1 \times_B n_2)}{(\det B)\, Q_B(n_1)\, Q_B(n_2)}.$$

The B-dihedral spread was introduced for the Euclidean setting in [63] as a rational analog to dihedral angles between planes in three dimensions. We will also define the B-**dihedral cross** between the same two planes Π_1 and Π_2 to be

$$C_B(\Pi_1, \Pi_2) \equiv 1 - E_B(\Pi_1, \Pi_2) = \frac{(n_1 \cdot_B n_2)^2}{Q_B(n_1)\, Q_B(n_2)}.$$

Note that the definitions of B-spread, B-cross, B-dihedral spread and B-dihedral cross are well-defined, i.e. the quantities do not vary when the vectors are varied under non-zero scalar multiplication.

Suppose now we have three concurrent lines l_1, l_2 and l_3 in $\mathbb{A}^3$ with respective direction vectors v_1, v_2 and v_3. We define the B-**solid spread** between l_1, l_2 and l_3 to be

$$\mathcal{S}_B(l_1, l_2, l_3) \equiv \frac{([v_1, v_2, v_3]_B)^2}{(\det B)\, Q_B(v_1)\, Q_B(v_2)\, Q_B(v_3)}.$$

With the inclusion of the factor $\det B$ in the denominator, we have that the B-solid spread is a well-defined quantity, i.e. for a non-zero scalar $\lambda \in \mathbb{F}$,

$$\begin{aligned}
\mathcal{S}_{\lambda B}(l_1, l_2, l_3) &= \frac{([v_1, v_2, v_3]_{\lambda B})^2}{(\det \lambda B)\, Q_{\lambda B}(v_1)\, Q_{\lambda B}(v_2)\, Q_{\lambda B}(v_3)} \\
&= \frac{\left([v_1, v_2, v_3]\left(\lambda^3 \det B\right)\right)^2}{\lambda^3\left(\lambda^3 \det B\right) Q_B(v_1)\, Q_B(v_2)\, Q_B(v_3)} \\
&= \frac{\lambda^6 ([v_1, v_2, v_3]_B)^2}{\lambda^6 (\det B)\, Q_B(v_1)\, Q_B(v_2)\, Q_B(v_3)} \\
&= \mathcal{S}_B(l_1, l_2, l_3),
\end{aligned}$$

and if we multiply any of the direction vectors of l_1, l_2 or l_3 by a non-zero scalar then the quantity remains invariant.

B-solid spreads are rational analogs of the solid angle, or the spherical excess, of a spherical triangle; a definition of Girard's but explained in English by Todhunter [53, pp. 72-73], the notion of a solid spread was introduced in [63] for the Euclidean setting. In the next chapter we will link the definition of solid spread to certain results in projective geometry.

We can also consider a rational analog of the solid angle of a dual spherical triangle [24]. Given the same lines l_1, l_2 and l_3 as before, we construct concurrent lines k_{12}, k_{13} and k_{23} with respective direction vectors

$$n_{12} \equiv v_1 \times_B v_2, \quad n_{13} \equiv v_1 \times_B v_3 \quad \text{and} \quad n_{23} \equiv v_2 \times_B v_3.$$

We define the B-**dual solid spread** between l_1, l_2 and l_3 to be

$$\mathcal{D}_B\left(l_1, l_2, l_3\right) \equiv \mathcal{S}_B\left(k_{12}, k_{13}, k_{23}\right).$$

Note again that this quantity is well-defined, since the solid spread is well-defined. So, the B-dual solid spread associated to a tripod is the B-solid spread associated to its dual tripod. This is a novel quantity in the affine case, for which its significance and its relevance to projective geometry will be highlighted later on.

1.6.2 Elementary results from affine rational trigonometry

We present some results with regards to the trigonometric quantities we have just defined. Here, we will restrict our presentation to the two-dimensional aspects of trigonometry, and create a separate chapter for a discussion on the trigonometry of the tetrahedron.

In what follows, we will consider a triangle $\overline{A_0A_1A_2}$ in $\mathbb{A}^3$ with B-quadrances

$$Q_{01} \equiv Q_B\left(A_0, A_1\right), \quad Q_{02} \equiv Q_B\left(A_0, A_2\right) \quad \text{and} \quad Q_{12} \equiv Q_B\left(A_1, A_2\right),$$

B-spreads

$$s_0 \equiv s_B\left(A_0A_1, A_0A_2\right), \quad s_1 \equiv s_B\left(A_0A_1, A_1A_2\right) \quad \text{and} \quad s_2 \equiv s_B\left(A_0A_2, A_1A_2\right),$$

and B-quadrea

$$\mathcal{A} \equiv \mathcal{A}_B\left(\overline{A_0A_1A_2}\right).$$

We illustrate the trigonometric quantities associated to $\overline{A_0A_1A_2}$ in Figure 1.7. Note that throughout the thesis we will follow Wildberger's notation of using small rectangles to denote B-quadrances in our diagrams, and straight line segments at vertices to denote B-spreads.

We draw on already-proven results from [59], but offer alternative proofs using the

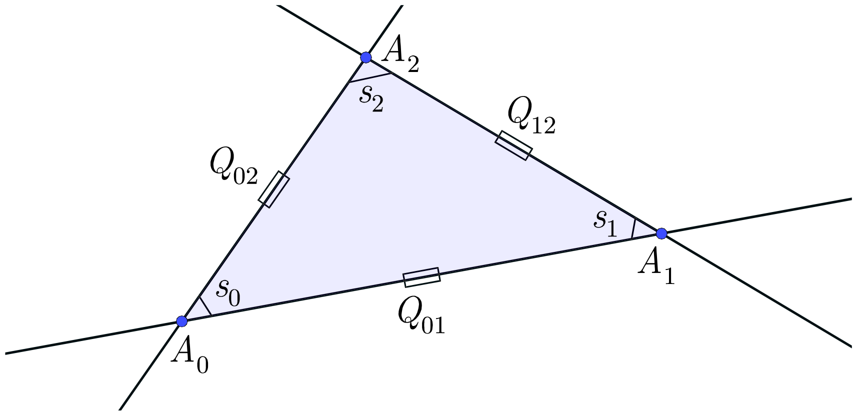

Figure 1.7: A triangle $\overline{A_0A_1A_2}$ with B-quadrances Q_{01}, Q_{02} and Q_{12}, and B-spreads s_0, s_1 and s_2 displayed

three-dimensional framework we have set up in this chapter. [59] dealt with the two-dimensional situation, and [55] discussed the same treatment in higher dimensions. In our three-dimensional framework, the B-vector product allows for a special treatment and we aim to prove the results now for an arbitrary symmetric bilinear form.

The first result is an analog of the cosine law in classical trigonometry.

Theorem 16 (Cross law) *For a triangle $\overline{A_0A_1A_2}$ in $\mathbb{A}^3$ with B-quadrances Q_{01}, Q_{02} and Q_{12}, and B-spreads s_0, s_1 and s_2, the relations*

$$(Q_{01}+Q_{02}-Q_{12})^2 = 4Q_{01}Q_{02}\,(1-s_0)\,,$$

$$(Q_{01}+Q_{12}-Q_{02})^2 = 4Q_{01}Q_{12}\,(1-s_1)$$

and

$$(Q_{02}+Q_{12}-Q_{01})^2 = 4Q_{02}Q_{12}\,(1-s_2)$$

are satisfied.

Proof. Let $v_1 \equiv \overrightarrow{A_0A_1}$ and $v_2 \equiv \overrightarrow{A_0A_2}$, so that by the polarisation formula,

$$v_1 \cdot_B v_2 = \frac{Q_{01}+Q_{02}-Q_{12}}{2}.$$

Rearrange this result to get

$$Q_{01}+Q_{02}-Q_{12} = 2\,(v_1 \cdot_B v_2)$$

and then square both sides to get

$$(Q_{01}+Q_{02}-Q_{12})^2 = 4\,(v_1 \cdot_B v_2)^2\,.$$

Since v_1 and v_2 are the respective direction vectors of the lines A_0A_1 and A_0A_2, the definition of the B-spread gives

$$s_0 = 1 - \frac{(v_1 \cdot_B v_2)^2}{Q_B(v_1) Q_B(v_2)}.$$

Rearrange this to get

$$(v_1 \cdot_B v_2)^2 = Q_B(v_1) Q_B(v_2)(1 - s_0) = Q_{01} Q_{02} (1 - s_0)$$

and so

$$(Q_{01} + Q_{02} - Q_{12})^2 = 4Q_{01}Q_{02}(1 - s_0).$$

The other relations follow by symmetry. ■

Note that we can rewrite the Cross law in terms of the crosses

$$c_0 \equiv 1 - s_0, \quad c_1 \equiv 1 - s_1 \quad \text{and} \quad c_2 \equiv 1 - s_2.$$

We can use the Cross law as a fundamental building block for a number of other results. For instance, we can express the B-quadrea of a triangle in terms of its B-quadrances and B-spreads.

Theorem 17 (Quadrea spread theorem) *For a triangle $\overline{A_0A_1A_2}$ in $\mathbb{A}^3$ with B-quadrances Q_{01}, Q_{02} and Q_{12}, B-spreads s_0, s_1 and s_2, and B-quadrea $\mathcal{A}$, we have*

$$\mathcal{A} = 4Q_{01}Q_{02}s_0 = 4Q_{01}Q_{12}s_1 = 4Q_{02}Q_{12}s_2.$$

Proof. From one of the Cross law relations

$$(Q_{01} + Q_{02} - Q_{12})^2 = 4Q_{01}Q_{02}(1 - s_0),$$

rearrange to get

$$4Q_{01}Q_{02} - (Q_{01} + Q_{02} - Q_{12})^2 = 4Q_{01}Q_{02}s_0.$$

Rewrite the left hand side to obtain

$$A(Q_{01}, Q_{02}, Q_{12}) = \mathcal{A} = 4Q_{01}Q_{02}s_0.$$

The other results follow by symmetry. ■

We can use the Quadrea spread theorem to determine whether three points in $\mathbb{A}^3$ lie on the same line.

Theorem 18 (Triple quad formula) *Let A_0, A_1 and A_2 be three points in $\mathbb{A}^3$ and*

$$Q_{01} \equiv Q_B(A_0, A_1), \quad Q_{02} \equiv Q_B(A_0, A_2) \quad \textit{and} \quad Q_{12} \equiv Q_B(A_1, A_2).$$

If A_0, A_1 and A_2 are collinear, then

$$\left(Q_{01}+Q_{02}+Q_{12}\right)^2 = 2\left(Q_{01}^2+Q_{02}^2+Q_{12}^2\right).$$

Proof. Consider the triangle $\overline{A_0A_1A_2}$ with B-quadrances Q_{01}, Q_{02} and Q_{12}, B-spreads s_0, s_1 and s_2, and B-quadrea $\mathcal{A}$. If A_0, A_1 and A_2 are collinear, then $s_0 = s_1 = s_2 = 0$ and thus $\mathcal{A} = 0$ by the Quadrea spread theorem. By the definition of the B-quadrea,

$$A\left(Q_{01},Q_{02},Q_{12}\right) = \left(Q_{01}+Q_{02}+Q_{12}\right)^2 - 2\left(Q_{01}^2+Q_{02}^2+Q_{12}^2\right) = 0.$$

Rearrange the equation to obtain

$$\left(Q_{01}+Q_{02}+Q_{12}\right)^2 = 2\left(Q_{01}^2+Q_{02}^2+Q_{12}^2\right).$$

Thus, the desired result is obtained. ■

The Cross law also gives the most important result in geometry and trigonometry: Pythagoras' theorem.

Theorem 19 (Pythagoras' theorem) *For a triangle $\overline{A_0A_1A_2}$ in $\mathbb{A}^3$ with B-quadrances Q_{01}, Q_{02} and Q_{12}, and B-spreads s_0, s_1 and s_2, $s_0 = 1$ precisely when*

$$Q_{01}+Q_{02} = Q_{12}.$$

Proof. Start with the Cross law relation

$$\left(Q_{01}+Q_{02}-Q_{12}\right)^2 = 4Q_{01}Q_{02}\left(1-s_0\right).$$

If $s_0 = 1$, then

$$4Q_{01}Q_{02}\left(1-s_0\right) = 0.$$

This gives

$$Q_{01}+Q_{02}-Q_{12} = 0$$

and hence

$$Q_{01}+Q_{02} = Q_{12}.$$

Conversely, if $Q_{01}+Q_{02} = Q_{12}$ then, $Q_{01}+Q_{02}-Q_{12} = 0$ and we thus have that

$$\begin{aligned}
\mathcal{A}_B\left(\overline{A_0A_1A_2}\right) &= \left(Q_{01}+Q_{02}+Q_{12}\right)^2 - 2\left(Q_{01}^2+Q_{02}^2+Q_{12}^2\right) \\
&= 4Q_{01}Q_{02} - \left(Q_{01}+Q_{02}-Q_{12}\right)^2 \\
&= 4Q_{01}Q_{02}.
\end{aligned}$$

Comparing this result with the Quadrea spread theorem, we must conclude that $s_0 = 1$ as required. ■

One other use of the Quadrea spread theorem is in determining ratios between B-

spreads and B-quadrances, which is a rational analog of the sine law in classical trigonometry.

Theorem 20 (Spread law) *For a triangle $\overline{A_0A_1A_2}$ with B-quadrances Q_{01}, Q_{02} and Q_{12}, B-spreads s_0, s_1 and s_2, and B-quadrea $\mathcal{A}$, the following relation is satisfied:*

$$\frac{s_0}{Q_{12}} = \frac{s_1}{Q_{02}} = \frac{s_2}{Q_{01}} = \frac{\mathcal{A}}{4Q_{01}Q_{02}Q_{12}}.$$

Proof. We start with the Quadrea spread theorem, which is

$$\mathcal{A} = 4Q_{01}Q_{02}s_0 = 4Q_{01}Q_{12}s_1 = 4Q_{02}Q_{12}s_2.$$

Compute s_0, s_1 and s_2 in terms of this result to get

$$s_0 = \frac{\mathcal{A}}{4Q_{01}Q_{02}}, \quad s_1 = \frac{\mathcal{A}}{4Q_{01}Q_{12}} \quad \text{and} \quad s_2 = \frac{\mathcal{A}}{4Q_{02}Q_{12}}.$$

Divide s_0, s_1 and s_2 by Q_{12}, Q_{02} and Q_{01} respectively to get

$$\frac{s_0}{Q_{12}} = \frac{s_1}{Q_{02}} = \frac{s_2}{Q_{01}} = \frac{\mathcal{A}}{4Q_{01}Q_{02}Q_{12}},$$

as required. ■

We will end this section by presenting a result that gives a relationship between the three B-spreads of a triangle, following the proof in [59, pp. 89-90] which was presented in the Euclidean setting.

Theorem 21 (Triple spread formula) *For a triangle $\overline{A_0A_1A_2}$ in $\mathbb{A}^3$ with B-spreads s_0, s_1 and s_2, we have the relation*

$$(s_0 + s_1 + s_2)^2 = 2\left(s_0^2 + s_1^2 + s_2^2\right) + 4s_0s_1s_2.$$

Proof. From the result of the Spread law, let

$$D \equiv \frac{4Q_{01}Q_{02}Q_{12}}{\mathcal{A}},$$

so that

$$Q_{01} = Ds_2, \quad Q_{02} = Ds_1 \quad \text{and} \quad Q_{12} = Ds_0.$$

Given the Cross law

$$(Q_{01} + Q_{02} - Q_{12})^2 = 4Q_{01}Q_{02}\left(1 - s_0\right),$$

we substitute our initial calculations and rearrange to get

$$D^2\left(s_0 + s_1 - s_2\right)^2 = 4D^2 s_0 s_1 \left(1 - s_2\right).$$

Divide by D^2 and rearrange this formula to obtain

$$4s_0s_1 - (s_0 + s_1 - s_2)^2 = 4s_0s_1s_2.$$

We use Archimedes' function to express this result as

$$(s_0 + s_1 + s_2)^2 - 2\left(s_0^2 + s_1^2 + s_2^2\right) = 4s_0s_1s_2$$

and rearrange to get

$$(s_0 + s_1 + s_2)^2 = 2\left(s_0^2 + s_1^2 + s_2^2\right) + 4s_0s_1s_2.$$

By symmetry, we will obtain the same result when using the other two results from the Cross law. ■

In classical Euclidean trigonometry, the Triple spread formula corresponds to the fact that angles in a triangle sum up to two right angles, as propositioned and proven in *Elements*. Here, we are generalising such a result to arbitrary geometries, where this simple fact does not hold.

1.6.3 Projective rational trigonometry

Rational trigonometry has an affine and projective version. The projective version is typically more algebraically involved. The distinction was first laid out in [64] by framing hyperbolic geometry in a projective setting. So, the projective results are the essential formulas for the rational trigonometry approach to both hyperbolic and spherical/elliptic trigonometry. For us, the spherical/elliptic interpretation, as seen in Moritz [42] and Todhunter [53], is key.

An important consequence of projective geometry is that any statement made about projective points will also hold with regard to projective lines; this is the **principle of duality** explained by Coxeter [16, pp. 15-16], which is the idea that there is a symmetry in the roles of projective points and projective lines in the projective plane.

In what follows, we already have a symmetric bilinear form equipped on $\mathbb{V}^3$ with matrix representation

$$B = \begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix}.$$

We define the B-**projective quadrance** between two projective points $p_1 \equiv [v_1]$ and $p_2 \equiv [v_2]$ in $\mathbb{P}^2$ to be

$$q_B(p_1, p_2) \equiv 1 - \frac{(v_1 \cdot_B v_2)^2}{Q_B(v_1)\, Q_B(v_2)} = \frac{Q_B(v_1 \times_B v_2)}{(\det B)\, Q_B(v_1)\, Q_B(v_2)}.$$

Dually, we will define the B-**projective spread,** between two projective lines $L_1 \equiv \langle n_1 \rangle$

and $L_2 \equiv \langle n_2 \rangle$ in $\mathbb{P}^2$ to be

$$S_B(L_1, L_2) \equiv 1 - \frac{(n_1 \cdot_B n_2)^2}{Q_B(n_1) Q_B(n_2)} = \frac{Q_B(n_1 \times_B n_2)}{(\det B) Q_B(n_1) Q_B(n_2)}.$$

It is important for us to observe the similarities in the definition of B-spread in affine rational trigonometry and the definition of B-projective quadrance above, as well as the similarities in the definition of B-dihedral spread and the definition of B-projective spread above. This is natural since projective points and lines can be associated with one-dimensional and two-dimensional subspaces of $\mathbb{V}^3$ respectively, and thus will also be associated with lines and planes in $\mathbb{A}^3$ respectively.

1.6.4 Elementary results from projective rational trigonometry

We now proceed to present results in projective rational trigonometry, which draw on the results from [55] and [60], but will be framed in the three-dimensional framework using B-vector products and a general symmetric bilinear form.

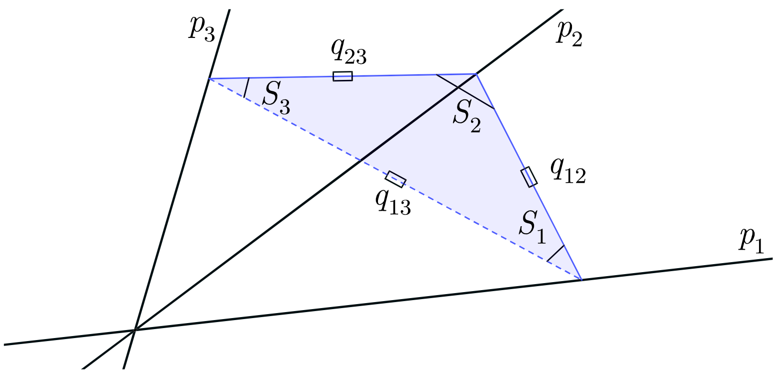

Figure 1.8: Tripod $\overline{p_1p_2p_3}$ with B-projective quadrances q_{12}, q_{13} and q_{23}, and B-projective spreads S_1, S_2 and S_3 displayed

In what follows, we have three linearly independent vectors $v_1, v_2, v_3 \in \mathbb{V}^3$ and take the tripod $\overline{p_1p_2p_3}$ with projective points

$$p_1 \equiv [v_1], \quad p_2 \equiv [v_2] \quad \text{and} \quad p_3 \equiv [v_3].$$

The B-projective quadrances of the tripod $\overline{p_1p_2p_3}$ are

$$q_{12} \equiv q_B(p_1, p_2), \quad q_{13} \equiv q_B(p_1, p_3) \quad \text{and} \quad q_{23} \equiv q_B(p_2, p_3)$$

and the B-projective spreads are

$$S_1 \equiv S_B(p_1p_2, p_1p_3), \quad S_2 \equiv S_B(p_1p_2, p_2p_3) \quad \text{and} \quad S_3 \equiv S_B(p_1p_3, p_2p_3).$$

We will also consider the B-dual tripod $\overline{r_1r_2r_3}$ of $\overline{p_1p_2p_3}$, where

$$r_1 \equiv p_2 \times_B p_3, \quad r_2 \equiv p_1 \times_B p_3 \quad \text{and} \quad r_3 \equiv p_1 \times_B p_2.$$

As a result of Proposition 11, the B-projective quadrances of $\overline{r_1r_2r_3}$ will be S_1, S_2 and S_3, and the B-projective spreads of $\overline{r_1r_2r_3}$ will be q_{12}, q_{13} and q_{23}. We illustrate the trigonometric quantities of $\overline{p_1p_2p_3}$ in Figure 1.8.

We now present a result regarding the ratio between B-projective spreads and B-projective quadrances.

Theorem 22 (Projective spread law) *Given a tripod $\overline{p_1p_2p_3}$ with B-projective quadrances q_{12}, q_{13} and q_{23}, and B-projective spreads S_1, S_2 and S_3, we have the relation*

$$\frac{S_1}{q_{23}} = \frac{S_2}{q_{13}} = \frac{S_3}{q_{12}}.$$

Proof. If the points of the tripod $\overline{p_1p_2p_3}$ are given by

$$p_1 \equiv [v_1], \quad p_2 \equiv [v_2] \quad \text{and} \quad p_3 \equiv [v_3],$$

we express the lines of it by

$$p_1p_2 = \langle n_{12} \rangle, \quad p_1p_3 = \langle n_{13} \rangle \quad \text{and} \quad p_2p_3 = \langle n_{23} \rangle,$$

where

$$n_{12} \equiv v_1 \times_B v_2, \quad n_{13} \equiv v_1 \times_B v_3 \quad \text{and} \quad n_{23} \equiv v_2 \times_B v_3.$$

Let $Q_i \equiv Q_B(v_i)$, for $i = 1, 2, 3$, and

$$N_{12} \equiv Q_B(n_{12}), \quad N_{13} \equiv Q_B(n_{13}) \quad \text{and} \quad N_{23} \equiv Q_B(n_{23}).$$

From the definition of the B-projective quadrance,

$$q_{12} = \frac{Q_B(v_1 \times_B v_2)}{(\det B)\, Q_B(v_1)\, Q_B(v_2)} = \frac{N_{12}}{(\det B)\, Q_1Q_2}$$

and similarly,

$$q_{13} = \frac{N_{13}}{(\det B)\, Q_1Q_3} \quad \text{and} \quad q_{23} = \frac{N_{23}}{(\det B)\, Q_2Q_3}.$$

From Corollary 8 and the definition of the B-projective spread,

$$\begin{aligned} S_1 &= \frac{Q_B(n_{12} \times n_{13})}{(\det B)\, Q_B(n_{12})\, Q_B(n_{13})} \\ &= \frac{Q_B\left((\det B)^2 (\det M)\, v_1\right)}{(\det B)\, N_{12}N_{13}} \\ &= \frac{(\det B)^3 (\det M)^2\, Q_1}{N_{12}N_{13}} \end{aligned}$$

and similarly,

$$S_2 = \frac{(\det B)^3 (\det M)^2 Q_2}{N_{12}N_{23}} \quad \text{and} \quad S_3 = \frac{(\det B)^3 (\det M)^2 Q_3}{N_{13}N_{23}}.$$

Hence, we deduce that

$$\frac{S_1}{q_{23}} = \frac{(\det B)^4 (\det M)^2 Q_1 Q_2 Q_3}{N_{12}N_{13}N_{23}} = \frac{S_2}{q_{13}} = \frac{S_3}{q_{12}}.$$

■

If we balance each side of the result of the Projective spread law to its lowest common denominator, multiplying through by the denominator motivates us to define the quantity

$$q_{12}q_{13}S_1 = q_{12}q_{23}S_2 = q_{13}q_{23}S_3 \equiv a_B\left(\overline{p_1p_2p_3}\right) \equiv a_B.$$

The quantity a_B will be called the B-**projective quadrea**, of the tripod $\overline{p_1p_2p_3}$.

There is a relationship between the B-projective quadrea and the B-projective quadrances of a tripod $\overline{p_1p_2p_3}$ discovered in [55], which is central to our study of projective rational trigonometry. We extend this result to an arbitrary symmetric bilinear form, using quite a different argument.

Theorem 23 (Projective cross law) *Given a tripod $\overline{p_1p_2p_3}$ with B-projective quadrances q_{12}, q_{13} and q_{23}, B-projective spreads S_1, S_2 and S_3, and B-projective quadrea a_B, the relation*

$$(a_B - q_{12} - q_{13} - q_{23} + 2)^2 = 4(1-q_{12})(1-q_{13})(1-q_{23})$$

is satisfied.

Proof. Let $Q_i \equiv Q_B(v_i)$, for $i = 1, 2, 3$. Use the Binet-Cauchy identity to get

$$(v_1 \times_B v_3) \cdot_B (v_2 \times_B v_3) = (\det B)\left[(v_1 \cdot_B v_2)(v_3 \cdot_B v_3) - (v_1 \cdot_B v_3)(v_2 \cdot_B v_3)\right].$$

Square both sides to obtain

$$\begin{aligned} & \left[(v_1 \times_B v_3) \cdot_B (v_2 \times_B v_3)\right]^2 \\ = \; & (\det B)^2 \left[(v_1 \cdot_B v_2)^2 Q_3^2 + (v_1 \cdot_B v_3)^2 (v_2 \cdot_B v_3)^2 - 2(v_1 \cdot_B v_2)(v_1 \cdot_B v_3)(v_2 \cdot_B v_3) Q_3\right]. \end{aligned}$$

As

$$S_3 = 1 - \frac{\left[(v_1 \times_B v_3) \cdot_B (v_2 \times_B v_3)\right]^2}{Q_B(v_1 \times_B v_3)\, Q_B(v_2 \times_B v_3)},$$

we rearrange this equation to obtain

$$\left[(v_1 \times_B v_3) \cdot_B (v_2 \times_B v_3)\right]^2 = Q_B(v_1 \times_B v_3)\, Q_B(v_2 \times_B v_3)(1 - S_3).$$

Using Lagrange's identity, this becomes

$$\begin{aligned}
\left[(v_1 \times_B v_3) \cdot_B (v_2 \times_B v_3)\right]^2 &= (\det B)^2 \left(Q_1 Q_3 - (v_1 \cdot_B v_3)^2\right)\left(Q_2 Q_3 - (v_2 \cdot_B v_3)^2\right)(1 - S_3) \\
&= (\det B)^2 Q_1 Q_2 Q_3^2 \left(1 - (1 - q_{13})\right)\left(1 - (1 - q_{23})\right)(1 - S_3) \\
&= (\det B)^2 Q_1 Q_2 Q_3^2 q_{13} q_{23} (1 - S_3).
\end{aligned}$$

Equate this result with our initial result and rearrange to get

$$\begin{aligned}
& 2(\det B)^2 Q_3 (v_1 \cdot_B v_2)(v_1 \cdot_B v_3)(v_2 \cdot_B v_3) \\
= {} & (\det B)^2 \left[Q_1 Q_2 Q_3^2 (1 - q_{12}) + Q_1 Q_2 Q_3^2 (1 - q_{13})(1 - q_{23}) - Q_1 Q_2 Q_3^2 q_{13} q_{23} (1 - S_3)\right] \\
= {} & (\det B)^2 Q_1 Q_2 Q_3^2 (S_3 q_{13} q_{23} - q_{12} - q_{13} - q_{23} + 2).
\end{aligned}$$

Divide both sides by $(\det B)^2 Q_3$ and then square each side to obtain

$$\begin{aligned}
& Q_1^2 Q_2^2 Q_3^2 (S_3 q_{13} q_{23} - q_{12} - q_{13} - q_{23} + 2)^2 \\
= {} & 4 (v_1 \cdot_B v_2)^2 (v_1 \cdot_B v_3)^2 (v_2 \cdot_B v_3)^2 \\
= {} & 4 Q_1^2 Q_2^2 Q_3^2 (1 - q_{12})(1 - q_{13})(1 - q_{23}).
\end{aligned}$$

Divide both sides by $(Q_1 Q_2 Q_3)^2$ to get

$$(S_3 q_{13} q_{23} - q_{12} - q_{13} - q_{23} + 2)^2 = 4(1 - q_{12})(1 - q_{13})(1 - q_{23}).$$

Since $a_B = S_3 q_{13} q_{23}$, we end up with

$$(a_B - q_{12} - q_{13} - q_{23} + 2)^2 = 4(1 - q_{12})(1 - q_{13})(1 - q_{23}),$$

as required. If we started with the B-quadruple scalar product $(v_1 \times_B v_2) \cdot_B (v_1 \times_B v_3)$ or $(v_1 \times_B v_2) \cdot_B (v_2 \times_B v_3)$, we would arrive at the same result by symmetry. ■

The Projective cross law can also be expressed in various asymmetric forms.

Corollary 24 *Given a tripod $\overline{p_1 p_2 p_3}$ with B-projective quadrances q_{12}, q_{13} and q_{23}, and B-projective spreads S_1, S_2 and S_3, the Projective cross law can be rewritten as either*

$$(q_{12} q_{13} S_1 - q_{12} - q_{13} + q_{23})^2 = 4 q_{12} q_{13} (1 - q_{23})(1 - S_1),$$

$$(q_{12} q_{23} S_2 - q_{12} + q_{13} - q_{23})^2 = 4 q_{12} q_{23} (1 - q_{13})(1 - S_2)$$

or

$$(q_{13} q_{23} S_3 + q_{12} - q_{13} - q_{23})^2 = 4 q_{13} q_{23} (1 - q_{12})(1 - S_3).$$

Proof. Substitute $a_B = S_1 q_{12} q_{13} = (1 - C_1) q_{12} q_{13}$ into the Projective cross law to get

$$((1 - C_1) q_{12} q_{13} - q_{12} - q_{13} - q_{23} + 2)^2 - 4(1 - q_{12})(1 - q_{13})(1 - q_{23}) = 0.$$

Expand the left-hand side and simplify the result as a polynomial in C_1 to obtain

$$\left(q_{12}^2 q_{13}^2\right) C_1^2 + 2q_{12}q_{13}\left(q_{12}+q_{13}+q_{23}-q_{12}q_{13}-2\right) C_1 + \left(q_{12}q_{13}-q_{12}-q_{13}+q_{23}\right)^2 = 0.$$

Add both sides by $4q_{12}q_{13}\left(1-q_{23}\right) C_1$ to get

$$\begin{aligned} & \left(q_{12}^2 q_{13}^2\right) C_1^2 + 2q_{12}q_{13}\left(q_{12}+q_{13}-q_{23}-q_{12}q_{13}\right) C_1 + \left(q_{12}q_{13}-q_{12}-q_{13}+q_{23}\right)^2 \\ = \; & 4q_{12}q_{13}\left(1-q_{23}\right) C_1. \end{aligned}$$

As the left-hand side is a perfect square, factorise this to get

$$\left(q_{12}+q_{13}-q_{23}-q_{12}q_{13}+q_{12}q_{13}C_1\right)^2 = 4q_{12}q_{13}\left(1-q_{23}\right) C_1.$$

Replace C_1 with $1-S_1$ and simplify to obtain

$$\begin{aligned} & \left(q_{12}+q_{13}-q_{23}-q_{12}q_{13}+\left(1-S_1\right) q_{12}q_{13}\right)^2 \\ = \; & \left(q_{12}+q_{13}-q_{23}-q_{12}q_{13}S_1\right)^2 = 4q_{12}q_{13}\left(1-q_{23}\right)\left(1-S_1\right). \end{aligned}$$

The other results follow by symmetry. ■

Note that the B-projective quadrea a_B also features in these asymmetric reformulation, and thus can replace the quantity $q_{12}q_{13}S_1$ (as well as its symmetrical reformulations).

In addition to the B-projective quadrea, we can also discuss the B-dual concept. This quantity is called the B-**quadreal** [64] and is defined by

$$l_B \equiv l_B\left(\overline{p_1p_2p_3}\right) \equiv q_{12}S_1S_2 = q_{13}S_1S_3 = q_{23}S_2S_3.$$

We can also say from Corollary 8 that l_B is the B-projective quadrea of the B-dual tripod $\overline{r_1r_2r_3}$ of $\overline{p_1p_2p_3}$ and a_B is the B-quadreal of $\overline{r_1r_2r_3}$. The following extends the result in [64] for B-quadratic forms.

Corollary 25 *For a tripod $\overline{p_1p_2p_3}$ with B-projective quadrances q_{12}, q_{13} and q_{23}, B-projective spreads S_1, S_2 and S_3, B-projective quadrea a_B and B-quadreal l_B,*

$$a_Bl_B = q_{12}q_{13}q_{23}S_1S_2S_3.$$

Proof. Given

$$a_B = q_{12}q_{13}S_1 = q_{12}q_{23}S_2 = q_{13}q_{23}S_3$$

and

$$l_B = q_{12}S_1S_2 = q_{13}S_1S_3 = q_{23}S_2S_3,$$

we get

$$\begin{aligned} a_B l_B &= (q_{12}q_{13}S_1)(q_{23}S_2S_3) = (q_{12}q_{23}S_2)(q_{13}S_1S_3) \\ &= (q_{13}q_{23}S_3)(q_{12}S_1S_2) = q_{12}q_{13}q_{23}S_1S_2S_3, \end{aligned}$$

as required. ■

The Projective cross law leads us to two more important results, which extends the results from [55] to general symmetric bilinear forms.

Theorem 26 (Projective triple quad formula) *Consider three projective points p_1, p_2 and p_3 in $\mathbb{P}^2$ with B-quadrances*

$$q_{12} \equiv q_B(p_1, p_2), \quad q_{13} \equiv q_B(p_1, p_3) \quad \text{and} \quad q_{23} \equiv q_B(p_2, p_3).$$

If p_1, p_2 and p_3 are collinear, then

$$(q_{12} + q_{13} + q_{23})^2 = 2\left(q_{12}^2 + q_{13}^2 + q_{23}^2\right) + 4q_{12}q_{13}q_{23}.$$

Proof. We start with the following result of the Projective cross law on the tripod $\overline{p_1p_2p_3}$:

$$(a - q_{12} - q_{13} - q_{23} + 2)^2 = 4(1 - q_{12})(1 - q_{13})(1 - q_{23}).$$

If p_1, p_2 and p_3 are collinear, then

$$S_1 = S_2 = S_3 = 0.$$

By the definition of the B-projective quadrea, $a_B = 0$ and thus substitute this result into our initial relation to get

$$(2 - q_{12} - q_{13} - q_{23})^2 = 4(1 - q_{12})(1 - q_{13})(1 - q_{23}).$$

By moving all the terms on the right-hand side over to the other side, we simplify the expression to obtain

$$\begin{aligned} 0 &= q_{12}^2 + q_{13}^2 + q_{23}^2 - 2q_{12}q_{13} - 2q_{12}q_{23} - 2q_{13}q_{23} + 4q_{12}q_{13}q_{23} \\ &= 2\left(q_{12}^2 + q_{13}^2 + q_{23}^2\right) - (q_{12} + q_{13} + q_{23})^2 + 4q_{12}q_{13}q_{23}. \end{aligned}$$

Now, rearrange to get the required result. ■

The Projective triple quad formula is analogous and parallel to the Triple spread formula in affine rational trigonometry. This is no coincidence, since the B-projective quadrance between two projective points is associated with the B-spread between two lines in affine rational trigonometry.

We now present the second result, which is a projective version of Pythagoras' theorem. This is an extension of the result in [55], [60] and [64] to arbitrary symmetric bilinear forms.

Theorem 27 (Projective Pythagoras' theorem) *Take a tripod $\overline{p_1p_2p_3}$ with B-projective quadrances q_{12}, q_{13} and q_{23}, and B-projective spreads S_1, S_2 and S_3. If $S_1 = 1$, then*

$$q_{23} = q_{12} + q_{13} - q_{12}q_{13}.$$

Proof. Start with the Projective cross law

$$(q_{12}q_{13}S_1 - q_{12} - q_{13} - q_{23} + 2)^2 = 4(1 - q_{12})(1 - q_{13})(1 - q_{23}).$$

Substitute $S_1 = 1$ and rearrange the result to get

$$(q_{12}q_{13} - q_{12} - q_{13} - q_{23} + 2)^2 - 4(1 - q_{12})(1 - q_{13})(1 - q_{23}) = 0.$$

The left-hand side can be factored as

$$(q_{12} + q_{13} - q_{23} - q_{12}q_{13})^2 = 0.$$

Solving for q_{23}, we get

$$q_{23} = q_{12} + q_{13} - q_{12}q_{13}.$$

■

Note the cross term $-q_{12}q_{13}$ involved in the Projective Pythagoras' theorem; this is not present in Pythagoras' theorem in affine rational trigonometry. As observed in [64], the Projective Pythagoras' theorem can be restated as

$$\begin{aligned} 1 - q_{23} &= 1 - q_{12} - q_{13} + q_{12}q_{13} \\ &= (1 - q_{12})(1 - q_{13}). \end{aligned}$$

As for the converse of the Projective Pythagoras' theorem, start with the asymmetric form of the Projective cross law

$$(q_{12}q_{13}S_1 - q_{12} - q_{13} + q_{23})^2 = 4q_{12}q_{13}(1 - q_{23})(1 - S_1).$$

If $q_{23} = q_{12} + q_{13} - q_{12}q_{13}$ then we have that

$$(q_{12}q_{13}(1 - S_1))^2 = 4q_{12}q_{13}(1 - q_{12})(1 - q_{13})(1 - S_1),$$

which can also be rearranged and factorised as

$$q_{12}q_{13}(1 - S_1)(4q_{12} + 4q_{13} - 3q_{12}q_{13} - S_1q_{12}q_{13} - 4) = 0.$$

Here, we see that $S_1 = 1$ is not the only solution; we can also have the solution

$$S_1 = \frac{4(q_{12} + q_{13} - 1)}{q_{12}q_{13}} - 3.$$

So, the converse of the Projective Pythagoras' theorem may not necessarily hold.

Chapter 2

Trigonometry of the tetrahedron

In this chapter, we begin to analyse our main object of interest: a general tetrahedron. Here, we consider the tetrahedron $\overline{A_0A_1A_2A_3}$ in $\mathbb{A}^3$, where

$$A_0 \equiv [x_0, y_0, z_0], \quad A_1 \equiv [x_1, y_1, z_1], \quad A_2 \equiv [x_2, y_2, z_2] \quad \text{and} \quad A_3 \equiv [x_3, y_3, z_3]$$

are its points. The lines, planes, sides, triangles, vertices and corners of $\overline{A_0A_1A_2A_3}$ have already been defined in the last chapter and are available to us.

We equip $\mathbb{V}^3$ with a non-degenerate symmetric bilinear form and represent this with the matrix

$$B = \begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix}$$

as before.

2.1 The Khafre Tetrahedron

We start with a historical example of a tetrahedron. The Khafre Pyramid at Giza, Egypt [44], pictured in Figure 2.1, is the second and central of the three pyramids and the one with the Sphinx in front of it. It has a base width of 411 cubits (approx. 216 metres) and a height of 274 cubits (approx. 143 metres) [20].

Noted by Gillings [23, p. 212] and Claggett [13, p. 90], the steepness of such a pyramid was described by the ancient Egyptians using the important ratio of run over rise, or *seqed* as the Egyptians called it, which in this case is

$$\frac{205.5}{274} = \frac{5.25}{7} = \frac{3}{4}.$$

This is indeed exactly 5 palms and 1 finger per cubit, as the Egyptians divided a cubit into 7 palms, and a palm into 4 fingers. The seqed was used by the Egyptians rather than slope or angle to measure steepness of a pyramid, and would have aided construction workers to ensure that the sides of the pyramid were inclined equally as it was being built.

Figure 2.1: The Khafre pyramid at Giza [32]

The quantity computed is the ratio $\frac{|OR|}{|OP|}$ in Figure 2.2, which has been suitably rescaled to make the crucial 3-4-5 triangle more visible.

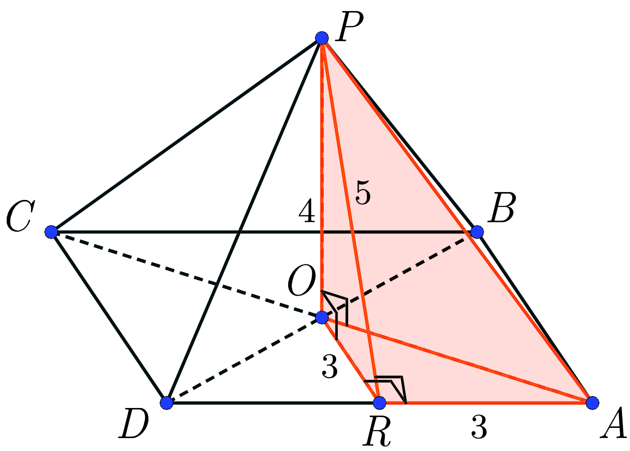

Figure 2.2: Geometry of the Khafre pyramid and tetrahedron with rescaled lengths

There is some remarkable geometry in this structure, some of which can be captured by the tetrahedron $\overline{ORAP}$ formed from the base triangle $\overline{ORA}$, where O is the center of the base, R the midpoint of the side $\overline{AD}$ of the base, and P the apex of the pyramid. We will call this tetrahedron the **Khafre tetrahedron**. This tetrahedron is nowhere near a general one; for example, it has all four of its faces as right triangles in the Euclidean sense, one of which (the base triangle) being also isosceles. Nevertheless we will see that we can use it to illustrate many of the relations that hold for a general tetrahedron as we progress through the thesis. When we discuss it, we will be using the standard Euclidean bilinear form, i.e. we set B to be the 3×3 identity matrix. By doing this, we are thus allowed to omit the B which is prefixed to any trigonometric quantity pertaining to the Khafre tetrahedron.

We use the diagram to establish coordinates. If we take O to be the origin of a coordinate system, with OR forming the x-axis, OP forming the z-axis, and the y-axis otherwise perpendicular to both, then we can define the points to be

$$O \equiv [0,0,0]\,, \quad R \equiv [3,0,0]\,, \quad A \equiv [3,3,0] \quad \text{and} \quad P \equiv [0,0,4]\,.$$

The signed volume of $\overline{ORAP}$ is

$$v\left(\overrightarrow{OR}, \overrightarrow{OA}, \overrightarrow{OP}\right) = \frac{1}{6}\det\begin{pmatrix} 3 & 0 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 6.$$

The quadrume of $\overline{ORAP}$ is then

$$\mathcal{V} = 144 \times 6^2 = 5184.$$

We can convert the length readings of the Khafre tetrahedron in Figure 2.2 to obtain its quadrances, which are

$$Q_{OR} = 9, \quad Q_{OA} = 18, \quad Q_{OP} = 16,$$

$$Q_{AP} = 34, \quad Q_{RP} = 25 \quad \text{and} \quad Q_{RA} = 9.$$

We see that Pythagoras' theorem allows us to easily determine all six quadrances (as seen in Figure 2.3), since all the triangles of the Khafre tetrahedron are right triangles in the Euclidean sense.

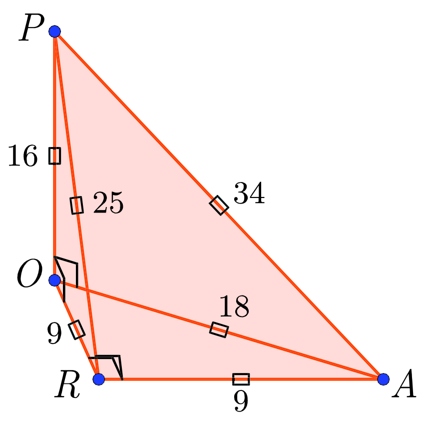

Figure 2.3: Quadrances of the Khafre tetrahedron

Denoting the triangles/faces of the Khafre tetrahedron by $\overline{ORA}$, $\overline{ORP}$, $\overline{OAP}$ and $\overline{ARP}$, the quadreas associated to them them will be denoted by and are evaluated as

$$\mathcal{A}_{ORA} = 324, \quad \mathcal{A}_{ORP} = 576, \quad \mathcal{A}_{OAP} = 1152 \quad \text{and} \quad \mathcal{A}_{ARP} = 900.$$

Since all the triangles are right-angled, we know that one of the spreads on each triangle is 1. If we define $s_{O;AR} \equiv s(OA, OR)$, $s_{O;AP} \equiv s(OA, OP)$, etc, then the spreads of the triangles $\overline{ORA}$, $\overline{ORP}$, $\overline{OAP}$ and $\overline{ARP}$ can be evaluated by the Quadrea spread theorem as

$$s_{O;RA} = \frac{1}{2}, \quad s_{R;OA} = 1, \quad s_{A;OR} = \frac{1}{2},$$
$$s_{O;RP} = 1, \quad s_{R;OP} = \frac{16}{25}, \quad s_{P;OR} = \frac{9}{25},$$
$$s_{O;AP} = 1, \quad s_{A;OP} = \frac{8}{17}, \quad s_{P;OA} = \frac{9}{17},$$
$$s_{A;RP} = \frac{25}{34}, \quad s_{R;AP} = 1 \quad \text{and} \quad s_{P;AR} = \frac{9}{34}.$$

In the next section, we will compute the other quantities associated to the Khafre tetrahedron, with a view of using it as a particularly simple test example to verify the results we will obtain.

2.2 The dihedral, solid and dual solid spreads

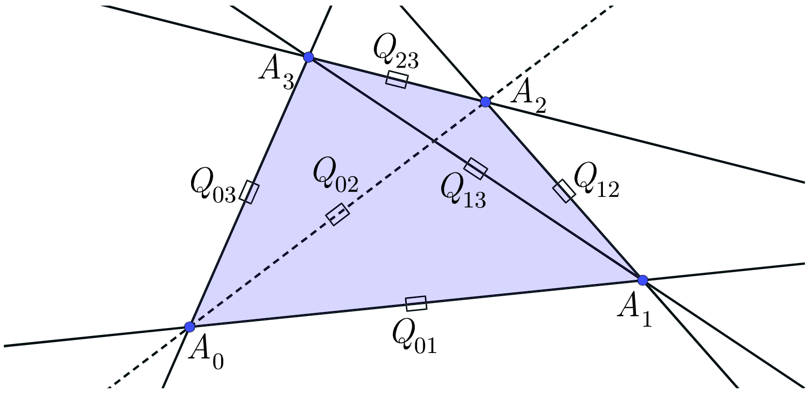

Figure 2.4: B-quadrances of the tetrahedron $\overline{A_0A_1A_2A_3}$

In what follows, we consider the general tetrahedron $\overline{A_0A_1A_2A_3}$ defined earlier, as in Figure 2.4. In this tetrahedron, the B-quadrances will be denoted by $Q_{ij} \equiv Q_B(A_i, A_j)$, for $0 \leq i < j \leq 3$. The B-quadreas of the triangles of $\overline{A_0A_1A_2A_3}$ will be denoted by

$$\mathcal{A}_{012} \equiv \mathcal{A}_B\left(\overline{A_0A_1A_2}\right), \quad \mathcal{A}_{013} \equiv \mathcal{A}_B\left(\overline{A_0A_1A_3}\right), \quad \mathcal{A}_{023} \equiv \mathcal{A}_B\left(\overline{A_0A_2A_3}\right)$$

and

$$\mathcal{A}_{123} \equiv \mathcal{A}_B\left(\overline{A_1A_2A_3}\right).$$

The B-quadrume of $\overline{A_0A_1A_2A_3}$ is denoted by $\mathcal{V} \equiv \mathcal{V}_B\left(\overline{A_0A_1A_2A_3}\right)$. Furthermore, we will denote $s_{i;jk}$ to be the B-spread between the lines A_iA_j and A_iA_k of $\overline{A_0A_1A_2A_3}$, for $0 \leq i \leq 3$ and $0 \leq j < k \leq 3$ not equal to i. In this section, we present methods for calculating the B-dihedral spreads, B-solid spreads and B-dual solid spreads of a tetrahedron.

2.2.1 Dihedral spreads

Given the tetrahedron $\overline{A_0A_1A_2A_3}$, we will denote the B-dihedral spreads by

$$E_{01} \equiv E_B\left(A_0A_1A_2, A_0A_1A_3\right), \quad E_{23} \equiv E_B\left(A_0A_2A_3, A_1A_2A_3\right),$$

$$E_{02} \equiv E_B\left(A_0A_1A_2, A_0A_2A_3\right), \quad E_{13} \equiv E_B\left(A_0A_1A_3, A_1A_2A_3\right),$$

$$E_{03} \equiv E_B\left(A_0A_1A_2, A_0A_2A_3\right) \quad \text{and} \quad E_{12} \equiv E_B\left(A_0A_1A_3, A_1A_2A_3\right).$$

We show a visual representation of the B-dihedral spreads in Figure 2.5. We can associate

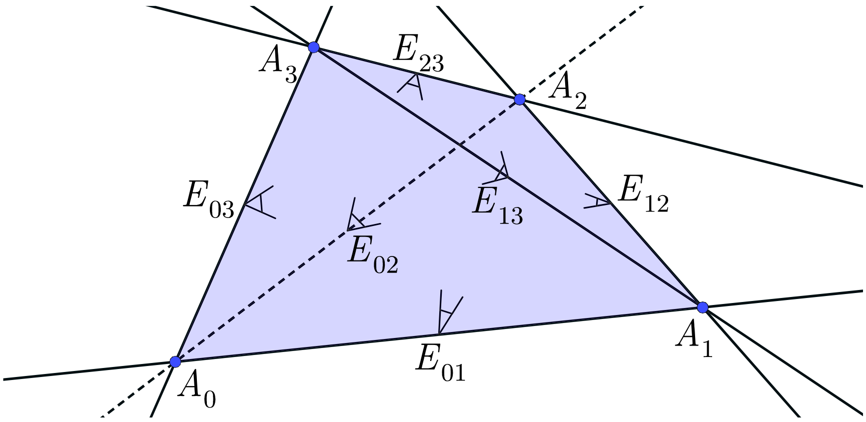

Figure 2.5: B-dihedral spreads of tetrahedron $\overline{A_0A_1A_2A_3}$

the B-dihedral spreads of $\overline{A_0A_1A_2A_3}$ to its edges, since any two planes of the tetrahedron will meet at one of its lines. The B-dihedral crosses will be denoted similarly, i.e. $C_{ij} \equiv 1 - E_{ij}$ for $0 \leq i < j \leq 3$. With that in mind, we present a simpler method for calculating the B-dihedral spreads of a tetrahedron by extending a result of [63].

Theorem 28 (Dihedral spread theorem) *For a tetrahedron $\overline{A_0A_1A_2A_3}$ with B-quadrances Q_{ij}, for $0 \leq i < j \leq 3$, B-quadreas $\mathcal{A}_{012}$, $\mathcal{A}_{013}$, $\mathcal{A}_{023}$ and $\mathcal{A}_{123}$, and B-quadrume $\mathcal{V}$, the B-dihedral spreads of $\overline{A_0A_1A_2A_3}$ are expressed as*

$$\begin{aligned} E_{01} &= \frac{4Q_{01}\mathcal{V}}{\mathcal{A}_{012}\mathcal{A}_{013}}, \quad E_{23} = \frac{4Q_{23}\mathcal{V}}{\mathcal{A}_{023}\mathcal{A}_{123}}, \\ E_{02} &= \frac{4Q_{02}\mathcal{V}}{\mathcal{A}_{012}\mathcal{A}_{023}}, \quad E_{13} = \frac{4Q_{13}\mathcal{V}}{\mathcal{A}_{013}\mathcal{A}_{123}}, \\ E_{03} &= \frac{4Q_{03}\mathcal{V}}{\mathcal{A}_{012}\mathcal{A}_{023}} \quad \text{and} \quad E_{12} = \frac{4Q_{12}\mathcal{V}}{\mathcal{A}_{013}\mathcal{A}_{123}}. \end{aligned}$$

Proof. Let $v_1 \equiv \overrightarrow{A_0A_1}$, $v_2 \equiv \overrightarrow{A_0A_2}$ and $v_3 \equiv \overrightarrow{A_0A_3}$, so that the B-normal vectors to the respective planes $A_0A_1A_2$ and $A_0A_1A_3$ are $v_1 \times_B v_2$ and $v_1 \times_B v_3$. By Lagrange's identity and the definition of the B-dihedral spread,

$$\begin{aligned} E_{01} &= 1-\frac{\left(\left(v_1\times_B v_2\right)\cdot_B\left(v_1\times_B v_3\right)\right)^2}{Q_B\left(v_1\times_B v_2\right)Q_B\left(v_1\times_B v_3\right)} \\ &= \frac{Q_B\left(\left(v_1\times_B v_2\right)\times_B\left(v_1\times_B v_3\right)\right)}{\left(\det B\right)Q_B\left(v_1\times_B v_2\right)Q_B\left(v_1\times_B v_3\right)}. \end{aligned}$$

By the Quadrea theorem,

$$\mathcal{A}_{012}=\frac{4Q_B\left(v_1\times_B v_2\right)}{\det B}\quad\text{and}\quad\mathcal{A}_{013}=\frac{4Q_B\left(v_1\times_B v_3\right)}{\det B}.$$

Hence,

$$\begin{aligned} \left(\det B\right)Q_B\left(v_1\times_B v_2\right)Q_B\left(v_1\times_B v_3\right) &= \left(\det B\right)\left(\frac{\left(\det B\right)\mathcal{A}_{012}}{4}\right)\left(\frac{\left(\det B\right)\mathcal{A}_{013}}{4}\right) \\ &= \frac{\left(\det B\right)^3\mathcal{A}_{012}\mathcal{A}_{013}}{16}. \end{aligned}$$

Now, consider the matrix

$$M\equiv\begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{pmatrix}.$$

From Corollary 8,

$$\begin{aligned} Q_B\left(\left(v_1\times_B v_2\right)\times_B\left(v_1\times_B v_3\right)\right) &= Q_B\left(\left[\left(\det B\right)^2\left(\det M\right)\right]v_1\right) \\ &= \left(\det B\right)^4\left(\det M\right)^2Q_B\left(v_1\right) \\ &= \left(\det B\right)^4\left(\det M\right)^2Q_{01}. \end{aligned}$$

By the Quadrume matrix product theorem,

$$Q_B\left(\left(v_1\times_B v_2\right)\times_B\left(v_1\times_B v_3\right)\right)=\frac{\left(\det B\right)^3Q_{01}\mathcal{V}}{4}$$

and thus

$$E_{01}=\frac{\left(\det B\right)^3Q_{01}\mathcal{V}}{4}\div\frac{\left(\det B\right)^3\mathcal{A}_{012}\mathcal{A}_{013}}{16}=\frac{4Q_{01}\mathcal{V}}{\mathcal{A}_{012}\mathcal{A}_{013}}.$$

The other results follow by symmetry. ■

The B-dihedral spread, a metrical affine quantity, can be viewed projectively as the B-projective spread between two projective lines. We now proceed to computing the dihedral spreads and crosses of the Khafre tetrahedron.

Example 29 (Dihedral spreads of the Khafre tetrahedron) *For the Khafre tetrahedron* $\overline{ORAP}$ *defined earlier, the dihedral spreads are denoted and evaluated by the Di-*

hedral spread theorem as

$$E_{OR} = \frac{4Q_{OR}\mathcal{V}}{\mathcal{A}_{OAR}\mathcal{A}_{ORP}} = \frac{4 \times 9 \times 5184}{324 \times 576} = 1,$$

$$E_{OA} = \frac{4Q_{OA}\mathcal{V}}{\mathcal{A}_{OAR}\mathcal{A}_{OAP}} = \frac{4 \times 18 \times 5184}{324 \times 1152} = 1,$$

$$E_{OP} = \frac{4Q_{OP}\mathcal{V}}{\mathcal{A}_{OAP}\mathcal{A}_{ORP}} = \frac{4 \times 16 \times 5184}{1152 \times 576} = \frac{1}{2},$$

$$E_{RA} = \frac{4Q_{AR}\mathcal{V}}{\mathcal{A}_{OAR}\mathcal{A}_{ARP}} = \frac{4 \times 9 \times 5184}{900 \times 324} = \frac{16}{25},$$

$$E_{RP} = \frac{4Q_{RP}\mathcal{V}}{\mathcal{A}_{ORP}\mathcal{A}_{ARP}} = \frac{4 \times 25 \times 5184}{900 \times 576} = 1$$

and

$$E_{AP} = \frac{4Q_{AP}\mathcal{V}}{\mathcal{A}_{OAP}\mathcal{A}_{ARP}} = \frac{4 \times 34 \times 5184}{1152 \times 900} = \frac{17}{25}.$$

We illustrate this in Figure 2.6, where the right angles on the edges denote that the dihedral spread is equal to 1*.*

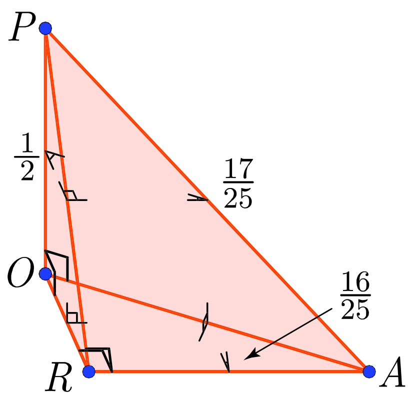

Figure 2.6: Dihedral spreads of the Khafre tetrahedron

Example 30 (Dihedral crosses of the Khafre tetrahedron) *Since the dihedral crosses are given by $C_{ij} = 1 - E_{ij}$, for distinct points i and j of the Khafre tetrahedron $\overline{ORAP}$, we have that*

$$C_{OA} = 0, \quad C_{OR} = 0, \quad C_{OP} = \frac{1}{2},$$

$$C_{RP} = 0, \quad C_{AP} = \frac{8}{25} \quad \textit{and} \quad C_{AR} = \frac{9}{25}.$$

2.2.2 Solid spreads

The B-solid spreads of the tetrahedron $\overline{A_0A_1A_2A_3}$ are denoted by

$$\mathcal{S}_0 \equiv \mathcal{S}_B\left(A_0A_1, A_0A_2, A_0A_3\right), \quad \mathcal{S}_1 \equiv \mathcal{S}_B\left(A_1A_0, A_1A_2, A_1A_3\right),$$

$$\mathcal{S}_2 \equiv \mathcal{S}_B\left(A_0A_2, A_1A_2, A_2A_3\right) \quad \text{and} \quad \mathcal{S}_3 \equiv \mathcal{S}_B\left(A_0A_3, A_1A_3, A_2A_3\right),$$

and are displayed in Figure 2.7. So we associate a B-solid spread to each corner of

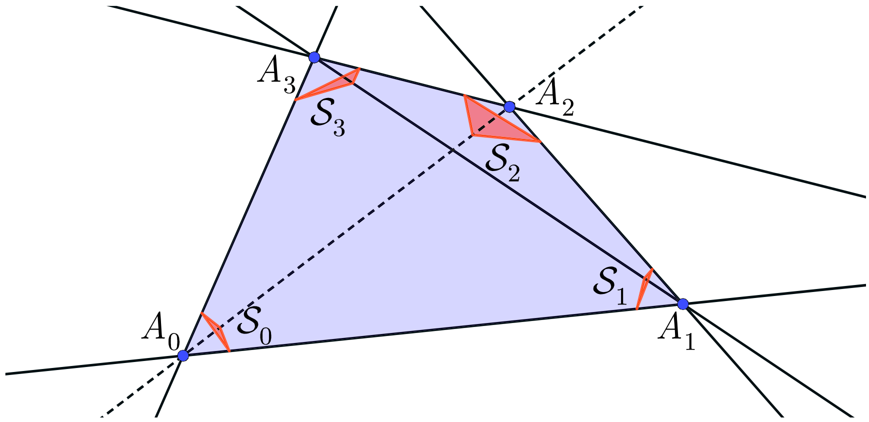

Figure 2.7: B-solid spreads of the tetrahedron $\overline{A_0A_1A_2A_3}$

$\overline{A_0A_1A_2A_3}$, and we can compute the B-solid spread for a tetrahedron using only our trigonometric quantities, as seen in the following two results, based on [63].

In our first result, we express the B-solid spreads of $\overline{A_0A_1A_2A_3}$ explicitly in terms of its B-quadrances and B-quadrume.

Theorem 31 (Solid spread theorem) *For a tetrahedron $\overline{A_0A_1A_2A_3}$ with B-quadrances Q_{ij}, for $0 \leq i < j \leq 3$, and B-quadrume $\mathcal{V}$, the B-solid spreads of $\overline{A_0A_1A_2A_3}$ are expressed as*

$$\mathcal{S}_0 = \frac{\mathcal{V}}{4Q_{01}Q_{02}Q_{03}}, \quad \mathcal{S}_1 = \frac{\mathcal{V}}{4Q_{01}Q_{12}Q_{13}},$$
$$\mathcal{S}_2 = \frac{\mathcal{V}}{4Q_{02}Q_{12}Q_{23}} \quad \textit{and} \quad \mathcal{S}_3 = \frac{\mathcal{V}}{4Q_{03}Q_{13}Q_{23}}.$$

Proof. Let v_1, v_2 and v_3 be the respective direction vectors of A_0A_1, A_0A_2 and A_0A_3, and define

$$M \equiv \begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{pmatrix}.$$

Using the Quadrume matrix product theorem, the definition of B-solid spread and the Scalar triple product theorem to obtain

$$\begin{aligned}
\mathcal{S}_0 &= \frac{\left(\left[v_1, v_2, v_3\right]_B\right)^2}{\left(\det B\right) Q_B\left(v_1\right) Q_B\left(v_2\right) Q_B\left(v_3\right)} \\
&= \frac{\left(\det\left(MB\right)\right)^2}{\left(\det B\right) Q_{01}Q_{02}Q_{03}} \\
&= \frac{\det\left(MBM^T\right)}{Q_{01}Q_{02}Q_{03}} \\
&= \frac{\mathcal{V}}{4Q_{01}Q_{02}Q_{03}}.
\end{aligned}$$

The other results follow by symmetry. ■

Alternatively, we can express the B-solid spreads of $\overline{A_0A_1A_2A_3}$ in terms of its B-spreads and B-dihedral spreads. We will call this theorem the Solid spread projective theorem, because of the similarities of the result with other results in projective geometry (which we will explain later).

Theorem 32 (Solid spread projective theorem) *For a tetrahedron $\overline{A_0A_1A_2A_3}$ with B-spreads $s_{i;jk}$, for $i = 0, 1, 2, 3$ and $0 \le j < k \le 3$ with $j, k \ne i$, and B-dihedral spreads E_{ij}, for $0 \le i < j \le 3$, the B-solid spreads are expressed as*

$$\mathcal{S}_0 = E_{01}s_{0;12}s_{0;13} = E_{02}s_{0;12}s_{0;23} = E_{03}s_{0;13}s_{0;23},$$

$$\mathcal{S}_1 = E_{01}s_{1;02}s_{1;03} = E_{12}s_{1;02}s_{1;23} = E_{13}s_{1;03}s_{1;23},$$

$$\mathcal{S}_2 = E_{02}s_{2;01}s_{2;03} = E_{12}s_{2;01}s_{2;13} = E_{23}s_{2;03}s_{2;13}$$

and

$$\mathcal{S}_3 = E_{03}s_{3;01}s_{3;02} = E_{13}s_{3;01}s_{2;12} = E_{23}s_{3;02}s_{2;12}.$$

Proof. Given the B-quadrances, B-quadreas and B-quadrume of $\overline{A_0A_1A_2A_3}$, use the Quadrea spread theorem and the Dihedral spread theorem to obtain

$$s_{0;12} = \frac{\mathcal{A}_{012}}{4Q_{01}Q_{02}}, \quad s_{0;13} = \frac{\mathcal{A}_{013}}{4Q_{01}Q_{03}} \quad \text{and} \quad E_{01} = \frac{4Q_{01}\mathcal{V}}{\mathcal{A}_{012}\mathcal{A}_{013}}.$$

So,

$$\begin{aligned} E_{01}s_{0;12}s_{0;13} &= \frac{\mathcal{A}_{012}}{4Q_{01}Q_{02}}\frac{\mathcal{A}_{013}}{4Q_{01}Q_{03}}\frac{4Q_{01}\mathcal{V}}{\mathcal{A}_{012}\mathcal{A}_{013}} \\ &= \frac{\mathcal{V}}{4Q_{01}Q_{02}Q_{03}}, \end{aligned}$$

which by the Solid spread theorem is equal to $\mathcal{S}_0$, as required. The other results follow by symmetry. ■

From the Solid spread projective theorem, the B-solid spread can be viewed projectively as the B-projective quadrea of a tripod given by three projective points. We now proceed to calculating the solid spreads of the Khafre tetrahedron.

Example 33 (Solid spreads of the Khafre tetrahedron) *By the Solid spread theorem, the solid spreads of the Khafre tetrahedron $\overline{ORAP}$ are*

$$\mathcal{S}_O = \frac{\mathcal{V}}{4Q_{OA}Q_{OR}Q_{OP}} = \frac{5184}{4 \times 18 \times 9 \times 16} = \frac{1}{2},$$

$$\mathcal{S}_R = \frac{\mathcal{V}}{4Q_{OR}Q_{AR}Q_{PR}} = \frac{5184}{4 \times 9 \times 9 \times 25} = \frac{16}{25},$$

$$\mathcal{S}_A = \frac{\mathcal{V}}{4Q_{OA}Q_{AR}Q_{PA}} = \frac{5184}{4 \times 18 \times 9 \times 34} = \frac{4}{17}$$

and

$$\mathcal{S}_P = \frac{\mathcal{V}}{4Q_{OP}Q_{PA}Q_{PR}} = \frac{5184}{4 \times 16 \times 34 \times 25} = \frac{81}{850}.$$

The Solid spread projective theorem can also be used to obtain the same values for the solid spreads, given the spreads and dihedral spreads of $\overline{ORAP}$. We illustrate the solid spreads of $\overline{ORAP}$ in Figure 2.8.

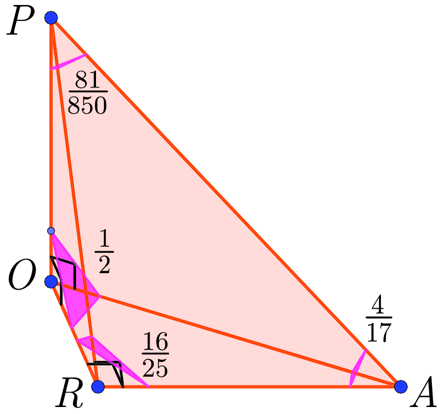

Figure 2.8: Solid spreads of the Khafre tetrahedron

2.2.3 Dual solid spreads

The B-dual solid spreads of the tetrahedron $\overline{A_0A_1A_2A_3}$ will be denoted by

$$\mathcal{D}_0 \equiv \mathcal{D}_B\left(A_0A_1, A_0A_2, A_0A_3\right), \quad \mathcal{D}_1 \equiv \mathcal{D}_B\left(A_0A_1, A_1A_2, A_1A_3\right),$$

$$\mathcal{D}_2 \equiv \mathcal{D}_B\left(A_0A_2, A_1A_2, A_2A_3\right) \quad \text{and} \quad \mathcal{D}_3 \equiv \mathcal{D}_B\left(A_0A_3, A_1A_3, A_2A_3\right).$$

They are displayed in Figure 2.9. As is the case with B-solid spreads, we associate the

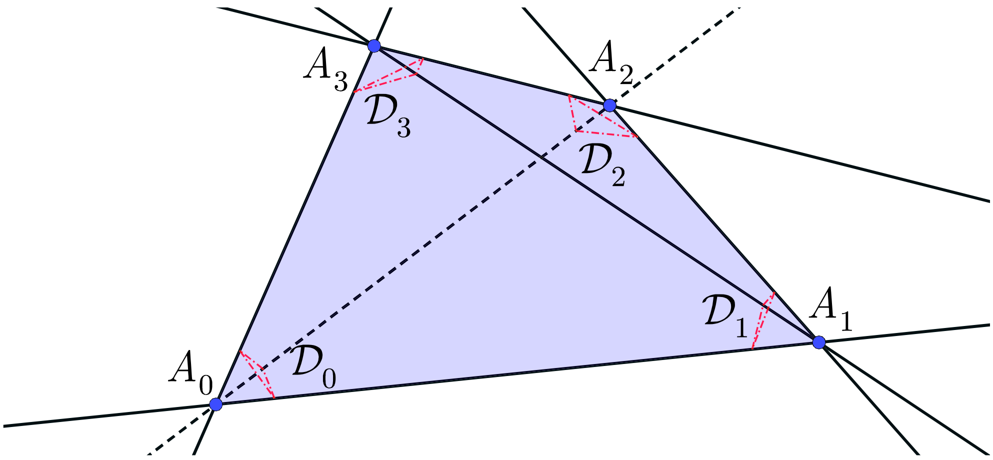

Figure 2.9: B-dual solid spreads of the tetrahedron $\overline{A_0A_1A_2A_3}$

B-dual solid spread to each corner of $\overline{A_0A_1A_2A_3}$, which can be calculated simply by the following two results. In the first one, we express the B-dual solid spread of $\overline{A_0A_1A_2A_3}$ explicitly in terms of its B-quadreas and B-quadrume.

Theorem 34 (Dual solid spread theorem) *For a tetrahedron $\overline{A_0A_1A_2A_3}$ with B-quadreas $\mathcal{A}_{012}$, $\mathcal{A}_{013}$, $\mathcal{A}_{023}$ and $\mathcal{A}_{123}$, and B-quadrume $\mathcal{V}$, the B-dual solid spreads of $\overline{A_0A_1A_2A_3}$ are expressed as*

$$\mathcal{D}_0 = \frac{4\mathcal{V}^2}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}}, \quad \mathcal{D}_1 = \frac{4\mathcal{V}^2}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{123}},$$
$$\mathcal{D}_2 = \frac{4\mathcal{V}^2}{\mathcal{A}_{012}\mathcal{A}_{023}\mathcal{A}_{123}} \quad \text{and} \quad \mathcal{D}_3 = \frac{4\mathcal{V}^2}{\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}}.$$

Proof. Suppose the direction vectors of l_1, l_2 and l_3 are v_1, v_2 and v_3 respectively; also define three concurrent lines k_{12}, k_{13} and k_{23} with direction vectors

$$n_{12} \equiv v_1 \times_B v_2, \quad n_{13} \equiv v_1 \times_B v_3 \quad \text{and} \quad n_{23} \equiv v_2 \times_B v_3.$$

Use the definition of the B-dual solid spread to get

$$\begin{aligned}\mathcal{D}_B(l_1, l_2, l_3) &= \mathcal{S}_B(k_{12}, k_{13}, k_{23}) \\ &= \frac{\left([n_{12}, n_{13}, n_{23}]_B\right)^2}{(\det B)\, Q_B(n_{12})\, Q_B(n_{13})\, Q_B(n_{23})}.\end{aligned}$$

Given the definitions of n_{12}, n_{13} and n_{23}, use the definition of the B-quadrea to deduce that

$$Q_B(n_{12}) = \frac{(\det B)\,\mathcal{A}_{012}}{4}, \quad Q_B(n_{13}) = \frac{(\det B)\,\mathcal{A}_{013}}{4} \quad \text{and} \quad Q_B(n_{23}) = \frac{(\det B)\,\mathcal{A}_{023}}{4}.$$

Hence,

$$(\det B)\, Q_B(n_{12})\, Q_B(n_{13})\, Q_B(n_{23}) = \frac{(\det B)^4}{64}\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}.$$

By the definition of the scalar triple product, expand $[n_{12}, n_{13}, n_{23}]_B$ using the definitions of n_{12}, n_{13} and n_{23} as

$$[n_{12}, n_{13}, n_{23}]_B = (v_1 \times_B v_2) \cdot_B \left[(v_1 \times_B v_3) \times_B (v_2 \times_B v_3)\right].$$

Define

$$M \equiv \begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{pmatrix},$$

so that by the Quadruple vector product theorem and the Scalar triple product theorem

$$\begin{aligned}[][n_{12}, n_{13}, n_{23}]_B &= (v_1 \times_B v_2) \cdot_B (\det B)\left([v_1, v_3, v_3]_B\, v_2 - [v_1, v_3, v_2]_B\, v_3\right) \\ &= (\det B)\left[(v_1 \times_B v_2) \cdot_B [v_1, v_2, v_3]_B\, v_3\right] \\ &= (\det M)(\det B)^2 [v_1, v_2, v_3]_B = (\det M)^2 (\det B)^3.\end{aligned}$$

By the Quadrume matrix product theorem,

$$\begin{aligned}([n_{12}, n_{13}, n_{23}]_B)^2 &= (\det M)^4 (\det B)^6 = (\det B)^4 \left[(\det M)^4 (\det B)^2\right] \\ &= (\det B)^4 \left(\frac{\mathcal{V}}{4}\right)^2 = \frac{(\det B)^4}{16}\mathcal{V}^2.\end{aligned}$$

So,

$$\mathcal{D}_B (l_{01}, l_{02}, l_{03}) = \frac{(\det B)^4 \mathcal{V}^2}{16} \div \frac{(\det B)^4 \mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}}{64} = \frac{4\mathcal{V}^2}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}}.$$

The other results follow by symmetry. ■

Next we use the B-spreads and B-dihedral spreads of $\overline{A_0A_1A_2A_3}$ to alternatively express its B-dual solid spreads. We will call this the Dual solid spread projective theorem for the same reasons as in the case of the Solid spread projective theorem.

Theorem 35 (Dual solid spread projective theorem) *For a tetrahedron $\overline{A_0A_1A_2A_3}$ with B-spreads $s_{i;jk}$, for $0 \leq i \leq 3$ and $0 \leq j < k \leq 3$ with $j, k \neq i$, and B-dihedral spreads E_{ij}, for $0 \leq i < j \leq 3$, the B-dual solid spreads of $\overline{A_0A_1A_2A_3}$ are expressed as*

$$\mathcal{D}_0 = s_{0;12}E_{01}E_{02} = s_{0;13}E_{01}E_{03} = s_{0;23}E_{02}E_{03},$$

$$\mathcal{D}_1 = s_{1;02}E_{01}E_{12} = s_{1;03}E_{01}E_{13} = s_{1;23}E_{12}E_{13},$$

$$\mathcal{D}_2 = s_{2;01}E_{02}E_{12} = s_{2;03}E_{02}E_{23} = s_{2;13}E_{12}E_{23}$$

and

$$\mathcal{D}_3 = s_{3;01}E_{03}E_{13} = s_{3;02}E_{03}E_{23} = s_{3;12}E_{13}E_{23}.$$

Proof. By the Dihedral spread theorem and the Quadrea spread theorem, we have that

$$s_{0,12}E_{01}E_{02} = \left(\frac{\mathcal{A}_{012}}{4Q_{01}Q_{02}}\right)\left(\frac{4Q_{01}\mathcal{V}}{\mathcal{A}_{012}\mathcal{A}_{013}}\right)\left(\frac{4Q_{02}\mathcal{V}}{\mathcal{A}_{012}\mathcal{A}_{023}}\right) = \frac{4\mathcal{V}^2}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}},$$

which, by the Dual solid spread theorem, is $\mathcal{D}_0$, as required. By symmetry, all the other results hold. ■

The Dual solid spread projective theorem tells us that the B-dual solid spread, a metrical affine quantity, is analogous to the B-quadreal of a tripod. We now proceed to compute the dual solid spreads of the Khafre tetrahedron.

Example 36 (Dual solid spreads of the Khafre tetrahedron) *By the Dual solid spread theorem, the dual solid spreads of the Khafre tetrahedron $\overline{ORAP}$ are*

$$\mathcal{D}_O = \frac{4\mathcal{V}^2}{\mathcal{A}_{OAR}\mathcal{A}_{OAP}\mathcal{A}_{ORP}} = \frac{4 \times (5184)^2}{324 \times 1152 \times 576} = \frac{1}{2},$$

$$\mathcal{D}_R = \frac{4\mathcal{V}^2}{\mathcal{A}_{OAR}\mathcal{A}_{ORP}\mathcal{A}_{ARP}} = \frac{4 \times (5184)^2}{324 \times 576 \times 900} = \frac{16}{25},$$

$$\mathcal{D}_A = \frac{4\mathcal{V}^2}{\mathcal{A}_{OAR}\mathcal{A}_{OAP}\mathcal{A}_{ARP}} = \frac{4 \times (5184)^2}{324 \times 1152 \times 900} = \frac{8}{25}$$

and

$$\mathcal{D}_P = \frac{4\mathcal{V}^2}{\mathcal{A}_{OAP}\mathcal{A}_{ORP}\mathcal{A}_{ARP}} = \frac{4 \times (5184)^2}{1152 \times 576 \times 900} = \frac{9}{50}.$$

We can also use the Dual solid spread projective theorem to arrive at the same answer, given we know the spreads and dihedral spreads of $\overline{ORAP}$. We display the dual solid spreads of $\overline{ORAP}$ in Figure 2.10.

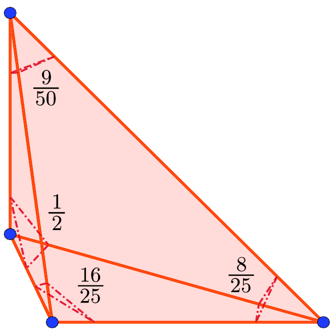

Figure 2.10: Dual solid spreads of the Khafre tetrahedron

2.3 Ratio theorems of a general tetrahedron

Here, we present some results with regards to ratios present in a tetrahedron. We consider the general tetrahedron $\overline{A_0A_1A_2A_3}$ along with all the trigonometric quantities defined in this chapter and in the previous chapter.

Firstly, we have a ratio theorem that gives a correspondence between products of opposing B-dihedral spreads and opposing B-quadrances, based on a result of Richardson [45] in the classical case.

Theorem 37 (Dihedral spread ratio theorem) *For a tetrahedron $\overline{A_0A_1A_2A_3}$ with B-quadrances Q_{ij}, B-quadreas $\mathcal{A}_{012}$, $\mathcal{A}_{013}$, $\mathcal{A}_{023}$ and $\mathcal{A}_{123}$, B-quadrume $\mathcal{V}$ and B-dihedral spreads E_{ij}, for $0 \leq i < j \leq 3$, the relation*

$$\frac{E_{01}E_{23}}{Q_{01}Q_{23}} = \frac{E_{02}E_{13}}{Q_{02}Q_{13}} = \frac{E_{03}E_{12}}{Q_{03}Q_{12}} = \frac{1}{K}$$

is satisfied, where

$$K \equiv \frac{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}}{16\mathcal{V}^2}.$$

Proof. By the Dihedral spread theorem,

$$E_{01}E_{23} = \frac{16\mathcal{V}^2}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}}Q_{01}Q_{23},$$

$$E_{02}E_{13} = \frac{16\mathcal{V}^2}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}} Q_{02}Q_{13}$$

and

$$E_{03}E_{12} = \frac{16\mathcal{V}^2}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}} Q_{03}Q_{12}.$$

Letting

$$K = \frac{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}}{16\mathcal{V}^2},$$

we then have

$$\frac{E_{01}E_{23}}{Q_{01}Q_{23}} = \frac{E_{02}E_{13}}{Q_{02}Q_{13}} = \frac{E_{03}E_{12}}{Q_{03}Q_{12}} = \frac{1}{K},$$

as required. ■

The quantity

$$K \equiv \frac{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}}{16\mathcal{V}^2}$$

is of importance in Richardson's paper [45] and has some significance for the study of the trigonometry of a general tetrahedron. We will call the constant K the **Richardson number**, which is named after the author of the cited paper. An immediate application of this is seen in the following result, which is an extension of another result of Richardson [45].

Theorem 38 (Dual solid spread ratio theorem) *For a tetrahedron $\overline{A_0A_1A_2A_3}$ with B-quadreas $\mathcal{A}_{012}$, $\mathcal{A}_{013}$, $\mathcal{A}_{023}$ and $\mathcal{A}_{123}$, B-quadrume $\mathcal{V}$, B-dual solid spreads $\mathcal{D}_0$, $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$, and Richardson number K, the relation*

$$\frac{\mathcal{D}_0}{\mathcal{A}_{123}} = \frac{\mathcal{D}_1}{\mathcal{A}_{023}} = \frac{\mathcal{D}_2}{\mathcal{A}_{013}} = \frac{\mathcal{D}_3}{\mathcal{A}_{012}} = \frac{1}{4K}$$

is satisfied.

Proof. By the Dual solid spread theorem,

$$\mathcal{D}_0 = \frac{4\mathcal{V}^2}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}}.$$

Divide through by $\mathcal{A}_{123}$ to get

$$\frac{\mathcal{D}_0}{\mathcal{A}_{123}} = \frac{4\mathcal{V}^2}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}} = \frac{1}{4K},$$

as required. The other results follow by symmetry. ■

A key consequence of this result is that a statement involving B-quadreas can also be equally applied to B-dual solid spreads, after taking into account the factor of $4K$. We will apply this in the next chapter.

Example 39 (Ratio theorems for Khafre tetrahedron) *For the Khafre tetrahedron $\overline{ORAP}$, we have*

$$\frac{E_{OR}E_{AP}}{Q_{OR}Q_{AP}} = \frac{17}{25 \times 9 \times 34} = \frac{1}{450},$$

$$\frac{E_{OA}E_{RP}}{Q_{OA}Q_{RP}} = \frac{1}{25 \times 18} = \frac{1}{450}$$

and

$$\frac{E_{OP}E_{AR}}{Q_{OP}Q_{AR}} = \frac{16}{50 \times 9 \times 16} = \frac{1}{450}.$$

Since

$$\frac{E_{01}E_{23}}{Q_{01}Q_{23}} = \frac{E_{02}E_{13}}{Q_{02}Q_{13}} = \frac{E_{03}E_{12}}{Q_{03}Q_{12}} = \frac{1}{450}$$

the Dihedral spread ratio theorem holds. Furthermore, we compute Richardson's number from the result of the Dihedral spread ratio theorem to be

$$K = 450.$$

Given the quadreas of the Khafre tetrahedron are known, we use the Dual solid spread ratio theorem to deduce that

$$\mathcal{D}_O = \frac{\mathcal{A}_{ARP}}{4K} = \frac{900}{4 \times 450} = \frac{1}{2},$$

$$\mathcal{D}_R = \frac{\mathcal{A}_{OAP}}{4K} = \frac{1152}{4 \times 450} = \frac{16}{25},$$

$$\mathcal{D}_A = \frac{\mathcal{A}_{ORP}}{4K} = \frac{576}{4 \times 450} = \frac{8}{25}$$

and

$$\mathcal{D}_P = \frac{\mathcal{A}_{OAR}}{4K} = \frac{324}{4 \times 450} = \frac{9}{50}.$$

Since these are the same quantities obtained by the Dual solid spread theorem, the Dual solid spread ratio theorem holds.

2.4 Skew quadrances of a tetrahedron

In this section, we aim to calculate the B-quadrances between opposite edges of a general tetrahedron $\overline{A_0A_1A_2A_3}$ in $\mathbb{A}^3$. This is a secondary trigonometric invariant, but can prove quite useful at times. Again, we assume that all the quantities of the tetrahedron have been previously defined for us to use without recall.

Two non-parallel lines in $\mathbb{A}^3$ are **skew** if they do not intersect. Two lines AB and CD, each with linearly independent direction vectors, are skew precisely when the points A, B, C and D are not coplanar, as seen in Hilbert and Cohn-Vossen [28, pp. 13-17]. Ideally this happens when $\overline{ABCD}$ is a non-null tetrahedron, i.e. when the quadrances of $\overline{ABCD}$ are all non-zero.

We will define the B-**skew quadrance** between two lines l and m with respective direction vectors u and v to be

$$Q_B\left(l, m\right) \equiv Q_B\left(\left(\operatorname{proj}_{u \times_B v} \overrightarrow{LM}\right)_B\right),$$

for arbitrary points L on l and M on m. If l and m are skew, then these two lines lie on parallel planes, both with common B-normal vectors $u \times_B v$. This quantity is thus invariant under selection of L and M. Note that we have previously defined

$$(\text{proj}_u\, v)_B = \left(\frac{u \cdot_B v}{u \cdot_B u}\right) u.$$

The following result, proven synthetically in the classical case by Richardson [45] and in more detail by Smith and Henderson [46] in the classical case as well, gives us the B-skew quadrances of the tetrahedron.

Theorem 40 (Skew quadrance theorem) *For a tetrahedron $\overline{A_0A_1A_2A_3}$ with B-quadrances Q_{ij} and B-quadrume $\mathcal{V}$, suppose*

$$R_{01;23} \equiv Q_B\left(A_0A_1, A_2A_3\right), \quad R_{02;13} \equiv Q_B\left(A_0A_2, A_1A_3\right)$$

and

$$R_{03;12} \equiv Q_B\left(A_0A_3, A_1A_2\right)$$

are the three B-skew quadrances of $\overline{A_0A_1A_2A_3}$. Then,

$$R_{01;23} = \frac{\mathcal{V}}{4Q_{01}Q_{23} - \left(Q_{02}+Q_{13}-Q_{03}-Q_{12}\right)^2},$$

$$R_{02;13} = \frac{\mathcal{V}}{4Q_{02}Q_{13} - \left(Q_{01}+Q_{23}-Q_{03}-Q_{12}\right)^2}$$

and

$$R_{03;12} = \frac{\mathcal{V}}{4Q_{02}Q_{13} - \left(Q_{01}+Q_{23}-Q_{03}-Q_{12}\right)^2}.$$

Proof. Let $v_i \equiv \overrightarrow{A_0A_i}$, for $i = 1,2,3$, and define

$$n \equiv v_1 \times_B \left(v_3 - v_2\right).$$

By definition, $R_{01;23}$ is given by the projection of a vector from one point on A_0A_1 to another point on A_2A_3 in the direction of n. It is convenient for us to choose the points B_0 and B_2, so that we can set $R_{01;23} \equiv Q_B\left(w\right)$, where

$$w \equiv \left(\text{proj}_n\, v_2\right)_B.$$

So,

$$\begin{aligned} R_{01;23} &= Q_B\left(\frac{v_2 \cdot_B n}{Q_B\left(n\right)} n\right) \\ &= \frac{\left(v_2 \cdot_B \left[v_1 \times_B \left(v_3 - v_2\right)\right]\right)^2}{Q_B\left(v_1 \times_B \left(v_3 - v_2\right)\right)}. \end{aligned}$$

We define

$$M \equiv \begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{pmatrix},$$

so that by the Scalar triple product theorem,

$$\begin{aligned} & (v_2 \cdot_B [v_1 \times_B (v_3 - v_2)])^2 \\ = \; & (v_2 \cdot_B [(v_1 \times_B v_3) - (v_1 \times_B v_2)])^2 \\ = \; & ([v_2, v_1, v_3]_B - [v_2, v_1, v_2]_B)^2 \\ = \; & [v_1, v_2, v_3]_B^2 \\ = \; & (\det B)^2 (\det M)^2 . \end{aligned}$$

By the Quadrume matrix product theorem,

$$(v_2 \cdot_B [v_1 \times_B (v_3 - v_2)])^2 = \frac{(\det B)\, \mathcal{V}}{4}.$$

We now use Lagrange's identity to get

$$\begin{aligned} & Q_B (v_1 \times_B (v_3 - v_2)) \\ = \; & (\det B) \left[Q_B (v_1)\, Q_B (v_3 - v_2) - (v_1 \cdot_B (v_3 - v_2))^2 \right] \\ = \; & (\det B) \left[Q_{01} Q_{23} - (v_1 \cdot_B (v_3 - v_2))^2 \right]. \end{aligned}$$

Given that

$$\begin{aligned} v_1 \cdot_B (v_3 - v_2) &= (v_1 \cdot_B v_3) - (v_1 \cdot_B v_2) \\ &= \frac{Q_{01} + Q_{03} - Q_{13}}{2} - \frac{Q_{01} + Q_{02} - Q_{12}}{2} \\ &= -\frac{1}{2} (Q_{02} + Q_{13} - Q_{03} - Q_{12}), \end{aligned}$$

we obtain

$$\begin{aligned} & Q_B (v_1 \times_B (v_3 - v_2)) \\ = \; & (\det B) \left[Q_{01} Q_{23} - \frac{(Q_{02} + Q_{13} - Q_{03} - Q_{12})^2}{4} \right] \\ = \; & \frac{\det B}{4} \left[4 Q_{01} Q_{23} - (Q_{02} + Q_{13} - Q_{03} - Q_{12})^2 \right]. \end{aligned}$$

So,

$$\begin{aligned} R_{01;23} &= \frac{(\det B)\, \mathcal{V}}{4} \times \frac{\det B}{4} \left[4 Q_{01} Q_{23} - (Q_{02} + Q_{13} - Q_{03} - Q_{12})^2 \right] \\ &= \frac{\mathcal{V}}{4 Q_{01} Q_{23} - (Q_{02} + Q_{13} - Q_{03} - Q_{12})^2}. \end{aligned}$$

The other results follow by symmetry. ■

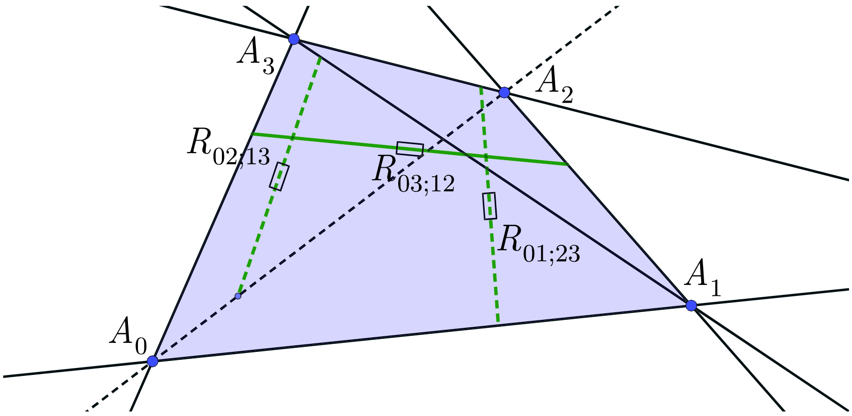

Figure 2.11: B-skew quadrances of the tetrahedron $\overline{A_0A_1A_2A_3}$

Figure 2.11 shows the B-skew quadrances present in a general tetrahedron $\overline{A_0A_1A_2A_3}$. It is also interesting for us to note that the denominator of our result is a rational form of Bretschneider's formula as seen in Bretschneider [6] and Coolidge [15], for the quadrea of a general quadrangle (a collection of four coplanar points) in terms of the six quadrances between any two of its points [68].

Example 41 (Skew quadrances of Khafre tetrahedron) *For the Khafre tetrahedron $\overline{ORAP}$, the skew quadrances between pairs of opposite lines of them are, by the Skew quadrance theorem,*

$$R_{OA;RP} = \frac{5184}{4 \times 18 \times 25 - (9 + 34 - 16 - 9)^2} = \frac{144}{41},$$

$$R_{OR;AP} = \frac{5184}{4 \times 9 \times 34 - (18 + 25 - 16 - 9)^2} = \frac{144}{25}$$

and

$$R_{OP;AR} = \frac{5184}{4 \times 16 \times 9 - (18 + 25 - 9 - 34)^2} = 9.$$

We display these skew quadrances in Figure 2.12 for the Khafre tetrahedron.

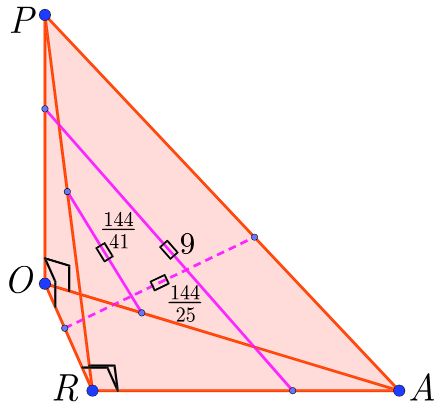

Figure 2.12: Skew quadrances of Khafre tetrahedron

Chapter 3

The Standard tetrahedron and its applications

This idea of using standard coordinates and a variable quadratic form to study general triangle geometry and trigonometry was developed in the hyperbolic case by Alkhaldi and Wildberger [67]. In Nguyen Le's doctoral thesis [35], she sets up a framework of affine triangle geometry by considering what is called a **Standard triangle**; the idea is to replace the study of a general triangle over a specific quadratic form with the study of a specific triangle over a general quadratic form. We draw motivation from the aforementioned ideas to create a similar framework with regards to a general tetrahedron.

Consider an affine map which sends a general tetrahedron $\overline{A_0A_1A_2A_3}$ to the tetrahedron $\overline{X_0X_1X_2X_3}$, where

$$X_0 \equiv [0,0,0]\,, \quad X_1 \equiv [1,0,0]\,, \quad X_2 \equiv [0,1,0] \quad \text{and} \quad X_3 \equiv [0,0,1]\,.$$

Such a tetrahedron will be called the **Standard tetrahedron** (see Figure 3.1). This affine

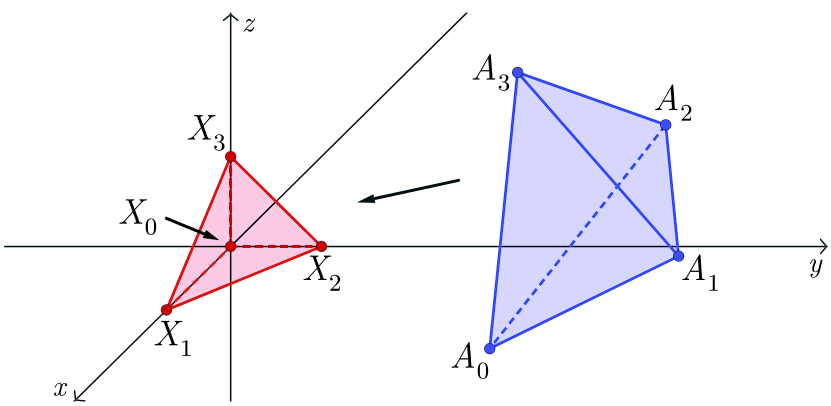

Figure 3.1: An affine map from $\overline{A_0A_1A_2A_3}$ to $\overline{X_0X_1X_2X_3}$

map can be defined by translating the point A_0 of the general tetrahedron $\overline{A_0A_1A_2A_3}$ to X_0 and then applying a linear map with matrix representation L to send the other three vertices to X_1, X_2 and X_3. If we equip $\mathbb{V}^3$ with a symmetric bilinear form with matrix representation C then this affine mapping induces a new symmetric bilinear form, defined by

$$\begin{aligned} u \cdot_C v &= uCv^T = u\left(LL^{-1}\right)C\left(LL^{-1}\right)^T v^T \\ &= (uL)\left[\left(L^{-1}\right)C\left(L^{-1}\right)^T\right](vL)^T. \end{aligned}$$

For $M \equiv L^{-1}$, we set the matrix MCM^T to be the matrix

$$B = \begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix}$$

that has been previously defined, so that

$$u \cdot_C v = (uL) \cdot_B (vL).$$

We use the matrix B to represent the induced symmetric bilinear form so that the adjugate matrix

$$\operatorname{adj} B = \begin{pmatrix} \alpha_1 & \beta_3 & \beta_2 \\ \beta_3 & \alpha_2 & \beta_1 \\ \beta_2 & \beta_1 & \alpha_3 \end{pmatrix}$$

will be available to us when we perform our calculations in this chapter. This matrix plays an important role in this method of studying the rational trigonometry of a general tetrahedron over a general metrical framework.

Example 42 (Induced symmetric bilinear form for the Khafre tetrahedron) *Consider the Khafre tetrahedron $\overline{ORAP}$ in $\mathbb{A}^3$, where*

$$O \equiv [0,0,0], \quad R \equiv [3,0,0], \quad A \equiv [3,3,0] \quad \text{and} \quad P \equiv [0,0,4],$$

and equip its associated vector space $\mathbb{V}^3$ with the Euclidean bilinear form, where its matrix representation is given by I, the 3×3 identity matrix. We consider an affine map which sends the points O, R, A and P respectively to the points X_0, X_1, X_2 and X_3, which ultimately degenerates to a linear map. Representing such a linear map by L, if $M \equiv L^{-1}$ then

$$M = \begin{pmatrix} 3 & 0 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

This linear map induces a new symmetric bilinear form with matrix representation

$$MM^T = \begin{pmatrix} 3 & 0 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}^T = \begin{pmatrix} 9 & 9 & 0 \\ 9 & 18 & 0 \\ 0 & 0 & 16 \end{pmatrix}.$$

So if we are to recalculate our trigonometric quantities for the Khafre tetrahedron with respect to the Standard tetrahedron $\overline{X_0X_1X_2X_3}$*, we simply need to calculate the trigonometric quantities of* $\overline{X_0X_1X_2X_3}$ *over this new symmetric bilinear form.*

Oftentimes a vector proof is an easy, time-saving process. When such a proof becomes too complicated we can use the above trigonometric quantities for the Standard tetrahedron to prove a result using brute force. This gives a powerful general technology for establishing results on this subject. Note that a proof involving substitution of the trigonometric quantities of the Standard tetrahedron is general in nature, since it can be obtained from a general tetrahedron by an affine map which preserves the geometric structure of the objects of interest. Because of this, we will see that such a technique will be applied when proving the subsequent results in this chapter. For the rest of this chapter, we will assume the Standard tetrahedron to be available to us with all the quantities defined and evaluated as below.

3.1 Trigonometric quantities of the Standard tetrahedron

In what follows, we define useful quantities associated to the matrices B and $\operatorname{adj} B$ as

$$D \equiv \alpha_1 + \alpha_2 + \alpha_3 + 2\beta_1 + 2\beta_2 + 2\beta_3,$$

$$r_1 \equiv a_2 + a_3 - 2b_1, \quad r_2 \equiv a_1 + a_3 - 2b_2, \quad r_3 \equiv a_1 + a_2 - 2b_3$$

and

$$\Delta \equiv \det B = a_1a_2a_3 + 2b_1b_2b_3 - a_1b_1^2 - a_2b_2^2 - a_3b_3^2.$$

Suppose $Q_{ij} \equiv Q_B(X_i, X_j)$ for $0 \le i < j \le 3$. Then,

$$\begin{aligned} Q_{01} &= Q_B(X_0, X_1) = Q_B\left(\overrightarrow{X_0X_1}\right) \\ &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = a_1. \end{aligned}$$

Similarly, we will have

$$Q_{02} = a_2 \quad \text{and} \quad Q_{03} = a_3.$$

We also have

$$\begin{aligned}
Q_{23} &= Q_B(X_2, X_3) = Q_B\left(\overrightarrow{X_2X_3}\right) \\
&= \begin{pmatrix} 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \\
&= a_2 + a_3 - 2b_1 = r_1,
\end{aligned}$$

and similarly,

$$Q_{13} = r_2 \quad \text{and} \quad Q_{12} = r_3.$$

Define the B-quadreas associated to each triangle of the Standard tetrahedron to be

$$\mathcal{A}_{012} \equiv \mathcal{A}_B\left(\overline{X_0X_1X_2}\right), \quad \mathcal{A}_{013} \equiv \mathcal{A}_B\left(\overline{X_0X_1X_3}\right), \quad \mathcal{A}_{023} \equiv \mathcal{A}_B\left(\overline{X_0X_2X_3}\right)$$

and

$$\mathcal{A}_{123} \equiv \mathcal{A}_B\left(\overline{X_1X_2X_3}\right).$$

By the definition of the B-quadrea,

$$\begin{aligned}
\mathcal{A}_{012} &= A(Q_{01}, Q_{02}, Q_{12}) = (a_1 + a_2 + r_3)^2 - 2\left(a_1^2 + a_2^2 + r_3^2\right) \\
&= 4\left(a_1a_2 - b_3^2\right) = 4\alpha_3.
\end{aligned}$$

Similarly,

$$\mathcal{A}_{013} = 4\alpha_2 \quad \text{and} \quad \mathcal{A}_{023} = 4\alpha_1.$$

Finally, we have

$$\begin{aligned}
\mathcal{A}_{123} &= A(r_1, r_2, r_3) = (r_1 + r_2 + r_3)^2 - 2\left(r_1^2 + r_2^2 + r_3^2\right) \\
&= 4(a_1 + a_2 + a_3 - b_1 - b_2 - b_3)^2 \\
&\quad -2\left((a_2 + a_3 - 2b_1)^2 + (a_1 + a_3 - 2b_2)^2 + (a_1 + a_2 - 2b_3)^2\right) \\
&= 4\left[\left(a_2a_3 - b_1^2\right) + \left(a_1a_3 - b_2^2\right) + \left(a_1a_2 - b_3^2\right)\right] \\
&\quad +8\left[(b_1b_2 - a_3b_3) + (b_1b_3 - a_2b_2) + (b_2b_3 - a_1b_1)\right] \\
&= 4(\alpha_1 + \alpha_2 + \alpha_3 + 2\beta_1 + 2\beta_2 + 2\beta_3) \\
&= 4D.
\end{aligned}$$

By the Quadrume theorem, the B-quadrume of the Standard tetrahedron $\overline{X_0X_1X_2X_3}$

is

$$\begin{aligned}
\mathcal{V} &\equiv \mathcal{V}_B\left(\overline{X_0X_1X_2X_3}\right) \\
&= \frac{1}{2}\det\begin{pmatrix} 2a_1 & a_1+a_2-r_3 & a_1+a_3-r_2 \\ a_1+a_2-r_3 & 2a_2 & a_2+a_3-r_1 \\ a_1+a_3-r_2 & a_2+a_3-r_1 & 2a_3 \end{pmatrix} \\
&= 4\det\begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix} = 4\Delta.
\end{aligned}$$

If $s_{i;jk} \equiv s_B\left(X_iX_j, X_iX_k\right)$ for $0 \le i \le 3$ and $0 \le j < k \le 3$ with j and k not equal to i, then by the Quadrea spread theorem,

$$s_{0;12} = \frac{\mathcal{A}_{012}}{4Q_{01}Q_{02}} = \frac{\alpha_3}{a_1a_2}.$$

Similarly,

$$s_{1;02} = \frac{\alpha_3}{a_1r_3} \quad \text{and} \quad s_{2;01} = \frac{\alpha_3}{a_2r_3},$$

and the remaining B-spreads of the Standard tetrahedron are

$$\begin{gathered}
s_{0;13} = \frac{\alpha_2}{a_1a_3}, \quad s_{1;02} = \frac{\alpha_2}{a_1r_2}, \quad s_{2;01} = \frac{\alpha_2}{a_3r_2}, \\
s_{0;23} = \frac{\alpha_1}{a_2a_3}, \quad s_{2;03} = \frac{\alpha_1}{a_2r_1}, \quad s_{3;02} = \frac{\alpha_1}{a_3r_1}, \\
s_{1;23} = \frac{D}{r_2r_3}, \quad s_{2;13} = \frac{D}{r_1r_3} \quad \text{and} \quad s_{3;12} = \frac{D}{r_1r_2}.
\end{gathered}$$

Using the Dihedral spread theorem,

$$\begin{aligned}
E_{01} &\equiv E_B\left(X_0X_1X_2, X_0X_1X_3\right) = \frac{4Q_{01}\mathcal{V}}{\mathcal{A}_{012}\mathcal{A}_{013}} \\
&= \frac{4a_1\left(4\Delta\right)}{\left(4\alpha_2\right)\left(4\alpha_3\right)} = \frac{a_1\Delta}{\alpha_2\alpha_3}.
\end{aligned}$$

By defining and calculating the other B-dihedral spreads of the Standard tetrahedron similarly, they will evaluate to

$$E_{02} = \frac{a_2\Delta}{\alpha_1\alpha_3}, \quad E_{03} = \frac{a_3\Delta}{\alpha_1\alpha_2},$$

$$E_{23} = \frac{r_1\Delta}{\alpha_1 D}, \quad E_{13} = \frac{r_2\Delta}{\alpha_2 D} \quad \text{and} \quad E_{12} = \frac{r_3\Delta}{\alpha_3 D}.$$

If $C_{ij} \equiv 1 - E_{ij}$ are the respective B-dihedral crosses of the Standard tetrahedron, then

$$\begin{aligned} C_{01} &= 1 - \frac{a_1 \Delta}{\alpha_2 \alpha_3} \\ &= \frac{\left(a_1 a_3 - b_2^2\right)\left(a_1 a_2 - b_3^2\right) - a_1\left(a_1 a_2 a_3 + 2 b_1 b_2 b_3 - a_1 b_1^2 - a_2 b_2^2 - a_3 b_3^2\right)}{\alpha_2 \alpha_3} \\ &= \frac{\left(a_1 b_1 - b_2 b_3\right)^2}{\alpha_2 \alpha_3} = \frac{\beta_1^2}{\alpha_2 \alpha_3}, \end{aligned}$$

and similarly

$$C_{02} = \frac{\beta_2^2}{\alpha_1 \alpha_3}, \quad C_{03} = \frac{\beta_3^2}{\alpha_1 \alpha_2},$$

$$C_{23} = \frac{\left(\alpha_1 + \beta_2 + \beta_3\right)^2}{\alpha_1 D}, \quad C_{13} = \frac{\left(\alpha_2 + \beta_1 + \beta_3\right)^2}{\alpha_2 D} \quad \text{and} \quad C_{12} = \frac{\left(\alpha_3 + \beta_1 + \beta_2\right)^2}{\alpha_3 D}.$$

Using the Solid spread theorem, the B-solid spread $\mathcal{S}_0 \equiv \mathcal{S}_B\left(X_0 X_1, X_0 X_2, X_0 X_3\right)$ evaluates to

$$\mathcal{S}_0 = \frac{\mathcal{V}}{4 Q_{01} Q_{02} Q_{03}} = \frac{\Delta}{a_1 a_2 a_3}.$$

We define the remaining B-solid spreads of the Standard tetrahedron similarly and obtain

$$\mathcal{S}_1 = \frac{\Delta}{a_1 r_2 r_3}, \quad \mathcal{S}_2 = \frac{\Delta}{a_2 r_1 r_3} \quad \text{and} \quad \mathcal{S}_3 = \frac{\Delta}{a_3 r_1 r_2}.$$

We can also use the Dual solid spread theorem to compute the B-dual solid spreads of the Standard tetrahedron. We will define and evaluate them as

$$\mathcal{D}_0 = \frac{\Delta^2}{\alpha_1 \alpha_2 \alpha_3}, \quad \mathcal{D}_1 = \frac{\Delta^2}{\alpha_2 \alpha_3 D}, \quad \mathcal{D}_2 = \frac{\Delta^2}{\alpha_1 \alpha_3 D} \quad \text{and} \quad \mathcal{D}_3 = \frac{\Delta^2}{\alpha_1 \alpha_2 D}.$$

Example 43 (Trigonometric quantities of the Khafre tetrahedron) *Recall from Example 42 that for the Khafre tetrahedron $\overline{ORAP}$, the induced symmetric bilinear form after a linear map from $\overline{ORAP}$ to the Standard tetrahedron $\overline{X_0 X_1 X_2 X_3}$ has matrix representation*

$$B = \begin{pmatrix} 9 & 9 & 0 \\ 9 & 18 & 0 \\ 0 & 0 & 16 \end{pmatrix}.$$

We take note of the fact that

$$\operatorname{adj} B = \begin{pmatrix} 288 & -144 & 0 \\ -144 & 144 & 0 \\ 0 & 0 & 81 \end{pmatrix},$$

as well as

$$D = 288 + 144 + 81 + 2 \times (-144) = 225,$$

$$r_1 = 18 + 16 = 34, \quad r_2 = 9 + 16 = 25, \quad r_3 = 9 + 18 - 18 = 9$$

and

$$\Delta = \det B = 1296.$$

By substitution of the relevant quantities into the above formulas, the Khafre tetrahedron $\overline{ORAP}$ *has quadrances*

$$Q_{OR} = 9, \quad Q_{OA} = 18, \quad Q_{OP} = 16,$$

$$Q_{AP} = 34, \quad Q_{RP} = 25 \quad \text{and} \quad Q_{RA} = 9,$$

quadreas

$$\mathcal{A}_{ORA} = 4 \times 81 = 324, \quad \mathcal{A}_{ORP} = 4 \times 144 = 576,$$

$$\mathcal{A}_{OAP} = 4 \times 288 = 1152 \quad \text{and} \quad \mathcal{A}_{ARP} = 4 \times 225 = 900,$$

quadrume

$$\mathcal{V} = 4 \times 1296 = 5184,$$

spreads

$$s_{O;RA} = \frac{81}{9 \times 18} = \frac{1}{2}, \quad s_{R;OA} = \frac{81}{9 \times 9} = 1, \quad s_{A;OR} = \frac{81}{9 \times 18} = \frac{1}{2},$$

$$s_{O;RP} = \frac{144}{9 \times 16} = 1, \quad s_{R;OP} = \frac{144}{9 \times 25} = \frac{16}{25}, \quad s_{P;OR} = \frac{144}{16 \times 25} = \frac{9}{25},$$

$$s_{O;AP} = \frac{288}{18 \times 16} = 1, \quad s_{A;OP} = \frac{288}{18 \times 34} = \frac{8}{17}, \quad s_{P;OA} = \frac{288}{16 \times 34} = \frac{9}{17},$$

$$s_{A;RP} = \frac{225}{34 \times 9} = \frac{25}{34}, \quad s_{R;AP} = \frac{225}{25 \times 9} = 1 \quad \text{and} \quad s_{P;AR} = \frac{225}{34 \times 25} = \frac{9}{34},$$

dihedral spreads

$$E_{OR} = \frac{9 \times 1296}{144 \times 81} = 1, \quad E_{AP} = \frac{34 \times 1296}{288 \times 225} = \frac{17}{25},$$

$$E_{OA} = \frac{18 \times 1296}{81 \times 288} = 1, \quad E_{RP} = \frac{25 \times 1296}{144 \times 225} = 1,$$

$$E_{OP} = \frac{16 \times 1296}{144 \times 288} = \frac{1}{2} \quad \text{and} \quad E_{RA} = \frac{9 \times 1296}{81 \times 225} = \frac{16}{25},$$

dihedral crosses

$$C_{OR} = 0, \quad C_{OA} = 0, \quad C_{OP} = \frac{1}{2},$$

$$C_{AP} = \frac{8}{25}, \quad C_{RP} = 0 \quad \text{and} \quad C_{RA} = \frac{9}{25},$$

solid spreads

$$\mathcal{S}_O = \frac{1296}{9 \times 18 \times 16} = \frac{1}{2}, \quad \mathcal{S}_R = \frac{1296}{9 \times 25 \times 9} = \frac{16}{25},$$

$$\mathcal{S}_A = \frac{1296}{34 \times 18 \times 9} = \frac{4}{17} \quad \text{and} \quad \mathcal{S}_P = \frac{1296}{34 \times 25 \times 16} = \frac{81}{850},$$

and dual solid spreads

$$\mathcal{D}_O = \frac{1296^2}{81 \times 144 \times 288} = \frac{1}{2}, \quad \mathcal{D}_R = \frac{1296^2}{81 \times 144 \times 225} = \frac{16}{25},$$

$$\mathcal{D}_A = \frac{1296^2}{81 \times 288 \times 225} = \frac{8}{25} \quad \text{and} \quad \mathcal{D}_P = \frac{1296^2}{144 \times 288 \times 225} = \frac{9}{50}.$$

These are the exact same quantities that we have obtained previously for $\overline{ORAP}$ *in Chapter 2.*

3.2 Tetrahedron cross law

The relationship between the B-quadreas associated to the faces of a general tetrahedron has been a study of interest for a long while, dating back to Richardson's paper [45] on the trigonometry of the tetrahedron in the classical framework. The Dual solid spread ratio theorem gave us the fact that the ratio of the B-quadreas to the B-dual solid spreads is constant; inspired by this, we aim to derive an algebraic relationship between the B-quadreas of a tetrahedron. While no direct relationship exists, we may derive a relationship by involving the three B-dihedral crosses of a tetrahedron which emanate from one of its points. Proven in the classical case by Lee [37], it is sufficient for us to prove this result in the rational case for the Standard tetrahedron $\overline{X_0X_1X_2X_3}$, as an affine map acting on this tetrahedron will leave the final result invariant.

Theorem 44 (Tetrahedron cross law) *For the tetrahedron* $\overline{A_0A_1A_2A_3}$ *with B-quadreas*

$$\mathcal{A}_{012} \equiv \mathcal{A}_B\left(\overline{A_0A_1A_2}\right), \quad \mathcal{A}_{013} \equiv \mathcal{A}_B\left(\overline{A_0A_1A_3}\right),$$

$$\mathcal{A}_{023} \equiv \mathcal{A}_B\left(\overline{A_0A_2A_3}\right) \quad \text{and} \quad \mathcal{A}_{123} \equiv \mathcal{A}_B\left(\overline{A_1A_2A_3}\right),$$

and B-dihedral crosses

$$C_{01} \equiv C_B\left(A_0A_1A_2, A_0A_1A_3\right), \quad C_{02} \equiv C_B\left(A_0A_1A_2, A_0A_2A_3\right)$$

and

$$C_{03} \equiv C_B\left(A_0A_1A_3, A_0A_2A_3\right),$$

we have the relation

$$\begin{aligned} &\left[\begin{array}{c} \left[(\mathcal{A}_{012} + \mathcal{A}_{013} + \mathcal{A}_{023} - \mathcal{A}_{123})^2 - 4\left(\mathcal{A}_{012}\mathcal{A}_{013}C_{01} + \mathcal{A}_{012}\mathcal{A}_{023}C_{02} + \mathcal{A}_{013}\mathcal{A}_{023}C_{03}\right)^2\right] \\ -64\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\left(\mathcal{A}_{012}C_{01}C_{02} + \mathcal{A}_{013}C_{01}C_{03} + \mathcal{A}_{023}C_{02}C_{03}\right) \end{array}\right]^2 \\ = \; & 4096\mathcal{A}_{012}^2\mathcal{A}_{013}^2\mathcal{A}_{023}^2C_{01}C_{02}C_{03}\left(\mathcal{A}_{012} + \mathcal{A}_{013} + \mathcal{A}_{023} - \mathcal{A}_{123}\right)^2. \end{aligned}$$

Proof. Define the **Tetrahedron cross function** to be

$$T(a,b,c,d,x,y,z) \equiv \left(\begin{array}{c} \left((a+b+c-d)^2 - 4(abx+acy+bcz)\right)^2 \\ -64abc(axy+bxz+cyz) \end{array} \right)^2 - 4096a^2b^2c^2xyz(a+b+c-d)^2,$$

so that we are required to prove the following:

$$T(\mathcal{A}_{012}, \mathcal{A}_{013}, \mathcal{A}_{023}, \mathcal{A}_{123}, C_{01}, C_{02}, C_{03}) = 0.$$

Substituting the quantities associated to the Standard tetrahedron, we obtain

$$\begin{aligned} & T(\mathcal{A}_{012}, \mathcal{A}_{013}, \mathcal{A}_{023}, \mathcal{A}_{123}, C_{01}, C_{02}, C_{03}) \\ = \; & 2^{16}\left[\left[(\alpha_1+\alpha_2+\alpha_3-D)^2 - 4\left(\beta_1^2+\beta_2^2+\beta_3^2\right)\right]^2 - 64\left(\beta_1^2\beta_2^2+\beta_1^2\beta_3^2+\beta_2^2\beta_3^2\right)\right]^2 \\ & -2^{28}\beta_1^2\beta_2^2\beta_3^2(\alpha_1+\alpha_2+\alpha_3-D)^2 \end{aligned}$$

We use the definition of D involved in the calculation of the trigonometric quantities of the Standard tetrahedron to obtain

$$\begin{aligned} & T(\mathcal{A}_{012}, \mathcal{A}_{013}, \mathcal{A}_{023}, \mathcal{A}_{123}, C_{01}, C_{02}, C_{03}) \\ = \; & 2^{16}\left(\begin{array}{c} \left[\left[4(\beta_1+\beta_2+\beta_3)^2 - 4\left(\beta_1^2+\beta_2^2+\beta_3^2\right)\right]^2 - 64\left(\beta_1^2\beta_2^2+\beta_1^2\beta_3^2+\beta_2^2\beta_3^2\right)\right]^2 \\ -2^{14}\beta_1^2\beta_2^2\beta_3^2(\beta_1+\beta_2+\beta_3)^2 \end{array} \right) \\ = \; & 2^{16}\left[2^{14}\beta_1^2\beta_2^2\beta_3^2(\beta_1+\beta_2+\beta_3)^2 - 2^{14}\beta_1^2\beta_2^2\beta_3^2(\beta_1+\beta_2+\beta_3)^2\right] = 0, \end{aligned}$$

as required. ■

Note that three similar relations can be obtained by permuting the indices. We now present a novel result (as a corollary) which can give us a reformulation of the Tetrahedron cross law in terms of the dual solid spreads instead of the quadreas. Recall that K is the Richardson number we defined in the previous chapter.

Corollary 45 *For a general tetrahedron $\overline{A_0A_1A_2A_3}$ with B-quadreas $\mathcal{A}_{012}$, $\mathcal{A}_{013}$, $\mathcal{A}_{023}$ and $\mathcal{A}_{123}$, B-dihedral crosses C_{01}, C_{02} and C_{03}, Richardson number K and B-dual solid spreads $\mathcal{D}_0$, $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$, the following relation is satisfied:*

$$T(\mathcal{A}_{012}, \mathcal{A}_{013}, \mathcal{A}_{023}, \mathcal{A}_{123}, C_{01}, C_{02}, C_{03}) = (4K)^8 \times T(\mathcal{D}_3, \mathcal{D}_2, \mathcal{D}_1, \mathcal{D}_0, C_{01}, C_{02}, C_{03}).$$

Proof. We take the reciprocal of each equality of the Dual solid spread theorem to obtain

$$\frac{\mathcal{A}_{123}}{\mathcal{D}_0} = \frac{\mathcal{A}_{023}}{\mathcal{D}_1} = \frac{\mathcal{A}_{013}}{\mathcal{D}_2} = \frac{\mathcal{A}_{012}}{\mathcal{D}_3} = 4K.$$

From this, we find that

$$
\begin{aligned}
& T\left(\mathcal{A}_{012}, \mathcal{A}_{013}, \mathcal{A}_{023}, \mathcal{A}_{123}, C_{01}, C_{02}, C_{03}\right) \\
= \; & T\left(4K\mathcal{D}_3, 4K\mathcal{D}_2, 4K\mathcal{D}_1, 4K\mathcal{D}_0, C_{01}, C_{02}, C_{03}\right) \\
= \; & (4K)^8 \left[\begin{array}{c} \left[\begin{array}{c} \left[\left(\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3-\mathcal{D}_0\right)^2 - 4\left(\mathcal{D}_2\mathcal{D}_3C_{01}+\mathcal{D}_1\mathcal{D}_3C_{02}+\mathcal{D}_1\mathcal{D}_2C_{03}\right)\right]^2 \\ -64\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3\left(\mathcal{D}_3C_{01}C_{02}+\mathcal{D}_2C_{01}C_{03}+\mathcal{D}_1C_{02}C_{03}\right) \end{array} \right]^2 \\ -4096\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3C_{01}C_{02}C_{03}\left(\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3-\mathcal{D}_0\right)^2 \end{array} \right] \\
= \; & (4K)^8 \times T\left(\mathcal{D}_3, \mathcal{D}_2, \mathcal{D}_1, \mathcal{D}_0, C_{01}, C_{02}, C_{03}\right),
\end{aligned}
$$

as required. ■

As a result of this corollary, we can replace the B-quadreas in the Tetrahedron cross law with the ppposing B-dual solid spreads, so that the Tetrahedron cross law holds for B-dual solid spreads also. We also note that by permuting the indices we obtain three similar relations to the above result. We now verify the Tetrahedron cross law for the Khafre tetrahedron given in Chapter 2.

Example 46 (Tetrahedron cross law on Khafre tetrahedron) *For the Khafre tetrahedron $\overline{ORAP}$, we note the important observation that $C_{OR} = C_{OA} = C_{RP} = 0$. This helps us greatly reduce our equations to*

$$
\begin{aligned}
& T\left(\mathcal{A}_{ORA}, \mathcal{A}_{ORP}, \mathcal{A}_{OAP}, \mathcal{A}_{RAP}, C_{OR}, C_{OA}, C_{OP}\right) \\
= \; & \left(\left(\mathcal{A}_{ORA}+\mathcal{A}_{ORP}+\mathcal{A}_{OAP}-\mathcal{A}_{RAP}\right)^2 - 4\mathcal{A}_{ORP}\mathcal{A}_{OAP}C_{OP}\right)^4,
\end{aligned}
$$

$$
\begin{aligned}
& T\left(\mathcal{A}_{ORA}, \mathcal{A}_{ORP}, \mathcal{A}_{RAP}, \mathcal{A}_{OAP}, C_{OR}, C_{RA}, C_{RP}\right) \\
= \; & \left(\left(\mathcal{A}_{ORA}+\mathcal{A}_{ORP}+\mathcal{A}_{RAP}-\mathcal{A}_{OAP}\right)^2 - 4\mathcal{A}_{ORA}\mathcal{A}_{RAP}C_{RA}\right)^4,
\end{aligned}
$$

$$
\begin{aligned}
& T\left(\mathcal{A}_{ORA}, \mathcal{A}_{OAP}, \mathcal{A}_{RAP}, \mathcal{A}_{ORP}, C_{OA}, C_{RA}, C_{AP}\right) \\
= \; & \left(\begin{array}{c} \left(\left(\mathcal{A}_{ORA}+\mathcal{A}_{OAP}+\mathcal{A}_{RAP}-\mathcal{A}_{ORP}\right)^2 - 4\left(\mathcal{A}_{ORA}\mathcal{A}_{RAP}C_{RA}+\mathcal{A}_{OAP}\mathcal{A}_{RAP}C_{AP}\right)\right)^2 \\ -64\mathcal{A}_{ORA}\mathcal{A}_{OAP}\mathcal{A}_{RAP}^2C_{RA}C_{AP} \end{array} \right)^2
\end{aligned}
$$

and

$$
\begin{aligned}
& T\left(\mathcal{A}_{ORP}, \mathcal{A}_{OAP}, \mathcal{A}_{RAP}, \mathcal{A}_{ORA}, C_{OP}, C_{RP}, C_{AP}\right) \\
= \; & \left(\begin{array}{c} \left(\left(\mathcal{A}_{ORP}+\mathcal{A}_{OAP}+\mathcal{A}_{RAP}-\mathcal{A}_{ORA}\right)^2 - 4\left(\mathcal{A}_{ORP}\mathcal{A}_{OAP}C_{OP}+\mathcal{A}_{OAP}\mathcal{A}_{RAP}C_{AP}\right)\right)^2 \\ -64\mathcal{A}_{ORP}\mathcal{A}_{OAP}^2\mathcal{A}_{RAP}C_{OP}C_{AP} \end{array} \right)^2.
\end{aligned}
$$

We substitute our required quantities to obtain

$$\begin{aligned} & T\left(\mathcal{A}_{ORA}, \mathcal{A}_{ORP}, \mathcal{A}_{OAP}, \mathcal{A}_{RAP}, C_{OR}, C_{OA}, C_{OP}\right) \\ = & \left((324+576+1152-900)^2 - 4\,(576)\,(1152)\left(\frac{1}{2}\right)\right)^4 = 0, \end{aligned}$$

$$\begin{aligned} & T\left(\mathcal{A}_{ORA}, \mathcal{A}_{ORP}, \mathcal{A}_{RAP}, \mathcal{A}_{OAP}, C_{OR}, C_{RA}, C_{RP}\right) \\ = & \left((324+576+900-1152)^2 - 4\,(324)\,(900)\left(\frac{9}{25}\right)\right)^4 = 0, \end{aligned}$$

$$\begin{aligned} & T\left(\mathcal{A}_{ORA}, \mathcal{A}_{OAP}, \mathcal{A}_{RAP}, \mathcal{A}_{ORP}, C_{OA}, C_{RA}, C_{AP}\right) \\ = & \left(\begin{array}{c} \left((324+1152+900-576)^2 - 4\left((324)\,(900)\left(\frac{9}{25}\right) + (1152)\,(900)\left(\frac{8}{25}\right)\right)\right)^2 \\ -64\,(324)\,(1152)\,(900)^2\left(\frac{9}{25}\right)\left(\frac{8}{25}\right) \end{array}\right)^2 \\ = & \left(1492\,992^2 - 2229\,025\,112\,064\right)^2 = 0 \end{aligned}$$

and

$$\begin{aligned} & T\left(\mathcal{A}_{ORP}, \mathcal{A}_{OAP}, \mathcal{A}_{RAP}, \mathcal{A}_{ORA}, C_{OP}, C_{RP}, C_{AP}\right) \\ = & \left(\begin{array}{c} \left((576+1152+900-324)^2 - 4\left((576)\,(1152)\left(\frac{1}{2}\right) + (1152)\,(900)\left(\frac{8}{25}\right)\right)\right)^2 \\ -64\,(576)\,(1152)^2\,(900)\left(\frac{1}{2}\right)\left(\frac{8}{25}\right) \end{array}\right)^2 \\ = & \left(2654\,208^2 - 7044\,820\,107\,264\right)^2 = 0. \end{aligned}$$

This completes the verification of all possible cases of the Tetrahedron cross law for the Khafre tetrahedron.

3.3 Dihedral cross relation

In this section we investigate the relation between the six B-dihedral crosses of a general tetrahedron. While introduced classically by Richardson [45], a rational version was proposed without proof in [63]. We now present this result with proof.

Theorem 47 (Dihedral cross relation) *For the tetrahedron $\overline{A_0A_1A_2A_3}$ with B-dihedral crosses*

$$C_{01} \equiv C_B\left(A_0A_1A_2, A_0A_1A_3\right), \quad C_{23} \equiv C_B\left(A_0A_2A_3, A_1A_2A_3\right),$$

$$C_{02} \equiv C_B\left(A_0A_1A_2, A_0A_2A_3\right), \quad C_{13} \equiv C_B\left(A_0A_1A_3, A_1A_2A_3\right),$$

$$C_{03} \equiv C_B\left(A_0A_1A_3, A_0A_2A_3\right) \quad \textit{and} \quad C_{12} \equiv C_B\left(A_0A_1A_2, A_1A_2A_3\right),$$

define the variables

$$X \equiv C_{01}C_{23}, \quad Y \equiv C_{02}C_{13}, \quad Z \equiv C_{03}C_{12},$$

$$x \equiv C_{01} + C_{23}, \quad y \equiv C_{02} + C_{13}, \quad z \equiv C_{03} + C_{12},$$

$$W \equiv C_{01}C_{02}C_{03} + C_{01}C_{12}C_{13} + C_{02}C_{12}C_{23} + C_{03}C_{13}C_{23},$$

$$P \equiv \frac{1}{2}\left(1 - x - y - z + X + Y + Z\right),$$

$$R \equiv P + z - Z, \quad S \equiv P + y - Y, \quad T \equiv P + x - X,$$

$$U \equiv \frac{1}{2}\left(P^2 - W + XY + XZ + YZ\right)$$

and

$$V \equiv \frac{1}{2}\left(U^2 - XYR^2 - XZS^2 - YZT^2\right).$$

Then, we have that

$$V^2 = XYZ\left(XR^2S^2 + YR^2T^2 + ZS^2T^2 + 2RSTU\right).$$

Proof. Given the B-dihedral crosses of the Standard tetrahedron $\overline{X_0X_1X_2X_3}$ that we have computed earlier, we can use a computer to obtain

$$\begin{aligned}
V^2 &= \frac{\beta_1^2\beta_2^2\beta_3^2}{\alpha_1^8\alpha_2^8\alpha_3^8D^8}\left(\det\left(\operatorname{adj}B\right)\right)^2\left(\alpha_1+\beta_2+\beta_3\right)^2\left(\alpha_2+\beta_1+\beta_3\right)^2\left(\alpha_3+\beta_1+\beta_2\right)^2 \\
&\quad \left(\begin{array}{c}
2\beta_1\beta_2\beta_3\left(\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3\right)\left(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3\right) \\
+2\beta_1\beta_2\beta_3\left(\alpha_1\beta_1^3+\alpha_2\beta_2^3+\alpha_3\beta_3^3\right)-\left(\alpha_1^2\beta_1^5+\alpha_2^2\beta_2^5+\alpha_3^2\beta_3^5\right) \\
-4\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3\left(\beta_1+\beta_2+\beta_3\right)-\alpha_1\left(\alpha_2^2\beta_2^4+\alpha_3^2\beta_3^4-\beta_1^4\left(\beta_2^2+\beta_3^2\right)\right) \\
-\alpha_2\left(\alpha_1^2\beta_1^4+\alpha_3^2\beta_3^4-\beta_2^4\left(\beta_1^2+\beta_3^2\right)\right)-\alpha_3\left(\alpha_1^2\beta_1^4+\alpha_2^2\beta_2^4-\beta_3^4\left(\beta_1^2+\beta_2^2\right)\right) \\
-\alpha_1\alpha_2\alpha_3\left(\alpha_1\alpha_2\left(\beta_1+\beta_2\right)^2+\alpha_1\alpha_3\left(\beta_1+\beta_3\right)^2+\alpha_2\alpha_3\left(\beta_2+\beta_3\right)^2\right) \\
-2\alpha_1\alpha_2\alpha_3\left(\beta_1^2\beta_2^2+\beta_1^2\beta_3^2+\beta_2^2\beta_3^2\right)-\alpha_1^2\beta_1\left(\alpha_2^2\alpha_3^2+\alpha_2^2\beta_2^2+\alpha_3^2\beta_3^2\right) \\
-\alpha_2^2\beta_2\left(\alpha_1^2\alpha_3^2+\alpha_1^2\beta_1^2+\alpha_3^2\beta_3^2\right)-\alpha_3^2\beta_3\left(\alpha_1^2\alpha_2^2+\alpha_1^2\beta_1^2+\alpha_2^2\beta_2^2\right) \\
-\beta_1\beta_2\beta_3\left(\beta_1\beta_2\left(\beta_1^2+\beta_2^2-2\beta_3^2\right)+\beta_1\beta_3\left(\beta_1^2-2\beta_2^2+\beta_3^2\right)+\beta_2\beta_3\left(-2\beta_1^2+\beta_2^2+\beta_3^2\right)\right)
\end{array}\right)^2 \\
&= XYZ\left(XR^2S^2+YR^2T^2+ZS^2T^2+2RSTU\right).
\end{aligned}$$

as required. ■

Note that if we are given five of the six B-dihedral crosses, solving for the other B-dihedral cross requires solving a degree 8 polynomial. We now proceed to verify the Dihedral cross relation for the Khafre tetrahedron.

Example 48 (Dihedral cross relation on Khafre tetrahedron) *For the Khafre tetrahedron $\overline{ORAP}$, the variables that we defined in the theorem become*

$$X = Y = x = W = P = T = U = V = 0,$$

as well as

$$Z = \frac{9}{50}, \quad y = \frac{8}{25}, \quad z = \frac{43}{50}, \quad R = \frac{17}{25} \quad \text{and} \quad S = \frac{8}{25}.$$

Since $X = Y = V = 0$, *we have that*

$$XYZ\left(XR^2S^2 + YR^2T^2 + ZS^2T^2 + 2RSTU\right) = 0 = V^2.$$

Thus the Dihedral cross relation is verified rather trivially for the Khafre tetrahedron.

Chapter 4

Special tetrahedra and their properties

We will now apply the framework from the previous chapters to verify our results with a variety of special tetrahedra. In particular, we will consider three specific examples of tetrahedra: the regular tetrahedron, the isosceles tetrahedron (or disphenoid) and the trirectangular tetrahedron. Furthermore, we will consider two further examples of a general tetrahedron, whereby in one case we consider a relativistic bilinear form equipped to the associated vector space and in the other case we consider an example over a finite field of 11 elements.

First, we will review some basic notation: a general tetrahedron $\overline{A_0A_1A_2A_3}$ has B-quadrances $Q_{ij} \equiv Q_B(A_i, A_j)$ (for $0 \leq i < j \leq 3$), B-quadreas

$$\mathcal{A}_{012} \equiv \mathcal{A}_B\left(\overline{A_0A_1A_2}\right), \quad \mathcal{A}_{013} \equiv \mathcal{A}_B\left(\overline{A_0A_1A_3}\right),$$

$$\mathcal{A}_{023} \equiv \mathcal{A}_B\left(\overline{A_0A_2A_3}\right) \quad \text{and} \quad \mathcal{A}_{123} \equiv \mathcal{A}_B\left(\overline{A_1A_2A_3}\right),$$

B-quadrume $\mathcal{V} \equiv \mathcal{V}_B\left(\overline{A_0A_1A_2A_3}\right)$, B-spreads $s_{i;jk} \equiv s_B(A_iA_j, A_iA_k)$ (for $i = 0,1,2,3$ and $0 \leq j < k \leq 3$ with $j,k \neq i$), B-dihedral spreads

$$E_{01} \equiv E_B(A_0A_1A_2, A_0A_1A_3), \quad E_{23} \equiv E_B(A_0A_2A_3, A_1A_2A_3),$$

$$E_{02} \equiv E_B(A_0A_1A_2, A_0A_2A_3), \quad E_{13} \equiv E_B(A_0A_1A_3, A_1A_2A_3),$$

$$E_{03} \equiv E_B(A_0A_1A_3, A_0A_2A_3) \quad \text{and} \quad E_{12} \equiv E_B(A_0A_1A_2, A_1A_2A_3),$$

B-dihedral crosses $C_{ij} \equiv 1 - E_{ij}$ (for $0 \leq i < j \leq 3$), B-solid spreads

$$\mathcal{S}_0 \equiv \mathcal{S}_B(A_0A_1, A_0A_2, A_0A_3), \quad \mathcal{S}_1 \equiv \mathcal{S}_B(A_0A_1, A_1A_2, A_1A_3),$$

$$\mathcal{S}_2 \equiv \mathcal{S}_B(A_0A_2, A_1A_2, A_2A_3) \quad \text{and} \quad \mathcal{S}_3 \equiv \mathcal{S}_B(A_0A_3, A_1A_3, A_2A_3),$$

and B-dual solid spreads

$$\mathcal{D}_0 \equiv \mathcal{D}_B\left(A_0A_1, A_0A_2, A_0A_3\right), \quad \mathcal{D}_1 \equiv \mathcal{D}_B\left(A_0A_1, A_1A_2, A_1A_3\right),$$

$$\mathcal{D}_2 \equiv \mathcal{D}_B\left(A_0A_2, A_1A_2, A_2A_3\right) \quad \text{and} \quad \mathcal{D}_3 \equiv \mathcal{D}_B\left(A_0A_3, A_1A_3, A_2A_3\right).$$

4.1 Regular tetrahedron

The **regular tetrahedron** is one of the five Platonic solids in Book XIII of *Elements* [26]. In our framework, we would like to generalise this by prescribing an arbitrary symmetric bilinear form so that we can take a regular tetrahedron in $\mathbb{A}^3$ to have the unique property that the B-quadrances between any two points of it are all equal. The solid is thus symmetrical and, as a result, it suffices for us to compute one of each trigonometric quantity (as the rest would be equal). Note that there is a subtle question of existence and uniqueness of regular tetrahedra (up to order of scale), but it will not be explored in this thesis.

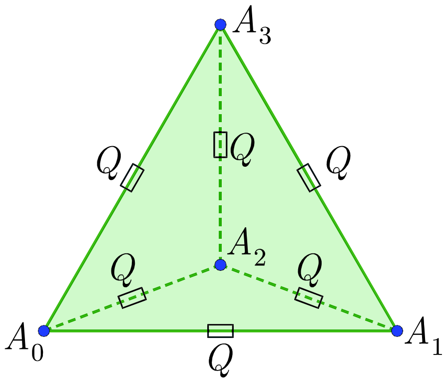

Figure 4.1: Regular tetrahedron

Suppose a regular tetrahedron $\overline{A_0A_1A_2A_3}$ has B-quadrances $Q_{ij} \equiv Q$ for $0 \leq i < j \leq 3$, as in Figure 4.1. The B-quadrume of $\overline{A_0A_1A_2A_3}$ is evaluated by the Quadrume theorem as

$$\mathcal{V} = \frac{1}{2}\begin{vmatrix} 2Q & Q & Q \\ Q & 2Q & Q \\ Q & Q & 2Q \end{vmatrix} = 2Q^3$$

and the B-quadrea of each of the four triangles of $\overline{A_0A_1A_2A_3}$ is

$$\mathcal{A} = A\left(Q, Q, Q\right) = (3Q)^2 - 2(3Q^2) = 3Q^2.$$

Suppose the B-spreads associated to a vertex of $\overline{A_0A_1A_2A_3}$ are equal to s. Then we use the Triple spread formula to obtain the equation

$$3s^2 - 4s^3 = s^2\left(3 - 4s\right) = 0.$$

Eliminating the trivial solution (which corresponds to a degenerate triangle of three collinear points) gives us the B-spreads of $\overline{A_0A_1A_2A_3}$, which are all equal to

$$s = \frac{3}{4}.$$

The B-spreads can also be obtained by use of the Quadrea spread theorem, and are precisely the B-spreads of an equilateral triangle.

By the Dihedral spread theorem we have that the B-dihedral spreads of $\overline{A_0A_1A_2A_3}$, denoted by E, are all equal to

$$E = \frac{4Q\mathcal{V}}{\mathcal{A}^2} = \frac{4Q\left(2Q^3\right)}{\left(3Q^2\right)^2} = \frac{8}{9}.$$

As a consequence the B-dihedral crosses of $\overline{A_0A_1A_2A_3}$, denoted by C, are all equal to

$$C = 1 - E = \frac{1}{9}.$$

By the Solid spread theorem we have that the B-solid spreads of $\overline{A_0A_1A_2A_3}$, denoted by $\mathcal{S}$, are all equal to

$$\mathcal{S} = \frac{\mathcal{V}}{4Q^3} = \frac{1}{2}.$$

We confirm this result by using the Solid spread projective theorem to arrive at the same answer:

$$\mathcal{S} = Es^2 = \frac{8}{9}\left(\frac{3}{4}\right)^2 = \frac{1}{2}.$$

If we denote the B-skew quadrances of $\overline{A_0A_1A_2A_3}$ (which are all equal) by R, then we can use the Skew quadrance theorem to obtain

$$R = \frac{\mathcal{V}}{4Q^2} = \frac{Q}{2}.$$

The ratio theorems in Chapter 3 will trivially hold due to the symmetry of the regular tetrahedron. The B-dual solid spreads of $\overline{A_0A_1A_2A_3}$, which are evaluated by the Dual solid spread theorem, will be denoted by $\mathcal{D}$ and are all equal to

$$\mathcal{D} = \frac{4\left(2Q^3\right)^2}{\left(3Q^2\right)^3} = \frac{16}{27}.$$

Confirming this result by using the Dual solid spread projective theorem,

$$\mathcal{D} = E^2 s = \left(\frac{8}{9}\right)^2 \frac{3}{4} = \frac{16}{27}$$

To verify all of the results of the Tetrahedron cross law for the regular tetrahedron $\overline{A_0A_1A_2A_3}$, we note that its symmetry implies that the four equations reduce to one single equation, namely

$$256\mathcal{A}^8\left((1-3C)^2-12C^2\right)^2=4096\mathcal{A}^6C^3\,(2\mathcal{A})^2=16\,384\mathcal{A}^8C^3.$$

We simplify this result to obtain an equation independent of $\mathcal{A}$, as follows:

$$\left(3C^2+6C-1\right)^2=64C^3.$$

We can now examine the possible solutions of this equation for C without any dependence on $\mathcal{A}$.

Given we know that $C=\frac{1}{9}$ for a regular tetrahedron, we can then deduce that

$$\left(3C^2+6C-1\right)^2=\left(\frac{1}{27}+\frac{2}{3}-1\right)^2=\frac{64}{729}=64\left(\frac{1}{9}\right)^3=64C^3.$$

So, $C=\frac{1}{9}$ is indeed a solution to this reduced equation and hence the Tetrahedron cross law is verified for a regular tetrahedron. Furthermore, we can factorise the difference between the two sides to get

$$\left(3C^2+6C-1\right)^2-64C^3=(9C-1)\,(C-1)^3\,.$$

This is zero precisely when $C=\frac{1}{9}$ or when $C=1$; while the former case has been discussed above, the latter case implies that $E=0$ and hence corresponds to the case to when the four points of the regular tetrahedron are coplanar.

In the case of the regular tetrahedron, where $C_{ij}=C=\frac{1}{9}$ for $0\le i<j\le 3$, we set

$$X\equiv C_{01}C_{23}=\frac{1}{81},\quad Y\equiv C_{02}C_{13}=\frac{1}{81},\quad Z\equiv C_{03}C_{12}=\frac{1}{81},$$

$$x\equiv C_{01}+C_{23}=\frac{2}{9},\quad y\equiv C_{02}+C_{13}=\frac{2}{9},\quad z\equiv C_{03}+C_{12}=\frac{2}{9},$$

$$W\equiv C_{01}C_{02}C_{03}+C_{01}C_{12}C_{13}+C_{02}C_{12}C_{23}+C_{03}C_{13}C_{23}=\frac{4}{729},$$

$$P\equiv\frac{1}{2}\,(1-x-y-z+X+Y+Z)=\frac{5}{27},$$

$$R\equiv P+z-Z=\frac{32}{81},\quad S\equiv P+y-Y=\frac{32}{81},\quad T\equiv P+x-X=\frac{32}{81},$$

$$U\equiv\frac{1}{2}\left(P^2-W+XY+XZ+YZ\right)=\frac{32}{2187}$$

and

$$V\equiv\frac{1}{2}\left(U^2-XYR^2-XZS^2-YZT^2\right)\frac{1024}{14\,348\,907}=\frac{2^{10}}{3^{15}}.$$

Then,

$$\begin{aligned}
& XYZ\left(XR^2S^2+YR^2T^2+ZS^2T^2+2RSTU\right) \\
= & \left(\frac{1}{81}\right)^3\left(\frac{1}{27}\left(\frac{32}{81}\right)^4+2\left(\frac{32}{2187}\right)\left(\frac{32}{81}\right)^3\right) \\
= & \frac{1048\,576}{205\,891\,132\,094\,649} \\
= & \frac{2^{20}}{3^{30}}=\left(\frac{2^{10}}{3^{15}}\right)^2=V^2
\end{aligned}$$

and we have the Dihedral cross relation for a regular tetrahedron.

4.2 Isosceles tetrahedron (Disphenoid)

As defined by Leech [38], a tetrahedron $\overline{A_0A_1A_2A_3}$ is an **isosceles tetrahedron** or a **disphenoid** precisely when

$$Q_{01}=Q_{23}, \quad Q_{02}=Q_{13} \quad \text{and} \quad Q_{03}=Q_{12}.$$

We can parameterise a disphenoid by defining

$$D_1\equiv Q_{01}=Q_{23}, \quad D_2\equiv Q_{02}=Q_{13} \quad \text{and} \quad D_3\equiv Q_{03}=Q_{12},$$

and illustrate this in Figure 4.2. If $D_1=D_2=D_3$, then a disphenoid degenerates to a regular tetrahedron.

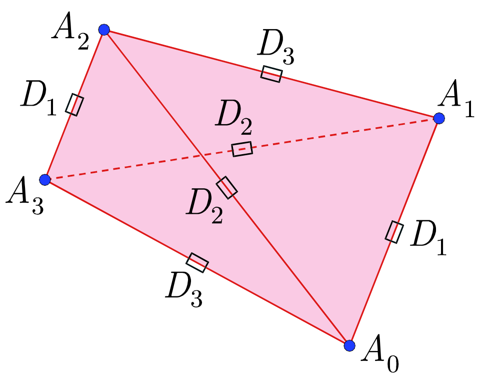

Figure 4.2: Isosceles tetrahedron (disphenoid)

As each triangle of the disphenoid has quadrances D_1, D_2 and D_3, the B-quadreas are all equal and evaluate to

$$\mathcal{A}=A\left(D_1,D_2,D_3\right)=\left(D_1+D_2+D_3\right)^2-2\left(D_1^2+D_2^2+D_3^2\right).$$

By the Quadrume theorem, the B-quadrume of the disphenoid is

$$\begin{aligned}\mathcal{V} &= 2\left(-D_1+D_2+D_3\right)\left(D_1-D_2+D_3\right)\left(D_1+D_2-D_3\right)\\ &= 16\left(\sigma-D_1\right)\left(\sigma-D_2\right)\left(\sigma-D_3\right),\end{aligned}$$

where

$$\sigma = \frac{D_1+D_2+D_3}{2}.$$

Observe the similarity between the result for the B-quadrume of a disphenoid and Heron's formula for the area of a triangle in the classical Euclidean framework, which is highlighted by Klain [33].

By the Spread law, we will have three unique B-spreads corresponding to each vertex of the disphenoid. By the Quadrea spread theorem, these are evaluated to be

$$s_{0;12} = s_{1;03} = s_{2;03} = s_{3;12} = \frac{\mathcal{A}}{4D_1D_2},$$

$$s_{0;13} = s_{1;02} = s_{2;13} = s_{3;02} = \frac{\mathcal{A}}{4D_1D_3}$$

and

$$s_{0;23} = s_{1;23} = s_{2;01} = s_{3;01} = \frac{\mathcal{A}}{4D_2D_3}.$$

By the Dihedral spread theorem, the B-dihedral spreads of the disphenoid are

$$E_{01} = E_{23} = \frac{4D_1\mathcal{V}}{\mathcal{A}^2}, \quad E_{02} = E_{13} = \frac{4D_2\mathcal{V}}{\mathcal{A}^2} \quad \text{and} \quad E_{03} = E_{12} = \frac{4D_3\mathcal{V}}{\mathcal{A}^2}.$$

Hence, the B-dihedral crosses $C_{ij} = 1 - E_{ij}$, for $0 \le i < j \le 3$, are

$$C_{01} = C_{23} = \frac{L_1^2}{\mathcal{A}^2}, \quad C_{02} = C_{13} = \frac{L_2^2}{\mathcal{A}^2} \quad \text{and} \quad C_{03} = C_{12} = \frac{L_3^2}{\mathcal{A}^2},$$

where

$$\begin{aligned}L_1 &\equiv -3D_1^2 + D_2^2 + D_3^2 + 2D_1D_2 + 2D_1D_3 - 2D_2D_3\\ &= \left(D_1+D_2+D_3\right)^2 - 4\left(D_1^2 + D_2D_3\right),\end{aligned}$$

$$\begin{aligned}L_2 &\equiv D_1^2 - 3D_2^2 + D_3^2 + 2D_1D_2 - 2D_1D_3 + 2D_2D_3\\ &= \left(D_1+D_2+D_3\right)^2 - 4\left(D_2^2 + D_1D_3\right)\end{aligned}$$

and

$$\begin{aligned}L_3 &\equiv D_1^2 + D_2^2 - 3D_3^2 - 2D_1D_2 + 2D_1D_3 + 2D_2D_3\\ &= \left(D_1+D_2+D_3\right)^2 - 4\left(D_3^2 + D_1D_2\right).\end{aligned}$$

We make the following observation with regards to the disphenoid.

Lemma 49 *If*

$$L_1 \equiv (D_1 + D_2 + D_3)^2 - 4\left(D_1^2 + D_2 D_3\right),$$

$$L_2 \equiv (D_1 + D_2 + D_3)^2 - 4\left(D_2^2 + D_1 D_3\right)$$

and

$$L_3 \equiv (D_1 + D_2 + D_3)^2 - 4\left(D_3^2 + D_1 D_2\right),$$

then

$$L_1 + L_2 + L_3 = \mathcal{A}.$$

Proof. Directly calculate the sum of L_1, L_2 and L_3 to obtain

$$\begin{aligned} L_1 + L_2 + L_3 &= 3\,(D_1 + D_2 + D_3)^2 - 4\left(D_1^2 + D_2^2 + D_3^2 + D_1 D_2 + D_1 D_3 + D_2 D_3\right) \\ &= (D_1 + D_2 + D_3)^2 - 2\left(D_1^2 + D_2^2 + D_3^2\right) \\ &= \mathcal{A}, \end{aligned}$$

as required. ■

By the Solid spread theorem, the B-solid spreads of a disphenoid are

$$\mathcal{S}_0 = \mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}_3 = \frac{\mathcal{V}}{4D_1 D_2 D_3}.$$

Given that

$$E_{01}E_{23} = \left(\frac{4D_1\mathcal{V}}{\mathcal{A}^2}\right)^2, \quad E_{02}E_{13} = \left(\frac{4D_2\mathcal{V}}{\mathcal{A}^2}\right)^2 \quad \text{and} \quad E_{03}E_{12} = \left(\frac{4D_3\mathcal{V}}{\mathcal{A}^2}\right)^2,$$

we have

$$\frac{E_{01}E_{23}}{Q_{01}Q_{23}} = \frac{E_{02}E_{13}}{Q_{02}Q_{13}} = \frac{E_{03}E_{12}}{Q_{03}Q_{12}} = \frac{16\mathcal{V}^2}{\mathcal{A}^4}.$$

Hence, the Dihedral spread ratio theorem is verified for $\overline{A_0A_1A_2A_3}$. Since the B-quadreas of the triangles of the disphenoid are equal, by the Dual solid spread theorem,

$$\mathcal{D}_0 = \mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = \frac{4\mathcal{V}^2}{\mathcal{A}^3}.$$

The Dual solid spread ratio theorem must hold by inspection, and the ratio of the B-dual solid spread to the quadrea is

$$\frac{\mathcal{D}}{\mathcal{A}} = \frac{4\mathcal{V}^2}{\mathcal{A}^4} = \frac{1}{4}\left(\frac{16\mathcal{V}^2}{\mathcal{A}^4}\right).$$

By the Skew quadrance theorem, the B-skew quadrances of a disphenoid are

$$R_{01;23} = \frac{\mathcal{V}}{4\left[D_1^2 - (D_2 - D_3)^2\right]} = \frac{-D_1 + D_2 + D_3}{2} = \sigma - D_1,$$

$$R_{02;13} = \frac{\mathcal{V}}{4\left[D_2^2 - (D_1 - D_3)^2\right]} = \frac{D_1 - D_2 + D_3}{2} = \sigma - D_2$$

and

$$R_{03;12} = \frac{\mathcal{V}}{4\left[D_3^2 - (D_1 - D_2)^2\right]} = \frac{D_1 + D_2 - D_3}{2} = \sigma - D_3.$$

As an aside, we observe that

$$R_{01;23} + R_{02;13} + R_{03;12} = 3\sigma - (D_1 + D_2 + D_3) = 3\sigma - 2\sigma = \sigma.$$

With the B-dihedral crosses and B-quadreas evaluated above, the four relations given in the Tetrahedron cross law reduce to the single result

$$256\left(\left(\mathcal{A}^2 - \left(L_1^2 + L_2^2 + L_3^2\right)\right)^2 - 4\left(L_1^2L_2^2 + L_1^2L_3^2 + L_2^2L_3^2\right)\right)^2 = 16\,384\mathcal{A}^2L_1^2L_2^2L_3^2,$$

which simplifies to

$$\left(\left(\mathcal{A}^2 - \left(L_1^2 + L_2^2 + L_3^2\right)\right)^2 - 4\left(L_1^2L_2^2 + L_1^2L_3^2 + L_2^2L_3^2\right)\right)^2 = 64\mathcal{A}^2L_1^2L_2^2L_3^2.$$

We can factor this result as

$$\begin{aligned} & \left(\left(\mathcal{A}^2 - \left(L_1^2 + L_2^2 + L_3^2\right)\right)^2 - 4\left(L_1^2L_2^2 + L_1^2L_3^2 + L_2^2L_3^2\right)\right)^2 - 64\mathcal{A}^2L_1^2L_2^2L_3^2 \\ = \; & \left(\mathcal{A} - L_1 + L_2 + L_3\right)\left(\mathcal{A} + L_1 + L_2 - L_3\right)\left(\mathcal{A} + L_1 - L_2 + L_3\right)\left(\mathcal{A} - L_1 - L_2 - L_3\right) \\ & \times \left(\mathcal{A} + L_1 + L_2 + L_3\right)\left(\mathcal{A} - L_1 + L_2 - L_3\right)\left(\mathcal{A} - L_1 - L_2 + L_3\right)\left(\mathcal{A} + L_1 - L_2 - L_3\right). \end{aligned}$$

The result of Lemma 49 implies that $\mathcal{A} - L_1 - L_2 - L_3 = 0$. So,

$$\left(\left(\mathcal{A}^2 - \left(L_1^2 + L_2^2 + L_3^2\right)\right)^2 - 4\left(L_1^2L_2^2 + L_1^2L_3^2 + L_2^2L_3^2\right)\right)^2 - 64\mathcal{A}^2L_1^2L_2^2L_3^2 = 0.$$

This gives us the desired result and hence the Tetrahedron cross law is verified for the disphenoid.

The variables in the Dihedral cross relation are

$$X \equiv C_{01}C_{23} = \left(\frac{L_1}{\mathcal{A}}\right)^4, \quad Y \equiv C_{02}C_{13} = \left(\frac{L_2}{\mathcal{A}}\right)^4, \quad Z \equiv C_{03}C_{12} = \left(\frac{L_3}{\mathcal{A}}\right)^4,$$

$$x \equiv C_{01} + C_{23} = 2\left(\frac{L_1}{\mathcal{A}}\right)^2, \quad y \equiv C_{02} + C_{13} = 2\left(\frac{L_2}{\mathcal{A}}\right)^2, \quad z \equiv C_{03} + C_{12} = 2\left(\frac{L_3}{\mathcal{A}}\right)^2,$$

$$W \equiv C_{01}C_{02}C_{03} + C_{01}C_{12}C_{13} + C_{02}C_{12}C_{23} + C_{03}C_{13}C_{23} = 4\left(\frac{L_1L_2L_3}{\mathcal{A}^3}\right)^2,$$

$$\begin{aligned} P &\equiv \frac{1}{2}(1-x-y-z+X+Y+Z) \\ &= \frac{1}{2\mathcal{A}^4}\left[\mathcal{A}^4 - 2\left(L_1^2+L_2^2+L_3^2\right)\mathcal{A}^2 + \left(L_1^4+L_2^4+L_3^4\right)\right], \end{aligned}$$

$$R \equiv P + z - Z = \frac{1}{2\mathcal{A}^4}\left[\mathcal{A}^4 - 2\left(L_1^2+L_2^2-L_3^2\right)\mathcal{A}^2 + \left(L_1^4+L_2^4-L_3^4\right)\right],$$

$$S \equiv P + y - Y = \frac{1}{2\mathcal{A}^4}\left[\mathcal{A}^4 - 2\left(L_1^2-L_2^2+L_3^2\right)\mathcal{A}^2 + \left(L_1^4-L_2^4+L_3^4\right)\right]$$

and

$$T \equiv P + x - X = \frac{1}{2\mathcal{A}^4}\left[\mathcal{A}^4 - 2\left(-L_1^2+L_2^2+L_3^2\right)\mathcal{A}^2 + \left(-L_1^4+L_2^4+L_3^4\right)\right].$$

Furthermore defining

$$U \equiv \frac{1}{2}\left(P^2 - W + XY + XZ + YZ\right)$$

and

$$V \equiv \frac{1}{2}\left(U^2 - XYR^2 - XZS^2 - YZT^2\right),$$

we can substitute the above variables and use a computer to obtain the remarkable factorisation

$$\begin{aligned} & V^2 - XYZ\left(XR^2S^2 + YR^2T^2 + ZS^2T^2 + 2RSTU\right) \\ = \; & \frac{1}{16384\mathcal{A}^{32}}\left(\mathcal{A}-L_1+L_2+L_3\right)\left(\mathcal{A}-L_1+L_2-L_3\right)\left(\mathcal{A}-L_1-L_2+L_3\right) \\ & \times\left(\mathcal{A}+L_1+L_2+L_3\right)\left(\mathcal{A}-L_1-L_2-L_3\right)\left(\mathcal{A}+L_1+L_2-L_3\right) \\ & \times\left(\mathcal{A}+L_1-L_2+L_3\right)\left(\mathcal{A}+L_1-L_2-L_3\right) \\ & \times\left(\mathcal{A}^2-(L_1+L_2)^2-L_3^2\right)^2\left(\mathcal{A}^2-(L_1+L_3)^2-L_2^2\right)^2 \\ & \times\left(\mathcal{A}^2-(L_2+L_3)^2-L_1^2\right)^2\left(\mathcal{A}^2-(L_1-L_2)^2-L_3^2\right)^2 \\ & \times\left(\mathcal{A}^2-(L_1-L_3)^2-L_2^2\right)^2\left(\mathcal{A}^2-(L_2-L_3)^2-L_1^2\right)^2. \end{aligned}$$

As Lemma 49 implies that $\mathcal{A} - L_1 - L_2 - L_3 = 0$ and thus

$$V^2 - XYZ\left(XR^2S^2 + YR^2T^2 + ZS^2T^2 + 2RSTU\right) = 0.$$

Thus we have the Dihedral cross relation for the disphenoid.

4.3 Trirectangular tetrahedron

For a general tetrahedron $\overline{A_0A_1A_2A_3}$, let $v_i \equiv \overrightarrow{A_0A_i}$ (for $i = 1, 2, 3$) and suppose that v_1, v_2 and v_3 are mutually B-perpendicular, i.e.

$$v_1 \cdot_B v_2 = v_1 \cdot_B v_3 = v_2 \cdot_B v_3 = 0.$$

Then we say that $\overline{A_0A_1A_2A_3}$ is a **trirectangular tetrahedron**. This tetrahedron is mentioned in Altshiller-Court [1, pp. 91-94]. With such a property, we use the definition of the B-spread to obtain

$$s_{0;12} = 1 - \frac{(v_1 \cdot_B v_2)^2}{Q_B(v_1)\,Q_B(v_2)} = 1,$$

and similarly

$$s_{0;13} = s_{0;23} = 1.$$

Furthermore, we use the Binet-Cauchy identity to obtain

$$\begin{aligned} E_{01} &= 1 - \frac{((v_1 \times_B v_2) \cdot_B (v_1 \times_B v_3))^2}{Q_B(v_1 \times_B v_2)\,Q_B(v_1 \times_B v_3)} \\ &= 1 - \frac{((v_1 \cdot_B v_1)(v_2 \cdot_B v_3) - (v_1 \cdot_B v_2)(v_1 \cdot_B v_3))^2}{Q_B(v_1 \times_B v_2)\,Q_B(v_1 \times_B v_3)} \\ &= 1, \end{aligned}$$

and similarly

$$E_{02} = E_{03} = 1.$$

Hence $\mathcal{S}_0 = 1$ by the Solid spread projective theorem.

We can parameterise the trirectangular tetrahedron $\overline{A_0A_1A_2A_3}$ by defining

$$Q_{01} \equiv G_1, \quad Q_{02} \equiv G_2 \quad \text{and} \quad Q_{03} \equiv G_3.$$

Because the edges emanating from A_0 are mutually B-perpendicular, we use Pythagoras' theorem to evaluate the other B-quadrances as

$$Q_{12} = G_1 + G_2, \quad Q_{13} = G_1 + G_3 \quad \text{and} \quad Q_{23} = G_2 + G_3.$$

We illustrate a trirectangular tetrahedron with these B-quadrances in Figure 4.3.

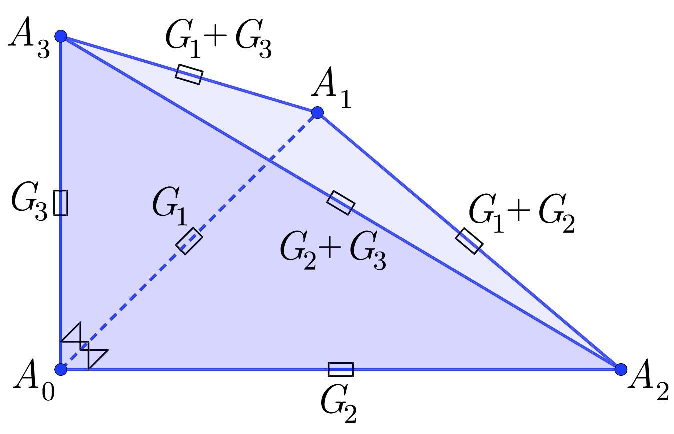

Figure 4.3: Trirectangular tetrahedron B-perpendicular at A_0

Using Pythagoras' theorem, we can simplify the calculations of the B-quadrume of $\overline{A_0A_1A_2A_3}$ to get

$$\mathcal{V} = \frac{1}{2}\begin{vmatrix} 2G_1 & 0 & 0 \\ 0 & 2G_2 & 0 \\ 0 & 0 & 2G_3 \end{vmatrix} = 4G_1G_2G_3.$$

We use the property that $s_{0;12} = s_{0;13} = s_{0;23} = 1$ and the Quadrea spread theorem to obtain

$$\mathcal{A}_{012} = 4G_1G_2, \quad \mathcal{A}_{013} = 4G_1G_3 \quad \text{and} \quad \mathcal{A}_{023} = 4G_2G_3.$$

To compute $\mathcal{A}_{123}$, we make a key observation regarding trirectangular tetrahedra, which extends a known result of de Gua de Malves (1783) [17] to arbitrary symmetric bilinear forms.

Theorem 50 (de Gua's theorem) *For a trirectangular tetrahedron $\overline{A_0A_1A_2A_3}$ which is B-perpendicular at A_0 with B-quadreas $\mathcal{A}_{012}$, $\mathcal{A}_{013}$, $\mathcal{A}_{023}$ and $\mathcal{A}_{123}$, the relation*

$$\mathcal{A}_{123} = \mathcal{A}_{012} + \mathcal{A}_{013} + \mathcal{A}_{023}$$

is satisfied.

Proof. We start with the Tetrahedron cross law

$$\begin{aligned} &\left(\begin{array}{c} \left(\begin{array}{c} (\mathcal{A}_{012} + \mathcal{A}_{013} + \mathcal{A}_{023} - \mathcal{A}_{123})^2 \\ -4\left(\mathcal{A}_{012}\mathcal{A}_{013}C_{01} + \mathcal{A}_{012}\mathcal{A}_{023}C_{02} + \mathcal{A}_{013}\mathcal{A}_{023}C_{03}\right) \end{array}\right)^2 \\ -64\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\left(\mathcal{A}_{012}C_{01}C_{02} + \mathcal{A}_{013}C_{01}C_{03} + \mathcal{A}_{023}C_{02}C_{03}\right) \end{array}\right)^2 \\ = \;& 4096\mathcal{A}_{012}^2\mathcal{A}_{013}^2\mathcal{A}_{023}^2C_{01}C_{02}C_{03}\left(\mathcal{A}_{012} + \mathcal{A}_{013} + \mathcal{A}_{023} - \mathcal{A}_{123}\right)^2. \end{aligned}$$

Since we have that $E_{01} = E_{02} = E_{03} = 1$, we can deduce that the B-dihedral crosses are

$$C_{01} = C_{02} = C_{03} = 0$$

and hence our result degenerates to

$$(\mathcal{A}_{012} + \mathcal{A}_{013} + \mathcal{A}_{023} - \mathcal{A}_{123})^8 = 0.$$

Solve for $\mathcal{A}_{123}$ to get

$$\mathcal{A}_{123} = \mathcal{A}_{012} + \mathcal{A}_{013} + \mathcal{A}_{023}.$$

■

This result is also alluded to by Cho in [10] and [11]. Defining

$$H \equiv G_1G_2 + G_1G_3 + G_2G_3,$$

we then have that

$$\mathcal{A}_{123} = 4H.$$

We use the Quadrea spread theorem to obtain the B-spreads for $\overline{A_0A_1A_2A_3}$, which are

$$s_{0;12} = 1, \quad s_{1;02} = \frac{G_2}{G_1+G_2}, \quad s_{2;01} = \frac{G_1}{G_1+G_2},$$

$$s_{0;13} = 1, \quad s_{1;03} = \frac{G_3}{G_1+G_3}, \quad s_{3;01} = \frac{G_1}{G_1+G_3},$$

$$s_{0;23} = 1, \quad s_{2;03} = \frac{G_3}{G_2+G_3}, \quad s_{3;02} = \frac{G_2}{G_2+G_3},$$

$$s_{1;23} = \frac{S}{(G_1+G_2)(G_1+G_3)}, \quad s_{2;13} = \frac{S}{(G_1+G_2)(G_2+G_3)}$$

and

$$s_{3;12} = \frac{S}{(G_1+G_3)(G_2+G_3)}.$$

By the Dihedral spread theorem, the B-dihedral spreads of $\overline{A_0A_1A_2A_3}$ are

$$\begin{aligned} E_{01} &= 1, \quad E_{23} = \frac{(G_2+G_3)\,G_1}{S}, \\ E_{02} &= 1, \quad E_{13} = \frac{(G_1+G_3)\,G_2}{S}, \\ E_{03} &= 1 \quad \text{and} \quad E_{12} = \frac{(G_1+G_2)\,G_3}{S}. \end{aligned}$$

By the Solid spread theorem, the B-solid spreads of $\overline{A_0A_1A_2A_3}$ are

$$\mathcal{S}_0 = 1, \quad \mathcal{S}_1 = \frac{G_2G_3}{(G_1+G_2)(G_1+G_3)},$$

$$\mathcal{S}_2 = \frac{G_1G_3}{(G_1+G_2)(G_2+G_3)} \quad \text{and} \quad \mathcal{S}_3 = \frac{G_1G_3}{(G_1+G_2)(G_2+G_3)}.$$

Given that

$$E_{01}E_{23} = \frac{(G_2+G_3)\,G_1}{S}, \quad E_{02}E_{13} = \frac{(G_1+G_3)\,G_2}{S} \quad \text{and} \quad E_{03}E_{12} = \frac{(G_1+G_2)\,G_3}{S},$$

we have

$$\frac{E_{01}E_{23}}{Q_{01}Q_{23}} = \frac{E_{02}E_{13}}{Q_{02}Q_{13}} = \frac{E_{03}E_{12}}{Q_{03}Q_{12}} = \frac{1}{S}.$$

As a result, we use the Dual solid spread ratio theorem to obtain the B-dual solid spread of the trirectangular tetrahedron, which are

$$\mathcal{D}_0 = 1, \quad \mathcal{D}_1 = \frac{G_2G_3}{S}, \quad \mathcal{D}_2 = \frac{G_1G_3}{S} \quad \text{and} \quad \mathcal{D}_3 = \frac{G_1G_2}{S}.$$

Observe that

$$\mathcal{D}_0 = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3.$$

This is a direct consequence of de Gua's theorem and the Dual solid spread ratio theorem.

By the Skew quadrance theorem, we can compute the B-skew quadrances of a trirectangular tetrahedron to be

$$R_{01;23} = \frac{4G_1G_2G_3}{4G_1(G_2+G_3)} = \frac{G_2G_3}{G_2+G_3},$$

$$R_{02;13} = \frac{4G_1G_2G_3}{4G_2(G_1+G_3)} = \frac{G_1G_3}{G_1+G_3}$$

and

$$R_{03;12} = \frac{4G_1G_2G_3}{4G_3(G_1+G_2)} = \frac{G_1G_2}{G_1+G_2}.$$

We saw that one of the results of the Tetrahedron cross law implied de Gua's theorem. As an exercise, let us verify one of the other results of the Tetrahedron cross law, say

$$\begin{aligned} &\left(\begin{array}{c} \left(\begin{array}{c} (\mathcal{A}_{012}+\mathcal{A}_{013}+\mathcal{A}_{123}-\mathcal{A}_{023})^2 \\ -4(\mathcal{A}_{012}\mathcal{A}_{013}C_{01}+\mathcal{A}_{012}\mathcal{A}_{123}C_{12}+\mathcal{A}_{013}\mathcal{A}_{123}C_{13}) \end{array}\right)^2 \\ -64\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{123}(\mathcal{A}_{012}C_{01}C_{12}+\mathcal{A}_{013}C_{01}C_{13}+\mathcal{A}_{123}C_{12}C_{13}) \end{array}\right)^2 \\ = &\ 4096\mathcal{A}_{012}^2\mathcal{A}_{013}^2\mathcal{A}_{123}^2C_{01}C_{12}C_{13}(\mathcal{A}_{012}+\mathcal{A}_{013}+\mathcal{A}_{123}-\mathcal{A}_{023})^2. \end{aligned}$$

Note if this result is true, the other two results follow by symmetry and hence the Tetrahedron cross law is verified for a trirectangular tetrahedron. We start with the B-dihedral crosses

$$C_{01} = 0, \quad C_{12} = \frac{G_1G_2}{S} \quad \text{and} \quad C_{13} = \frac{G_1G_3}{S}.$$

We deduce that

$$\mathcal{A}_{012}+\mathcal{A}_{013}+\mathcal{A}_{123}-\mathcal{A}_{023} = 8(G_1G_2+G_1G_3),$$

$$\mathcal{A}_{012}\mathcal{A}_{013}C_{01}+\mathcal{A}_{012}\mathcal{A}_{123}C_{12}+\mathcal{A}_{013}\mathcal{A}_{123}C_{13} = 16\left(G_1^2G_2^2+G_1^2G_3^2\right)$$

and

$$\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{123}(\mathcal{A}_{012}C_{01}C_{12}+\mathcal{A}_{013}C_{01}C_{13}+\mathcal{A}_{123}C_{12}C_{13}) = 256G_1^4G_2^2G_3^2.$$

Since $C_{01}=0$,

$$\begin{aligned} &\left(\begin{array}{c} \left(\begin{array}{c} (\mathcal{A}_{012}+\mathcal{A}_{013}+\mathcal{A}_{123}-\mathcal{A}_{023})^2 \\ -4(\mathcal{A}_{012}\mathcal{A}_{013}C_{01}+\mathcal{A}_{012}\mathcal{A}_{123}C_{12}+\mathcal{A}_{013}\mathcal{A}_{123}C_{13}) \end{array}\right)^2 \\ -64\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{123}(\mathcal{A}_{012}C_{01}C_{12}+\mathcal{A}_{013}C_{01}C_{13}+\mathcal{A}_{123}C_{12}C_{13}) \end{array}\right)^2 \\ = &\ 2^{24}\left(\left((G_1G_2+G_1G_3)^2-\left(G_1^2G_2^2+G_1^2G_3^2\right)\right)^2-4G_1^4G_2^2G_3^2\right)^2 = 0 \\ = &\ 4096\mathcal{A}_{012}^2\mathcal{A}_{013}^2\mathcal{A}_{123}^2C_{01}C_{12}C_{13}(\mathcal{A}_{012}+\mathcal{A}_{013}+\mathcal{A}_{123}-\mathcal{A}_{023})^2. \end{aligned}$$

So, the result holds and by symmetry the Tetrahedron cross law is verified for a trirectangular tetrahedron.

Define the variables

$$X \equiv C_{01}C_{23} = 0, \quad Y \equiv C_{02}C_{13} = 0, \quad Z \equiv C_{03}C_{12} = 0,$$

$$x \equiv C_{01} + C_{23} = \frac{G_1G_2}{S}, \quad y \equiv C_{02} + C_{13} = \frac{G_1G_3}{S}, \quad z \equiv C_{03} + C_{12} = \frac{G_2G_3}{S},$$
$$W \equiv C_{01}C_{02}C_{03} + C_{01}C_{12}C_{13} + C_{02}C_{12}C_{23} + C_{03}C_{13}C_{23} = 0,$$
$$P \equiv \frac{1}{2}(1 - x - y - z + X + Y + Z) = 0,$$
$$R \equiv P + z - Z = \frac{G_1G_2}{S}, \quad S \equiv P + y - Y = \frac{G_1G_3}{S}, \quad T \equiv P + x - X = \frac{G_2G_3}{S},$$
$$U \equiv \frac{1}{2}\left(P^2 - W + XY + XZ + YZ\right) = 0$$

and

$$V \equiv \frac{1}{2}\left(U^2 - XYR^2 - XZS^2 - YZT^2\right) = 0.$$

Thus

$$V^2 = XYZ\left(XR^2S^2 + YR^2T^2 + ZS^2T^2 + 2RSTU\right) = 0.$$

So, the Dihedral cross relation trivially holds for a trirectangular tetrahedron.

4.4 A relativistic example

Consider $\mathbb{A}^3$ over the rational number field, and equip a symmetric bilinear form on its associated vector space $\mathbb{V}^3$ (over the rational number field) defined by

$$(x_1, y_1, z_1) \cdot_B (x_2, y_2, z_2) = x_1x_2 + y_1y_2 - z_1z_2.$$

This scalar product is called the **relativistic scalar product** or the **Minkowski scalar product** [40], and we can represent this symmetric bilinear form by the matrix

$$B \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Suppose we have a tetrahedron $\overline{A_0A_1A_2A_3}$ in $\mathbb{A}^3$ with points

$$A_0 \equiv [0,0,0], \quad A_1 \equiv [1,2,3], \quad A_2 \equiv [-2,1,-1] \quad \text{and} \quad A_3 \equiv [0,-2,1].$$

The B-quadrance Q_{01} is

$$Q_{01} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -4.$$

Similarly, the remaining B-quadrances of $\overline{A_0A_1A_2A_3}$ are

$$Q_{02} = 4, \quad Q_{03} = 3, \quad Q_{23} = 9, \quad Q_{13} = 13 \quad \text{and} \quad Q_{12} = -6.$$

The B-quadreas of $\overline{A_0A_1A_2A_3}$ are then

$$\mathcal{A}_{012} = (-4+4-6)^2 - 2\,(16+16+36) = -100$$

and similarly

$$\mathcal{A}_{013} = -244, \quad \mathcal{A}_{023} = 44 \quad \text{and} \quad \mathcal{A}_{123} = -316.$$

The B-quadrume of $\overline{A_0A_1A_2A_3}$ is

$$\mathcal{V} = \frac{1}{2}\begin{vmatrix} -8 & 0+6 & -1-13 \\ 0+6 & 8 & 7-9 \\ -1-13 & 7-9 & 6 \end{vmatrix} = -900.$$

By the Quadrea spread theorem,

$$s_{0;12} = \frac{-100}{4\times(-4)\times 4} = \frac{25}{16}.$$

The remaining B-spreads of $\overline{A_0A_1A_2A_3}$ are

$$s_{1;02} = -\frac{25}{24}, \quad s_{2;01} = \frac{25}{24},$$

$$s_{0;13} = \frac{61}{12}, \quad s_{1;03} = \frac{61}{52}, \quad s_{3;01} = -\frac{61}{39},$$
$$s_{0;23} = \frac{11}{12}, \quad s_{2;03} = \frac{11}{36}, \quad s_{3;02} = \frac{11}{27},$$
$$s_{1;23} = \frac{79}{78}, \quad s_{2;13} = \frac{79}{54} \quad \text{and} \quad s_{3;12} = -\frac{79}{117}.$$

By the Dihedral spread theorem, we have that

$$E_{01} = \frac{4\times(-4)\times(-900)}{(-100)\,(-244)} = \frac{36}{61},$$

and similarly the remaining B-dihedral spreads of $\overline{A_0A_1A_2A_3}$ are

$$E_{02} = \frac{36}{11}, \quad E_{03} = \frac{675}{671},$$

$$E_{23} = \frac{2025}{869}, \quad E_{13} = -\frac{2925}{4819} \quad \text{and} \quad E_{12} = \frac{54}{79}.$$

The B-dihedral crosses $C_{ij} \equiv 1 - E_{ij}$ of $\overline{A_0A_1A_2A_3}$, for $0 \le i < j \le 3$, are

$$\begin{aligned} C_{01} &= \frac{25}{61}, \quad C_{02} = -\frac{25}{11}, \quad C_{03} = -\frac{4}{671}, \\ C_{23} &= -\frac{1156}{869}, \quad C_{13} = \frac{7744}{4819} \quad \text{and} \quad C_{12} = \frac{25}{79}. \end{aligned}$$

By the Solid spread theorem, the B-solid spread $\mathcal{S}_0$ is

$$\mathcal{S}_0 = \frac{-900}{4 \times (-4) \times 4 \times 3} = \frac{75}{16}.$$

Similarly, the remaining B-solid spreads of $\overline{A_0A_1A_2A_3}$ are

$$\mathcal{S}_1 = -\frac{75}{104}, \quad \mathcal{S}_2 = \frac{25}{24} \quad \text{and} \quad \mathcal{S}_3 = -\frac{25}{39}.$$

Note the non-standard outputs obtained in our calculations above; in the relativistic bilinear form, we have B-quadrance outputs that end up being less than 0 and B-spread outputs outside the usual range of 0 to 1, which is normally the case in Euclidean geometry. This is an important possibility of working with arbitrary symmetric bilinear forms.

We compute

$$E_{01}E_{23} = \frac{72\,900}{53\,009}, \quad E_{02}E_{13} = -\frac{105\,300}{53\,009} \quad \text{and} \quad E_{03}E_{12} = \frac{36\,450}{53\,009},$$

as well as

$$Q_{01}Q_{23} = -36, \quad Q_{02}Q_{13} = 52 \quad \text{and} \quad Q_{03}Q_{12} = -18,$$

so that

$$\frac{E_{01}E_{23}}{Q_{01}Q_{23}} = \frac{E_{02}E_{13}}{Q_{02}Q_{13}} = \frac{E_{03}E_{12}}{Q_{03}Q_{12}} = -\frac{2025}{53\,009}.$$

We have thus verified the Dihedral spread ratio theorem for $\overline{A_0A_1A_2A_3}$, and from this we can deduce the Richardson number to be

$$K = -\frac{53009}{2025}.$$

We use the Dual solid spread ratio theorem to obtain

$$\mathcal{D}_0 = \left(-\frac{2025}{53009}\right) \times (-79) = \frac{2025}{671},$$

$$\mathcal{D}_1 = \left(-\frac{2025}{53009}\right) \times 11 = -\frac{2025}{4819},$$

$$\mathcal{D}_2 = \left(-\frac{2025}{53009}\right) \times (-61) = \frac{2025}{869}$$

and

$$\mathcal{D}_3 = \left(-\frac{2025}{53009}\right) \times (-25) = \frac{50\,625}{53\,009}.$$

We use the Skew quadrance theorem to obtain

$$R_{01;23} = \frac{-900}{4 \times (-36) - (17+3)^2} = \frac{225}{136},$$

$$R_{02;13} = \frac{-900}{4 \times 52 - (5+3)^2} = -\frac{25}{4}$$

and

$$R_{03;12} = \frac{-900}{4 \times (-18) - (5-17)^2} = \frac{25}{6}.$$

We proceed to verify one of the results of the Tetrahedron cross law, say

$$\begin{aligned}
&\left(\begin{array}{c} \left(\begin{array}{c} (\mathcal{A}_{012} + \mathcal{A}_{013} + \mathcal{A}_{023} - \mathcal{A}_{123})^2 \\ -4\left(\mathcal{A}_{012}\mathcal{A}_{013}C_{01} + \mathcal{A}_{012}\mathcal{A}_{023}C_{02} + \mathcal{A}_{013}\mathcal{A}_{023}C_{03}\right) \end{array}\right)^2 \\ -64\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\left(\mathcal{A}_{012}C_{01}C_{02} + \mathcal{A}_{013}C_{01}C_{03} + \mathcal{A}_{023}C_{02}C_{03}\right) \end{array}\right)^2 \\
= {} & 4096\mathcal{A}_{012}^2\mathcal{A}_{013}^2\mathcal{A}_{023}^2 C_{01}C_{02}C_{03}\left(\mathcal{A}_{012} + \mathcal{A}_{013} + \mathcal{A}_{023} - \mathcal{A}_{123}\right)^2.
\end{aligned}$$

With the quantities we evaluated above, we have that

$$\mathcal{A}_{012} + \mathcal{A}_{013} + \mathcal{A}_{023} - \mathcal{A}_{123} = 16,$$

$$\mathcal{A}_{012}\mathcal{A}_{013}C_{01} + \mathcal{A}_{012}\mathcal{A}_{023}C_{02} + \mathcal{A}_{013}\mathcal{A}_{023}C_{03} = 20\,064,$$

$$\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\left(\mathcal{A}_{012}C_{01}C_{02} + \mathcal{A}_{013}C_{01}C_{03} + \mathcal{A}_{023}C_{02}C_{03}\right) = 101\,280\,000$$

and

$$\mathcal{A}_{012}^2\mathcal{A}_{013}^2\mathcal{A}_{023}^2 C_{01}C_{02}C_{03}\left(\mathcal{A}_{012} + \mathcal{A}_{013} + \mathcal{A}_{023} - \mathcal{A}_{123}\right)^2 = 1638\,400\,000\,000.$$

So

$$\begin{aligned}
&\left(\begin{array}{c} \left(\begin{array}{c} (\mathcal{A}_{012} + \mathcal{A}_{013} + \mathcal{A}_{023} - \mathcal{A}_{123})^2 \\ -4\left(\mathcal{A}_{012}\mathcal{A}_{013}C_{01} + \mathcal{A}_{012}\mathcal{A}_{023}C_{02} + \mathcal{A}_{013}\mathcal{A}_{023}C_{03}\right) \end{array}\right)^2 \\ -64\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\left(\mathcal{A}_{012}C_{01}C_{02} + \mathcal{A}_{013}C_{01}C_{03} + \mathcal{A}_{023}C_{02}C_{03}\right) \end{array}\right)^2 \\
= {} & \left(\left(256 - 4\,(20064)\right)^2 - 64\,(101280000)\right)^2 \\
= {} & 6710\,886\,400\,000\,000 = 4096 \times 1638\,400\,000\,000 \\
= {} & 4096\mathcal{A}_1^2\mathcal{A}_2^2\mathcal{A}_3^2 C_{01}C_{02}C_{03}\left(\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 - \mathcal{A}_0\right)^2.
\end{aligned}$$

We perform similar calculations for the remaining results of the Tetrahedron cross law, which then enables us to verify the Tetrahedron cross law for the tetrahedron $\overline{A_0A_1A_2A_3}$.

The variables required for the Dihedral cross relation are

$$X \equiv C_{01}C_{23} = -\frac{28\,900}{53\,009}, \quad Y \equiv C_{02}C_{13} = -\frac{17\,600}{4819}, \quad Z \equiv C_{03}C_{12} = -\frac{100}{53\,009},$$

$$x \equiv C_{01} + C_{23} = -\frac{48\,791}{53\,009}, \quad y \equiv C_{02} + C_{13} = -\frac{35\,291}{53\,009}, \quad z \equiv C_{03} + C_{12} = \frac{16\,459}{53\,009},$$

$$W \equiv C_{01}C_{02}C_{03} + C_{01}C_{12}C_{13} + C_{02}C_{12}C_{23} + C_{03}C_{13}C_{23} = \frac{3325\,473\,256}{2809\,954\,081},$$

$$P \equiv \frac{1}{2}(1-x-y-z+X+Y+Z) = -\frac{50\,984}{53\,009}, \quad R \equiv P+z-Z = -\frac{34\,425}{53\,009},$$
$$S \equiv P+y-Y = \frac{107\,325}{53\,009}, \quad T \equiv P+x-X = -\frac{70\,875}{53\,009},$$
$$U \equiv \frac{1}{2}\left(P^2 - W + XY + XZ + YZ\right) = \frac{2445\,592\,500}{2809\,954\,081}$$

and

$$V \equiv \frac{1}{2}\left(U^2 - XYR^2 - XZS^2 - YZT^2\right) = -\frac{35\,463\,189\,150\,000\,000}{717\,803\,812\,484\,414\,051}.$$

Thus,

$$\begin{aligned} V^2 &= \frac{1257\,637\,784\,688\,677\,722\,500\,000\,000\,000\,000}{515\,242\,313\,217\,159\,849\,022\,981\,796\,806\,230\,601} \\ &= XYZ\left(XR^2S^2 + YR^2T^2 + ZS^2T^2 + 2RSTU\right). \end{aligned}$$

Hence, the Dihedral cross relation holds for the tetrahedron $\overline{A_0A_1A_2A_3}$.

4.5 An example over $\mathbb{F}_{11}$

Consider the affine 3-space ($\mathbb{A}^3$) over the finite field $\mathbb{F}_{11}$, whose elements will be represented by integers between -5 and 5 and whose operations will be represented by the main integer operations modulo 11. Equip its associated vector space with the symmetric bilinear form defined by the Euclidean scalar product

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = x_1x_2 + y_1y_2 + z_1z_2.$$

Here, we will consider the same tetrahedron $\overline{A_0A_1A_2A_3}$ with the same points as in the previous example, where

$$A_0 \equiv [0,0,0], \quad A_1 \equiv [1,2,3], \quad A_2 \equiv [-2,1,-1] \quad \text{and} \quad A_3 \equiv [0,-2,1].$$

As is the case with the rational trigonometry of the Khafre tetrahedron, we are allowed to omit the B prefix from the trigonometric quantities because we are dealing with the Euclidean scalar product, where B is the 3×3 identity matrix.

The quadrance Q_{01} is given by

$$Q_{01} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 + 4 - 2 = 3.$$

Similarly, the remaining quadrances of the tetrahedron $\overline{A_0A_1A_2A_3}$ are

$$Q_{02} = -5, \quad Q_{03} = 5, \quad Q_{23} = -5, \quad Q_{13} = -1 \quad \text{and} \quad Q_{12} = 4.$$

Furthermore, we have that

$$\begin{aligned}\mathcal{A}_{012} &= A(3,-5,4) = 2^2 - 2(-2+3+5)\\ &= 4-2(-5) = 14 = 3,\end{aligned}$$

and similarly the remaining quadreas of $\overline{A_0A_1A_2A_3}$ are

$$\mathcal{A}_{013} = 1, \quad \mathcal{A}_{023} = -4 \quad \text{and} \quad \mathcal{A}_{123} = -3.$$

The quadrume of $\overline{A_0A_1A_2A_3}$ is

$$\mathcal{V} = \frac{1}{2}\begin{vmatrix} -5 & 3-5-4 & 3+5+1 \\ 3-5-4 & 1 & -5+5+5 \\ 3+5+1 & -5+5+5 & -1 \end{vmatrix} = \frac{-455}{2} = \frac{-4}{2} = -2.$$

By the Quadrea spread theorem,

$$s_{0;12} = \frac{3}{4\times 3\times(-5)} = \frac{1}{2} = -5.$$

The remaining spreads of $\overline{A_0A_1A_2A_3}$ are

$$s_{1;02} = \frac{1}{5} = -2, \quad s_{2;01} = -\frac{3}{3} = -1,$$

$$s_{0;13} = \frac{1}{5} = -2, \quad s_{1;03} = -\frac{1}{1} = -1, \quad s_{3;01} = \frac{1}{2} = -5,$$
$$s_{0;23} = \frac{1}{3} = 4, \quad s_{2;03} = -\frac{1}{3} = -4, \quad s_{3;02} = \frac{1}{3} = 4,$$
$$s_{1;23} = \frac{3}{5} = 5, \quad s_{2;13} = \frac{3}{3} = 1 \quad \text{and} \quad s_{3;12} = \frac{3}{2} = -4.$$

By the Dihedral spread theorem, we have that

$$E_{01} = \frac{4\times 3\times(-2)}{3\times 1} = -8 = 3,$$

as well as

$$E_{02} = \frac{1}{3} = 4, \quad E_{03} = 10 = -1, \quad E_{23} = \frac{10}{3} = -4,$$
$$E_{13} - = -\frac{8}{3} = 1 \quad \text{and} \quad E_{12} = \frac{32}{9} = -5.$$

From the above results, the dihedral crosses $C_{ij} \equiv 1 - E_{ij}$ of $\overline{A_0A_1A_2A_3}$, for $0 \le i < j \le 3$, are

$$\begin{aligned} C_{01} &= -2, \quad C_{02} = -3, \quad C_{03} = 2,\\ C_{23} &= 5, \quad C_{13} = 0 \quad \text{and} \quad C_{12} = -5.\end{aligned}$$

By the Solid spread theorem,

$$\mathcal{S}_0 = \frac{-2}{4 \times 3 \times (-5) \times 5} = \frac{1}{-4} = -3$$

and thus the remaining solid spreads of $\overline{A_0A_1A_2A_3}$ are

$$\mathcal{S}_1 = \frac{1}{-3} = -4, \quad \mathcal{S}_2 = \frac{1}{3} = 4 \quad \text{and} \quad \mathcal{S}_3 = \frac{1}{5} = -2.$$

Given that

$$E_{01}E_{23} = -1, \quad E_{02}E_{13} = 4 \quad \text{and} \quad E_{03}E_{12} = 5,$$

we observe that

$$\frac{E_{01}E_{23}}{Q_{01}Q_{23}} = \frac{1}{4} = 3,$$

$$\frac{E_{02}E_{13}}{Q_{02}Q_{13}} = \frac{4}{5} = 3$$

and

$$\frac{E_{03}E_{12}}{Q_{03}Q_{12}} = \frac{1}{4} = 3.$$

As all ratios are equal, the Dihedral spread ratio theorem is verified for $\overline{A_0A_1A_2A_3}$. Given that

$$K = \frac{1}{3} = 4$$

and

$$\frac{1}{4K} = \frac{1}{4 \times 4} = -2,$$

we have by the Dual solid spread ratio theorem

$$\mathcal{D}_0 = \frac{-3}{-2} = -4$$

and similarly the remaining dual solid spreads of $\overline{A_0A_1A_2A_3}$ are

$$\mathcal{D}_1 = 2, \quad \mathcal{D}_2 = 5 \quad \text{and} \quad \mathcal{D}_3 = 4.$$

We will now verify one of the results of the Tetrahedron cross law, say

$$\begin{aligned} & \left(\begin{array}{c} \left(\begin{array}{c} (\mathcal{A}_{012} + \mathcal{A}_{013} + \mathcal{A}_{023} - \mathcal{A}_{123})^2 \\ -4\left(\mathcal{A}_{012}\mathcal{A}_{013}C_{01} + \mathcal{A}_{012}\mathcal{A}_{023}C_{02} + \mathcal{A}_{013}\mathcal{A}_{023}C_{03}\right) \end{array} \right)^2 \\ -64\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\left(\mathcal{A}_{012}C_{01}C_{02} + \mathcal{A}_{013}C_{01}C_{03} + \mathcal{A}_{023}C_{02}C_{03}\right) \end{array} \right)^2 \\ = \ & 4096\mathcal{A}_{012}^2\mathcal{A}_{013}^2\mathcal{A}_{023}^2C_{01}C_{02}C_{03}\left(\mathcal{A}_{012} + \mathcal{A}_{013} + \mathcal{A}_{023} - \mathcal{A}_{123}\right)^2, \end{aligned}$$

With the quantities we defined in this section, we deduce that

$$\mathcal{A}_{012} + \mathcal{A}_{013} + \mathcal{A}_{023} - \mathcal{A}_{123} = 3,$$

$$\mathcal{A}_{012}\mathcal{A}_{013}C_{01} + \mathcal{A}_{012}\mathcal{A}_{023}C_{02} + \mathcal{A}_{013}\mathcal{A}_{023}C_{03} = 0,$$

$$\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\left(\mathcal{A}_{012}C_{01}C_{02} + \mathcal{A}_{013}C_{01}C_{03} + \mathcal{A}_{023}C_{02}C_{03}\right) = -5,$$

and

$$4096\mathcal{A}_1^2\mathcal{A}_2^2\mathcal{A}_3^2C_{01}C_{02}C_{03}\left(\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 - \mathcal{A}_0\right)^2 = 3.$$

So,

$$4096\mathcal{A}_{012}^2\mathcal{A}_{013}^2\mathcal{A}_{023}^2C_{01}C_{02}C_{03}\left(\mathcal{A}_{012} + \mathcal{A}_{013} + \mathcal{A}_{023} - \mathcal{A}_{123}\right)^2 = \left(4 + 2\times(-5)\right)^2 = 36 = 3$$
$$= \left(\begin{array}{c}\left(\left(\mathcal{A}_{012} + \mathcal{A}_{013} + \mathcal{A}_{023} - \mathcal{A}_{123}\right)^2 - 4\left(\mathcal{A}_{012}\mathcal{A}_{013}C_{01} + \mathcal{A}_{012}\mathcal{A}_{023}C_{02} + \mathcal{A}_{013}\mathcal{A}_{023}C_{03}\right)\right)^2 \\ -64\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\left(\mathcal{A}_{012}C_{01}C_{02} + \mathcal{A}_{013}C_{01}C_{03} + \mathcal{A}_{023}C_{02}C_{03}\right)\end{array}\right)^2.$$

We can perform similar calculation for the other results, which would enable us to verify the Tetrahedron cross law for $\overline{A_0A_1A_2A_3}$.

Define the variables

$$X \equiv C_{01}C_{23} = (-2)\times 5 = 1, \quad x \equiv C_{01} + C_{23} = -2 + 5 = 3,$$

$$Y \equiv C_{02}C_{13} = 0\times(-3) = 0, \quad y \equiv C_{02} + C_{13} = 0 - 3 = -3,$$

$$Z \equiv C_{03}C_{12} = 2\times(-5) = 1, \quad z \equiv C_{03} + C_{12} = 2 - 5 = -3,$$

$$W \equiv C_{01}C_{02}C_{03} + C_{01}C_{12}C_{13} + C_{02}C_{12}C_{23} + C_{03}C_{13}C_{23} = 1 - 2 = -1,$$

$$P \equiv \frac{1}{2}\left(1 - x - y - z + X + Y + Z\right) = \frac{1-(-3)+2}{2} = 3,$$

$$R \equiv P + z - Z = 3 - 3 - 1 = -1, \quad S \equiv P + y - Y = 3 - 3 + 0 = 0,$$

$$T \equiv P + x - X = 3 + 3 - 1 = 5,$$

$$U \equiv \frac{1}{2}\left(P^2 - W + XY + XZ + YZ\right) = \frac{-2+1+1}{2} = 0$$

and

$$V \equiv \frac{1}{2}\left(U^2 - XYR^2 - XZS^2 - YZT^2\right) = 0,$$

As $Y = 0$,

$$V^2 = XYZ\left(XR^2S^2 + YR^2T^2 + ZS^2T^2 + 2RSTU\right) = 0.$$

So, the Dihedral cross relation holds for the tetrahedron $\overline{A_0A_1A_2A_3}$ over $\mathbb{F}_{11}$.

Afterword

We are able to extend the framework in this thesis to set up a framework for trigonometry over higher-dimensional spaces. Most naturally, we can start with the four-dimensional affine space $\mathbb{A}^4$ over a field $\mathbb{F}$ with characteristic not equal to 2. While the definitions of quadrance, quadrea and quadrume naturally extend from the contents of this thesis, we are also able to talk about a hyperquadrume associated to a 4-simplex in $\mathbb{A}^4$. The Cayley-Menger determinant, as discussed earlier, provides a natural framework by which we can define metrical quantities not only in $\mathbb{A}^4$ but also for higher-dimensional affine spaces $\mathbb{A}^4$ over arbitrary fields not of characteristic 2. As for the spreads and their higher-dimensional counterparts, the recent paper by Wildberger in 2017 [62] gives an insight as to how to calculate spreads between planes in four-dimensional space; we may naturally extend such a concept to calculating spreads between objects of k dimensions in n-dimensional space, for $k \leq n$.

Furthermore, we can extend the framework in this thesis to understand the trigonometry of a projective or hyperbolic tetrahedron. Here, we can build on from the framework of projective planar trigonometry in this thesis to discuss the projective tetrahedron in the three-dimensional projective space $\mathbb{P}^3$. One aspect of setting up this framework involves calculation of the various types of spreads for this tetrahedron. Of more interest to current literature, however, is the calculation of the projective quadrume, or in the classical case the volume, of the projective/hyperbolic tetrahedron. This concept has been explored in independent works by Cho and Kim [12], Derevnin and Mednykh [18], Horváth [29], Molnár [41], Murakami and Yano [43], and Ushijima [54]. Whether there is a rational analog to this quantity has yet to be explored.

An interesting problem that one may come up with from reading this thesis is to find a possible relationship between the solid spreads of a tetrahedron in three-dimensional affine space, in a similar flavour to that of the Dihedral cross relation. With the framework set up in this thesis, one is poised to apply the techniques and tools here to various modern three-dimensional problems in robotics, animation, video games, physics, engineering and surveying.

Bibliography

[1] Altshiller-Court, N. (1979). *Modern Pure Solid Geometry.* New York, NY: Chelsea Pub. Co.

[2] Anton, H. & Rorres, C. (2005). *Elementary Linear Algebra with Applications* (9th ed.). Hoboken, NJ: John Wiley & Sons, Inc.

[3] Audet, D. (2011). Déterminants sphérique et hyperbolique de Cayley-Menger. *Bulletin AMQ*, **51**(2), 45-52.

[4] Belifante, J. G. F. & Kolman, B. (1972). *A Survey of Lie Groups and Lie Algebras with Applications and Computational Methods.* Philadelphia, PA: Society for Industrial and Applied Mathematics.

[5] Binet, J. F. M. (1812). Mémoire sur un systeme de formules analytiques, et leur application à des considerations géométriques. *Journal École Polytechnique*, **9**, 280-302.

[6] Bretschneider, C. A. (1842). Untersuchung der trigonometrischen Relationen des geradlinigen Viereckes. *Archiv der Mathematik und Physik*, **2**, 225-261.

[7] Brualdi, R. A. & Schneider, H. (1983). Determinantal identities: Gauss, Schur, Cauchy, Sylvester, Kronecker, Jacobi, Binet, Laplace, Muir, and Cayley. *Linear Algebra and its Applications.* **52-53**, 769-791.

[8] Cauchy, A. (1815). Memoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et des signes contraires par suite des transpositions opérées entre les variables qu'elles renferment. *Journal École Polytechnique*, **17**, 29-112.

[9] Chapman, S. & Milne, E. A. (1939). The proof of the formula for the vector triple product. *The Mathematical Gazette*, **23**(253), 35-38.

[10] Cho, E. C. (1991). The generalised cross product and the volume of a simplex. *Applied Mathematics Letters*, **4**(6), 51-53.

[11] Cho, E. C. (1995). The volume of a tetrahedron. *Applied Mathematics Letters*, **8**(2), 71-73.

[12] Cho, Y. & Kim, H. (1999). On the volume formula for hyperbolic tetrahedra. *Discrete Computational Geometry*, **22**, 347-366.

[13] Claggett, M. (1999). *Ancient Egyptian Science: A Source Book, Vol III: Ancient Egyptian Mathematics*. Philadelphia, PA: American Philosophical Society.

[14] Collomb, C. (n.d.). *A tutorial on inverting* 3 *by* 3 *matrices with cross products*. Retrieved from http://www.emptyloop.com/technotes.

[15] Coolidge, J. L. (1939). A historically interesting formula for the area of a quadrilateral. *The American Monthly*, **46**(6), 345-347.

[16] Coxeter, H. S. M. (1993). *The Real Projective Plane* (3rd ed.). New York, NY: Springer-Verlag.

[17] de Gua de Malves, J. P. (1783). Propositions neuves & non moins utiles que curieuses, sur le tètraédre. *Histoire de l'Academie royale des sciences* (volume for 1783), 363-402.

[18] Derevnin, D. A. & Mednykh, A. D. (2005). A formula for the volume of a hyperbolic tetrahedron. *Russian Mathematical Surveys*, **60**(2), 159-160.

[19] Dörrie, H. (1965). *100 Great Problems of Elementary Mathematics* (D. Antin, Trans.). Toronto, ON: Dover Publications Inc. (Original work published 1958).

[20] Egyptian Architecture - Khafre's Pyramid, Giza. (n.d.). Retrieved from http://www.essential-architecture.com/ASIA-WEST/NA-EG/NA-EG-026.htm.

[21] Gantmacher, F. R. (2000). *The Theory of Matrices* (Vol. 1) (K. A. Hirsch, Trans.). Providence, RI: AMS Chelsea Publishing. (Original work published 1959).

[22] Gibbs, J. W. & Wilson, E. B. (1901). *Vector Analysis: A text-book for the use of students of mathematics and physics founded upon the lectures of J. Willard Gibbs*. New Haven, CT: Yale University Press.

[23] Gillings, R. J. (1982). *Mathematics in the Time of the Pharaohs*. New York, NY: Dover Publications.

[24] Granville, W. A. (1909). Duality in the formulas of spherical trigonometry. *The American Mathematical Monthly*, **16**(3), 47-53.

[25] Havlicek, H. & Weiß, G. (2003). Altitudes of a tetrahedron and traceless quadratic forms. *American Mathematical Monthly*, **110**(8), 679-693.

[26] Heath, T. L. (1956). *The Thirteen Books of Euclid's Elements*. New York, NY: Dover Publishing Inc.

[27] Heath, T. L. (1921). *A History of Greek Mathematics, Volume 2: From Aristarchus to Diophantus*. Oxford, England: Clarendon Press.

[28] Hilbert, D. & Cohn-Vossen, S. (1952). *Geometry and the Imagination* (P. Nemenyi, Trans.). Providence, RI: American Mathematical Society Chelsea Publishing.

[29] Horváth, Á. G. (2012). Formulas on hyperbolic volume. *Aequationes mathematicae*, **83**, 97-116

[30] Humphreys, J. E. (1972). *Introduction to Lie Algebras and Representation Theory.* New York, NY: Springer-Verlag.

[31] Jacobi, C. G. J. (1829). *Fundamenta Nova Theoriae Functionum Ellipticarum.* Paris, France: Pontrieu & Co. and Treuttel & Wuerz.

[32] Khafre (n.d.). In *Encyclopaedia Britannica online.* Retrieved from https://www.britannica.com/place/Khafre-pyramid-Egypt.

[33] Klain, D. A. (2004). An intuitive derivation of Heron's formula. *American Mathematical Monthly*, **111**(8), 709-712.

[34] Lagrange, J. L. (1773). Solutions analytiques de quelques problèmes sur les pyramides triangulaires. *Oeuvres de Lagrange*, **3**, 661-692.

[35] Le, N. H. (2015). *Four-Fold Symmetry in Universal Triangle Geometry.* Doctoral thesis for the School of Mathematics & Statistics, UNSW Sydney.

[36] Le, N. H. & Wildberger, N. J. (2014). Incenter circles, chromogeometry, and the Omega triangle. *KoG*, **18**, 5-18.

[37] Lee, J. R. (1997). The law of cosines in a tetrahedron. *Journal of the Korean Mathematical Society, Education Series B: The Pure and Applied Mathematics*, **4**(1), 1-6.

[38] Leech, J. (1950). Some properties of the isosceles tetrahedron. *The Mathematical Gazette*, **24**(310), 269-271.

[39] Milnor, J. & Husemoller, D. (1973) *Symmetric Bilinear Forms.* Berlin, Germany: Springer-Verlag.

[40] Minkowski, H. (1907). Das Relativitätsprinzip. *Annalen der Physik*, **352**(15), 927-938.

[41] Molnár, E. (1989) Projective metrics and hyperbolic space. *Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae: Sectio Mathematica*, **32**, 127-157.

[42] Moritz, R. E. (1913). *A Text-book on Spherical Trigonometry.* New York, NY: John Wiley & Sons Inc.

[43] Murakami, J. & Yano, M. (2005). On the volume of a hyperbolic and spherical tetrahedron. *Communications in Analysis and Geometry*, **13**(2), 379-400.

[44] Pyramids of Giza (n.d.). In *Encyclopaedia Britannica online*. Retrieved from https://www.britannica.com/topic/Pyramids-of-Giza.

[45] Richardson, G. (1902). The trigonometry of a tetrahedron. *The Mathematical Gazette*, **2**(32), 149-158.

[46] Smith, J. & Henderson, M. (1985). Tetrahedra, skew lines, and volume. *The College Mathematics Journal*, **16**(2), 138-140.

[47] Sommerville, D. M. Y. (1958). *An Introduction to the Geometry of n Dimensions*. New York, NY: Dover Publishing Inc.

[48] Spiegel, M. R. (1959). *Schaum's Outline of Vector Analysis*. New York, NY: McGraw-Hill Inc.

[49] Steele, J. M. (2004). *The Cauchy-Schwartz Master Class*. Cambridge, England: Cambridge University Press.

[50] Stillwell, J. (2002). *Mathematics and its history* (2nd ed.). New York, NY: Springer.

[51] Strang, G. (1988). *Linear Algebra and its Applications* (3rd ed.). San Diego, CA: Harcourt Brace Jovanovich.

[52] Taylor, B. (1715). *Methodus Incrementorum Directa et Inversa*. London, England: Impensis Gulielmi Innys.

[53] Todhunter, I. (1886). *Spherical Trigonometry: For the Use of Colleges and Schools*. London, England: MacMillan & Co.

[54] Ushijima, A. (2006). A volume formula for generalised hyperbolic tetrahedra. *Non-Euclidean Geometries: János Bolyai Memorial Volume*, 249-265.

[55] Wildberger, N. J. (2006). Affine and projective universal geometry. Retrieved from https://arxiv.org/abs/math/0612499.

[56] Wildberger, N. J. (2015). An introduction to rational trigonometry and chromogeometry. *G*, **12**(23), 45-61.

[57] Wildberger, N. J. (2010). Chromogeometry. *The Mathematical Intelligencer*, **32**(1), 26-32.

[58] Wildberger, N. J. (2009). Chromogeometry and relativistic conics. *KoG*, **13**, 43-50.

[59] Wildberger, N. J. (2005). *Divine Proportions: Rational Trigonometry to Universal Geometry*. Sydney, NSW: Wild Egg Books.

[60] Wildberger, N. J. (2018). Projective and spherical trigonometry. Retrieved from https://www.researchgate.net/publication/268286292_Projective_and_spherical_trigonometry.

[61] Wildberger, N. J. (2007). One dimensional metrical geometry. *Geometriae Dedicata*, **128**(1), 145-166.

[62] Wildberger, N. J. (2017). Rational trigonometry in higher dimensions and a diagonal rule for 2-planes in four-dimensional space. *KoG*, **21**, 47-54.

[63] Wildberger, N. J. (2012). Three dimensional geometry, ZOME, and the elusive tetrahedron [Video file]. Retrieved from https://www.youtube.com/watch?v=rzUtdwh2rj0&t=1s.

[64] Wildberger, N. J. (2013). Universal hyperbolic geometry I: trigonometry. *Geometriae Dedicata*, **163**(1), 215-274.

[65] Wildberger, N. J. (2010). Universal hyperbolic geometry II: a pictorial overview. *KoG*, **14**, 3-24.

[66] Wildberger, N. J. (2011). Universal hyperbolic geometry III: first steps in projective triangle geometry. *KoG*, **15**, 25-49.

[67] Wildberger, N. J. & Alkhaldi, A. (2012). Universal hyperbolic geometry IV: sydpoints and twin circumcircles. *KoG*, **16**, 43-62.

[68] Wildberger, N. J. (2011). WildLinAlg: A geometric course in Linear Algebra [Video playlist]. Retrieved from https://www.youtube.com/playlist?list=PL01A21B9E302D50C1.