A generalization of the Beurling-Hedenmalm uncertainty principle

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Publication Date:
2016

DOI:
https://doi.org/10.26190/unsworks/19044

A generalization of the Beurling-Hedenmalm uncertainty principle

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A thesis in fulfillment of the requirements for the degree of

Doctor of Philosophy

School of Mathematics and Statistics
Faculty of Science

March 2016

PLEASE TYPE

THE UNIVERSITY OF NEW SOUTH WALES
Thesis/Dissertation Sheet

Surname or Family name: Gao

First name: Xin

Other name/s:

Abbreviation for degree as given in the University calendar: PHD

School: School of Mathematics and Statistics

Faculty: Faculty of Science

Title:

Abstract 350 words maximum: (PLEASE TYPE)

We study the uncertainty principles of Hardy and of Beurling, and functions that "only just" satisfy the inequalities of uncertainty principles. More specifically, we show that if a function and its Fourier transform have nearly gaussian decay, then the coefficients of its Hermite expansion decay fast, and vice versa. We give a new and simple proof of a generalisation of Beurling's uncertainty principle first in R using complex analysis. Then we generalise to R^n using various techniques. Also we illustrate connections with the classical moment problem.

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Acknowledgments

I would like to thank Michael Cowling for being a great supervisor and the School of Mathematics and Statistics at the University of New South Wales for their generous support throughout my candidature.

Contents

Chapter 1

Introduction

Definition 1.0.1. *Define the Fourier transform $\hat{f}$ of a function $f \in L_1(\mathbb{R})$ by*

$$\mathcal{F}(f)(y) = \hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ixy}\, dx \tag{1.1}$$

for all $y \in \mathbb{R}$.

The uncertainty principle is the phenomenon that a function f and its Fourier transform $\hat{f}$ can not both decay rapidly.

To interpret the uncertainty principle rigorously, we need to have a precise definition of what the decay of functions f and $\hat{f}$ means. Different measurements of the decay of functions f and $\hat{f}$ give us different theorems that demonstrate the uncertainty principle.

Suppose that $\alpha > 1$ and define the decay of f to be

$$\mathcal{D}_f = \frac{\| |x|^\alpha f(x) \|_{L_2(\mathbb{R})}}{\| f(x) \|_{L_2(\mathbb{R})}}. \tag{1.2}$$

Then the uncertainty principle becomes a mild generalization of the Heisenberg-Pauli-Weyl inequality (see Cowling and Price [6]):

$$\mathcal{D}_f \mathcal{D}_{\hat{f}} \geq C. \tag{1.3}$$

In particular if $\alpha = 1$ then the above inequality is the Heisenberg-Pauli-Weyl uncertainty principle (see Dym and McKean [7]).

Also, suppose that the decay of f is defined as

$$\mathcal{D}_f(\alpha) = \left\| f(x) e^{\alpha x^2/2} \right\|_\infty \tag{1.4}$$

Then Hardy's uncertainty principle says that if $\mathcal{D}_f(\alpha)$ and $\mathcal{D}_{\hat{f}}(\beta)$ are both finite and $\alpha\beta = 1$, then there exists a constant C such that $f(x) = Ce^{-\alpha x^2/2}$. Also if $\mathcal{D}_f(\alpha)$

and $\mathcal{D}_{\hat{f}}(\beta)$ are both finite and $\alpha\beta > 1$, then f is zero (see Hardy [11]).

Similarly, suppose that we define the decay of f to be

$$\mathcal{D}_f(\alpha) = \left\| f(x) e^{\alpha x^2/2} \right\|_{L_2(\mathbb{R})}. \tag{1.5}$$

If $\mathcal{D}_f(\alpha)$ and $\mathcal{D}_{\hat{f}}(\beta)$ are both finite and $\alpha\beta \geq 1$, then f is zero (see Cowling and Price [5]).

Moreover, the Morgan-Gel'fand-Shilov type uncertainty principle interprets the decay of functions f and $\hat{f}$ by

$$\mathcal{D}_f(\alpha) = \int_{\mathbb{R}^n} \frac{|f(x)|\, e^{\alpha^p |x|^p/p}}{(1+|x|)^N}\, dx, \tag{1.6}$$

$$\mathcal{D}_{\hat{f}}(\beta) = \int_{\mathbb{R}^n} \frac{\left|\hat{f}(x)\right|\, e^{\beta^q |x|^q/q}}{(1+|x|)^N}\, dx, \tag{1.7}$$

where $1/p+1/q = 1$. Gel'fand and Shilov [10] extended the work of Morgan [13] and proved that if $\mathcal{D}_f(\alpha)$ and $\mathcal{D}_{\hat{f}}(\beta)$ are both finite and $\alpha\beta \geq 1/4$, then f is zero unless $\alpha\beta = 1/4$ and $p = q = 2$, in which case $f(x) = P(x)e^{-\alpha^2 x^2/2}$ where P is a polynomial.

Suppose $N \geq 0$ and $1 \leq p, q < \infty$. Then in the Cowling-Price type uncertainty principle, the decay of the functions f and $\hat{f}$ may be defined by

$$\mathcal{D}_f(\alpha) = \int_{\mathbb{R}^n} \left(\frac{|f(x)|\, e^{\alpha |x|^2/2}}{(1+|x|)^N} \right)^p dx \tag{1.8}$$

.

$$\mathcal{D}_{\hat{f}}(\beta) = \int_{\mathbb{R}^n} \left(\frac{\left|\hat{f}(x)\right|\, e^{\beta |x|^2/2}}{(1+|x|)^N} \right)^q dx. \tag{1.9}$$

If $\mathcal{D}_f(\alpha)$ and $\mathcal{D}_{\hat{f}}(\beta)$ are both finite and $\alpha\beta > 1$, then f is zero. Also if $\alpha\beta = 1$ then $f(x) = P(x)e^{-\alpha^2 x^2/2}$, where P is a polynomial (see Cowling and Price [6]).

We notice that the Hardy style theorems are about a pair α and β such that $\mathcal{D}_f(\alpha)$ and $\mathcal{D}_{\hat{f}}(\beta)$ are finite. We can ask a natural problem: if we define

$$\mathcal{D}_f = \sup_{\alpha} \{\alpha \mid \mathcal{D}_f(\alpha) < \infty\} \tag{1.10}$$

and

$$\mathcal{D}_{\hat{f}} = \sup_{\beta} \left\{\beta \mid \mathcal{D}_{\hat{f}}(\beta) < \infty\right\}, \tag{1.11}$$

what can we say about $\mathcal{D}_f$ and $\mathcal{D}_{\hat{f}}$? Generally if $\mathcal{D}_f\mathcal{D}_{\hat{f}} > 1$, then f is zero. So the interesting case is when $\mathcal{D}_f\mathcal{D}_{\hat{f}} = 1$. The growth of $\mathcal{D}_f(\alpha)\mathcal{D}_{\hat{f}}(\beta)$ when $\alpha\beta \to 1$ plays an important part here and this leads to our first problem, as follows.

Problem 1.0.2. *Suppose that $f \in L_1(\mathbb{R})$,*

$$\int_{\mathbb{R}} |f(x)|\, e^{sx^2/2}\, dx \leq C(1-s)^{-(N+1)/2} \tag{1.12}$$

and

$$\int_{\mathbb{R}} \left|\hat{f}(y)\right| e^{sy^2/2} dy \leq C(1-s)^{-(N+1)/2} \tag{1.13}$$

when $0 \leq s < 1$. What can we say about the function f ?

Answer*: If f satisfies the above conditions, then there exists a polynomial P of degree at most N such that $f(x) = P(x)e^{-x^2/2}$. Part of Problem 1.0.2 can be solved via standard arguments that already exist in Hardy's paper* [11]. *In this thesis, it is also implied by Theorem 4.4.4 in Chapter 4.*

In Beurling's uncertainty principle, $f(x)$ and $\hat{f}(y)$ are considered together, and the decay is interpreted by

$$\mathcal{D}_f = \iint_{\mathbb{R}^2} \left|f(x)\, \hat{f}(y)\right| e^{|xy|}\, dx\, dy. \tag{1.14}$$

Beurling's uncertainty principle says if the above integral is finite, then f is equal to zero. This was generalized by Bonami, Demange and Jaming, who proved that if

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\frac{f(x)\hat{f}(y)}{(1+|x|+|y|)^N}\right| e^{|\langle x,y\rangle|}\, dx\, dy < \infty, \tag{1.15}$$

then $f(x) = P(x)e^{-\langle Ax,x\rangle}$ where $P(x)$ is a polynomial and A is a positive definite matrix (see Bonami, Demange and Jaming [3]).

Similarly, in Beurling's uncertainty principle, we can define

$$\mathcal{D}_f = \sup_{\alpha} \left\{\alpha \mid \mathcal{D}_f(\alpha) < \infty\right\}, \tag{1.16}$$

where

$$\mathcal{D}_f(\alpha) = \iint_{\mathbb{R}^2} \left|f(x)\, \hat{f}(y)\right| e^{\alpha|xy|}\, dx\, dy. \tag{1.17}$$

When $\mathcal{D}_f > 1$, f is zero by Beurling's uncertainty principle. Thus the interesting problem is what we can say about f when $\mathcal{D}_f = 1$. Again the growth of $\mathcal{D}_f(\alpha)$ when α goes to 1 plays an important part here. In this thesis, the following problem regarding this uncertainty principle is discussed and solved (Also my paper "On

Beurling's uncertainty principle" was accepted and published by the Bulletin of the London Mathematical Society).

Problem 1.0.3. *If a function f on $\mathbb{R}$ is such that*

$$\iint_{\mathbb{R}^2} \left|f(x)\,\hat{f}(y)\right| e^{\lambda|xy|}\,dx\,dy = O((1-\lambda)^{-N}) \tag{1.18}$$

as $\lambda \to 1-$, what can we say about the function f ?

***Answer**: If f satisfies the above condition, then f is the product of a polynomial of degree of at most $\lfloor N - 1/2 \rfloor$ and a gaussian. Problem 1.0.3 was open and by solving this problem we developed an approach that can actually treat Problem 1.0.2 and Problem 1.0.3 in a uniform way in Chapter 4. Also in Chapter 5, this problem is generalized into higher dimensions by using an Radon-transform reduction.*

Before we state the last problem, we define the Mellin transform $\mathcal{M}_f^k$ of a function f by

$$\mathcal{M}_f^k(z) = \int_{\mathbb{R}} f(x)\,\mathrm{sgn}^k(x)\,|x|^{z-1/2}\,dx.$$

Problem 1.0.4. *Suppose that f and $\hat{f}$ are of gaussian decay and a sequence of complex numbers $\{z_n\}$ satisfies the condition*

$$\sum_{n=1}^{\infty} \frac{1}{z_n^2} = \infty. \tag{1.19}$$

Is f uniquely decided by the values of $\mathcal{M}_f^k(z_n)$?

***Answer**: This can be answered affirmatively by further exploring the approach in Chapter 4. In Chapter 6, I showed that the value of $\mathcal{M}_f^k(z_n)$ uniquely decides the function f.*

In the literature, a useful approach to understand the uncertainty principle for a given function f is to construct an analytic auxiliary function based on f and then try to conclude certain properties of that auxiliary function which imply useful properties of f.

In this thesis three different analytic functions built from f are discussed. The most obvious one is the natural analytic extension of the Fourier transform, defined as follows.

Definition 1.0.5 (Analytic continuation of the Fourier transform)**.** *Given a function f in $L_1(\mathbb{R})$, we define the analytic extension of f to be*

$$f(z) = \int_{\mathbb{R}} \hat{f}(y) e^{zyi}\,dy, \tag{1.20}$$

for all complex z for which the integral is defined.

The second way of constructing an analytic function from f is by the Bargmann transform.

Definition 1.0.6 (Bargmann transform)**.** *Given a function f in $L_1(\mathbb{R})$, we define its Bargmann transform B_f as follows.*

$$B_f(z) = \left\langle f(\cdot)\exp((\cdot)^2/2), \exp\left(-(\cdot - z/2)^2\right)\right\rangle \tag{1.21}$$

for all complex z for which $B_f(z)$ is defined. Here $\langle\cdot,\cdot\rangle$ denotes the usual $L_2(\mathbb{R})$ inner product.

The third method of getting an analytic function from f is the Θ transform which is defined as follows (a formal definition is found in Chapter 3.

Definition 1.0.7 (Θ transform of f)**.** *Given a function f in $L_1(\mathbb{R})$, we define*

$$\Theta_f^k(z) = \frac{\mathcal{M}_f^k(z)}{\Gamma(\frac{1}{4} + \frac{z}{2} + \frac{k}{2})}, \tag{1.22}$$

for all $z \in \mathbb{C}$ for which $\mathcal{M}_f^k(z)$ is defined.

This transform was not in the literature before and is my main contribution in this thesis. This thesis is dedicated to showing the idea behind the solution of all above three problems by examining properties of Θ_f^k and how these properties can be used. Roughly speaking, our result about Θ_f^k can be stated as follows.

Observation 1.0.8. *Suppose that there exist positive numbers c, d, α and β such that $|f(x)| \le ce^{-\alpha x^2/2}$ and $\left|\hat{f}(y)\right| \le de^{-\beta y^2/2}$ for all $x, y \in \mathbb{R}$. Then Θ_f^k is an analytic function of order 1 and, by the Hadamard factorization theorem (see Chapter 2),*

$$\Theta_f^k(z) = z^m e^{az+b} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n}\right) \tag{1.23}$$

for some fixed values m, a and b.

Chapter 2

Background material

2.1 The Fourier transform and Schwartz space

Definition 2.1.1. *Define the Fourier transform $\hat{f}$ of a function $f \in L_1(\mathbb{R}^n)$ by*

$$\mathcal{F}(f)(y) = \hat{f}(y) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(x) e^{-i\langle x,y\rangle}\, dx, \tag{2.1}$$

for all $y \in \mathbb{R}^n$.

In this thesis we denote by γ_a the gaussian function $e^{-ax^2/2}$ on $\mathbb{R}$, and then

$$\hat{\gamma_a}(y) = \sqrt{\frac{1}{a}}\gamma_{1/a}(y). \tag{2.2}$$

Definition 2.1.2. *The Schwartz space on $\mathbb{R}^n$ is the function space*

$$\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta} < \infty \quad \forall \alpha, \beta \in \mathbb{N}^n \right\},$$

where $C^\infty(\mathbb{R}^n)$ is the set of smooth functions from $\mathbb{R}^n$ to $\mathbb{C}$, and

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta f(x) \right|,$$

where sup *denotes the supremum, while x^α is the monomial and D^β is the partial derivative given by the standard multi-index notation.*

It is well known that the Fourier transformation is a bijection on $\mathcal{S}(\mathbb{R}^n)$, and hence we may extend the definition of the Fourier transformation to the dual space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions.

2.2 The Hermite functions

Definition 2.2.1. *We denote by h_n the nth Hermite functions* [18]*, defined as follows:*

$$h_n(x) = \left(\frac{1}{\sqrt{\pi}2^n n!}\right)^{1/2} (-1)^n \frac{d^n(e^{-x^2})}{dx} e^{x^2/2}. \tag{2.3}$$

It is easy to check that, when $m < n$,

$$\int_{\mathbb{R}} x^m h_n(x) e^{-x^2/2}\, dx = C \int_{\mathbb{R}} D^n(e^{-x^2}) x^m\, dx = 0, \tag{2.4}$$

by repeated integration by parts, where D denotes differentiation. Thus $\int_{\mathbb{R}} h_n(x)h_m(x)\, dx = 0$ when $m \neq n$, and the $\{h_n\}$ form an orthonormal basis in $L_2(\mathbb{R})$. Also h_n is a polynomial multiplied by $\gamma_1(x)$ and we denote by H_n the Hermite polynomials, defined as follows.

Definition 2.2.2.

$$H_n(x) = \frac{d^n(e^{-x^2})}{dx} e^{x^2}. \tag{2.5}$$

It is easy to check that

$$h_n(x) = \left(\frac{1}{\sqrt{\pi}2^n n!}\right)^{1/2} (-1)^n H_n(x) e^{-x^2/2}. \tag{2.6}$$

Moreover (see Thangavelu [18])

$$\mathcal{F}(\gamma_1 H_n) = (-i)^n \gamma_1 H_n. \tag{2.7}$$

2.3 The Mellin transform

Suppose that $f : \mathbb{R}^+ \to \mathbb{C}$. Then we can define a function $\tilde{f}(x) = f(e^x)$ and by direct computation

$$\begin{aligned} \mathcal{F}(\tilde{f})(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(e^x) e^{-i\xi x}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} f(t) t^{-i\xi} \frac{dt}{t}. \end{aligned} \tag{2.8}$$

This leads to the traditional definition of the Mellin transform, as follows.

Definition 2.3.1. *Suppose that $\int_{\mathbb{R}^+} |f(x)|\, x^{s-1/2}\, dx < \infty$ when $s \in [\alpha, \beta] \subseteq \mathbb{R}$. Then the Mellin transform $\mathcal{M}_f$ is defined by*

$$\mathcal{M}_f(z) = \sqrt{2\pi} \mathcal{F}(\tilde{f}) \left(iz - \frac{i}{2}\right) = \int_{\mathbb{R}^+} f(t) t^{z-1/2}\, dt \tag{2.9}$$

when $\operatorname{Re} z \in [\alpha, \beta]$.

If $f(t)$ is in $L_1(\mathbb{R}^+, dt/t)$, then $f \circ \exp$ is in $L_1(\mathbb{R})$. Also we can check that by the above definition

$$\mathcal{F}(\tilde{f})(x) = \frac{1}{\sqrt{2\pi}} \mathcal{M}_f(-1/2 - ix).$$

Thus by the Fourier inversion formula we can compute $f(x)$ when $x \geq 0$:

$$\begin{aligned}
f(x) &= \mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}} \mathcal{M}_f(-1/2 - i(\cdot))\right)(\log x) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{M}_f(-ti - 1/2) e^{i \log x t}\, dt \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{M}_f(-ti - 1/2) x^{it}\, dt \\
&= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \mathcal{M}_f(y) x^{-y-1/2}\, dy,
\end{aligned} \tag{2.10}$$

where the last integral is a complex line integral.

Definition 2.3.2. *Define the Mellin inverse transform of a complex function f defined on the strip $\{z \in \mathbb{C} : |\mathrm{Re}(z)| < c\}$ by*

$$\mathcal{M}_f^{-1}(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} f(y) x^{-y-1/2}\, dy \tag{2.11}$$

where $x > 0$ and the integral is a complex line integral.

It follows that, by defining the Mellin inverse transform $\mathcal{M}^{-1}$ as above, we can reconstruct half of the function f on $\mathbb{R}^+$. To reconstruct the whole function f on $\mathbb{R}$ we will also need $\mathcal{M}_{f^\vee} = \mathcal{F}(\widetilde{f^\vee})$, where $f^\vee(t) = f(-t)$.

For convenience, to uniquely reconstruct f via $\mathcal{M}^{-1}$, we adjust the definition of $\mathcal{M}_f$ as follows.

Definition 2.3.3. *Given a function $f \in L_1(\mathbb{R})$, define*

$$\mathcal{M}_f^k(z) = \int_{\mathbb{R}} f(x)\, \mathrm{sgn}^k(x)\, |x|^{z-1/2}\, dx, \tag{2.12}$$

for all $z \in \mathbb{C}$ for which $\mathcal{M}_f^k(z)$ is defined.

It is easy to check that if f is even, then $\mathcal{M}_f^0$ is twice the traditional Mellin transform of f. Similarly if f is odd, then $\mathcal{M}_f^1$ is twice the traditional Mellin transform of f. Also $\mathcal{M}_f^0 = \mathcal{M}_{f^\vee}^0$ and $\mathcal{M}_f^1 = -\mathcal{M}_{f^\vee}^1$.

A function f can be split into an even part f_e and an odd part f_o such that $f = f_e + f_o$. Then we can reconstruct f on $\mathbb{R}$ by

$$f(x) = \frac{1}{2}\left((\mathcal{M}^{-1}\mathcal{M}_f^0)(|x|) + \mathrm{sgn}(x)(\mathcal{M}^{-1}\mathcal{M}_f^1)(|x|)\right) \tag{2.13}$$

for all x in $\mathbb{R}$.

2.4 Multiplicative convolution

Definition 2.4.1. *Suppose that $|f|$ and $|g|$ are bounded and in $L_1(\mathbb{R}^+, dt/t)$. Then we define the multiplicative convolution $f \star_M g$ between f and g by*

$$f \star_M g(x) = \int_{\mathbb{R}^+} f(y) g\left(\frac{x}{y}\right) \frac{dy}{y} \tag{2.14}$$

for all $x \in \mathbb{R}$.

Because we can check that, when $u > 0$,

$$\begin{aligned} \mathcal{M}_{f \circ \delta_u}(z) &= \int_{\mathbb{R}^+} f(ux) x^{z-1/2}\, dx \\ &= \int_{\mathbb{R}^+} u^{-1/2-z} f(t) t^{z-1/2}\, dt \\ &= u^{-1/2-z} \mathcal{M}_f(z), \end{aligned} \tag{2.15}$$

where $f \circ \delta_u(x) = f(ux)$, we can find the formula for the traditional Mellin transform of a multiplicative convolution as follows:

$$\begin{aligned} \mathcal{M}_{f \star_M g}(z) &= \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^+} f(y) g\left(\frac{x}{y}\right) \frac{dy}{y} \right) x^{z-1/2}\, dx \\ &= \int_{\mathbb{R}^+} \frac{f(y)}{y} \left(\int_{\mathbb{R}^+} g\left(\frac{x}{y}\right) x^{z-1/2}\, dx \right) dy \\ &= \int_{\mathbb{R}^+} f(y) y^{-1/2+z} \mathcal{M}_g(z)\, dy \\ &= \mathcal{M}_f(z) \mathcal{M}_g(z). \end{aligned} \tag{2.16}$$

In this thesis we define an operator $\star_m$ that is similar to multiplicative convolution as follows.

Definition 2.4.2. *Suppose that functions f and g are in $L_2(\mathbb{R})$. Then we define $f \star_m g$ by*

$$(f \star_m g)(x) = \int_{\mathbb{R}} f(y) g(xy)\, dy. \tag{2.17}$$

From the above definition, we get a similar equation regarding the Mellin transform as follows.

Lemma 2.4.3. *Suppose that f and g satisfy the integrability condition*

$$\int_{\mathbb{R}} |f(x)|\, x^{-1/2+s} < \infty \tag{2.18}$$

and

$$\int_{\mathbb{R}} |g(x)|\, x^{-1/2+s} < \infty \tag{2.19}$$

for all $s \in [-a, a]$. *Then*

$$\mathcal{M}^k_{f *_m g}(z) = \mathcal{M}^k_f(-z)\mathcal{M}^k_g(z) \tag{2.20}$$

when $\mathrm{Re}(z) \in [-a, a]$.

Proof. When $\mathrm{Re}\, z \in [-a, a]$ we have, by computation,

$$\begin{aligned}
\mathcal{M}^k_{f\circ\delta_\alpha}(z) &= \int_{\mathbb{R}} \mathrm{sgn}^k(x) f(\alpha x) |x|^{z-1/2}\, dx \\
&= \int_{\mathbb{R}} |\alpha|^{-z-1/2} \mathrm{sgn}^k(t/\alpha) f(t) |t|^{z-1/2}\, dt \\
&= \mathrm{sgn}^k(\alpha) |\alpha|^{-z-1/2} \mathcal{M}^k_f(z).
\end{aligned} \tag{2.21}$$

It follows that when $\mathrm{Re}(z) \in [-a, a]$

$$\begin{aligned}
\mathcal{M}^k_{f \star_m g}(z) &= \int_{\mathbb{R}} \mathrm{sgn}^k(x) \left(\int_{\mathbb{R}} f(y) g(xy)\, dy \right) |x|^{z-1/2}\, dx \\
&= \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} \mathrm{sgn}^k(x) g(yx) |x|^{z-1/2}\, dx \right) dy \\
&= \int_{\mathbb{R}} f(y)\, \mathrm{sgn}^k(y) |y|^{-z-1/2}\, dy\, \mathcal{M}^k_f(z) \\
&= \mathcal{M}^k_f(-z)\mathcal{M}^k_g(z),
\end{aligned} \tag{2.22}$$

as required. ∎

2.5 The Γ function

The Γ function shows up in the Mellin transforms of certain functions in this thesis. Here we recall several useful formulas that are needed to compute the inverse Mellin transform.

We first define the Γ function as follows.

Definition 2.5.1. *For all* $z \in \mathbb{C}$ *such that* $\mathrm{Re}(z) > 0$, *define*

$$\Gamma(z) = \int_{\mathbb{R}^+} x^{z-1} e^{-x}\, dx. \tag{2.23}$$

To extend the Γ function to the whole plane we use the following reflection formula, due to Euler.

Lemma 2.5.2. *For all $z \in \mathbb{C}\backslash\mathbb{Z}$,*

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}. \tag{2.24}$$

We can thus quickly conclude that Γ function has no zeros anywhere in the complex plane; we will need this property in Chapter 4. Also by computing the Mellin transform of gaussian functions, we get the following result.

Lemma 2.5.3. *For all $z \in \mathbb{C}$,*

$$\mathcal{M}_{(\cdot)^k\gamma_a}(z) = \frac{1}{2}\left(\frac{a}{2}\right)^{-(z/2+1/4+k/2)} \Gamma\left(\frac{z}{2}+\frac{1}{4}+\frac{k}{2}\right). \tag{2.25}$$

Proof.

$$\begin{aligned}\mathcal{M}_{(\cdot)^k\gamma_a}(z) &= \int_{\mathbb{R}^+} e^{-ax^2/2} x^{z-1/2+k}\,dx \\ &= \frac{1}{2}\int_{\mathbb{R}^+} e^{-y}\left(\frac{2}{a}\right)^{z/2+1/4+k/2} y^{z/2-3/4+k/2}\,dy \\ &= \frac{1}{2}\left(\frac{a}{2}\right)^{-(z/2+1/4+k/2)} \Gamma\left(\frac{z}{2}+\frac{1}{4}+\frac{k}{2}\right),\end{aligned} \tag{2.26}$$

as required. □

By changing a to $2e^{-t}$ we notice that

$$e^{-t(1/4+k/2)}\mathcal{M}_{(\cdot)^k\gamma_{2e^{-t}}}(z) = \frac{1}{2}e^{zt/2}\,\Gamma\left(\frac{z}{2}+\frac{1}{4}+\frac{k}{2}\right). \tag{2.27}$$

It follows that

$$\frac{\partial^n}{\partial t}(e^{-t(1/4+k/2)}\mathcal{M}_{(\cdot)^k\gamma_{2e^{-t}}})(z) = \frac{1}{2}(\frac{z}{2})^n e^{zt/2}\Gamma\left(\frac{z}{2}+\frac{1}{4}+\frac{k}{2}\right). \tag{2.28}$$

Thus we have the following lemma to compute the Mellin inverse of functions of the form $\mathcal{M}_f^k(z) = P(z/2)e^{bz}\,\Gamma\left(\frac{z}{2}+\frac{1}{4}+\frac{k}{2}\right)$ for $z \in \mathbb{C}$, where P is a polynomial.

Lemma 2.5.4. *Suppose that the Mellin transform $\mathcal{M}_f(z)$ satisfies*

$$\mathcal{M}_f(z) = P(z/2)e^{bz}\,\Gamma\left(\frac{z}{2}+\frac{1}{4}+\frac{k}{2}\right), \tag{2.29}$$

where $k = 0, 1$, $z \in \mathbb{C}$ and P is a polynomial of degree d. Then

$$f(x) = 2\left[P\left(\frac{\partial}{\partial t}\right)\right]_{t=2b}\left(x^k e^{-t(1/4+k/2)}\exp(e^{-t}x^2)\right) \tag{2.30}$$

for $x \in \mathbb{R}$. Hence f is a polynomial of degree $2d+k$ times the gaussian function $\gamma_{2e^{-2b}}$.

Proof. Combine Lemma 2.5.3 and the above arguments. □

Suppose that f is a function on the complex plane, analytic except at the points $\pm i$, of the form

$$f(z) = \sum_{k=0}^{1} \sum_{j=0}^{N} \frac{a_{j,k} z^k}{(1+z^2)^{j+1/2}}. \tag{2.31}$$

In order to get the Mellin transform of f on the real line, we need to compute the Mellin transform of each term in the above equation on the real line. For convenience, we define $\theta_{k,n}$ as follows.

Definition 2.5.5. *We define*

$$\theta_{k,n}(x) = \frac{x^k}{(1+x^2)^{n+1/2}} \tag{2.32}$$

for all $x \in \mathbb{R}$.

It is easy to compute the Mellin transform of each $\theta_{n,k}$ and we have the following results.

$$\begin{aligned}
\mathcal{M}_{\theta_{k,n}}(z) &= \int_{\mathbb{R}^+} \frac{x^{k+z-1/2}}{(1+x^2)^{n+1/2}}\, dx \\
&= \int_0^{\pi/2} \frac{(\tan\theta)^{k+z-1/2}}{(1+\tan^2\theta)^{n+1/2}} \frac{1}{\cos^2\theta}\, d\theta \\
&= \int_0^{\pi/2} (\sin\theta)^{k+z-1/2} (\cos\theta)^{2n-1/2-k-z}\, d\theta \\
&= \frac{1}{2\Gamma(n+1/2)} \Gamma\left(n + \frac{1}{4} - \frac{k+z}{2}\right) \Gamma\left(\frac{1}{4} + \frac{k+z}{2}\right). \\
&= C_n \Gamma\left(n + \frac{1}{4} - \frac{k+z}{2}\right) \Gamma\left(\frac{1}{4} + \frac{k+z}{2}\right).
\end{aligned} \tag{2.33}$$

It follows that, for f of the form (2.31),

$$\begin{aligned}
\mathcal{M}_f^0(z) &= \frac{1}{2} \sum_{n=0}^{N} a_{n,0} C_n \Gamma\left(n + \frac{1}{4} - \frac{z}{2}\right) \Gamma\left(\frac{1}{4} + \frac{z}{2}\right) \\
&= P(z) \Gamma\left(\frac{1}{4} - \frac{z}{2}\right) \Gamma\left(\frac{1}{4} + \frac{z}{2}\right),
\end{aligned} \tag{2.34}$$

where P is a polynomial of degree at most N. Also similarly

$$\begin{aligned}
\mathcal{M}_f^1(z) &= \frac{1}{2} \sum_{n=0}^{N} a_{n,1} C_n \Gamma\left(n - \frac{1}{4} - \frac{z}{2}\right) \Gamma\left(\frac{3}{4} + \frac{z}{2}\right) \\
&= Q(z) \Gamma\left(\frac{3}{4} - \frac{z}{2}\right) \Gamma\left(\frac{3}{4} + \frac{z}{2}\right),
\end{aligned} \tag{2.35}$$

where Q is a polynomial of degree at most $N-1$.

2.6 The Hadamard factorization theorem

In this section we give a quick review of some classical results about analytic functions.

Suppose that f is analytic in $B(0,r)$, the ball around 0 of radius r, and that f has no zeros in $B(0,r)$. Then $\log|f|$ is also a harmonic function in $B(0,r)$ and by the mean value theorem we have

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\theta})\right| \, d\theta. \tag{2.36}$$

Jensen's formula says that if f is analytic in the ball around zero $B(0,r)$ and f has zeros $\{z_k\}$, then

$$\log|f(0)| = \sum_{k=1}^{N} \log\left(\frac{z_k}{r}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\theta})\right| \, d\theta. \tag{2.37}$$

From Jensen's formula, we can derive a lemma regarding the connection between the number of zeros of an analytic function f and the growth of f.

Lemma 2.6.1. *If f is an entire function, $B(r)$ is an increasing function defined on $\mathbb{R}^+$ and $|f(z)| \leq B(|z|)\,|f(0)|$ where $f(0)$ is not zero, then*

$$n(r) \leq \frac{\log B(2r)}{\log 2}, \tag{2.38}$$

where $n(r)$ is the number of zeros in the unit ball $B(0,r)$.

Proof. See Conway [4, p. 282]. □

If we know that $\log B(2r) \leq r \log 2$, then the number of zeros $n(r)$ of function f in the ball $B(r,0)$ is less than r by applying the above estimate. It follows that if we randomly pick $\{z_n\}$ such that

$$|\,\{z_k : |z_k| < r\}\,| \leq r,$$

where $|S|$ indicates the cardinality of a set S, then the values $f(z_k)$ uniquely determine the function f.

Definition 2.6.2 (order of an entire function)**.** *The order (at infinity) of an entire* *function $f(z)$ is defined by*

$$\rho = \limsup_{r\to\infty} \frac{\log(\log \|f\|_{B_r})}{\log\, r},$$

where B_r is the disk of radius r and $\|f\|_{B_r}$ is the maximum value of $|f(x)|$ for all $x \in B_r$. Also it is equivalent to the infimum of all m such that $f(z) = O(e^{|z|^m})$ as $z \to \infty$.

Definition 2.6.3 (rank of an entire function)**.** *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is an entire function with zeros at z_n and a zero of order m at 0. If there exists an integer p such that*

$$\sum_{n=1}^{\infty} |z_n|^{-p-1} < \infty \tag{2.39}$$

where $\{z_n\}$ is sorted so that $|z_n|$ increases, then f is defined to be of finite rank. The smallest p such that the above inequality holds is defined to be the rank of f. If f has only finite many zeros, f has rank 0.

We recall the Weierstrass factorization theorem.

Theorem 2.6.4 (Weierstrass Factorization Theorem)**.** *Suppose that f is an entire function with zeros at z_n where $|z_n| \neq 0$ and a zero of order m at 0 and there exists a sequence $\{p_n\}$ such that*

$$P(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right) \tag{2.40}$$

converges, where

$$E_p(z) = (1 - z)\exp\left(\sum_{n=1}^{p} \frac{z^n}{n}\right). \tag{2.41}$$

Then

$$f(z) = z^m e^{g(z)} P(z). \tag{2.42}$$

In the Weierstrass factorization theorem g is not necessary a polynomial. However if g is a polynomial of degree p and the rank (defined by Definition 2.6.3) is q, we can define the genus μ of the entire function f to be $\max(p, q)$. The Hadamard factorization theorem says that if f is of order λ, then $\mu \leq \lambda$.

Theorem 2.6.5 (Hadamard factorization theorem)**.** *Suppose that f is an entire function of finite order λ. Then f also has finite rank p and f has a canonical form*

$$f(z) = z^m e^{g(z)} P(z), \tag{2.43}$$

where g is a polynomial of finite degree q and $P(z)$ is as in Theorem 2.6.4. Moreover

$$\lambda - 1 \leq \max(p, q) \leq \lambda. \tag{2.44}$$

Proof. See Conway [4, p. 289]. $\square$

2.7 Notation

Suppose that f and g are functions with domain $\mathbb{D}$. We say $|f(x)| \lesssim |g(x)|$ when there exists a constant C such that $|f(x)| \leq C\,|g(x)|$ for all $x \in \mathbb{D}$. We denote by $\circ$ the composition operator such that $(f \circ g)(x) = f(g(x))$. Also we denote by δ_u the dilation function such that $\delta_u(x) = ux$.

Chapter 3

Hardy's uncertainty principle revisited

3.1 Introduction

Hardy's uncertainty principle on $\mathbb{R}$ is a classical result in harmonic analysis. Hardy initially states his result by saying that if

$$|f(x)| \lesssim (1+|x|)^m \gamma_a(x) \text{ and } \left|\hat{f}(y)\right| \lesssim (1+|y|)^m \gamma_b(y)$$

for all $x, y \in \mathbb{R}$ and if $ab = 1$, then f is equal to $P\gamma_a$ where P is a polynomial. Also if $ab > 1$, then both f and $\hat{f}$ are null. Moreover if $ab < 1$ then there are infinitely many linearly independent functions that satisfy the condition.

In this chapter, I will firstly review this classical result and then use similar techniques to get estimates of derivatives of functions that satisfy Hardy's condition. With the estimates of derivatives we will be able to estimate the coefficients for Hermite expansion of function f. It turns out that the estimating of coefficients for Hermite expansion gives us a new and quicker way to prove Hardy's uncertainty principle.

In Garg and Thangavelu [8], it is proved that if $f \in L_1(\mathbb{R})$ and satisfies estimates $|f(x)| \leq Ce^{-ax^2/2}$ and $\left|\hat{f}(y)\right| \leq Ce^{-ay^2/2}$ for some $0 < a < 1$, then

$$|\langle f, h_k \rangle| \leq C(2k+1)e^{(2k+1)t/2}$$

where t is determined by the condition $\tanh(2t) = a/2$. Also, conversely, if $|\langle f, h_k\rangle| \leq C(2k+1)e^{(2k+1)t/2}$, then $|f(x)| \leq Ce^{-\tanh(t)|x|^2/2}$. In this chapter, we consider the related condition

$$|f(x)| \leq Ce^{-x^2/2}e^{t|x|} \qquad \text{and} \qquad \left|\hat{f}(y)\right| \leq Ce^{-y^2/2}e^{t|y|} \tag{3.1}$$

for small t. We show that this condition implies that

$$|\langle f, h_n\rangle| \lesssim \frac{nt^n}{\sqrt{n!}}.$$

Conversely, we show that if $|\langle f, h_n\rangle| \lesssim nt^n/\sqrt{n!}$, then

$$|f(x)| \lesssim (t\,|x| + 1)e^{-x^2/2}e^{\sqrt{2}t|x|}.$$

3.2 Analytic continuation of the Fourier transform

3.2.1 Hardy's uncertainty principle

There are many ways to rephrase Hardy's uncertainty principle. The simplest way is the following.

Theorem 3.2.1. *Suppose that $|f(x)| \lesssim e^{-x^2/2}$ and $\left|\hat{f}(y)\right| \lesssim e^{-y^2/2}$. Then there exists a constant C such that $f(x) = Ce^{-x^2/2}$.*

The above theorem has two different proofs in Hardy's initial paper (see Hardy [11]), but the usual proof is the following.

Proof. Let $z = x + yi$. Then we have

$$\begin{aligned} \left|\hat{f}(z)\right| = \left|\hat{f}(x+yi)\right| &= \left|\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-xti+yt}dt\right| \\ &\le \frac{C_a}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-t^2/2}e^{yt}dt \\ &= C_a e^{y^2/2}. \end{aligned} \tag{3.2}$$

Thus $\hat{f}$ is an entire function on the whole complex plane and $\hat{f}(z)e^{z^2/2}$ is bounded on both the real and imaginary axes. We can check that the auxiliary function F_f^δ, given by

$$F_f^\delta(z) = \hat{f}(z)e^{z^2/2}e^{i\delta z^2/2}, \tag{3.3}$$

is bounded on the line $z = re^{\theta i}$ when $\theta = 0$. Indeed, when $r > 0$ and $\theta = \pi/2 -$

$\arctan\delta$, we have

$$
\begin{aligned}
\left|F_f^\delta(re^{\theta i})\right| &= \left|\hat{f}(re^{\theta i})e^{(re^{\theta i})^2/2}e^{i\delta(r\cos\theta+ir\sin\theta)^2/2}\right| \\
&\le C_a e^{r^2\sin^2\theta/2}e^{(-r^2\sin^2\theta+r^2\cos^2\theta)/2}e^{-\delta r^2\sin\theta\cos\theta} \\
&= C_a e^{r^2\cos^2\theta/2}e^{-\delta r^2\sin\theta\cos\theta} \\
&= C_a e^{(r^2\cos^2\theta)(1/2-\delta\tan\theta)} \\
&= C_a e^{-(r^2\cos^2\theta)/2} \\
&\le C_a.
\end{aligned}
$$

Also F_f^δ is of order two in the region

$$\left\{z \in \mathbb{C} : \operatorname{Arg}(z) \in \left[0, \frac{\pi}{2} - \arctan\delta\right]\right\}.$$

Thus by applying the Phragmen-Lindelöf principle to the auxiliary function F_f^δ, we know that it is bounded where $\operatorname{Arg}(z) \in \left[0, \frac{\pi}{2} - \arctan\delta\right]$. Because $\arctan\delta \to 0$ when $\delta \to 0$, by letting δ go to zero, we conclude that F_f^0 is bounded in the first quadrant. Using a similar technique on the other three quadrants of the complex plane, we can show that $F_f^0(z)$ is bounded on the whole complex plane, and thus must be constant. Thus $f = C\gamma_1$. □

Although this proof is simple, it is based on two important techniques. First, we find an analytic auxiliary function F_f^0 based on f and extended it to the entire complex plane. Second we find attributes of the auxiliary function F_f^0 based on the decay of f. In the above example we conclude that the decay of F_f^0 on the real axis is controlled by the decay of $\hat{f}$ by definition while the decay of F_f^0 on the imaginary axis is controlled by the decay of f.

Since we are using the most straightforward way to construct an analytic function A based on f by letting $A(z) = F_f^0(z)$ for all $z \in \mathbb{C}$, we only require $f(x)e^{\lambda x}$ to be integrable on $\mathbb{R}$ for all positive λ. Thus if we replace the condition of Theorem 3.2.1 by the following looser pair of conditions

$$
\begin{aligned}
|f(x)| &\lesssim e^{-x^2/2}\,|\psi(x)| \\
\left|\hat{f}(y)\right| &\lesssim e^{-y^2/2}\,|\psi(y)|,
\end{aligned}
\tag{3.4}
$$

and pick $\psi(x)$ carefully, we can still get an analytic function $F_f^0(z)$.

Remark: If we carefully check the definition of the Bargmann transform in Definition 1.0.6, we will find that the Bargmann transform of f is very similar to F_f^0. It is because that if a function f is not of gaussian decay, we can always multiply $f(x)$ by $e^{-x^2/2}$ to make a function of gaussian decay. Thus B_f can be treated as a normalized version of $F_f^0(fe^{-(\cdot)^2/2})$. This kind of relationship explains why they can both be used to estimate the coefficients of Hermite expansions of f in the following sections.

3.2.2 Estimation of Hermite expansions (first approach)

We will develop properties of F_f^0 and their applications in this section.

Lemma 3.2.2. *Suppose that $r > 0$, $\lambda > 0$ and n is a positive integer. Then*

$$\min_r \frac{e^{\lambda r}}{r^n} = \max \frac{r^n}{e^{\lambda r}} = \left(\frac{\lambda e}{n}\right)^n.$$

Proof. Notice that

$$\frac{d(e^{\lambda r}/r^n)}{dr} = \frac{\lambda e^{\lambda r} r^n - n r^{n-1} e^{\lambda r}}{r^{2n}} = \frac{(\lambda r - n)e^{\lambda n}}{r^{n+1}}.$$

Thus $\dfrac{d(e^{\lambda r}/r^n)}{dr} < 0$ when $r < n/\lambda$ and $\dfrac{d(e^{\lambda r}/r^n)}{dr} \geq 0$ when $r \geq n/\lambda$. So when $r = n/\lambda$, $\dfrac{e^{\lambda r}}{r^n}$ is minimal and is equal to $\left(\dfrac{\lambda e}{n}\right)^n$, as required. $\square$

Lemma 3.2.3. *Suppose that f satisfies the conditions* (3.4) *and $|\psi(x)| \lesssim e^{t|x|}$ where $0 \leq t \leq 1$. Then $\left|F_f^0(z)\right| \lesssim e^{\sqrt{2}t|z|}$.*

Proof. Let $u, v \in \mathbb{R}$, we have

$$\begin{aligned}
\left|\hat{f}(u+vi)\right| &= \left|\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix(u+iv)}\, dx\right| \\
&= \left|\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-uxi+vx}\, dx\right| \\
&\lesssim \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-x^2/2} e^{(vx+t|x|)}\, dx \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-x^2/2} e^{(|v||x|+t|x|)}\, dx \\
&\leq \frac{1}{\sqrt{2\pi}} e^{(|v|+t)^2/2} \int_{\mathbb{R}} e^{-(|x|+|v|+t)^2/2}\, dx \\
&\lesssim e^{v^2/2} e^{t|v|}.
\end{aligned} \tag{3.5}$$

It follows that $F_f^\delta(z)e^{-tz+itz}$, given by

$$F_f^\delta(z)e^{-tz+itz} = e^{-tz+itz}\hat{f}(z)e^{z^2/2}e^{i\delta z^2/2}, \tag{3.6}$$

satisfies the following: when $z = r \in \mathbb{R}$ and $r \geq 0$,

$$\left|F_f^\delta(z)e^{-tz+itz}\right| = e^{-tr}\left|\hat{f}(r)e^{r^2/2}\right| \leq C.$$

Also when $z = re^{\theta i}$ where $\theta = \pi/2 - \arctan\delta$ and $r \geq 0$

$$
\begin{aligned}
&\left|F_f^\delta(z)e^{-tz+itz}\right| \\
&= e^{-tr(\cos\theta+\sin\theta)}\left|\hat{f}(re^{\theta i})e^{(re^{\theta i})^2/2}e^{i\delta(e^{\theta i})^2/2}\right| \\
&\lesssim e^{-tr(\cos\theta+\sin\theta)}e^{r^2\sin^2\theta/2}e^{tr\sin\theta}e^{(-r^2\sin^2\theta+r^2\cos^2\theta)/2}e^{-\delta r^2\sin\theta\cos\theta} \\
&\lesssim e^{r^2\cos^2\theta/2}e^{-\delta r^2\sin\theta\cos\theta} \\
&= e^{(r^2\cos^2\theta)(1/2-\delta\tan\theta)} \\
&= e^{-(r^2\cos^2\theta)/2} \\
&\leq 1.
\end{aligned}
$$

Thus $F_f^\delta(z)e^{-tz+itz}$ is bounded on the half line $z = r$ where $r > 0$ and on the half line $z = re^{\theta i}$ where $\theta = \frac{\pi}{2} - \arctan\delta$ and $r > 0$. Also $F_f^\delta(z)e^{-tz+itz}$ is of order two in the region

$$\left\{z \in \mathbb{C} : \operatorname{Arg}(z) \in \left[0, \frac{\pi}{2} - \arctan\delta\right]\right\}.$$

Thus by applying the Phragmen-Lindelöf principle to the auxiliary function $F_f^\delta(z)e^{-tz+itz}$, we know that it is bounded where $\operatorname{Arg}(z) \in \left[0, \frac{\pi}{2} - \arctan\delta\right]$. Because $\arctan\delta \to 0$ when $\delta \to 0$, by letting δ go to zero, we conclude that $F_f^0(z)e^{-tz+itz}$ is bounded in the first quadrant. Hence

$$\left|F_f^0(z)\right| \leq C\left|e^{zt(1-i)}\right| \leq Ce^{t(x+y)}$$

where $z = x + iy$, that is, $F_f^0(z) \leq Ce^{\sqrt{2}t|z|}$. By using the same techniques on the other three quadrants of the complex plane, we get the result. □

A direct consequence of the above estimate is that when t approaches 0, the space of functions satisfying condition (3.4) approaches the one dimensional space $\{\mathbb{C}\gamma_1\}$. Moreover we can use Lemma 3.2.3 to estimate the growth of not only the function F_f^0 itself, but also its derivatives, by analyticity.

Lemma 3.2.4. *Suppose that g is an analytic function such that $|g(w)| \leq e^{\lambda|w|}$ for all $w \in \mathbb{C}$. Then*

$$|(D^n g)(w)| \leq n\lambda^n e^{\lambda|w|}. \tag{3.7}$$

Proof. Because g is analytic,

$$|(D^n g)(w)| = \frac{n!}{2\pi}\left|\int_\Gamma \frac{g(z)}{(z-w)^{n+1}}\,dz\right|, \tag{3.8}$$

where Γ is the circle of radius r around w. Because $|z| \leq |w| + r$ for y on this circle, we have the following estimate:

$$
\begin{aligned}
|(D^n g)(w)| &= \frac{n!}{2\pi}\left|\int_\Gamma \frac{g(z)}{(z-w)^{n+1}}\, dz\right| \\
&\le \min_r\left(\frac{n!}{2\pi}\frac{2\pi r e^{\lambda(|w|+r)}}{r^{n+1}}\right) \\
&\le \min_r\left(\frac{e^{\lambda r}}{r^n}\right) n! e^{\lambda|w|} \\
&= n!\left(\frac{\lambda e}{n}\right)^n e^{\lambda|w|},
\end{aligned}
\tag{3.9}
$$

where the last line follows from Lemma 3.2.2. Because

$$
\sum_{k=1}^n \log k \le \int_0^n \log(x+1)\, dx,
$$

it follows that

$$
\begin{aligned}
n! = \exp\left(\sum_{k=1}^n \log k\right) &\le \exp\left(\int_0^n \log(x+1)\, dx\right) \\
&= \exp\left((n+1)\log(n+1) - n\right) \\
&= e\left(\frac{n+1}{e}\right)^{n+1}.
\end{aligned}
\tag{3.10}
$$

Thus

$$
\left(\frac{e}{n+1}\right)^{n+1} \le \frac{e}{n!}. \tag{3.11}
$$

It follows that $\left(\frac{e}{n}\right)^n \le \frac{ne}{n!}$ and we can simplify the above inequality to get $|(D^n g)(w)| \le n\lambda^n e^{\lambda|w|}$. □

By combining Lemma 3.2.3 and Lemma 3.2.4, we get the following.

Lemma 3.2.5. *Suppose that f satisfies the conditions* (3.4) *and $|\psi(x)| \lesssim e^{t|x|}$. Then $\left|(D^n F_f^0)(z))\right| \lesssim n(\sqrt{2}t)^n e^{\sqrt{2}t|z|}$ for all $z \in \mathbb{C}$.*

A straightforward application of this estimate is that we can estimate inner products between $F_f^0(x)$ and $e^{x^2/2}D^n(e^{-x^2})$ in $L_2(\mathbb{R})$ by the following lemma.

Lemma 3.2.6. *Suppose that $\lambda > 0$ and f has an analytic extension to the whole complex plane such that*

$$
\left|f(w)e^{w^2/2}\right| \le e^{\lambda|w|} \tag{3.12}
$$

for all $w \in \mathbb{C}$. Then for all $n \in \mathbb{N}$,

$$
|\langle f, h_n\rangle| \le 2ne^{\lambda^2/4}\pi^{1/4}\frac{\lambda^n}{\sqrt{2^n n!}}, \tag{3.13}
$$

where

$$h_n(x) = \frac{\pi^{-1/4}}{\sqrt{2^n n!}} e^{x^2/2} D^n(e^{-x^2}) \tag{3.14}$$

is the Hermite function defined by Definition 2.2.1.

Proof. Using 3.2.3 and setting g in Lemma 3.2.4 to be $f\gamma_{-1}$, we have

$$D^n(f(x)e^{x^2/2}) \leq n\lambda^n e^{\lambda|x|}$$

on the real line. Thus

$$\begin{aligned} \left|\int_{\mathbb{R}} f(x)h_n(x)\,dx\right| &= \left|\frac{\pi^{-1/4}}{\sqrt{2^n n!}} \int_{\mathbb{R}} f(x)e^{x^2/2} D^n(e^{-x^2})\,dx\right| \\ &= \left|\frac{\pi^{-1/4}}{\sqrt{2^n n!}} \int_{\mathbb{R}} D^n(f(x)e^{x^2/2})e^{-x^2}\,dx\right| \\ &\leq \frac{\pi^{-1/4}}{\sqrt{2^n n!}} n\lambda^n \int_{\mathbb{R}} e^{-x^2} e^{\lambda|x|}\,dx \\ &\leq 2ne^{\lambda^2/4}\pi^{1/4} \frac{\lambda^n}{\sqrt{2^n n!}}, \end{aligned} \tag{3.15}$$

as required. □

The reason that we can integrate by parts is that $P(x)e^{\lambda|x|}/e^{x^2}$ always tends to zero as $x \to \pm\infty$. Thus we can now conclude the following theorem.

Theorem 3.2.7. *Suppose that f satisfies the conditions* (3.4) *and* $|\psi(x)| \lesssim e^{t|x|}$ *for all* $x \in \mathbb{R}$. *Then*

$$|\langle f, h_n\rangle| \lesssim \frac{nt^n}{\sqrt{n!}}, \tag{3.16}$$

for all $n \in \mathbb{N}$ *when* $0 < t \leq 1$.

Proof. By Lemma 3.2.6 and Lemma 3.2.3. □

Because the Hermite functions form an orthogonal basis in $L_2(\mathbb{R})$, the above estimate leads to a new proof of the following form of Hardy's uncertainty principle as follows.

Theorem 3.2.8. *Suppose that* $|f(x)| \lesssim (1+|x|)^N\gamma_1(x)$ *and* $\left|\hat{f}(y)\right| \lesssim (1+|y|)^N\gamma_1(y)$ *for all* $x, y \in \mathbb{R}$. *Then there exists a polynomial P of degree at most N such that* $f = P\gamma_1$.

Proof. By Lemma 3.2.2, $|x|^N \leq (N/te)^N e^{t|x|}$ for all $x \in \mathbb{R}$, when $0 \leq t \leq 1$. Thus by assumption,

$$|f(x)| \lesssim \left(\frac{N}{te}\right)^N e^{t|x|}\gamma_1(x) \tag{3.17}$$

and

$$\left|\hat{f}(y)\right| \leq \left(\frac{N}{te}\right)^N e^{t|y|}\gamma_1(y) \tag{3.18}$$

for all $x, y \in \mathbb{R}$. From Theorem 3.2.7 we can conclude that

$$|\langle f, h_n\rangle| \lesssim \left(\frac{N}{te}\right)^N \frac{nt^n}{\sqrt{n!}}. \tag{3.19}$$

It follows that $|\langle f, h_n\rangle| \lesssim nt^{n-N}/\sqrt{n!}$ for all $n > N$. Because this inequality holds for all $0 \leq t \leq 1$, by letting t approach zero, we see that $|\langle f, h_n\rangle| = 0$. So there exists a polynomial P of degree at most N such that

$$f(x) = \sum_{k=0}^{N} a_k h_k(x) = P(x)\gamma_1(x), \tag{3.20}$$

as required. □

So far we have shown that $|f(x)| \lesssim e^{-x^2/2}e^{t|x|}$ and $\left|\hat{f}(y)\right| \lesssim e^{-y^2/}e^{t|y|}$ for small t implies that

$$|\langle f, h_n\rangle| \lesssim \frac{nt^n}{\sqrt{n!}}.$$

Moreover we claim that if $|\langle f, h_n\rangle| \lesssim \frac{nt^n}{\sqrt{n!}}$, then

$$|f(x)| \lesssim (t\,|x| + 1)e^{-x^2/2}e^{\sqrt{2}t|x|}.$$

Theorem 3.2.9. *Suppose that f is in $L_2(\mathbb{R})$ and satisfies the estimate*

$$|\langle f, h_n\rangle| \lesssim \frac{nt^n}{\sqrt{n!}}$$

where $0 < t < 1$. Then

$$|f(x)| \lesssim (t\,|x| + 1)e^{-x^2/2}e^{\sqrt{2}t|x|}.$$

Proof. First we observe that, when $0 \leq \rho \leq 1$,

$$\sum_{n=0}^{\infty} \frac{n^2\rho^n}{n!} = \rho\frac{d}{d\rho}\rho\frac{d}{d\rho}\sum_{n=0}^{\infty}\frac{\rho^n}{n!} = \rho\frac{d}{d\rho}\rho\frac{d}{d\rho}e^{\rho} = (\rho^2 + \rho)e^{\rho}.$$

We also recall a particular case of Mehler's formula (see Thangavelu [18, p. 8]): when $0 \leq \rho < 1$,

$$\sum_{n=0}^{\infty}\left(\frac{\rho}{2}\right)^n \frac{|H_n(x)|^2}{n!} = \frac{1}{(1-\rho^2)^{1/2}}e^{2\rho x^2/(1+\rho)}.$$

By definition of h_n, we have

$$h_n(x) = \left(\frac{1}{\sqrt{\pi}2^n n!}\right)^{1/2} H_n(x)e^{-x^2/2}.$$

Now suppose that $|\langle f, h_n\rangle| \lesssim \frac{nt^n}{\sqrt{n!}}$. Then when $|x| \geq 1$ and $0 < t \leq 1$, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
\left|f(x)e^{-x^2/2}\right| &= \left|e^{x^2/2}\sum_{n=0}^{\infty}\langle f, h_n\rangle h_n(x)\right| \\
&\leq C\sum_{n=0}^{\infty} t^n \frac{n}{2^{n/2}n!}\left|H_n(x)\right| \\
&\leq C\left(\sum_{n=0}^{\infty}(\sqrt{2}t\left|x\right|)^n\frac{n^2}{n!}\right)^{1/2}\left(\sum_{n=0}^{\infty}\left(\frac{t}{2\sqrt{2}\left|x\right|}\right)^n\frac{\left|H_n(x)\right|^2}{n!}\right)^{1/2} \\
&= C\frac{(2t^2\left|x\right|^2+\sqrt{2}t\left|x\right|)^{1/2}}{(1-t^2/(2\left|x\right|^2))^{1/4}}\exp\left(\frac{t\left|x\right|}{\sqrt{2}}\right)\exp\left(\frac{t\left|x\right|}{\sqrt{2}(1+t/\left|x\right|)}\right) \\
&\lesssim (t\left|x\right|+1)e^{\sqrt{2}t|x|/2},
\end{aligned}$$

as required. □

3.3 Analytic continuation of the Bargmann transform

In this section we introduce another way to produce an analytic auxiliary function. Let

$$B_f(z) = \int_{\mathbb{R}} f(x)e^{x^2/2}e^{-(x-z/2)^2}\,dx, \tag{3.21}$$

where $z \in \mathbb{C}$. This auxiliary function B_f is called the Bargmann transform of f and can be defined by standard convolution: $B_f(z) = (f\gamma_{-1} * \gamma_2)(z/2)$.

Recall that in the previous sections we have discussed the auxiliary function $F_f^0(z) = \hat{f}(z)e^{z^2/2}$. Notice that

$$\begin{aligned}
B_f(\sqrt{2}w) &= \int_{\mathbb{R}} f(x)e^{-x^2/2}e^{\sqrt{2}xw-w^2/2}\,dx \\
&= \int_{\mathbb{R}} f(t/\sqrt{2})e^{-t^2/4}e^{tw-w^2/2}\,dx \\
&= \sqrt{\pi}F_g^0(-iw),
\end{aligned}$$

where $g(x) = f(x/\sqrt{2})e^{-x^2/4}$. Thus although the Bargmann transform is usually treated as a separate topic in the literature, it is related to F.

3.3.1 Estimation of Hermite expansions (second approach)

Using the Bargmann transform we can prove Theorem 3.2.7 in a different way. (Recall that Theorem 3.2.7 says that if $|f(x)| \lesssim \gamma_1(x)e^{t|x|}$ and $\left|\hat{f}(y)\right| \lesssim \gamma_1(y)e^{t|y|}$, then $\langle f, h_n\rangle \lesssim nt^n/\sqrt{n!}$.)

Let $g(x) = f(x/\sqrt{2})e^{-x^2/4}$. Because $|f(x)| \lesssim \gamma_1(x)e^{t|x|}$ and $\left|\hat{f}(y)\right| \lesssim \gamma_1(y)e^{t|y|}$, it follows that

$$\begin{aligned}
|g(x)| &= \left|f(x/\sqrt{2})e^{-x^2/4}\right| \\
&\lesssim e^{-x^2/2}e^{\sqrt{2}t|x|/2} \\
&\lesssim e^{-x^2/2}e^{t|x|}
\end{aligned}$$

and

$$\begin{aligned}
|\hat{g}(x)| &= \left|\left(\sqrt{2}\hat{f}(\sqrt{2}(\cdot)) * e^{-(\cdot)^2}\right)(x)\right| \\
&\leq \left(e^{-(\cdot)^2}e^{\sqrt{2}t|\cdot|} * e^{-(\cdot)^2}\right)(x) \\
&\leq 2\int_{\mathbb{R}^+} e^{-y^2}e^{\sqrt{2}ty}e^{-(x-y)^2}\,dy \\
&= 2e^{-x^2}\int_{\mathbb{R}^+} \exp(-2y^2+\sqrt{2}ty+2xy)\,dy \\
&= 2e^{-x^2}\int_{\mathbb{R}^+} \exp\left(-2\left(y-\frac{\sqrt{2}t+2x}{4}\right)^2+\frac{(\sqrt{2}t+2x)^2}{8}\right)dy \\
&= 2e^{t^2/4}e^{-x^2/2}e^{\sqrt{2}t|x|/2}\int_{\mathbb{R}^+} \exp\left(-2\left(y-\frac{\sqrt{2}t+2x}{4}\right)^2\right)dy \\
&\lesssim e^{-x^2/2}e^{t|x|}.
\end{aligned}$$

Thus, by Lemma 3.2.5, $\left|(D^nF_g^0)(z))\right| \lesssim n(\sqrt{2}t)^n e^{\sqrt{2}t|z|}$ for all $z \in \mathbb{C}$. It follows that

$$|(D^nB_f)(z))| = \left|(D^nF_g^0)\left(\frac{-zi}{\sqrt{2}}\right)\right| \lesssim 2^{-n/2}nt^ne^{t|z|}.$$

If $B_f(z) = \sum_n c_nz^n$, then $|c_n| \lesssim 2^{-n/2}nt^n/n!$. By using the fact that

$$\begin{aligned}
B_{h_n}(z) &= e^{-z^2/4}\left(\frac{1}{\sqrt{\pi}2^nn!}\right)^{1/2}(-1)^n\int_{\mathbb{R}}\frac{d^n(e^{-x^2})}{dx}e^{xz} \\
&= (-1)^n2^{-n/2}\frac{z^n}{\sqrt{n!}}
\end{aligned}$$

and $f = \sum_{n=0}^{\infty}\langle f, h_n\rangle h_n$, we have

$$B_f(z) = \sum_{n=0}^{\infty}\langle f, h_n\rangle(-1)^n2^{-n/2}\frac{z^n}{\sqrt{n!}}.$$

Thus

$$|\langle f, h_n\rangle| = \left|\frac{D^n(B_f)(0)}{D^n(B_{h_n})(0)}\right| = \frac{n!\,|c_n|}{2^{-n/2}\sqrt{n!}} \lesssim \frac{nt^n}{\sqrt{n!}}, \tag{3.22}$$

when $0 \le t < 1$.

Remark: A similar theorem can be found in Garg and Thangavelu [8] where they used Bargmann transform as a tool to estimate the coefficients of Hermite expansions of f when f and $\hat{f}$ are both of gaussian decay. Also in [9], they proved that if the following Beurling style inequality holds:

$$\iint_{\mathbb{R}^2} \left|f(x)\,\hat{f}(y)\right| e^{\lambda|xy|}\,dx\,dy < \infty, \tag{3.23}$$

then $\langle f, h_n\rangle$ is of exponential decay with a rate that depends on λ.

3.3.2 Application of the Bargmann transform

In the above proof, we used the estimate of the derivative of B_f to estimate $\langle f, h_n\rangle$ while, in the proof of Theorem 3.2.7, we use the estimate of derivative of $f(x)\gamma_1(x)$ at $x = 0$. Both approaches lead to the same result. Because B_f can be treated as a normalized version of F_f^0, we claim that we can use both approaches to prove the following uncertainty principle regarding functions in $\mathcal{S}'$.

Theorem 3.3.1 (Bonami et al. [3]). *Suppose that $e^{x^2/2}f(x) \in \mathcal{S}'$ and that $e^{y^2/2}\hat{f}(y) \in \mathcal{S}'$. Then $f(x) = P(x)\gamma_1(x)$ for some polynomial P.*

Proof. We will prove this by estimating the derivative of the Bargmann transform of f:

$$B_f(z) = \langle f e^{|\cdot|^2/2}, e^{-(\cdot - z/2)^2}\rangle, \tag{3.24}$$

where $z \in \mathbb{C}$. Because $e^{x^2/2}f(x)$ and $e^{y^2/2}\hat{f}(y)$ are tempered distributions, there exists an integer N such that

$$\begin{aligned} B_f(z) &= \langle f e^{|\cdot|^2/2}, e^{-(\cdot - z/2)^2}\rangle \\ &\le C \max_{\alpha+\beta\le N} \left|x^\alpha \partial^\beta e^{-(x-z/2)^2}\right| \\ &\le C_N\,|1+z|^N \\ &\lesssim \left(\frac{1}{te}\right)^N e^{t|z|} \end{aligned} \tag{3.25}$$

for all $t \in [0,1]$. If $B_f(z) = \sum_n c_n z^n$, then we have

$$|c_n| \lesssim e^{tr} r^{-n} \left(\frac{1}{te}\right)^N. \tag{3.26}$$

By computing the maximum value of the right hand side of the above estimate we get

$$|c_n| \lesssim \left(\frac{te}{n}\right)^n \left(\frac{1}{te}\right)^N. \tag{3.27}$$

Therefore for any $0 \leq t < 1$ we have

$$\begin{aligned} |\langle f, h_n \rangle| &= \sqrt{n!}\,|c_n| \\ &\lesssim \sqrt{n!} \left(\frac{te}{n}\right)^n \left(\frac{1}{te}\right)^N \\ &\lesssim \frac{t^n}{\sqrt{n!}} \left(\frac{1}{te}\right)^N. \end{aligned} \tag{3.28}$$

Because the above estimate holds for all $t \in [0, 1]$, we can conclude that $\langle f, h_n \rangle = 0$ when $n > N$ by letting t go to zero. Thus $f(x) = P(x)\gamma_1(x)$. $\square$

Chapter 4

Beurling's uncertainty principle and its generalization

4.1 Introduction

Beurling's version of the uncertainty principle states that if

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left| f(x)\hat{f}(y) \right| e^{|xy|} \, dx \, dy < \infty, \tag{4.1}$$

then $f = 0$. This was generalized by Bonami et al. [3], who proved that if

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \frac{f(x)\hat{f}(y)}{(1 + |x| + |y|)^N} \right| e^{|\langle x,y \rangle|} \, dx \, dy < \infty, \tag{4.2}$$

then $f(x) = P(x)e^{-\langle Ax,x \rangle}$ where $P(x)$ is a polynomial and A is a positive definite matrix.

Hedenmalm extended Beurling's result in a different way.

Definition 4.1.1 (Hedenmalm auxiliary function). *Given function $f \in L_2(\mathbb{R})$, define the Hedenmalm auxiliary function F_f by*

$$F_f(\lambda) = \int_{\mathbb{R}} \bar{f}(x) f(\lambda x) \, dx. \tag{4.3}$$

Using this auxiliary function, Hedenmalm showed that $F(\lambda) = c_0(1 + \lambda^2)$ for some constant c_0 if f satisfies (4.1). Thus f is an even function and

$$\left| \mathcal{M}_f^0(z) \right| = C \left| \Gamma \left(\frac{1}{4} + \frac{z}{2} \right) \right|, \tag{4.4}$$

where $z \in i\mathbb{R}$ and $\mathcal{M}_f^0$ is the Mellin transform of the even part of f [12].

Although $f(x) = Ce^{-ax^2}$ is a solution to equation (4.4) when $C \in \mathbb{C}$ and $a > 0$, it is not clear whether this is the only form of solution for equation (4.4). In this chapter I will show that if $f \in L_1(\mathbb{R})$, $\hat{f} \in L_1(\mathbb{R})$, and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left| f(x)\hat{f}(y) \right| e^{\lambda|xy|}\, dx\, dy \leq \frac{C}{(1-\lambda)^{N+1/2}} \tag{4.5}$$

for all $\lambda \in [0,1)$, then there exists $a \in \mathbb{R}^+$ and a polynomial P of degree at most N such that $f = P\gamma_a$. Also in Chapter 6, I will show that a similar theorem also holds in $\mathbb{R}^n$.

Remarks: Some original ideas of this chapter are from Hedenmalm[12]. I revisited and translated them into operational statements to obtain new estimates whose interest is obvious.

4.2 The Θ transform

Recall that we define the Mellin transform of a function f on $\mathbb{R}$ by

$$\mathcal{M}_f^k(z) = \int_{\mathbb{R}} f(x)\, \mathrm{sgn}(x)^k\, |x|^{-1/2+z}\, dx \tag{4.6}$$

for all $z \in \mathbb{C}$ where the right hand side is defined. It is obvious that given a function f, the Mellin transform $\mathcal{M}_f^k$ might not be defined on the whole complex plane. However if f is bounded and for all $N > 0$, there exists C_N such that $|f(x)| \leq C_N x^{-N}$ when $x \to \infty$, and then $\mathcal{M}_f^k$ will be defined on the half plane $\mathrm{Re}(z) > -1/2$.

However, even if $\mathcal{M}_f^k$ is defined on the half plane $\mathrm{Re}(z) > -1/2$, it does not have an analytic extension to the whole complex plane unless $|f(x)| < C_N x^N$ when $|x| \to 0$ for all N. The condition $|f(x)| < C_N x^N$ when $|x| \to 0$ is too strong to be useful. It can be shown that if f is C^∞ near 0, then $\mathcal{M}_f^0$ has a meromorphic extension to $\mathbb{C}$ with possible poles at $-1/2 - 2k$ where k is a non-negative integer, and $\mathcal{M}_f^1$ has a meromorphic extension to $\mathbb{C}$ with possible poles at $-3/2 - 2k$ where k is a non-negative integer.

Thus we introduce a transform Θ_f as follows.

Definition 4.2.1. *Given $f \in L_1(\mathbb{R})$, define*

$$\Theta_f^k(z) = \frac{\mathcal{M}_f^k(z)}{\Gamma\left(\frac{1}{4} + \frac{z}{2} + \frac{k}{2}\right)} \tag{4.7}$$

for all $z \in \mathbb{C}$ such that $\mathcal{M}_f(z)$ is defined.

Because Γ has no zeros in the whole complex plane, Θ_f is defined wherever $\mathcal{M}_f$ is defined. In particular, if $\mathcal{M}_f(z)$ is defined where $\mathrm{Re}(z) > -1/2$, $\Theta_f(z)$ is also defined when $\mathrm{Re}(z) > -1/2$.

So far we still can not extend $\Theta_f(z)$ to the points where $\mathcal{M}_f(z)$ is not defined. But we can hope that the singularities of $\mathcal{M}_f(z)$ at $\mathrm{Re}(z) < -1/2$ are canceled by the singularities of the Γ function. More precisely, we will show that $\Theta_f(-z) = \Theta_{\hat{f}}(z)$, when $\mathcal{M}_f(z)$ and $\mathcal{M}_{\hat{f}}(z)$ are both defined, that is, when $|\mathrm{Re}(z)| < 1/2$. Thus we can extend $\Theta_f(z)$ to $\mathrm{Re}(z) < 0$ by extending $\Theta_{\hat{f}}(z)$ to $\mathrm{Re}(z) > 0$, which only requires that $\hat{f}(x)$ has good decay at $x \to \infty$.

Lemma 4.2.2. *When* $\mathrm{Re}(z) \in (-1/2, 1/2)$,

$$\left(|\cdot|^{-1/2+z}\right)^{\wedge} = \frac{\Gamma(1/4 + z/2)}{\Gamma(1/4 - z/2)} |\cdot|^{-1/2-z}$$

and

$$\left(\mathrm{sgn}(\cdot)\,|\cdot|^{-1/2+z}\right)^{\wedge} = \frac{\Gamma(3/4 + z/2)}{\Gamma(3/4 - z/2)} \mathrm{sgn}(\cdot)\,|\cdot|^{-1/2-z}.$$

Proof. See Theorem 4.1 in Stein and Weiss [16]. ◻

Lemma 4.2.3. *Suppose that f and $\hat{f}$ are both integrable. Then $\Theta^k_f(-z) = \Theta^k_{\hat{f}}(z)$ when* $|\mathrm{Re}(z)| < 1/2$.

Proof. Because f and $\hat{f}$ are both integrable, $\hat{f}$ and f are both bounded. Thus $\mathcal{M}^0_f(z)$, $\mathcal{M}^1_f(z)$, $\mathcal{M}^0_{\hat{f}}(z)$ and $\mathcal{M}^1_{\hat{f}}(z)$ are defined when $-1/2 < \mathrm{Re}(z) < 1/2$. By direct computation we have

$$\begin{aligned}
\Theta^0_{\hat{f}}(z) &= \int_{\mathbb{R}} \hat{f}(x)\,|x|^{-1/2+z}\,\Gamma(1/4+z/2)^{-1}\,dx \\
&= \frac{1}{\Gamma(1/4+z/2)} \int_{\mathbb{R}} f(x)\Gamma(1/4+z/2)\Gamma(1/4-z/2)^{-1}\,|x|^{-1/2-z}\,dx \\
&= \frac{1}{\Gamma(1/4-z/2)} \int_{\mathbb{R}} f(x)\,|x|^{-1/2-z}\,dx \\
&= \Theta^0_f(-z),
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
\Theta^1_{\hat{f}}(z) &= \int_{\mathbb{R}} \hat{f}(x)\,\mathrm{sgn}(x)\,|x|^{-1/2+z}\,\Gamma(1/4+z/2)^{-1}\,dx \\
&= \frac{1}{\Gamma(3/4+z/2)} \int_{\mathbb{R}} \frac{\Gamma(3/4+z/2)}{\Gamma(3/4-z/2)} f(x)\,\mathrm{sgn}(x)\,|x|^{-1/2-z}\,dx \\
&= \frac{1}{\Gamma(3/4-z/2)} \int_{\mathbb{R}} f(x)\,\mathrm{sgn}(x)\,|x|^{-1/2-z}\,dx \\
&= \Theta^1_f(-z),
\end{aligned} \tag{4.9}$$

as claimed. ◻

Corollary 4.2.4. *Suppose that $f(\cdot)e^{\alpha|\cdot|} \in L_1(\mathbb{R})$ and $\hat{f}(\cdot)e^{\beta|\cdot|} \in L_1(\mathbb{R})$ for some $\alpha > 0$ and $\beta > 0$. Then $\Theta_f(z)$, initially defined on* $\mathrm{Re}(z) > -1/2$*, has an analytic* *continuation to the whole complex plane. Also we have*

$$\Theta_f(-z) = \Theta_{\hat{f}}(z) \tag{4.10}$$

for all $z \in \mathbb{C}$.

Proof. Since $\Theta_f(z)$ is defined on $\mathrm{Re}(z) > -1/2$, by Lemma 4.2.2 we know that when $|\mathrm{Re}(z)| < 1/2$,

$$\Theta_f(-z) = \Theta_{\hat{f}}(z). \tag{4.11}$$

Because the right hand side of the above equation is defined for all $\mathrm{Re}(z) > 0$, $\Theta_f(z)$ can be extended analytically to the whole half plane $\mathrm{Re}(z) < 0$. □

4.3 Estimate of the growth of the Θ transform

Suppose that f and $\hat{f}$ are bounded and $f(\cdot)e^{\alpha|\cdot|} \in L_1(\mathbb{R})$ and $\hat{f}(\cdot)e^{\beta|\cdot|} \in L_1(\mathbb{R})$ for some $\alpha > 0$ and $\beta > 0$. We have shown that $\Theta_f(s)$ is an entire function. Now we will show that $\Theta_f(s)$ is an analytic function of order at most 1.

Lemma 4.3.1. *If f is a bounded function and $f(\cdot)e^{\alpha|\cdot|} \in L_1(\mathbb{R})$ for some $\alpha > 0$, then $\Theta_f(z)$ is defined for all z with $\mathrm{Re}(z) > -1/2$, and there is a constant A, which depends on α, such that when $\mathrm{Re}(z) \geq 0$*

$$|\Theta_f(z)| \lesssim e^{A|z|\log(1+|z|)}. \tag{4.12}$$

Proof. From the definition, we know that $\mathcal{M}_f$ is defined when $\mathrm{Re}(z) \geq 0$. When $|z|$ is sufficiently large and $\mathrm{Re}(z) \geq 1$ we have

$$\begin{aligned} |\mathcal{M}_f(z)| &\lesssim \left(\int_{\mathbb{R}} |f(x)|^2 e^{\alpha|x|}\,dx\right)^{1/2} \left(\int_{\mathbb{R}} e^{-\alpha|x|}\,|x|^{-1+2\,\mathrm{Re}(z)}\,dx\right)^{1/2} \\ &\lesssim \|f\|_\infty^{1/2} \left\|fe^{\alpha|\cdot|}\right\|_1^{1/2} \left(\int_{\mathbb{R}+} e^{-\alpha x} x^{-1+2\,\mathrm{Re}(z)}\,dx\right)^{1/2} \\ &\lesssim \alpha^{-\mathrm{Re}(z)}\Gamma(2\,\mathrm{Re}(z))^{1/2} \\ &\lesssim \left(\frac{2\,|z|}{\alpha e}\right)^{\mathrm{Re}(z)}, \end{aligned} \tag{4.13}$$

where the last inequality follows from the Stirling's approximation. Thus

$$|\Theta_f(z)| \leq \left|\frac{\mathcal{M}_f(z)}{\Gamma(1/4 + z/2)}\right| \lesssim e^{A|z|\log(1+|z|)}, \tag{4.14}$$

where A is a constant that depends on α. When $0 \le \mathrm{Re}(z) \le 1$ we have

$$\begin{aligned} |\mathcal{M}_f(z)| &\lesssim \int_{\mathbb{R}+} e^{-\alpha|x|} x^{-1/2+\mathrm{Re}(z)}\, dx \\ &\lesssim \int_0^1 e^{-\alpha|x|} x^{-1/2}\, dx + \int_1^\infty e^{-\alpha|x|} x^{1/2}\, dx \\ &< \infty. \end{aligned} \tag{4.15}$$

Thus there exists a constant A such that

$$|\Theta_f(z)| \lesssim e^{A|z|\log(1+|z|)}, \tag{4.16}$$

when $\mathrm{Re}(z) \ge 0$, as claimed. □

Lemma 4.3.2. *If $\hat{f}$ is a bounded function and $\hat{f}(\cdot)e^{\beta|\cdot|} \in L_1(\mathbb{R})$ for some $\beta > 0$, then $\Theta_f(z)$ is defined for all z with $\mathrm{Re}(z) \le 0$, and there is a constant A which depends on β such that*

$$|\Theta_f(z)| \lesssim e^{A|z|\log(1+|z|)}, \tag{4.17}$$

for all z when $\mathrm{Re}(z) \le 0$.

Proof. This result is a combination of Lemma 4.3.1 and Lemma 4.2.3. □

Now we can conclude the following theorem.

Theorem 4.3.3. *Suppose that f is a bounded function and $f(\cdot)e^{\alpha|\cdot|} \in L_1(\mathbb{R})$ and $\hat{f}(\cdot)e^{\beta|\cdot|} \in L_1(\mathbb{R})$ for some $\alpha > 0$ and $\beta > 0$. Then $\Theta_f(z)$, initially defined on $\mathrm{Re}(z) > -1/2$, extends analytically to the whole complex plane and is of order 1. Moreover if $\Theta_f(z)$ has finitely many zeros, then the degree of the canonical Weierstrass form of $\Theta_f(z)$ is finite, thus it must be of the form $P(z)e^{az}$ where P is a polynomial.*

Proof. This is a result that follows from Lemma 4.2.4, Lemma 4.3.1 and Lemma 4.3.2, and the fact that the degree of the canonical Weierstrass form of an entire function with finite order is an integer if it has finite many zeros (see Conway [4]). □

4.4 A generalized Beurling style theorem

In this section, we use Hedenmalm's ideas but define a slightly different auxiliary function F_f based on $f \in L_2(\mathbb{R})$ as follows:

$$F_f(\lambda) = \int_{\mathbb{R}} f(x)f(\lambda x)\, dx \tag{4.18}$$

for all $\lambda \in \mathbb{R}$.

Lemma 4.4.1. *Suppose that f is a bounded and integrable function, $C > 0$, $N > 0$ and*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(x)\hat{f}(y)\right| e^{\lambda|xy|}\,dx\,dy \le \frac{C}{(1-\lambda)^{N+1/2}}, \tag{4.19}$$

when $0 \le \lambda < 1$. Then $\Theta_f^k(z)$ is entire and can have at most $(N-k)/2$ zeros.

Proof. Suppose that f satisfies the condition (4.19). Then for $\lambda = 1/2$, the following holds:

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(x)\hat{f}(y)\right| e^{|xy|/2}\,dx\,dy < \infty.$$

Thus

$$\int_{\mathbb{R}} \left|f(x)\hat{f}(y)\right| e^{|xy|/2}\,dx < \infty$$

for almost all $y \in \mathbb{R}$. Hence there exists $\alpha > 0$ such that

$$\int_{\mathbb{R}} \left|f(x)e^{-\alpha|x|}\right|\,dx < \infty. \tag{4.20}$$

Similarly there exists $\beta > 0$ such that

$$\int_{\mathbb{R}} \left|\hat{f}(y)e^{-\beta|y|}\right|\,dy < \infty. \tag{4.21}$$

Now we are going to show that there exist constants $C_k \in \mathbb{C}$ and $D_k \in \mathbb{C}$ such that for all $\lambda \in \mathbb{R}$

$$F(\lambda) = \int_{\mathbb{R}} f(x)f(\lambda x)\,dx = \sum_{k=0}^{N} C_k(1+\lambda^2)^{-k-1/2} + \sum_{k=1}^{N} D_k\lambda(1+\lambda^2)^{-k-1/2}. \tag{4.22}$$

This is a generalized version of the result of Hedenmalm [12]. To start, we define $G(z)$ where $z \in \mathbb{R}$ as follows:

$$G(z) = \sqrt{1+z^2}\int_{\mathbb{R}} f(x)f(zx)\,dx. \tag{4.23}$$

Now $G(z)$ extends analytically to the strip $\{z \in \mathbb{C} : |\mathrm{Im}(z)| < 1\}$, because

$$G(z) = \sqrt{1+z^2}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)\hat{f}(y)e^{izxy}\,dy\,dx,$$

and

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(x)\hat{f}(y)e^{izxy}\right|\,dy\,dx \\
&\lesssim \int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(x)\hat{f}(y)\right| e^{|\mathrm{Im}(z)xy|}\,dy\,dx \\
&\lesssim \frac{1}{(1-|\mathrm{Im}(z)|)^{N+1/2}}.
\end{aligned} \tag{4.24}$$

Secondly we can verify from the definition (4.23) that

$$G(z) = G\left(\frac{1}{z}\right), \tag{4.25}$$

initially for $z \in \mathbb{R} \setminus \{0\}$ and then by analytic continuation wherever both sides are defined. Because the strip $\{z \in \mathbb{C} : |\operatorname{Im}(z)| < 1\}$ contains all points of the closed unit ball except $\pm i$, G will continue analytically to the whole complex plane with possible singularities at $\pm i$, and is bounded at infinity.

Third, we show that G is meromorphic by showing that $G(z)(1+z^2)^{2N}$ stays bounded when z tends to $\pm i$. When z tends to $\pm i$ inside the closed unit ball,

$$\begin{aligned}\limsup_{\substack{z\to\pm i\\|z|\le 1}} \left|G(z)(1+z^2)^{2N}\right| &\lesssim \limsup_{\substack{z\to\pm i\\|z|\le 1}} \left|\frac{(1+z^2)^{2N}}{(1-|\operatorname{Im}(z)|)^N}\right| \\ &\le \sup_{|z|\le 1}\left(\frac{|1+z^2|^2}{1-|\operatorname{Im}(z)|}\right)^N.\end{aligned} \tag{4.26}$$

So it suffices to show that $\dfrac{|1+z^2|^2}{1-|\operatorname{Im}(z)|}$ is bounded in the unit ball. Write $z = u+iv$. Then $|u| \le \sqrt{1-v^2}$ when $|z| \le 1$, and

$$\begin{aligned}\frac{|1+z^2|^2}{1-|\operatorname{Im}(z)|} &\le \frac{|z-i|^2\,|z+i|^2}{1-|\operatorname{Im}(z)|} \\ &\le \frac{4\,|u|^2 + 4\,|1-|v||^2}{1-|v|} \\ &\le 8.\end{aligned} \tag{4.27}$$

Thus

$$\sup_{|z|\le 1}\left|G(z)(1+z^2)^{2N}\right| < \infty. \tag{4.28}$$

When z goes to $\pm i$ from outside the unit ball, by setting $w = 1/z$, we see that

$$\limsup_{\substack{z\to\pm i\\|z|\ge 1}} \left|G(z)(1+z^2)^{2N}\right| = \limsup_{\substack{w\to\mp i\\|w|\le 1}} \left|\frac{G(w)(1+w^2)^{2N}}{|w|^{2N}}\right| < \infty.$$

Thus $G(z)$ is meromorphic and we can conclude from (4.24) that the degrees of the poles of $G(z)$ at $\pm i$ are at most N. Thus

$$\int_{\mathbb{R}} f(x)f(zx)\,dx = \frac{G(z)}{\sqrt{1+z^2}} = \sum_{k=0}^{N} C_k(1+z^2)^{-k-1/2} + \sum_{k=1}^{N} D_k z(1+z^2)^{-k-1/2}. \tag{4.29}$$

Notice that by (2.34) the Mellin transform $\mathcal{M}_t^0$ of the term

$$t(\lambda) = \frac{1}{(1+\lambda^2)^{1/2+k}} \tag{4.30}$$

satisfies

$$\begin{aligned} \mathcal{M}_t^0(z) &= C_k \Gamma\left(\frac{1}{4} - \frac{z}{2} + k\right) \Gamma\left(\frac{1}{4} + \frac{z}{2}\right) \\ &= P^0(z) \Gamma\left(\frac{1}{4} - \frac{z}{2}\right) \Gamma\left(\frac{1}{4} + \frac{z}{2}\right), \end{aligned} \tag{4.31}$$

where $P^0(z)$ is a polynomial of degree at most k. It follows that

$$\mathcal{M}_F^0(z) = Q^0(z) \Gamma\left(\frac{1}{4} - \frac{z}{2}\right) \Gamma\left(\frac{1}{4} + \frac{z}{2}\right),$$

where $Q^0(z)$ is a polynomial of degree at most N. Because

$$\mathcal{M}_F^0(z) = \mathcal{M}_f^0(z) \mathcal{M}_f^0(-z) \qquad \forall z \in \mathbb{C},$$

it follows that

$$\Theta_f^0(z) \Theta_f^0(-z) = Q^0(z) \qquad \forall z \in \mathbb{C}. \tag{4.32}$$

Because $\Theta_f^0(z)$ is entire by Corollary 4.2.4, $\Theta_f^0(z)$ has no poles. Thus every zero of $Q^0(z)$ is a zero of $\Theta_f^0(z)$ or $\Theta_f^0(-z)$. It follows that $\Theta_f^0(z)$ can have at most $N/2$ zeros. To show that the result also holds for Θ_f^1 we need to check the Mellin transform $\mathcal{M}_t^1$ of the following term

$$t(\lambda) = \frac{\lambda}{(1+\lambda^2)^{1/2+k}}, \tag{4.33}$$

where $k > 0$. By (2.35), its Mellin transform $\mathcal{M}_t^1$ satisfies

$$\begin{aligned} \mathcal{M}_t^1(z) &= C_k \Gamma\left(-\frac{1}{4} - \frac{z}{2} + k\right) \Gamma\left(\frac{3}{4} + \frac{z}{2}\right) \\ &= P^1(z) \Gamma\left(\frac{3}{4} - \frac{z}{2}\right) \Gamma\left(\frac{3}{4} + \frac{z}{2}\right), \end{aligned} \tag{4.34}$$

where $P^1(z)$ is a polynomial of degree at most $k-1$. It follows that

$$\mathcal{M}_F^1(z) = Q^1(z) \Gamma\left(\frac{3}{4} - \frac{z}{2}\right) \Gamma\left(\frac{3}{4} + \frac{z}{2}\right),$$

where $Q^1(z)$ is a polynomial of degree at most $N-1$. Because $\mathcal{M}_F^1(z) = \mathcal{M}_f^1(z)\mathcal{M}_f^1(-z)$, it follows that

$$\Theta_f^1(z) \Theta_f^1(-z) = Q^1(z). \tag{4.35}$$

Because $\Theta_f^1(z)$ is entire by Corollary 4.2.4, $\Theta_f^1(z)$ has no poles. Thus every zero of $Q^1(z)$ is a zero of $\Theta_f^1(z)$ or $\Theta_f^1(-z)$. It follows that $\Theta_f^1(z)$ can have at most $(N-1)/2$ zeros. $\square$

Lemma 4.4.2. *Suppose that f is a bounded real function on $\mathbb{R}$ and*

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|f(x)\hat{f}(y)\right| e^{\lambda|xy|}\,dx\,dy \leq \frac{C}{(1-\lambda)^{N+1/2}} \tag{4.36}$$

for all $\lambda \in [0,1)$. Then there exist a,b in $\mathbb{R}^+$ and two polynomials G and H of degree at most N such that $f = G\gamma_a + H\gamma_b$. In particular H is even and G is odd.

Proof. From Lemma 4.4.1 we know that $\Theta_f^k(z)$ can have at most $(N-k)/2$ zeros. From Lemma 4.3.1 and Lemma 4.3.2 we know that $\Theta_f(z)$ is of order 1. Thus there exist $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$ and polynomials $P(z)$, $Q(z)$ such that

$$\Theta_f^0(z) = P(z)e^{\alpha z} \text{ and } \Theta_f^1(z) = Q(z)e^{\beta z}, \tag{4.37}$$

where $P(z)$ is of degree at most $N/2$ and and $Q(z)$ is of degree at most $(N-1)/2$. Notice that when $\mathrm{Re}(z) = 0$ and f is a real function, by definition of the Θ transform we have

$$\Theta_f^0(z) = \overline{\Theta_f^0(-z)} \text{ and } \Theta_f^1(z) = \overline{\Theta_f^1(-z)}. \tag{4.38}$$

It follows that

$$P(i)e^{\alpha i} = \overline{P(-i)e^{-\alpha i}} \text{ and } Q(i)e^{\beta i} = \overline{Q(-i)e^{-\beta i}}. \tag{4.39}$$

Thus α and β are real. It follows that there exist $\{C_k\}$ and $\{D_k\}$ such that

$$\begin{aligned}
\Theta_f^0(z) &= \left(\sum_{k=0}^{N/2} C_k \frac{\partial^k}{\partial \alpha^k}\right) e^{\alpha z} \\
\Theta_f^1(z) &= \left(\sum_{k=0}^{(N-1)/2} D_k \frac{\partial^k}{\partial \beta^k}\right) e^{\beta z}
\end{aligned} \tag{4.40}$$

and

$$\begin{aligned}
\mathcal{M}_f^0(z) &= \left(\sum_{k=0}^{N/2} C_k \frac{\partial^k}{\partial \alpha^k}\right) (\Gamma(1/4 + z/2)e^{\alpha z}) \\
\mathcal{M}_f^1(z) &= \left(\sum_{k=0}^{(N-1)/2} D_k \frac{\partial^k}{\partial \beta^k}\right) \left(\Gamma(3/4 + z/2)e^{\beta z}\right).
\end{aligned} \tag{4.41}$$

Thus by Lemma 2.5.4

$$
\begin{aligned}
\frac{f(x)+f(-x)}{2} &= \left(\sum_{k=0}^{N/2} C_k \frac{\partial^k}{\partial \alpha^k}\right)\left(\frac{1}{2e^{\alpha/2}} e^{-(e^{\alpha}x)^2}\right) \\
\frac{f(x)-f(-x)}{2} &= \left(\sum_{k=0}^{(N-1)/2} D_k \frac{\partial^k}{\partial \beta^k}\right)\left(\frac{x}{2e^{\beta/2}} e^{-(e^{\beta}x)^2}\right).
\end{aligned}
\tag{4.42}
$$

So we can conclude that there exist $a = 2e^{\alpha} > 0$, $b = 2e^{\beta} > 0$ and polynomials $H(x)$, $G(x)$ of degree at most N such that

$$
\frac{f(x)+f(-x)}{2} = H(x)e^{-ax^2/2} \tag{4.43}
$$

and

$$
\frac{f(x)-f(-x)}{2} = G(x)e^{-bx^2/2}. \tag{4.44}
$$

So $f = G\gamma_a + H\gamma_b$. $\square$

Lemma 4.4.3. *Fix $n \in \mathbb{N}$. If $f = g\gamma_a + h\gamma_b$, g and h are polynomials, and*

$$
\int_{\mathbb{R}}\int_{\mathbb{R}} \left| f(x)\hat{f}(y) \right| y^n e^{\lambda|xy|}\, dx\, dy \le \frac{C}{(1-\lambda)^{N+1/2}}, \tag{4.45}
$$

for all $\lambda \in [0,1)$, then $a = b$.

Proof. Assume that $a \le b$, then $\hat{f} = \tilde{G}\,\gamma_{1/a} + \tilde{H}\,\gamma_{1/b}$, where $\tilde{G}$ and $\tilde{H}$ are also polynomials and $0 < 1/b \le 1/a$. Therefore there exists a positive constant R such that $\left|f(x)\right| \ge c\,\gamma_a$ when $x \ge R$ and $\left|\hat{f}(y)\right| \ge d\,\gamma_{1/b}$ when $y \ge R$. Further, the first quadrant is the disjoint union of three sets; the first where $0 \le x < R$ and $y \ge 0$; the second where $x \ge R$ and $0 \le y < R$, and the third where $x \ge R$ and $y \ge R$. When $0 \le \lambda \le 1$, it is clear that

$$
\int_0^{\infty}\int_0^{R} \gamma_a(x)\,\gamma_{1/b}(y) y^n\, e^{\lambda xy}\, dx\, dy \lesssim R\int_0^{\infty} \gamma_{1/b}(y) y^n\, e^{Ry}\, dy < \infty
$$

and

$$
\int_0^{R}\int_R^{\infty} \gamma_a(x)\,\gamma_{1/b}(y) y^n\, e^{\lambda xy}\, dx\, dy \le R^{n+1}\int_R^{\infty} \gamma_a(x)\, e^{Rx}\, dx < \infty,
$$

and so

$$
\int_R^{\infty}\int_R^{\infty} \gamma_a(x)\,\gamma_{1/b}(y) y^n\, e^{\lambda xy}\, dx\, dy < \infty
$$

if and only if

$$
\int_0^{\infty}\int_0^{\infty} \gamma_a(x)\,\gamma_{1/b}(y) y^n\, e^{\lambda xy}\, dx\, dy < \infty.
$$

Moreover,

$$\int_0^\infty \int_0^\infty \gamma_a(x)\,\gamma_{1/b}(y) y^n\, e^{\lambda xy}\, dx\, dy = a^n \int_0^\infty \int_0^\infty \gamma_c(x)\gamma_c(y) y^n\, e^{\lambda xy}\, dx\, dy$$

where $c = (a/b)^{1/2}$; this integral is finite for all $\lambda \in [0, 1)$ if and only if $c \geq 1$. So (4.45) implies that $a = b$. $\square$

Now we can conclude the following final result of this chapter.

Theorem 4.4.4. *If $f \in L_1(\mathbb{R})$, $\hat{f} \in L_1(\mathbb{R})$ and*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left| f(x)\hat{f}(y) \right| e^{\lambda|xy|}\, dx\, dy \leq \frac{C}{(1-\lambda)^{N+1/2}}, \tag{4.46}$$

when $0 \leq \lambda < 1$, then there exist a in $\mathbb{R}^+$ and a polynomial P of degree at most N such that $f = P\gamma_a$.

Proof. We denote by f_r and f_i the real and imaginary parts of f. It is easy to check that f_r and f_i both satisfy (4.46). Thus, from Lemma 4.4.2, we know that there exist positive numbers a and b such that $f_r = G_r\gamma_a + H_r\gamma_b$, where G_r and H_r are polynomials of degree at most N. Thus $a = b$ by Lemma 4.45. Thus $f_r = P_r\gamma_a$ where P_r is a polynomial of degree at most N. Similarly, there exists $c > 0$ such that $f_i = P_i\gamma_c$ where P_i is a polynomial of degree at most N. Thus $f = P_r\gamma_a + P_i\gamma_c$. Again $a = c$ by Lemma 4.4.3. Thus there exists a polynomial P of degree at most N such that $f = P\gamma_a$. $\square$

4.5 Applications

Suppose that f satisfies the assumption of Bonami, Demange and Jaming [3] (see (4.2)). Then for $\lambda = 1/2$, the following holds:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\left| f(x)\hat{f}(y) \right|}{(1+|x|+|y|)^N} e^{|xy|/2}\, dx\, dy < \infty.$$

Thus

$$\int_{\mathbb{R}} \left| f(x)\hat{f}(y) \right| e^{|xy|/3}\, dx < \infty$$

for almost every $y \in \mathbb{R}$ when $f(y) \neq 0$. Thus there exists $\alpha > 0$ such that

$$\int_{\mathbb{R}} \left| f(x)e^{-\alpha|x|} \right| dx < \infty. \tag{4.47}$$

Similarly there exists $\beta > 0$ such that

$$\int_{\mathbb{R}} \left| \hat{f}(y) e^{-\beta|y|} \right| \, dy < \infty. \tag{4.48}$$

Using the fact that

$$D^n(\hat{f}) = \mathcal{F}(f(\cdot)(\cdot)^n),$$

we deduce that $\hat{f}$ and f are real analytic. Thus they will not have compact support. By using the assumption of (4.2) again, we see that

$$\int_{\mathbb{R}} |f(x)| \, e^{\epsilon|x|/2} \, dx < \infty$$

for all $\epsilon > 0$. So f and $\hat{f}$ are bounded and

$$\iint_{\min\{|x|,|y|\}\leq 1} \left| f(x) \, \hat{f}(y) \right| e^{\lambda|xy|} \, dx \, dy < \infty$$

for all $\lambda \in \mathbb{R}^+$. Moreover,

$$\begin{aligned}
&\iint_{\min\{|x|,|y|\}\geq 1} \left| f(x) \, \hat{f}(y) \right| e^{\lambda|xy|} \, dx \, dy \\
&\quad = O((1-\lambda)^N) \iint_{\min\{|x|,|y|\}\geq 1} \frac{\left| f(x) \, \hat{f}(y) \right|}{(1+|x|+|y|)^N} e^{|xy|} \, dx \, dy,
\end{aligned}$$

since

$$\max_{\min\{|x|,|y|\}\geq 1} e^{(\lambda-1)|xy|} (1+|x|+|y|)^N = O((1-\lambda)^N).$$

Indeed, on a line segment where $|x| + |y|$ is constant, $e^{(\lambda-1)|xy|}$ is maximum when $|x| = 1$ or $|y| = 1$.

Hence our arguments also imply the result of Bonami, Demange, and Jaming [3]. An argument we give later (see 5.1.4) shows that in fact these results are equivalent in $\mathbb{R}$. Unlike Hardy's uncertainty principle, Beurling's uncertainty does not has L_p versions in the literature. However by using the generalized version of Beurling's uncertainty principle (Theorem 4.4.4) we are able to prove the following theorems.

Theorem 4.5.1. *Suppose that $1 < p < \infty$, that f and $\hat{f}$ are in $L_p(\mathbb{R})$, and*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left| f(x) \hat{f}(y) \right|^p e^{p|xy|} \, dx \, dy < \infty. \tag{4.49}$$

Then f is 0.

Proof. Observe that if $0 < \lambda < 1$, then

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(x)\hat{f}(y)\right| e^{\lambda|xy|}\,dx\,dy \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(x)\hat{f}(y)\right|^{\lambda} e^{\lambda|xy|} \left|f(x)\hat{f}(y)\right|^{1-\lambda} dx\,dy \\
&\leq \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(x)\hat{f}(y)\right|^{p} e^{p|xy|}\,dx\,\,dy\right)^{\lambda/p} \\
&\quad \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(x)\hat{f}(y)\right|^{\frac{p(1-\lambda)}{p-\lambda}} dx\,dy\right)^{1-\lambda/p} \\
&\lesssim \left(\int_{\mathbb{R}} |f(x)|^{\frac{p(1-\lambda)}{p-\lambda}}\,dx\right)^{1-\lambda/p} \left(\int_{\mathbb{R}} \left|\hat{f}(y)\right|^{\frac{p(1-\lambda)}{p-\lambda}}\,dy\right)^{1-\lambda/p}.
\end{aligned} \tag{4.50}$$

Because

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(x)\hat{f}(y)\right|^{p} e^{p|xy|}\,dx\,dy < \infty, \tag{4.51}$$

there exist two positive numbers α, β such that

$$\int_{\mathbb{R}} |f(x)|^{p} e^{\alpha|x|}\,dx < \infty \tag{4.52}$$

and

$$\int_{\mathbb{R}} \left|\hat{f}(y)\right|^{p} e^{\beta|y|} dy < \infty, \tag{4.53}$$

by the same argument used to prove (4.47) and (4.48). Thus

$$\begin{aligned}
\int_{\mathbb{R}} |f(x)|^{\frac{p(1-\lambda)}{p-\lambda}}\,dx &\leq \int_{\mathbb{R}} |f(x)|^{\frac{p(1-\lambda)}{p-\lambda}} e^{\frac{1-\lambda}{p-\lambda}\alpha|x|} e^{-\frac{1-\lambda}{p-\lambda}\alpha|x|}\,dx \\
&\leq \left(\int_{\mathbb{R}} |f(x)|^{p} e^{\alpha|x|}\,dx\right)^{\frac{1-\lambda}{p-\lambda}} \left(\int_{\mathbb{R}} e^{-\frac{1-\lambda}{p-1}\alpha|x|}\,dx\right)^{\frac{p-1}{p-\lambda}} \\
&\lesssim \left(\frac{1}{1-\lambda}\right)^{\frac{p-1}{p-\lambda}}.
\end{aligned} \tag{4.54}$$

Similarly

$$\begin{aligned}
\int_{\mathbb{R}} \left|\hat{f}(y)\right|^{\frac{p(1-\lambda)}{p-\lambda}}\,dy &\leq \int_{\mathbb{R}} \left|\hat{f}(y)\right|^{\frac{p(1-\lambda)}{p-\lambda}} e^{\frac{1-\lambda}{p-\lambda}\beta|y|} e^{-\frac{1-\lambda}{p-\lambda}\beta|y|} dy \\
&\leq \left(\int_{\mathbb{R}} \left|\hat{f}(y)\right|^{p} e^{\beta|y|}\,dy\right)^{\frac{1-\lambda}{p-\lambda}} \left(\int_{\mathbb{R}} e^{-\frac{1-\lambda}{p-1}\beta|y|}\,dy\right)^{\frac{p-1}{p-\lambda}} \\
&\lesssim \left(\frac{1}{1-\lambda}\right)^{\frac{p-1}{p-\lambda}}.
\end{aligned} \tag{4.55}$$

Thus by combining the above inequalities there exists a positive number N such that

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(x)\hat{f}(y)\right| e^{\lambda|xy|}\,dx\,dy \\
&\lesssim \left(\int_{\mathbb{R}} |f(x)|^{\frac{p(1-\lambda)}{p-\lambda}}\,dx\right)^{\frac{p-\lambda}{p}} \left(\int_{\mathbb{R}} \left|\hat{f}(y)\right|^{\frac{p(1-\lambda)}{p-\lambda}}\,dy\right)^{\frac{p-\lambda}{p}} \qquad (4.56)\\
&\lesssim \frac{1}{(1-\lambda)^N}.
\end{aligned}$$

It follows that there exists a constant t such that $f(x) = P(x)e^{-tx^2}$ where P is a polynomial of degree at most N. By checking (4.49) we can conclude that $P(x) = 0$. □

Theorem 4.5.2. *Suppose that f and $\hat{f}$ are in $L_p(\mathbb{R})$ and there exists a positive number N such that*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(x)\hat{f}(y)\right|^p e^{\lambda p|xy|}\,dx\,dy \lesssim \frac{1}{(1-\lambda)^N}, \qquad (4.57)$$

where $p > 1$. Then f is a polynomial of degree at most $N+1$ times a gaussian

Proof. By picking positive numbers λ and σ such that $\lambda < \sigma < 1$, we have, by Hölder's inequality,

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(x)\hat{f}(y)\right| e^{\lambda|xy|}\,dx\,dy \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(x)\hat{f}(y)\right|^{\sigma} e^{\lambda|xy|} \left|f(x)\hat{f}(y)\right|^{1-\sigma}\,dx\,dy \\
&\le \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(x)\hat{f}(y)\right|^p e^{\lambda p|xy|/\sigma}\,dx\,dy\right)^{\sigma/p} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(x)\hat{f}(y)\right|^{\frac{p(1-\sigma)}{p-\sigma}}\,dx\,dy\right)^{\frac{p-\sigma}{p}} \\
&\lesssim \left(\frac{1}{(1-\lambda/\sigma)^N}\right)^{\frac{\sigma}{p}} \left(\int_{\mathbb{R}} |f(x)|^{\frac{p(1-\sigma)}{p-\lambda}}\,dx\right)^{\frac{p-\sigma}{p}} \left(\int_{\mathbb{R}} \left|\hat{f}(y)\right|^{\frac{p(1-\sigma)}{p-\sigma}}\,dy\right)^{\frac{p-\sigma}{p}}.
\end{aligned} \qquad (4.58)$$

Because

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(x)\hat{f}(y)\right|^p e^{\sigma p|xy|}\,dx\,dy < \infty, \qquad (4.59)$$

there exist two positive numbers α, β such that

$$\int_{\mathbb{R}} |f(x)|^p e^{\alpha|x|}\,dx < \infty \qquad (4.60)$$

and

$$\int_{\mathbb{R}} \left|\hat{f}(y)\right|^p e^{\beta|y|}dy < \infty. \qquad (4.61)$$

Thus

$$
\begin{aligned}
\int_{\mathbb{R}} |f(x)|^{\frac{p(1-\sigma)}{p-\sigma}} \, dx &\leq \int_{\mathbb{R}} |f(x)|^{\frac{p(1-\sigma)}{p-\sigma}} e^{\frac{1-\sigma}{p-\sigma}\alpha|x|} e^{-\frac{1-\sigma}{p-\sigma}\alpha|x|} \, dx \\
&\leq \left(\int_{\mathbb{R}} |f(x)|^p e^{\alpha|x|} \, dx\right)^{\frac{1-\sigma}{p-\sigma}} \left(\int_{\mathbb{R}} e^{-\frac{1-\sigma}{p-1}\alpha|x|} \, dx\right)^{\frac{p-1}{p-\sigma}} \\
&\lesssim \left(\frac{1}{1-\sigma}\right)^{\frac{p-1}{p-\sigma}}.
\end{aligned}
\tag{4.62}
$$

Similarly

$$
\begin{aligned}
\int_{\mathbb{R}} \left|\hat{f}(y)\right|^{\frac{p(1-\sigma)}{p-\sigma}} \, dy &\leq \int_{\mathbb{R}} \left|\hat{f}(y)\right|^{\frac{p(1-\sigma)}{p-\sigma}} e^{\frac{1-\sigma}{p-\sigma}\beta|y|} e^{-\frac{1-\sigma}{p-\sigma}\beta|y|} \, dy \\
&\leq \left(\int_{\mathbb{R}} \left|\hat{f}(y)\right|^p e^{\beta|y|} \, dy\right)^{\frac{1-\sigma}{p-\sigma}} \left(\int_{\mathbb{R}} e^{-\frac{1-\sigma}{p-1}\beta|y|} \, dy\right)^{\frac{p-1}{p-\sigma}} \\
&\lesssim \left(\frac{1}{1-\sigma}\right)^{\frac{p-1}{p-\sigma}}.
\end{aligned}
\tag{4.63}
$$

Thus by combining the above inequalities, for all $\lambda < \sigma < 1$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} \left|f(x)\hat{f}(y)\right| e^{\lambda|xy|} \, dx \, dy \\
&\lesssim \left(\frac{1}{(1-\lambda/\sigma)^N}\right)^{\frac{\sigma}{p}} \left(\int_{\mathbb{R}} |f(x)|^{\frac{p(1-\sigma)}{p-\sigma}} \, dx\right)^{\frac{p-\sigma}{p}} \left(\int_{\mathbb{R}} \left|\hat{f}(y)\right|^{\frac{p(1-\sigma)}{p-\sigma}} \, dy\right)^{\frac{p-\sigma}{p}} \\
&\lesssim \left(\frac{1}{(1-\lambda/\sigma)^N}\right)^{\frac{\sigma}{p}} \frac{1}{(1-\sigma)^{2(p-1)/p}}.
\end{aligned}
\tag{4.64}
$$

By picking $\sigma = \frac{1+\lambda}{2}$, it follows that,

$$
\begin{aligned}
\int_{\mathbb{R}} \int_{\mathbb{R}} \left|f(x)\hat{f}(y)\right| e^{\lambda|xy|} \, dx \, dy &\lesssim \left(\frac{(1+\lambda)^N}{(1-\lambda)^N}\right)^{\frac{1+\lambda}{2p}} \left(\frac{2}{1-\lambda}\right)^{\frac{2(p-1)}{p}} \\
&\lesssim \frac{1}{(1-\lambda)^{N+2}}.
\end{aligned}
\tag{4.65}
$$

Thus by Theorem 4.4.4, there exists a constant t such that $f(x) = P(x)e^{-tx^2}$ where P is a polynomial. $\square$

Chapter 5

The Beurling-Hedenmalm problem on $\mathbb{R}^n$

5.1 Introduction

From the previous chapters, there are strong reasons for us to believe that if the generalized Beurling style inequality on $\mathbb{R}^n$ holds:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|f(x)\hat{f}(y)\right| e^{\lambda|\langle x,y\rangle|}\,dx\,dy \lesssim \frac{1}{(1-\lambda)^N} \tag{5.1}$$

for all $0 \le \lambda < 1$, then f is a polynomial times a gaussian in $\mathbb{R}^n$.

In this chapter we show that the generalized Beurling-Hedenmalm uncertainty principle with assumption (5.1) is true. Moreover at the end of this chapter we will show that if we weaken the assumption (5.1) as follows

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|f(x)\hat{f}(y)\right| e^{\lambda|x||y|}\,dx\,dy \lesssim \frac{1}{(1-\lambda)^N} \tag{5.2}$$

for all $\lambda \in [0,1)$, then we can use the tools of spherical harmonics and Bessel functions to prove that f is a polynomial times $e^{-a|\cdot|^2}$, where $a \ge 0$.

Before we start our proofs in $\mathbb{R}^n$, we first generalize the auxiliary function that we used in $\mathbb{R}$.

Definition 5.1.1. *Given a bounded $L_1(\mathbb{R})$ function f with exponential decay at infinity, we define the auxiliary function F_f^n as follows:*

$$F_f^n(s) = \int_{\mathbb{R}} r^{n-1} f(r) f(sr)\,dr \tag{5.3}$$

for all $s \in \mathbb{R}$.

Recall that in Definition 2.3.3 we defined $\mathcal{M}_f^k$ to be

$$\mathcal{M}_f^k(z) = \int_{\mathbb{R}} f(x)\,\mathrm{sgn}^k(x)\,|x|^{z-1/2}\,dx.$$

Thus when f is bounded and of exponential decay at infinity, $\mathcal{M}_f^k(z)$ is defined when $\mathrm{Re}(z) \in (-1/2, \infty)$ and $\mathcal{M}_f^k(-z+(n-1))$ is defined when $\mathrm{Re}(z) \in (-\infty, n-1/2)$. Also we can check that

$$\mathcal{M}_{F_f^n}^k(z) = \mathcal{M}_f^k(-z+(n-1))\mathcal{M}_f^k(z), \tag{5.4}$$

when $\mathrm{Re}(z) \in (-1/2, n-1/2)$.

Lemma 5.1.2. *Suppose that real functions f and $\hat{f}$ are both of exponential decay at infinity and $F_f^n(s)$, which is initially defined for $s \in \mathbb{R}$, extends analytically to $F_f^n(z)$ to the strip $\{z \in \mathbb{C} : |\mathrm{Im}(z)| < 1\}$, and satisfies*

$$\left|F_f^n(z)\right| \lesssim \left(\frac{1}{1-|\mathrm{Im}(z)|}\right)^N \tag{5.5}$$

there. Then there exist polynomials p_k of degree at most $\lfloor N - n/2 \rfloor - k$ such that

$$\Theta_f^k(z)\Theta_f^k(-z+n-1) = p_k(z).$$

Thus there exist a positive number a and a polynomial P of degree at most $\lfloor N - n/2 \rfloor$ such that

$$f(x) = P(x)e^{-ax^2/2}.$$

Proof. Let $F(z) = (1+z^2)^{n/2}F_f^n(z)$, where we take a branch of $(z^2+1)^{n/2}$ that is positive on the real axis and analytic when $|\mathrm{Im}(z)| < 1$. Also when $z \neq 0$ and $z \in \mathbb{R}$,

$$\begin{aligned} F\left(\frac{1}{z}\right) &= \left(\frac{1+z^2}{z^2}\right)^{n/2} F_f^n\left(\frac{1}{z}\right) \\ &= \left(\frac{1+z^2}{z^2}\right)^{n/2} \int_{\mathbb{R}} r^{n-1} f(r) f\left(\frac{r}{z}\right) dr \\ &= \left(\frac{1+z^2}{z^2}\right)^{n/2} \int_{\mathbb{R}} z^n t^{n-1} f(t) f(zt)\,dt \\ &= (1+z^2)^{n/2} F_f^n(z) \\ &= F(z). \end{aligned} \tag{5.6}$$

Thus by analytic continuation $F(z) = F(1/z)$ holds when $|\mathrm{Im}(z)| < 1$ and $|\mathrm{Im}(1/z)| < 1$. Because the union $\{z : |\mathrm{Im}(1/z)| < 1\} \cup \{z : |\mathrm{Im}(z)| < 1\}$ covers the whole complex plane except the points $\pm i$, F can be extended to a meromorphic function on

$\mathbb{C}$ with poles at $\pm i$, and by using a similar argument to that in Chapter 4 we find the degree of each pole is at most $\lfloor N - n/2 \rfloor$. By restricting $F(z)$ to the real line we get

$$F_f^n(s) = \sum_{k=0}^{\lfloor N-n/2 \rfloor} a_k (1+s^2)^{-k-n/2} + \sum_{k=1}^{\lfloor N-n/2 \rfloor} b_k s (1+s^2)^{-k-n/2},$$

and there exist polynomials p of degree at most $\lfloor N - n/2 \rfloor$ and q of degree at most $\lfloor N - n/2 \rfloor - 1$ such that

$$\mathcal{M}^0_{F_f^n}(z) = p(z)\Gamma\left(\frac{z}{2}+\frac{1}{4}\right)\Gamma\left(\frac{1}{4}-\frac{z}{2}+\frac{n-1}{2}\right) \tag{5.7}$$

and

$$\mathcal{M}^1_{F_f^n}(z) = q(z)\Gamma\left(\frac{z}{2}+\frac{3}{4}\right)\Gamma\left(\frac{3}{4}-\frac{z}{2}+\frac{n-1}{2}\right), \tag{5.8}$$

for all z where $\mathcal{M}^k_{F_f^n}(z)$ are defined. In particular we notice that because $F_f^n(s)$ is bounded, the right hand sides of the above equations are both defined when $|\mathrm{Re}(z)| < 1/2$. By Theorem 4.3.3, $\Theta_f^k(z)$ is analytic on the whole complex plane, thus by analytic continuation and equation (5.4), we can conclude that

$$\Theta_f^0(z)\Theta_f^0(-z+n-1) = p(z)$$

and

$$\Theta_f^1(z)\Theta_f^1(-z+n-1) = q(z),$$

for all $z \in \mathbb{C}$. By using same arguments as those in Chapter 4, we can conclude that there exist a positive number a and a polynomial P of degree at most $\lfloor N - n/2 \rfloor$ such that

$$f(x) = P(x)e^{-ax^2/2},$$

as required. $\square$

Lemma 5.1.3. *Suppose that $M > 0$. Then*

$$e^{-sr}(1+r^M) \lesssim s^{-M}$$

for all $r \geq 0$ and all $s \in (0,1]$.

Proof. When $0 \leq r < 1$,

$$e^{-sr}(1+r^M) \leq 2e^{-sr} \leq 2.$$

When $r \geq 1$, by Lemma 3.2.2,

$$e^{-sr}(1+r^M) \leq 2e^{-sr}r^M \leq 2\frac{M^M}{e^M}\frac{1}{s^M}.$$

Thus the lemma follows. $\square$

Lemma 5.1.4. *Suppose that $f \in L_1(\mathbb{R}^n)$, $N > 0$ and N is an integer. Then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|f(x)\hat{f}(y)\right| e^{\lambda|\langle x,y\rangle|}\,dx\,dy \lesssim (1-\lambda)^{-N} \tag{5.9}$$

for all $\lambda \in [0,1)$ implies that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left|f(x)\hat{f}(y)\right| e^{|\langle x,y\rangle|}}{1+|\langle x,y\rangle|^{N+1}}\,dx\,dy < \infty. \tag{5.10}$$

Conversely, if

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left|f(x)\hat{f}(y)\right| e^{|\langle x,y\rangle|}}{1+|\langle x,y\rangle|^{N}}\,dx\,dy < \infty, \tag{5.11}$$

then (5.9) *holds.*

Proof. (5.9) $\Rightarrow$ (5.10):

If we integrate both sides of (5.9) $N+1$ times with respect to λ and change the order of integration, we see that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|f(x)\hat{f}(y)\right| \int_0^1\int_0^{\lambda}\cdots\int_0^{\lambda_{N-1}} e^{\lambda_N|\langle x,y\rangle|}\,dx\,dy\,d\lambda_N\cdots d\lambda_1\,d\lambda\\
&= \int_0^1\int_0^{\lambda}\cdots\int_0^{\lambda_{N-1}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|f(x)\hat{f}(y)\right| e^{\lambda_N|\langle x,y\rangle|}\,dx\,dy\,d\lambda_N\cdots d\lambda_1\,d\lambda\\
&\lesssim \int_0^1\int_0^{\lambda}\cdots\int_0^{\lambda_{N-1}} C(1-\lambda_N)^{-N}\,d\lambda_N\cdots d\lambda_1\,d\lambda\\
&< \infty.
\end{aligned}$$

By changing the order of integration, we also see that

$$\begin{aligned}
&\int_0^1\int_0^{\lambda}\cdots\int_0^{\lambda_{N-1}} e^{\lambda_N|\langle x,y\rangle|}\,d\lambda_N\cdots d\lambda_1\,d\lambda\\
&= \int_0^1\int_{\lambda_N}^1\cdots\int_{\lambda_1}^1 e^{\lambda_N|\langle x,y\rangle|}\,d\lambda\,d\lambda_1\cdots d\lambda_N\\
&= \int_0^1\int_{\lambda_N}^1\cdots\int_{\lambda_2}^1 (1-\lambda_1)e^{\lambda_N|\langle x,y\rangle|}\,d\lambda_1\cdots d\lambda_N\\
&= \cdots\\
&= \int_0^1 \frac{(1-\lambda_N)^N}{N!} e^{\lambda_N|\langle x,y\rangle|}\,d\lambda_N\\
&= I,
\end{aligned}$$

say. If $|\langle x,y\rangle| \le 1$, then

$$I \ge \int_0^1 \frac{(1-\lambda)^N}{N!}\,d\lambda = \frac{1}{(N+1)!} \ge \frac{e^{|\langle x,y\rangle|}}{e(N+1)!} \ge \frac{e^{|\langle x,y\rangle|}}{e(N+1)!(1+|\langle x,y\rangle|^N)},$$

while if $|\langle x,y\rangle| > 1$ then

$$\begin{aligned} I &= \int_0^1 \frac{\lambda^N}{N!} e^{(1-\lambda)|\langle x,y\rangle|}\,d\lambda \\ &= \frac{e^{|\langle x,y\rangle|}}{N!}\int_0^1 \lambda^N e^{-\lambda|\langle x,y\rangle|}\,d\lambda \\ &= \frac{e^{|\langle x,y\rangle|}}{N!\,|\langle x,y\rangle|^{N+1}}\int_0^{|\langle x,y\rangle|} t^N e^{-t}\,dt \\ &\ge \frac{e^{|\langle x,y\rangle|}}{eN!\,|\langle x,y\rangle|^{N+1}}\int_0^1 t^N\,dt \\ &= \frac{e^{|\langle x,y\rangle|}}{e(N+1)!\,|\langle x,y\rangle|^{N+1}} \\ &\ge \frac{e^{|\langle x,y\rangle|}}{e(N+1)!(1+|\langle x,y\rangle|^{N+1})}. \end{aligned}$$

(5.11) $\Rightarrow$ (5.9): From Lemma 5.1.3 we know that

$$(1-\lambda)^N \lesssim \frac{e^r}{e^{\lambda r}(1+r^N)},$$

when $\lambda \in [0,1)$ and $r > 0$. It follows that

$$\begin{aligned} (1-\lambda)^N \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|f(x)\hat{f}(y)\right| e^{\lambda|\langle x,y\rangle|}\,dx\,dy &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left|f(x)\hat{f}(y)\right| e^{|\langle x,y\rangle|}}{1+|\langle x,y\rangle|^N}\,dx\,dy \\ &< \infty. \end{aligned}$$

Thus the lemma follows. $\square$

Lemma 5.1.5. *Suppose that $f \in L_1(\mathbb{R}^n)$ and that $N > 0$. If*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|f(x)\hat{f}(y)\right| e^{\lambda|\langle x,y\rangle|}\,dx\,dy \lesssim (1-\lambda)^{-N}$$

for all $0 \le \lambda < 1$, then

$$\int_{\mathbb{R}^n} |f(x)|\, e^{r|\langle x,w\rangle|} < \infty$$

for all $w \in S^{n-1}$ and all $r \ge 0$. Thus f and $\hat{f}$ are analytic functions on $\mathbb{C}^n$.

Proof. Because $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|f(x)\hat{f}(y)\right|\,dx\,dy < \infty$, f and $\hat{f}$ are integrable and hence $\hat{f}$ and f are continuous. Provided that $\hat{f}(y_0) \ne 0$ for some fixed number $y_0 = |y_0|\,w_0$, there exists a closed ball $\bar{B}(y_0, \epsilon)$ such that $\hat{f}(y) \neq 0$ for all $y \in \bar{B}(y_0, \epsilon)$. Fix λ in $[0, 1)$ close to 1. Then for almost all $y \in \bar{B}(y_0, \epsilon)$,

$$\int_{\mathbb{R}^n} \left| f(x)\hat{f}(y) \right| e^{\lambda |\langle x, y\rangle|}\, dx < \infty. \tag{5.12}$$

Notice that for all $w \in S^{n-1}$, $y_0 + \operatorname{sgn}(\langle x, y_0\rangle) \operatorname{sgn}(\langle x, w\rangle)\epsilon w$ is in $\bar{B}(y_0, \epsilon)$ and

$$|\langle x, y_0 + \operatorname{sgn}(\langle x, y_0\rangle) \operatorname{sgn}(\langle x, w\rangle)\epsilon w\rangle| = |\langle x, y_0\rangle| + |\epsilon\langle x, w\rangle| \geq \epsilon\, |\langle x, w\rangle|\, .$$

We may choose y in $\bar{B}(y_0, \epsilon)$ arbitrary close to $y_0 + \operatorname{sgn}(\langle x, y_0\rangle) \operatorname{sgn}(\langle x, w\rangle)\epsilon w$ so that (5.12) holds, and thus

$$\int_{\mathbb{R}^n} |f(x)|\, e^{\epsilon |\langle x, w\rangle|/2}\, dx < \infty.$$

It follows that $\hat{f}$ is analytic on $\mathbb{R}^n$, thus must have countably many zeros. So for all $r_0 > 0$ there exists $r \geq r_0$ such that $\hat{f}(y) \neq 0$ for all $y \in \bar{B}(y_0, r)$. Therefore by the same argument

$$\int_{\mathbb{R}^n} |f(x)|\, e^{r|\langle x, w\rangle|}\, dx < \infty,$$

and the lemma follows. □

5.2 The Beurling-Hedenmalm uncertainty principle on $\mathbb{R}^n$

Definition 5.2.1. *Given a continuous function $f \in L_1(\mathbb{R}^n)$, we define the Radon transform of f relative to the nonzero vector $w \in \mathbb{R}^n$ to be*

$$\mathcal{R}_f^w(r) = \int_{\langle w, x\rangle = r} f(x)\, dx,$$

where dx is the $n-1$ dimensional Lebesgue measure and $r \in \mathbb{R}$.

Lemma 5.2.2. *Suppose that f is a continuous function in $L_1(\mathbb{R}^n)$ and N is a positive integer such that*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left| f(x)\hat{f}(y) \right|}{1 + |\langle x, y\rangle|^N} e^{|\langle x, y\rangle|}\, dx\, dy < \infty.$$

Then for almost all vectors w on the unit sphere S^{n-1},

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\left| \mathcal{R}_f^w(r) \widehat{\mathcal{R}_f^w}(s) \right| e^{|rs|} s^{n-1}}{1 + |rs|^N}\, dr\, ds < \infty.$$

Proof. Using the assumption

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left|f(x)\hat{f}(y)\right|}{1+\left|\langle x,y\rangle\right|^N}\, e^{|\langle x,y\rangle|}\, dx\, dy < \infty,$$

we can show that, by changing to polar coordinates, that

$$\int_{S^{n-1}}\int_{\mathbb{R}}\int_{\mathbb{R}^n} s^{n-1}\frac{\left|f(x)\hat{f}(sw)\right|}{1+\left|\langle x,sw\rangle\right|^N}\, e^{|\langle x,sw\rangle|}\, dx\, ds\, dw < \infty,$$

hence for almost all vectors w on the unit sphere S^{n-1} we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} s^{n-1}e^{s|\langle x,w\rangle|}\frac{\left|f(x)\hat{f}(sw)\right|}{1+\left|\langle x,sw\rangle\right|^N}\, ds\, dx < \infty.$$

Also we can verify that

$$\begin{aligned}\hat{\mathcal{R}_f^w}(s) &= \int_{\mathbb{R}}\left(\int_{\langle x,w\rangle=t} f(x)\, dx\right) e^{-its}\, dt\\ &= \int_{\mathbb{R}^n} f(x)e^{-i\langle x,sw\rangle}\, dx\\ &= \hat{f}(sw).\end{aligned} \tag{5.13}$$

So it follows that

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\left|\mathcal{R}_f^w(r)\hat{\mathcal{R}_f^w}(s)\right|}{1+\left|rs\right|^N} s^{n-1}e^{|rs|}\, dr\, ds\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\left|\int_{\langle w,x\rangle=r} f(x)\, dx\right| \frac{\left|\hat{f}(sw)\right|}{1+\left|rs\right|^N}\, s^{n-1}e^{|rs|}\, dr\, ds\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\left|\int_{\langle w,x\rangle=r} \frac{f(x)e^{|\langle w,x\rangle s|}}{1+\left|s\langle w,x\rangle\right|^N}\hat{f}(sw)\, dx\right| s^{n-1}\, dr\, ds\\ &\le \int_{\mathbb{R}}\int_{\mathbb{R}^n} \frac{\left|f(x)\hat{f}(sw)\right|}{1+\left|s\langle w,x\rangle\right|^N}\, s^{n-1}e^{|\langle w,x\rangle s|}\, dx\, ds\\ &< \infty\end{aligned}$$

for almost all vectors w on the unit sphere S^{n-1}. □

Corollary 5.2.3. *Suppose that f is a continuous function in $L_1(\mathbb{R}^n)$ and N is a positive integer such that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left|f(x)\hat{f}(y)\right|}{1+\left|\langle x,y\rangle\right|^N}\, e^{|\langle x,y\rangle|}\, dx\, dy < \infty.$$

Then for almost all vectors w on the unit sphere S^{n-1},

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{R}_f^w(r)\hat{\mathcal{R}_f^w}(s)\right| e^{\lambda|rs|}s^{n-1}\,dr\,ds \lesssim \frac{1}{(1-\lambda)^N}$$

for all $\lambda \in [0,1)$.

Proof. This follows from the previous lemma and Lemma 5.1.3. From Lemma 5.1.3 we know that $\frac{1+|rs|^N}{e^{(1-\lambda)|rs|}} \lesssim \frac{1}{(1-\lambda)^N}$, when $\lambda \in [0,1)$. Thus

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{R}_f^w(r)\hat{\mathcal{R}_f^w}(s)\right| s^{n-1}e^{\lambda|rs|}\,dr\,ds \\
&\lesssim \frac{1}{(1-\lambda)^N}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\left|\mathcal{R}_f^w(r)\hat{\mathcal{R}_f^w}(s)\right|}{1+|rs|^N}s^{n-1}e^{|rs|}\,dr\,ds \\
&\lesssim \frac{1}{(1-\lambda)^N}
\end{aligned}$$

as required. □

Lemma 5.2.4. *Fix $N \in \mathbb{N}$, suppose that $f_1, f_2, \dots f_n (n = 1, 2, \dots)$ are bounded analytic functions on $[0,\infty)$, and that $f_n \to f$ locally uniformly as $n \to \infty$. If each f_n is of the form $p_n(x)e^{-\lambda_n x^2/2}$, where p_n is a polynomial of degree at most N and $\lambda_n \in \mathbb{R}^+$, then so is f.*

Proof. By assumption, we may write

$$f_n(x) = p_n(x)e^{-\lambda_n x^2/2}.$$

By passing to a sub-sequence if necessary, we may assume that $\lambda_n \to \lambda$ in $[0,+\infty]$ as $n \to \infty$.

Suppose first that $\lambda \to 0$ as $n \to \infty$. Then $e^{\lambda_n x^2/2}f_n(x) \to f(x)$ locally uniformly as $n \to \infty$, so

$$p_n(x) \to f(x).$$

Thus f is a polynomial of degree at most N. Since f is bounded, f is constant, and is of the required form.

To deal with the possibility that $\lambda_n \to \infty$ as $n \to \infty$, recall the Markov lemma for polynomials of degree at most N:

$$\max\left\{\left|p^{(k)}(x)\right| : 0 \le x \le 1\right\} \le C_N \max\{|p(y)| : 0 \le y \le 1\}.$$

This implies that

$$|p(x)| \le C_N \max\{|p(y)| : 0 \le y \le 1\}(1+x)^N$$

for all $x \in \mathbb{R}^+$. In our situation, this implies that

$$\begin{aligned}|f_n(x)| &\leq |p_n(y)|\, e^{-\lambda_n x^2/2} \\ &\leq C_n \max\{|p_n(y)| : 0 \leq y \leq 1\}(1+x)^N e^{-\lambda_n x^2/2} \\ &\leq C_n \max\{|f_n(y)| : 0 \leq y \leq 1\}(1+x)^N e^{(1-\lambda_n)x^2/2} \\ &\leq C(1+x)^N e^{(1-\lambda_n)x^2/2},\end{aligned}$$

whence $f_n(x) \to 0$ uniformly on $[2, \infty)$, and hence $f = 0$.

Finally, if $\lambda_n \to \lambda \in \mathbb{R}^+$, then $p_n(x) \to f(x)e^{\lambda x^2/2}$ locally uniformly, whence $f(x)e^{\lambda x^2/2}$ is a polynomial, and f is of the required form. $\square$

Lemma 5.2.5. *Given a $f \in L_1(\mathbb{R}^n)$ and a positive integer N. If*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left| f(x)\hat{f}(y) \right| e^{\lambda|\langle x,y\rangle|}\, dx\, dy \lesssim (1-\lambda)^{-(N-1)}$$

for all $0 \leq \lambda < 1$, then $\mathcal{R}_f^w$ is equal to a gaussian multiplied by a polynomial of degree at most $\lfloor N - n/2 \rfloor$ for all $w \in S^{n-1}$.

Proof. From Lemma 5.1.4 and Lemma 5.2.3 we can conclude that for almost all vectors w on the unit sphere S^{n-1},

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left| \mathcal{R}_f^w(r)\hat{\mathcal{R}_f^w}(s) \right| e^{\lambda|rs|} s^{n-1}\, dr\, ds \lesssim \frac{1}{(1-\lambda)^N} \tag{5.14}$$

for all $\lambda \in [0, 1)$. As before, it is sufficient to treat the case when $\mathcal{R}_f^w$ is a real function. Because $\mathcal{R}_f^w(t) = \hat{f}(wt)$, it is equivalent to prove the case when $\hat{f}$ is real.

Now by Lemma 5.1.2, for almost all vectors w on the unit sphere S^{n-1}, there exists a polynomial $\tilde{p}_w$ of degree at most $\lfloor N - n/2 \rfloor$ such that

$$\Theta^k_{\mathcal{R}_f^w}(z)\Theta^k_{\mathcal{R}_f^w}(-z+n-1) = \tilde{p}_w(z). \tag{5.15}$$

Also by Lemma 5.1.5, for all $w \in S^{n-1}$,

$$\begin{aligned}\int_{\mathbb{R}} e^{|t|} \left|\mathcal{R}_f^w(t)\right| dt &\leq \int_{\mathbb{R}} e^{|t|} \int_{\langle w,x\rangle = t} |f(x)|\, dx\, dt \\ &\leq \int_{\mathbb{R}^n} e^{|\langle w,x\rangle|} |f(x)|\, dx \\ &< \infty,\end{aligned} \tag{5.16}$$

and for almost $w \in S^{n-1}$

$$\begin{aligned}\int_{\mathbb{R}} e^{|t|} \left|\hat{\mathcal{R}_f^w}(t)\right| dt &\leq \int_{\mathbb{R}} e^{t} \left|\hat{f}(wt)\right| dt \\ &< \infty.\end{aligned} \tag{5.17}$$

So by Theorem 4.3.3, $\Theta^k_{\mathcal{R}^w_f}(z)$ is of order one for almost all $w \in S^{n-1}$. Thus

$$\Theta^k_{\mathcal{F}(\mathcal{R}^w_f)}(z) = \Theta^k_{\mathcal{R}^w_f}(-z) = p_w(z)e^{q(w)z} \tag{5.18}$$

for almost all $w \in S^{n-1}$, where $p_w(z)$ is a polynomial of degree at most $\lfloor N - n/2 \rfloor/2$. It follows that, by the arguments to prove Lemma 4.4.2,

$$\mathcal{F}(\mathcal{R}^w_f(t)) = P_w(t)e^{-Q_w t^2/2}$$

for almost all $w \in S^{n-1}$, where P_w is a polynomial of degree at most $\lfloor N - n/2 \rfloor$ and $Q_w > 0$. Now $\mathcal{F}(\mathcal{R}^w_f(t))$ is locally uniformly continuous in the variables w and t because $\mathcal{F}(\mathcal{R}^w_f(t)) = \hat{f}(wt)$ and $\hat{f}(wt)$ is continuous. Thus by Lemma 5.2.4,

$$\mathcal{F}(\mathcal{R}^w_f(t)) = P_w(t)e^{-Q_w t^2/2}$$

for all $w \in S^{n-1}$, where P_w is a polynomial of degree at most $\lfloor N - n/2 \rfloor$ and $Q_w \geq 0$. □

Lemma 5.2.6. *Suppose f is a smooth function on $\mathbb{R}^n$, and is homogeneous of degree M. Then f is a polynomial of degree M.*

Proof. We prove this by induction on the degree of f. Firstly, if f is homogeneous of degree 0, then

$$f(w) = \lim_{t \to 0} \left(\frac{1}{t}\right)^0 f(tw) = \lim_{t \to 0} f(tw) = f(0).$$

Thus f is a constant which is of degree 0. Secondly, assume that a smooth homogeneous function on $\mathbb{R}^n$ of degree k is a polynomial of degree k. Then given a smooth function f which is homogeneous of degree $k+1$, we can check that each of its partial derivatives is homogeneous of order k and must be a polynomial of degree k by assumption. So

$$f(x) = \int_0^1 \frac{d(f(tx))}{dt}\, dt = \int_0^1 \sum_{j=1}^n \frac{\partial f}{\partial x_j}(tx) x_j \, dt$$

is a polynomial of degree $k+1$. □

Theorem 5.2.7. *Suppose that $f \in L_1(\mathbb{R}^n)$ and that $N > 0$. If*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left| f(x)\hat{f}(y) \right| e^{\lambda|\langle x,y\rangle|}\, dx\, dy \lesssim (1-\lambda)^{-(N-1)}$$

for all $0 \leq \lambda < 1$, then there exists a polynomial P and a homogeneous nonnegative *polynomial Q of degree 2 such that*

$$\hat{f}(x) = P(x)e^{-Q(x)/2}$$

for all $x \in \mathbb{R}^n$.

Proof. Let $m = \lfloor N - n/2 \rfloor$. From Lemma 5.2.5 we know that for all w on the unit sphere S^{n-1},

$$\begin{aligned}\hat{f}(tw) &= P_w(t)e^{-Q(w)t^2} \\ &= \left(a_0(w) + a_1(w)t + \cdots + a_m(w)t^m\right) e^{-Q(w)t^2}\end{aligned}$$

for all $t \in \mathbb{R}$ where $a(w)$ is not always 0. Because $\hat{f}(tw)$ has an analytic extension to the whole complex plane in variable w, for all $w \in \mathbb{R}^n \setminus \{0\}$ we may also write

$$\begin{aligned}\hat{f}(tw) &= \left(a_0\left(\frac{w}{|w|}\right) + a_1\left(\frac{w}{|w|}\right)|w|\, t + \cdots + a_m\left(\frac{w}{|w|}\right)|w|^m\, t^m\right) e^{-Q(w)t^2} \\ &= \left(a_0(w) + a_1(w)t + \cdots + a_m(w)t^m\right) e^{-Q(w)t^2},\end{aligned}$$

where $t \in \mathbb{R}$. So we can extend the domain of a_j to $\mathbb{R}^n \setminus \{0\}$ by letting $a_0(w) = a_0(w/|w|)$, $a_1(w) = a_1(w/|w|)\,|w|\,t$, and so on. Thus $a_j(w)$ is homogeneous of degree j. Similarly we can extend the domain of Q to $\mathbb{R}^n \setminus \{0\}$ by letting $Q(w) = Q(w/|w|)\,|w|^2$. Now

$$\hat{f}(tw) = \left(a_0(w) + a_1(w)t + \cdots + a_m(w)t^m\right) e^{-Q(w)t^2},$$

for all $w \in \mathbb{R}^n \setminus \{0\}$. Also we can check that

$$\hat{f}(wz)\hat{f}(-wz)\hat{f}(iwz)\hat{f}(-iwz) = P_w(z)P_w(-z)P_w(iz)P_w(-iz).$$

Thus by taking derivatives of z on both side for $4m$ times and let $z = 0$, we get

$$m!a_m^4(w) = \frac{\partial^{4m}(\hat{f}(zw)\hat{f}(-zw)\hat{f}(izw)\hat{f}(-izw)}{\partial z^{4m}}(0).$$

Thus $a_m^4(w)$ is analytic. Similarly

$$\begin{aligned}&\frac{d\hat{f}(zw)}{dz}\hat{f}(-zw)\hat{f}(izw)\hat{f}(-izw) \\ &= \left(\frac{d}{dz}P_w(z) - 2P_w(z)Q(w)z\right) P_w(-z)P_w(iz)P_w(-iz).\end{aligned}$$

Thus by taking derivatives of z on both side for $4m + 1$ times and letting $z = 0$, we get

$$2m!a_m^4(w)Q(w) = \frac{\partial^{4m+1}(\hat{f}'(wz)\hat{f}(-wz)\hat{f}(iwz)\hat{f}(-iwz))}{\partial z^{4m+1}}(0).$$

Thus $Q(w)$ is also real analytic. Because $\hat{f}(wz) = P_w(z)e^{-Q(w)z^2}$, $P_w(z)$ is analytic in the variable w as well and

$$a_k(w) = \left(\frac{d^k}{dz^k}P_w(z)\right)(0)$$

is also real analytic. Thus by Lemma 5.2.6, $a_k(w)$ is a homogeneous polynomial of degree k. Thus

$$\begin{aligned}\hat{f}(y) &= P_y(y/|y|)e^{-Q(y)}\\ &= \sum_{k=0}^{m} a_k(y)e^{-Q(y)}\\ &= P(y)e^{-Q(y)},\end{aligned}$$

where P is polynomial of degree m as required. □

We remark that a similar argument was given by [3].

5.3 The Mellin transform and spherical harmonics

In this section we weaken the assumption (5.1) as follows:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|f(x)\hat{f}(y)\right|e^{\lambda|x||y|}\,dx\,dy \lesssim \frac{1}{(1-\lambda)^N} \tag{5.19}$$

when $0 \le \lambda < 1$. By Theorem 5.2.7 we know that f is a polynomial times a gaussian in $\mathbb{R}^n$. In the rest of this chapter we will prove this weak result in a different way by using spherical harmonics and Bessel functions.

Spherical harmonics are the restriction of homogeneous harmonic polynomials to the unit sphere of $\mathbb{R}^n$.

Definition 5.3.1. *Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$ and $p(x)$ is a harmonic homogeneous polynomial on $\mathbb{R}^n$ of degree k. Then the restriction of $p(x)$ on S^{n-1} is called a spherical harmonic of degree k. We denote by $S_k(z)$ a spherical harmonic of degree k.*

Definition 5.3.2. *The space Ω_k is defined to be the set of all the spherical harmonics of degree k and we take $\{S_{k,j}\}$ to be an orthonormal basis of Ω_k. Also for convenience we denote by S_k a spherical harmonic in Ω_k.*

Definition 5.3.3. *Given a function f in $L_1(\mathbb{R}^n)$ and a spherical harmonic S_m of degree m, we define f_m by*

$$f_m(r) = \int_{S^{n-1}} f(rx')S_m(x')\,dx'. \tag{5.20}$$

Definition 5.3.4. *Given a function f in $L_1(\mathbb{R}^n)$, define the Mellin transform of f related to spherical harmonic $S_{m,j}$ as follows*

$$\mathcal{M}_f^{m,j}(z) = \int_{\mathbb{R}^+} r^{z-1/2} \int_{S^{n-1}} f(rx')S_{m,j}(x')\,dx'dr. \tag{5.21}$$

Because

$$f(x) = \sum_{m=1}^{\infty}\sum_{j=1}^{\infty} S_{m,j}(x) \int_{S^{n-1}} f(rx')S_{m,j}(x')\,dx',$$

we have

$$f(x) = \sum_{m=1}^{\infty}\sum_{j=1}^{\infty} S_{m,j}(x)\mathcal{M}^{-1}(\mathcal{M}_f^{m,j})(|x|). \tag{5.22}$$

It follows that (5.21) can be rewritten as follows.

$$\mathcal{M}_f^m(z) = \mathcal{M}_{f_m}(z). \tag{5.23}$$

Definition 5.3.5. *Given a function $f \in L_2(\mathbb{R}^n)$, we define an operator G^m of f as follows*

$$G_f^m(r) = r^{-m}f_m(r) = r^{-m} \int_{S^{n-1}} f(rx')S_m(x')\,dx'. \tag{5.24}$$

In most cases, we consider G_f^m to be a radial function on $\mathbb{R}^{n+2m}$ and we have the following lemma regarding the Fourier transform of G_f^m as a function in $\mathbb{R}^{n+2m}$.

Lemma 5.3.6. *Suppose that f is in $L_2(\mathbb{R}^n)$ and $g_m(x)$ is a radial function in $\mathbb{R}^{n+2m}$ such that $g_m(x) = G_f^m(|x|)$. Then*

$$\mathcal{F}(g_m)(y) = G_{\hat{f}}^m(|y|). \tag{5.25}$$

Proof. See Stein and Weiss [16]. □

Because $g_m(x)$ is a radial function on $\mathbb{R}^{n+2m}$,

$$\begin{aligned}
&\int_{\mathbb{R}^+} r^{n+2m-1} G_f^m(r)G_f^m(\lambda r)\,dr \\
&= \omega_{2n+m-1} \int_{\mathbb{R}^{n+2m}} g_m(x)g_m(\lambda x)\,dx \\
&= \omega_{2n+m-1} \int_{\mathbb{R}^{n+2m}} \int_{\mathbb{R}^{n+2m}} g_m(x)\hat{g_m}(y)e^{i\lambda\langle x,y\rangle}\,dx\,dy
\end{aligned} \tag{5.26}$$

Recall that the weaker version of the Beurling-Hedenmalm uncertainty principle assumes the following weaker inequality that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|f(x)\hat{f}(y)\right| e^{\lambda|x||y|}\,dx\,dy \lesssim \frac{1}{(1-\lambda)^N} \tag{5.27}$$

when $0 \le \lambda < 1$. To prove that f must be a polynomial times a gaussian if f satisfies the above inequality, we need to establish the relationship between the Beurling style inequality of f on $\mathbb{R}^n$ and the Beurling style inequality of $G_f^m(r)$ on $\mathbb{R}$.

Definition 5.3.7. *Given a positive number r, define*

$$\Omega_n(r) = \omega_{n-2}\int_0^1 e^{rs}(1-s^2)^{(n-3)/2}\,ds. \tag{5.28}$$

and

$$\psi^n(r) = \int_{S^{n-1}} e^{r|\langle w_x, w_y\rangle|}\,dw_y, \tag{5.29}$$

where w_x is a vectors on the unit sphere S^{n-1}.

Lemma 5.3.8. *Suppose that $r > 0$, and ψ^n, Ω_n are functions defined in Definition 5.3.7. Then*

$$\psi^n(r) = 2\Omega_n(r).$$

Proof. Notice that ψ is a function does not depends on the selection of w_x, we can evaluate the inner integral of ψ^n by first integrate over the parallel $L_\theta = \{w_y \in S^{n-1}; \langle w_x, w_y\rangle = \cos\theta\}$ orthogonal to w_x. Because $e^{r|\langle w_x, w_y\rangle|}$ is constant on L_θ and the measure of L_θ is $\omega_{n-2}(\sin\theta)^{n-2}$, we have

$$\psi^n(r) = \int_0^\pi \omega_{n-2}(\sin\theta)^{n-2}e^{r|\cos\theta|}\,d\theta.$$

Let $s = \cos\theta$, we have

$$\begin{aligned}\psi^n(r) &= \omega_{n-2}\int_{-1}^1 (1-s^2)^{(n-3)/2}e^{r|s|}\,ds \\ &= 2\omega_{n-2}\int_0^1 (1-s^2)^{(n-3)/2}e^{r|s|}\,ds \\ &= 2\Omega_n(r),\end{aligned} \tag{5.30}$$

as required. □

Lemma 5.3.9. *Suppose that $r > 0$, and ψ, Ω are functions defined in Definition 5.3.7. Then*

$$\Omega_{n+2m}(r) \le \frac{(2\pi)^m}{r^m}\Omega_n(r). \tag{5.31}$$

Thus by Theorem 5.3.8,

$$\psi^{n+2m}(r) \lesssim r^{-m}\psi^n(r). \tag{5.32}$$

Proof.

$$\begin{aligned}
\Omega_n(r) &= \omega_{n-2}\sum_{k=1}^{\infty}\int_0^1 \frac{(rs)^k}{k!}(1-s^2)^{(n-3)/2}\,ds \\
&= \omega_{n-2}\sum_{k=1}^{\infty} r^k \frac{\Gamma(1/2+k/2)}{\Gamma(k+1)}\frac{\Gamma(n/2-1/2)}{\Gamma(k/2+n/2)} \\
&= \omega_{n-2}\sum_{k=1}^{\infty} \frac{2^{-k}r^k}{\Gamma(k/2+1)}\frac{\Gamma(n/2-1/2)}{\Gamma(k/2+n/2)},
\end{aligned} \tag{5.33}$$

where the last equation follows from the fact $\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$ for all $z \in \mathbb{C}$. Thus, it follows that

$$\begin{aligned}
\Omega_{n+2m}(r) &= \omega_{n+2m-2}\sum_{k=1}^{\infty} \frac{2^{-k}r^k}{\Gamma(k/2+1)}\frac{\Gamma(n/2-1/2+m)}{\Gamma(k/2+n/2+m)} \\
&\leq \omega_{n+2m-2}\sum_{k=1}^{\infty} \frac{2^{-k}r^k}{\Gamma(k/2+m/2+1)}\frac{\Gamma(n/2-1/2+m)}{\Gamma(k/2+n/2+m/2)} \\
&\leq \omega_{n+2m-2}\sum_{j=m+1}^{\infty} \frac{2^m}{r^m}\frac{2^{-j}r^j}{\Gamma(j/2+1)}\frac{\Gamma(n/2-1/2+m)}{\Gamma(j/2+n/2)} \\
&\leq \omega_{n+2m-2}\frac{2^m}{r^m}\frac{\Gamma(n/2-1/2+m)}{\Gamma(n/2-1/2)}\sum_{j=m+1}^{\infty} \frac{2^{-j}r^j}{\Gamma(j/2+1)}\frac{\Gamma(n/2-1/2)}{\Gamma(j/2+n/2)} \\
&\leq \frac{\omega_{n+2m-2}}{\omega_{n-2}}\frac{2^m}{r^m}\frac{\Gamma(n/2-1/2+m)}{\Gamma(1/2-n/2)}\Omega_n(r) \\
&= \frac{(2\pi)^m}{r^m}\Omega_n(r),
\end{aligned} \tag{5.34}$$

as required. (Remark: The first inequality holds because

$$\Gamma(a)\Gamma(b+2c) \geq \Gamma(a+c)\Gamma(b+c),$$

when $0 < a < b$ and c is a positive integer.) □

Lemma 5.3.10. *Suppose that f is a function in $L_2(\mathbb{R}^n)$, S_m is a spherical harmonic of degree m and*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|f(x)\hat{f}(y)\right| e^{\lambda|x||y|}\,dx\,dy \lesssim \left(\frac{1}{1-\lambda}\right)^N, \tag{5.35}$$

for all $\lambda \in [0,1)$. Then

$$\int_{\mathbb{R}^{n+2m}}\int_{\mathbb{R}^{n+2m}} |g_m(x)\hat{g_m}(y)|\, e^{\lambda|\langle x,y\rangle|}\,dx\,dy \lesssim \left(\frac{1}{1-\lambda}\right)^N, \tag{5.36}$$

where $g_m(x)$ is a function in $\mathbb{R}^{n+2m}$ and $g_m(x) = G_f^m(|x|)$.

Proof. Let

$$\begin{aligned} I &= \int_{\mathbb{R}^{n+2m}} \int_{\mathbb{R}^{n+2m}} |g_m(x)\hat{g_m}(y)|\, e^{\lambda|\langle x,y\rangle|}\, dx\, dy \\ &= \frac{1}{\omega_{n+2m-1}} \int_{\mathbb{R}^+} r^{n+m-1}\, |f_m(r)| \\ &\qquad \int_{S^{n+2m-1}} \int_{\mathbb{R}^{n+2m}} |\hat{g}_m(y)|\, e^{\lambda r|\langle w_x,y\rangle|}\, dy\, dw_x\, dr \end{aligned} \tag{5.37}$$

Because $\hat{g}_m$ is a radial function, $\int_{\mathbb{R}^{n+2m-1}} |\hat{g_m}(y)|\, e^{\lambda r|\langle w_x,y\rangle|} dy$ is a function does not depends on w_x. Thus

$$\begin{aligned} I &= \int_{\mathbb{R}^+} r^{n+m-1}\, |f_m(r)| \left(\int_{\mathbb{R}^{n+2m}} |\hat{g_m}(y)|\, e^{\lambda r|\langle w_x,y\rangle|}\, dy \right) dr \\ &= \frac{1}{\omega_{n+2m-1}} \int_{\mathbb{R}^+} r^{n+m-1}\, |f_m(r)| \\ &\qquad \left(\int_{\mathbb{R}^+} s^{n+m-1} \left| \hat{f}_m(s) \right| \psi^{n+2m}(\lambda r s)\, ds \right) dr, \end{aligned}$$

By Lemma 5.3.9, $\psi^{n+2m}(\lambda rs) \lesssim (\lambda rs)^{-m} \psi^n(\lambda rs)$. Thus after putting this inequality back, we get

$$\begin{aligned} I &\lesssim \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} s^{n-1} r^{n-1} \left| f_m(r)\hat{f}_m(s) \right| \psi^n(\lambda rs)\, dr\, ds \\ &\lesssim \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} s^{n-1} r^{n-1} e^{\lambda rs} \\ &\qquad \int_{S^{n-1}} \int_{S^{n-1}} \left| f(rw_x) S(w_x) \hat{f}(sw_y) S(w_y) \right| dw_x\, dw_y\, dr\, ds \\ &\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| f(x)\hat{f}(y) \right| e^{\lambda|x||y|}\, dx\, dy \\ &\lesssim (\frac{1}{1-\lambda})^N, \end{aligned}$$

as required. $\square$

Lemma 5.3.11. *Suppose that f is a function in $L_2(\mathbb{R}^n)$, S_m is a spherical harmonic of degree m in $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| f(x)\hat{f}(y) \right| e^{\lambda|x||y|}\, dx\, dy \lesssim \left(\frac{1}{1-\lambda} \right)^N, \tag{5.38}$$

for all $\lambda \in [0, 1)$. Then

$$\int_{S^{n-1}} f(rx') S_m(x')\, dx' = 0 \tag{5.39}$$

and

$$\int_{S^{n-1}} \hat{f}(rx') S_m(x')\, dx' = 0 \tag{5.40}$$

when $N - n/2 < m$.

Proof. As in Lemma 4.4.2 and Lemma 5.3.11, we can conclude that

$$\int_{\mathbb{R}} r^{n+2m-1} G_f^m(r) G_f^m(\lambda r) dr \lesssim \left(\frac{1}{1-\lambda}\right)^N.$$

Thus by Lemma 5.1.2, result follows. □

Lemma 5.3.12. *Suppose that f is a function in $L_2(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| f(x) \hat{f}(y) \right| e^{\lambda |x||y|} \, dx \, dy \lesssim \left(\frac{1}{1-\lambda}\right)^N, \tag{5.41}$$

for all $\lambda \in [0,1)$. Then for each m, when $m \leq \lfloor N - n/2 \rfloor$, there exists a positive number a such that

$$\int_{S^{n-1}} f(rx') S_m(x') \, dx' = P_m(r) e^{-ar^2}, \tag{5.42}$$

$$\int_{S^{n-1}} \hat{f}(rx') S_m(x') \, dx' = Q_m(r) e^{-r^2/4a}, \tag{5.43}$$

where P_m and Q_m are polynomials of degree at most $\lfloor N - n/2 \rfloor$.

Proof. As in Lemma 4.4.2 and Lemma 5.3.11, we can conclude that

$$\int_{\mathbb{R}} r^{n+2m-1} G_f^m(r) G_f^m(\lambda r) dr \lesssim \left(\frac{1}{1-\lambda}\right)^N.$$

Thus by Lemma 5.1.2, we can conclude that there exists a positive number a such that

$$G_f^m(r) = P(r) e^{-ar^2} \tag{5.44}$$

where P is a polynomial of degree at most $\lfloor N - n/2 - m \rfloor$. By the definition of G_f^m we know there exists a polynomial P of degree at most $\lfloor N - n/2 - m \rfloor$ such that

$$\int_{S^{n-1}} f(rx') S_m(x') \, dx' = r^m P(r) e^{-ar^2}. \tag{5.45}$$

By a similar argument, we can conclude that for any S_m there exists a polynomial Q of degree at most $\lfloor N - n/2 - m \rfloor$ such that

$$\int_{S^{n-1}} \hat{f}(rx') S_m(x') \, dx' = r^m Q(r) e^{-r^2/4a}. \tag{5.46}$$

The lemma follows by the fact that $\lfloor N - n/2 - m \rfloor + m \leq \lfloor N - n/2 \rfloor$, when N,m are all integers. □

5.4 A weak uncertainty principle on $\mathbb{R}^n$

Now we are ready to prove the weak Beurling-Hedenmalm theorem on $\mathbb{R}^n$.

Theorem 5.4.1. *Suppose that $f \in L_2(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|f(x)\hat{f}(y)\right| e^{\lambda|x||y|}\, dx\, dy \lesssim \frac{1}{(1-\lambda)^N}, \tag{5.47}$$

for all $0 \le \lambda < 1$. Then there exists positive constant a such that $f(x) = P(x)e^{-ax^2}$ where $P(x)$ is a polynomial of degree at most $N - n/2$.

Proof. By Lemma 5.3.12 we know that for all m there exists P_m and Q_m and a_m such that

$$\int_{S^{n-1}} f(rx')S_m(x')\, dx' = r^m P_m(r)e^{-a_m r^2} \tag{5.48}$$

$$\int_{S^{n-1}} \hat{f}(rx')S_m(x')\, dx' = r^m Q_m(r)e^{-\frac{1}{4a_m}r^2} \tag{5.49}$$

where P_m and Q_m are polynomials of degree at most $\lfloor N-n/2-m\rfloor$. Also by Lemma 5.3.11 we get that if $m > N - \lfloor n/2\rfloor$, then $P_m = Q_m = 0$. So by reconstructing f via its projection to S_m we get the following.

$$f(x) = \sum_{m=0}^{\lfloor N-n/2\rfloor} f_m(|x|)S_m(x) = \sum_{m=0}^{\lfloor N-n/2\rfloor} P_m(|x|)e^{-a_m|x|^2}S_m(x) \tag{5.50}$$

and

$$\hat{f}(x) = \sum_{m=0}^{\lfloor N-n/2\rfloor} \hat{f}_m(|x|)S_m(x) = \sum_{m=0}^{\lfloor N-n/2\rfloor} Q_m(|x|)e^{-|x|^2/4a_m}S_m(x). \tag{5.51}$$

Denote $\lfloor N - n/2\rfloor$ by M. Then we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)f(y)|\, e^{\lambda|x||y|}\, dx\, dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\sum_i^M\sum_j^M S_i(x)S_j(y)P_i(x)Q_j(y)e^{\lambda|x||y|}e^{-a_i|x|^2-|y|^2/4a_j}\right| dx\, dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\sum_i^M\sum_j^M \frac{S_i(x)S_j(y)P_i(x)Q_j(y)}{e^{a_i(|x|-\lambda/2a_i)^2}e^{(1/4a_j-\lambda^2/4a_i)|y|^2}}\right| dx\, dy
\end{aligned} \tag{5.52}$$

Because P_i and Q_j are of finite degree, the above integral converges if and only if

$$\lim_{\lambda\to 1}\max_{i,j}\left(\frac{1}{a_j} - \frac{\lambda^2}{a_i}\right) \ge 0. \tag{5.53}$$

Thus $a_j = a_i$. So there exists a positive constant a such that

$$f(x) = \sum_{m=0}^{\lfloor N-n/2 \rfloor} P_m(|x|)e^{-a|x|^2} S_m(x), \tag{5.54}$$

as required. $\square$

Chapter 6

Connection to the moment problem

In this chapter we are going to develop a generalized result about the moment problem.

6.1 Introduction

Suppose that μ is a measure on the interval $I \subseteq \mathbb{R}$. Then the nth moment of μ on I is defined by

$$\mathcal{M}_\mu(n) = \int_I x^n \, d\mu. \tag{6.1}$$

For a positive sequence $\{s_n\}$, if there exists a positive measure μ such that $\mathcal{M}_\mu(n) = s_n$ for all n on I, then we say that μ is a solution to the moments $\{s_n\}$ on I.

Definition 6.1.1. *We denote by $\mathcal{M}_{(\cdot,I)}(n) = s_n$ the classical moment problem of finding a positive measure μ that has moments $\{s_n\}$ on I.*

If there is only one solution to the classical moment problem $\mathcal{M}_{(\cdot,I)}(n) = s_n$, then we say this moment problem is determinate. Otherwise we call this moment problem indeterminate.

The classical moment problem was studied in great depth by Akhiezer and Kemmer [1]. In this chapter we give a quick overview of the standard moment problem, and we notice that most of the results in the classical moment problems are about the moments of a positive measure μ with its moments on $I \subseteq \mathbb{R}$. By comparing the definition of the moments and the Mellin transform $\mathcal{M}_f^k$, we notice that the nth moment $\mathcal{M}_\mu(n)$ of an absolutely continuous measure μ is just the value of $\mathcal{M}_f(z)$ at $z = n + 1/2$, where f is the density function of μ. Thus a natural problem is raised: whether we can replace the nth moment $\mathcal{M}_\mu(n)$ by $\mathcal{M}_f(z_n)$ in the classical problem and get similar results.

In this chapter we discuss the following generalized moment problem regarding the Mellin transform of f.

Definition 6.1.2. *Suppose that $\{z_n : n = 0, 1, \cdots\}$ and $\{m_n : n = 0, 1, \cdots\}$ are two sequences in $\mathbb{C}$. If there exists a function f defined on $\mathbb{C}$ such that $\mathcal{M}_f(z_n) = m_n$, then we say f is a solution to the moment problem $\mathcal{M}_{(\cdot,\mathbb{C})}(z_n) = m_n$. If this solution is unique, then we say that $\mathcal{M}_{(\cdot,\mathbb{C})}(z_n) = m_n$ is determinate. Otherwise we say that $\mathcal{M}_{(\cdot,\mathbb{C})}(z_n) = m_n$ is indeterminate.*

It is worth mentioning that in the above definition we do not require f to be a density function of some measure μ.

6.2 Review of the moment problem

Here we recall three classical moment problems, by considering three different types of closed intervals $I \subseteq \mathbb{R}$. When $I = \mathbb{R}$, $\mathcal{M}_{(\cdot,\mathbb{R})}(n) = m_n$ is called the Hamburger moment problem. When $I = \mathbb{R}^+$, $\mathcal{M}_{(\cdot,\mathbb{R}^+)}(n) = m_n$ is called the Stieltjes moment problem. When $I = [0, 1]$, $\mathcal{M}_{(\cdot,[0,1])}(n) = m_n$ is called the Hausdorff moment problem. Following are some basic observations.

- A positive measure with finite support is determined by its moments.
- The Hausdorff moment problem is determinate. More generally moment problems associated to positive measures with compact support are determinate.

Lemma 6.2.1 (Carleman)**.** *Suppose that $0 \le m_n \lesssim R^n n!$, where $R > 0$. Then the Hamburger moment problem $\mathcal{M}_{(\cdot,\mathbb{R})}(n) = m_n$ is determinate. Also if there exists R such that $0 \le m_n \lesssim R^n(2n)!$, then the Stieltjes moment problem $\mathcal{M}_{(\cdot,\mathbb{R}^+)}(n) = m_n$ is determinate.*

Proof. See Shohat and Tamarkin [14]. See this reference for more details about Carleman's condition. $\square$

Lemma 6.2.2 (Krein)**.** *Suppose that μ is a solution of the Hamburger moment problem $\mathcal{M}_{(\cdot,\mathbb{R})}(n) = m_n$ and*

$$\int_{\mathbb{R}} \frac{\log f(x)}{1 + x^2}\, dx < \infty, \tag{6.2}$$

where f is the density function of μ. Then $\mathcal{M}_{(\cdot,\mathbb{R})}(n) = m_n$ is indeterminate. Also if μ is a solution of the Stieltjes moment problem $\mathcal{M}_{(\cdot,\mathbb{R}^+)}(n) = m_n$ and

$$\int_{\mathbb{R}^+} \frac{\log f(x)}{1 + x} \frac{\sqrt{x}}{x}\, dx < \infty, \tag{6.3}$$

where f is the density function of μ, then $\mathcal{M}_{(\cdot,\mathbb{R}^+)}(n) = m_n$ is indeterminate.

Proof. See Simon [15]. $\square$

Here is an example of an indeterminate moment problem.

Lemma 6.2.3. *The Stieltjes moment problem of finding a positive measure μ with given moments m_n such that*

$$m_n = \int_{\mathbb{R}^+} x^{k/4} e^{-x^{1/4}} x^n \, dx$$

where $(0 \leq k < 4)$ is indeterminate.

Proof. This can be easily proved by Krein's Condition (see Stoyanov [17]). For simplicity we prove the lemma by showing that

$$\left(1 + \frac{1}{2}\sin(x^{1/4})x^{-k/4}\right) x^{k/4} e^{-x^{1/4}} \tag{6.4}$$

has the same moment sequence and is positive. It is equivalent to observe that

$$\begin{aligned}
&\int_0^\infty x^n \sin(x^{1/4}) e^{-x^{1/4}} \, dx \\
&= \int_0^\infty 4x^{4n+3} \sin(x) e^{-x} \, dx \\
&= \frac{4}{2i} \int_0^\infty x^{4n+3} e^{-(1-i)x} \, dx - \int_0^\infty x^{4n+3} e^{-(1+i)x} \, dx \\
&= \frac{4}{2i} \left((1-i)^{-4(n+1)} - (1+i)^{-4(n+1)}\right) \int_0^\infty x^{4n+3} e^{-x} \, dx \\
&= \frac{4}{2i} \left((-4)^{1-n} - (-4)^{1-n}\right) \Gamma\left(4(n+2)\right) \\
&= 0.
\end{aligned} \tag{6.5}$$

In addition to (6.5), we also have

$$\left|\frac{1}{2}\sin(x^{1/4})x^{-k/4}\right| < 1,$$

for all $x \geq 0$, which means that $\left(1 + \frac{1}{2}\sin(x^{1/4})x^{-k/4}\right) x^{k/4} e^{-x^{1/4}}$ is still positive on $\mathbb{R}^+$. $\square$

Suppose that u is a positive measure on $\mathbb{R}$. Then we can construct a sequence of polynomials $\{p_k\}$ based on μ the via Gram-Schmidt process as follows:

$$
\begin{aligned}
&q_0(x) = \frac{1}{\int_I d\mu}, \quad p_0 = \frac{q_0}{\langle q_0, q_0\rangle} \\
&q_1(x) = x - \langle x, p_0\rangle p_0, \quad p_1 = \frac{q_1}{\langle q_1, q_1\rangle} \\
&q_2(x) = x^2 - \langle x^2, p_1\rangle p_1 - \langle x^2, p_0\rangle p_0, \quad p_2 = \frac{q_2}{\langle q_2, q_2\rangle} \\
&\quad \cdots \\
&q_n(x) = x^n - \sum_{k=0}^{n-1} \langle x^n, p_k\rangle p_k, \quad p_n = \frac{q_n}{\langle q_n, q_n\rangle}.
\end{aligned} \tag{6.6}
$$

It follows that p_n is of degree n and

$$
\begin{aligned}
\int_I p_n^2(x)\, d\mu(x) = 1, &\quad \text{thus } \{p_n\} \text{ are normalized.} \\
\int_I p_i(x) p_j(x)\, d\mu(x) = 0, &\quad \text{thus } \{p_n\} \text{ are orthonormal.}
\end{aligned} \tag{6.7}
$$

Once the sequence $\{p_n\}$ is decided, we can define two closely related matrices, $\mathbf{P}$ and $\mathbf{M}$ related to the sequence of moments $m_n = \mathcal{M}_\mu(n)$ as follows.

Definition 6.2.4. *Suppose that $\{m_n\}$ is the sequence of moments of some positive measure and $\{p_n\}$ is the normalized sequence of orthonormal polynomials constructed via Gram-Schmidt procedure defined in* (6.6). *Then for all $p_n(x)$ there exists a sequence $\{c_{n,i}\}$ such that*

$$
p_n(x) = \sum_{j=0}^{n} c_{n,j} x^j. \tag{6.8}
$$

We define the infinite matrix $\mathbf{P}$ to be the matrix with elements $c_{i,j}$ in the i,j position. Also we define the Hankel matrix $\mathbf{M}$ to be the matrix with elements m_{i+j} in the i,j position.

From the definition of the infinite matrices $\mathbf{P}$ and $\mathbf{M}$, we can verify that $\mathbf{PMP}^T = \mathbf{I}$. Thus $\mathbf{M}$ is uniquely decided by $\mathbf{P}$. So it is natural to expect that properties of $\{p_n\}$ imply the determinism of $\mathbf{M}$. In fact we have the following (see Simon [15] for stronger theorems).

Theorem 6.2.5. *If the Stieltjes moment problem $\mathcal{M}_{(\cdot,\mathbb{R}^+)}(n) = m_n$ is indeterminate, u is one of its solution and p_k is the normalized sequence of orthonormal polynomials constructed via the Gram-Schmidt procedure related to u, then for any $z \in \mathbb{C}$, the sequence $p_n(z)$ is in l^2.*

Proof. See Simon [15]. □

Here we present a quicker way to show the inverse of an indeterminate moment matrix must have bounded entries on its diagonal.

Lemma 6.2.6. *Suppose that* $\mathbf{M}$ *is a infinite Hankel matrix whose elements are moments of a measure* μ*. Then* $(\mathbf{M}^{-1})_{k,k}$ *are bounded if the moment sequence of* u *is indeterminate.*

Proof. Suppose that $p_i(x) = \sum P_{i,j}x^j$ are the normalized orthonormal polynomials related to μ . Then $\mathbf{PMP}^T = \mathbf{I}$ which means $\mathbf{M}^{-1} = \mathbf{P}^T\mathbf{P}$. So

$$\begin{aligned}(\mathbf{M}^{-1})_{k,k} = \sum_n^\infty C_{n,k}^2 =& \frac{1}{2\pi}\sum_n^\infty \left(\int_0^{2\pi} p_n(e^{i\theta})e^{-ik\theta}d\theta\right)^2 \\ \leq& \frac{1}{2\pi}\sum_n^\infty \left(\int_0^{2\pi} |p_n(e^{i\theta})|d\theta\right)^2 \\ \leq& \sum_n^\infty \left(\int_0^{2\pi} |p_n(e^{i\theta})|^2 d\theta\right) \\ =& \int_0^{2\pi} \left(\sum_n^\infty |p_n(e^{i\theta})|^2 d\theta\right).\end{aligned} \tag{6.9}$$

Because μ is indeterminate, $\sum_n |p_n(z)|^2$ converges on the whole complex plane (see Simon [15]). Thus the above term is bounded. □

It is worth mention that the above lemma is also implied by Berg, Chen and Ismail [2], who proved a strong theorem saying that the minimal eigenvalue of M converges to a strict positive number.

6.3 A generalized moment problem

By the definition of Mellin transform, we notice that a necessary condition for a moment problem $\mathcal{M}_{(\cdot,\mathbb{C})}(z_n) = m_n$ to be determinate is that $\mathcal{M}_f^k$ being uniquely decided by the moments m_n. However $\mathcal{M}_f^k$ is not necessarily an analytic function thus we can not use the argument in Lemma 2.6.1. So we need to build a analytic function from $\mathcal{M}_f^k$, and again we use the Θ transform of f to achieve this. In Chapter 4 we have showed that $\Theta_f^k(z)$ is of order one when f and $\hat{f}$ are of exponential decay. Thus we claim that f can be determined by $\Theta_f^k(z_n)$ where $\{z_n\}$ is a sequence of complex numbers whose modulus are of certain linear growth.

Suppose both $f \in L_2(\mathbb{R})$ and $\hat{f} \in L_2(\mathbb{R})$ are of exponential decay at infinity. Then, in Chapter 4, we have showed that Θ_f^k can be extended to the whole complex plane and is of order one. Thus by the Hadamard factorization theorem (see Theorem (2.6.5)), we can give out a very specific form of Θ_f.

Theorem 6.3.1. *Suppose that there exist positive numbers α and β such that $f(x) \lesssim e^{-\alpha x^2}$ and $\hat{f}(\xi) \lesssim e^{-\beta x^2}$. Then there exist a complex sequence $\{z_n : n = 0, 1, \cdots\}$ such that*

$$\Theta_f(z) = z^m e^{az+b} \prod_{n=0}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n}\right) \tag{6.10}$$

Proof. This follows from the Hadamard factorization theorem and Theorem 4.3.3. □

Now we are ready to conclude the final result for this chapter. It tells what kind of point sequence $\{z_n\}$ on the complex plane can be picked such that the values of $\mathcal{M}_f$ on these points uniquely decide a function f that satisfies Hardy's condition.

Theorem 6.3.2. *Suppose that f and $\hat{f}$ are of exponential decay and a sequence of nonzero complex numbers $\{z_n : n = 0, 1, \cdots\}$ that satisfies the condition*

$$\sum_{n=0}^{\infty} \frac{1}{|z_n|^2} = \infty. \tag{6.11}$$

Then the value of $\mathcal{M}_f^k(z_n)$ uniquely decides the function f.

Proof. Suppose that f, $\hat{f}$, g and $\hat{g}$ are of exponential decay, and $\mathcal{M}_f(z_k) = \mathcal{M}_g(z_k)$. Then both $f - g$ and $\hat{f} - \hat{g}$ are of gaussian decay as well. By Theorem 6.3.1 we know that if Θ_{f-g} is not constantly zero then

$$\Theta_{(f-g)}(z) = z^m e^{az+b} \prod_{k=0}^{\infty} (1 - \frac{z}{v_k}) \exp(\frac{z}{v_k}) \prod_{n=0}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n}\right) \tag{6.12}$$

where z_n and v_k are zeros of $\Theta_{(f-g)}(z)$. Thus

$$\sum_{n=0}^{\infty} (r/z_n^2) + \sum_{k=0}^{\infty} (r/v_k^2) < \infty$$

for all $r > 0$. Thus it contradicts the assumption that $\sum_{n=0}^{\infty}(1/z_n^2) = \infty$. It follows that $\Theta_{(f-g)}(z) = 0$ for all z in the complex plane. So $f = g$ as required. □

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