Levy stochastic differential geometry with applications in derivative pricing

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Publication Date:
2010

DOI:
https://doi.org/10.26190/unsworks/4818

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PhD Thesis

Lévy Stochastic Differential Geometry with Applications in Derivative Pricing

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Supervisor: Professor A.H. Dooley

September 20, 2010

Acknowledgements

I enjoyed the help and support from numerous friends and faculty members during whilst writing this thesis. My greatest debt however, goes to my supervisor, Professor Tony Dooley, who initiated me to the fantastic field of stochastic analysis. With his incredible breadth and depth of knowledge and intuition, he guided me to structure and clarify my thought and suggested valuable insightful comments. This thesis would not have been possible to write without his help and support.

I am also indebted to express my sincere thankfulness to A/Prof Ben Goldys, A/Prof Marek Rutkowski, who were very kind and supportive to us during the year. Prof Jerzy Zabczyk also deserves a special mention, for the recommendation of a number of useful references, and I very much enjoyed the discussions I had with him during my visit to Warsaw. I would also like to thank Prof Nolan Wallach for the inspirational dialogue exchanges at the 2007 ICE-EM Australian Graduate Winter School in Brisbane.

Finally, I would like to thank everyone in the School of Mathematics and Statistics, UNSW and ARC Centre of Excellence for Mathematics and Statistics of Complex Systems (MASCOS) for the support over the past three years. And most important of all, my beloved fiancee, Annie, for all the encouragement and support she gave at all times.

Han Zhang, November 2007.

Dedication

To Annie, with Love.

Preface

A defining characteristic of Lévy processes on $\mathbb{R}^d$ is the property of independent increments [49], which implies that the associated probability distribution is a convolution power of another probability distribution. This is known as the infinitely divisible property of stochastic analysis, and acts as a bridge to connect stochastic analysis with harmonic analysis.

If we replace $\mathbb{R}^d$ by a general Riemannian manifold, it is not straightforward to see what is the correct notion of a Lévy process, as there is not usually a notion of convolution nor of harmonic analysis. However, if the manifold is a symmetric space, both these features are available.

In particular, in the case of a Riemannian symmetric space $\mathbb{M} = G/K$, Gangolli (c.f. [25] and [26]) was able to completely classify the family of spherically symmetric infinitely divisible measures, and recovered the Lévy-Khintchine formula for spherical Lévy processes on $\mathbb{M}$.

Gangolli's construction could not be possibly be extended to a fully fledged Lévy-Khintchine formula for a general Lévy process at the time, as the theory of non-commutative harmonic analysis was inadequate. Geometric and harmonic analysis have undergone a quantum leap in the past forty years (c.f. [28] , [29] , [30]), and the tools to generalize Gangolli's results are now available. We note that there is already a significant amount of progress made in extending the results of Hunt and Gangolli in other directions, such as, to locally compact groups (c.f. [32] , [5], [12]) and to hypergroups (c.f. [11]).

Another way of interpreting an $\mathbb{M}$-valued Lévy process was initiated by Applebaum in [3] , where the author attempted to generalize the Eels-Elworhty construction in [20] of a Brownian motion to a Lévy process. However, it was found in [3] that the resulting process was in general not a Markov process. Following [4] and

[6], it is gradually understood that the Eels-Elworthy construction will only give a Markov process when the underlying Lévy process is spherical.

This aim of this thesis is to apply the more recent tools of geometric and harmonic analysis to solve the puzzle that was outstanding for more than forty years. More specifically, we will

1. Compute a new Lévy-Khintchine formula for general Lévy processes using Eisenstein integrals.
2. Classify the family of general infinitely divisible measures on the Riemannian symmetric space G/K.
3. Provide some ideas on future directions of research, such as extending the present construction to pseudo-Riemannian manifolds.

Finally, we will demonstrate the importance of manifold-valued stochastic processes to the modeling of market volatility. It is known that the volatility structure of financial market is non-flat, as observed by the volatility smile and the volatility surface. In the final chapter of this thesis, we argue that the volatility structure only appears to non-flat: Under a "uniform flow of information", it is the asset dynamics themselves that are evolving under a non-flat background. We conclude by showing how the new Lévy-Khintchine formula we derived can be applied to option pricing, when the underlying asset dynamics is evolving on a manifold.

The structure of the thesis is as follows.

Chapter 1 is an introduction to basic concepts in stochastic analysis. It is only intended to be a reference.

Chapter 2 is a background chapter to manifold theory and stochastic differential geometry. It ends with Applebaum's construction of a horizontal Lévy process on a manifold, and showing how the resulting process fails to be Markov.

Chapter 3 is another background chapter to Lie groups and Lévy processes on Lie groups. The ideas developed in Chapter 3, especially the Hunt's formula and the Lévy-Itô decomposition, are central to later chapters.

Chapter 4 illustrates how one could compute the spherical Lévy-Khintchine formula from Harish-Chandra's theory of spherical harmonic analysis.

Chapter 5 illustrates how the techniques in chapter 4, combined with the more recently developed theory of Eisenstein integrals can lead to a general Lévy-Khintchine formula on a Riemannian symmetric space. We then explore some important consequences of this formula, such as a classification of infinitely divisible measures on G/K.

Chapter 6 attempts to apply the theory developed in chapter 5 to option pricing and the modeling of implied volatility.

Notation

The subject of study in this thesis overlaps stochastic analysis, differential geometry, harmonic analysis and representation theory. To avoid confusion, I will once and for all fix

Spaces:

- E - Topological space.
- G - Lie group.
- $\mathfrak{g}$ - Lie algebra.
- G/H - Affine symmetric space.
- G/K - Symmetric space
- K - Compact subgroup of G
- $\mathbb{M}$ - Manifold
- $(\Omega, \mathcal{F}, \mathbb{P})$ - The default probability space.

Function spaces:

- $B(E)$ - Borel measurable functions on E.
- $B_b(E)$ - Bounded Borel measurable functions on E
- $C(E)$ - continuous functions on E.
- $C^k(E)$ - k times differentiable functions on E.
- $C^\natural(\mathbb{M})$ - space of spherically symmetric continuous functions on $\mathbb{M}$
- $L^p(E)$ - functions with $\int |f|^p d\mu < \infty$.
- $\mathcal{M}(E)$ - spaces of all measures on E.
- $\mathcal{M}(E)$ - spaces of all probability measures on E.
- $\mathcal{M}^\natural(\mathbb{M})$ - space of spherically symmetric measures on $\mathbb{M}$.

Contents

Chapter 1

Lévy Processes

In this chapter, we examine the basic facts about infinitely divisible random variables and Lévy processes. First, we will define the Lévy process and show its connection with infinitely divisible random variables, and then present the classical Lévy-Khintchine representation and Lévy-Itô decomposition formulae. These objects form the basic building blocks of any Lévy theory, and these are all well known in the literature.

The first section will give a brief survey of results for general stochastic processes with càdlàg paths following the spirit of chapter 1 of [50] and [47]. The concept of Lévy processes will be introduced in the second section in accordance with [2] (chap 1, 2), [10] (chap 1) and [49] (chap 2), for the case when $E = \mathbb{R}^d$. This will give us an intuitive idea of the kind of random variables and stochastic processes that we are interested in studying.

Readers are highly recommended to read [47] for a general theory of stochastic analysis and semimartingales with processes of càdlàg paths.

1.1 Stochastic processes

This section contains a whirlwind tour of general stochastic processes and stochastic analysis, for processes with càdlàg paths.

1.1.1 Stochastic Processes

For a separable topological space $(E, \mathcal{O}(E))$, the symbol $\mathcal{B}(E)$ will mean the **Borel σ-algebra** of E. That is, the σ-algebra of E generated by the open sets of $\mathcal{O}(E)$. We complete the σ-algebra with respect to some measure μ by adding all subsets of sets N with $\mu(N) = 0$ to the σ-algebra, when there is no risk of confusion (that is, when all the measures we are interested in are absolutely continuous with respect to one another).

Definition 1.1.1. An E-valued **random variable** is a measurable map, $X : \Omega \to E$. That is, we require $X^{-1}(A) \in \mathcal{F}$, for every $A \in \mathcal{B}(E)$.

Remark 1.1.2. Let E' be another topological space, not necessarily distinct from E, and $f : E \to E'$ be a Borel measurable map. Then the composed map $f(X) = f \circ X : \Omega \to E'$ is a E'-valued random variable.

Definition 1.1.3. If E is equipped with a preference relation $\succ$, we will declare two random variables X and Y to have the relation "$X \succ Y$" if $X(\omega) \succ Y(\omega)$ holds $\mathbb{P}$-a.s.

When we deal with random variables, very rarely we end up working with the maps $X : \Omega \to E$ themselves. In general, this is due to the lack of structure of Ω. We introduce the concept of the distribution of a random variable.

Definition 1.1.4. Let X be an E-valued random variable. The **induced measure** or the **distribution** of X is a probability measure defined on E, denoted by μ_X, such that $\mu_X(A) = (\mathbb{P} \circ X^{-1})(A))$, for every $A \in \mathcal{F}$.

We will declare two measures μ and ν to be equal if,

- μ and ν are defined on the same measurable space $(\Omega, \mathcal{F})$.
- $\mu(A) = \nu(A)$ for every $A \in \mathcal{F}$.

Definition 1.1.5. ([49], pp. 7) Let $\{\mu_k\}$, $k = 1, 2, ...$ be a sequence of probability measures on E.

- Let $f \in B_b(E)$, then we write $\mu(f)$ to mean $\int_E f(x) d\mu(x)$.
- We say $\{\mu_k\}$ **converges** or **converges strongly** to a probability measure μ, written as $\mu_n \to \mu$ as $n \to \infty$ if for every $f \in B_b(E)$, $\mu_n(f) \to \mu(f)$.
- We say $\{\mu_k\}$ **converges weakly** to μ, $\mu_n(f) \to \mu(f)$ for each $f \in B_c(E)$.
- We say $\{\mu_k\}$ is **Bernoulli convergent** to μ, $\mu_n(f) \to \mu(f)$ for each bounded $f \in B_b(E)$.

Remark 1.1.6. It is also common to write $\mu_X(A) = \mathbb{P}(X \in A)$, where $A \subset \mathcal{B}(E)$. Here, A can be understood as the image of A under X in Definition 1.1.2. Without further warning, we will be using both of these interchangably.

Definition 1.1.7. The **expectation** of a random variable, X, with respect to a probability measure $\mathbb{P}$ is defined by,

$$\mathbb{E}X = \int_\Omega x d\mathbb{P}(x).$$

Remark 1.1.8. One may recover the induced measure of X from the expectation by noting that $\mathbb{E}\mathbf{1}_A(X) = \mu_X(A)$.

Let μ_X be the induced measure of a random variable X defined on a measurable space $(E, \mathcal{B}(E))$ and f is a $\mathcal{B}(E)$-measurable function. We will use $\mu_X(f)$ and the probabilistic notation $\mathbb{E}(f(X))$ interchangably without further warning to mean $\int_E f(x)d\mu(x)$, whenever this integral exist.

Definition 1.1.9. Let X be a random variable with $\mathbb{E}|X| < \infty$, and $\mathcal{G} \subseteq \mathcal{F}$. The **conditional expectation** of X on $\mathcal{G}$, written as $\mathbb{E}(X|\mathcal{G})$, is a random variable that satisfies

- $\mathbb{E}(X|\mathcal{G})$ is $\mathcal{G}$-measurable.
- For every $A \in \mathcal{G}$,

$$\int_A \mathbb{E}(X|\mathcal{G})d\mathbb{P} = \int_A X d\mathbb{P}.$$

Remark 1.1.10. It is well known that the conditional expectation exists and is unique up to the $\mathbb{P}$-a.s. equivalence of random variables (c.f. [2], p. 9). That is, if both Y and Z satisfies the conditional expectation definition for $\mathbb{E}(X|\mathcal{G})$, then $Y = Z$ almost surely. Readers are referred to Appendix A of [44] for the details.

Stochastic processes are mathematical models of time evolution of random phenomena. Therefore, we need the concept of a filtered probability space to model the flow of information with respect to time. This inspires the following definition of a stochastic basis.

Definition 1.1.11. A **stochastic basis**, denoted by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is a probability space equipped with a **filtration** $\mathcal{F}_t$, a sequence of σ-algebras $\{\mathcal{F}_t\}_{0\leq t\leq\infty}$, such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ whenever $s \leq t$.

It is convenient to assume that the filtration is *right continuous* and *complete*. More precisely, this means

- $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$.
- $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s, \forall t \geq 0$.

Definition 1.1.12. Let $(E, \mathcal{B}(E))$ be a measurable space. An E-valued **stochastic process** defined over $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a family of maps, $X = \{X_t\}_{t\in\mathbb{R}_+}$. If in addition, the maps $X_t : \Omega \to E$ are $\mathcal{F}_s$-measurable for every $s \leq t$, we say $\{X_t\}$ is an $\mathcal{F}_t$-**adapted** process.

Definition 1.1.13. Let I be a set. A filtration $\{\mathcal{F}_t\}$ is said to be **generated** by a collection of processes $\{X_t^\alpha\}$, $\alpha \in I$ if $\mathcal{F}_t = \sigma(\{X_u^\alpha, 0 \leq u \leq t\})$. That is, $\mathcal{F}_t$ is the smallest σ-algebra such that X_u^α are $\mathcal{F}_t$-measurable for every $\alpha \in I$ and every $0 \leq u \leq t$.

The filtration generated by a single process is called the **natural filtration** of that process. A process X_t will always be adapted to its own filtration, $\sigma(X_t)$.

Definition 1.1.14. (c.f. [47], p. 7) Let $\{M_t\}$ be a stochastic process defined on a $\mathbb{R}^d$, we say M_t is a **martingale** with respect to $\mathcal{F}_t$ if for each $t \geq 0$,

- M_t is $\mathcal{F}_t$-adapted.
- $\mathbb{E}(|M_t|) < \infty$
- $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$, for every $s \leq t$.

If in addition, we have $\mathbb{E}(|M_t|^2) < \infty$, for every $t \geq 0$, we call M_t a **square-integrable** martingale.

Definition 1.1.15. A $\{\mathcal{F}_t\}$-**stopping time** τ is a random variable taking values in $\mathbb{R}^+ \cup \{\infty\}$, such that the set $\{\tau \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$.

For a stopping time τ,

$$\mathcal{F}_\tau = \{B \in \mathcal{F}_\infty | B \cap \{\tau \leq t\} \in \mathcal{F}_t \forall t \geq 0\}$$

is a σ-algebra, and $\mathcal{F}_\tau = \mathcal{F}_t$ if $\tau = t$. Moreover, if $\tau \leq \sigma$ are two stopping times, then $\mathcal{F}_\tau \subset \mathcal{F}_\sigma$.

An important class of stochastic processes is the class of Markov processes. Lévy processes, and solutions to stochastic differential equations driven by Lévy processes generally belong to this class. In this thesis, we are interested in studying càdlàg Markov process on E. It is convenient to further assume that E is metrisable by a complete metric (i.e., we assume E is a Polish space), as convergence issues will play an important role.

Definition 1.1.16. ([21] , p. 521) Let E be a Polish space and ρ be its metric, a function $\varphi : [0,\infty) \to \Psi$ is **càdlàg**(continu à droite, limites à gauche) if it is right continuous and the limit

$$\lim_{s \uparrow t} \phi(t)$$

exists for all $t \in (0,\infty)$. A **càdlàg process** is a random process whose trajectory, viewed as a function in time, is a càdlàg function. Similarly, a function $\varphi : [0,\infty) \to E$ is **càglàd** if it is left continuous with left limit, and a process is càglàd if its sample paths are càglàd functions in time.

Definition 1.1.17. A process X_t taking values in E is called a **Markov process** if for every $f \in B_b(E)$, and for every $0 \leq s \leq t$, $\mathbb{E}(f(X_t)|\mathcal{F}_s) = \mathbb{E}(f(X_s) \mid X_s)$.

With each Markov process X, we associate a family of operators $\{T_{s,t}, 0 \leq s \leq t\}$, mapping from $B_b(E)$ to the Banach space of all bounded functions on E, by

$$(T_{s,t}f)(x) = \mathbb{E}(f(X_t)|X_s = x)$$

for each $f \in B_b(E), x \in E$.

Definition 1.1.18. We say the Markov process X is **normal** if $T_{s,t}(B_b(E)) \subseteq B_b(E)$, for each $0 \leq s \leq t$.

The following theorem summarises the key results about the operators $T_{s,t}$. Note that only property (3) requires the Markov property; all other properties of $T_{s,t}$ hold for general stochastic processes.

Theorem 1.1.19. *([2], p. 121) If X is a normal Markov process, then*

- $T_{s,t}$ *is a linear operator on* $B_b(E)$ *for each* $0 \leq s \leq t$.
- $T_{s,s}f = f$ *for each* $s \geq 0$ *and* $f \in B_b(E)$.
- $T_{s,t}T_{t.u} = T_{s,u}$ *whenever* $0 \leq s \leq t \leq u$.
- $f \geq 0 \Rightarrow T_{s,t}f \geq 0$ *for all* $0 \leq s \leq t, f \in B_b(E)$.
- $T_{s,t}$ *is a contraction, i.e.* $||T_{s,t}f||_{B_b(E)} \leq ||f||_{B_b(E)}$ *for each* $0 \leq s \leq t$ *and* $f \in B_b(E)$.
- $T_{s,t}(1) = 1, \forall t \geq 0$.

The situation simplifies if we assume X is **time homogeneous**, that is when $T_{s,t} = T_{0,t-s}$ for all $0 \leq s \leq t$. An alternative notation of $T_{0,t-s}$ for time homogeneous processes is to simply write $T_{t-s} = T_{s,t} = T_{0,t-s}$.

Definition 1.1.20. Let $(\mathcal{E}, \mathcal{A})$ be a measurable space. A **probability kernel** on $(\mathcal{E}, \mathcal{A})$ is a map $K : \mathcal{E} \times \mathcal{A} \to \mathbb{R}_+$, such that $K(x, .)$ is a probability measure for each fixed $x \in \mathcal{E}$, and $x \mapsto K(x, A)$ is $\mathcal{A}$-measurable for each fixed $A \in \mathcal{A}$.

Notice that for an $\mathcal{A}$-measurable function f, we may write,

$$Kf(x) = \int_{\mathcal{E}} f(y)K(x, dy)$$

and therefore think of K as an operator on some function space, where $K(f)$ makes sense (e.g. the bounded Borel functions on E).

Definition 1.1.21. A **semigroup** of probability kernels on $(\mathcal{E}, \mathcal{A})$ is a set of probability kernels $\{P_t\}_{t \geq 0}$, which in addition, satisfies

- $P_0(x, .) = \delta_x$ (identity element property), and
- $P_{s+t} = P_t P_s$ (semigroup property)

It can be easily checked that when X is a homogeneous Markov process, $(T_t f)(x, A) = P_t(x, A)$ for $f(x) = \mathbf{1}_A(x)$.

Remark 1.1.22. The transition semigroup is useful when studying Markov processes, as the distribution of a Markov process is determined completely by the transition semigroup P_t and the initial distribution ν. One obtains,

$$\mathbb{P}(f_1(X_{t_1}) \in A_1, ..., f_n(X_{t_n}) \in A_n) = \prod_{k=1}^{n} \int_{f^{-1}(A_k)} f(z_k) P_{t_k - t_{k-1}}(x, dz_k)$$

where the f_k's are integrable.

An important class of Markov processes comprises the Feller processes. Let E be a locally compact Hausdorff space with a countable base of open sets. Let $C_0(E)$ be the space of continuous functions on E vanishing at ∞ is a Banach space under the sup-norm, $||f|| = \sup_{x \in E} |f(x)|$.

Definition 1.1.23. A semigroup of probability kernels P_t on E is called a **Feller semigroup** if

- $C_0(E)$ is invariant under P_t, that is $P_t f \in C_0(E)$ for $f \in C_0(E)$ and $t \in \mathbb{R}_+$.

- $P_t f \to f$ in $C_0(E)$ as $t \to 0$.

Given such a semigroup and a probability measure ν on E, there is a càdlàg Markov process Z_t with P_t as transition semigroup and ν as the initial distribution. Such a process is called a **Feller process**. We now wish to define a linear operator A for which "$P_t = e^{tA}$" can be meaningful. In general, we expect A to be unbounded, and hence we need to first define its domain D_A.

Definition 1.1.24. Suppose $\{P_t\}_{t\geq 0}$ is an a semigroup of probability kernels in the Banach space $C_0(E)$. We define

$$D_A = \left\{ \psi \in C_0(E) : \exists \phi_\psi \in C_0(E) : \lim_{t\downarrow 0} \left|\left| \frac{P_t\psi - \psi}{t} - \phi_\psi \right|\right| = 0 \right\}$$

One can verify that D_A is a linear space.

Definition 1.1.25. We define a linear operator A on $C_0(E)$ with domain D_A by the prescription $A\psi = \phi_\psi$, so that for each $\psi \in D_A$,

$$A\psi = \lim_{t\downarrow 0} \frac{P_t\psi - \psi}{t}.$$

We call A the **generator** of the semigroup $\{P_t, t \geq 0\}$. When $\{P_t\}$ is the Feller semigroup associated with a Feller process $\{Z_t\}_{t\geq 0}$, we may also call A the generator of Z.

1.1.2 Stochastic integration

We aim to develop the theory of stochastic integration with respect to semimartingales following the spirit of [47]. For a given stochastic process X, we think of an integral with respect to X, as simply a map $I_X : \mathbb{X} \to \mathbf{L}^0$, where $\mathbb{X}$ is some space of processes that X belongs to. But, for this map to be a "reasonable" definition of an integral, it should be linear and also satisfy some version of the bounded convergence theorem. A particularly weak form of this theorem is that the uniform convergence of processes $H^n \to H$ implies only convergence in probability of $I_X(H^n) \to I_X(H)$.

Inspired by the above, we will proceed to formulate a version of I_X that is suitable to be called the "stochastic integral", and give an abstract definition of the semimartingale accordingly.

Definition 1.1.26. ([47], p. 51) A process H is said to be **simple predictable** if H has a representation of the form

$$H_t = H_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=1}^{n} H_i \mathbf{1}_{(T_i, T_{i+1})}(t)$$

where $0 = T_1 \leq \cdots \leq T_{n+1} < \infty$, is a finite sequence of stopping times, H_i are $\mathcal{F}_{T_i}$-measurable, with $|H_i| < \infty$ a.s., $0 \leq i \leq n$. The set of all simple predictable processes is denoted as $\mathbf{S}$

We give $\mathbf{S}$ a topology by uniform convergence in (t, ω), and we denote the resulting topological space by $\mathbf{S}_u$. We write $\mathbf{L}^0$ for the space of finite-valued random variables, with topology induced by convergence in probability. For a given process X, we define a linear map $I_X : \mathbf{S}_u \to \mathbf{L}^0$ by setting,

$$I_X(H) = H_0 X_0 + \sum_{i=1}^{n} H_i (X_{T_{i+1}} - X_{T_i})$$

where the H_i's correspond to that in Definition 1.0.1.

Definition 1.1.27. (c.f. [47], p. 52)

- A process X is a **total semimartingale** if X is càdlàg, adapted and $I_X : \mathbf{S}_u \to \mathbf{L}^0$ is continuous.
- A process X is a **semimartingale** if for each $T \in (0, \infty)$, $(X_{t \wedge T})_{t \geq 0}$ is a total semimartingale.

The above definition defines semimartingales as the set of "good candidates of integrators" of simple predictable processes. Next, we will extend the space of integrands to a larger class that includes many more interesting processes. The basic idea is simply complete the space. Before doing so, we will need a new topology that is suitable for such a completion.

Definition 1.1.28. A sequence of processes $(H^n)_{n \geq 1}$ converges to a process H **Uniformly on Compacta in Probability** (abbreviated as *ucp* if, for each $t > 0$, $\sup_{0 \leq s \leq t} |H_s^n - H_s| \to 0$ in probability.

Definition 1.1.29. Let

- $\mathbb{D}$ be the set of adapted processes with càdlàg paths.
- $\mathbb{L}$ be the set of adapted processes with càglàd paths.
- $L_b \subset \mathbb{L}$ with bounded paths.

All of the above spaces are equipped with the *ucp* topology.

Theorem 1.1.30. *(c.f. [47] , p. 57) The space* $\mathbf{S}$ *is dense in* $\mathbb{L}$ *under the ucp topology*

Proof. Let $Y \in \mathbb{L}$, $R_n = \inf\{t : |Y_t| > n\}$. Then R_n is a stopping time and $Y^n = Y^{R^n}\mathbf{1}_{\{R_n>0\}} \in L_b$, and converges to Y in *ucp*. Thus, L_b is dense in $\mathbb{L}$. Without loss of generality, we assume $Y \in L_b$. By the càglàdicity of Y, we can define Z by $Z_t = \lim_{u\downarrow t} Y_u$, so that Z is càdlàg. For $\varepsilon > 0$, define

- $T_0^\varepsilon = 0$
- $T_{n+1}^\varepsilon = \inf\{t : t > T_n^\varepsilon \text{ and } |Z_t - Z_{T_n^\varepsilon}| > \varepsilon\}$

Since Z is càdlàg, the T_n^ε are stopping times increasing to ∞ a.s. as n increases. Let $Z^\varepsilon = \sum_n Z_{T_n^\varepsilon}\mathbf{1}_{[T_n^\varepsilon, T_{n+1}^\varepsilon)}$, for each $\varepsilon > 0$. Then Z^ε are bounded and converge uniformly to Z as $\varepsilon \to 0$. Let

$$U^\varepsilon = Y_0\mathbf{1}_{\{0\}} + \sum_n Z_{T_n^\varepsilon}\mathbf{1}_{[T_n^\varepsilon, T_{n+1}^\varepsilon)}.$$

Then, $U^\varepsilon \in L_b$ and the preceding argument implies that U^ε converges uniformly on compacta to $Y_0\mathbf{1}_{\{0\}} + Z_- = Y$. Finally, define

$$Y^{n,\varepsilon} = Y_0\mathbf{1}_{\{0\}} + \sum_{i=1}^{n} Z_{T_i^\varepsilon}\mathbf{1}_{(T_i^\varepsilon \wedge n, T_{i+1}^\varepsilon \wedge n]}.$$

This can be made arbitrarily close to $Y \in L_b$ by taking ε small enough and n large enough. □

The operator I_X maps processes to random variables, and plays the role of a definite integral. Next, we define an operator J_X that maps processes to processes, which plays the role of an indefinite integral.

Definition 1.1.31. For $H \in \mathbf{S}$ and X a càdlàg process, we define $J_X : \mathbf{S} \to \mathbb{D}$ by

$$J_X(H) = H_0 X_0 + \sum_{i=1}^{n} H_i (X^{T_{i+1}} - X^{T_i})$$

for H in $\mathbf{S}$ with the representation

$$H = H_0 \mathbf{1}_{\{0\}} + \sum_{i=1}^{n} H_1 \mathbf{1}_{(T_i, T_{i+1}]},$$

where the H_i's are $\mathcal{F}_{T_i}$-measurable random variables, and $0 = T_0 \leq T_1 \leq ... \leq T_{n+1} < \infty$ are stopping times.

Moreover, we call $J_X(H)$ the **stochastic integral of H with respect to X.** We will use the following notations interchangeably,

$$J_X(H) = \int H_t dX_t = (H.X)$$

Remark 1.1.32. Notice the following relations between I_X and J_X:

- If we consider $I_{X^t}(H)$ as a process indexed by t, then $I_{X^t}(H) = J_X(H)_t$.
- On the other hand, $I_X(H) = \int_0^\infty H_t dX_t$.

Theorem 1.1.33. *(c.f. [47], p. 58) Let X be a semimartingale, then the mapping $J_X : \mathbf{S}_u \to \mathbb{D}$ is continuous under the ucp topology.*

Proof. Without loss of generality, we can take X to be a total semimartingale, as we are only dealing with convergence on compact sets. First, let us suppose that $\mathbf{S} \ni H^k \to 0$ uniformly and is uniformly bounded. We will show that $J_X(H^k) \to 0$ in the *ucp* topology.

Let $\delta > 0$ be given, and define stopping times τ^k by

$$\tau^k = \inf\{t \geq 0 : |(H^k.X)_t| \geq \delta\}.$$

Then, $H^k \mathbf{1}_{[0,\tau^k]} \in \mathbf{S}$ and tends to 0 uniformly as $k \to \infty$. Hence for every t,

$$\begin{aligned}\mathbb{P}((H^k.X)_t^* > \delta) &\leq \mathbb{P}(|H^k.X_{T^k \wedge t}| \geq \delta) \\ &= \mathbb{P}(|(H^k \mathbf{1}_{[0,T^k]}.X)_t| \geq \delta) \\ &= \mathbb{P}(|I_X(H^k 1_{[0,T^k \wedge t]})| \geq \delta) \\ &\to 0\end{aligned}$$

as $k \to \infty$ by the definition of a total semimartingale.

Hence, what we have shown is that $J_X : \mathbf{S}_u \to \mathbb{D}$ is continuous. Next, we show that $J_X : \mathbf{S}_u \to \mathbb{D}$ is continuous.

Suppose $H^k \to 0$ in *ucp*, and let $\delta > 0, \varepsilon > 0$ and $t > 0$. Then that there exists η, so that $||H||_u \leq \eta$ implies $\mathbb{P}(J_X(H)_t^* > \delta) < \frac{1}{2}\varepsilon$. Let $R_k = \inf\{s : |H_s^k| > \eta\}$, and set $\tilde{H}^k = H^k \mathbf{1}_{[0,R_k]} \mathbf{1}_{R_k > 0}$. Then, $\tilde{H}^k \in \mathbf{S}$ and $||\tilde{H}^k||_u \leq \eta$ by left continuity. Since $R^k \geq t$ implies $(\tilde{H}^k.X)_t^* = (H^k.X)_t^*$, we have

$$\begin{aligned}\mathbb{P}((H^k.X)_t^* > \delta) &\leq \mathbb{P}((\tilde{H}^*.X)_t^* > \delta) + \mathbb{P}(R^k < t) \\ &\leq \frac{1}{2}\varepsilon + \mathbb{P}((H^k)_t^* > n) \\ &< \varepsilon\end{aligned}$$

for k large enough as $\lim_{k \to \infty} \mathbb{P}((H^k)_t^* \geq \eta) = 0$. □

We have now established that if X is a semimartingale, the integration operator J_X is continuous on $\mathbf{S}_u$, and that $\mathbf{S}_u$ is dense in $\mathbb{L}$. The obvious thing to do now is to extend J_X from $\mathbf{S}_u$ to $\mathbb{L}$ by continuity. This is a valid extension as $\mathbb{D}$ is a complete metric space. Hence, we have the following definition.

Definition 1.1.34. Let X be a semimartingale. The continuous linear map $J_X : \mathbb{L} \to \mathbb{D}$ obtained from extending $J_X : \mathbf{S} \to \mathbb{D}$, is called the **stochastic integral** or the **Itô integral**.

One important consequence of Theorem 1.1.33 is that the integrand of an Itô integral must be a predictable process (c.f. [47], p. 101 - 103), and the process we are integrating with respect to must be a semimartingale (c.f. [47] p. 129 - 133, p. 146 - 148). If a stochastic process $\{X_t\}_{t \geq 0}$ has càdlàg paths, then in general $\{X_t\}_{t \geq 0}$

is not predictable (with respect to its own filtration). To overcome this problem, in most cases, we can simply write $X_{t-} = \lim_{s\uparrow t} X_s$. Then, it can shown that X_{t-} is predictable and we put X_{t-} in the integrand instead of X_t.

Remark 1.1.35.

- If the trajectory of X is continuous over some interval $[a, b]$, then $X_{t-} = X_t$ for all $t \in (a, b]$. In particular, if X is a continuous process, then X itself is automatically predictable.
- In cases when X contain jumps, it is convenient to define the difference operator $\Delta X_t = X_t - X_{t-}$[1] to extract the jump component of X. In particular, $\Delta X_t = 0$ if X is continuous at t and ΔX_t equals the jump size if X jumps at t.

We will now focus on the case when $E = \mathbb{R}^d$. First, we define the concept of a covariation.

Definition 1.1.36. Let X, Y be semimartingales, the **quadratic variation process** of X, denoted by $\{< X, X >_t\}_{t\geq 0}$ is defined by

$$< X, X >_t = X_t^2 - 2\int_0^t X_{s-} dX_s,$$

where $Z_{s-} = \lim_{u\uparrow s} Z_u$. The **covariation**, or the **bracket** of X, Y, is defined by

$$< X, Y >_t = \frac{1}{4}(< X + Y, X + Y >_t - < X - Y, X - Y >_t).$$

for each $t \geq 0$.

The next theorem is summarizes some important properties of the covariation (c.f. [47], p. 66-68, 77).

Theorem 1.1.37. *Let X, Y be semimartingales, H a càdlàg adapted process and for any process $\{Z_t\}_{t\geq 0}$, let $\Delta Z_t = Z_t - Z_{t-}$. The following summarises the key results regarding quadratic variations and covariations*

1. *The quadratic variation process $< X, X >$ is adapted, càdlàg and increasing. In particular, $< X, X >$ and $< X, Y >$ are processes of bounded variation, hence $\int f(t) d < X, Y >_t$ can be interpreted in the Lebesgue-Stieltjes sense.*

[1] whenever the underlying state space E has enough algebraic structure to take differences

2. $< X, X >_0 = X_0^2$ and $\Delta < X, X >_t = (\Delta X_t)^2$. *Similarly,* $< X, Y >_0 = X_0 Y_0$, $\Delta < X, Y >_t = \Delta X_t \Delta Y_t$.

3. *If* σ_n *is a sequence of random partitions tending to the identity, then*

$$X_0^2 + \sum_i (X^{\tau_{i+1}^n} - X^{\tau_i^n})^2 \to < X, X >$$

$$X_0 Y_0 + \sum_i (X^{\tau_{i+1}^n} - X^{\tau_i^n})(Y^{\tau_{i+1}^n} - Y^{\tau_i^n}) \to < X, Y >,$$

and

$$\sum_i H_{\tau_i^n}(X^{\tau_{i+1}^n} - X^{\tau_i^n})(Y^{\tau_{i+1}^n} - Y^{\tau_i^n}) \to \int H_{s-} d[X, Y]_s$$

all converge under the ucp topology, where σ_n *is the sequence* $0 = \tau_0^n \leq \tau_1^n \leq \ldots \leq \tau_i^n \leq \ldots \leq \tau_{k_n}^n$, *and where* τ_i^n *are stopping times.*

Next, we state Itô's formula for càdlàg semimartingales.

Theorem 1.1.38. *Let* $X = (X^1, \ldots, X^d)$ *be d-tuple of semimartingales, and let* $f \in C^2(\mathbb{R}^d)$, *then* $f(X)$ *is a semimartingale, and the following formula holds:*

$$\begin{aligned} f(X_t) = f(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(X_{s-}) dX_s^i + \frac{1}{2} \sum_{i,j}^{d} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s-}) d < X^i, X^j >_s^c \\ + \sum_{0<s<t} \left(f(X_s) - f(X_{s-}) - \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(X_{s-}) \Delta X_s^i \right) \end{aligned} \tag{1.1}$$

We now define two alternative types of integrals that will be useful in stochastic differential geometry, namely the Fisk-Stratonovich integral, and the Marcus canonical integral.

Definition 1.1.39. Let X, Y be semimartingales, we define the **Fisk-Stratonovich integral** of Y with respect to X by

$$\int_0^t Y_{s-} \circ dX_s := \int_0^t Y_s dX_s + \frac{1}{2} < Y, X >_t^c .$$

The Fisk-Stratonovich integral is sometimes called the Stratonovich integral. It can be easily checked that if $f \in C^3(\mathbb{R}^d)$ and X is a continuous semimartingale, then

$$f(X_t) = f(X_0) + \sum_i \int_0^t \frac{\partial f}{\partial x_i}(X_s) \circ dX_s^i. \tag{1.2}$$

Therefore, this allows us to recover the change of variable formula in a coordinate free fashion. This is an advantage over the Itô integral in stochastic differential geometry.

If X is a càdlàg semimartingale, the Stratonovich integral no longer gives us a "chain-rule" type of coordinate free transformation formula. Hence, we need a separate approach to take care of the jumps. We will follow the approach of [2] to define the Marcus canonical integral for integrands of the form $\{f(s, X_{s-})\}_{s\geq 0}$, where $f : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ is such that $s \mapsto f(s, X_{s-})$ is predictable and the Itô integrals $\int_0^t f_i(s, X_{s-})dX_s^i$ exists for all $t \geq 0$ and $i = 1, ..., d$. We assume that there exists a measurable map

$$\Phi : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$

such that for each $s \geq 0, x, y \in \mathbb{R}^d$, the following holds:

- $u \mapsto \Phi(s, u, x, y)$ is continuously differentiable.
- $\frac{\partial \Phi}{\partial u}(s, u, x, y) = \sum_i x^i f_i(s, y + ux)$ for each $u \in \mathbb{R}$.
- $\Phi(s, 0, x, y) = \Phi(s, 0, x, 0)$.

Such a Φ is called the **Marcus canonical form**. Given such a map, we define the **Marcus canonical integral** as follows. For each $t \geq 0$,

$$\begin{aligned}\int_0^t \langle f(s, X_{s-}), \diamond dX_s \rangle_{\mathbb{R}^d} &:= \int_0^t \langle f(s, X_{s-}), \circ dX_s^c \rangle_{\mathbb{R}^d} + \int_0^t \langle f(s, X_{s-}), dX_s^d \rangle_{\mathbb{R}^d} \\ &+ \sum_{0 \leq s \leq t} \left(\Phi(s, 1, X_{s-}, \Delta X_s) - \Phi(s, 0, X_{s-}, \Delta X_s) - \frac{\partial \Phi}{\partial u}(s, 0, Y_{s-}, \Delta Y_s) \right)\end{aligned} \tag{1.3}$$

It then can be checked that if X is a càdlàg semimartingale, and $f \in C^3(\mathbb{R}^d)$, then

$$f(X_t) - f(X_0) = \sum_i \int_0^t \frac{\partial f}{\partial x_i}(X_{s-}) \diamond dX_s^i, \tag{1.4}$$

for each $t \geq 0$ with probability 1.

1.2 Euclidean Lévy processes

The two most studied examples of stochastic processes are the Brownian motion and the Poisson process. The former is thought of as the building blocks of processes of continuous paths, while the latter is held in similar regard for pure jump processes. The theory of Lévy processes aim to combine both of the two. The following properties of stochastic processes will be central to the definition of a Lévy process.

Definition 1.2.1. A stochastic process $\{X_t\}$ on $\mathbb{R}^d$ is said to have:

- **independent increments**, if for any choice of $n \geq 1$, and $0 \leq t_0 < t_1 < ... < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ are independent.
- **stationary increments**, if the distribution of $X_{s+t} - X_t$ is independent of t.

Definition 1.2.2. A stochastic process $\{X_t\}$ on $\mathbb{R}^d$ is said to be **stochastically continuous** if for every $t \geq 0$ and $\varepsilon > 0$,

$$\lim_{s \to t} \mathbb{P}(|X_s - X_t| > \varepsilon) = 0.$$

Now we are ready to give the definition of a Lévy process.

Definition 1.2.3. A stochastic process $\{Z_t\}_{t \geq 0}$ on $\mathbb{R}^d$ is a called **Lévy Process** if the following conditions are satisfied.

- $Z_0 = 0$ a.s.
- Z has independent and stationary increments.
- Z is stochastically continuous.

If $Z_0 \neq 0$, and all else are true, we call Z_t a **shifted Lévy Process**. Sometimes we will also consider the case when Z_0 is random, so that μ_{Z_0} is not a point mass, we call such a process a **randomly shifted Lévy process**.

The defining properties of Lévy processes, the stationary and independent increments assumptions, are closely related to another concept in probability theory, infinite divisibility.

Now, observe that for two $\mathbb{R}^d$-valued independent random variables X, Y, and $A \subset \mathbb{R}^d$, $\mu_{X+Y}(A) = \int_{\mathbb{R}^d} \mu_X(A-z)d\mu_Y(z)$ where $A - z = \{y - z : y \in A\}$. Inspired by this, we introduce the concept of a convolution.

Definition 1.2.4. Let μ and ν be probability measures on $\mathbb{R}^d$. The **convolution** $\mu * \nu$ of the measures is defined by,

$$(\mu * \nu)(A) = \int_{\mathbb{R}^d} \mu(A-z)d\nu(z)$$

for every $A \in \mathcal{B}(\mathbb{R}^d)$, here $A - z = \{y - z : y \in A\}$.

Remark 1.2.5. The following facts about convolutions are well-known, (c.f. [2], pp. 20 - 21)

- The convolution $\mu * \nu$ is a probability measure on $\mathbb{R}^d$.
- Let δ_0 be the Dirac measure centered at 0, then $\delta_0 * \mu = \mu$ for all $\mu \in L(\mathbb{R}^d)$.
- If $f \in B_b(\mathbb{R}^d)$, then for all $\mu_i \in L(\mathbb{R}^d)$, $i = 1, 2, 3$,
  - $\mu_1 * \mu_2 = \mu_2 * \mu_1$
  - $(\mu_1 * \mu_2) * \mu_3 = \mu_1 * (\mu_2 * \mu_3)$
- The space of probability measures over R^d forms a commutative algebra under the convolution product.
- If $f \in B_b(\mathbb{R}^d)$, and $X_i : \Omega \to \mathbb{R}^d$ are independent random variables, with induced measures μ_i, $i = 1, ...n$, $n \in \mathbb{N}$ then,

$$\mathbb{E}f(X_1 + ... + X_n) = \int_{R^d} f d\mu_1 * ... * \mu_n.$$

An immediate consequence of this is that

$$\mu_{X_1+...+X_n} = \mu_{X_1} * ... * \mu_{X_n}$$

if we pick $f(x) = \mathbf{1}_A(x)$.

An infinitely divisible random variable is a random variable that can be written as the sum of arbitrarily many independent and identical random variables. This induces a restriction on the underlying probability measure: it must be arbitrarily high convolution powers of some other probability measures.

Definition 1.2.6. Let μ be a probability measure on $\mathbb{R}^d$, we define the **n-th convolution power** of μ, denoted by μ^{*n} by

$$\mu^{*n} = \mu * ... * \mu \qquad (n \text{ times})$$

Definition 1.2.7. A $\mathbb{R}^d$-valued random variable X is said to be **infinitely divisible** if for any $n \in \mathbb{N}$, there exists ν depending on n, such that $\nu^{*n} = \mu_X$.

Corollary 1.2.8. *Let $\{Z_t\}_{t\geq 0}$ be a Lévy process. Then the random variables Z_t, $t \geq 0$ are infinitely divisible.*

Proof. For every $n \in \mathbb{N}$, we can write Z_t as,

$$\begin{aligned} Z_t &= Z_0 + (Z_{t_1} - Z_{t_0}) + ... + (Z_{t_n} - Z_{t_{n-1}}) \\ &= Z_0 + Y_1 + ... + Y_n. \end{aligned}$$

where $t_k = \frac{tk}{n}$, $k = 0, ..., n$ and $Y_k = Z_k - Z_{k-1}$. By the independent and stationary increment property, Y_k and Y_l are independent and identically distributed, and hence, $\mu_{Z_t} = (\mu_Y)^{*n}$. □

Next, we introduce the characteristic function of a measure.

Definition 1.2.9. Let μ be a probability measure, the **characteristic function** (or the **Fourier transform**) of μ, denoted by $\hat{\mu}$ is defined by.

$$\hat{\mu}(\lambda) := \int_{\mathbb{R}^d} e^{i\langle\lambda,x\rangle_{\mathbb{R}^d}} d\mu(x).$$

with $\lambda \in \mathbb{R}^d$.

In order to make inferences about μ from knowledge of $\hat{\mu}$, we need the following results, all of these results are well-known.

Theorem 1.2.10. *[[49], pp. 8 - 10, [2], pp. 15 - 17]*

- *Let $\mu, \nu \in L(\mathbb{R}^d)$, then $\widehat{\mu * \nu} = \hat{\mu}\hat{\nu}$. In particular, $\mu^{*n} = (\hat{\mu})^n$.*
- *(Glivenko) If $\hat{\mu}_n$ and $\hat{\mu}$ are characteristic functions of probability distributions μ_n and μ respectively, for each $n \in \mathbb{N}$, then $\hat{\mu}_n(\lambda) \to \hat{\mu}(\lambda)$ for all $\lambda \in \mathbb{R}^d$ implies $\mu_n \to \mu$ weakly as $n \to \infty$.*
- *(Lévy's continuity theorem) If $\{\phi_n\}_{n\in\mathbb{N}}$ is a sequence of characteristic functions, and there exists a function $\phi : \mathbb{R}^d \to \mathbb{C}$, such that for all $\lambda \in \mathbb{R}^d$, $\phi_n(\lambda) \to \phi(\lambda)$ as $n \to \infty$, and ϕ is continuous at 0, then ϕ is the characteristic function of a probability distribution.*
- *(Bochner) Let $\mu \in L(\mathbb{R}^d)$, we have that*
  - *$\hat{\mu}(0) = 1$, and*
  - *$|\hat{\mu}(z)| \leq 1$, and*
  - *$\hat{\mu}(z)$ is uniformly continuous and nonnegative definite in the sense that, for each $n = 1, 2, ...$*

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\hat{\mu}(z_j - z_k)\xi_j\bar{\xi}_k \geq 0$$

    for $z_1, ..., z_n \in \mathbb{R}^d$, $\xi_1, ..., \xi_n \in \mathbb{C}$.

  Conversely, if a complex valued function ϕ satisfies (i) - (iii), then $\phi(z)$ is the characteristic function of a distribution on $\mathbb{R}^d$.

The following theorem gives a representation of characteristic functions of all infinitely divisible distributionos. It is called the **Lévy Khintchine formula**. It was first obtained on $\mathbb{R}$ around 1930 by de Finetti and Kolmogorov in special cases, and then by Lévy in the general case. Immediately, the formula was extended to $\mathbb{R}^d$. A much simpler proof on $\mathbb{R}$ was given by Khintchine.

Definition 1.2.11. A measure ν on $\mathbb{R}^d$ is said to be a **Lévy measure** if it satisfies

- $\nu(\{0\}) = 0$, and
- $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)d\nu(x) < \infty$.

Theorem 1.2.12. *(Lévy-Khintchine representation) If μ is an infinitely divisible probability measure, then there exists $\gamma \in \mathbb{R}^d$, a nonnegative definite symmetric $d \times d$ matrix A, and a Lévy measure ν, such that $\hat{\mu}(\lambda) = \exp \psi(\lambda)$, where*

$$\psi(\lambda) = i \langle \gamma, \lambda \rangle - \frac{1}{2} \langle \lambda, A\lambda \rangle + \int_{\mathbb{R}^d} (e^{i\langle \lambda, x \rangle} - 1 - i \langle \lambda, x \rangle \mathbf{1}_{\{|x| \leq 1\}}(x)) d\nu(x) \qquad (1.5)$$

Moreover, this representation is unique.

Conversely, for any choice of γ, A, ν as above, there exists an infinitely divisible distribution μ with characteristic function given by (1.5).

Let $\{Z_t\}_{t \geq 0}$ be a Lévy process. If we regard μ as above to be the probability distribution of Z_1, with characteristic function $\hat{\mu}(\lambda) = \exp(\psi(\lambda))$, then $\hat{\mu}_{Z_t} = (\hat{\mu}_{Z_1})^t = \exp(t\psi(\lambda))$. Hence, we have obtained a Lévy-Khintchine representation for Lévy processes.

Corollary 1.2.13. *Keeping the notations in Theorem 1.2.12, let $\{Z_t\}_{t \geq 0}$ be a Lévy process. There exists unique γ, A and ν, such that*

$$\hat{\mu}_{Z_t}(\lambda) = \exp \left[t \left(i \langle \gamma, \lambda \rangle - \frac{1}{2} \langle \lambda, A\lambda \rangle + \int_{\mathbb{R}^d} (e^{i\langle \lambda, x \rangle} - 1 - i \langle \lambda, x \rangle \mathbf{1}_{\{|x| \leq 1\}}(x)) d\nu(x) \right) \right] \qquad (1.6)$$

From the Lévy-Khintchine representation, it makes sense to guess that a Lévy process can be decomposed as the sum of a continuous process and a jump process. We will make this observation precise by studying the famous Lévy-Itô decomposition. We need to introduce a new concept, a Poisson random measure, before this can be done.

Definition 1.2.14. (c.f. [2] , p. 89) Let $(\mathcal{E}, \mathcal{A})$ be a measurable space. A **random measure** M on $(\mathcal{E}, \mathcal{A})$ is a collection of random variables $\{M(A)\}_{A \in \mathcal{A}}$, such that

- $M(\emptyset) = 0$;
- Given any sequence $\{A_n\}_{n \in \mathbb{N}}$ of mutually disjoint sets in $\mathcal{A}$, we have

$$M\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} M(A_n) \qquad a.s.$$

This is called the σ-additive property.

- For each disjoint family of sets $A_1, ..., A_n \in \mathcal{A}$, the random variables $M(A_1), ..., M(A_n)$ are mutually independent.

Definition 1.2.15. Let M be a random measure, we say M is a **Poisson random measure** if for every A such that $M(A) < \infty$, there exists $\lambda \in \mathbb{R}$, such that for all $k \in \mathbb{N}$

$$\mathbb{P}(M(A) = k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

for every $k \in \mathbb{N}$.

Remark 1.2.16. It can be shown that $\lambda(A) = \mathbb{E}M(A)$. Conversely, given a σ-finite measure λ on a measurable space $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$, there exists a Poisson random measure M on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\lambda(A) = \mathbb{E}M(A)$ for every $A \in \mathcal{A}$.

Next, we introduce the counting measure of a stochastic process as a random measure. Recall that if X_t be a stochastic process with càdlàg paths, the *jump process* of X_t, denoted by ΔX_t as

$$\Delta X_t = X_t - X_{t-}$$

where $X_{t-} = \lim_{s\uparrow t} X_s$.

Definition 1.2.17. Let $A \in \mathcal{B}(\mathbb{R}^d - \{0\})$. We define the **counting measure** of X as

$$N_X(t, A) = \#\{0 \leq s \leq t; \Delta X_s \in A\} = \sum_{0\leq s\leq t} \mathbf{1}_A(\Delta X_s).$$

We call the map $\lambda_X : A \mapsto \mathbb{E}N_X(1, A)$, the **intensity measure** of X.

Remark 1.2.18. When there is no risk of confusion, one can write $N(t, A)$ and $\lambda(A)$ to denote the counting measure, and the intensity measure of X.

Definition 1.2.19. For each $t \geq 0$, we define the **compensated Poisson random measure** $\{\tilde{N}(t, A)\}_{t\geq 0}$ by

$$\tilde{N}(t, A) = N(t, A) - t\lambda(A),$$

It is clear that if X_t is time homogeneous, then $\tilde{N}(t, A)$ becomes a martingale, as elementary computation shows that $\mathbb{E}(t, A) = t\mathbb{E}(N(1, A)) = t\lambda(A)$.

Remark 1.2.20.

- When X_t is a Poisson process with intensity λ, we can identify $X_t = N(t, A)$ where $\lambda = \mathbb{E}N(1, A) > 0$.
- When X_t is a compound Poisson process, that is, we can write $X_t = X_1 + ... + X_{N_t}$, $X_1, X_2, ...$ are independent and identical random variables, and N_t is a Poisson process independent to $X_i, i = 1, 2, ...$ Then, $N(t, A)$ counts the number of jumps of size $x \in A$ up to time t.

Now, we are ready to define the Poisson integral.

Definition 1.2.21. Let $(E, \mathcal{A})$ be a measurable space, $f : E \to \mathbb{R}^d$ be a Borel measurable function, and let $A \in \mathcal{A}$ be such that $N(t, A)$ is bounded below for every $t > 0$. Then for each $t > 0, \omega \in \Omega$, we define the **Poisson integral** of f as a random sum by,

$$\int_A f(x)N(t, dx)(\omega) = \sum_{x \in A} f(x)N(t, \{x\})(\omega).$$

For each t fixed, $\int_A f(x)N(t, dx)$ is a random variable, and it becomes a càdlàg process if we allow t to vary. If $N(t, A)$ is the counting measure of a certain càdlàg process X_t, then $N(t, \{x\}) \neq 0 \iff \Delta X(u) = x$ for at least one $0 \leq u \leq t$. Hence,

$$\int_A f(x)N_X(t, dx) = \sum_{0 \leq u \leq t} f(\Delta X(u))\mathbf{1}_A(\Delta X(u)).$$

Theorem 1.2.22. *([2], p. 91) Let $A \subseteq \mathbb{R}^d$ be such that $N(t, A) > 0$, and for each $t > 0$, we have*

- *$\int_A f(x)N(t, dx)$ has a compound Poisson distribution if we fix t, and it becomes a compound Poisson process if we allow t to vary.*
- *For each $u \in \mathbb{R}^d$,*

$$\mathbb{E}\left(\exp\left[i\left\langle u, \int_A f(x)N(t, dx)\right\rangle\right]\right) = \exp\left[t\int_A (e^{i\langle u,x\rangle} - 1)d\lambda_f(x)\right]$$

where $\lambda_f = \lambda \circ f^{-1}$.

Now we are ready to state the Lévy-Itô decomposition theorem.

Theorem 1.2.23. *([2], p. 108) If Z_t is a Lévy process, then there exists $b \in \mathbb{R}^d$, a Wiener process W_t with covariance matrix A, and an independent Poisson random measure N on $\mathbb{R}_+ \times (\mathbb{R}^d - \{0\})$ such that for each $t \geq 0$,*

$$Z_t = bt + W_t + \int_{|z|<1} z\tilde{N}(t, dz) + \int_{|x|>1} xN(t, dx).$$

Notice that the above decomposition suggests that any Lévy process Z_t is very close to an independent sum of a Brownian motion and a compound Poisson process. In fact, we can make this idea precise by considering the interlacing construction (c.f. p. 47 of [2]). We now focus on Lévy processes whose jumps are bounded above by 1. By the Lévy-Itô decomposition, we have

$$Y_t = bt + W_t + \int_{|x|<1} x\tilde{N}(t, dx).$$

We define a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ that decreases monotonically to zero by,

$$\varepsilon_n = \sup\left\{y \geq 0 : \int_{0<|x|<y} x^2 d\nu(x) \leq \frac{1}{8^n}\right\}$$

where ν is the Lévy measure of Y. Now, we define $Y^n = \{Y_t^n\}_{t\geq 0}$ as follows,

$$\begin{aligned} Y_t^n &= bt + W_t + \int_{\varepsilon_n \leq |x|<1} x\tilde{N}(t, dx) \\ &= C_t^n + \int_{\varepsilon \leq |x|<1} xN(t, dx), \end{aligned}$$

where for each $n \in \mathbb{N}$, C_n is the continuous process given by

$$C_t^n = W_t + t\left(b - \int_{\varepsilon_n \leq |x|<1} x d\nu(x)\right).$$

By theorem 1.2.22, $\int_{\varepsilon_n \leq |x|<1} xN(t, dx)$ is a compound Poisson process, with jumps ΔY_t at times $\{T_n^m, m \in \mathbb{N}\}$. This allows us to reconstruct Y_t^n by **interlacing**, namely,

$$Y_t^n = \begin{cases} C_t^n, & \text{for } 0 \leq t < T_n^1; \\ C_{T_n^1}^n + \Delta Y_{T_n^1}, & t = T_n^1; \\ Y_{T_n^1}^n + C_t^n - C_{T_n^1}^n, & T_n^1 < t < T_n^2; \\ Y_{T_n^2-}^n + \Delta Y_{T_n^2}, & t = T_2. \end{cases}$$

and so on, recursively. The main theorem regarding interlacing is as follows,

Theorem 1.2.24. *(c.f. [2]) For each $t \geq 0$, $\lim_{n\to\infty} Y_t^n = Y_t$ a.s., and the convergence is uniform on compacta.*

The significance of this construction is that it allows us to write down precisely how we can regard Lévy processes as limits of compound Poisson processes. On the other hand, this also allows us to intuitively construct Lévy processes by taking limits of compound Poisson processes when the underlying state space is no longer $\mathbb{R}^d$.

We end this section by giving a brief survey of the main theorems concerning stochastic differential equations driven by Lévy processes. Readers should refer to Chapter 4 of [2] and Chapter 5 of [47] for more details.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space, let $\{W_t\}_{t\geq 0}$ be a d-dimensional Wiener process and N an independent Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R}^d - \{0\})$ with associated compensator $\tilde{N}$ and intensity measure ν. We assume that ν is a Lévy measure. We are interested in studying equations of the form,

$$\begin{aligned} dX_t = \mu(X_{t-})dt + \sigma(Y_{t-})dW_t &+ \int_{|y|<c} F(X_{t-}, x)\tilde{N}(dt, dy) \\ &+ \int_{|y|\geq c} G(Y_{t-}, x)N(dt, dx) \end{aligned} \tag{1.7}$$

where X_t is a $\mathbb{R}^d$-valued process, $\mu \in \mathbb{R}^d$, $\sigma \in \mathrm{End}(\mathbb{R}^n, \mathbb{R}^d)$, $c > 0$, and W_t is an n-dimensional Wiener process. For the sake of simplicity, we will assume $n = d$ and so σ becomes a $d \times d$ symmetric matrix.

Definition 1.2.25. A **strong solution** to (1.7) is a process Y_t adapted to $\mathcal{F}_t$ such that for each $t \geq 0, 1 \leq i \leq d$,

$$Y(t) = Y(0) + \int_0^t \mu(Y_{t-})dt + \int_0^t \sigma(Y_{t-})dW_t + \int_0^t \int_{|y|<c} F(Y_{t-}, y)\tilde{N}(dt, dy)$$
$$+ \int_0^t \int_{|y|\geq c} G(Y_{t-}, y)N(dt, dy). \tag{1.8}$$

The final term in (1.2) involving large jumps that is controlled by G can be handled using interlacing (c.f. [2], p. 302), and we will formulate a sufficient condition for the existence of strong solutions to the **modified equation**,

$$dX_t = \mu(X_{t-})dt + \sigma(X_{t-})dW_t + \int_{|y|<c} F(X_{t-}, x)\tilde{N}(dt, dy), \tag{1.9}$$

with initial condition $X_0 = 0$. For $x, y \in \mathbb{R}^d$, let $a(x, y) = \sigma(x)\sigma(y)^*$, so that

$$a^{ik}(x, y) = \sum_{j=1}^{d} \sigma_j^i(x)\sigma_j^k(y)$$

for each $1 \leq i, k \leq d$. We will equip such matrices with a seminorm given by $||a|| = \sum_i |a_i^i|$.

We formulate the following two conditions:

- **Lipschitz condition:** There exists $K_1 > 0$ such that for all $y_1, y_2 \in \mathbb{R}^d$,

$$|\mu(y_1) - \mu(y_2)|^2 + ||a(y_1, y_1) - 2a(y_1, y_2) + a(y_2, y_2)|| \tag{1.10}$$
$$+ \int_{|x|<c} |F(y_1, x) - F(y_2, x)|^2 d\nu(x) \leq K_1|y_1 - y_2|^2.$$

- **Growth condition:** There exists $K_2 > 0$ such that for all $y \in \mathbb{R}^d$,

$$|b(y)|^2 + ||a(y, y)|| + \int_{|x|<c} |F(y, x)|^2 d\nu(x) \leq K_2(1 + |y|^2). \tag{1.11}$$

Remark 1.2.26. ([2] p. 303) A straightforward calculation yields

$$||a(y_1, y_1) - 2a(y_1, y_2) + a(y_2, y_2)|| = \sum_{i,j} [\sigma_{ij}(y_1) - \sigma_{ij}(y_2)]^2.$$

Theorem 1.2.27. *Assume the Lipschitz (1.10) and growth conditions (1.11) stated as above. There exists a unique strong solution $\{Y_t\}$ to the modified stochastic differential equation (1.9). Moreover, the process $\{Y_t\}$ is adapted and càdlàg.*

We have now covered the basic concepts of stochastic analysis and Lévy processes on an Euclidean space. The majority part of this thesis is to attempt to extend as much of this as possible to non-flat spaces.

Chapter 2

Basic Stochastic Differential Geometry

The central aim of this thesis is to generalize the theory introduced in the previous chapter to the study of Lévy processes on non-Euclidean spaces. However, it is far from obvious how to give a reasonable definition of a Lévy process on a general manifold. One major problem is the formulation of the independent increment property when the underlying state space lacks a group structure.

Brownian motion is a stochastic process possessing the independent increment property, and its construction has been successfully carried over on a manifold via the Laplace-Beltrami operator (chapter 2 - 4,[33]). In this chapter, we will study this construction in detail, and explain precisely where the problem lie in the case of general Lévy processes.

The structure of this chapter are as follows. Section 2.1 will give a brief introduction of all the basic manifold theory needed in this thesis following chapter 1 of [28]. Section 2.2 will look more deeply into the concept of "rolling without slipping" in the deterministic case first, and then extending it to the stochastic case following chapter 2 of [33]. In section 2.3, we give a brief survey of the papers [3], [6]. In these papers, Applebaum first showed that the techniques of section 2.2 applied to Levy processes, will in general, produce a manifold valued process that is non-Markovian. Then in [6], Applebaum and Estrade showed that the construction is problem free if the Levy process is isotropic.

2.1 Manifolds

Intuitively, a manifold $\mathbb{M}$ is a Hausdorff space equipped with some differentiable structure, while a Riemannian manifold is a manifold whose tangent spaces are equipped with an inner product at every point. This purpose of this section is to make the above intuitive ideas precise, and develop the necessary geometric theory to define certain classes stochastic processes on manifolds. It also serves as an opportunity to standardize notations.

Let $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ denote two Euclidean spaces of dimensions d_1 and d_2 respectively. Let O and O' be open subsets, $O \subset \mathbb{R}^{d_1}$, $O' \subset \mathbb{R}^{d_2}$ and suppose ϕ is a mapping of O into O'. The mapping ϕ is called **differentiable** if the coordinates $\phi_j(p)$ of $\phi(p)$ are differentiable functions of the coordinates $x_i(p)$ of $p \in O, i = 1, ...d_i, j = 1, ..., d_2$. The mapping is called **analytic**, if for each point $p \in O$ there exists a neighbourhood U of p and d_2 power series P_j, $j = 1, ..., d_2$ in d_1 variables such that

$$\phi_j(q) = P_j(x_1(q) - x_1(p), ..., x_m(q) - x_m(p))$$

converges absolutely, for every $j = 1, ..., d_2$ and $q \in U$. A differentiable mapping $\phi : O \to O'$ is called a **diffeomorphism** of O onto O' if ϕ is bijective and the inverse mapping ϕ^{-1} is differentiable.

Definition 2.1.1. ([28], p. 4) Let $\mathbb{M}$ be a Hausdorff topological space. An **open chart** on $\mathbb{M}$ is a pair (U, ϕ), where U is an open subset of $\mathbb{M}$ and ϕ is a homeomorphism of U onto an open subset of $\mathbb{R}^d$. A C^k **differentiable structure** on M of dimension d is a collection of open charts $(U_\alpha, \phi_\alpha)_{\alpha \in A}$ on $\mathbb{M}$, where A is an index set and $\phi_\alpha(U_\alpha)$ is an open subset of $\mathbb{R}^d$ that satisfies the following conditions,

1. $\mathbb{M} = \bigcup_{\alpha \in A} U_\alpha$.
2. For each pair $\alpha, \alpha' \in A$, the mapping $\phi_{\alpha'} \circ \phi_\alpha^{-1}$ is a k-times differentiable mapping of $\phi_\alpha(U_\alpha \cap U_{\alpha'})$ onto $\phi_{\alpha'}(U_\alpha \cap U_{\alpha'})$.
3. The collection $(U_\alpha, \phi_\alpha)_{\alpha \in A}$ is a maximal family of open charts for which (1) and (2) hold,

for $k \in \mathbb{N} \cup \{\infty\}$. A C^k**-manifold** of dimension d is a Hausdorff space with a differentiable structure of dimension d.

An **analytic structure** of dimension d can be defined just as above, except that we need to replace the word "differentiable" by the word "analytic". Similarly, an **analytic manifold** is just a Hausdorff space with an analytic structure. A **complex manifold** can also be defined analogously, where in addition we need to replace every "$\mathbb{R}^d$" by "$\mathbb{C}^d$", and the charts are complex analytic.

Definition 2.1.2. Let A be an algebra over a field $\mathbf{F}$, a **derivation** of A is a mapping $D : A \to A$ such that

- $D(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 D f_1 + \alpha_2 D f_2$ for $\alpha_1, \alpha_2 \in k$ and $f, g \in A$.
- $D(fg) = f(Dg) + (Df)g$ for $f, g \in A$.

A **vector field** on a C^r manifold $\mathbb{M}$ is a derivation of the algebra $C^r(\mathbb{M})$. The set of all vector fields on $\mathbb{M}$ is denoted by $\mathcal{D}^1(\mathbb{M})$.

Let $X, Y \in \mathcal{D}^1(\mathbb{M})$. Then $XY - YX$ is also a derivation of $C^\infty(\mathbb{M})$, and it is denoted by the bracket $[X, Y]$. It is common to write $\mathrm{ad}(X)(Y) = [X, Y]$, and $\mathrm{ad}(X)$ is called the **Lie derivative** with respect to X, or the **adjoint map**. It can be checked that the bracket satisfies the **Jacobi identity**,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

The above considerations turn $\mathcal{D}^1(\mathbb{M})$ into a module over $C^\infty(\mathbb{M})$.

If $f \in C^k(\mathbb{M})$ and $X, Y \in \mathcal{D}^1(\mathbb{M})$, then we will interpret the vector fields fX as $fX : g \mapsto f(Xg)$, and $X + Y$ as $(X + Y) : g \mapsto Xg + Yg$ for $g \in C^k(M)$.

Definition 2.1.3. For $x \in \mathbb{M}$ and $X \in \mathcal{D}^1(\mathbb{M})$, let $X_x : C^k(x) \to \mathbb{R}$ denote the linear map, $X_x : f \mapsto (Xf)(x)$, where $C^k(x)$ is the set of k-times differentiable functions at x.

The set $\{X_x : X \in \mathcal{D}^1(\mathbb{M})\}$ is called the **tangent space** to $\mathbb{M}$ at x, denoted by $T_x\mathbb{M}$. Its elements are called **tangent vectors** to M at x. Finally, the **tangent bundle** is given by $T\mathbb{M} = \bigcup_{x \in M} T_x M$.

It can be shown that $T_x\mathbb{M}$ is a vector space over $\mathbb{R}$ spanned by d linearly independent vectors

$$\partial_i : f \mapsto \frac{\partial f^*}{\partial x_i}|_{\phi(x)}$$

for $f \in C^r(\mathbb{M})$, $f^* = f \circ \phi^{-1}$ being the **pull back** of f and x_i is a basis of $O = \phi(U)$ where U is an open set containing x. (c.f. [28], p. 10)

Remark 2.1.4. Another way of looking at the set of vector fields on $\mathbb{M}$ is to identify it with the space $\Gamma(T\mathbb{M})$ of smooth sections of the tangent bundle, these are the smooth maps $X : \mathbb{M} \mapsto T\mathbb{M}$. In fact, we will use $\Gamma(T\mathbb{M})$ and $\mathcal{D}^1(\mathbb{M})$ to denote vector fields on $\mathbb{M}$ interchangably without further explanation.

Let R be a commutative ring and A a module over R. Let A^* denote the set of all R-linear maps of A into R. Then A^* is an R-module and it is called the **dual** of A.

Definition 2.1.5. Let $\mathbb{M}$ be a C^k-manifold and let $\mathcal{D}_1(\mathbb{M})$ denote the dual of $\mathcal{D}^1(\mathbb{M})$. The elements of $\mathcal{D}_1(\mathbb{M})$ are called the **differential 1-forms** on $\mathbb{M}$ (or just **1-forms**).

Remark 2.1.6. For each $x \in \mathbb{M}$, let $T_x^*\mathbb{M} = (T_x\mathbb{M})^*$ be the cotangent space at x, the dual space to $T_x(\mathbb{M})$. The cotangent bundle $T^*\mathbb{M} = \bigcup_{x\in\mathbb{M}} T_x^*\mathbb{M}$ is a differentiable manifold, and it can be checked that a smooth section $\theta \in \Gamma(T^*\mathbb{M})$ is a differential 1-form. Hence, the smooth sections $\Gamma(T^*\mathbb{M})$ and $\Gamma(T\mathbb{M})$ are also duals of each other.

For a C^k-manifold $\mathbb{M}$, consider the $C^k(\mathbb{M})$-module

$$\mathcal{D}^1 \times ... \times \mathcal{D}^1 \qquad r \text{ times}$$

and let $\mathcal{D}_r$ denote the $C^k(\mathbb{M})$-module of all $C^k(\mathbb{M})$-multilinear maps of $\mathcal{D}^1 \times ... \times \mathcal{D}^1$ into $C^k(\mathbb{M})$. Similarly, let $\mathcal{D}^s$ denote the set of all $C^k(\mathbb{M})$-multilinear maps. Finally, let $\mathcal{D}_s^r$ denote the set of all $C^k(\mathbb{M})$-multilinear maps of

$$\mathcal{D}_1 \times ... \times \mathcal{D}_1 \times \mathcal{D}^1 \times ... \times \mathcal{D}^1$$

into $C^k(\mathbb{M})$, where the above product contains r superscripts and s subscripts.

Definition 2.1.7. A **tensor field** Θ on $\mathbb{M}$ of type (r, s) is an element of $\mathcal{D}_s^r(\mathbb{M})$. Moreover, Θ is said to be **contravariant** of degree r and **covariant** of degree s.

Tensor fields of type $(0,0)$, $(1,0)$ and $(0,1)$ on $\mathbb{M}$ are the differentiable functions, the vector fields, and the 1-forms on $\mathbb{M}$ respectively. In general, the bundle of (r, s)-tensors is

$$T^{r,s}\mathbb{M} = \bigcup_{x\in\mathbb{M}} T_x\mathbb{M}^{\otimes r} \otimes_{\mathbb{R}} T_x^*\mathbb{M}^{\otimes s}.$$

An (r, s)-tensor Θ on $\mathbb{M}$ is a smooth section of the vector bundle $T^{r,s}\mathbb{M}$. For each $x \in \mathbb{M}$, the value of the tensor field $\Theta_x \in \mathrm{Hom}_{\mathbb{R}}(T_x\mathbb{M}^{\otimes s}, T_x\mathbb{M}^{\otimes r})$.

In local coordinates, $x = \{x^r\}$ with $X_i = \partial/\partial x_i$, we denote the frame bundle on T^*M dual to $\{X_i\}$ by $\{dx^i\}$, so $dx^i(X_j) = \delta^i_j$. In terms of this basis, an (r, s)-tensor locally can be written as

$$\Theta = \Theta^{i_1,\dots,i_r}_{j_1,\dots,j_s} X_{i_1} \otimes \dots \otimes X_{i_r} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}.$$

Definition 2.1.8. An **affine connection** on a manifold $\mathbb{M}$ is a rule ∇ which assigns to each vector field X a linear map $\nabla_X : \Gamma(T\mathbb{M}) \to \Gamma(T\mathbb{M})$, satisfying the following conditions,

1. $\nabla_{fX+gY} = f\nabla_X + g\nabla_Y$.
2. $\nabla_X(fY) = f\nabla_X(Y) + (Xf)Y$

for $f, g \in C^k(\mathbb{M}), X, Y \in \Gamma(T\mathbb{M})$. The operator ∇_X is called **covariant differentiation** with respect to X.

In local coordinates, a connection can be expressed in terms of its Christoffel symbols.

Definition 2.1.9. Let $x \in \mathbb{M}$ with $x = \{x^1, \dots, x^d\}$ as the local chart on an open subset O of M, so the vector fields $X_i = \partial/\partial x_i$ spans the tangent space $T_x\mathbb{M}$ for each $x \in O$. The **Christoffel symbols** Γ^k_{ij} are functions on O defined uniquely by the relation $\nabla_{X_i} X_j = \Gamma^k_{ij} X_k$, and the Einstein summation convention is applied over k.

Definition 2.1.10. A vector field V along a curve $\{x_t\}$ on M is said to be **parallel** along the curve if $\nabla_{\dot{x}} V = 0$ at every point of the curve. The vector V_{x_t} at x_t is said to be the **parallel transport** of V_{x_0} along the curve.

In local coordinates, if $x_t = \{x^i_t\}$ and $V_{x_t} = v^i(t)X_i$, then V is parallel if and only if $v^i(t)$ satisfies the following system of equations,

$$\dot{v}^k(t) + \Gamma^k_{jl}(x_t)\dot{x}^j_t v^l(t) = 0.$$

Definition 2.1.11. A curve $\{\gamma_t\}$ on M is called a **geodesic** if $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ along $\{\gamma_t\}$.

In local coordinates, γ_t is a geodesic if it satisfies,

$$\ddot{\gamma}_t^k + \Gamma_{jl}^k(\gamma_t)\dot{\gamma}_t^j\dot{\gamma}_t^l = 0.$$

Note that a geodesic is uniquely determined by its initial position γ_0 and its initial direction $\dot{\gamma}_0$.

Theorem 2.1.12. *([28], p. 32) Let $\mathbb{M}$ be a manifold with an affine connection. Let x be any point in $\mathbb{M}$, and γ_t^X be a geodesic starting at x in the direction of X. Then there exists an open neighbourhood N_0 of 0 in $T_x\mathbb{M}$ and an open neighbourhood N_x in $\mathbb{M}$ such that the map $X \mapsto \gamma_1^X$ is a diffeomorphism of N_0 onto N_x.*

Definition 2.1.13. The map $X \mapsto \gamma_1^X$ in the previous theorem is called the **exponential map** at x, and it will be denoted by Exp (or Exp_x, if there is ambiguity about the starting point).

Definition 2.1.14. Let $\mathbb{M}$ be a manifold with an affine connection and x a point on $\mathbb{M}$. An open neighbourhood N_0 of the origin in $T_x\mathbb{M}$ is said to be **normal** if

- The mapping Exp is a diffeomorphism of N_0 onto an open neighbourhood of N_x of x in $\mathbb{M}$.
- If $X \in N_0$, and $0 \leq t \leq 1$, then $tX \in N_0$.
- A neighbourhood N_x of $x \in \mathbb{M}$ is called a **normal neighbourbood** of x if $N_x = \mathrm{Exp}N_0$ where N_0 is a normal neighbourhood of 0 in $T_x\mathbb{M}$.
- Let $X_1, ..., X_m$ denote some basis of $T_x\mathbb{M}$, the inverse map

$$\mathrm{Exp}_x(a_1X_1 + ... + a_mX_m) \to (a_1, ..., a_m)$$

of N_x into $\mathbb{R}^d$ is called a system of **normal coordinates** at x.

Theorem 2.1.15. *Let $\mathbb{M}$ be a C^k manifold with an affine connection. Then each point $x \in \mathbb{M}$ has a normal neighbourhood N_x which is a normal neighbourhood of each of its points. In particular, two points in N_x can be joined by exactly one geodesic segment, up to a linear change of parameter, contained in N_x.*

Let $\mathbb{M}$ be a manifold with an affine connection ∇. We put

$$T(X,Y) = \nabla_X(Y) - \nabla_Y(X) - [X,Y]$$

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$$

for all $X, Y \in \mathcal{D}^1$. In differential geometry, the tensors $T(X,Y) \in \mathcal{D}^1_2(\mathbb{M})$ and $R(X,Y) \in \mathcal{D}^1_3(\mathbb{M})$ are called the **torsion tensor** and the **Riemann curvature tensor** respectively [1].

Definition 2.1.16. Let $\mathbb{M}$ be a C^∞-manifold. A **Riemannian structure** or **Riemannian metric** on $\mathbb{M}$ is a (0,2)-type tensor field g that satisfies,

- $g(X,Y) = g(Y,X)$ for all $X, Y \in \mathcal{D}^1(\mathbb{M})$.
- For each $x \in \mathbb{M}$, g_x is a nondegenerate bilinear form on $T_x\mathbb{M} \times T_x\mathbb{M}$.
- g_x to be positive definite for every $x \in \mathbb{M}$.

A **Riemannian manifold** is a connected C^∞-manifold with a Riemannian structure.

Remark 2.1.17. An intuitive interpretation of the Riemannian metric is as a collection of inner products on the tangent spaces $T_x\mathbb{M}$ for each $x \in \mathbb{M}$. For this reason, we will use $g(X,Y)$ / $\langle X, Y\rangle$ and $g_x(X,Y)$ / $\langle X, Y\rangle_x$ interchangably without further warning.

Theorem 2.1.18. *(c.f. [28], p. 48) Let $\mathbb{M}$ be a Riemannian manifold. There exists a unique affine connection on $\mathbb{M}$ satisfying the following conditions:*

- *The torsion tensor T is identically zero, i.e. $\nabla_X Y - \nabla_Y X = [X,Y]$.*
- *The parallel displacement preserves the inner product on tangent spaces, i.e. $\nabla_Z g = 0$.*

Definition 2.1.19. Let $\mathbb{M}$ be a Riemannian manifold, $x_0, x_1 \in \mathbb{M}$. Suppose further that there exists a geodesic curve γ such that $\gamma_0 = x_0$ and $\gamma_1 = x_1$. The **distance** between x_0 and x_1, denoted by $d(x_0, x_1)$ is defined by

$$d(x_0, x_1) = \int_0^1 \sqrt{g_{\gamma_t}(\dot\gamma_t, \dot\gamma_t)}dt.$$

[1]For details of the above claims, see [28] , p. 44.

Finally, we will conclude this introductory section by stating a couple of embedding theorems regarding Riemannian manifolds.

Theorem 2.1.20. *(Whitney Embedding Theorem) Let $\mathbb{M}$ be a connected smooth d-dimensional manifold, then $\mathbb{M}$ can be smoothly embedded in Euclidean $2m$-space. That is, there exists a smooth injective map $\phi : \mathbb{M} \to \mathbb{R}^{2d}$.*

Remark 2.1.21. This is the best linear bound on the smallest dimensional Euclidean space that all d-dimensional manifolds embed in. For example, the d-dimensional real projective plane cannot be embedded into Euclidean $(2d-1)$-space if $d = 2^n$, for some $n \in \mathbb{N}$.

2.2 Rolling without slipping

The intuitive idea behind the "rolling without slipping" is to curl a curve that lives on $\mathbb{R}^d$ onto a d-dimensional manifold via the so called "development map". First of all, we will revise how a deterministic path can be rolled onto a manifold. This is done by first "lifting" the curve up to a "frame bundle", and then projecting it back down to the manifold. This process can be constructed by solving a system of ODEs.

By stochastic development, we really mean reproducing the same process, except we will end up solving systems of Stratonovich equations instead of ODEs. Stochastic development is one common way of constructing continuous stochastic processes on manifolds from their $\mathbb{R}^d$ counterparts. However, we will see that this technique will break down when the corresponding process is càdlàg. In particular, the manifold-valued Lévy process obtained in this fashion will not be a Markov process unless it is isotropically distributed (c.f. [3] , [6]).

We will follow the approach of [33] , §2.1 and §2.3.

2.2.1 Deterministic development

Definition 2.2.1. A **frame** at x is an $\mathbb{R}$-linear isomorphism $u : \mathbb{R}^d \to T_x\mathbb{M}$. The set of all frames at x is denoted by $\mathfrak{F}_x(\mathbb{M})$, while the **frame bundle** is $\mathfrak{F}(\mathbb{M}) = \bigcup_{x\in\mathbb{M}} \mathfrak{F}_x(\mathbb{M})$.

The frame bundle can be made into a differentiable manifold of dimension $d+d^2$, and the canonical projection $\pi : \mathfrak{F}(\mathbb{M}) \to \mathbb{M}$ is a smooth map. The group $GL(d,\mathbb{R})$ acts on $\mathfrak{F}(\mathbb{M})$ fibre-wise in the sense that each fibre $\mathfrak{F}_x(\mathbb{M})$ is diffeomorphic to $GL(d,\mathbb{R})$, and $M = \mathfrak{F}(\mathbb{M})/GL(d,\mathbb{R})$ in the sense of diffeomorphism. This makes the triplet $(\mathfrak{F}(\mathbb{M}), \mathbb{M}, GL(d,\mathbb{R}))$ into a principal bundle with structure group $GL(d,\mathbb{R})$. Moreover, we can easily see that the tangent bundle has the form

$$TM = \mathfrak{F}(M) \times_{GL(d,\mathbb{R})} \mathbb{R}^d$$

by the action $(u,e) \mapsto ue$, with $u \in GL(d,\mathbb{R}), e \in \mathbb{R}^d$ and ue is simply the matrix u acting on e.

Definition 2.2.2. Let $T_u\mathfrak{F}(\mathbb{M})$ be the tangent space of the frame bundle at u. A vector $X \in T_u\mathfrak{F}(\mathbb{M})$ is called **vertical** if it is tangent to the fibre $\mathfrak{F}(\mathbb{M})_{\pi u}$.

The space of vertical vectors at u is denoted by $V_u\mathfrak{F}(M)$; it is a subspace of $T_u\mathfrak{F}(\mathbb{M})$ of dimension d^2, while the total dimension of $T\mathfrak{F}(\mathbb{M})$ is $d + d^2$. If $\mathbb{M}$ is equipped with a connection ∇, we can interpret a curve $\{u_t\}$ in $\mathfrak{F}(\mathbb{M})$ as a smooth choice of frames at each point of the curve $\{\pi u_t\}$ on $\mathbb{M}$.

Definition 2.2.3. The curve $\{u_t\}$ is called **horizontal** if for each $e \in \mathbb{R}^d$, the vector field $\{u_t e\}$ is parallel along $\{\pi u_t\}$. A tangent vector $X \in T_u\mathfrak{F}(\mathbb{M})$ is called **horizontal** if it is the tangent vector of a horizontal curve from u. The space of horizontal vectors at u is denoted by $H_u\mathfrak{F}(\mathbb{M})$.

Now we will make the connection from the horizontal frames back to the manifold itself. Note that $H_u\mathfrak{F}(\mathbb{M})$ is a d-dimensional subspace of $T_u\mathfrak{F}(\mathbb{M})$, and we have the decomposition,

$$T_u\mathfrak{F}(\mathbb{M}) = V_u\mathfrak{F}(\mathbb{M}) \oplus H_u\mathfrak{F}(\mathbb{M}).$$

Hence, the canonical projection $\pi : \mathfrak{F}(\mathbb{M}) \to \mathbb{M}$ gives an isomorphism π_* between $H_u\mathfrak{F}(\mathbb{M})$ and $T_{\pi u}\mathbb{M}$. Moreover, for each $X \in T_x\mathbb{M}$, and a frame u at x, there is a unique horizontal vector X^*, the **horizontal lift** of X to u, such that $\pi_* X^* = X$. In fact, for every vector field X on $\mathbb{M}$, X^* is a horizontal vector field on $\mathfrak{F}(\mathbb{M})$. Moreover, given a curve $\{x_t\}$ and a frame u_0 and x_0, there is a unique horizontal curve $\{u_t\}$ such that $\pi u_t = x_t$. It is called the **horizontal lift** of x_t from u_0. The linear map

$$\tau_{t_0 t_1} = u_{t_1} u_{t_0}^{-1} : T_{x_{t_0}}\mathbb{M} \to T_{x_{t_1}}\mathbb{M}$$

is independent of the choice of the initial frame u_0 and is called the **parallel transport** along $\{x_t\}$.

Remark 2.2.4. The horizontal lift $\{u_t\}$ of $\{x_t\}$ is obtained by solving an ordinary differential equation. A detailed proof of why the solution never blows up can be found in [38]. However, the stochastic version of this fact will need a proof when we consider stochastic developments and stochastic horizontal lifts in the next section.

A local chart $x = \{x^i\}$ on a neighbourhood $O \subseteq \mathbb{M}$ induces a local chart $\tilde{O} = \pi^{-1}(O)$ in $\mathfrak{F}(\mathbb{M})$ as follows. Let $X_i = \partial/\partial x^i, i = 1, ..., d$, be the moving frame defined by the local chart. For a frame $u \in \tilde{O}$, we have $ue_i = \langle e_i, X\rangle_{\mathbb{R}^d}$ where e is some matrix $e = (e^i_j) \in GL(d, \mathbb{R})$. Then, $(x, e) = (x^i, e^i_j) \in \mathbb{R}^{d+d^2}$ is a local chart for $\tilde{O}$. In terms of this chart, the vertical subspace $V_u\mathfrak{F}(\mathbb{M})$ is spanned by $X_{kj} = \partial/\partial e^k_j$, $1 \leq j, k \leq d$, and the vector fields $\{X_i, X_{kj}, 1 \leq i, j \leq d\}$ span the tangent space $T_u\mathfrak{F}(\mathbb{M})$ for every $u \in \tilde{O}$. We will need the local expression for the fundamental horizontal vector field H_i.

Proposition 2.2.5. *In terms of the local chart on $\mathfrak{F}(\mathbb{M})$ described above, at $u = (x, e) = (x^i, e^k_j) \in \mathfrak{F}(\mathbb{M})$, we have*

$$H_i(u) = e^j_i X_j - e^j_i e^l_m \Gamma^k_{jl}(x) X_{km},$$

where

$$X_i = \frac{\partial}{\partial x_i}, X_{km} = \frac{\partial}{\partial e^k_m}.$$

and we use the Einstein summation convention.

Let $\{u_t\}$ be a horizontal lift of a differentiable curve $\{x_t\}$ on $\mathbb{M}$. Since $\dot{x}_t \in T_{x_t}\mathbb{M}$, we have $u_t^{-1}\dot{x}_t \in \mathbb{R}^d$. This motivates the following definition.

Definition 2.2.6. The **anti-development** of the curve $\{x_t\}$, or of the horizontal curve $\{u_t\}$, is a curve $\{w_t\}$ in $\mathbb{R}^d$ defined by

$$w_t = \int_0^t u_s^{-1}\dot{x}_s ds.$$

Note that w depends on the choice of u_0 and x_0, in a simple way. Take another horizontal lift $\{v_t\}$ of $\{x_t\}$, and suppose that $u_0 = v_0 g$ for $g \in GL(d, \mathbb{R})$, then the anti-development of $\{v_t\}$ is $\{gw_t\}$.

Observe that $u_t \dot{w}_t = \dot{x}_t$, so we obtain that $H_{\dot{w}_t}(u_t) = \widetilde{u_t \dot{w}_t} = \tilde{\dot{x}}_t = \dot{u}_t$. So $\{w_t\}, \{u_t\}$ and $\{x_t\}$ on $\mathbb{M}$ are related by an ordinary differential equation on $\mathfrak{F}(\mathbb{M})$

$$\dot{u}_t = \langle H(u_t), \dot{w}_t \rangle_{\mathbb{R}^d}$$

If $\mathbb{M}$ is a Riemannian manifold with Riemannian metric $g(.,.)$, then we can restrict ourselves to a smaller set of frames, the orthogonal frames.

Definition 2.2.7. The **orthogonal frame bundle**, denoted by $\mathcal{O}(\mathbb{M})$ is the set of all Euclidean isometries $u : \mathbb{R}^d \to T_x\mathbb{M}$.

The action group is therefore reduced from $GL(d, \mathbb{R})$ to $O(d)$, and $\mathcal{O}(\mathbb{M})$ is a principal fibre bundle with structure group $O(d)$, in the sense of [33]. A connection ∇ is compatible with a Riemannian metric if and only if

$$\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(X, \nabla_Y Z) \tag{2.1}$$

for every triple of vector fields X, Y, Z. A connection that is compatible with the Riemannian metric preserves orthogonality of the frame. This connection is called the Levi-Civita connection and the existence and uniqueness of such a connection is given by the *Fundamental theorem of Riemannian geometry* which we state below.

Theorem 2.2.8. *(Fundamental theorem of Riemannian geometry) Let $\mathbb{M}$ be a Riemannian manifold with metric g, then there is a unique connection ∇ such that (2.1) holds for every $X, Y, Z \in \mathcal{D}^1(\mathbb{M})$, and in addition $\nabla_X Y - \nabla_Y X = [X, Y]$.*

Hence, if a Riemannian manifold is equipped with such a connection, then everything we said about $\mathfrak{F}(\mathbb{M})$ carries over to the orthogonal frame bundle $\mathcal{O}(\mathbb{M})$. Also note that although each individual X_{ij} may not be tangent to the fibre $\mathcal{O}(\mathbb{M})_x$, the linear combination $e^l_m \Gamma^k_{jl} X_{km}$ are in fact tangent. This makes $H_i(u)$ a vector field on $\mathcal{O}(\mathbb{M})$ for each $i = 1, ..., d$.

2.2.2 Stochastic development

Now, we are going to "roll" a continuous semimartingale onto a manifold using the strategy described in the previous subsection. However, before this can be done, we will need to understand the stochastic version of

$$\dot{u}_t = \langle H(u_t), \dot{w}_t \rangle_{\mathbb{R}^d}.$$

This will involve solving stochastic differential equations on manifolds, and the solution will typically be a manifold valued semimartingale. Hence, we make the following definitions.

Definition 2.2.9. Let $\mathbb{M}$ be a differentiable manifold equipped with a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Let τ be a $\mathcal{F}_t$-stopping time. A **semimartingale** on $\mathbb{M}$ up to a stopping time τ is a stochastic process X on $\mathbb{M}$, such that for every $f \in C^\infty(\mathbb{M})$, $f(X)$ is a real-valued semimartingale on $[0, \tau)$.

Definition 2.2.10. A **stochastic differential equation**, abbreviated as **SDE**, on $\mathbb{M}$ is defined by l-vector fields $V = (V_1, ..., V_l)$ on $\mathbb{M}$, an $\mathbb{R}^l$-valued semimartingale Z referred to as the **driving** semimartingale, and an $\mathbb{M}$-valued random variable $X_0 \in \mathcal{F}_0$, written (symbolically) as

$$dX_t = \left\langle V(X_t), \Box dZ_t^i \right\rangle_{\mathbb{R}^l} \tag{2.2}$$

An $\mathbb{M}$-valued semimartingale X, defined up to a stopping time τ is a **solution** to this equation up to τ, if for all $f \in C^\infty(\mathbb{M})$, and for each $0 \leq t < \tau$,

$$f(X_t) = f(X_0) + \int_0^t \left\langle V_i f(X_s), \Box dZ_s^i \right\rangle_{\mathbb{R}^l}. \tag{2.3}$$

and where $\Box = ., \circ, \diamond$ to denote the Itô, Fisk-Stratonovich and Marcus canonical integral respectively.

Definition 2.2.11. Let $\mathbb{M}$ be a manifold equipped with a connection ∇, and a frame bundle $\mathfrak{F}$. Let U_t be an $\mathfrak{F}$-valued process and Z_t a d-dimensional semimartingale. Consider the following SDE,

$$dU_t = \langle H(U_t), \circ dZ_t \rangle_{\mathbb{R}^d} \tag{2.4}$$

1. An $\mathfrak{F}(\mathbb{M})$-valued semimartingale U is said to be **horizontal** if there exists $\mathbb{R}^d$-valued semimartingale Z such that (2.4) holds. The unique Z is called the **anti-development** of U, or of its projection $X = \pi U$.
2. Let Z be an $\mathbb{R}^d$-valued semimartingale and U_0 an $\mathfrak{F}(\mathbb{M})$-valued $\mathcal{F}_0$-measurable random variable. The solution U of (2.4) is called a **stochastic development** of Z in $\mathfrak{F}(\mathbb{M})$. Its projection $X = \pi U$ is called a **stochastic development** of Z in $\mathbb{M}$.
3. Let X be an $\mathbb{M}$-valued semimartingale. An $\mathfrak{F}(\mathbb{M})$-valued horizontal semimartingale U such that its projection $\pi U = X$ is called a **stochastic horizontal lift** of X.

In order to establish that the above definitions are well defined, especially the transitions $X \to U$ and $U \to W$, we will prove the existence of a horizontal lift by deriving a stochastic differential equation for it on the frame bundle $\mathfrak{F}(\mathbb{M})$ driven by X. By Whitney's embedding theorem, we may regard $\mathbb{M}$ as a closed submanifold of $\mathbb{R}^N$, and regard X as an $\mathbb{R}^N$-valued semimartingale. We will prove the following lemma.

Lemma 2.2.12. *Let $\mathbb{M}$ be a closed submanifold of $\mathbb{R}^N$. For each $x \in \mathbb{M}$, let $P(x) : \mathbb{R}^N \to T_x\mathbb{M}$ be the orthogonal projection from $\mathbb{R}^n$ onto the subspace $T_x\mathbb{M} \subseteq \mathbb{R}^N$, where $n > N$. If X is an $\mathbb{M}$-valued semimartingale, then we have on $\mathbb{R}^N$,*

$$X_t = X_0 + \int_0^t P(X_s) \circ dX_s$$

Proof. Let $\{e_i\}, i = 1, 2, ..., N$ be the canonical basis for $\mathbb{R}^N$, and define

$$P_i(x) = P(x)e_i, Q_i(x) = e_i - P(x)e_i,$$

so that $P_i(x)$ is tangent to $\mathbb{M}$, and $Q_i(x)$ is normal to $\mathbb{M}$, and $P_ix + Q_ix = e_i$ for all $x \in \mathbb{R}^N$. Let

$$Y_t = X_0 + \int_0^t P_i(X_s) \circ dX_s^i.$$

We first show that Y is a process on $\mathbb{M}$. Let f be a smooth nonnegative function on $\mathbb{R}^N$ that vanishes only on $\mathbb{M}$. By Itô's formula,

$$f(Y_t) = f(X_0) + \int_0^t P_i f(X_s) \circ dX_t^i.$$

But if $x \in \mathbb{M}$, then $P_i(x) \in T_x\mathbb{M}$ and $P_i f(x) = 0$. Hence $P_i f(X_t) = 0$ and $f(Y_t) = 0$. This implies that $Y_t \in \mathbb{M}$ for each $t \geq 0$.

Now, for each $x \in \mathbb{R}^N$, define $h(x) = \inf_{y \in M} d(x, y)$. Since $\mathbb{M}$ is a closed submanifold of $\mathbb{R}^N$, $h : \mathbb{R}^N \to \mathbb{M} \subseteq \mathbb{R}^N$ is a well defined smooth function in a neighbourhood of M, and is constant on each line segment perpendicular to $\mathbb{M}$. Hence, we get that $Q_i h(x) = 0$ for each $x \in \mathbb{M}$ and each $i = 1, 2, ..., N$, as the Q_i's are normal to the manifold. Hence, if we regard e_i as a vector field on $\mathbb{R}^N$, we have

$$P_i h(x) = P_i h(x) + Q_i h(x) = e_i h(x), x \in \mathbb{M}.$$

Using the above and the fact that $X_t, Y_t \in \mathbb{M}$, we have

$$\begin{aligned} Y_t = h(Y_t) &= X_0 + \int_0^t P_i h(X_s) \circ dX_s^i \\ &= X_0 + \int_0^t e_i h(X_s) \circ dX_s^i \\ &= h(X_t) = X_t. \end{aligned}$$

This completes the proof. □

We now prove that the solution to the above equation is indeed the horizontal lift of X. We begin by proving another simple geometrical fact. First, let $f = \{f^i\} : \mathbb{M} \to \mathbb{R}^N$ be the coordinate function. Its lift $\tilde{f} : \mathfrak{F}(\mathbb{M}) \to \mathbb{R}^N$ defined by $\tilde{f}(u) = f(\pi u) = \pi u \in \mathbb{M} \subseteq \mathbb{R}^N$, is just the projection $\pi : \mathfrak{F}(\mathbb{M}) \to \mathbb{M}$ written as an $\mathbb{R}^N$-valued function on $\mathfrak{F}(\mathbb{M})$.

Lemma 2.2.13. *Let e_i be the i-th coordinate of $\mathbb{R}^d$, $\tilde{f} : \mathfrak{F}(\mathbb{M}) \to \mathbb{M} \subseteq \mathbb{R}^N$ be the projection function. The following two identities hold on $\mathfrak{F}(\mathbb{M})$:*

$$P_i^* \tilde{f}(u) = P_i(\pi u),$$

$$\left\langle P(\pi u), H_i \tilde{f}(u) \right\rangle_{\mathbb{R}^N} = ue_i$$

for $i = 1, ..., d$.

Proof. If $\{u_t\}$ is the horizontal lift from $u_0 = u$ of a curve $\{x_t\}$ with $\dot{x}_0 = P_\alpha(\pi u)$, then $P^*_\alpha(u) = \dot{u}_0$. Hence,

$$P^*_\alpha f(u) = \frac{d\tilde{f}(u_t)}{dt} = \frac{d\pi u_t}{dt} = \dot{x}_0 = P_\alpha(\pi u).$$

This establishes the first identity, and the proof of the second identity is very similar. Take $\{v_t\}$ to be the horizontal lift from $v_0 = u$ of a curve $\{y_t\}$ on M with $\dot{y}_0 = ue_i$. Then,

$$H_i \tilde{f}(u) = \frac{d\tilde{f}(v_t)}{dt} = \frac{d(\pi v_t)}{dt} = \dot{y}_0 = ue_i.$$

This means $H_i \tilde{f}(u) \in T_{\pi u}\mathbb{M}$ for each $i = 1, ..., d$. Hence, $P(\pi u) H_i \tilde{f}(u) = H_i \tilde{f}(u)$ and we have

$$\left\langle P(\pi u), H_i \tilde{f}(u) \right\rangle_{\mathbb{R}^N} = P(\pi u) H_i \tilde{f}(u) = H_i \tilde{f}(u) = ue_i,$$

which proves the second identity. □

Now we can prove the main theorems for this section.

Theorem 2.2.14. *(c.f. Chapter 2 of [33]) A horizontal semimartingale U on the frame bundle $\mathfrak{F}(\mathbb{M})$ has a unique anti-development W. In fact,*

$$W_t = \int_0^t \left\langle U_s^{-1} P(X_s), \circ dX_s \right\rangle,$$

where $X_t = \pi U_t$.

Proof. W is the $\mathbb{R}^d$-valued semimartingale defined to satisfy,

$$dU_t = \langle H(U_t), \circ dW_t \rangle_{\mathbb{R}^d}.$$

Let $\tilde{f}$ be the canonical projection, such that $\tilde{f}(U_t) = \pi U_t = X_t$, we have for every $\alpha = 1, ..., N$

$$dX_t^\alpha = \left\langle H\tilde{f}^\alpha(U_t), \circ dW_t \right\rangle_{\mathbb{R}^d}.$$

Multiply both sides by $U^{-1}P_\alpha(X_t) \in \mathbb{R}^d$, and using the second identity from Lemma 2.2.12, note that since

$$\left\langle P(\pi u), H_i \tilde{f}^\alpha(u) \right\rangle_{\mathbb{R}^N} = \sum_{\alpha=1}^{N} P_\alpha(\pi u) H_i \tilde{f}^\alpha(u) = ue_i,$$

we have

$$U_t^{-1}P_\alpha(X_t) \circ dX_t^\alpha = \left\langle U_t^{-1}P_\alpha(X_t) H\tilde{f}^\alpha(U_t), \circ dW_t \right\rangle_{\mathbb{R}^d} = \left\langle e_i, dW_t^i \right\rangle.$$

□

Theorem 2.2.15. *Suppose that $X = \{X_t\}_{0\le t<\tau}$ is a semimartingale on $\mathbb{M}$ up to a stopping time τ, and U_0 an $\mathfrak{F}(\mathbb{M})$-valued $\mathcal{F}_0$-measurable random variable, such that $\pi U_0 = X_0$. Then, there is a unique horizontal lift $\{U_t\}_{0\le t<\tau}$ of X starting from U_0. Moreover, the horizontal lift is also defined up to τ.*

Proof. Following Remark 2.2.4, we will first prove that without assuming uniqueness of U, a horizontal lift U of X is always defined up to τ, i.e. there is no explosion in the vertical direction.

By a stopping time argument, we can assume without loss of generality that $\tau = \infty$, so the semimartingale X is defined on all of $[0, \infty)$. We can also assume that there is a relatively compact neighbourhood O covered by a local chart $x = \{x^i\}$, such that $X_t \in O$ for all $t \ge 0$, and $u = \{x^i, e^i_j\}$ be the corresponding local charts defined on $\mathfrak{F}(\mathbb{M})$. Let

$$X_i = \frac{\partial}{\partial x_i}, X_{ij} = \frac{\partial}{\partial e^i_j},$$

these are vector fields on $\mathfrak{F}(\mathbb{M})$. Define,

$$h(u) = \sum_{i,j}^{d} |e^i_j|^2,$$

it is enough to show that $h(U_t)$ does not explode.

One way of doing this is by first writing the horizontal lift P^*_α of P_α in the local coordinates, and then deriving a stochastic differential equation for $h(U_t)$. We proceed as follows. Since $ue_i = e_i^j X_j$ by definition, we have $X_q = f_q^i ue_i$, where $\{f_i^j\}$ is the inverse matrix of $\{e_i^j\}$. The horizontal lift H_i of ue_i is given by

$$H_i(u) = e_i^j X_j - e_i^j e_m^l \Gamma_{jl}^k(x) X_{km},$$

and hence the horizontal lift of X_q is

$$X_q^* = X_q - e_m^l \Gamma_{ql}^k X_{km}.$$

If $P_\alpha(x) = p_\alpha^q(x) X_q$, then the horizontal lift of $P_\alpha(x)$ is

$$P_\alpha^*(u) = p_\alpha^q X_q - p_\alpha^q e_m^l \Gamma_{qj}^k(x) X_{km}.$$

Using Itô's formula and the fact that $dU_t = P_\alpha^*(U_t) \circ dX_t^\alpha$, we obtain that

$$\begin{aligned} h(U_t) &= h(U_0) + \int_0^t P_\alpha^* h(U_s) dX_s^\alpha + \frac{1}{2}\int_0^t P_\alpha^* P_\beta^* h(U_s) d\left\langle X^\alpha, X^\beta \right\rangle_s \\ &\leq h(U_0) + \int_0^t Ch(U_s) dX_s^\alpha + \frac{1}{2}\int_0^t Ch(U_s) d\left\langle X^\alpha, X^\beta \right\rangle_s . \end{aligned}$$

As the coefficients of P_α^*, Γ_{ij}^k and p_α^i are uniformly bounded on the relatively compact neighbourhood O; from the definition of h, we can assert that there exists a constant C such that $|P_\alpha^* h| \leq Ch$ and $|P_\alpha^* P_\beta^* h| \leq Ch$.

Now, the problem has been reduced to problem of determining the explosion time of a solution to a real valued stochastic differential equation, and standard theory tells us $h(U_t)$ never explodes. Hence our first proof is finished.

Next, we determine the uniqueness of the horizontal lift. From Lemma 2.2.10, a good candidate for the horizontal lift U of X is the unique solution of the following equation on $\mathfrak{F}(\mathbb{M})$,

$$dU_t = \langle P^*(U_t), \circ dX_t \rangle ,$$

where $P^*(u)$ is the horizontal lift of $P(\pi u)$. First, we will need to verify that this is indeed a horizontal lift of X. Since U is obviously horizontal, it suffices to show that $\pi U = X$. Put $Y_t = \pi(U_t)$, we have by Lemma 2.2.12,

$$dY_t = P_\alpha^* \pi(U_t) \circ dX_t^\alpha = P_\alpha(Y_t) \circ dX_t^\alpha,$$

with initial condition $Y_0 = \pi(U_0) = X_0$. On the other hand, by lemma 2.2.11, X is a solution of the same equation. Hence by uniqueness, we have $X = Y$ and this establishes that U is a horizontal lift of X.

Now, let Π be another horizontal lift of X. Since Π is horizontal, there exists an $\mathbb{R}^d$-valued semimartingale W such that

$$d\Pi_t = H_i(\Pi_t) \circ dW_t^i. \tag{2.5}$$

By theorem 2.2.13, the anti-development is given by

$$dW_t = \Pi_t^{-1} P_\alpha(X_t) \circ dX_t^\alpha.$$

Substituting this back into (2.5), and using the fact that the horizontal lift $P_\alpha^*(\Pi_t)$ of $P_\alpha(X_t)$ is given by

$$P_\alpha^*(\Pi_t) = \sum_{i=1}^{d} (\Pi^{-1} P_\alpha(X_t))^i H_i(\Pi_t).$$

We find that $d\Pi_t = P_\alpha^*(\Pi_t) \circ dX_t^\alpha$, and so Π satisfies the same equation as U. By uniqueness of strong solutions to SDE's, we have $\Pi = U$ up to indistinguishability.

□

2.3 Horizontal Lévy processes on Riemannian manifolds

There have been several attempts in applying the techniques suggested in the previous section to the construction of a Lévy process on a manifold $\mathbb{M}$. Most notably, through the papers [3] and [6], Applebaum and Estrade were able to extend the Eels-Elworthy construction to construct isotropic Lévy processes on Riemannian manifolds. In this section, we will briefly outline the construction of [3], and highlight the problems that prevented this technique from being extended to general Lévy processes as it was pointed out by Applebaum in [3].

Let $\mathbb{M}$ be a d-dimensional connected Riemannian manifold, and let $\mathcal{O}(\mathbb{M})$ be the bundle of orthogonal frames over $\mathbb{M}$ with the canonical surjection $\pi : \mathcal{O}(\mathbb{M}) \to \mathbb{M}$. Let $r_p \in \mathcal{O}(\mathbb{M})$ with $\pi(r_p) = p$. We may regard $r_p = (r_p^1, ..., r_p^d)$ as a linear isometry $r_p : \mathbb{R}^d \to T_p(\mathbb{M})$, via the action $r_p(x) = \sum_j x_j r_p^j$ for each $x \in \mathbb{R}^d$.

We note that $\mathbb{M}$ equipped with its Riemannian connection enables us to write the Whitney direct sum

$$T(\mathcal{O}(\mathbb{M})) = H(\mathbb{M}) \oplus V(\mathbb{M}),$$

where $H(\mathbb{M})$ and $V(\mathbb{M})$ are the subbundles of the horizontal and vertical fibres respectively.

Let $\mathbb{X} = \{X(x), x \in \mathbb{R}^d\}$ denote the **canonical horizontal vector fields** on $\mathcal{O}(\mathbb{M})$. They are characterised by

- Each $X(x)(r) \in H_r(\mathbb{M})$.
- $d\pi_r(F(x)) = r_p(x)$.

for each $x \in \mathbb{R}^d, r \in \mathcal{O}(\mathbb{M})$ with $\pi(r) = p$. Here, $d\pi_r : T_r(\mathcal{O}(\mathbb{M})) \to T_p(\mathbb{M})$ is the differential of π. $\mathbb{X}$ will be given the smallest topology for which the maps $x \mapsto X(x)$ are continuous.

Sometimes it is convenient to write $X_i = X(e_i)$, where $\{e_i, 1 \leq i \leq d\}$ is the natural basis for $\mathbb{R}^d$. We will assume that each X on $\mathcal{O}(\mathbb{M})$ is complete, so that $\mathbb{M}$ is geodesically complete. For each $u \in \mathbb{R}, r \in \mathcal{O}(\mathbb{M}), x \in \mathbb{R}^d$, the exponential map $r_x(u) = \exp(uX(x))(r)$ is the unique solution of the differential equation in $\mathcal{O}(\mathbb{M})$,

$$\frac{\partial r_x(u)}{\partial u} = X(x)(r_x(u)),$$

with initial condition $r_x(0) = r$. The family $\{r_t(u)\}$ forms a continuous one-parameter group of diffeomorphisms generated by each $X(x)$.

Let Y be a d-dimensional Lévy process on some complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. By theorem 1.2.23, there exists an n-dimensional Wiener process $\{W_t\}$ with $n \leq d$, a Poisson random measure N on $\mathbb{R}^+ \times (\mathbb{R}^n - \{0\})$, which is independent of W and has associated Lévy measure ν on $\mathbb{R}^n - \{0\}$ given by $\mathbb{E}N(t, A) = t\nu(A)$ for all $t \in \mathbb{R}^+, A \in \mathcal{B}(\mathbb{R}^n - \{0\})$, such that $Z = (Z^1, ..., Z^n)$ has

Lévy-Itô decomposition

$$Y_t = ct + \sigma W_t + \int_0^t \int_{||y|| \geq 1} yN(dt, dy) + \int_0^t \int_{||y||<1} y\tilde{N}(dt, dy) \tag{2.6}$$

for $1 \leq j \leq n, t \geq 0$. Here $c \in \mathbb{R}^d, \sigma \in \mathrm{End}(\mathbb{R}^n, \mathbb{R}^d)$, and $\tilde{N}$ is the compensated process $\tilde{N}(t, A) = N(t, A) - t\nu(A)$.

Definition 2.3.1. (c.f. [6] , p. 173) Let $R = \{R_t\}_{0 \leq t \leq \eta}$ be a càdlàg semimartingale on $\mathcal{O}(\mathbb{M})$ where η is an explosion time. We say that R is a **horizontal Lévy process** starting at $r \in \mathcal{O}(\mathbb{M})$ if R solves the following stochastic differential equation in $\mathcal{O}(\mathbb{M})$,

$$\begin{aligned} f(R_t) =& f(r) + \int_0^t \sum_i X_i f(R_{s-}) dY_s^i + \int_0^t \frac{1}{2} \sum_{j,k} a_{jk} X_j X_k f(R_u) ds \\ &+ \sum_{s \leq t} \left[f((\exp(X(\Delta Z_s)))(R_{s-})) - f(R_{s-}) - \sum_i X_i f(R_{s-}) \Delta Y_u^i \right] \end{aligned} \tag{2.7}$$

for all $f \in C^\infty(\mathcal{O}(\mathbb{M})), t \geq 0$.

By the Lévy-Itô decomposition (2.6), we can write this as

$$\begin{aligned} f(R_t) =& f(r) + \int_0^t \sum_i X_i f(R_s) dW_s^i + \int_0^t A_{\mathcal{O}(\mathbb{M})}(f)(R_{s-}) ds \\ &+ \int_0^t \int_{0<||y||<1} [f(\exp(X(y))(R_{s-})) - f(R_{s-})] \tilde{N}(ds, dy) \\ &+ \int_0^t \int_{||y||>1} [f(\exp(X(y))(R_{s-})) - f(R_{s-})] N(ds, dy) \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} A_{\mathcal{O}(\mathbb{M})}(f)(r) =& \sum_i c^i X_i f(r) + \frac{1}{2} \sum_{jk} a_{jk} X_j X_k f(r) \\ &+ \int_{\mathbb{R}^d - \{0\}} [f(\exp(X(y)))(r) - f(r) - 1_{||y||<1} X(y) f(r)] d\nu(y). \end{aligned} \tag{2.9}$$

for $f \in C^(\mathcal{O}(\mathbb{M})), r \in \mathcal{O}(\mathbb{M})$, where $a = (a^{ij})$ is the non-negative definite matrix $\sigma\sigma^*$.

It was shown in Theorem 2.1 and Theorem 2.2 of [3] that (2.7) has a unique càdlàg solution $R = \{R_t\}_{0 \leq t < \eta}$, up to explosion time η. [6] limited the construction to compact manifolds, where it can be shown that $\eta = \infty$ a.s. It was claimed by the authors that their technique in [6] could also be extended to processes on noncompact manifolds up to some explosion time η.

If we let $\{T_t\}_{t\geq 0}$ be the Markov semigroup on $C_0(\mathcal{O}(\mathbb{M}))$, the space of continuous functions on $\mathcal{O}(\mathbb{M})$ vanishing at infinity, defined by

$$T_t(f)(r) = \mathbb{E}(f(R_t)|R_0 = r)$$

for $f \in C_0(\mathcal{O}(\mathbb{M})), r \in \mathcal{O}(\mathbb{M})$. Then, it can be shown (c.f. [3] p.177) that $A_{\mathcal{O}(\mathbb{M})}$ is the generator of the semigroup T_t. Next, we consider the càdlàg process $\zeta_t = \pi(r_t)$, and we define the linear operator $A_{\mathbb{M}}(r)g(p) = A_{\mathcal{O}(\mathbb{M})}(g \circ \pi)(r)$. Then for $g \in C^\infty(\mathbb{M}), p = \pi(r)$,

$$\begin{aligned} A_{\mathbb{M}}(r)g(p) &= \sum_i c^i R_i(g)(p) + \frac{1}{2}\Delta_{\mathbb{M}}(g)(p) \\ &+ \int_{\mathbb{R}^d - \{0\}} \left[g\left(\exp\left(\sum_i y^i R_i\right)\right)(p) - g(p) - 1_{||y||<1} \sum_i y_i R_i(g)(p)\right] d\nu(y), \end{aligned} \tag{2.10}$$

where $R_j = r(e_j) \in T_p(\mathbb{M})$.

Applebaum pointed out in [3], p. 178 that, the frame dependence of the operator $A_{\mathbb{M}}$ indicates that the semigroup $\{T_t\}_{t\geq 0}$ on $C_0(\mathcal{O}(\mathbb{M}))$ does not, in general, project to a semigroup on $C_0(\mathbb{M})$. Hence, our candidate, ζ_t, for the Levy process on $\mathbb{M}$ is in general, not a Markov process, as the law of ζ_t in general may depend on the choice of the initial frame $R_0 = r$ at $\zeta_0 = p$.

Let us write $R_t = R(r, Y, t)$ for the solution of (2.7) and $\zeta_t = \zeta(r, Y, t)$ for its projection on $\mathbb{M}$. Let r' be another orthogonal frame at $p = \pi(r')$ and let O be the orthogonal transformation such that $r' = rO$ (c.f. [6] p. 175). It is easy to show that $\zeta(r', Y, t) = \zeta(r, OY, t)$ using the relation $d\pi_{rO}(f(y)) = d\pi_r(f(Oy)), \forall y \in \mathbb{R}^d$. Hence, the processes $\zeta_t = \zeta(r, Y, t)$ and $\zeta'_t = \zeta(r, OY, t)$ will agree in law if the processes Y and OY agree in law for all $O \in O(d)$[2].

Now if we write $R^d = M(d)/O(d)$, where $M(d)$ is the group of all isometries of $\mathbb{R}^d$, then we see that we require Y to be a spherically symmetric Lévy process

[2] $O(d)$ is the orthogonal group on $\mathbb{R}^d$.

on $\mathbb{R}^d$ in the sense of [25] , [26] and [4]. It follows that Y is characterized by the Lévy-Khintchine formula

$$\mathbb{E}e^{i\langle u,Y_t\rangle} = \exp\left[t\left(-\frac{1}{2}a||u||^2 + \int_{\mathbb{R}^d-\{0\}}(e^{i\langle u,y\rangle} - 1 - i\langle u,y\rangle 1_{||y||\leq 1})\right)d\nu(y)\right]$$

where $a = \sigma\sigma^*$ of the σ that appears in (2.6) and the Lévy measure ν satisfies $\nu(OA) = \nu(A)$ for all $A \in \mathcal{B}(\mathbb{R}^d - \{0\})$. In the case where Y is spherically symmetric as above, the process $\zeta_t = \zeta(r, Y, t)$ will always be a Markov process (c.f. [6]). We call it the **isotropic Lévy process on** $\mathbb{M}$. Theorem 3.1 of [6] gives sufficient conditions for a general càdlàg semimartingale to be an isotropic Lévy process.

Remark 2.3.2. Concluding Remarks

In this chapter, we studied some standard techniques that allow one to construct a continuous trajectory semimartingale on a manifold by solving a set of stochastic differential equations. As long as the solutions to these SDE's exists and are unique, the manifold valued process resembles everything we expect from its Euclidean analogue.

Such a set of techniques directly applied to the construction of a manifold-valued Levy process proved to be problematic. Even if the SDE's on the orthogonal frame bundle have a unique solution, it may not project back as a Markov process on the manifold unless it is isotropic. This assumption is very restrictive, as it rules out any processes that has a drift.

There is obviously a large missing piece of the puzzle to understand what is really happening in the anisotropic case. The rest of the thesis will be pursuing this direction.

Chapter 3

Lie Groups and Symmetric Spaces

In the previous chapter, we have seen how Brownian motion and Lévy processes can be "rolled" onto manifolds via stochastic development. However, in general the stochastic development of a anisotropic Lévy process fails to be even a Markov process. In this chapter, we will focus on the group-theoretic aspects of isotropic Lévy processes on symmetric spaces in the spirit of [25] and [26]. This will lay the groundwork for the next chapter, where Fourier analysis techniques used by previous authors, such as Applebaum [4], Gangolli [25] and Liao [41], will be generalised to give the Lévy Khintchine formula for an arbitrary Lévy process on a symmetric space. Furthermore, this will shed some light on how one may proceed to construct a general Lévy process on a general Riemannian manifold.

This chapter will first recall that a wide class of Riemannian manifolds can be thought of as the homogeneous space of a Lie group. We will then give a brief survey of the relevant theorems about Lie groups, and the definition of a Lie group-valued Lévy process. We will regard a manifold-valued (or symmetric space-valued) Lévy process as a coset-valued Lévy process in a Lie group. Finally, we will establish a Lévy-Khintchine formula for the special class of "isotropic" or "spherical" symmetric space valued Lévy processes.

For a Lie group[1] G, let $C(G), C_c(G), C^\infty(G), C_0(G)$ be the spaces of continuous functions, continuous functions with compact support, the infinitely differentiable functions on G, and continuous functions vanishing at ∞ respectively. We set $C_c^\infty(G) = C^\infty(G) \cap C_c(G)$. Let $\mathbf{D}(G)$ and $\mathcal{D}(G)$, respectively, be the algebra of all differential operators on $C^\infty(G)$, and the elements of $\mathbf{D}(G)$ which commute with the left action of G on $C^\infty(G)$.

[1] See Definition 3.2.1

We fix a left-invariant Haar measure dx on G, and let $L^1(G)$ be the algebra, with convolution as the product operation, of Borel measurable functions integrable with respect to dx. Elements of this algebra are equivalent classes of functions which are equal almost surely. Let $\mathcal{M}(G)$ and $\mathcal{M}_1(G)$ be the set of finite regular non-negative Borel measures and probability measures on G respectively. Both $\mathcal{M}(G)$ and $\mathcal{M}_1(G)$ are semigroups with respect to the convolution product.

For $g \in G$, we denote the left translation induced by g, $h \mapsto gh$ of G by γ_g, and the right translation $h \mapsto hg$ of G by κ_g. γ_g, κ_g induce transformations f^{γ_g} and f^{κ_g} on $C(G), C_c(G), C^\infty(G)$ by the rules, $f^{\gamma_g}(h) = f \circ \gamma_g^{-1}(g) = f(g^{-1}h)$ and $f^{\kappa_g}(h) = f \circ \kappa_g^{-1}(h) = f(hg^{-1})$. Similarly, we extend these conventions to other function spaces such as $L_1(G)$ in the obvious way.

3.1 Basic facts about Lie groups and symmetric spaces

In this section, we will show that a wide class of Riemannian manifolds may be thought of as a quotient space of the form G/K, where G is a Lie group and K is a compact subgroup of G. We begin by listing some basic facts about Lie groups and Lie algebras that can be found in any graduate level textbook. This section is intended to act as an "dictionary" for the terminologies that will be used beyond this point. I will be mainly following [28] , [39] and [41].

Definition 3.1.1. A **Lie group** G is a group and a manifold such that both the product map $G \times G \ni (g, h) \mapsto gh \in G$ and the inverse map $G \ni g \mapsto g^{-1} \in G$ are smooth. The identity element of G is denoted by e.

Definition 3.1.2. ([39], p. 24) Let k be a field, (in this thesis, we consider only $k = \mathbb{R}$ or $\mathbb{C}$). An **Lie algebra** $\mathfrak{g}$ is a vector space over k, with a product $[.,.]$, linear in each variable, which satisfies

- $[X, X] = 0$, for all $X \in \mathfrak{g}$ (and hence $[X, Y] = -[Y, X]$), and
- the **Jacobi identity**

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

To every Lie group, there is an associated Lie algebra, whose vector space structure is the tangent space of G at the identity element, and the Lie bracket captures the "local structure" of the group. If $h \in G$, then the translation maps $\gamma_h : g \mapsto hg$ and $\kappa_h : g \mapsto gh$ are diffeomorphisms from G onto itself. Therefore, so is the conjugation map $\Psi_h = \gamma_h \circ \kappa_h^{-1} : g \mapsto hgh^{-1}$. The conjugation map fixes the identity element e and therefore its tangent map at e is a linear automorphism of T_eG. Hence, $T_e\Psi_h \in \mathrm{End}(T_eG)$.

Definition 3.1.3. Let V be a finite dimensional vector space over a field k, G a Lie group, and $\mathfrak{g}$ a Lie algebra. A **representation** δ of G on V is a homomorphism $\delta : G \to \mathrm{End}_k V$. A **representation** ρ of $\mathfrak{g}$ on V is a homomorphism of Lie algebras $\mathfrak{g} \to \mathrm{End}_k V$, with

$$\rho([X,Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X), \text{ forall } X, Y \in \mathfrak{g}.$$

If G is non-compact, we define its **representation** on a Hilbert space V being a group homomorphism $\delta : G \to B(V)$, where $B(V)$ is the group of bounded linear operators of V which have a bounded inverse such that the map $G \times V \to V$ given by $(g, v) \to \Psi(g)v$ is continuous.

If $g \in G$, we define $\mathrm{Ad}_G(g) \in GL(T_eG)$ by $\mathrm{Ad}_G(g) = T_e\Psi_g$. The map $\mathrm{Ad}_G : G \to \mathrm{End}(T_eG)$ is an homomorphism and it is called the **adjoint representation** of G (c.f. [28], p. 127). Let $h \in G$, the mapping $\Psi(h) : g \mapsto hgh^{-1}$ is an analytic isomorphism of G onto itself. We put $\mathrm{ad}_G(h) = d\Psi(h)_e$, the derivative of Ψ at identity. Elementary computation shows that if $X, Y \in T_eG$, then

$$(\mathrm{ad}_G X)Y = \frac{d}{dt}(e^{tX}Ye^{-tX})\Big|_{t=0} = XY - YX.$$

Hence, $\mathrm{ad}_G : T_eG \to T_eG$ is an Lie algebra automorphism in the sense that ad_G is a linear transformation, and it also preserves the commutator on T_eG. Hence, we call the tangent space T_eG equipped with the commutator bracket **the Lie algebra associated with** G.

Definition 3.1.4. For any $D \in \mathrm{End}_k\mathfrak{g}$ for which

$$D[X,Y] = [X, DY] + [DX, Y]$$

is said to be a **derivation**.

The derivation defined here is in fact consistent with the derivation of an algebra in 2.1.2. If we assume V is in addition an algebra, and $X, Y \in V$, then we have $[X, Y] = XY - YX$. Hence,

$$\begin{aligned} D[X, Y] &= D(XY - YX) \\ &= (DX)Y + X(DY) - (DY)X - YD(X) \\ &= [X, DY] + [DX, Y], \end{aligned}$$

which agrees with Definition 3.1.4.

For a Lie algebra $\mathfrak{g}$, we get a linear map $\mathrm{ad} : \mathfrak{g} \mapsto \mathrm{End}_k\mathfrak{g}$ given by $(\mathrm{ad}X)(Y) = [X, Y]$. From the Jacobi identity, we see that

$$(\mathrm{ad}Z)[X, Y] = [X, (\mathrm{ad}Z)Y] + [(\mathrm{ad}Z)X, Y].$$

This shows that $\mathrm{ad}X$ is a derivation for every $X \in \mathfrak{g}$.

The following definition summarises some of the key concepts and definitions about Lie groups and Lie algebras that will be used throughout this thesis. Readers are referred to Chapter 1 of [39] for more details.

Definition 3.1.5.

1. If $\mathfrak{a}$, $\mathfrak{b}$ are subsets of $\mathfrak{g}$, we write

$$[\mathfrak{a}, \mathfrak{b}] = \mathrm{span}\{[X, Y] : X \in \mathfrak{a}, Y \in \mathfrak{b}\}.$$

2. A **(Lie) subalgebra** $\mathfrak{h}$ of $\mathfrak{g}$ is a subspace satisfying $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, so $\mathfrak{h}$ itself is a Lie algebra. A **(Lie) subgroup** H of a (Lie) group G is a subset of G, that is a (Lie) group on its own right.
3. An **ideal** $\mathfrak{h}$ in $\mathfrak{g}$ is a subspace satisfying $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$; an ideal is automatically a subalgebra.
4. Let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$. Then $\mathfrak{g}/\mathfrak{a}$ as a vector space becomes a Lie algebra under the definition $[X + \mathfrak{a}, Y + \mathfrak{a}] = [X, Y] + \mathfrak{a}$. This is called the **quotient algebra** of $\mathfrak{g}$ and $\mathfrak{a}$.
5. The Lie algebra $\mathfrak{g}$ is **Abelian** if $[\mathfrak{g}, \mathfrak{g}] = 0$, and the Lie group G is **Abelian** if $gh = hg$ for every $g, h \in G$.

6. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. If $X, Y \in \mathfrak{g}$, then $\mathrm{ad}X\mathrm{ad}Y$ is a linear transformation from $\mathfrak{g}$ to itself. We define the **Killing form** of $\mathfrak{g}$, denoted B to be,
$$B(X,Y) = \mathrm{Tr}(\mathrm{ad}X\mathrm{ad}Y)$$
7. A Lie group G is said to be **compact** if it is compact as a topological space.
8. If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{s} \subset \mathfrak{g}$, then
$$Z_{\mathfrak{g}}(\mathfrak{s}) = \{X \in \mathfrak{g} : [X,Y] = 0 \ \forall\ Y \in \mathfrak{s}\}$$
is the **centraliser** of $\mathfrak{s}$ in $\mathfrak{g}$. Similarly, if G is a Lie group and $S \subset G$, then
$$Z_G(S) = \{g \in G : gs = sg, \forall s \in S\}$$
9. If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$, then
$$N_{\mathfrak{g}}(\mathfrak{s}) = \{X \in \mathfrak{g} : [X,Y] \in \mathfrak{s} \forall Y \in \mathfrak{s}\}$$
is the **normaliser** of $\mathfrak{s}$ in $\mathfrak{g}$. Similarly, if G is a Lie group and S a subgroup of G, then
$$N_G(S) = \{g \in G : gsg^{-1} \in S, \forall s \in S\}.$$

Ideals and homomorphisms for Lie algebras have a number of properties in common with ideals and homomorphisms of rings. It is left to the reader to check the Lie algebraic version of the isomorphism theorems for more details.

Proposition 3.1.6. *([39] , p. 30) If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in a Lie algebra, then so are $\mathfrak{a}+\mathfrak{b}, \mathfrak{a}\cap\mathfrak{b}$ and $[\mathfrak{a},\mathfrak{b}]$.*

We now define recursively $\mathfrak{g}^0 = \mathfrak{g}, \mathfrak{g}^1 = [\mathfrak{g},\mathfrak{g}], \mathfrak{g}^{j+1} = [\mathfrak{g}^j, \mathfrak{g}^j]$ and $\mathfrak{g}_0 = \mathfrak{g}, \mathfrak{g}_1 = [\mathfrak{g},\mathfrak{g}], \mathfrak{g}_{j+1} = [\mathfrak{g},\mathfrak{g}_j]$. Each $\mathfrak{g}^j$ and $\mathfrak{g}_j$ are ideals of $\mathfrak{g}$.

Definition 3.1.7. We call the sequence
$$\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \dots$$

the **commutator series** for $\mathfrak{g}$, and the sequence

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \cdots$$

the **lower central series** for $\mathfrak{g}$.

Definition 3.1.8.

- We say that $\mathfrak{g}$ is **solvable** if $\mathfrak{g}^j = \{0\}$ for some j.
- We say that $\mathfrak{g}$ is **nilpotent** if $\mathfrak{g}_j = \{0\}$ for some j.

Proposition 3.1.9. *If $\mathfrak{g}$ is a finite dimensional Lie algebra, then there exists a unique solvable ideal $\mathfrak{r}$ of $\mathfrak{g}$ containing all solvable ideals in $\mathfrak{g}$.*

Definition 3.1.10.

1. The ideal $\mathfrak{r}$ in Proposition 3.1.9 is called the **radical** of $\mathfrak{g}$, and is denoted $\mathrm{rad}(\mathfrak{g})$.
2. A finite dimensional Lie algebra $\mathfrak{g}$ is **simple** if $\mathfrak{g}$ is nonabelian and $\mathfrak{g}$ has no proper nonzero ideals.
3. A finite dimensional Lie algebra $\mathfrak{g}$ is **semisimple** if $\mathfrak{g}$ has no nonzero solvable ideals, i.e., if $\mathrm{rad}(\mathfrak{g}) = \{0\}$.
4. A Lie group is said to be **solvable**, **nilpotent**, or **semisimple** if it is connected and if its Lie algebra is solvable, nilpotent or semisimple, respectively.

Proposition 3.1.11. *([39] , p. 30)*

- *In a simple Lie algebra $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Since $\mathrm{rad}(\mathfrak{g}) \neq \mathfrak{g}$, every simple Lie algebra is in fact semisimple.*
- *If $\mathfrak{g}$ is a finite dimensional Lie algebra, then $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple.*
- *(Cartan's criterion): A Lie algebra $\mathfrak{g}$ is semi-simple if and only if its Killing form B is non-degenerate, that is, if $B(X, .)$ is not identically zero for any non-zero $X \in \mathfrak{g}$.*

We identify a complex semisimple Lie algebra $\mathfrak{g}$ with the complexification $\mathfrak{g}_o + i\mathfrak{g}_o$ of $\mathfrak{g}_o$ where $\mathfrak{g}_o$ is a real semisimple Lie algebra. Let $\mathfrak{g}^*$ the vector space dual (considered over the field of complex numbers) of the finite dimensional vector space $\mathfrak{g}$. For $X \in \mathfrak{g}$, let $\theta(X) = -X^*$, where X^* is the dual of X. It can be verified that θ is an **involution**, that is an automorphism of the Lie algebra with square equal to the identity. To see that θ respects brackets, we have

$$\theta[X,Y] = -[X,Y]^* = -[Y^*,X^*] = [-X^*,-Y^*] = [\theta(X),\theta(Y)].$$

Let B be the Killing form. The involution θ has the property that $B_\theta(X,Y) := -B(X,\theta Y)$ is symmetric and positive definite. Hence, we make the following definition,

Definition 3.1.12. An involution θ of a real semisimple Lie algebra $\mathfrak{g}_o$ such that the symmetric bilinear form $B_\theta(X,Y) = -B(X,\theta Y)$ is positive definite is called a **Cartan involution**.

Remark 3.1.13. $B_\theta(.,.)$ can also act as an inner product on $\mathfrak{g}$. We will be using the notations $\langle .,. \rangle_{\mathfrak{g}}$ and $B_\theta(.,.)$ interchangably without further warning.

It is known that if $\mathfrak{g}_o$ is a real semisimple Lie algebra, then $\mathfrak{g}_o$ has a Cartan involution ([39] , p.p 358). Moreover, a Cartan involution θ of $\mathfrak{g}_o$ yields an eigenspace decomposition

$$\mathfrak{g}_o = \mathfrak{k}_o \oplus \mathfrak{p}_o$$

of $\mathfrak{g}_o$ into +1 and -1 eigenspaces, since θ is an involution. Moreover, these eigenspaces must bracket according to the rules

$$[\mathfrak{k}_o,\mathfrak{k}_o] \subseteq \mathfrak{k}_o, \quad [\mathfrak{k}_o,\mathfrak{p}_o] \subseteq \mathfrak{p}_o, \quad [\mathfrak{p}_o,\mathfrak{p}_o] \subseteq \mathfrak{k}_o$$

This decomposition is called the **Cartan decomposition**. If $X \in \mathfrak{k}_o$ and $Y \in \mathfrak{p}_o$, it can be checked that $\langle X,Y \rangle_{\mathfrak{g}_o} = 0$. Hence, we say $\mathfrak{k}_o$ and $\mathfrak{p}_o$ are orthogonal under the inner product $\langle , \rangle_{\mathfrak{g}_o} = B_\theta$. The Cartan decomposition yields the following theorem on the Lie group.

Definition 3.1.14. A **Cartan subalgebra** is a nilpotent subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ that is self-normalising (if $[X,Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, then $Y \in \mathfrak{h}$).

Theorem 3.1.15. *([39], p. 362) Let G be a semisimple Lie group, let θ be a Cartan involution of its Lie algebra $\mathfrak{g}_o$, let $\mathfrak{g}_o = \mathfrak{k}_o \oplus \mathfrak{p}_o$ be the corresponding Cartan decomposition, and let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}_o$. Then,*

1. *there exists a Lie group automorphism Θ of G with differential θ, and Θ has $\Theta^2 = 1$, Θ is called the* **global Cartan involution**,
2. *the subgroup of G fixed by Θ is K,*
3. *the mapping $K \times \mathfrak{p}_o \to G$ given by $(k, X) \mapsto k \exp X$ is a diffeomorphism onto, this is called the* **global Cartan involution**,
4. *K is closed,*
5. *K contains the center Z of G,*
6. *K is compact if and only if Z is finite, and*
7. *when Z is finite, K is a maximal compact subgroup of G.*

We now pause for a moment on Lie theory, and turn our attention to discuss its impact on Riemannian manifolds.

Definition 3.1.16.

- A Riemannian manifold $\mathbb{M}$ is called a **Riemannian locally symmetric space**, if for each $x \in \mathbb{M}$, there exists a normal neighbourhood of p on which the geodesic symmetry with respect to p is an isometry.
- Let $\mathbb{M}$ be an analytic Riemannian manifold, $\mathbb{M}$ is called a **Riemannian globally symmetric space** if each $x \in M$ is an isolated fixed point of an involutive isometry s_p of M.

Theorem 3.1.17. *(c.f. [28])*

1. *Let $\mathbb{M}$ be a complete*[2]*, simply connected Riemannian locally symmetric space. Then, M is Riemannian globally symmetric.*
2. *Let M be a Riemannian globally symmetric space, and x_0 any point in M. If $G = I_0(M)$*[3]*, and K is the subgroup of G which leaves p_0 fixed, then K is a compact subgroup of the connected group G and G/K is analytically diffeomorphic to M under the mapping $gK \to g.p_0, g \in G$.*
3. *The mapping $\sigma : g \mapsto s_{p_0} g s_{p_0}$ is an involutive automorphism of G, such that K lies between the closed group K_σ of all fixed points of σ and the identity component of K_σ. The group K contains no normal subgroup of G other than the identity.*

[2]Completeness of $\mathbb{M}$ just means every Cauchy sequence in $\mathbb{M}$ converges to a limit.

[3]Connected component of the isometry group containing the identity element.

Definition 3.1.18. Let G be a connected Lie group and H a closed subgroup. The pair (G, H) is called a *symmetric pair* if there exists an involutive analytic automorphism σ of G, such that $(H_\sigma)_0 \subset H \subset H_\sigma$, where H_σ is the set of fixed points of σ and $(H_\sigma)_0$ is the identity component of H_σ. If in addition, the group $Ad_G(H)^4$ is compact, (G, H) is said to be a Riemannian symmetric pair.

Examples of such spaces include

- The spheres, $S^{n-1} = SO(n)/SO(n-1)$.
- The hyperbolic plane, $H = SU(1,1)/SO(2)$.

Readers may also consult chapter 10 of [28] for a complete classification of Riemannian symmetric spaces.

When G is semisimple, K contains no normal subgroups of G. Let $\pi : G \to \mathbb{M}$ be the canonical surjection so that $\pi(g) = gK$ for each $g \in G$. We denote by γ the left action of G on $\mathbb{M}$ so that $\gamma(g_1)g_2K = g_1g_2K$, for each $k_1, k_2 \in K$. Note that $\pi \circ L_g = \gamma(g) \circ \pi$ for each $g \in G$.

Let $\mathfrak{g}_o, \mathfrak{k}_o$ be the Lie algebras of G, K, Ad denotes the adjoint representation of G on $\mathfrak{g}_o$. We can assert the existence of a Cartan involution, an involution θ of the Lie algebra $\mathfrak{g}_o$, such that the symmetric bilinear form B_θ (c.f. Definition 3.2.12) is non-negative definite. Now, let $\mathfrak{k}_o$ and $\mathfrak{p}_o$ be the eigenspaces corresponding to the eigenvalues 1 and -1 of θ respectively, so that we have the Cartan decomposition of the Lie algebra, $\mathfrak{g}_o = \mathfrak{k}_o + \mathfrak{p}_o$.

In this decomposition, $\mathfrak{p}_o$ can be identified in a natural way with the tangent space to the coset space G/K at $\pi(e) \in G/K$, where $\pi : G \to G/K$ is the natural projection map. $\mathfrak{k}_o$ and $\mathfrak{p}_o$ are orthogonal under the Cartan-Killing form B of $\mathfrak{g}_o$, and B is negative definite on $\mathfrak{k}_o \times \mathfrak{k}_o$, and positive definite on $\mathfrak{p}_o \times \mathfrak{p}_o$.

We summarise the impact of this discussion on the theory of Riemannian manifold in the following theorem.

Theorem 3.1.19.

Let $G = I_0(\mathbb{M})$ and $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively. Then $\mathfrak{k} = \{X \in \mathfrak{g} : (d\sigma)_e X = X\}$ and if $\mathfrak{p} = \{X \in \mathfrak{g} : (d\sigma)_e X = -X\}$, we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let π denote the natural mapping $g \to g.p_0$ of G onto M. Then,

[4] Here, $Ad_G(H)$ means the Lie subgroup of $Ad_G(G)$ which is the image of H under Ad_G.

$(d\pi)_e$ maps $\mathfrak{k}$ into $\{0\}$ and $\mathfrak{p}$ isomorphically onto M_{p_0}. If $X \in \mathfrak{p}$, then the geodesic emanating from p_0 with tangent vector $(d\pi)_e X$ is given by

$$\gamma_{d\pi.X}(t) = \exp tX.p_0, \qquad (d\pi = (d\pi)_e).$$

Moreover, if $Y \in M_{p_0}$, then $(d\exp tX)_{p_0}(Y)$ is the parallel translate of Y along the geodesic.

Let $\mathfrak{a}_{\mathfrak{p}_o}$ be a maximal abelian subspace of $\mathfrak{p}_o$ and $\mathfrak{a}_o$ a maximal abelian subalgebra of $\mathfrak{g}_o$ containing $\mathfrak{a}_{\mathfrak{p}_o}$. Then, $\mathfrak{a}_o = \mathfrak{a}_o \cap \mathfrak{p}_o + \mathfrak{a}_o \cap \mathfrak{k}_o$ and $\mathfrak{a}_{\mathfrak{p}_o} = \mathfrak{a}_o \cap \mathfrak{p}_o$. We write $\mathfrak{a}_{\mathfrak{k}_o} = \mathfrak{a}_o \cap \mathfrak{k}_o$. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_o$ and let $\mathfrak{a}, \mathfrak{a}_{\mathfrak{p}}, \mathfrak{a}_{\mathfrak{k}}, \mathfrak{k}, \mathfrak{p}$ etc. denote the subspaces of $\mathfrak{g}$ generated by $\mathfrak{a}_o, \mathfrak{a}_{\mathfrak{p}_o}, \mathfrak{a}_{\mathfrak{k}_o}, \mathfrak{k}_o, \mathfrak{p}_o$ respectively. Then, $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$, and it is a maximal abelian subspace of $\mathfrak{p}$. A simple calculation (c.f. [39] , p. 360) shows that $(\mathrm{ad}X^*) = -\mathrm{ad}(\theta X), \forall X \in \mathfrak{g}_o$, where the adjoint $(.)^*$ is relative to the inner product $\langle .,. \rangle_{\mathfrak{g}_o}$. Hence, the set $\{\mathrm{ad}(H) \mid H \in \mathfrak{a}\}$ is a commuting family of self adjoint transformations of $\mathfrak{g}$. It follows that, $\mathfrak{g}$ is the orthogonal direct sum of simultaneous eigenspaces, all the eigenvalues being real. If we fix such an eigenspace and if λ_H is the eigenvalue of $\mathrm{ad}H$, then the equation $\mathrm{ad}(H)X = \lambda_H X$ shows that λ_H is linear in H. Hence, the simultaneous eigenvalues are members of the dual space $\mathfrak{a}^*$.

Definition 3.1.20.

For $\lambda \in \mathfrak{a}^*$, we write

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid \mathrm{ad}(H)X = \lambda(H)X, \forall H \in \mathfrak{a}\}.$$

If $\mathfrak{g}_\lambda \neq \{0\}$ and $\lambda \neq 0$, we call λ a **root** of $\mathfrak{g}$. The set of **restricted roots** is denoted by Δ. Any nonzero $\mathfrak{g}_\lambda$ is called a **root space**, and each member of $\mathfrak{g}_\lambda$ is called a **root vector** for the root λ. The dimension of $\mathfrak{g}_\lambda$ is called the **multiplicity** of the root λ.

Let $\mathfrak{m}$ and M be the centralisers of $\mathfrak{a}$ in $\mathfrak{k}$ and in K respectively, that is

$$\mathfrak{m} = \{X \in \mathfrak{k} : \mathrm{ad}(X)H = 0, H \in \mathfrak{a}\}$$

and

$$M = \{k \in K : \mathrm{Ad}(k)H = H, H \in \mathfrak{a}\}.$$

M is a closed Lie subgroup of K with Lie algebra $\mathfrak{m}$, and it is also the centraliser of A in K. Moreover, the center Z of G is contained in M.

Theorem 3.1.21. *([39], p. 370)*

The roots and the root spaces have the following properties:

1. *$\mathfrak{g}$ is the orthogonal direct sum $\mathfrak{g} = \mathfrak{g}_\circ \oplus \bigoplus_{\lambda\in\Delta} \mathfrak{g}_\lambda$. This is called the* ***root space decomposition*** *of $\mathfrak{g}$.*
2. *$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$ (as a consequence of the Jacobi identity),*
3. *$\theta\mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$, and hence $\lambda \in \Delta$ implies $-\lambda \in \Delta$.*
4. *$\mathfrak{g}_\circ = \mathfrak{a} \oplus \mathfrak{m}$ orthogonally.*

For each root λ, the equation $\lambda = 0$ determines a subspace of $\mathfrak{a}^{\mathbb{C}}$ of codimension 1. These subspaces divide $\mathfrak{a}$ into several open convex cones, called **Weyl chambers**. Fix a Weyl chamber $\mathfrak{a}_+$, let $\overline{\mathfrak{a}_+}$ denote the topological **closure** of $\mathfrak{a}_+$. A root λ is called **positive** if it is positive on $\mathfrak{a}_+$. Since a root cannot vanish anywhere on $\mathfrak{a}_+$, a nonpositive root must be negative and for every negative root $-\lambda$, by Theorem 3.2.21 (3), λ is a positive root. Note that the positivity of the roots depends on the choice of the Weyl chamber. We choose a positive Weyl chamber and fix it once and for all.

Proposition 3.1.22. *([41] , p. 107) The set of positive roots span the dual space of $\mathfrak{a}$.*

Let λ be a positive root and $X \in \mathfrak{g}_\lambda$, since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $[\mathfrak{k}, \mathfrak{k}] \in \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}] \in \mathfrak{p}$, we may write $X = Y + Z$ with $Y \in \mathfrak{p}$ and $Z \in \mathfrak{k}$, such that $\forall H \in \mathfrak{a}$, $\mathrm{ad}(H)Y = \lambda(H)Z$, and $\mathrm{ad}(H)Z = \lambda(H)Y$. It follows that $\forall H \in \mathfrak{a}, \mathrm{ad}(H)^2Y = \lambda(H)^2Y$ and $\mathrm{ad}(H)^2Z = \lambda(H)^2Z$. For each positive root λ, let

$$\mathfrak{p}_\lambda = \{Y \in \mathfrak{p} : \mathrm{ad}(H)^2Y = \lambda(H)^2Y, H \in \mathfrak{a}\}$$

and

$$\mathfrak{k}_\lambda = \{Z \in \mathfrak{k} : \mathrm{ad}(H)^2Z = \lambda(H)^2Z, H \in \mathfrak{a}\}.$$

We have the following proposition relates $\mathfrak{g}_\lambda$ with $\mathfrak{p}_\lambda$ and $\mathfrak{k}_\lambda$.

Proposition 3.1.23. *([41] , p. 107)*

1. $\mathfrak{p} = \mathfrak{a} \oplus \bigoplus_{\lambda\in\Delta^+} \mathfrak{p}_\lambda$, *and* $\mathfrak{k} = \mathfrak{m} \oplus \bigoplus_{\lambda\in\Delta^+} \mathfrak{k}_\lambda$. *Moreover, the components under the direct sums are mutually orthogonal under* $\langle .,. \rangle_{\mathfrak{g}}$ *in the sense that* $\langle \mathfrak{p}_\lambda, \mathfrak{p}_{\lambda'} \rangle = 0$ *and* $\langle \mathfrak{k}_\lambda, \mathfrak{k}'_\lambda \rangle = 0$ *for* $\lambda \neq \lambda'$.
2. *For every positive root* λ, $\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda} = \mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda$.
3. *For every positive root* λ *and* $H \in \mathfrak{a}_+$, *the maps*

$$\mathfrak{p}_\lambda \ni Y \mapsto \mathrm{ad}(H)Y \in \mathfrak{k}_\lambda \quad \text{and} \quad \mathfrak{k}_\lambda \ni Z \mapsto \mathrm{ad}(H)Z \in \mathfrak{p}_\lambda$$

are linear bijections.

Now we define

$$\mathfrak{n}_+ = \sum_{\lambda\in\Delta^+} \mathfrak{g}_\lambda \quad \text{and} \quad \mathfrak{n}_- = \sum_{\lambda\in\Delta^+} \mathfrak{g}_{-\lambda}.$$

Since $[\mathfrak{g}_\lambda, \mathfrak{g}'_\lambda] \subset \mathfrak{g}_{\lambda+\lambda'}$, both $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are closed under Lie brackets and therefore are Lie subalgebras of $\mathfrak{g}$. Moreover, since there are only finitely many roots, that there exists a k large enough, such that $[\mathrm{ad}_{\mathfrak{n}_+}(Y)]^k = 0$ and $[\mathrm{ad}_{\mathfrak{n}_-}(Y)]^k = 0$. This shows both $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are nilpotent Lie subalgebras. Let N_+ and N_- be the (connected) Lie subgroups of G generated by $\mathfrak{n}_+$ and $\mathfrak{n}_-$ respectively.

Proposition 3.1.24. *([41] , p. 109) The exponential maps* $\exp : \mathfrak{n}_+ \longrightarrow N_+$ *and* $\exp : \mathfrak{n}_- \longrightarrow \mathfrak{n}_-$ *are diffeomorphisms.*

Theorem 3.1.25. *([39] , p. 373 - 374 , Iwasawa Decomposition)*

1. $\mathfrak{g}$ *is a vector space direct sum* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. *Here* $\mathfrak{a}$ *is abelian,* $\mathfrak{n}$ *is nilpotent,* $\mathfrak{a} \oplus \mathfrak{n}$ *is a solvable Lie subalgebra of* $\mathfrak{g}$ *and* $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$. *This is called the* ***Iwasawa decomposition*** *of* $\mathfrak{g}$.
2. *Let* G *be a semisimple Lie group, let* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ *be an Iwasawa decomposition of the Lie algebra of* $\mathfrak{g}$ *of* G, *and let* A *and* N *be the analytic subgroups of* G *with Lie algebras* $\mathfrak{a}$ *and* $\mathfrak{n}$. *Then the multiplication map* $K \times A \times N \rightarrow G$ *given by* $(k, a, n) \mapsto kan$ *is a diffeomorphism onto. The groups* A *and* N *are simply connected. This is called the* ***Iwasawa decomposition*** *of* G

Let M' be the normaliser of $\mathfrak{a}$ in K, and recall the M is the centraliser of $\mathfrak{a}$ in K. It is clear that M' is a closed subgroup of K and is also the normaliser of A in K.

Proposition 3.1.26. *([41], p. 112)*

1. *M and M' have the same Lie algebra $\mathfrak{m}$.*
2. *M is a closed normal subgroup of M', and the quotient group $W = M'/M$ is a finite group. W is called the* **Weyl group**.

Remark 3.1.27. If the center Z of G is finite, then by Proposition 3.1.15 the subgroup K of G is compact, and therefore M and M' are both compact.

For $w = m_w M \in W$, $\mathrm{Ad}(m_w) : \mathfrak{a} \to \mathfrak{a}$ is a linear map that does not depend on the coset representation $m_w \in M'$; therefore, $W \ni w \mapsto \mathrm{Ad}(m_w) \in GL(\mathfrak{a})$ is a faithful representation of W on $\mathfrak{a}$, where $GL(\mathfrak{a})$ is understood as the group of linear automorphisms on $\mathfrak{a}$.

Proposition 3.1.28. *([41] , p. 112)*

1. *W permutes the Weyl chambers and is simply transitive on the set of Weyl chambers in the sense that for all pairs of Weyl chambers C_1 and C_2, there exists $w \in W$ such that $w(C_1) = C_2$, and if $w \neq e_W$ (e_W is the identity element of W), then for all Weyl chambers C, $w(C) \neq C$.*
2. *For any $H \in \overline{\mathfrak{a}_+}$ (closure of $\mathfrak{a}_+$), the orbit $\{wH : w \in W\}$ intersects $\overline{\mathfrak{a}_+}$ only at H.*
3. *For $w \in W$ and $\lambda \in \Delta$, $\lambda \circ w \in \Delta$ and if $w \neq e_W$, then for some $\lambda \in \Delta$, $\lambda \circ w \neq \lambda$.*
4. *For $w \in W$ and $\lambda \in \Delta$, $\mathrm{Ad}(m_w)\mathfrak{g}_\lambda = \mathfrak{g}_{\lambda \circ w^{-1}}$.*
5. *Let $w \in W$. If $w(\mathfrak{a}_+) = -\mathfrak{a}_+$, then $\mathrm{Ad}(m_w)n_+ = n_-$.*

Definition 3.1.29. A positive root is called **simple** if it is not the sum of two positive roots.

Proposition 3.1.30. *([28] , p. 292) Let $\Sigma = \{\beta_1, ..., \beta_l\}$ be the set of all simple roots. The number of simple roots l, is equal to $\dim(\mathfrak{a})$ and any positive root can be written as $\lambda = \sum_{i=1}^{l} c_i \beta_i$ where the coefficients c_i are nonnegative integers.*

The preceding proposition allows us to identify $\mathfrak{a}$ with $\mathbb{R}^l$. For a root λ, let $s_\lambda \in W$ be the reflection about the hyperplane $\lambda = 0$ in $\mathfrak{a}$, with respect to the inner product $\langle ; \rangle_{\mathfrak{g}}$. This is a linear map $\mathfrak{a} \to \mathfrak{a}$ given by,

$$s_\lambda(H) = H - 2\frac{\lambda(H)}{\lambda(H_\lambda)} H_\lambda, H \in \mathfrak{a},$$

where H_λ is the element of $\mathfrak{a}$ representing λ; that is, $\lambda(H) = \langle H, H_\lambda \rangle_{\mathfrak{g}}$ for $H \in \mathfrak{a}$. We state two theorems from [28] relevant to this construction.

Proposition 3.1.31.

1. *([28], p. 289) The Weyl group W is generated by $\{s_\lambda : \lambda \in \Delta_+\}$.*
2. *([28], p. 292) Let s_i be the reflection in $\mathfrak{a}$ about the hyperplane $\beta_i = 0$, where β_i is a simple root. Then s_i permutes all the roots in Δ_+ that are not proportional to β_i, that is, the map $\lambda \mapsto \lambda \circ s_i$ permutes all the roots $\alpha \in \Delta_+$ not proportional to β_i.*

3.2 An Example: The Hyperbolic Plane H^2

Let $\mathbb{M}$ be the open disc $\{|z| < 1\} \subset \mathbb{C}$ with the Riemannian structure

$$\langle u, v \rangle_z = \frac{\langle u, v \rangle_{\mathbb{R}^2}}{(1 - |z|^2)^2}$$

for u, v being tangent vectors at $z \in \mathbb{M}$. This setup is usually called the Poincaré model of the hyperbolic plane H^2. We will first state some geometric properties of this space.

Consider the group

$$SU(1,1) = \{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \},$$

which acts on $\mathbb{M}$ by the map,

$$g : z \mapsto \frac{az + b}{\bar{z} + \bar{a}}, |z| < 1. \tag{3.1}$$

The action is transitive, and the subgroup fixing o is $SO(2)$, so we have the identification

$$\mathbb{M} = SU(1,1)/SO(2).$$

The Riemannian structure is preserved by the action of (3.1). To see this, let $z(t)$ be a curve with $z(0) = z$ and $z'(0) = u$. Then,

$$gu = \left(\frac{d}{dt} g(z(t)) \right)_{t=0} = \frac{z'(0)}{(\bar{z} + \bar{a})^2}$$

at $g(z)$, and from a simple computation we obtain the relation

$$\langle gu, gu \rangle_{g(z)} = \langle u, u \rangle_z .$$

Moreover, the mapping defined by (3.1) is conformal, and therefore maps circles (and lines) into circles and lines. Hence, the geodesics of $\mathbb{M}$ are the circular arcs perpendicular to the boundary $|z| = 1$, and also straight lines through the origin, o.

Notice that another way of writing our Riemannian structure is

$$ds^2 = \sum_{i,j} g_{ij} dx_i dx_j, \quad g_{ij} = (1 - |z|^2)\delta_{ij},$$

and we put as usual $\bar{g} = |\det(g_{ij})|$ and $g^{ij} = g_{ij}^{-1}$. Then, the Riemannian measure

$$f \mapsto \int_{\mathbb{M}} f \sqrt{\bar{g}} dx_1 ... dx_n,$$

and the Laplace-Beltrami operator

$$L : f \mapsto \frac{1}{\sqrt{\bar{g}}} \sum_k \partial_k \left(\sum_i g^{ik} \sqrt{\bar{g}} \partial_i f \right)$$

on $\mathbb{M}$ becomes, respectively,

$$dz = (1 - x^2 - y^2)^{-2} dx dy,$$

$$L = (1 - x^2 - y^2)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

One can check by direct computation that they are invariant under all isometries. It is also easy to prove directly that each $SU(1,1)$-invariant differential operator on $\mathbb{M}$ is a polynomial in L.

3.3 Lévy processes on Lie groups

We now give an introduction to Lie group valued Lévy processes. The study of Lévy processes on Lie groups is already quite well established. It began in 1956 with [34], where G. Hunt was able to classify all convolution semigroups on a Lie group, and derived a Lévy-Khintchine formula in a Lie group setting. In 1967, Parthasarathy in [45] conjectured that every family of infinitely divisible measures on a Lie group can be embedded in a convolution semigroup of measures. Hunt's formula will be central to the rest of this thesis. In this section, we will be mainly following the treatment of [41].

Before we can define a Lévy process on a Lie group G, we must first formulate the concept of independent and stationary increments using the group actions of G. Let G be a Lie group, and let $\{g_t\}_t$ be a stochastic process in G. For $s < t$, we call $g_s^{-1}g_t$ the **right increment** and $g_t g_s^{-1}$ the **left increment** of the process g_t.

Definition 3.3.1.

- The process g_t is said to have **independent right increments** if for any $0 < t_1 < ... < t_n$,

$$g_0, g_0^{-1}g_{t_1}, ..., g_{t_{n-1}}^{-1}g_{t_n}$$

  are independent, and **independent left increments** if

$$g_0, g_{t_1}g_0^{-1}, ..., g_{t_n}g_{t_{n-1}}^{-1}$$

  are independent.
- The process g_t is said to have **stationary right increment** if for every $0 \leq s \leq t$, $g_s^{-1}g_t = g_0^{-1}g_{t-s}$ in distribution, and **stationary left increment** if $g_t g_s^{-1} = g_{t-s}g_0^{-1}$ in distribution.
- A stochastic process X_t in G is called **càdlàg** if its paths $t \mapsto g_t$ are càdlàg a.s., with respect to the topology of G.

Definition 3.3.2.

A G-valued stochastic process $\{g_t\}_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is called a **left Lévy process** if it is $\mathcal{F}_t$-adapted, and it has independent and stationary **right** increments; and **right Lévy process** if it has stationary and independent **left** increments.

Remark 3.3.3.

- The above definition may seem unnatural at first glance, but the reason why we call a process with independent and stationary right increments a left Lévy process is that it turns out its transition semigroup and generator are invariant under left translations.

- The definition of a left/right G-valued Lévy process did not assume $g_0 = e$, where e is the identity element of G. If g_t is a process where $g_0 \neq e$, then we can define $g_t^e = g_0^{-1} g_t$, so that g_t^e is a process starting at the identity.
- If g_t is a left-Lévy process, then g_t^{-1} will be a right Lévy process, and the map $g \mapsto g^{-1}$ is a Lie group automorphism. This gives rise to a duality between left and right Lévy processes, in the sense that any theorem (preserved under the $g \mapsto g^{-1}$ automorphism) regarding a left Lévy process will have a counterpart relating to a right Lévy process. Hence, it is enough for our purposes to only concentrate on the case when g_t is a left Lévy process.

Let $\mathcal{B}(G)$ be the σ-algebra on G and let $B(G)_+$ be the space of nonnegative Borel functions on G. For $t \in \mathbb{R}_+, g \in G$ and $f \in B(G)$, we define $P_t f(g) = \mathbb{E} f(g g_t^e)$. Then, for each $t > s$, we have

$$\mathbb{E}(f(g_t)|\mathcal{F}_s) = \mathbb{E}(f(g_s g_s^{-1} g_t)|\mathcal{F}_s) = \mathbb{E} f(h g_{t-s}^e)|_{h=g_s} = P_{t-s} f(g_s).$$

Taking expectations gives, $P_t f(g) = P_s P_{t-s} f(g)$. This means that $\{P_t\}_{t \geq 0}$ is a semigroup of probability kernels on G, and that g_t is a time homogeneous Markov process with transition semigroup P_t.

Definition 3.3.4.

A time homogeneous Markov process on a manifold $\mathbb{M}$ (c.f. Definition 1.1.20) is said to be a **Feller process** if

- $P_t : C_0(\mathbb{M}) \subseteq C_0(\mathbb{M}), \forall t \geq 0.$
- $\lim_{t \to 0} ||P_t f - f||_{C_0(\mathbb{M})} = 0$ for all $f \in C_0(\mathbb{M})$.

Moreover, since $\{g_t\}$ is right continuous, it follows that P_t is a Feller semigroup and is left invariant on G, that is $P_t f(g'g) = P_t(f \circ g')(g)$ for all $g' \in G$. Therefore, $\{g_t\}$ is a left invariance Feller process on G.

Next, we introduce the idea of a convolution on a Lie group G. The role of convolution in studying Lévy processes on G is analogous to that on $\mathbb{R}^d$ explained in section 1.2.

Definition 3.3.5.

- Let μ and ν be two probability measures on $(G, \mathcal{B}(G))$, where $\mathcal{B}(G)$ is the Borel σ-algebra of G. Their convolution $\mu * \nu$ on G is defined by,

$$(\mu * \nu)(A) = \int_G \mu(Ag^{-1}) d\nu(g)$$

for every $A \in \mathcal{B}(G)$, and we interpret $Ag^{-1} = \{h \in G | hg \in A\}$

- Let μ and ν be measures on G, and f a bounded $\mathcal{B}(G)$ measurable function on G. If we interpret $\mu(f) = \int_G f d\mu$, then we have

$$(\mu * \nu)(f) = \int \int_G f(gh) d\mu(g) d\nu(h).$$

The convolution $*$ is associative, that is, $(\mu_1 * \mu_2) * \mu_3 = \mu_1 * (\mu_2 * \mu_3)$ provided the integrals are well defined.

Definition 3.3.6.

A **convolution semigroup** of probability measures is a family $\{\mu_t\}_{t \geq 0}$ of probability measures such that

- $\mu_0 = \delta_e$
- $\mu_t * \mu_s = \mu_{s+t}$.

for every $s, t \geq 0$. If in addition, we have $\mu_t \to \delta_e$ weakly as $t \downarrow 0$ (and hence $\mu_t \to \mu_s$ weakly as $t \downarrow s$), then we say $\{\mu_t\}_{t \geq 0}$ is a **continuous** semigroup.

Let g_t be a left Lévy process in G, and let $\{\mu_t\}_{t \geq 0}$ be the family of the marginal distribution of the process g_t^e, that is $\mu_t = P \circ (g_t^e)^{-1}$ where the "-1" should be understood as the inverse map of the random variable $(g_t^e) : \Omega \to G$, for every $t \geq 0$. Then, $\{\mu_t\}_{t \geq 0}$ is a continuous convolution semigroup of probability measures on G and

$$P_t f(g) = \int_G f(gh) d\mu_t(h).$$

Conversely, if we start with a family of continuous semigroups of probability measures on G, then $\{P_t\}_{t \geq 0}$ is a left invariance Feller semigroup (c.f. Definition 1.1.23). There exists a càdlàg Markov process g_t with transition semigroup P_t and an arbitrary initial distribution (c.f. [41] p. 251). By the Markov property of g_t, we obtain for all $s < t$,

$$E(f(g_s^{-1} g_t) | \mathcal{F}_s) = P_{t-s} f(g g_s)\big|_{g = g_s^{-1}} = \mu_{t-s}(f),$$

almost surely, and where $\mathcal{F}_s$ is the natural filtration of g_t. This shows that g_t has independent and stationary increments, and therefore a Lévy process in G.

In 1956, Hunt [34], completely characterised left invariant Feller semigroups of probability kernels in G, or equivalently, Lévy processes in G by their generators (c.f. Definition 1.1.25). To state the result, we need to fix a basis $\{X_1, ..., X_d\}$ of $\mathfrak{g}$. There are functions $x_1, ..., x_d \in C_c^\infty(G)$, such that $x_i(e) = 0$ and $X_j x_k = \delta_{jk}$. These functions may be used as local coordinates in a neighbourhood of e, with $X_i = \frac{\partial}{\partial x_i}$ at e, and hence, will be called the coordinate functions associated to the basis $\{X_1, ..., X_d\}$. In a neighbourhood U of e, x_i may be defined to satisfy

$$g = \exp\left(\sum_{i=1}^{d} x_i(g) X_i\right)$$

for $g \in U$. The coordinate functions are not uniquely determined by the basis, but if $\{x'_1, ..., x'_d\}$ form another set of coordinate functions associated to the same basis, then $x'_i = x_i + O(|x|^2)$ on some neighbourhood of e, where $|x|^2 = \sum_{i=1}^{d} x_i^2$.

Any $X \in \mathfrak{g}$ induces a left invariant vector field X^l on G defined by $X^l(g) = D\gamma_g(X)$, where $D\gamma_g$ is the differential map of γ_g. It also induces a right invariant vector field $X^r(g) = D\kappa_g(X)$. For any integer $k \geq 0$, let $C_0^{k,l}(G) = C^k(G) \cap C_0(G)$, such that

$$Y_1^l f \in C_0(G), \quad Y_1^l Y_2^l f \in C_0(G), \quad ..., \quad Y_1^l ... Y_k^l f \in C_0(G)$$

for any $Y_1, Y_2, ..., Y_k \in \mathfrak{g}$, and the space $C_0^{k,r}(G)$ is defined similarly with right invariant vector fields, and with Y_i^l replaced by Y_i^r.

Theorem 3.3.7. *(c.f. [41] , p. 11)*

Let A be the generator of a left invariant Feller semigroup of probability kernels on a Lie group G. Then, its domain $\mathrm{Dom}(A)$ *contains $C_0^{2,l}(G)$, and $\forall f \in C_0^{2,l}(G)$ and $g \in G$,*

$$\begin{aligned} Af(g) = \frac{1}{2} \sum_{j,k=1}^{d} a_{jk} X_j^l X_k^l f(g) + \sum_{i=1}^{d} c_i X_i^l f(g) \\ + \int_{G-\{e\}} \left(f(gh) - f(g) - \sum_{i=1}^{d} x_i(h) X_i^l f(g) \right) d\nu(h), \end{aligned} \tag{3.2}$$

where a_{jk}, c_i are constants with $\{a_{jk}\}$ being a nonnegative definite symmetric matrix, and ν is a measure on G satisfying

$$\nu(\{e\}) = 0, \quad \nu(\sum_i x_i^2) < \infty, \qquad \text{and } \nu(U^c) < \infty \tag{3.3}$$

for any neighbourhood U of G with U^c being the complement of U in G.

Conversely, if the matrix $\{a_{jk}\}$ and the measure ν satisfy conditions (3.2) and (3.3), and c_i are arbitrary constants, then there exists a unique left invariant Feller semigroup P_t of probability kernels on G whose generator A restricted to $C_0^{2,l}(G)$ is given by (3.2).

Definition 3.3.8. A measure ν on G satisfying (3.3) is called a **Lévy measure** on G (c.f. Definition 1.2.11 for the Lévy measure on $\mathbb{R}^d$).

Proposition 3.3.9. *(c.f. [41] , p. 13)*

The differential operator

$$D = \frac{1}{2} \sum_{j,k} a_{jk} X_j^l X_k^l$$

on $C^2(G)$ and the Lévy measure ν given in Theorem 3.3.7 are completely determined by the generator A, and are independent of the basis $\{X_1, ..., X_d\}$ of $\mathfrak{g}$ and the associated coordinate functions x_i and coefficients a_{jk}.

Remark 3.3.10. (c.f. [41], p. 14)

- The second order differential operator D appearing in the preceding proposition will be called the **diffusion part** of A.
- The coefficients c_i of (3.2), in general, will depend on the choice of basis of $\mathfrak{g}$ and the associated coordinate functions.

If the Lévy measure ν satisfies the following finite first moment condition:

$$\int \sum_{i=1}^{d} |x_i(g)| d\nu(g) < \infty, \tag{3.4}$$

then the integral $\int_G [f(gh) - f(g)]d\nu(h)$ exists, and the formula (3.2) simplifies to

$$Af(g) = \frac{1}{2}\sum_{j,k=1}^{d} a_{jk}X_j^l X_k^l f(g) + \sum_{i=1}^{d} b_i X_i^l f(g) + \int_G [f(gh) - f(g)]d\nu(h), \tag{3.5}$$

for $f \in C_0^{2,l}(G)$, and where

$$b_i = c_i - \int_G x_i(h)d\nu(h).$$

In this case, there is no need to introduce the coordinate functions $x_1, ..., x_d$. Note that condition (3.4) is independent of the choice of basis and the associated coordinate functions.

We will now present the Lie group version of the "Lévy-Itô decomposition" formula. It characterises a Lévy process in G by a stochastic integral equation involving stochastic integrals with respect to a Brownian motion, and a Poisson random measure. This was originally due to Applebaum and Kunita in [7].

Theorem 3.3.11. *(c.f. [41], p. 19) Let g_t be a Lévy process in G. Assume its generator A restricted to $C_0^{2,l}(G)$ is given by (3.2) with coefficients a_{jk}, c_i and the Lévy measure ν. Let N be the counting measure of the right jumps of g_t given in Definition 1.2.17, and $\{\mathcal{F}_t^e\}$ be the natural filtration of the process $g_t^e = g_0^{-1}g_t$. Then, there exists a d-dimensional $\{\mathcal{F}_t^e\}$-Wiener process motion $W_t = (W_t^1, ..., W_t^d)$ with covariance matrix $\{a_{jk}\}$, such that it is independent of N under $\{\mathcal{F}_t^e\}$ and $\forall f \in C_0^{2,l}(G)$,*

$$\begin{aligned} f(g_t) =& f(g_0) + \sum_{i=1}^{d}\int_0^t X_i^l f(g_{s-}) \circ dB_s^i + \sum_{i=1}^{d} c_i \int_0^t X_i^l f(g_{s-})ds \\ &+ \int_0^t \int_G [f(g_{s-}h) - f(g_{s-})]\tilde{N}(ds\, dh) \\ &+ \int_0^t \int_G \left[f(g_{s-}h) - f(g_{s-}) - \sum_{i=1}^{d} x_i(h)X_i^l f(g_{s-}) \right] ds\, d\nu(h) \end{aligned} \tag{3.6}$$

Conversely, given a G-valued random variable u, a d-dimensional Wiener process W_t with covariance matrix $\{a_{jk}\}$, constants c_i and a Poisson random measure N on $\mathbb{R}_+ \times G$ whose characteristic measure ν is a Lévy measure, such that u, $\{W_t\}$ and N are independent, then there is a unique càdlàg process g_t in G with $g_0 = u$, adapted *to the filtration $\{\mathcal{F}_t\}$ generated by $u, \{B_t\}$ and N, such that (3.6) is satisfied for any $f \in C_0^{2,l}(G)$. Moreover, g_t is a left Lévy process in G whose generator restricted to $C^{2,l}(G)$ is given by (3.2).*

In the computation of expectations, it is often convenient to work with Itô integrals. We note that if g_t is a left Lévy process that satisfies (3.4) for any $f \in C_c^\infty(G)$, then for any $f \in C^1(G)$, we have for every $j = 1, ..., d$,

$$\int_0^t f(g_{s-}) \circ dB_s^j = \int_0^t f(g_{s-}) dB_s^j + \frac{1}{2} \sum_{k=1}^d \int_0^t X_k^l f(g_s) a_{jk} ds. \tag{3.7}$$

The proof of this claim can be found in [41], p. 20. Replacing f by $X_j^l f$ in (3.7), we see that (3.6) can be rewritten as:

$$f(g_t) = f(g_0) + M_t^f + \int_0^t Af(g_s) ds, \tag{3.8}$$

where A is given by (3.5) and

$$M_t^f = \sum_{j=1}^d \int_0^t X_j^l f(g_{s-}) dB_s^j + \int_0^t \int_G [f(g_{s-}h) - f(g_{s-})] \tilde{N}(ds\ dh)$$

is an L^2-martingale.

3.4 Lévy Processes on Manifolds: One-Point Motions

We have shown in section 3.1 that a large class of Riemannian manifolds, specifically, the Riemannian symmetric spaces can be presented in the form of $\mathbb{M} = G/K$, where G is a Lie group and K is a maximally compact subgroup of G. In section 3.3, we surveyed the main ideas of a Lévy process on a Lie group. In this section, we will combine these ideas to present the Lévy process on $\mathbb{M}$ as an "one-point motion". In relation to the material developed in Chapter 2, the Lie group G is a subbundle of the orthonormal frame bundle of G/K. For example, when $K = U(n)$ we have the unitary bundle and when $K = Sp(n)$ we have the symplectic bundle. Hence, a G-valued stochastic process in relation to a process on $\mathbb{M} = G/K$ is like a stochastic process on a frame bundle of a Riemannian manifold to a process on the manifold itself.

Definition 3.4.1. Let G be a Lie group that acts transitively on a manifold $\mathbb{M} = G/K$ on the left, and let g_t be a stochastic process in G. For any point $x \in \mathbb{M}$, we call the process $x_t = g_t x$ the **one-point motion** of g_t in $\mathbb{M}$ starting from x.

In general, the one-point motion of a Markov process in G is not a Markov process in M, except when $\{g_t\}_{t\geq 0}$ is a right Lévy process. We require g_t to be a right Lévy process, as g_t acts on the left of x. Hence for $f \in \mathcal{B}(\mathbb{M})$,

$$\mathbb{E}(f(x_t)|\mathcal{F}_s) = \mathbb{E}(f(g_t x)|\mathcal{F}_s) = \mathbb{E}(f(g_t g_s^{-1} g_s x)|\mathcal{F}_s) = \mathbb{E}(f(g_t g_s^{-1} x_s)|\mathcal{F}_s) = \mathbb{E}(f(g_{t-s} x_s)).$$

Let $P_t^M f(x) = \mathbb{E}(f(g_t x))$ for $f \in C_0(\mathbb{M})$, then $P_t^M f(x)$ is a Feller semigroup for x_t.

Remark 3.4.2. If $\{g_t\}_{t\geq 0}$ was a left Lévy process, then the incremental action $g_s^{-1} g_t$ on the manifold could be viewed as $g_s^{-1} g_t x = g_s^{-1}(g_t x) = g_s(x')$, where $x' = g_t x \in \mathbb{M}$. However, $g_s(x')$ is in general not measurable with respect to $\mathcal{F}_s$, as x' has embedded information up to time $t > s$.

Equation (3.6) is the combination of the Lévy-Itô decomposition and Itô's formula for left Lévy processes. By duality, if g_t is a right Lévy process, then for any $f \in C_b(G) \cap C^2(G)$ and with $a_{jk}, b_i, c_i, N, W_j, X_i, x_i$ having the same meaning as in Theorem 3.3.11 and (3.6), g_t solves the stochastic differential equation,

$$f(g_t) = f(g_0) + M_t^f + \int_0^t Af(g_s)ds, \tag{3.9}$$

with

$$\begin{aligned} Af(g) &= \frac{1}{2}\sum_{j,k=1}^{d} a_{jk} X_j^r X_k^r f(g) + \sum_{i=1}^{d} c_i X_i^r f(g) \\ &+ \int_{G-\{e\}} \left(f(hg) - f(g) - \sum_{i=1}^{d} x_i(h) X_i^r f(g) \right) d\nu(h) \end{aligned}$$

If, in addition, ν satisfies (3.4), the integral $\int_G [f(gh) - f(g)]d\nu(h)$ exists, and A takes the simpler form

$$Af(g) = \frac{1}{2}\sum_{j,k=1}^{d} a_{jk} X_j^r X_k^r f(g) + \sum_{i=1}^{d} b_i X_i^r f(g) + \int_G [f(hg) - f(g)]d\nu(h), \tag{3.10}$$

where $b_i = c_i - \int_G x_i(h) d\nu(h)$, and

$$M_t^f = \sum_{j=1}^d \int_0^t X_j^r f(g_{s-}) dW_s^j + \int_0^t \int_G [f(g_{s-}h) - f(g_{s-})] \tilde{N}(ds\ dh)$$

is an L^2-martingale.

Every $X \in \mathfrak{g}$ induces a vector field X^* on $\mathbb{M}$ by

$$X^* f(x) = \frac{d}{dt} f(e^{tX} x)|_{t=0}$$

for any $f \in C^1(\mathbb{M})$ and $x \in \mathbb{M}$. Let $\pi_x : G \to \mathbb{M}$ be the map given by $\pi_x(g) = gx$. If $f \in C_c^2(\mathbb{M})$, then $f \circ \pi_x \in C_b(G) \cap C^2(G)$ with $X^r(f \circ \pi_x) = (X^* f) \circ \pi_x$ for $X \in \mathfrak{g}$. Therefore, we obtain the following stochastic differential equation for the one-point motion $x_t = g_t x$ of g_t in $\mathbb{M}$. For $f \in C^2(\mathbb{M})$,

$$f(x_t) = f(x) + M_t^f + \int_0^t A_{\mathbb{M}} f(g_s) ds, \tag{3.11}$$

with

$$\begin{aligned} A_{\mathbb{M}} f(x) &= \frac{1}{2} \sum_{j,k=1}^d a_{jk} X_j^* X_k^* f(x) + \sum_{i=1}^d c_i X_i^* f(x) \\ &+ \int_{G-\{e\}} \left(f(hx) - f(x) - \sum_{i=1}^d x_i(h) X_i^* f(x) \right) d\nu(h). \end{aligned}$$

Here ν is a measure on G satisfying (3.3), and if in addition ν satisfies (3.4), $A_{\mathbb{M}}$ takes the simpler form

$$A_{\mathbb{M}} f(x) = \frac{1}{2} \sum_{j,k=1}^d a_{jk} X_j^r X_k^r f(g) + \sum_{i=1}^d b_i X_i^r f(g) + \int_G [f(hx) - f(x)] d\nu(h), \tag{3.12}$$

where $b_i = c_i - \int_G x_i(h) d\nu(h)$, and

$$M_t^f = \sum_{j=1}^d \int_0^t X_j^* f(x_s) dW_s^j + \int_0^t \int_G [f(hx_{s-}) - f(x_{s-})] \tilde{N}(ds\ dh).$$

In the forthcoming chapter, we will combine the Lévy-Khintchine decomposition on $\mathbb{M} = G/K$ with Fourier analysis to compute the probability density function of Z_t

for every $t \geq 0$.

CHAPTER 4

Spherical Lévy Processes

In this chapter, we describe how techniques from spherical harmonic analysis can be applied to calculate the probability distribution of a spherical Lévy process on a symmetric spaces of the form $\mathbb{M} = G/K$.

Section 4.1 introduces the algebra $\mathcal{D}(G/K)$, and then proceeds to establish the isomorphism $\Gamma : \mathcal{D}(G/K) \to \mathcal{D}_W(A)$. Section 4.2 introduces the spherical functions as K-invariant joint eigenfunctions of differential operators on $\mathbb{M}$, with eigenvalues $\Gamma(D)(i\lambda)$, where Γ is the isomorphism introduced in section 4.1. Section 4.3 establishes a Lévy-Khintchine formula for isotropic Lévy processes on $\mathbb{M}$ using the spherical transform. Section 4.4 introduces the inversion formula, and then uses it to obtain the law of a Lévy process. Section 4.5 contains an intuitive discussion on convolution, spherical transforms, the compound Poisson process and their interrelationships.

This will set us up for the following chapter, where we will demonstrate how one can apply the "δ-spherical transform" to obtain a Lévy-Khintchine formula for general Lévy processes on G/K.

The key references to this chapter are [4] , [25] , [29] and [30].

We will use notation and definition from Chapter 3.

4.1 Differential Operators

For $\mathbb{M} = G/K$ be a Riemannian globally symmetric space with G semisimple and K maximally compact. Let

$$C(\mathbb{M}) = C(G/K) = \{f | f \in C(G), f^{\kappa_k} = f, \forall k \in K\}$$

$$C^\natural(\mathbb{M}) = C(K\backslash G/K) = \{f | f \in C(G), f^{\kappa_k} = f^{\gamma_k} = f, \forall k \in K\},$$

respectively be the space of smooth functions on $\mathbb{M}$ and those that are K-invariant. The spaces $C_c(\mathbb{M}), C_c^\natural(\mathbb{M}), C^\infty(\mathbb{M}), C^{\infty\natural}(\mathbb{M})$ are defined analogously. The spaces $C(\mathbb{M}), C_c(\mathbb{M})$ and $C^\infty(\mathbb{M})$ may be thought of as the function spaces on the symmetric space $\mathbb{M} = G/K$, or as functions on cosets of G (constant on K). Let $\mathbb{D}(\mathbb{M}) = C_c^\infty(\mathbb{M}) = C^\infty(\mathbb{M}) \cap C_c(\mathbb{M})$. The reason for such a choice of notation is that we are also interested in $\mathbb{D}'$, the dual of $\mathbb{D}$. We denote the spaces of measure on $\mathbb{M}$ by

$$\mathcal{M}(\mathbb{M}) = \{\mu | \mu \in \mathcal{M}(G); \mu^{\kappa_k} = \mu, \forall k \in K\}$$

and the K-invariant measures on $\mathbb{M}$ by

$$\mathcal{M}^\natural(\mathbb{M}) = \{\mu | \mu \in \mathcal{M}(G); \mu^{\gamma_k} = \mu^{\kappa_k} = \mu, \forall k \in K\}.$$

Let $\mathbf{D}(\mathbb{M})$ and $\mathcal{D}(\mathbb{M})$, respectively, be the algebra of differential operators on $C^\infty(G)$, and those that commutes with the left action of G on $C^\infty(\mathbb{M})$. Let $\mathcal{D}_0(G)$ be the subalgebra of $\mathcal{D}(G)$ that also commutes with the right action of K. Each $D \in D_0(G)$ leaves $C^\infty(\mathbb{M})$ invariant. It can be shown (c.f. [29] p. 239 - 241) that the algebra of restrictions of operators $D \in D_0(G)$ is isomorphic with the algebra $\mathcal{D}(\mathbb{M})$ of differential operators on $C^\infty(\mathbb{M})$ which commute with left translations by elements of G.

For an n-tuple of integers $\alpha = (\alpha_1, ..., \alpha_n)$, with $\alpha_i \geq 0$, we put

$$D^\alpha = \partial_1^{\alpha_1} ... \partial_n^{\alpha_n}$$

and

$$|\alpha| = \alpha_1 + ... + \alpha_n.$$

We will topologise $C^\infty(\mathbb{M})$ and $C_c^\infty(\mathbb{M})$ by the seminorms,

$$||f||_k^C = \sum_{|\alpha| \leq k} \sup_{x \in C} |(D^\alpha f)(x)|,$$

as C runs through the compact subsets of $\mathbb{M}$ and k runs through $\mathbb{N}$. If (U, φ) is a local coordinate system on $\mathbb{M}$, this gives a topology on $C^\infty(U)$ with the property that a sequence $\{f_n\}$ in $C^\infty(U)$ converges to 0 if and only if for each differential operator D on U, $Df_n \to 0$ uniformly on each compact subset of U. In particular, the topology of $C^\infty(U)$ is independent of the coordinate system.

The space $C^\infty(\mathbb{M})$ is now given the weakest topology for which all the restriction maps $f \to f|_U$, as (U, φ) runs through all local coordinate systems on $\mathbb{M}$, are continuous. We will assume that $\mathbb{M}$ has a countable base of open sets, so that we can restrict the charts (U, φ) to a countable family of $(U_j, \varphi_j), j = 1, 2, ...$ Since each $C^\infty(U_j)$ is a Fréchet space, it follows that $C^\infty(\mathbb{M})$ is also a Fréchet space, and again, a sequence (f_n) in $C^\infty(\mathbb{M})$ converges to 0 if and only if for each differential operator D on $\mathbb{M}$, $Df_n \to 0$ uniformly on each compact subset of $\mathbb{M}$.

Writing $\mathbb{M}$ as the union of an increasing sequence of relatively compact open sets, we see that $\mathbb{D}(\mathbb{M})$ is dense in $C^\infty(\mathbb{M})$. For each compact set $K \subset \mathbb{M}$, let $\mathbb{D}_K(\mathbb{M})$ denote the set of function in $\mathbb{D}(\mathbb{M})$ with support in K, we give $\mathbb{D}_K(\mathbb{M})$ the induced topology of $\mathbb{C}^\infty(\mathbb{M})$. It is a closed subspace of $C^\infty(\mathbb{M})$, and hence a Fréchet space.

Definition 4.1.1. ([29] , p. 240)

- A continuous linear functional T on $C_c(\mathbb{M})$ is called a **distribution**. The set of all such distributions is denoted by $C_c'(\mathbb{M})$.
- A linear functional on $\mathbb{D}(\mathbb{M})$ is called a **distribution** if for any compact set $K \subset \mathbb{M}$, the restriction of T to $\mathbb{D}_K(\mathbb{M})$ is continuous. The set of all such distributions is denoted by $\mathbb{D}'(\mathbb{M})$.

There is a close connection between the set of distributions on $\mathbb{M}$ and the set of measures on $\mathbb{M}$. More specifically, let X be a Hausdorff topological space and μ a measure on the Borel σ-algebras of X. The measure μ is **tight** if $\mu(B) = \sup\{\mu(K) : K \text{ compact } \subseteq B\}$, **locally finite** if every point has a neighborhood of finite measure, and a **Radon** measure if it is tight and locally finite. It is known that probability measures on Borel σ-algebras of every separable completely metrisable topological space[1] are Radon measures (c.f. [32], p.17).

Hence, if μ is a probability measure on $\mathbb{M}$, then the mapping $I : f \mapsto \int_{\mathbb{M}} f(x) d\mu(x)$ is a continuous positive linear map from $C_c(\mathbb{M}) \to \mathbb{R}$, where positivity means that $I(f) \geq 0$ whenever f is a non-negative function. On the other hand, by the Riesz representation theorem we know that for every continuous positive linear functional

[1]These are also known as Polish spaces.

T on $C_c(\mathbb{M})$, there exists a Radon measure μ, such that $T(f) = \int_{\mathbb{M}} f(x)d\mu(x)$ for every $f \in C_c(\mathbb{M})$. Therefore, every probability measure μ on $\mathbb{M}$ defines an element of $C_c'(\mathbb{M})$, and every element of $C_c'(\mathbb{M})$ corresponds to a probability measure.

To extend the above conclusion to $\mathbb{D}'(\mathbb{M})$, we will not distinguish between T and μ if

$$T(f) = \int_K f(x)d\mu(x)$$

for every compact $K \subset \mathbb{M}$, and every $f \in \mathbb{D}(\mathbb{M})$ with $\operatorname{supp}(f) \subset K$. In such cases, we write

$$T(f) = \int_{\mathbb{M}} f(x)dT(x) := \int_{\mathbb{M}} f(x)d\mu(x).$$

We now give $\mathbb{D}(\mathbb{M})$ the **inductive limit topology** of the spaces $\mathbb{D}_K(\mathbb{M})$, by taking as a fundamental system of neighbourhoods of 0 the convex sets W, such that for each compact subset $K \subset \mathbb{M}$, $W \cap \mathbb{D}_K(\mathbb{M})$ is a neighbourhood of 0 in $\mathbb{D}_K(\mathbb{M})$. With this topology of $\mathbb{D}(\mathbb{M})$, the continuous linear functionals T are precisely the distributions on $\mathbb{M}$ [2]. Thus, $\mathbb{D}'(\mathbb{M})$ is just the dual space of $\mathbb{D}(\mathbb{M})$.

Recall that $\mathcal{D}(G/K)$ the algebra of all differential operators on G/K which are invariant under all the transformations $\gamma(g) : xK \mapsto gxK$ of G/K onto itself. The algebra $\mathcal{D}(G/K)$ will play a central role in the remainder of this thesis. We will now describe the algebra $\mathbf{D}(G/K)$ in terms of the Lie algebras G and K.

First, let us consider the case when $H = \{e\}$ and write $\mathcal{D}(G)$ for $\mathcal{D}(G/\{e\})$, the set of left invariant differential operators on G.

Definition 4.1.2. ([29] , p. 280) If V is a finite dimensional vector space over $\mathbb{R}$, the **symmetric algebra** $S(V)$ over V is defined as the algebra of complex-valued polynomial functions on the dual space V^*. If $X_1, ..., X_n$ is a basis of V, $S(V)$ can be identified with the commutative algebra of polynomials

$$\sum_{(k)} a_{k_1,...,k_n} X_1^{k_1} ... X_n^{k_n}.$$

[2]If $r > 0$, $T^{-1}(B_r(0))$ is a convex set containing 0, so the continuity assumption amounts to the definition of a distribution.

Let $\mathfrak{g}$ denote the Lie algebra of G and $\exp : \mathfrak{g} \to G$ the exponential mapping. If $X \in \mathfrak{g}$, let $\tilde{X}$ denote the vector field on G given by

$$(\tilde{X}f)(g) = X(f \circ \gamma_g) = \left\{ \frac{d}{dt} f(g \exp tX) \right\}_{t=0}, f \in C^\infty(G),$$

where γ_g denotes the left translation of $x \mapsto gx$ of G onto itself. Then, $\tilde{X}$ is a differential operator on G, and if $h \in G$ then

$$(\tilde{X}^{\gamma_h} f)(g) = (\tilde{X}(f \circ \gamma_h))(h^{-1}g) = (\tilde{X}f)(g),$$

so $\tilde{X} \in \mathcal{D}(G)$. The group $\mathrm{Ad}(G)$ operates on $S(\mathfrak{g})$ by extension of the action of these groups on $\mathfrak{g}$ and let $I(\mathfrak{g}) \subset S(\mathfrak{g})$ be the subset of $\mathrm{Ad}(G)$-invariants. The following theorem connects the symmetric algebra $S(\mathfrak{g})$ and $\mathcal{D}(G)$, In particular, it shows that $\mathcal{D}(G)$ is generated by $\{\tilde{X} : X \in \mathfrak{g}\}$.

Theorem 4.1.3. *([29] , p. 280) Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $S(\mathfrak{g})$ denote the symmetric algebra of the vector space $\mathfrak{g}$. Then there exists a unique linear bijection $\lambda : S(\mathfrak{g}) \to \mathcal{D}(G)$, such that $\lambda(X^m) = \tilde{X}^m, X \in \mathfrak{g}, m \in \mathbb{N}$. If $X_1, ..., X_n$ is any basis of $\mathfrak{g}$ and $P \in S(\mathfrak{g})$, then*

$$(\lambda(P)f)(g) = \{P(\partial_1, ..., \partial_n) f(g \exp(t_1 X_1 + ... + t_n X_n))\}_{t=0}$$

where $f \in C^\infty(G)$, $\partial_i = \partial/\partial t_i$ and $t = (t_1, ..., t_n)$. Moreover, $\lambda(I(\mathfrak{g})) = \mathcal{Z}(G)$, where $\mathcal{Z}(G)$ is the center of $\mathcal{D}(G)$

Definition 4.1.4. ([29], p. 282) The mapping λ is called **symmetrisation map.**

The mapping λ has the following property. If $Y_1, ..., Y_p \in \mathfrak{g}$, then

$$\lambda(Y_1, ..., Y_p) = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \tilde{Y}_{\sigma(1)} ... \tilde{Y}_{\sigma(p)}$$

where $\mathfrak{S}_p$ is the symmetric group on p letters.

Now, let us consider $\mathbb{M} = G/K$ to be a symmetric space of the noncompact type, that is, G is a connected semisimple Lie group with finite center and K is a maximal compact subgroup. We begin with the following proposition.

Proposition 4.1.5. *([29], p. 288) Let $\mathbb{M} = G/K$ where $G = I(\mathbb{M})$, the group of isometries over $\mathbb{M}$. Then, $\mathcal{D}(G/K)$ consists of the polynomials in the Laplace-Beltrami operator.*

Let E be a Euclidean space, let $\mathcal{S}(E)$ be the space of rapidly decreasing C^∞ functions on E. We have the following chain of inclusions that $\mathbb{D}(E) \subset \mathcal{S}(E) \subset C^\infty(E)$. Let the subscripts K and W denote K-invariant and W-invariant functions respectively, where W is the Weyl group. Let ϖ be the canonical homomorphism from $\mathcal{D}_K(G)$ to $\mathcal{D}(\mathbb{M})$. We are interested in the following chains of inclusions

$$\mathcal{D}_K(\mathfrak{p}) \subset \mathcal{S}_K(\mathfrak{p}) \subset C_K^\infty(\mathfrak{p}) \subset C^\infty(\mathfrak{p})$$

and

$$\mathcal{D}_W(\mathfrak{a}) \subset \mathcal{S}_W(\mathfrak{a}) \subset C_W^\infty(\mathfrak{a}) \subset C^\infty(\mathfrak{a}).$$

Theorem 4.1.6. *([29], p. 295) The restriction to $\mathfrak{a}$ is an isomorphism of $\mathcal{D}_K(\mathfrak{p})$ onto $\mathcal{D}_W(\mathfrak{a})$. Moreover, it induces the following isomorphisms from $C_K^\infty(\mathfrak{p})$ onto $C_W^\infty(\mathfrak{a})$ and from $\mathcal{S}_K(\mathfrak{p})$ and $\mathcal{S}_W(\mathfrak{a})$.*

Consider now the bijection $\lambda : S(\mathfrak{g}) \to \mathcal{D}(G)$ from Theorem 4.1.3. It identifies the commutative algebras $S(\mathfrak{a})$ and $\mathcal{D}(A)$, and identifies the set $I(\mathfrak{a})$ of W-invariants in $S(\mathfrak{a})$ with the set $\mathcal{D}_W(A)$ of W-invariant differential operators on $A.o$ with constant coefficients. The following proposition can be thought of as an "operator version" of the Iwasawa decomposition.

Proposition 4.1.7. *([29] , p. 302) For each $D \in \mathcal{D}(G)$, there exists a unique element $D_\lambda \in \mathcal{D}(A)$ such that*

$$D - D_\lambda \in \mathfrak{n}\mathcal{D}(G) + \mathcal{D}(G)\mathfrak{k}.$$

Moreover, $(D\phi)|_A = D_\lambda \phi|_A$ whenever $\phi \in C^\infty(G)$ satisfies $\phi(ngk) = \phi(g)$ $(n \in N, g \in G, k \in K)$.

As usual, let $\mathfrak{a}^*$ be the dual of $\mathfrak{a}$ and $\mathfrak{a}_{\mathbb{C}}^*$ be the set of all linear functionals of $\mathfrak{a} \to \mathbb{C}$. Let $\varrho = \frac{1}{2}\sum_{\lambda \in \Delta_+} m_\lambda \lambda$, where m_λ is the multiplicity of λ.

Theorem 4.1.8. *([29], p. 305 - 306)*

- *The mapping*

$$\vartheta : D \mapsto e^{-\varrho} D_\lambda e^{\varrho}$$

is a homomorphism of $\mathcal{D}_K(G)$ *onto* $\mathcal{D}_W(A)$ *with kernel* $\mathcal{D}_K(G) \cap \mathcal{D}(G)\mathfrak{k}$.

- *The mapping*

$$\Gamma : \mathcal{D}(G/K) \to \mathcal{D}_W(A)$$

given by $\Gamma(\varpi(D)) = \vartheta(D)$ *for* $D \in \mathcal{D}_K(G)$ *is a surjective isomorphism.*

Remark 4.1.9. $\mathcal{D}(A)$ is a commutative polynomial ring, as each linear mapping $v : \mathfrak{a} \to \mathbb{C}$ extends uniquely to a homomorphism of $\mathcal{D}(A)$ into $\mathbb{C}$, denoted $D \mapsto D(v)$. Let $I(\mathfrak{a}_{\mathfrak{p}_o})$ be the polynomial functions on $\mathfrak{a}_{\mathfrak{p}_o}$ which are invariant under W. Therefore, $I(\mathfrak{a}_{\mathfrak{p}_o})$ and $\mathcal{D}_W(A)$ are isomorphic as algebras, and thus $I(\mathfrak{a}_{\mathfrak{p}_o})$ and $\mathcal{D}(G/K)$ are also isomorphic as algebras.

4.2 Eigenspaces and Spherical Functions

We begin with two important definitions.

Definition 4.2.1.

- A function $\phi : G \to \mathbb{C}$ is said to be **spherical (or K-spherical)** if $\phi \circ \gamma_k = \phi \circ \kappa_k = \phi$ for each $k \in K$.
- It is said to be an **elementary spherical function**, if it satisfies in addition

$$\int_K \phi(xky)dk = \phi(x)\phi(y)$$

and $\phi(e) = 1$. Here, dk stands for the normalised Haar measure of K.

Definition 4.2.2. A **joint eigenfunction** f on $\mathbb{M} = G/K$ is an eigenfunction of each of the operators $D \in \mathcal{D}(\mathbb{M})$. Let $\Lambda : \mathcal{D}(\mathbb{M}) \to \mathbb{C}$ be a homomorphism and let

$$\mathcal{E}_\Lambda(\mathbb{M}) = \{f \in C^\infty(\mathbb{M}) : Df = \Lambda(D)f, \forall D \in \mathcal{D}(\mathbb{M})\}.$$

The joint eigenfunctions of G/K are characterised as follows,

Proposition 4.2.3.

- *Let ϕ_1 and ϕ_2 be two spherical functions on G, such that $D\phi_1 = \lambda_D\phi_1$ and $D\phi_2 = \lambda_D\phi_2$ for every $D \in \mathcal{D}_K(G)$. Then, $\phi_1 = \phi_2$.*
- *Each joint eigenspace $E_\Lambda(\mathbb{M}) \neq \{0\}$ contains exactly one spherical function, which we will call ϕ_Λ.*
- *The members f of E_Λ are characterised by the equation,*

$$\int_K f(xkyK)dk = f(xK)\phi_\Lambda(yK), x, y \in G.$$

The elementary spherical functions can be shown to be analytic and can be equivalently characterised by the following properties:

- $\phi(e) = 1$
- $\phi \in C^{\infty\natural}(\mathbb{M})$
- ϕ is an eigenfunction of each $D \in \mathfrak{D}(\mathbb{M})$.

For $g \in G$, let $A(g)$ denote the unique element of $\mathfrak{a}_{\mathfrak{p}_0}$ such that $g = n \exp A(g)k$ where $k \in K, n \in N$ in the Iwasawa decomposition. A fundamental result due to Harish-Chandra, says

Theorem 4.2.4. *([29] , p. 418) As λ runs through $\mathfrak{a}^*_{\mathbb{C}}$, the functions*

$$\phi_\lambda(g) = \int_K e^{(i\lambda-\varrho)A(kg)}dk, g \in G,$$

exhaust the class of elementary spherical functions on G. Moreover, two such functions ϕ_λ and $\phi_{\lambda'}$ are identical if and only if $\lambda' = w\lambda$, where w is an element in the Weyl group.

Let $\mathfrak{a}^*$ be the space of real valued linear functionals on $\mathfrak{a}$, so that $\mathfrak{a}^*_{\mathbb{C}} = \mathfrak{a}^* + i\mathfrak{a}^*$. If $\lambda \in \mathfrak{a}^*$, then $\phi_\lambda(x)$ is positive definite. As a consequence, $\phi_\lambda(e) = 1, \phi_\lambda(x^{-1}) = \overline{\phi_\lambda(x)}$ and $|\phi_\lambda(x)| \leq 1$ for $x \in G$, $\lambda \in E_R$.

Finally, we need to note that if $D \in \mathcal{D}(G/K)$, then $D\phi_\lambda = \Gamma(D)(i\lambda)\phi_\lambda$, where Γ is the isomorphism of $\mathcal{D}(\mathbb{M})$ onto $I(\mathfrak{a}_{\mathfrak{p}_o})$ in Remark 4.1.9, and $\Gamma(D)(i\lambda)$ stands for the value of the polynomial function $\Gamma(D)$ at the point $i\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. This makes sense in view of the identification of $\mathfrak{a}^*_{\mathfrak{p}_o}$ with $\mathfrak{a}_{\mathfrak{p}_o}$ via the Cartan-Killing form.

In particular, if we choose a basis $\Lambda_1, ..., \Lambda_l$ for $\mathfrak{a}^*$ and let $\lambda = \sum_{j=1}^{l} \lambda_j\Lambda_j$, then the eigenvalue corresponding to ϕ_λ of any operator $D \in \mathcal{D}(\mathbb{M})$ is a polynomial in

$\lambda_1, ..., \lambda_l$. It is also clear that $\phi_{-i\varrho} = 1$ so that a $D \in \mathcal{D}(\mathbb{M})$ annihilates constant functions if and only if $\Gamma(D)(\varrho) = 0$. It is easy to conclude from these facts and from the fact that $I(\mathfrak{a}_{\mathfrak{p}_o})$ contains no linear polynomials, that second degree polynomials in $I(\mathfrak{a}_{\mathfrak{p}_o})$ which correspond to second order operators in $\mathcal{D}(\mathbb{M})$ which annihilates constants are of the form

$$\sum_{u,v=1}^{l} Q_{uv} H_u H_v - \sum_{u,v=1}^{l} Q_{uv} \varrho(H_u) \varrho(H_v),$$

where $H_u, u = 1, ..., l$ is an orthonormal basis for $\mathfrak{a}_{\mathfrak{p}_o}$, and $\{Q_{uv}\}$ is invariant under W. Further, such an element corresponds to an elliptic operator if and only if $\{Q_{uv}\}$ is a non-negative definite matrix.

4.3 The Spherical Lévy-Khintchine Formula

The construction of a general Lévy process on a Riemannian manifold, or a Riemannian symmetric space was well known to be problematic. Several authors, most notably Applebaum ([3], [4], [6]) and Gangolli ([25], [26]), were able to obtain significant partial results by restricting the class of processes to be isotropic or spherical. By this, we mean:

Definition 4.3.1.

- A measure μ on G is said to be K-**spherical** if $\mu(k_1 A k_2) = \mu(A)$ for every $k_1, k_2 \in K$ and $A \in \mathcal{B}(G)$. A G-valued stochastic process $\{Z_t\}_{t \geq 0}$ is said to be **spherical** if its induced measure μ_{Z_t} is a spherical measure for every $t > 0$.
- Similarly, a measure μ on $\mathbb{M} = G/K$ is said to be K-**spherical** if $\mu(kA) = \mu(A)$ for every $k \in K$ and $A \in \mathcal{B}(\mathbb{M})$. A $\mathbb{M}$-valued stochastic process $\{Z_t\}_{t \geq 0}$ is said to be **spherical** if its induced measure μ_{Z_t} is a spherical measure for every $t > 0$.

Proposition 4.3.2. *(c.f. [2]) There is a one-to-one correspondence between spherical processes in G and those in M by $Z_t(\omega) = \pi(\zeta_t(\omega))$, where $\pi : G \to \mathbb{M}$ is the canonical projection.*

We are now ready to state the Lévy-Khintchine formula for spherical processes on $\mathbb{M}$. This formula is sometimes also referred as the Lévy-Khintchine-Hunt-Gangolli formula. The proof presented here is based on section 4 of [4].

Theorem 4.3.3. *Let ζ_t be a spherical Lévy process on $\mathbb{M} = G/K$ with Lévy measure ν, and $\phi_{-\lambda}$ an elementary spherical function, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then,*

$$\mathbb{E}(\phi_{-\lambda}(\zeta_t)) = \exp\left[t\left(c\beta(\Delta_{\mathbb{M}}, \phi_{-\lambda}) + \int_{G-\{e\}} (\phi_{-\lambda}(h) - 1)\, d\nu(h)\right)\right], \tag{4.1}$$

where c is a constant and $\beta(\Delta_{\mathbb{M}}, \phi_{-\lambda})$ is the eigenvalue of the Laplacian $\Delta_{\mathbb{M}}$ corresponding to the eigenfunction $\phi_{-\lambda}$.

Proof. In Section 3.3, we developed a formula for the Lévy-Itô decomposition of the one-point motion (c.f. (3.10)): For $f \in C^2(\mathbb{M})$,

$$f(x_t) = f(x) + M_t^f + \int_0^t A_{\mathbb{M}} f(g_s) ds, \tag{4.2}$$

with

$$\begin{aligned} A_{\mathbb{M}} f(x) = &\frac{1}{2} \sum_{j,k=1}^d a_{jk} X_j^* X_k^* f(x) + \sum_{i=1}^d c_i X_i^* f(x) \\ &+ \int_{G-\{e\}} \left(f(hx) - f(x) - \sum_{i=1}^d x_i(h) X_i^* f(x) \right) d\nu(h), \end{aligned}$$

where ν is a measure on G satisfying (3.3). If in addition ν satisfies (3.4), $A_{\mathbb{M}}$ takes the simpler form

$$A_{\mathbb{M}} f(x) = \frac{1}{2} \sum_{j,k=1}^d a_{jk} X_j^* X_k^* f(x) + \sum_{i=1}^d b_i X * r_i f(x) + \int_G [f(hx) - f(x)] d\nu(h), \tag{4.3}$$

where $b_i = c_i - \int_G x_i(h) d\nu(h)$, and

$$M_t^f = \sum_{j=1}^d \int_0^t X_j^* f(x_s) dW_s^j + \int_0^t \int_G [f(hx_{s-}) - f(x_{s-})] \tilde{N}(ds\ dh).$$

The symbols $a_{jk}, b_i, c_i, N, W_j, X_i, x_i$ having the same meaning as they were in (3.10) of Section 3.3.

If the underlying Lévy process ζ_t is spherical, then it cannot possess a drift term [3]. Hence, we have $b_i = 0$ for $i = 1, 2, ..., d$, and $\sum_{j,k} a_{jk} X_i^* X_j^*$ can be recognized as $c\Delta_{\mathbb{M}}$ where c is a constant. Therefore, the generator $A_{\mathbb{M}}$ in the Lévy-Itô decomposition for spherical Lévy processes becomes,

$$A_{\mathbb{M}} f(x) = c\Delta_{\mathbb{M}} f(x) + \int_{G-\{e\}} (f(hx) - f(x))\, d\nu(h)$$

for $f \in C^{2\natural}(\mathbb{M})$.

Now, we shall apply this formula to derive the Lévy-Khintchine formula for a spherical Lévy process. The Lévy-Itô decomposition for spherical processes holds for all $f \in C^{2\natural}$, and in particular it holds for $f = \phi_{-\lambda}$, where $\phi_{-\lambda}$ is the elementary spherical function with eigenvalue λ. Hence,

$$\phi_{-\lambda}(\zeta_t) = \phi_{-\lambda}(\zeta_0) + M_t^{\phi_{-\lambda}} + \int_0^t A_{\mathbb{M}} \phi_{-\lambda}(\zeta_s) ds.$$

Therefore,

$$\mathbb{E}\phi_{-\lambda}(\zeta_t) = \phi_{-\lambda}(\zeta_0) + M_0^{\phi_{-\lambda}} + \int_0^t A_{\mathbb{M}} \mathbb{E}(\phi_{-\lambda}(\zeta_s)) ds.$$

The commutativity between the operators $A_{\mathbb{M}}$ and $\mathbb{E}$ is guaranteed as $\phi_{-\lambda}$ is bounded, and integration under $\mathbb{E}$ is taken with respect to a probability measure. Hence, we can now differentiate both sides to obtain

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}(\phi_{-\lambda}(\zeta_t)) &= A_{\mathbb{M}} \mathbb{E}(\phi_{-\lambda}(\zeta_t)) \\
&= c\mathbb{E}(\Delta_{\mathbb{M}} \phi_{-\lambda}(\zeta_t)) + \int_{G-\{e\}} (\mathbb{E}(\phi_{-\lambda}(h\zeta_t)) - \mathbb{E}(\phi_{-\lambda}(\zeta_t)))\, d\nu(h) \\
&= c\beta(\Delta_{\mathbb{M}}, \phi_{-\lambda}) \mathbb{E}\phi_{-\lambda}(\zeta_t) + \int_{G-\{e\}} (\mathbb{E}(\phi_{-\lambda}(h\zeta_t)) - \mathbb{E}(\phi_{-\lambda}(\zeta_t)))\, d\nu(h)
\end{aligned}$$

[3]This can be verified by taking the Lévy-Itô decomposition on G for spherical processes, and then projecting it to $\mathbb{M}$ via the Cartan decomposition, c.f. [2].

Since K is compact, we can let dk be the normalised Haar measure on K, so

$$\begin{aligned}\int_{G-\{e\}} \mathbb{E}(\phi_{-\lambda}(h\zeta_t))d\nu(h) &= \int_{G-\{e\}} \int_K \mathbb{E}(\phi_{-\lambda}(h\zeta_t))dkd\nu(h) \\ &= \int_{G-\{e\}} \mathbb{E}\left[\int_K (\phi_{-\lambda}(hk\zeta_t))dk\right] d\nu(h) \\ &= \int_{G-\{e\}} \mathbb{E}(\phi_{-\lambda}(\zeta_t))\phi_{-\lambda}(h)d\nu(h),\end{aligned}$$

where the second equality follows from that $\phi_{-\lambda}$ is spherical (K-bi-invariant), and the third equality follows from the defining identity of elementary spherical functions (c.f. Definition 4.2.1): for every $g, h \in G$,

$$\int_K \phi(gkh)dk = \phi(g)\phi(h).$$

Hence,

$$\begin{aligned}\frac{d}{dt}\mathbb{E}(\phi_{-\lambda}(\zeta_t)) &= A_{\mathbb{M}}\mathbb{E}(\phi_{-\lambda}(\zeta_t)) \\ &= c\beta(\Delta_{\mathbb{M}}, \phi_{-\lambda})\mathbb{E}\phi_{-\lambda}(\zeta_t) + \int_{G-\{e\}} \left(\mathbb{E}(\phi_{-\lambda}(\zeta_t))\phi_{-\lambda}(h) - \mathbb{E}(\phi_{-\lambda}(\zeta_t))\right) d\nu(h) \\ &= c\beta(\Delta_{\mathbb{M}}, \phi_{-\lambda})\mathbb{E}\phi_{-\lambda}(\zeta_t) + \mathbb{E}(\phi_{-\lambda}(\zeta_t)) \int_{G-\{e\}} (\phi_{-\lambda}(h) - 1)\, d\nu(h) \\ &= \mathbb{E}(\phi_{-\lambda}(\zeta_t)) \left(c\beta(\Delta_{\mathbb{M}}, \phi_{-\lambda})\mathbb{E}\phi_{-\lambda}(\zeta_t) + \int_{G-\{e\}} (\phi_{-\lambda}(h) - 1)\, d\nu(h)\right)\end{aligned}$$

with initial condition $\mathbb{E}(\phi_{-\lambda}(\xi_0)) = 1$. We have now reduced the problem of finding $\mathbb{E}(\phi_{-\lambda}(\xi_t))$ into a first order separable equation. Exponentiating both sides gives the desired result. □

4.4 The Spherical and Inverse Spherical Transform

In the previous section, we derived an explicit formula for $\mathbb{E}\phi_{-\lambda}(\zeta_t)$, where $\{\zeta_t\}$ is a spherical Lévy process. This is called the spherical Lévy-Khintchine formula. This section will be devoted to explaining the significance of this formula via the theory of spherical transforms. In principle, knowledge of $\mathbb{E}\phi_{-\lambda}(\zeta_t)$ will (under certain regularity conditions) allow us to compute the probability distribution of ζ_t at any point in time.

We begin by describing the theory of spherical transforms for K-bi-invariant functions on G, or K-invariant functions on $\mathbb{M} = G/K$, and then extend our theory to K-bi-invariant distributions[4].

Definition 4.4.1. (c.f. [29])

Let f be a K-bi-invariant function on G, we define

$$\tilde{f}(\lambda) := \int_G f(x)\phi_{-\lambda}(z)dz$$

whenever the above integral makes sense.

If $f(x) = f_X(x)$ is the probability density function of a random variable X, then the above definition can be thought of as $\tilde{f}(\lambda) = \mathbb{E}(\phi_{-\lambda}(X))$. The spherical transform is a well-developed theory in modern harmonic analysis. The central theorems of interest are the Plancherel formula, the Parsevel's identity and the inversion formula. The formulation of these theorems depends critically on Harish-Chandra's $\mathbf{c}$-function.

To keep this thesis self-contained, we will begin by giving a brief survey on the $\mathbf{c}$-function. This will allow us to state the Plancherel formula, Parsevel's identity and the inversion formula for the spherical transform. Finally, we will demonstrate how it relates to the Lévy-Khintchine formula derived in the previous section.

The Harish-Chandra's $\mathbf{c}$-function in a nutshell, is a measure of the asymptotic behavior of the spherical functions $\phi_\lambda(x)$ as $|x| \to \infty$.

Definition 4.4.2. The $\mathbf{c}$ function is given by the prescription

$$\mathbf{c}(\lambda) := \lim_{t\to\infty} e^{(-i\lambda+\rho)(tH)}\phi_\lambda(\exp tH)$$

for $H \in \mathfrak{a}^+$ arbitrary and $\mathrm{Re}(i\lambda) \in \mathfrak{a}^*_+$.

The set $\{\lambda \in \mathfrak{a}^* : \mathrm{Re}(i\lambda) \in \mathfrak{a}^*_+\}$ mentioned in the above definition is also the range of λ for which the limit is defined (c.f.[29]). The function $\mathbf{c}(\lambda)$ extends to the

[4]All probability measures defined on the Borel σ-algebra of Polish spaces can be regarded as distributions via the Riesz Representation theorem (c.f. Section 4.1).

meromorphic function

$$\mathbf{c}(\lambda) = c_0 \prod_{\alpha \in \Sigma_0^+} \frac{2^{-i\langle i\lambda, \alpha_0 \rangle} \Gamma(\langle i\lambda, \alpha_0 \rangle)}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + 1 + \langle i\lambda, \alpha_0 \rangle))\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + \langle i\lambda, \alpha_0 \rangle))}$$

on $\mathfrak{a}_{\mathbb{C}}^*$. Here, Σ_0^+ is the set of positive indivisible roots, α_0 is the normalised root $\alpha / \langle \alpha, \alpha \rangle$, and the constant c_0 is given by the condition $\mathbf{c}(-i\rho) = 1$. By Proposition 7.2 of Chapter 4 of [29], we have

$$|\mathbf{c}(\lambda)|^{-1} \leq C_1 + C_2 |\lambda|^{m/2}$$

for constants C_1 and C_2 and where $m = \dim(\mathfrak{n})$. We are now ready to state the main theorems regarding the spherical transform.

Theorem 4.4.3. *There exists a constant c, such that the following hold:*

1. *Inversion formula: For $f \in \mathcal{D}^\natural(\mathbb{M})$, we have the inversion formula*

$$f(x) = c \int_{\mathfrak{a}^*} \hat{f}(\lambda) \phi_\lambda(x) |c(\lambda)|^{-2} d\lambda. \tag{4.4}$$

2. *Plancherel formula: For $f \in L^2(\mathbb{M})$, we have*

$$\int_{\mathbb{M}} |f(x)|^2 dx = c \int_{\mathfrak{a}^*} |\hat{f}(\lambda)|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda,$$

where in the above formulae,

$$c = \frac{2^m}{(2\pi)^k |K/M| |W|}$$

with $m = \dim(\mathfrak{n}), k = \dim(\mathfrak{a}), |K/M|$ is the volume of K/M under the K-invariant Riemannian metric on K/M induced by the inner product on $\mathfrak{k}$, and $|W|$ is the cardinality of W.

3. *The image $\mathcal{D}^{\natural'}(\mathbb{M})$ is dense in $L^2(\mathfrak{a}^*/W, |\mathbf{c}(\lambda)|^{-2} d\lambda)$, and here the normalisation of $d\lambda$ and dg can be arbitrary.*

Now we may combine the inversion formula (Theorem 4.3.3) with the Lévy-Khintchine formula (3.10) to obtain:

Theorem 4.4.4. *Let ζ_t be a spherical Lévy process defined on a manifold $\mathbb{M} = G/K$ of non-compact type, μ_{ζ_t} be the probability distribution of ζ_t and we assume that μ_{ζ_t} is square-integrable. Then,*

$$\mu_{\zeta_t}(x) = c\int_{\mathfrak{a}^*} \hat{\mu}_t(\lambda)\phi_\lambda(x)|\mathbf{c}(\lambda)|^{-2}d\lambda,$$

where $\hat{\mu}_t = \mathbb{E}(\phi_{-\lambda}(\zeta_t))$.

Proof. The mapping $\mu_{\zeta_t} \mapsto \mathbb{E}(\phi_{-\lambda}(\zeta_t))$ is the spherical transform of μ_{ζ_t} as

$$\mathbb{E}(\phi_{-\lambda}(\zeta_t)) = \int_M \phi_{-\lambda}(x)\mu_{\zeta_t}(dx).$$

We regard μ_{ζ_t} as a distribution on $\mathbb{M}$, in the sense that

$$\int_{\mathbb{M}} \phi_{-\lambda}(x)\mu_{\zeta_t}(dx) = \int_{\mathbb{M}} \phi_{-\lambda}(x)\mu_{\zeta_t}(x)dx.$$

The image $\mathcal{D}^\natural(\mathbb{M})$ is dense in $L^2(\mathfrak{a}^*/W, |\mathbf{c}(\lambda)|^{-2}d\lambda)$, and hence the inversion formula can be extended to $\mathcal{D}^{\natural'}(\mathbb{M})$ via the Plancherel formula. Hence, an application of the inversion formula gives what we desire. □

Remark 4.4.5. Theorem 4.4.4 does not necessarily require ζ_t to have a smooth density. It is a basic application of the Fourier inversion to the formula derived in Theorem 4.3.3, which according to the Paley-Wiener theorem (c.f. Theorem 7.1 [29], p 450), does not require f to be smooth. When f is non-smooth, $\mu_{\zeta_t}(x)$ in Theorem 4.4.4 will become a distribution on $\mathbb{M}$, as opposed to being a density function. This is precisely why we are viewing probability measures through the lens of distribution theory in section 4.1.

4.5 Convolution and Compound Poisson Processes

We introduced the Lévy-Khintchine formula in the previous section, and deduced the law of a $\mathbb{M}$-valued spherical Lévy process using the spherical inversion formula. In this section, we will focus on a special case when ζ_t is a compound Poisson process, where we will gain a more intuitive understanding of what makes the techniques in the previous section work.

Let X, Y be two independent random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is well known that when X and Y are both real valued, then $\mu_{X+Y} = \mu_X * \mu_Y$, where $*$ denotes the **convolution** between μ_X and μ_Y. For sake of simplicity, we will always assume unless stated otherwise, that the random variables on $\mathbb{R}^d$, G and $\mathbb{M}$ possesses a density with respect to the Lebesgue measure, Haar measure or Riemannian volume measure respectively. We point out that most of our results do not need this assumption.

If μ_X and μ_Y are measures on G, then by Definition 3.2.5,

$$(\mu_X * \mu_Y)(A) = \int_G \mu(Ag^{-1}) d\nu(g)$$

for every $A \in \mathcal{B}(G)$, and

$$(\mu_X * \mu_Y)(f) = \int_{G\times G} f(gh)\mu(dg)\nu(dh)$$

for every Borel measurable function f, provided if the integral exists. If f_X and f_Y are the densities of μ_X and μ_Y, then

$$(f_X * f_Y)(X) = \int_G f_1(g) f_2(g^{-1}x) dg.$$

Let f and μ be a function and a measure on $\mathbb{M}$ respectively. Denote by f^* (and μ^*) the unique function (and measure) on G, such that $f^*(gk) = f^*(g)$ (and $\mu^*(Ak) = \mu^*(A)$) for $g \in G$, $k \in K$ (and $A \in \mathcal{B}(G)$), and $\pi(f^*) = f$ (and $\pi(\mu^*) = \mu$) where $\pi : G \to G/K$ is the canonical projection.

Let μ_X and μ_Y be measures on $\mathbb{M}$,

$$(\mu_X * \mu_Y)(A) = \pi\left(\int_G \mu^*(Ag^{-1}) d\nu^*(g)\right)$$

for every $A \in \mathcal{B}(G)$, and

$$\begin{aligned}(\mu_X * \mu_Y)(f) &= \pi\left(\int_{G\times G} f^*(gh)\mu_X^*(dg)\mu_Y^*(dh)\right)\\ &= \int_{\mathbb{M}}\left(\int_G f^*(gy)\mu_X^*(dg)\right)\mu_Y(dy)\end{aligned}$$

for every Borel measurable function f, provided if the integral exists. Here $x = gK \in \mathbb{M}, y = hK \in \mathbb{M}$. If f_X and f_Y are the densities of μ_X and μ_Y, then

$$(f_X * f_Y)(X) = \pi\left(\int_G f_1^*(g) f_2(g^{-1}x)dg\right).$$

Suppose X, Y are two K-bi-invariant G-valued random variables. Then,

$$\mu_{XY}(A) = \mu_{XY}(\mathbf{1}_{\{XY \in A\}}) = \int_{G\times G} \mathbf{1}_{\{XY\in A\}}(x, y) d\mu_X(x) d\mu_Y(y)$$

for every $A \in \mathcal{B}(G)$. Since the choice of A was arbitrary, we have $\mu_{XY} = \mu_X * \mu_Y = \mu_Y * \mu_X$. It can be further checked that the convolution defined here is associative, and the space $L^{1\natural}(\mathbb{M})$ of spherical random variables with finite mean can be made into a commutative algebra under the convolution product. We summarize this discussion in the following theorem.

Theorem 4.5.1. *([29] , p. 408) Let $\mathbb{M} = G/K$ be a Riemannian symmetric space, then $C_c^\natural(\mathbb{M})$ (and consequently $C_c^{\natural'}(\mathbb{M}) = \mathcal{M}_c^\natural(\mathbb{M})$) are commutative algebras under convolution.*

A consequence of this theorem is that every irreducible representation of $C_c^\natural(\mathbb{M})$ is one dimensional, and our analysis boils down to the study of $\mathrm{Hom}(C_c^\natural(G), \mathbb{C})$, the set of continuous homomorphisms of the algebra $C_c^\natural(G)$ onto $\mathbb{C}$. The following theorem classifies the set of all such homomorphisms.

Theorem 4.5.2. *([29] , p. 409) Let $f \in C_c^\natural(\mathbb{M})$, the mappings*

$$f \mapsto \int_{\mathbb{M}} f(x)\phi(x)dx,$$

exhaust $\mathrm{Hom}(C_c^\natural(\mathbb{M}), \mathbb{C})$, where ϕ is a bounded spherical function on G.

Definition 4.5.3. Let V be a Hilbert space, and ρ a representation of G on V with each $\rho(x)$ unitary, the ρ is called a **unitary representation.**

A well known property of a spherical function ϕ is that it is positive definite (c.f. [29], p. 389) in the sense that

$$\sum_{i,j=1}^{n} \phi(x_i^{-1}x_j)\alpha_i\bar{\alpha}_j \geq 0$$

for all finite sets $x_1, ..., x_n$ of elements in G and any complex numbers $\alpha_1, ..., \alpha_n$.

Theorem 4.5.4. *([29] , p. 390, 410)*

1. *Let ρ be a unitary representation of G on a Hilbert space $\mathcal{H}$. For each vector $e \in \mathcal{H}$, the function $x \mapsto \langle e, \rho e \rangle$ is a positive definite function on G. Conversely, to any positive definite function $\phi \not\equiv 0$ on G corresponds a unitary representation of ρ of G, such that $\phi(x) = \langle e, \rho(x)e \rangle$ for a suitable vector e. We shall call ρ the unitary representation* **associated** *to ϕ.*
2. *Let $\phi \not\equiv 0$ be a positive definite spherical function on G, and let ρ be the unitary representation of G associated to ϕ. Then ρ is irreducible and spherical.*
3. *Conversely, if ρ is an irreducible unitary spherical representation of G and e a unit vector left fixed by all $\rho(k), k \in K$, then the function $\langle e, \rho(x)e \rangle$ is a positive definite spherical function on G.*

Inspired by the above theorems, it is natural to define a spherical transform as per Definition 4.1.1 (c.f. [29] , p. 399). For $\mu \in S^\natural(\mathbb{M})$, we define $\hat{\mu}$ by the homomorphism

$$\hat{\mu}(\lambda) = \int_{\mathbb{M}} \phi_{-\lambda}(x) d\mu(x), \lambda \in \mathfrak{a}_{\mathbb{C}}^*,$$

and if μ possesses a density $f \in C_c^\natural(\mathbb{M})$, then we identify $\hat{\mu}(\lambda)$ with $\hat{f}(\lambda)$

$$\hat{\mu}(\lambda) = \int_{\mathbb{M}} f(x)\phi_{-\lambda}(x) dx.$$

By Theorem 4.5.2, the spherical transform is an one dimensional representation of $C_c^\natural(\mathbb{M})$. Hence if $f, g \in C_c^\natural(\mathbb{M})$, we automatically get $\widehat{(f * g)} = \hat{f}\hat{g}$. Further, it can be shown that if $\mu, \nu \in \mathcal{M}^\natural(\mathbb{M})$, $(\widehat{\mu * \nu})(\lambda) = \hat{\mu}(\lambda)\hat{\nu}(\lambda), \lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Given $\mu, \nu \in \mathcal{M}^\natural(\mathbb{M})$, we shall write $\mu\nu$ for the convolution of μ and ν, the order being immaterial in view of the commutativity of $\mathcal{M}^\natural(\mathbb{M})$. μ^j will stand for the j-fold convolution of μ with itself. Under this notation, we have $\widehat{\mu\nu} = \hat{\mu}\hat{\nu}$, with the product on the right being pointwise.

Let $\{Z_t\}$ be a stochastic process with increments that are independent and identical. We partition $[0, t]$ by $\mathcal{P}_n([0, t]) = \{0 = t_0 < t_1 < ... < t_n = t\}$, with $t_j = jt/n$; then, $\widehat{Z}_t = (\widehat{Z_{t/n}})^n$ in distribution. Hence, we have the following definition.

Definition 4.5.5. A measure $\mu \in \mathcal{M}_1^\natural(\mathbb{M})$ is said to be **infinitely divisible** if for each positive integer j, there exists a measure $\nu \in \mathcal{M}_1^\natural(\mathbb{M})$ such that $\nu^j = \mu$.

Proposition 4.5.6. *The convolution product of two infinitely divisible measures $\in S_0^\natural(\mathbb{M})$ is again infinitely divisible.*

Proof. Let μ and ν be two infinitely divisible measures. Then, there exists μ_n and ν_n such that $\mu_n^n = \mu$ and $\nu_n^n = \nu$ for every $n = 1, 2, 3, \ldots$. Therefore,

$$\widehat{(\mu * \nu)} = \hat{\mu}\hat{\nu} = \hat{\mu}_n^n \hat{\nu}_n^n = (\widehat{\mu_n * \nu_n})^n.$$

□

Proposition 4.5.7. *Suppose $\mu_j \in \mathcal{M}_0^\natural(\mathbb{M}), j = 1, 2, \ldots; \mu_j \to \mu$ weakly as $j \to \infty$, and μ_j is infinitely divisible for each j. Then μ is infinitely divisible.*

The proof of this proposition is based on the union of the proofs of several lemmas in section 5 of [25].

Proof. We need to show that for every $n = 1, 2, \ldots$, the sequence $\mu_j^{1/n}$ converges weakly to some $\nu \in \mathcal{M}_0^\natural(\mathbb{M})$, and that $\nu^n = \mu$.

Lemma 4.5.8. *Let f be an analytic function, and $m_j \in \mathcal{M}_0^\natural(\mathbb{M})$ be a sequence of measures that converges weakly to m, then the sequence $f(m_j)$ converges weakly to $f(m)$*[5].

Proof of lemma: Since f is analytic, we have $\widehat{f(m_j)} = f(\widehat{m_j}) \to f(m)$ as $j \to \infty$.

Now we go back to the proof of the proposition. The "f" in Proposition 4.5.6 in context of Lemma 4.5.7 is $f(x) = x^{1/n}$. This function is analytic everywhere except when $x = 0$, for every $n = 1, 2, \ldots$. Hence, it suffices to prove that if m is infinitely divisible, then $\widehat{m}$ is never zero.

For any $m \in \mathcal{M}_0^\natural(\mathbb{M})$, we define its adjoint m^* by $m^*(B) = m(B^{-1}) = m(\{b \in G : b^{-1} \in B\})$. Then clearly, $m^* \in \mathcal{M}_0^\natural(\mathbb{M}), (m^*)^* = m$. Observe that $(mm^*)^* = m^*m = mm^*$, so (mm^*) is self-adjoint. Furthermore, if m is infinitely divisible, then there is, for each j, a measure $n_j \in \mathcal{M}_0^\natural(\mathbb{M})$ such that $m = (n_j)^j$. Hence, $mm^* = (n_j n_j^*)^j$ so that (mm^*) is infinitely divisible. Its also easily checked that $\widehat{mm^*}(\lambda) = |\hat{m}(\lambda)|^2$ so that $\hat{m}(\lambda) = 0$ if and only if $\widehat{mm^*}(\lambda) = 0$.

[5] The terms in the power series $f(m) = a_0 + a_1 m + a_2 m^2 + \ldots$ should be understood as convolution powers

Hence, we may assume without loss of generality that m is self adjoint, and for each j, there exists a self-adjoint measure $n_j \in \mathcal{M}_0^\natural(\mathbb{M})$ such that $m = (n_j)^j$. Note that $\hat{m}(\lambda), \hat{n}_j(\lambda)$ are now real valued for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Therefore, since $\hat{m}(\lambda) = (\hat{n}_j(\lambda))^j$, and $|\hat{m}(\lambda)| \leq 1$, we have $(\hat{n}_j(\lambda))^2 = ((\hat{m}(\lambda))^2)^{1/j} \to \beta(\lambda)$, where $\beta(\lambda) = 0$ or 1 according to whether $\hat{m}(\lambda) = 0$ or $\hat{m}(\lambda) \neq 0$. It now suffices to prove that $\beta(\lambda) = 1$. Now,

$$\hat{m}(0) = \int_G e^{-\varrho} H(x) dm(x)$$

and so $\hat{m}(0) > 0$; therefore $\beta(0) = 1$. But, $n_j^2 \in \mathcal{M}_0^\natural(\mathbb{M})$, hence $\hat{n}_j^2(-i\varrho) = 1$. By the Banach-Alaoglu theorem[6] , we are guaranteed that for a subsequence $\{n_{j_\alpha}^2\}, \alpha = 1, 2, ...$ we have $n_{j_\alpha}^2 \to n^2 \in \mathcal{M}^\natural(\mathbb{M})$; and since the n_j^2's are probability measures, we have $\hat{n}_{j_\alpha}^2(\lambda) \to \hat{n}^2(\lambda), \lambda \in \mathfrak{a}_{\mathbb{C}}^*$. But since $\hat{n}_j^2(\lambda) \to \beta(\lambda)$, it follows that $\hat{n}^2(\lambda) = \beta(\lambda)$. If $n_{j_{\alpha'}}^2$ is another weakly convergent subsequence of $\{n_j^2\}$, with limit n_0^2, then we shall have similarly $\hat{n}(\lambda) = \beta(\lambda) = \hat{n}_0^2(\lambda)$, forcing $n = n_0$. Hence, each convergent subsequence of $\{n_j^2\}$ has the same limit n^2, such that $\hat{n}^2 = \beta$.

This in particular shows that, β is continuous on $\mathfrak{a}_{\mathbb{C}}^*$, and since $\mathfrak{a}_{\mathbb{C}}^*$ is connected and $\beta(0) = 1$, it follows that $\beta(\lambda) = 1$ for every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Hence, $\hat{\mu}(\lambda)$ cannot be zero for any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, and hence this establishes the proposition. □

The convolution theory introduced above allows us to study compound Poisson processes from a more intuitive point of view. Let N_t be a Poisson process on the integers with intensity parameter c. A compound Poisson process on $\mathbb{R}$ is a stochastic process $\zeta_t = X_1 + ... + X_{N_t}$, where $X_1, X_2, ...$ are independent and identical random variables.

Now we wish to mimic this idea for a spherical Poisson process on a Riemannian symmetric space $\mathbb{M} = G/K$. Let $X_1, X_2, ...$ be spherical random variables on $\mathbb{M}$, simultaneously regarded as K-bi-invariant random variables[7] on G. Let $\zeta_t = X_1 X_2 ... X_{N_t}$ or $X_{N_t} X_{N_t - 1} .. X_1$ depending on whether we want a right or left (respectively) compound Poisson process - but for now, let us assume that it is a left process. Then,

$$\mu_{\zeta_t} = \sum_{j=0}^{\infty} \mathbb{P}(N_t = j) \mu_X^j = \sum_{j=0}^{\infty} \frac{1}{j!} e^{-ct} (ct)^j \mu_X^j,$$

[6] c.f. *http://en.wikipedia.org/wiki/Banach-Alaoglu_theorem*

[7] By this, we mean random variables on G whose law is K-bi-invariant.

where the powers in the above series are understood as convolution powers. Note that the above power series converges absolutely, and hence the spherical transform of μ_{ζ_t} is given by

$$\begin{aligned}\hat{\mu}_{\zeta_t}(\lambda) &= \sum_{j=0}^{\infty} \frac{1}{j!} e^{-ct} (ct)^j \hat{\mu}_X^j(\lambda) \\ &= e^{-ct} \sum_{j=0}^{\infty} \frac{1}{j!} (ct)^j (\widehat{\mu_X})^j(\lambda) \\ &= \exp(ct(\widehat{\mu_X(\lambda)} - 1))\end{aligned}$$

for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. This formula, similar to the Lévy-Khintchine formula, allows us to compute the probability distribution of ζ_t by Fourier inversion. The following theorem summarizes this discussion.

Theorem 4.5.9. *Let $\zeta_t = X_{N_t} X_{N_t - 1} \ldots X_1$ be a compound Poisson process with intensity c ($c \in \mathbb{R}, c > 0$). Then, inheriting the notation from Theorem 4.4.4, the law of ζ_t is given by*

$$\mu_{\zeta_t}(x) = c_0 \int_{\mathfrak{a}_{\mathbb{C}}^*} \exp(ct(\widehat{\mu_X(\lambda)} - 1)) \phi_{-\lambda}(x) |\mathbf{c}(\lambda)|^{-2} d\lambda,$$

where c_0 is a constant to ensure that $\int_{\mathbb{M}} \mu_{\zeta_t}(dx) = 1$.

Now we apply the above ideas to construct the Poisson measure. For $g \in G$, consider the set $\tilde{x} = \{k_1 g k_2 | k_1, k_2 \in K\}$. Let dk be the Haar measure on K and $dk \times dk$ be the measure on $K \times K$. Under the map $K \times K \to KgK$ given by $(k_1, k_2) \mapsto k_1 g k_2$, $dk \times dk$ induces a measure on $\tilde{x}$ in the natural way, which we shall call μ_x. μ_x may clearly be regarded as a measure on G by setting it 0 outside $\tilde{x}$. We call this extended measure μ_x also. It is clear that $\mu_x \in S_0^{\natural}(\mathbb{M})$.

Definition 4.5.10. Let $x \in \mathbb{M}$, The measure

$$P_{x,c} = \sum_{j=0}^{\infty} \exp(-c) c^j \mu_x^j / j!$$

will be called the Poisson measure with jump size x and jump rate c. Here $x \in G$, c is a real number ≥ 0.

It is clear that $P_{x,c} \in S_0^\natural(\mathbb{M})$ for each $x \in G, c \geq 0, P_{x,0}$ being μ_e identically for all $x \in G$. It is an easy computation to verify that $\hat{\mu}_x(\lambda) = \phi_{-\lambda}(x), \lambda \in \mathfrak{a}_{\mathbb{C}}^*$, and that $\hat{P}_{x,c}(\lambda) = \exp(c(\phi_{-\lambda}(x) - 1))$. It is also clear that $P_{x,c}P_{x,d} = P_{x,c+d}$, and that $P_{x,c} = P_{x,c/n}^n$ making it infinitely divisible. The following theorem shows that in fact all infinitely divisible measures are derived from the Poisson measure.

Theorem 4.5.11. *A measure $\mu \in \mathcal{M}_0^\natural(\mathbb{M})$ is infinitely divisible if and only if there exists a sequence $\{\mu_j\} \in \mathcal{M}_0^\natural(\mathbb{M})$ such that each μ_j is a convolution of a finite number Poisson measures, and $\mu_j \to \mu$.*

Proof. If μ is infinitely divisible, then for each n, there is a ν_n such that $(\nu_n)^n = \mu$, and so $\hat{\mu}(\lambda) = \hat{\nu}_n(\lambda)^n$. Since $\hat{\mu}(\lambda) \neq 0$ for any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$[8], we have

$$n(\hat{\mu}(\lambda)^{1/n} - 1) = n(\hat{\nu}_n(\lambda) - 1) \to \log \hat{\mu}(\lambda)$$

as $n \to \infty$. Hence, $\hat{\mu}(\lambda) = \lim_{n\to\infty} \exp[n(\hat{\nu}_n(\lambda) - 1)]$. Since

$$\hat{\nu}_n(\lambda) = \int_{\mathbb{M}} \phi_{-\lambda}(x) d\nu_n(x),$$

and $\int_{\mathbb{M}} d\nu_m(x) = 1$, we have

$$\hat{\mu}(\lambda) = \lim_{n\to\infty} \exp\left(n \int_{\mathbb{M}} (\phi_{-\lambda}(x) - 1) d\nu_n(x)\right).$$

Our assertion follows by writing the integral as a limit of Riemann sums and noting that $\hat{P}_{x,c}(\lambda) = \exp[c(\phi_{-\lambda}(x) - 1)]$. □

Corollary 4.5.12. *$\mu \in \mathcal{M}_0^\natural(\mathbb{M})$ is infinitely divisible if and only if*

$$\hat{\mu}(\lambda) = \lim_{j\to\infty} \exp(-\psi_j(\lambda)),$$

where $\psi_j(\lambda) = \int_{\mathbb{M}} [1 - \phi_{-\lambda}(x)] d\nu_j(x)$ with $\nu_j \in \mathcal{M}^\natural(\mathbb{M})$.

The above corollary combines with the following theorem (Thm 6.1 of [25]) to give the Lévy-Khintchine formula for spherical infinitely divisible probability measures.

[8]This was established in the proof of Proposition 4.5.6.

Theorem 4.5.13. *Let*

$$\psi_j(\lambda) = \int_{\mathbb{M}} (1 - \phi_{-\lambda}(x)) d\nu_j(x), \nu_j \in \mathcal{M}_0^{\natural}(\mathbb{M}), j = 1, 2, ...$$

and suppose that $\lim_{j\to\infty} \psi_j(\lambda) = \psi(\lambda)$. *Then there exists a constant* c, *a spherical measure* ν *and a second order elliptic differential operator* $L \in \mathcal{D}(\mathbb{M})$ *which annihilates constants, such that*

$$\psi(\lambda) = c - \beta(L, \phi_{-\lambda}) + \int_{|x|>0} (1 - \phi_{-\lambda}(x)) d\nu(x),$$

where $\beta(L, \phi_{-\lambda})$ *is the eigenvalue of* L *corresponding to the eigenfunction* $\phi_{-\lambda}$. *Further,*

$$\int_{\mathbb{M}} \frac{|x|^2}{1 + |x|^2} d\nu(x) < \infty.$$

For such a $\psi(\lambda)$, *if* $\hat{\mu}(\lambda) = \exp(-\psi(\lambda))$, *then* $\mu \in \mathcal{M}_0^{\natural}$ *if and only if* $c = 0$. *Conversely, given a second order elliptic differential operator* $L \in \mathcal{D}(\mathbb{M})$, *with* $L\phi_{-\lambda} = \beta(L, \phi_{-\lambda})\phi_{-\lambda}$ *and a spherical measure* ν *satisfying the above conditions, the function*

$$\beta(L, \phi_{-\lambda}) + \int_{|x|>0} [1 - \phi_{-\lambda}(x)] d\nu(x)$$

is the limit as $j \to \infty$ *of functions* $\psi_j(\lambda)$ *which arise from measures* $\nu_j \in \mathcal{M}^{\natural}(\mathbb{M})$ *according to the prescription*

$$\psi_j(\lambda) = \int [1 - \phi_{-\lambda}(x)] d\nu_j(x).$$

For proof, see section 6 of [25].

4.6 An Example: Isotropic Lévy Processes on H^2

Recall from section 3.2 that H^2 is the open disc $\{z = x + iy : |z|^2 = |x|^2 + |y|^2 < 1\} \subset \mathbb{C}$ with the Riemannian structure

$$\langle u, v \rangle_z = \frac{\langle u, v \rangle_{\mathbb{R}^2}}{(1 - |z|^2)^2}$$

for u, v being tangent vectors at $z \in X$ via the Poincaré model. Moreover, we can write H^2 as the quotient of the symmetric pair $\mathbb{M} = SU(1,1)/SO(2)$. The Laplace-Beltrami operator

$$L : f \mapsto \frac{1}{\sqrt{\bar{g}}} \sum_k \partial_k \left(\sum_i g^{ik} \sqrt{\bar{g}} \partial_i f \right)$$

on $\mathbb{M}$ becomes, respectively,

$$L = (1 - x^2 - y^2)^{-2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

A spherical function on $\mathbb{M}$ is a radial eigenfunction of L. In geodesic polar coordinates, L has the form

$$L = \frac{\partial^2}{\partial r^2} + 2 \coth 2r \frac{\partial}{\partial r} + 4 \sinh^{-2}(2r) \frac{\partial^2}{\partial \theta^2}.$$

Therefore, the spherical function ϕ satisfies

$$\frac{d^2 \phi}{dr^2} + 2 \coth 2r \frac{d\phi}{dr} = -(\lambda^2 + 1)\phi.$$

We note that the spherical functions are non-zero at $z = o$, and this allows us to assume without loss of generality that $\phi_{-\lambda}(o) = 1$. Moreover, the spherical functions satisfy the relation $\phi_{-\lambda} = \phi_{-\lambda}$.

We now derive an explicit formula for the elementary spherical functions in H^2. Let $d(z_1, z_2)$ denote the Riemannian distance between the points $z_1, z_2 \in \mathbb{M}$. Since every straight line through the origin is a geodesic,

$$d(o, x) = \int_0^1 \frac{|x|}{1 - t^2 x^2} dt = \frac{1}{2} \log \frac{1 + |x|}{1 - |x|} = \tanh^{-1}(r) \tag{4.5}$$

where $r = |x|$. The abelian component of $SU(1,1)$ under the Iwasawa decomposition is given by the one-parameter group,

$$a_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \in SU(1,1), t \in \mathbb{R}.$$

It acts on the unit disc by the map

$$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cosh t + y \sinh t \\ x \sinh t + y \cosh t \end{pmatrix}$$

where $x + iy = z$. Now, $d(o, a_t.o) = d(o, (\tanh(t), 0)) = t$.

If we substitute $z = |z|e^{i\theta} = (\tanh r)e^{i\theta}$, then one can obtain ([29], p. 38)

$$\phi_\lambda(a_r.o) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cosh(2r) - \sinh(2r)\cos\theta)^{-\frac{1}{2}(i\lambda+1)} d\theta. \tag{4.6}$$

Recall that if $f \in D^\natural(H^2)$, its spherical transform is given by the prescription

$$\tilde{f}(\lambda) = \int_{\mathbb{M}} f(z)\phi_{-\lambda}(z)dz,$$

whenever this integral exists. Harish-Chandra's $\mathbf{c}$-function, is given by

$$\mathbf{c}(\lambda) = \lim_{r\to\infty} e^{(-i\lambda+1)r} \phi_\lambda(a_r.o).$$

This limit exists when $\mathrm{Re}(i\lambda) > 0$, and it can be extended to the meromorphic function

$$\mathbf{c}(\lambda) = \pi^{-\frac{1}{2}} \frac{\Gamma(\frac{1}{2} i\lambda)}{\Gamma(\frac{1}{2}(i\lambda + 1))}.$$

The spherical transform $f \to \tilde{f}$ is inverted by the formula

$$f(z) = \frac{1}{2\pi^2} \int_{\mathbb{R}} \tilde{f}(\lambda)\phi_\lambda(z)|\mathbf{c}(\lambda)|^{-2} d\lambda, f \in D^\natural(D).$$

A spherical Lévy process ζ_t on $\mathbb{M} = H^2$ is given by the one-point motion of a $SO(2)$-bi-invariant Lévy process ζ_t on $G = SU(1,1)$. Thus by Theorem 4.3.4,

$$\begin{aligned} \mathbb{E}(\phi_{-\lambda}(\zeta_t)) &= \exp\left[t \left(c\beta(\Delta_{\mathbb{M}}, \phi_{-\lambda}) + \int_{SU(1,1)-\{e\}} (\phi_{-\lambda}(h) - 1) d\nu(h) \right) \right]. \\ &= \exp\left[t \left(c(\lambda^2 - 1) + \int_{\mathbb{M}} \int_{SO(2)-\{e\}} (\phi_{-\lambda}(zk) - 1) d\nu(z) dk \right) \right] \\ &= \exp\left[t \left(c(\lambda^2 - 1) + \int_{\mathbb{M}} (\phi_{-\lambda}(z) - 1) d\nu(z) \right) \right] \end{aligned}$$

where direct computation shows that $\beta(\Delta_{\mathbb{M}}, \phi_{-\lambda}) = \lambda^2 - 1$. Now, since $\phi_{-\lambda}(z)$ is spherical, we have $\phi_{-\lambda}(z) = \phi_{-\lambda}(r)$, where $|z| = r$. Combining this with (4.6), we have

$$\phi_{-\lambda}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\cosh(2r) - \sinh(2r)\cos\theta)^{\frac{1}{2}(-i\lambda+1)}d\theta.$$

Therefore, the characteristic function of ζ_t is given by

$$\begin{aligned}\mathbb{E}(\phi_{-\lambda}(\zeta_t)) = \exp\Bigg[t\Bigg(c(\lambda^2-1) + \frac{1}{2\pi}\int_{\mathbb{M}}\Bigg(\int_{-\pi}^{\pi}(\cosh(2r) - \sinh(2r)\cos\theta)^{\frac{1}{2}(-i\lambda+1)}d\theta \\ - 2\pi\Bigg)d\nu(z)\Bigg)\Bigg].\end{aligned}$$

Chapter 5

Construction of a General Lévy Process on G/K

In chapter 2, we gave an exposition of the rolling without slipping process and showed how it can be applied to construct any semimartingale with continuous trajectories on a Riemannian manifold from one defined on $\mathbb{R}^d$ (c.f. [20] and [33]). Then, we have shown how this procedure fails to apply to Lévy processes in general (c.f. [3] and [6]): it only works when the Lévy process is isotropic. In chapter 3, we introduced the definition of Lévy processes on Lie groups, and in chapter 4, we were able to establish the Lévy-Khintchine formula for K-bi-invariant Lévy processes (c.f. [4] , [25] and [26]). These processes correspond to the isotropic Lévy processes considered in Chapter 2, when one consider $\mathbb{M} = G/K$, $K = O(n)$ and G the orthogonal frame bundle of $\mathbb{M}$.

The purpose of this chapter is to introduce some new techniques (c.f. [30] and [54]) that have not appeared in the stochastic analysis literature to the author's knowledge which allow one to establish a version of the Lévy-Khintchine formula for general Lévy processes on Riemannian symmetric space. The harmonic analytic tools used here are well established, and we will be mainly following the ideas developed in [29] and [30].

We believe that our technique can be generalised to reductive symmetric spaces of the form $\mathbb{M} = G/H$, by applying more recent work of Delorme, Schlichtkrull and van den Ban (c.f. [54]). Here, G is a reductive Lie group and H is a closed subgroup of G. We will, however, only study the Riemannian case.

The structure of this chapter are is follows: We begin with some discussions in Section 5.1. Section 5.2 is a discussion on how the Lévy-Khintchine formula of a spherical Lévy process on H^2 (c.f. section 4.6) can be extended to general Lévy processes on H^2. A particular focus will be the theory of horocycles. Section

5.3 will extend the ideas from harmonic analysis in section 5.1 to the setting of a general G/K. In particular, we study Eisenstein integrals which serve as generalized versions of spherical functions, and the relation to the full Fourier transform. A new Lévy-Khintchine formula, generalizing Gangolli's ([25] , [26]) will be introduced in section 5.4.

5.1 Motivation

Let $\mathbb{M} = G/K$ and consider the algebra $\mathcal{D}(\mathbb{M})$ of all left invariant differential operators on $\mathbb{M}$. A function on $\mathbb{M}$ which is an eigenfunction of each $D \in \mathcal{D}(\mathbb{M})$ will be called a *joint eigenfunction* of $\mathcal{D}(\mathbb{M})$. Given a homomorphism $\chi : \mathcal{D}(\mathbb{M}) \to \mathbb{C}$, recall from Definition 4.2.2, the space

$$\mathcal{E}_\chi(\mathbb{M}) = \{f \in C^\infty(\mathbb{M}) : Df = \chi(D)f, \forall D \in \mathcal{D}(\mathbb{M})\}$$

is called a joint eigenspace. Let T_χ denote the natural representation of G on $\mathcal{E}_\chi(\mathbb{M})$, that is, $(T_\chi(g)f)(x) = f(g^{-1}x)$. These representations are called *eigenspace representations.* By harmonic analysis on $\mathbb{M}$, we normally wish to seek answer to the following problems,

1. Describe the joint eigenspaces $\mathcal{E}_\chi(\mathbb{M})$ of $\mathcal{D}(\mathbb{M})$.
2. Decompose any "reasonably nice" function on $\mathbb{M} = G/K$ into joint eigenfunctions of $\mathcal{D}(\mathbb{M})$.
3. Determine for which χ the eigenspace representation T_χ is irreducible.

In Lévy process theory, these problems correspond to decomposing the probability distribution of the process ζ_t into Eisenstein integrals, which are joint eigenspaces of the Laplacian.

To obtain a generalized Lévy-Khintchine formula for $\mathbb{M}$, we are really trying to find a suitable set of "basis" functions $\{\varphi_\alpha\}$, so that from knowledge of $\mathbb{E}(\varphi_\alpha(\zeta_t))$, we can reconstruct the law of ζ_t. As soon as it becomes clear what φ_α ought to be, $\mathbb{E}(\varphi_\alpha(\zeta_t))$ can be computed via the Lévy-Itô decomposition, similarly to the way Theorem 4.4 was obtained.

5.2 An Example: H^2

We begin with the example of the hyperbolic plane that illustrates the idea at an intuitive level. The material covered in this section is quite routine, so we will be closely following the development of [29] p.33 - 36.

Given $a, b \geq 0$, let $E_{a,b}$ denote the space of holomorphic functions on $\mathbb{C} - \{0\}$ satisfying

$$||f||_{a,b} = \sup_z(|f(z)|e^{-a|z|-b|z|^{-1}}) < \infty.$$

Then, $E_{a,b}$ is a Banach space with the norm $||.||_{a,b}$. Observe that $E_{a,b} \subset E_{a',b'}$ if and only if $a \leq a'$, $b \leq b'$, and in this case $||f||_{a,b} \geq ||f||_{a',b'}$, so that the injection of $E_{a,b}$ into $E_{a',b'}$ is continuous. We can give the union

$$E = \bigcup_{a.b} E_{a,b},$$

the inductive limit topology. We identify the members of E with their restrictions to the unit circle S^1 and call the members of the dual space E' **entire functionals** on S^1. Since these generalise measures, we will denote the dual pairing as follows:

$$T(f) = \int_{S^1} f(\omega)dT(\omega), f \in E, T \in E'.$$

Now we are ready to state our first theorem.

Theorem 5.2.1. *The eigenfunctions of the Laplacian on $\mathbb{R}^2$ are precisely the harmonic functions*[1] *and the functions*

$$f(x) = \int_{S^1} e^{i\lambda\langle x,\omega\rangle} dT(\omega),$$

where $\lambda \in \mathbb{C} - \{0\}$ and T is an entire functional on S^1.

Remark 5.2.2. The right hand side of the above is well defined. If $x = (x_1, x_2)$, the integrand is the restriction to S^1 of the function

$$f(z) = \exp\left(\frac{1}{2}(i\lambda)x_1(z + z^{-1}) + \frac{1}{2}\lambda x_2(z - z^{-1})\right),$$

which belongs to E.

[1] These functions are annihilated by the Laplacian, and thus correspond to $\lambda = 0$.

With these preparations, we can now describe a system of harmonic analysis on H^2 that will be applied to obtain the Lévy-Khintchine formula for a H^2-valued Lévy process. Observe from the Euclidean case, if $\mu \in \mathbb{C}$ and $\omega \in S^{n-1}$, the function $x \mapsto e^{\mu\langle x,\omega\rangle_{\mathbb{R}^n}}$ has the following properties,

1. It is an eigenfunction of the Laplacian $L_{\mathbb{R}^n}$ on $\mathbb{R}^n$.
2. It is constant on each hyperplane perpendicular to ω.

Now, a hyperplane is orthogonal to a family of parallel lines. The geometric analogue of this when $\mathbb{R}^d$ is replaced by $\mathbb{M}$ is called a **horocycle**. In the $\mathbb{M} = H^2$ case, this is a circle ξ on $\mathbb{M}$ tangential to the boundary $B = \partial\mathbb{M}$. All such circles are orthogonal to the geodesics of $\mathbb{M}$ tending to the point of contact $b \in B$.

For $z \in \xi$, we let $< z, b >$ be the Riemannian distance from o to ξ. This distance is negative if o lies inside ξ. This "inner product" $< z, b >$ is an non-Euclidean analogue of $\langle x, \omega\rangle_{\mathbb{R}^n}$, which geometrically means the (signed) distance from $0 \in \mathbb{R}^n$ to the hyperplane through x with normal ω. We shall write $\xi(z, b)$ for the above horocycle through z and b.

By this analogy, we have the following lemma.

Lemma 5.2.3. *([29] , pp 32) We consider the function $e_{\mu,b} : z \mapsto e^{\mu<z,b>}, z \in \mathbb{M}$. For the Laplacian L, we have $Le_{\mu,b} = \mu(\mu - 2)e_{\mu,b}$.*

Proof. For $t \in \mathbb{R}$, let

$$a_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh h \end{pmatrix} \in SU(1,1),$$

and let $d(z_1, z_2)$ denote the Riemannian distance between the points $z_1, z_2 \in \mathbb{M}$. Since every straight line through the origin is a geodesic,

$$d(o, x) = \int_0^1 \frac{|x|}{1 - t^2x^2}dt = \frac{1}{2}\log\frac{1 + |x|}{1 - |x|} = \tanh^{-1}(r) \quad (3)$$

where $r = |x|$. Then,

$$d(o, a_t.o) = d(o, \tanh(t)) = t.$$

If b_0 is the point of B on the positive x-axis, we therefore have $e_{\mu,b_0}(\tanh(t)) = e^{\mu t}$. The horocycles tangential to B at b_0 are the orbits of the group

$$N = \left[n_s = \begin{pmatrix} 1+is & -is \\ is & 1-is \end{pmatrix} : s \in \mathbb{R}\right]$$

The orbit $N.o$ is such a horocycle and so is the orbit $Na_t.o = a_tN.o$ (a_t normalises N). Now consider functions on $\mathbb{M}$ that are constant on each of these horocycles. Because of its N-invariance, L maps the class of such functions into itself. The restriction of L to such functions is a differential operator L_0 in the variable t. Since the isometry a_t satisfies $a_t \tanh\tau = \tanh(t+\tau)$, the invariance of L under a_t means that L_0 is invariant under the translation $\tau \mapsto \tau + t$. Therefore, L_0 has constant coefficients, so $e^{\mu t}$ is an eigenfunction for it, and hence e_{μ,b_0} is an eigenfunction of L. The eigenvalue can be calculated by expressing L in the coordinates $a_tn_s.o \mapsto (t,s)$.
□

Looking at the above lemma, it is natural to reformulate it by "completing the square". We obtain that

$$L(e^{(i\lambda+1)<z,b>}) = -(\lambda^2+1)e^{(i\lambda+1)<z,b>}, \lambda \in \mathbb{C}.$$

Inspired by the above, we define the Fourier transform on $\mathbb{M} = H^2$ as follows:

Definition 5.2.4. If f is a complex-valued function on $\mathbb{M}$, its **Fourier transform** is defined by

$$\hat{f}(\lambda, b) = \int_{\mathbb{M}} f(z)e^{(-i\lambda+1)<z,b>}dz$$

for all $\lambda \in \mathbb{C}, b \in B$, for which this integral exists, where dz is the Riemannian measure surface on $\mathbb{M}$ given by $dz^2 = (1-x^2-y^2)^{-2}dxdy$.

Definition 5.2.5. We call a C^∞ function $\psi(\lambda, b)$ on $\mathbb{C}\times B$, which is holomorphic in λ, a **holomorphic function of uniform exponential type** R if for each $N \in \mathbb{N}$,

$$\sup_{\lambda\in\mathbb{C}, b\in B} e^{-R|\mathrm{Im}\lambda|}(1+|\lambda|)^N|\psi(\lambda, b)| < \infty.$$

Now we can state our main theorem regarding the Fourier transform on H^2.

Theorem 5.2.6. *(c.f. [29] , p. 33)*

1. *If $f \in \mathcal{D}(\mathbb{M})$, then*

$$f(z) = \frac{1}{4\pi} \int_{\mathbb{R}} \int_B \hat{f}(\lambda, b) e^{(i\lambda+1)<z,b>} \lambda \tanh\left(\frac{\pi\lambda}{2}\right) d\lambda db,$$

where db is the circular measure on B normalised by $\int db = 1$.

2. *The mapping $f \mapsto \hat{f}$ is a bijection of $D(\mathbb{M})$ onto the space of holomorphic functions $\psi(\lambda, b)$ of uniform exponential type satisfying the functional equation*

$$\int_B e^{(i\lambda+1)<z,b>} \psi(\lambda, b) db = \int_B e^{(-i\lambda+1)<z,b>} \psi(-\lambda, b) db.$$

3. *The mapping $f \mapsto \hat{f}$ extends to an isometry of $L^2(\mathbb{M})$ onto*

$$L^2\left(\mathbb{R}^+ \times B, (2\pi)^{-1} \lambda \tanh(\frac{1}{2}\pi\lambda d\lambda db)\right).$$

Remark 5.2.7. The main difference between the present Fourier transform and the spherical transform considered in the previous chapter, is that now $\hat{f}$ is a function of two variables λ and b, as the kernel of the Fourier transform is in two variables. This requires us to nest another layer of Fourier analysis in the b direction on top of the Fourier transform.

Let $\mathcal{A}(B)$ denote the space of analytic functions on the boundary B, considered as an analytic manifold. Let U be an open annulus containing B, $\mathcal{H}(U)$ the space of holomorphic functions on U topologised by uniform convergence on compact subsets. Since each analytic function on B extends to a function in $\mathcal{H}(U)$ for a suitably chosen U, we can identify $\mathcal{A}(B)$ with the union $\bigcup_U \mathcal{H}(U)$ and give it the inductive limit topology.

Definition 5.2.8. The elements in the dual space $\mathcal{A}'(B)$ are called **hyperfunctions**. Since they generalise measures, it is convenient to write

$$T(f) = \int_B f(b) dT(b), f \in \mathcal{A}(B), T \in \mathcal{A}'(B).$$

For $\lambda \in \mathbb{C}$, let $\mathcal{E}_\lambda(\mathbb{M}) = \{f \in C^\infty(\mathbb{M}) : Lf = -(\lambda^2+1)f\}$, with the topology induced from $C^\infty(\mathbb{M})$.

Theorem 5.2.9. *The eigenfunctions of the Laplace-Beltrami operator L on $\mathbb{M}$ are precisely the functions*

$$f(z) = \int_B e^{(i\lambda+1)<z,b>} dT(b),$$

where $\lambda \in \mathbb{C}$ and $T \in \mathcal{A}'(B)$. Moreover, if $i\lambda \neq -1, -3, -5, \ldots$, then the mapping $T \mapsto f$ is a bijection of $\mathcal{A}'(B)$ onto $\mathcal{E}_\lambda(D)$.

Definition 5.2.10. For $\lambda \in \mathbb{C}$, let T_λ denote the representation of $SU(1,1)$ on the eigenspace $\mathcal{E}_\lambda(\mathbb{M})$.

We then have the following result.

Theorem 5.2.11. *The eigenspace representation T_λ is irreducible if and only if $i\lambda + 1 \notin 2\mathbb{Z}$.*

Next, we revise the classical spherical transform on H^2, and we will show how it is related to the Fourier transform.

Definition 5.2.12. The point $\lambda \in \mathbb{C}$ is *simple* if the mapping $F \mapsto f$ of $L^2(B) \to C^\infty(D)$ given by

$$f(z) = \int_B e^{(i\lambda+1)<z,b>} F(b)db$$

is one-to-one.

Theorem 5.2.13. *If $-\lambda$ is simple, then the function space on B, $\{\tilde{f}(\lambda, .) : f \in D(\mathbb{M})\}$ is dense in $L^2(B)$.*

The preceding theorems tells us that Fourier transforms are elements of $L^2(B)$ if we fix λ. We need to establish yet another Fourier theory for $L^2(B)$ in order to understand the full Fourier transform on $\mathbb{M}$.

Theorem 5.2.14. *Let $m \in \mathbb{Z}$. The eigenfunctions f of L satisfying the homogeneity condition,*

$$f(e^{i\theta}z) = e^{im\theta} f(z)$$

are the constant multiples of the functions

$$\Phi_{\lambda,m}(z) = \int_B e^{(i\lambda+1)<z,b>} \chi_m(b)db,$$

where $\lambda \in \mathbb{C}$ *and the characters* $\chi_m(e^{i\varphi}) = e^{im\varphi}$. *Moreover, we have the relation*

$$e^{(i\lambda+1)<z,b>} = \sum_m \Phi_{\lambda,m}(z)\chi_{-m}(b).$$

The $\Phi_{\lambda,m}$'s are called **generalised spherical functions**, or **Eisenstein integrals**.

We obtain the following Fourier expansion from the theorem above,

$$\begin{aligned}\hat{f}(\lambda, b) &:= \int_{\mathbb{M}} f(z)e^{(-i\lambda+1)<z,b>}dz \\ &= \int_{\mathbb{M}} f(z)\sum_m \Phi_{-\lambda,m}(z)\chi_{-m}(b)dz \\ &= \sum_m a_m\chi_{-m}(b)\end{aligned}$$

where,

$$a_m = \int_{\mathbb{M}} f(z)\Phi_{-\lambda,m}(z)dz.$$

Interchanging the order of summation and integral is valid if we assume $\int_{\mathbb{M}} |f(z)|dz = 1$. This assumption is not over-restrictive for us, because we are primarily interested in using f as the probability density function of a random variable.

This calculation motivates the following definition,

Definition 5.2.15. Let $f \in D(\mathbb{M})$, the m**-spherical transform** of f is given by,

$$\tilde{f}_m(\lambda) = \int_{\mathbb{M}} f(z)\Phi_{-\lambda,m}(z)dz.$$

This suggests that to obtain a Lévy-Khintchine formula for general Lévy processes on H^2, $\{\zeta_t\}_{t\geq 0}$, the quantity we should compute is $\mathbb{E}(\Phi_{-\lambda,m}(\zeta_t))$. This allows us to recover the Fourier transform of the law $\hat{\mu}_{\zeta_t}(\lambda, b)$ as

$$\hat{\mu}_{\zeta_t}(\lambda, b) = \sum_m \mathbb{E}(\Phi_{-\lambda,m}(\zeta_t))\chi_{-m}(b).$$

Applying the inversion formula of Theorem 5.2.6 to $\hat{\mu}_{\zeta_t}(\lambda, b)$ recovers the law of ζ_t.

5.3 A survey of Eisenstein integrals

From the example in the previous section, we have obtained:

1. The full Fourier transform and its inversion formula on $\mathbb{M}$ by using the $< .,. >$ in place of $\langle .,. \rangle$.
2. The full Fourier transform is a function in λ and b, it needs another layer of Fourier analysis in the b direction before it can be useful for our purposes. To that regard, we were able to write the full Fourier transform as

$$\hat{f}(\lambda, b) = \sum_m \tilde{f}_m(\lambda)\chi_{-m}(b),$$

   where $\tilde{f}_m(\lambda)$ is the generalized spherical transform of order m.
3. In the case where $\mathbb{M} = H^2$, we have $\chi_{-m}(\varphi) = e^{im\varphi}$. We will need to find a suitable candidate for χ_m in the general $\mathbb{M} = G/K$ case.
4. The Lévy-Khintchine formula will be the m-spherical transform, $\mathbb{E}(\Phi_{-\lambda,m}(\zeta_t))$ for an $\mathbb{M}$-valued Lévy process ζ_t. We will need to decide what $\Phi_{-\lambda,m}$ is for the general $\mathbb{M} = G/K$.
5. We can recover the probability distribution of ζ_t by applying the Fourier inversion formula to

$$\hat{f}(\lambda, b)(t) = \sum_m (\Phi_{-\lambda,m}(\zeta_t))\chi_{-m}(b).$$

In this section, these ideas will be extended to the setting of general G/K. The most significant change is that representations of K will be used instead of characters for general symmetric spaces. Recall that $\mathbb{M} = H^2$ can be represented as the unit disc $\{|z| < 1\}$ with the Riemannian metric $ds^2 = (1-x^2-y^2)^{-2}(dx^2+dy^2)$. We introduced the horocycles of this space in the previous section, they are the circles in $\mathbb{M}$ tangential to the boundary $\{B : |z| = 1\}$. Now, we will write $\mathbb{M}$ as

$$\mathbb{M} = SU(1,1)/SO(2)$$

and we shall describe the horocycles in a group theoretic way. This will suggest the generalisation to a symmetric space of the non-compact type.

Recall that the Lie algebra $\mathfrak{su}(1,1)$ of $SU(1,1)$ is given by

$$\mathfrak{su}(1,1) = \left[\begin{pmatrix} i\lambda & \beta \\ \bar{\beta} & -i\lambda \end{pmatrix} : \lambda \in \mathbb{R}, \beta \in \mathbb{C} \right]$$

and an Iwasawa decomposition by $\mathfrak{su}(1,1) = \mathfrak{g}_o = \mathfrak{k}_o + \mathfrak{a}_o + \mathfrak{n}_o$, where

$$\mathfrak{k}_o = \left[t \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} : t \in \mathbb{R} \right] \quad \mathfrak{a}_o = \left[t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : t \in \mathbb{R} \right] \quad \mathfrak{n}_o = \left[t \begin{pmatrix} i & -i \\ i & -i \end{pmatrix} : t \in \mathbb{R} \right]$$

The subgroup $N_0 = \exp \mathfrak{n}_o$ of $SU(1,1)$ equals

$$N_0 = \left[\begin{pmatrix} 1+it & -it \\ it & 1-it \end{pmatrix} : t \in \mathbb{R} \right]$$

and the horocycle with $0 \leq x < 1$ as diameter equals the orbit $\xi_0 = N_0.0$. Any other horocycle ξ can be written in the form $ka.\xi_0$ ($k \in K_0 = \exp \mathfrak{k}_o$ and $a \in A_0 = \exp \mathfrak{a}_o$). But, $\xi = kaN_0(ka)^{-1}.(ka.0)$ so ξ is an orbit of a group conjugate to N_0. Conversely, if $g \in G_0 = SU(1,1), z \in \mathbb{M}$, and let $g^{-1}.z = na$, then the orbit $gN_0g \in .z$ is the horocycle $ga.\xi_0$.

This situation can be generalised to $\mathbb{M} = G/K$, where G is a connected semisimple Lie groups G with a finite center, and K is a maximal compact subgroup of G. As usual practice, let $G = KAN$ be an Iwasawa decomposition, and let M denote the centraliser of A in K, let $o = \{K\} \in \mathbb{M}$, and ξ_0 the orbit $N.o$.

Definition 5.3.1. A **horocycle** in $\mathbb{M}$ is any orbit $N'.x$ where $x \in X$ and N' is a subgroup of G conjugate to N.

Remark 5.3.2. The above definition looks as if it depends on the actual choice of the Iwasawa decomposition. However, the choice of the Iwasawa decomposition is immaterial[2], as all such decompositions are conjugate.

First, we state some basic facts about horocycles ([30] , pp. 77).

Theorem 5.3.3.

- *Each horocycle is a closed submanifold of* X.

[2] Immaterial in the sense that it does not affect the geometry. c.f. [30] , pp. 77.

- *The group G acts transitively on the set of horocycles in X. The subgroup of G which maps the horocycle ξ_0 into itself is equal to MN.*

The second part of the theorem motivates the following definition.

Definition 5.3.4. The set of horocycles in X with the differentiable structure of G/MN will be called the **dual space** of $\mathbb{M}$, denoted by Ξ.

The following theorem generalises the idea of the classical "polar coordinate" representation to a symmetric space.

Theorem 5.3.5.

- *$B = K/M$ can be identified with the set of all Weyl chambers in $\mathfrak{p}$. The map $\eta : (kM, a) \mapsto kaK$ is a differentiable mapping of $B \times A \to X$, and the map $\iota : (kM, a) \mapsto kaMN$ is a diffeomorphism of $(B \times A$ onto Ξ. If $A^+ = \exp \mathfrak{a}^+$, the restriction of η to $B \times A^+$ is a diffeomorphism onto the regular set $\mathbb{M}' \subset \mathbb{M}$, and moreover, $\mathbb{M}' = K\bar{A^+}.o$.*
- *Given $x \in \mathbb{M}, b \in B$ fixed, there exists a unique horocycle passing through x with normal b.*
- *Given a horocycle ξ and a point $x \in \xi$, there exist exactly $|W|$ distinct horocycles passing through x with tangent space at x equal to $T_x\xi$.*

By the above theorem, we see that each $\xi \in \Xi$ can be written as $\xi = kaMN$, where $kM \in B = K/M$ and $a \in A$ are unique. Under this decomposition, we say the Weyl chamber kM is **normal** to ξ, and call $A(x, b) = \log a$ the **composite distance from o to ξ**. More generally, if $x = g_1K \in X$ and $\xi = g_2MN \in \Xi$, we write $< x, \xi >= H(g_1^{-1}g_2)$ and call it the **composite distance from x to ξ**, where $H(g) \in \mathfrak{a}$ is given by $g \in K \exp H(g)N$ under the Iwasawa decomposition. $< x, \xi >$ is well defined and invariant under the action of G. This is the symmetric space analogue of the scalar product $\langle y, w\rangle$ on $\mathbb{R}^d$, where $||w||_{\mathbb{R}^d} = 1$. The vector-valued inner product $A(x, b)$ also generalises the inner product $<,>$ on H^2 considered in the previous section.

In terms of the Iwasawa decomposition, $G = KAN = NAK$, if $g \in G$, $H(g) \in \mathfrak{a}$, and $A(g) \in \mathfrak{a}$ are determined by $g = k_1 \exp H(g)n_1 = n_2 \exp A(g)k_2$ with $k_1, k_2 \in K, n_1, n_2 \in N$, then $A(g) = -H(g^{-1})$. If $x = gK \in \mathbb{M}, b = kM \in B$, then $g^{-1}k \in K \exp(H(g^{-1}k)N)$, so the point gK lies on the horocycle

$k\exp(-H(g^{-1}k)).\xi_0$. Therefore, we obtain a rather explicit formula for our inner product,

$$A(gK, kM) = A(k^{-1}g) = -H(g^{-1}k).$$

Theorem 5.3.6. *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*, b \in B$. The function*

$$e_{\lambda,b} : x \mapsto e^{(i\lambda+\varrho)A(x,b)}$$

is a joint eigenfunction of $\mathcal{D}(\mathbb{M})$. In fact, $e_{\lambda,b}$ belongs to the joint eigenspace

$$\mathcal{E}_\lambda(\mathbb{M}) = \{f \in \mathcal{E}(\mathbb{M}) : Df = \Gamma(D)(i\lambda)f \text{ for } D \in \mathcal{D}(\mathbb{M})\}.$$

where Γ is the Γ introduced in Theorem 4.1.8.

The analogy between $\langle x, \omega\rangle_{\mathbb{R}^d}$ and $A(x, b)$ motivates the following definition of the Fourier transform on $\mathbb{M} = G/K$.

Definition 5.3.7. ([30], p. 223) If f is a function on $\mathbb{M}$, the **Fourier transform** $\hat{f}$ is defined by

$$\hat{f}(\lambda, b) = \int_{\mathbb{M}} f(x)e^{(-i\lambda+\varrho)A(x,b)}dx$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*, b \in B$ for which this integral exists.

Remark 5.3.8. If $f \in D^\natural(\mathbb{M})$, we can replace f with $f^{\tau(k)}$ (by the K-invariance property), where $f^{\tau(k)}(x) = f(k^{-1}x)$. Since $A(k.x, b) = A(x, k^{-1}b)$, the Fourier transform becomes

$$\begin{aligned}\hat{f}(\lambda, b) = \int_{\mathbb{M}} f(x)e^{(-i\lambda+\varrho)A(x,b)}dx &= \int_{\mathbb{M}} f^{\tau(k)}(kx)e^{(-i\lambda+\varrho)A(kx,b)}dx \\ &= \int_{\mathbb{M}} f(x)e^{(-i\lambda+\varrho)A(x,k^{-1}b)}dx \\ &= \hat{f}(\lambda, k^{-1}b),\end{aligned}$$

and hence $\hat{f}(\lambda, b)$ is independent of b. Integrating both sides of the Fourier transform equation with respect to the normalised Haar measure db we get,

$$\begin{aligned}\hat{f}(\lambda) &= \int_B \int_{\mathbb{M}} f(x) e^{(-i\lambda+\varrho)A(x,b)} dx db \\ &= \int_{\mathbb{M}} f(x) \left(\int_B e^{(-i\lambda+\varrho)A(x,b)} db \right) dx \\ &= \int_{\mathbb{M}} f(x) \phi_{-\lambda}(x) dx\end{aligned}$$

This shows that the Fourier transform considered here reduces to the classical spherical transform for K-invariant functions f.

We now state the Fourier inversion formula (c.f. [30], p. 225).

Theorem 5.3.9. *For each $f \in \mathcal{D}(\mathbb{M})$,*

$$f(x) = |W|^{-1} \int_{a^*} \int_B e^{(i\lambda+\varrho)A(x,b)} \tilde{f}(\lambda, b) |c(\lambda)|^{-2} d\lambda db$$

for $x \in \mathbb{M}$ and $\tilde{f}$ is the Fourier transform of f according to Definition 5.2.4.

The Plancherel formula allows us to extend the Fourier transform from $D(\mathbb{M})$ to $L^2(\mathbb{M})$. To state the result, we first define $\mathfrak{a}^*_+ = \{\lambda \in \mathfrak{a}^* : A_\lambda \in \mathfrak{a}^+\}$ of the positive Weyl chamber $\mathfrak{a}^+$.

Theorem 5.3.10. *(c.f. [30], p. 227) The Fourier transform $f(x) \mapsto \hat{f}(\lambda, b)$ extends to an isometry of $L^2(\mathbb{M})$ onto $L^2(a^*_+ \times B)$ (with the measure $|c(\lambda)|^{-2} d\lambda db$ on $a^*_+ \times B$. Moreover,*

$$\int_X f_1(x)\overline{f_2(x)} dx = w^{-1} \int_{a^* \times B} \tilde{f}_1(\lambda, b) \overline{\tilde{f}_2(\lambda, b)} |c(\lambda)|^{-2} d\lambda db.$$

Analogous to the H^2 example, the Fourier transform of $f \in L^2(\mathbb{M})$ is a function of $b \in B$ for each fixed $\lambda \in \mathfrak{a}^*$. What we need now is to develop another layer of Fourier theory on functions on B, and nest that into our general Fourier transform.

Definition 5.3.11. The element $\lambda \in \mathfrak{a}^*_c$ is said to be **simple** if and only if the **Poisson transform** $P_\lambda : F \in L^2(B) \mapsto f \in \mathcal{E}_\lambda(\mathbb{M})$ given by

$$f(x) = \int_B e^{(i\lambda+\varrho)A(x,b)} F(b) db$$

is injective.

Theorem 5.3.12. *Suppose $-\lambda \in \mathfrak{a}_c^*$ is simple, then the space of functions, $b \mapsto \tilde{f}(\lambda, b)$, is dense in $L^2(B)$ as f runs through $D(\mathbb{M})$.*

The following result applies Fourier decomposition on the compact subgroup K to obtain a classification of the joint eigenfunctions of $\mathcal{D}(\mathbb{M})$.

Theorem 5.3.13. *Let f be an arbitrary joint eigenfunction of $\mathcal{D}(\mathbb{M})$. Then, there exists a $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and a hyperfunction T on B, such that*

- *f is given by the Poisson transform of T,*

$$f(x) = \int_B e^{(i\lambda+\varrho)A(x,b)} dT(b).$$

- *T has the Fourier series*

$$T \sim \sum_{\delta \in \hat{K}_M} d(\delta)\mathrm{Tr}(A_\delta \delta(k)),$$

in the sense that

$$A_\delta = \int_K \delta(k^{-1}) d\tilde{T}(k),$$

where $\tilde{T}$ being the lift of T to K.

- *$f(x)$ has a convergent expansion in $C^\infty(\mathbb{M})$,*

$$f(x) = \sum_{\delta \in \hat{K}_M} d(\delta)\mathrm{Tr}(A_\delta \Phi_{\lambda,\delta}(x)).$$

The full Fourier transform allows us to decompose an "arbitrary function" along the harmonics of the λ direction. Theorems 4.2.12 and 4.2.13, allows us to perform the decomposition likewise the b direction. We will now obtain an explicit description of the structure of the range $\widetilde{\mathcal{D}(\mathbb{M})}$ of $\mathcal{D}(\mathbb{M})$ under the Fourier transform.

Definition 5.3.14. A C^∞ function $\psi(\lambda, b)$ on $\mathfrak{a}_{\mathbb{C}}^* \times B$, holomorphic in λ is called a **holomorphic function of uniform exponential type** if there exists a constant $R \geq 0$ such that for each $N \in \mathbb{Z}^+$,

$$\sup_{\lambda \in \mathfrak{a}_{\mathbb{C}}^*, b \in B} e^{-R|\mathrm{Im}\lambda|}(1+|\lambda|)^N |\psi(\lambda, b)| < \infty.$$

Here, $\mathrm{Im}\lambda = y$ if $\lambda = x + iy; x, y \in \mathfrak{a}^*$, and $|\lambda|^2 = (|x|^2_{\mathfrak{a}^*} + |y|^2_{\mathfrak{a}^*})$.

The space of all such functions will be denoted by $\mathcal{H}^R(\mathfrak{a}^* \times B)$, and $\mathcal{H}(\mathfrak{a}^* \times B) = \bigcup_{R\geq 0} \mathcal{H}^R(\mathfrak{a}^* \times B)$. Finally, we take $\mathcal{H}(\mathfrak{a}^* \times B)_W$ to be the space of functions $\psi \in \mathcal{H}(\mathfrak{a}^* \times B)$ satisfying,

$$\int_B e^{(is\lambda+\varrho)A(x,b)}\psi(s\lambda, b)db = \int_B e^{(i\lambda+\varrho)A(x,b)}\psi(\lambda, b)db$$

for $s \in W$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}, x \in \mathbb{M}$. The space of W-invariant holomorphic functions $\lambda \mapsto \psi(\lambda) \in \mathcal{H}(\mathfrak{a}^* \times B)$ (independent of b) is denoted by $\mathcal{H}_W(\mathfrak{a}^*_{\mathbb{C}})$.

Theorem 5.3.15. *The Fourier transform $f(x) \mapsto \tilde{f}(\lambda, b)$ is a bijection of $\mathcal{D}(\mathbb{M})$ onto $\mathcal{H}(\mathfrak{a}^* \times B)_W$. Moreover, $\psi = \tilde{f} \in \mathcal{H}(\mathfrak{a}^* \times B)$ if and only if $supp(f) \subset Cl(B_R(o))$.*

Recall that $B = K/M$. Let $\hat{K}$ denote the set of equivalence classes of unitary irreducible representations on K. For each $\delta \in \hat{K}$, let V_δ be a finite dimensional inner product vector space on which a representation of class δ is realised, let such a representation also be denoted by δ. Let $\hat{K}_M$ denote the set of elements $\delta \in \hat{K}$ for which the subspace

$$V_\delta^M = \{v \in V_\delta : \delta(m)v = v, m \in M\} \neq \{0\}.$$

If $\delta \in \hat{K}_M$, the **contragredient representation** $\check{\delta}$ also belongs to $\hat{K}_M$, where $\check{\delta}(g) := \delta(g^{-1})^*$. Finally, we put $d(\delta) = \dim V_\delta$ and $l(\delta) = \dim V_\delta^M$.

Definition 5.3.16. For $\delta \in \hat{K}_M, \lambda \in \mathfrak{a}^*_{\mathbb{C}}$, the function

$$\Phi_{\lambda,\delta}(x) = \int_K e^{(i\lambda+\varrho)A(x,kM)}\delta(k)dk, x \in \mathbb{M},$$

is called the **Eisenstein integral** of class δ. These are also known as the **generalised spherical functions**.

The function $\Phi_{\lambda,\delta} : X \to \mathrm{Hom}(V_\delta, V_\delta)$, satisfies the following conditions,

- $\Phi_{\lambda,\delta}(k.x) = \delta(k)\Phi_{\lambda,\delta}(x), \Phi_{\lambda,\delta}(x)\delta(m) = \Phi_{\lambda,\delta}(x)$.
- $D\Phi_{\lambda,\delta} = \Gamma(D)(i\lambda)\Phi_{\lambda,\delta}, D \in \mathcal{D}(\mathbb{M})$.

If δ is the trivial representation, then $\Phi_{\lambda,\delta}$ reduces to the (zonal) spherical function ϕ_λ. Let $f \in C^\infty(\mathbb{M})$, we put

$$f^\delta(x) = d(\delta) \int_K f(kx)\delta(k^{-1})dk,$$

where the right hand side is a Bochner integral. Then, f^δ is a C^∞ map from $\mathbb{M}$ to $\mathrm{Hom}(V_\delta, V_\delta)$ satisfying

$$f^\delta(kx) = \delta(k)f^\delta(x).$$

Hence, if $f^\delta \not\equiv 0$, then $f^\delta(a.o) \neq 0$ for some $a \in A$, and hence the space V_δ^M of $\hat{K}_M$-fixed vectors is non-zero. The functions f^δ determine f by the Peter-Weyl theorem for vector valued functions (c.f. [29], Chapter 5, Corollary 3.4)

$$f(x) = \sum_{\delta \in \hat{K}_M} d(\delta) \int_K \chi_\delta(k^{-1}) f(kx)dk = \sum_{\delta \in \hat{K}_M} \mathrm{Tr}(f^\delta).$$

where χ_δ is the character of $\delta = \mathrm{Tr}(\delta)$, and the last equality follows from $f^\delta = 0$ if $V_\delta^M = \{0\}$.

With $\delta \in \hat{K}$ acting on V_δ, consider the space $\mathcal{D}(\mathbb{M}, \mathrm{Hom}(V_\delta, V_\delta))$ of C^∞ functions on $\mathbb{M}$ of compact support having values in $\mathrm{Hom}(V_\delta, V_\delta)$. We define,

$$\mathcal{D}^\delta(\mathbb{M}) := \{F \in \mathcal{D}(X, \mathrm{Hom}(V_\delta, V_\delta)) : F(kx) = \delta(k)F(x)\}.$$

This space carries a natural topology in which it is an LF-space [3]. In fact, it is the inductive limit of the Fréchet spaces

$$\mathcal{D}_R(\mathbb{M}, \mathrm{Hom}(V_\delta, V_\delta)), R = 0, 1, 2, ...$$

of functions in $\mathcal{D}(\mathbb{M}, \mathrm{Hom}(V_\delta, V_\delta))$, having support in $Cl(B^R(o))$. Let $\mathcal{D}_\delta(\mathbb{M})$ denote the space of K-finite functions in $\mathcal{D}(\mathbb{M})$ of type δ. The spaces $\mathcal{D}^\delta(\mathbb{M})$ and $\mathcal{D}_\delta(\mathbb{M})$ are given the topologies induced from the space $\mathcal{D}(\mathbb{M}, \mathrm{Hom}(V_\delta, V_\delta))$ and the space $\mathcal{D}(\mathbb{M})$ respectively. For $\varphi \in C(K), f \in C(\mathbb{M})$, we write

$$(\varphi * f)(x) = \int_K \varphi(k) f(k^{-1}x)dk, x \in X.$$

[3] An LF-space is a topological vector space that is a countable strict inductive limit of Fréchet spaces. See [29] p. 398 for more details.

Theorem 5.3.17.

- *The map* $Q : F(x) \mapsto \mathrm{Tr}(F(x))$ *is a homomorphism of* $\mathcal{D}^{\delta}(\mathbb{M})$ *onto* $\mathcal{D}_{\check{\delta}}(\mathbb{M})$ *and its inverse is given by* $f \mapsto f^{\delta}$.
- *The maps*

$$p : f \in \mathcal{D}(\mathbb{M}) \mapsto d(\delta)\chi_{\delta} * f \in \mathcal{D}_{\check{\delta}}(\mathbb{M})$$

$$q : f \in \mathcal{D}(\mathbb{M}) \mapsto f^{\delta} \in D^{\delta}(\mathbb{M})$$

are continuous open surjections and the images $\mathcal{D}_{\check{\delta}}(\mathbb{M})$ *and* $\mathcal{D}^{\delta}(\mathbb{M})$ *are LF-spaces, closed in* $\mathcal{D}(\mathbb{M})$ *and* $\mathcal{D}(X, \mathrm{Hom}(V_{\delta}, V_{\delta}))$*, respectively.*

If $f \in \mathcal{D}_{\check{\delta}}(\mathbb{M})$, then $f = d(\delta)\chi_{\delta} * f$, so

$$\tilde{f}(\lambda, kM) = d(\delta) \int_{\mathbb{M}} f(x) \left(\int_K e^{(-i\lambda+\varrho)A(x,vK)} \chi_{\delta}(kv^{-1}) dv \right) dx.$$

This leads to the following definition of the δ-spherical transform.

Definition 5.3.18. For $f \in \mathcal{D}_{\check{\delta}}(\mathbb{M})$, the **$\delta$-spherical transform** $\tilde{f} \in \mathcal{H}(\mathfrak{a}^*_{\mathbb{C}}, \mathrm{Hom}(V_{\delta}, V_{\delta}^M))$ is defined by

$$\hat{f}_{\delta}(\lambda) = d(\delta) \int_X f(x) \Phi_{\bar{\lambda},\delta}(x)^* dx,$$

where

$$\Phi_{\bar{\lambda},\delta}(x)^* = \int_K e^{(-i\lambda+\varrho)A(x,kM)} \delta(k^{-1}) dk.$$

Theorem 5.3.19. *([30], p.290) The δ-spherical transform is inverted by*

$$f(x) = |W|^{-1} \mathrm{Tr} \left[\int_{\mathfrak{a}^*} \Phi_{\lambda,\delta}(x) \hat{f}_{\delta}(x) |c(\lambda)|^{-2} d\lambda \right], f_{\delta} \in \mathcal{D}_{\check{\delta}}(\mathbb{M}),$$

where $|W|$ *is the cardinality of the Weyl group. Moreover,*

$$\int_{\mathbb{M}} |f(x)|^2 dx = |W|^{-1} d(\delta)^{-1} \int_{\mathfrak{a}^*} \mathrm{Tr}(\tilde{f}(\lambda)\tilde{f}(\lambda)^*) |c(\lambda)|^{-2} d\lambda.$$

If $\delta \equiv 1$, then $f \mapsto \hat{f}_1$ is just the spherical transform of functions in $\mathcal{D}^{\natural}(\mathbb{M})$. In general, $\delta(m)\hat{f}_{\delta}(\lambda) = \hat{f}_{\delta}(\lambda)$, and $\hat{f}_{\delta} : \mathfrak{a}^*_{\mathbb{C}} \mapsto \mathrm{Hom}(V_{\delta}, V_{\delta}^M)$ is a smooth function. The following theorem summarizes the relationship between the δ-spherical transform and the full Fourier transform.

Theorem 5.3.20. *([30], p.286) The δ-spherical transform $\hat{f}$ is related to the Fourier transform $\tilde{f}$ by*

$$\hat{f}_\delta(\lambda) = d(\delta)\int_K \tilde{f}(\lambda, kM)\delta(k^{-1})dk, \quad \tilde{f}(\lambda, kM) = \sum_{\delta\in\hat{K}_M} \mathrm{Tr}(\delta(k)\hat{f}_\delta(\lambda)).$$

5.4 The Lévy Khintchine Formula for $\mathbb{M} = G/K$

Let G be a Lie group which acts transitively on a manifold $\mathbb{M} = G/K$ on the left, and let g_t be a stochastic process in G. For any point $x \in \mathbb{M}$, we call the process $x_t = g_t x$ the **one-point motion** of g_t on $\mathbb{M}$ starting from x.

In general, the one-point motion of a Markov process in G is not a Markov process in M, except when g_t is a right Lévy process. We require g_t to be a right Lévy process(c.f. [41]), as g_t acts on the left of x. Hence for $f \in C_c(\mathbb{M})$,

$$\mathbb{E}(f(x_t)|\mathcal{F}_s) = \mathbb{E}(f(g_t x)|\mathcal{F}_s) = \mathbb{E}(f(g_t g_s^{-1} g_s x)|\mathcal{F}_s) = \mathbb{E}(f(g_t g_s^{-1} x_s)|\mathcal{F}_s) = \mathbb{E}(f(g_{t-s} x_s)).$$

Let $P_t^M f(x) = \mathbb{E}(f(g_t x))$ for $f \in C_0(\mathbb{M})$, then $P_t^M f(x)$ is a Feller semigroup[4] for x_t.

Recall the Lévy-Itô decomposition equation (3.6). This is a Lévy-Itô decomposition for left Lévy processes. By duality, if g_t is a right Lévy process, then for any $f \in C_b(G) \cap C^2(G)$ and with $a_{jk}, b_i, c_i, N, W_j, X_i, x_i$ having the same meaning as in Theorem 3.3.11 and (3.8). Then g_t solves the stochastic differential equation,

$$f(g_t) = f(g_0) + M_t^f + \int_0^t Af(g_s)ds, \tag{5.1}$$

[4] c.f. the discussion on page 54 of section 3.3.

with

$$\begin{aligned} Af(g) &= \frac{1}{2}\sum_{j,k=1}^{d} a_{jk}X_j^rX_k^rf(g) + \sum_{i=1}^{d} c_iX_i^rf(g) \\ &+ \int_{G-\{e\}} \left(f(hg) - f(g) - \sum_{i=1}^{d} x_i(h)X_i^rf(g) \right) d\nu(h) \end{aligned}$$

where ν is a measure on G satisfying (3.2). If in addition ν satisfies (3.3), the integral $\int_G[f(gh) - f(g)]d\nu(h)$ exists, A takes a simpler form

$$Af(g) = \frac{1}{2}\sum_{j,k=1}^{d} a_{jk}X_j^rX_k^rf(g) + \sum_{i=1}^{d} b_iX_i^rf(g) + \int_G [f(hg) - f(g)]d\nu(h), \tag{5.2}$$

where $b_i = c_i - \int_G x_i(h)d\nu(h)$, and

$$M_t^f = \sum_{j=1}^{d} \int_0^t X_j^rf(g_{s-})dW_s^j + \int_0^t \int_G [f(g_{s-}h) - f(g_{s-})]\tilde{N}(ds\ dh)$$

is an L^2-martingale.

Every $X \in \mathfrak{g}$ induces a vector field X^* on $\mathbb{M}$ by

$$X^*f(x) = \frac{d}{dt}f(e^{tX}x)|_{t=0}$$

for any $f \in C^1(\mathbb{M})$ and $x \in \mathbb{M}$. Let $\pi_x : G \to \mathbb{M}$ be the map $\pi_x(g) = gx$. If $f \in C_c^2(\mathbb{M})$, then $f \circ \pi_x \in C_b(G) \cap C^2(G)$ with $X^r(f \circ \pi_x) = (X^*f) \circ \pi_x$ for $X \in \mathfrak{g}$. Therefore, we obtain the following stochastic differential equation for the one-point motion $x_t = g_tx$ of g_t in $\mathbb{M}$. For $f \in C^2(\mathbb{M})$,

$$f(x_t) = f(x) + M_t^f + \int_0^t A_{\mathbb{M}}f(g_s)ds, \tag{5.3}$$

with

$$\begin{aligned} A_{\mathbb{M}}f(x) &= \frac{1}{2}\sum_{j,k=1}^{d} a_{jk}X_j^*X_k^*f(x) + \sum_{i=1}^{d} c_iX_i^*f(x) \\ &+ \int_{G-\{e\}} \left(f(hx) - f(x) - \sum_{i=1}^{d} x_i(h)X_i^*f(x) \right) d\nu(h). \end{aligned}$$

Here ν is a measure on G satisfying (3.2). If in addition ν satisfies (3.3), $A_{\mathbb{M}}$ takes the simpler form

$$A_{\mathbb{M}}f(x) = \frac{1}{2}\sum_{j,k=1}^{d} a_{jk}X_j^r X_k^r f(g) + \sum_{i=1}^{d} b_i X_i^r f(g) + \int_G [f(hx) - f(x)]d\nu(h), \quad (5.4)$$

where $b_i = c_i - \int_G x_i(h)d\nu(h)$, and

$$M_t^f = \sum_{j=1}^{d}\int_0^t X_j^* f(x_s)dW_s^j + \int_0^t\int_G [f(hx_{s-}) - f(x_{s-})]\tilde{N}(ds\, dh).$$

In section 5.3, we have established that the knowledge of $\mathbb{E}(\Phi^*_{\bar{\lambda},\delta}(\zeta_t))$ is sufficient to determine the law of ζ_t. Hence, the δ-spherical transform of the law of ζ_t amounts to a reasonable candidate to be called a "Lévy-Khintchine" formula for the K-Types. This result will then be summed over the irreducible representations to give the general Fourier transform of the Lévy process, and then applying the Fourier inversion to recover the probability distribution of the underlying process.

Let ζ_t be a Lévy process on $\mathbb{M}$ starting at x (regarded as the one-point motion), $\Phi_{\hat{\lambda},\delta}$ an Eisenstein integral (c.f. Definition 5.3.16), and $\phi_{\hat{\lambda}}$ an elementary spherical function (c.f. Definition 4.2.1). Now, $\Phi_{\hat{\lambda},\delta}$ is an eigenfunction of the Laplacian $\Delta_{\mathbb{M}}$, and a joint eigenfunction of the left-invariant differential operators $D \in \mathcal{D}(\mathbb{M})$ (c.f. [30] p 234, p244). To this extent, let $\mathcal{E}(\Delta_{\mathbb{M}}, \Phi_{\hat{\lambda},\delta})$ and $\mathcal{E}(D, \Phi_{\hat{\lambda},\delta})$ respectively denote the eigenvalue of $\Delta_M M$ and D corresponding to the eigenfunction $\Phi_{\hat{\lambda},\delta}$. With respect to the isomorphism $\Gamma : \mathcal{D}(\mathbb{M}) \to \mathcal{D}_W(A)$ introduced in Theorem 4.1.18, we have $\Gamma(D)(i\bar{\lambda}) = \mathcal{E}(D, \Phi_{\hat{\lambda},\delta})$ as $D\Phi_{\bar{\lambda},\delta} = \Gamma(D)(i\lambda)\Phi_{\bar{\lambda},\delta}$ (c.f. Discussion following Definition 5.3.16).

Theorem 5.4.1. *Let c be a fixed constant, $D = \sum_i b_i X_i$, $\kappa(\lambda) = c\mathcal{E}(\Delta_{\mathbb{M}}, \Phi_{\bar{\lambda},\delta}) + \mathcal{E}(D, \Phi_{\bar{\lambda},\delta})$, $\iota(\lambda, h^*) = \sum_{i=1}^{d} x_i^\natural(h^*K)\mathcal{E}(X_i^*, \Phi_{\bar{\lambda},\delta})$ where $x_i^\natural(h^*K) = \int_K x_i(h^*k)dk$. Then:*

- *The* δ*-**spherical transform**,* $\mathbb{E}(\Phi_{\bar{\lambda},\delta}(\zeta_t))$ *is determined by the following initial value problem:*

$$\frac{d}{dt}\beta(t)^* = \kappa(\lambda)\beta(t)^* + \int_{\mathbb{M}} \left(\Phi_{\bar{\lambda},\delta}(h^*K)\alpha(t) - (1 + \iota(\lambda, h^*))\beta(t)^* \right) d\nu(h^*)$$
$$\beta(0) = \Phi_{\bar{\lambda},\delta}(x)$$

  where $x = \zeta_0$ *and* $\alpha(t) = \mathbb{E}(\phi_{\bar{\lambda}}(\zeta_t))$*.*

- *If the Lévy measure* ν *is finite on* $\mathbb{M}$*, we can assume that* $\nu(\mathbb{M}) = 1$*, and we obtain the following closed form solution for the* δ*-spherical transform* $\mathbb{E}(\Phi_{\bar{\lambda},\delta}(\zeta_t)^*)$*:*

$$\mathbb{E}(\Phi_{\bar{\lambda},\delta}(\zeta_t)^*) = e^{(\tilde{\kappa}(\lambda)-1)t)}\Phi_{\bar{\lambda},\delta}(x) + \int_0^t e^{(\tilde{\kappa}(\lambda)-1)(t-u)}\varphi(u)du \tag{5.5}$$

  where $\tilde{\kappa}(\lambda) = \kappa(\lambda) - \int_{\mathbb{M}} \iota(\lambda, h^*)d\nu(h^*)$*,* $h^*K \in \mathbb{M}$ *is simultaneously understood as a coset of* G*, and*

$$\varphi(t) = \mathbb{E}(\phi_{\hat{\lambda}}(\zeta_t)) \int_{\mathbb{M}} \Phi_{\bar{\lambda},\delta}(h^*K)d\nu(h^*).$$

Proof. For each $\delta \in \hat{K}$, every matrix entry in $\Phi_{\bar{\lambda},\delta}$ is at least twice differentiable. This allows us to put $f = \Phi^*_{\bar{\lambda},\delta}$, and $\beta(t) = \mathbb{E}(\Phi_{\bar{\lambda},\delta}(\zeta_t)^*)$, and apply (5.3) on an entry-by-entry basis. Hence, we obtain

$$\begin{aligned}
\beta(t) &= \mathbb{E}(\Phi_{\bar{\lambda},\delta}(\zeta_t)^*) \\
&= \mathbb{E}\left(\Phi_{\bar{\lambda},\delta}(Z_0)^* + M_t^{\Phi^*_{\bar{\lambda},\delta}} + \int_0^t A_{\mathbb{M}}\Phi_{\bar{\lambda},\delta}(\zeta_s)^* ds \right) \\
&= \Phi_{\bar{\lambda},\delta}(o)^* + M_0^{\Phi^*_{\bar{\lambda},\delta}} + \mathbb{E}\Bigg(\int_0^t \Bigg[\frac{1}{2}\sum_{j,k=1}^d a_{jk}X_j^*X_k^*\Phi_{\bar{\lambda},\delta}(\zeta_s)^* + \sum_{i=1}^d b_iX_i^*\Phi_{\bar{\lambda},\delta}(\zeta_s)^* \\
&\quad + \int_G \left(\Phi_{\bar{\lambda},\delta}(h\zeta_s)^* - \Phi_{\bar{\lambda},\delta}(\zeta_s)^* - \sum_{i=1}^d x_i(h)X_i^*\Phi_{\bar{\lambda},\delta}(\zeta_s) \right) d\nu(h) \Bigg] ds \Bigg)
\end{aligned}$$

where the third line was obtained using the martingale property of $M_t^{\Phi^*_{\bar{\lambda},\delta}}$. Now, let $\Delta_{\mathbb{M}}$ be the Laplacian on $\mathbb{M}$ and D be the differential operator $\sum_i b_iX_i$. We differentiate both sides with respect to t and taking the transpose of both sides,

$$
\begin{aligned}
\frac{d}{dt}\beta(t)^* &= \mathbb{E}\bigg(\frac{1}{2}\sum_{j,k=1}^{d} a_{jk}X_j^*X_k^*\Phi_{\bar{\lambda},\delta}(\zeta_t) + \sum_{i=1}^{d} b_i X_i^*\Phi_{\bar{\lambda},\delta}(\zeta_t) \\
&\quad + \int_G \Big[\Phi_{\bar{\lambda},\delta}(h\zeta_t) - \Phi_{\bar{\lambda},\delta}(\zeta_t) - \sum_{i=1}^{d} x_i(h)X_i^*\Phi_{\bar{\lambda},\delta}(\zeta_t)\Big] d\nu(h)\bigg) \\
&= \mathbb{E}\bigg(c\Delta_{\mathbb{M}}\Phi_{\bar{\lambda},\delta}(\zeta_t) + D\Phi_{\bar{\lambda},\delta}(\zeta_t) \\
&\quad + \int_G \Big[\Phi_{\bar{\lambda},\delta}(h\zeta_t) - \Phi_{\bar{\lambda},\delta}(\zeta_t) - \sum_{i=1}^{d} x_i(h)X_i^*\Phi_{\bar{\lambda},\delta}(\zeta_t)\Big] d\nu(h)\bigg) \\
&= \mathcal{E}(c\Delta_{\mathbb{M}}, \Phi_{\bar{\lambda},\delta})\beta(t)^* + \mathcal{E}(D, \Phi_{\bar{\lambda},\delta})\beta(t)^* \\
&\quad + \mathbb{E}\bigg(\int_G \Big[\Phi_{\bar{\lambda},\delta}(h\zeta_t) - \Phi_{\bar{\lambda},\delta}(\zeta_t) - \sum_{i=1}^{d} x_i(h)X_i^*\Phi_{\bar{\lambda},\delta}(\zeta_t)\Big] d\nu(h)\bigg) \\
&= \kappa(\lambda)\beta(t)^* + \mathbb{E}\bigg(\int_{\mathbb{M}} \Big[\int_K \Phi_{\bar{\lambda},\delta}(h^*k\zeta_t)dk - \int_K \Phi_{\bar{\lambda},\delta}(\zeta_t)dk \\
&\quad - \int_K \sum_{i=1}^{d} x_i(h^*k)X_i^*\Phi_{\bar{\lambda},\delta}(\zeta_t)dk\Big] d\nu(h^*)\bigg) \\
&= \kappa(\lambda)\beta(t)^* + \mathbb{E}\bigg(\int_{\mathbb{M}} \Big[\Phi_{\bar{\lambda},\delta}(h^*K)\phi_{\bar{\lambda}}(\zeta_t) - \Phi_{\bar{\lambda},\delta}(\zeta_t) \\
&\quad - \sum_{i=1}^{d} x_i^{\natural}(h^*K)X_i^*\Phi_{\bar{\lambda},\delta}(\zeta_t)\Big] d\nu(h^*)\bigg)
\end{aligned}
$$

where h^* in the third line is the unique element in $\mathbb{M}$ such that $h = h^*k$, and dk is the normalised Haar measure on K and $x_i^{\natural}(h^*K) = \int_K x_i(h^*k)dk$. The last equality in the preceding computation follows from the fact that $\Phi_{\bar{\lambda},\delta}(.)^*$ is a joint eigenfunction of left-invariant differential operators and Proposition 4.2.4.

Since $d\nu(h^*)$ is a Lévy measure on $\mathbb{M}$, we have

$$\int_{\mathbb{M}} \bigg|\bigg|\Phi_{\bar{\lambda},\delta}(h^*K)\phi_{\bar{\lambda}}(\zeta_t) - \Phi_{\bar{\lambda},\delta}(\zeta_t) - \sum_{i=1}^{d} x_i^{\natural}(h^*)X_i^*\Phi_{\bar{\lambda},\delta}(\zeta_t)\bigg|\bigg| d\nu(h^*) < \infty$$

and hence we may exchange the order of expectation and integral to get

$$\frac{d}{dt}\beta(t)^* = \kappa(\lambda)\beta(t)^* + \int_{\mathbb{M}} \mathbb{E}\Big[\Phi_{\bar{\lambda},\delta}(h^*K)\phi_{\bar{\lambda}}(\zeta_t) - \Phi_{\bar{\lambda},\delta}(\zeta_t) - \sum_{i=1}^{d} x_i^{\natural}(h^*k)X_i^*\Phi_{\bar{\lambda},\delta}(\zeta_t)\Big] d\nu(h^*)$$

Now, since $\Phi_{\bar{\lambda},\delta}$ is a joint eigenfunction of left-invariant differential operators, $X_i^*\Phi_{\bar{\lambda},\delta} = \mathcal{E}(X_i^*, \Phi_{\bar{\lambda},\delta})\Phi_{\bar{\lambda},\delta}$. Therefore, the above equation simplifies to

$$\begin{aligned}\frac{d}{dt}\beta(t)^* &= \kappa(\lambda)\beta(t)^* + \int_{\mathbb{M}} \left(\Phi_{\bar{\lambda},\delta}(h^*K)\alpha(t) - \beta(t)^* - \sum_{i=1}^{d} x_i^{\natural}(h^*k)\mathcal{E}(X_i^*, \Phi_{\bar{\lambda},\delta})\beta(t)^*\right) d\nu(h^*) \\ &= \kappa(\lambda)\beta(t)^* + \int_{\mathbb{M}} \left(\Phi_{\bar{\lambda},\delta}(h^*K)\alpha(t) - (1+\iota(\lambda, h^*))\beta(t)^*\right) d\nu(h^*)\end{aligned}$$

We have now established the first part of the theorem. Now, if ν is a finite measure on $\mathbb{M}$, we can assume that $\int_{\mathbb{M}} d\nu(h^*) = 1$. Under this assumption we can split up the integral against the Lévy measure to solve the initial value problem as a first order linear ODE. Moreover, the finiteness of $\nu(\mathbb{M})$ allows us to set $\tilde{\kappa}(\lambda) = \kappa(\lambda) - \int_{\mathbb{M}} \iota(\lambda, h^*)d\nu(h^*)$, which simplifies the above ODE as

$$\begin{aligned}\frac{d}{dt}\beta(t)^* &= \tilde{\kappa}(\lambda)\beta(t)^* + \left(\int_{\mathbb{M}} (\Phi_{\bar{\lambda},\delta}(h^*K)\alpha(t) - \beta(t)^*)d\nu(h^*)\right) \\ &= (\tilde{\kappa}(\lambda) - 1)\beta(t)^* + \varphi(t)\end{aligned}$$

where

$$\varphi(t) = \alpha(t)\int_{\mathbb{M}} \Phi_{\bar{\lambda},\delta}(h^*K)d\nu(h^*).$$

We can obtain an explicit expression for $\alpha(t)$, and hence for $\varphi(t)$ as follows. Put $f = \phi_\lambda$ into (5.3), and analogous to the previous calculation for $\beta(t)$, we proceed as follows,

$$\begin{aligned}\frac{d}{dt}\alpha(t) &= \tilde{\kappa}(\lambda)\alpha(t) + \mathbb{E}\left(\int_{\mathbb{M}} \left[\int_K [\phi_{\bar{\lambda}}(h^*k\zeta_t) - \phi_{\bar{\lambda}}(\zeta_t)]dk\right] d\nu(h^*)\right) \\ &= \tilde{\kappa}(\lambda)\alpha(t) + \mathbb{E}\left(\int_{\mathbb{M}} [\phi_{\bar{\lambda}}(h^*)\phi_{\bar{\lambda}}(\zeta_t) - \phi_{\bar{\lambda}}(\zeta_t)]d\nu(h^*)\right) \\ &= \left(\tilde{\kappa}(\lambda) + \int_{\mathbb{M}} [\phi_{\bar{\lambda}}(h^*) - 1]d\nu(h)\right)\alpha(t)\end{aligned}$$

with initial condition $\alpha(0) = 1$. Integrating both sides gives,

$$\alpha(t) = \exp\left[t\left(\tilde{\kappa}(\lambda) + \int_{\mathbb{M}} [\phi_{\bar{\lambda}}(h^*) - 1]d\nu(h^*)\right)\right],$$

and hence

$$\varphi(t) = \exp\left[t\left(\tilde{\kappa}(\lambda) + \int_{\mathbb{M}} [\phi_{\bar{\lambda}}(h^*) - 1] d\nu(h^*)\right)\right] \int_{\mathbb{M}} \Phi_{\bar{\lambda},\delta}(h^* K) d\nu(h^*).$$

Since we have now obtained an explicit formula for $\varphi(t)$, we may regard it as a term independent of β, and therefore we have reduced the original problem to the following first order linear ODE,

$$\begin{aligned} &\frac{d}{dt}\beta(t)^* - (\tilde{\kappa}(\lambda) - 1)\beta(t)^* = \varphi(t) \\ &\beta(0)^* = \Phi_{\bar{\lambda},\delta}(x), \end{aligned} \tag{5.6}$$

assuming that $\zeta_0 = x$. We solve this by multiplying the integrating factor $e^{-(\tilde{\kappa}(\lambda)-1)t}$ on both sides of (5.6) to get

$$e^{-(\tilde{\kappa}(\lambda)-1)t}\left(\frac{d}{dt}\beta(t)^* - (\tilde{\kappa}(\lambda) - 1)\beta(t)^*\right) = e^{-(\tilde{\kappa}(\lambda)-1)t}\varphi(t)$$

and we recognize the left hand side

$$e^{-(\tilde{\kappa}(\lambda)-1)t}\left(\frac{d}{dt}\beta(t)^* - (\tilde{\kappa}(\lambda) - 1)\beta(t)^*\right) = \frac{d}{dt}(e^{-(\tilde{\kappa}(\lambda)-1)t}\beta(t)^*).$$

Hence, combining the above with the initial condition $\beta(0)^* = \Phi_{\bar{\lambda},\delta}(x)$, we have

$$e^{-(\tilde{\kappa}(\lambda)-1)t}\beta(t)^* = \Phi_{\bar{\lambda},\delta}(x) + \int_0^t e^{-(\tilde{\kappa}(\lambda)-1)u}\varphi(u)du.$$

Therefore,

$$\beta(t)^* = e^{(\tilde{\kappa}(\lambda)-1)t)}\Phi_{\bar{\lambda},\delta}(x) + \int_0^t e^{(\tilde{\kappa}(\lambda)-1)(t-u)}\varphi(u)du$$

We have now completed the proof of Theorem 5.4.1. □

As a consequence of Theorem 5.4.1, we are able to develop the following theorem that gives the Fourier transform and the underlying probability distribution of a general Lévy process on $\mathbb{M}$.

Theorem 5.4.2. *Inheriting all the notations from Theorem 5.4.1 and section 5.3, and assuming $\nu(\mathbb{M}) = 1$, the Fourier Transform*[5] *of the $\mathbb{M}$-valued Lévy process is given by*

$$\tilde{f}_{\zeta_t}(\lambda, b) = \sum_{\delta \in \hat{K}_M} \mathrm{Tr}\left[\delta(k)\left(e^{(\tilde{\kappa}(\lambda)-1)t)}\Phi_{\bar{\lambda},\delta}(x) + \int_0^t e^{(\tilde{\kappa}(\lambda)-1)(t-u)}\varphi(u)du\right)\right],$$

where $b = kM \in K/M$. The density function is given by

$$f_{\zeta_t}(x) = \frac{1}{|W|}\int_{\mathfrak{a}^*}\int_B e^{(i\lambda+\varrho)A(x,b)}|\mathbf{c}(\lambda)|^{-2} \sum_{\delta \in \hat{K}_M} \mathrm{Tr}\left[\delta(k)\left(e^{(\tilde{\kappa}(\lambda)-1)t)}\Phi_{\bar{\lambda},\delta}(x)\right.\right.$$
$$\left.\left. + \int_0^t e^{(\tilde{\kappa}(\lambda)-1)(t-u)}\varphi(u)du\right)\right] d\lambda db.$$

Here, W is the Weyl group, and $\mathbf{c}$ is the Harish-Chandra's $\mathbf{c}$-function.

Proof. The first part of the theorem uses the fact that $\tilde{f}_{\zeta_t}(\lambda, b) = \sum_{\delta\in\hat{K}_M} \mathrm{Tr}(\delta(k)\hat{f}_{\zeta_t}(\lambda))$ (c.f. Theorem 5.3.20), and $\hat{f}_{\zeta_t}(\lambda) = \mathbb{E}(\Phi_{\bar{\lambda},\delta}(\zeta_t))$. Then, application of (5.5) to $\mathbb{E}(\Phi_{\bar{\lambda},\delta}(\zeta_t))$ yield the desired result. Direct application of the Fourier Inversion (Theorem 5.3.10) gives the second part of the theorem. □

Remark 5.4.3. For the same reason in Remark 4.4.5, Theorem 5.4.2 does not necessarily require ζ_t to have a smooth density. The only difference is that the version of Fourier inversion and Paley-Wiener theorem we are referring to are Theorem 5.3.9 and Theorem 1.5 of [30], p.227 respectively. When f is non-smooth, $\mu_{\zeta_t}(x)$ in Theorem 5.4.2 will become a distribution on $\mathbb{M}$.

5.5 Application of Theorems 5.4.1 and 5.4.2

In the previous section, we have obtained an explicit form of the Lévy Khintchine formula for a general Lévy process on a Riemannian symmetric space $\mathbb{M} = G/K$ via Theorem 5.4.1. To recover underlying probability distribution from (5.5), we had to first reconstruct the full Fourier transform, via the relation $\tilde{f}(\lambda, b) = \sum_{\delta\in\hat{K}_M} \mathrm{Tr}(\delta(k)\mathbb{E}(\Phi_{\bar{\lambda},\delta}(\zeta_t))$, where $b = kM$ (c.f. Theorem 5.3.20 and Theorem 5.4.2), and then apply the Fourier inversion formula (Theorem 5.3.10, Theorem 5.4.2 and

[5]We are really talking about the Fourier Transform of the probability distribution

Theorem 5 of [8]). On the other hand, one can apply Theorem 5.3.19 to first invert the δ-spherical transform to obtain f^δ, and then recover f by applying the Paley-Wiener theorem, where $f = \sum_{\delta \in \hat{K}_M} \mathrm{Tr}(f^\delta)$. Moreover, the harmonic analysis theory tells us that both of the methods will yield the same result (c.f. Chapter 4, [30]).

We can also decompose any tempered distribution f on $\mathbb{M}$ into its f^δ parts via the prescription

$$f^\delta(x) = d(\delta) \int_K f(k.x)\delta(k^{-1})dk$$

where δ is any irreducible representation of K on a vector space V_δ of dimension $d(\delta)$. While each f^δ is a matrix with dimension $d(\delta)$, $\mathrm{Tr}(f^\delta)$ can be thought of as a function on $\mathbb{M}$. Now, f^δ respects the action of δ in the sense that $f^\delta(k.x) = \delta(k)f^\delta(x)$ for every $k \in K$ and $x \in \mathbb{M}$. One can recover f from f^δ by the well known Paley-Wiener decomposition formula

$$f = \sum_{\delta \in \hat{K}_M} \mathrm{Tr}(f^\delta). \tag{5.7}$$

This allows us to complete extend the computation of the example in section 4.6 and section 5.2 to a general Lévy process on H^2.

By theorem 5.4.1, we know that for each fixed δ, the δ-spherical transform of the density $\beta(t) = \mathbb{E}(\Phi_{\lambda,\delta}(\zeta_t))$ satisfies the initial value problem:

$$\frac{d}{dt}\beta(t) = \kappa(\lambda)\beta(t) + \int \ldots \, d\nu$$
$$\beta(0) = \Phi_{\lambda,\delta}(x)$$

where $x \in G/K$ is the starting value of the Levy process. In the case of a Brownian motion with drift, the solution is just $\beta(t) = \mathbb{E}(\Phi_{\lambda,\delta}(\zeta_t)) = e^{\kappa(\lambda)t}\Phi_{\lambda,\delta}(x)$.

Now, $\kappa(\lambda) = c\mathcal{E}(\Delta_{\mathbb{M}}, \Phi_{\lambda,\delta}) + \mathcal{E}(D, \Phi_{\lambda,\delta})$, where $D = \sum_i b_i X_i$ and the X_i's are a set of local coordinates.

To compute the first eigenvalue $\mathcal{E}(\Delta_{\mathbb{M}}, \Phi_{\lambda,\delta})$, we use the following facts from Helgason (GGA, p 49):

On the unit disc model of $H^2 = SU(1,1)/SO(2)$,

- The Eisenstein integrals satisfies the "homogeneity condition": $f(e^{i\theta}z) = e^{im\theta}f(z)$, where m is the index of the character $\chi_m(e^{i\theta}) = e^{im\phi}$ (i.e. it is a substitute for δ).
- The function $F(r) = f(\tanh r)$ satisfies

$$F"(r) + 2\coth(2r)F'(r) - 4m^2\sinh^{-2}(2r)F + (\lambda^2+1)F = 0.$$

The Laplace-Beltrami operator on H^2 is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + 2\coth(2r)\frac{\partial}{\partial r} + 4\sinh^{-2}(2r)\frac{\partial^2}{\partial \theta^2},$$

Hence,

$$\begin{aligned}
\Delta(\Phi_{\lambda,m}(\tanh re^{i\theta})) &= \Delta(e^{im\theta}\Phi_{\lambda,m}(\tanh r)) \\
&= \left(\frac{\partial^2}{\partial r^2} + 2\coth(2r)\frac{\partial}{\partial r} + 4\sinh^{-2}(2r)\frac{\partial^2}{\partial\theta^2}\right)(e^{im\theta}\Phi_{\lambda,m}(\tanh r)) \\
&= e^{im\theta}\left(\frac{\partial^2}{\partial r^2} + 2\coth(2r)\frac{\partial}{\partial r}\right)(\Phi_{\lambda,m}(\tanh r)) \\
&\quad + 4\sinh^{-2}(2r)\Phi_{\lambda,m}(\tanh r)\frac{\partial^2}{\partial\theta^2}e^{im\theta} \\
&= (4m^2\sinh^{-2}(2r) - (\lambda^2+1))e^{im\theta}\Phi_{\lambda,m}(\tanh r) \\
&\quad - 4m^2e^{im\theta}\sinh^{-2}(2r)\Phi_{\lambda,m}(\tanh r) \\
&= -(\lambda^2+1)\Phi_{\lambda,m}(\tanh re^{i\theta})
\end{aligned}$$

So the eigenvalue $\mathcal{E}(\Delta, \Phi_{\lambda,m}) = -(\lambda^2+1)$.

Now, to compute $\mathcal{E}(D, \Phi_{\lambda,\delta})$. In general, $D = \sum_i b_i X_i$ where X_i are just the local coordinates. In this case, it basically boils down to $X_1 = d/dr$ and $X_2 = d/d\theta$. With $\mathcal{E}(d/d\theta, \Phi_{\lambda,m})$ it's actually quite simple, because

$$\begin{aligned}
\frac{d}{d\theta}\Phi_{\lambda,m}(\tanh re^{i\theta}) &= \Phi_{\lambda,m}(\tanh r)\frac{d}{d\theta}e^{im\theta} \\
&= ime^{im\theta}\Phi_{\lambda,m}(\tanh r) \\
&= im\Phi_{\lambda,m}(\tanh re^{i\theta})
\end{aligned}$$

So the eigenvalue here is just im.

The computation of $\mathcal{E}(d/dr, \Phi_{\lambda,m})$ is rather involved, but the basic idea is to differentiate directly using the formula given by [29], p.60:

$$\Phi_{\lambda,m}(r) = (1-r^2)^v r^{|m|} \frac{\Gamma(|m|+v)}{\Gamma(v)|m|!} F(v, |m|+v; |m|+1; r^2)$$

where $v = \frac{1}{2}(i\lambda + 1)$ and $F(,;;)$ here is the hypergeometric function (c.f. [29] p 50).

To summarise, the delta-spherical transform of the density function is given by the expression:

$$\begin{aligned}\mathbb{E}(\Phi_{\lambda,m}(\zeta_t)) &= e^{\kappa(\lambda)t}\Phi_{\lambda,\delta}(x) \\ &= e^{-(\lambda^2+1)+im+\mathcal{E}(d/dr,\Phi_{\lambda,m})}\Phi_{\lambda,m}(x)\end{aligned}$$

Hence, the density function of ζ_t can be obtained by applying Fourier inversion to

$$\tilde{f} = \sum_{m\in\mathbb{Z}} \mathbb{E}(\Phi_{\lambda,m}(\zeta_t)) = e^{-(\lambda^2+1+im+\mathcal{E}(d/dr,\Phi_{\lambda,m}))t}\Phi_{\lambda,m}(x).$$

Chapter 6

Volatility Modeling

In this chapter, we aim to apply the ideas developed in the previous chapters to problems in mathematical finance. It is written more for a purpose of illustrating the potential applications of the main theorems discovered in this thesis, as opposed to be a complete piece work on its own right.

Although financial application of stochastic differential geometry is a relatively new field, there is already a substantial literature in the area (c.f. [14], [22], [23]). Most of these ideas are based on the following two-step observation:

- Traditional models have observed empirical defects.
- Tweak the traditional models with some curvature helps to (partially) resolve the defects.

However to date, there still lacks a unified theory to explain exactly what exactly is the role of curvature playing in the realm of finance. This chapter is aimed as a forum for brief discussion, the topic we are attempting to cover is so large that perhaps it deserves a full thesis on its own.

6.1 Introduction to Volatility Modeling

The holder of an option to an asset S, has the right but not the obligation to buy/sell the stock at a pre-determined price K, called the strike price, and at a pre-determined time T, called the maturity day. If the option gives you the right to buy, it is called a call option, and if it gives you the right to sell, it is called a put option. In their seminal paper [15], Black and Scholes showed that by holding a certain portfolio that consists of the underlying asset and cash, one can completely replicate the risk profiles of the option under certain idealistic assumptions. In particular, they assumed that the asset price dynamics are governed by the SDE,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

where W_t is a Wiener process.

A fundamental property that is required with any mathematical model, is that all underlying assets in the resulting market dynamics must be arbitrage free, so the price of the option must be the price of constructing the replicating portfolio.

Definition 6.1.1. Let $\{\phi_t\}$ be a stochastic process adapted to $\mathcal{F}_t$. We say $\{\phi_t\}$ is an **admissible strategy** over the period $[t, T]$ if

$$\int_t^T |\phi_s| ds < \infty.$$

Now, suppose that the process $\{S_t\}_{t\geq 0}$ is the price of a traded asset, and $\{\phi_t\}$ an admissible trading strategy with respect to the same filtration as S_t. Let $0 = t_0 < t_1 < ... < t_n = T$ be a partition of $[0, t]$ and let $I_i = (t_{i-1}, t_i]$, $i = 1, 2, ..., n$. Assume that ϕ is constant on each I_i, then the amount of capital gain over I_i is given by $\phi_{t_{i-1}}(S_{t_i} - S_{t_{i-1}})$. Let $V(\phi)$ be the value of the investment summing over all periods, and let our partition become infinitely fine as $n \to \infty$, we obtain the Riemann sum,

$$V(\phi) = \sum_{i=1}^{n} \phi_{t_{i-1}}(S_{t_i} - S_{t_{i-1}}) \to \int_0^T \phi_t dS_t.$$

Definition 6.1.2. An **arbitrage strategy** ϕ is an admissible strategy such that,

- $V_0 = 0$,
- $\mathbb{P}(V(\phi) < 0) = 0$ and
- $\mathbb{P}(V(\phi) > 0) > 0$.

In practice, the existence of arbitrage strategies depend on opportunities that are transient, they can only arise in market inefficiencies. It is undesirable to allow such an opportunity to occur systematically when we are building a mathematical model. Hence, we need to impose an extra condition called the "arbitrage free" condition to prohibit arbitrage opportunities arise systematically in the model dynamics. The only tradable asset here are the underlying asset S_t and the its options C_t.

Theorem 6.1.3. *(Fundamental Theorem of Asset Pricing)*

Let $\{S_t\}_{t\geq 0}$ be the price dynamics of a tradable asset, modeled as a stochastic process adapted to the filtration $\mathcal{F}_t$. Then, the induced market is free of arbitrage opportunities if and only if the discounted price with respect to numeraire B_t[1]*, S_t/B_t, is a local martingale under a measure $\mathbb{Q}$ that is equivalent with respect to the real world measure $\mathbb{P}$.*

As a corollary to the above theorem, we can write down the option price as an expectation under the equivalent martingale measure $\mathbb{Q}$.

Corollary 6.1.4. *The price of any tradable V maturing at time T, valued at time t, is given by $\frac{B_T}{B_t}\mathbb{E}_{\mathbb{Q}}(V|\mathcal{F}_t)$.*

In the case when V is an call option on the underlying S_t with strike K, and B is a risk free cash account with interest rate r, then $V = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}((S_T - K)^+|\mathcal{F}_t)$ with the dynamics of S_t under $\mathbb{Q}$ being

$$\frac{dS_t}{S_t} = (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t.$$

Computing this expectation gives us the celebrated Black-Scholes formula.

Theorem 6.1.5.

For an asset with price dynamics following a geometric brownian motion with (constant) volatility σ and initial value S_0, the call option with strike K and maturity T under the risk free rate r, at time t has price,

$$BS(S_0, r, \sigma, K, T-t) := S_0\Phi(d_1) - e^{r(T-t)}\Phi(d_2) \tag{6.1}$$

where $\Phi(.) = \int_{-\infty}^{.} \frac{1}{\sqrt{2\pi}}e^{-1/2x^2}dx$, and

$$d_1 = \frac{\log(S_0/K) + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \text{ and } d_2 = d_1 - \sigma\sqrt{t}.$$

[1] Here, we may regard B_t as the our cash account, so $B_t = e^{-rt}$.

In practice however, the above formula rarely gives the actively traded option prices if we substitute σ with historical volatility. On the other hand, given the traded option prices, we can reverse engineer the "correct" σ so that the option price that the Black Scholes formula gives actually agrees with the market prices. The following definition formalises the above discussion.

Definition 6.1.6. Let p be the observed derivative price and $BS(S_0, K, \mu, \sigma, T-t)$ be the formula (6.1). The **implied volatility** of this derivative[2] is given by the solving for σ in the equation

$$p = B(S_0, r, \sigma, K, T - t).$$

It is commonly observed in the market, that for the same underlying S, the implied volatility for is the lowest when the option is at the money, and it becomes higher when its deeper-in or deeper-out of the money. Thus, plotting the implied volatility against the strike level will typically produces a figure that looks like a "smile", and hence this curve is given the name the **implied volatility smile**.

There is already a huge amount of work published in the literature as to model volatility, and recover the volatility smile in their models (c.f. . These include local volatility models, stochastic volatility models, and asset dynamics driven by non-log-normal processes. However, there are much fewer publications that addresses the underlying reason why do we observe the volatility smiles, or search for an unification to the existing models and explanations of this phenomenon. In fact, one of the authors of the Black-Scholes paradigm had made it very clear: Implied volatility is the wrong volatility substituted into the wrong formula to obtain the right option price asked by the financial markets.

In this section, rather than trying to answer directly "why do we observe a smile", we are going to turn the question back to itself: *There is really no smile, the smile we observe is just an illusion.* Then, we will see if we can substitute the right volatility into the right formula to get the right price!

[2]Here, we distinguish derivatives by their payoff, underlying, strike and maturity.

6.2 Information Geometry

The philosophy behind stochastic modeling lies deeply in the idea that *randomness arises in the absence of information.* For example, in any filtered probability space $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{P})$, a common interpretation is that $(\mathcal{F}_t)$ controls the flow of information. When one lives long enough to observe $(\mathcal{F}_\infty)$, that person would enter the mind of God and every event would be measurable with respect to his/her σ-algebra. For any $t < \infty$, the filtration $(\mathcal{F}_t)$ hides all the information that would flow in beyond time t, and hence creating randomness for snap shots of any stochastic process beyond time t. We will now talk about how to quantitatively assess the amount of information contained in an observation. Readers are referred to [51] for a comprehensive treatment of information geometry in statistics, and [14] for its role in interest rate models.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(E, \mathcal{B}(E))$ be a measurable space and $X : \Omega \to E$ be a random variable (measurable map). Now, let $A \in \mathcal{F}$ and denote "the amount of information contained in A about X" as $I_X(A)$, where I can be regarded as a map $I : Map(\Omega, E) \times \mathcal{F} \to [0, \infty]$. Note that if $A \subseteq B$, then we should expect $I_X(A) > I_X(B)$, as A gives more "finer details" about X than B. We assume the following properties with I that can be regarded as axioms,

1. $I_X(\emptyset) = \infty$ and $I_X(\Omega) = 0$
2. If $A \subseteq B$, then $I_X(A) > I_X(B)$.
3. Let $A, B \in \mathcal{F}$ be independent with respect to $\mathbb{P}$, then $I_X(A \cap B) = I_X(A) + I_X(B)$.

Solving the Cauchy-functional equation induced by property 3, we have that the only possible candidate for $I_X(A)$ is $I_X(A) = -\log(\mathbb{P}(X^{-1}(A)))$. From here on, we will treat this as the defining property of I. In practice, one often consider a family of random variables X parametrized by $\theta = \{\theta_i\}_{i=1,2,..,n}$ with each $\theta_i \in \Theta$, $i = 1, 2, .., n$, where Θ is a parameter set, with an assumed conditional density $f(x, \theta) = f(x, \theta_1, ..., \theta_n)$. The central idea behind Fisher information is to measure how much information is revealed by an additional increment of the parameter θ, on the underlying random variable X.

Definition 6.2.1. The **Fisher information** of a family of random variables $X_{\theta_1,...,\theta_n}$ parametrized by $\theta_i \in \Theta$, $i = 1, 2, ..., n$ is given by

$$I_X(\theta_1, ..., \theta_n) = \mathbb{E}(H(\log f(x|\theta_1, ..., \theta_n))),$$

where H is the Hessian in the directions $\theta_1, ..., \theta_n$.

Now following [14], we find that the Fisher information can be considered as a metric to the Riemannian manifold $\mathbb{M}_X = \{X_\theta : \theta \in \Theta\}$, the space of all random variables X parameterized by θ, with conditional density $f(x, \theta)$. We will simply write $\mathbb{M}_X$ as $\mathbb{M}$ when there is no risk of confusion.

Definition 6.2.2. The Riemannian metric can be written as

$$g_{ij}(\theta) = \mathbb{E}(\partial_i \log f(x,\theta))\partial_j \log f(x,\theta)|\theta) = \int_{-\infty}^{\infty} f(x,\theta)\partial_i(\log f(x,\theta))\partial_j(\log f(x,\theta))dx \tag{6.2}$$

so that the infinitesimal line segment on $\mathbb{M}$ is given by

$$ds^2 = \sum_{i,j} g_{ij} d\theta_i d\theta_j$$

where the vector fields $\partial_i = \frac{\partial}{\partial\theta_i}$, $\partial_j = \frac{\partial}{\partial\theta_j}$ and $\partial_{ij} = \frac{\partial^2}{\partial\theta_i\partial\theta_j}$.

It can be shown by tensor transformation laws that this metric is invariant under re-parameterization. Furthermore, the Riemannian metric defined here gives us a sense of "distance", "volume", etc. In particular, we are interested in the geodesics that are induced with respect to a given metric g_{ij}. These are the curves $C(u) = \{C^i(u)\}$, $i = 1, 2, ..., \dim(\mathbb{M})$. In local coordinates, they are the solutions of the differential equation

$$\frac{dC^i}{du^2} + \sum_{j,k} \Gamma^i_{jk} \frac{dC^j}{du}\frac{dC^k}{du},$$

where Γ^i_{jk} are the Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2}\sum_l g^{il}(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}),$$

and where g^{il} is the inverse metric of g_{il} in the sense of inverse matrices: $g^{il}g_{lm} = \delta^i_m$. The geodesics of $\mathbb{M}$ measures the minimum amount of Fisher information that is required to get from one point to another. A Brownian motion on $\mathbb{M}$ will best describe the diffusion process with respect to a uniform stream of information entropy.

In option pricing theory, one normally models the dynamics of the underlying asset $\{S_t\}_{t\geq 0}$ and its volatility as stochastic differential equations driven by Wiener processes. Thus, it is of central interest to us to derive the information manifold of a Gaussian distribution with parameters μ and σ, where

$$f(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right].$$

Now, let us compute the induced Riemannian metric:

$$\log f(x, \mu, \sigma) = -\frac{1}{2}\log(2\pi) - \log\sigma - \frac{1}{2}\left(\frac{x^2 - 2x\mu + \mu^2}{\sigma^2}\right),$$

so

$$\begin{aligned}
\frac{\partial}{\partial\mu}\log f(x, \mu, \sigma) &= -\frac{1}{\sigma^2}(x-\mu) \\
\frac{\partial}{\partial\sigma}\log f(x, \mu, \sigma) &= -\frac{1}{\sigma} + \frac{1}{\sigma^3}(x-\mu)^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
g_{\mu\mu} &= \mathbb{E}\left(-\frac{1}{\sigma^4}(X-\mu)^2 \middle| \mu, \theta\right) = \frac{1}{\sigma^2} \\
g_{\sigma\sigma} &= \mathbb{E}\left(\left(-\frac{1}{\sigma} + \frac{1}{\sigma^3}(X-\mu)^2\right)^2 \middle| \mu, \theta\right) = \frac{2}{\sigma^2} \\
g_{\mu\sigma} &= \mathbb{E}\left(\left(-\frac{1}{\sigma^2}(X-\mu)\right)\left(-\frac{1}{\sigma} + \frac{1}{\sigma^3}(X-\mu)^2\right) \middle| \mu, \theta\right) = 0.
\end{aligned}$$

Therefore, the resulting Riemannian metric is given by

$$ds^2 = \frac{1}{\sigma^2}(d\mu^2 + 2d\sigma^2)$$

defined on the upper half plane $\{(\mu, \sigma) : \sigma > 0\}$, which we recognize as the upper half plane model of the hyperbolic plane, H^2. The upper half plane model and the Poincaré model of the hyperbolic plane are related by the Cayley transform, where

Hence, everything we said about H^2 via the Poincaré model in sections 3.2, 4.6 and 5.1 will hold on the upper half plane model via the conformal mapping property of the Cayley transform. In particular, this means the geodesics of H^2 on the upper half plane will consists of

- Semicircles with both ends on the horizontal axis, and
- Straight lines perpendicular to the horizontal axis,

where the horizontal axis is parameterized by μ and verticle axis parameterized by σ. In particular, the horizontal lines are **not** geodesics. Any horizontal line that is tangential to a geodesic semi-circle will always be tangential at the mean, and lie above every other point in that semi-circle.

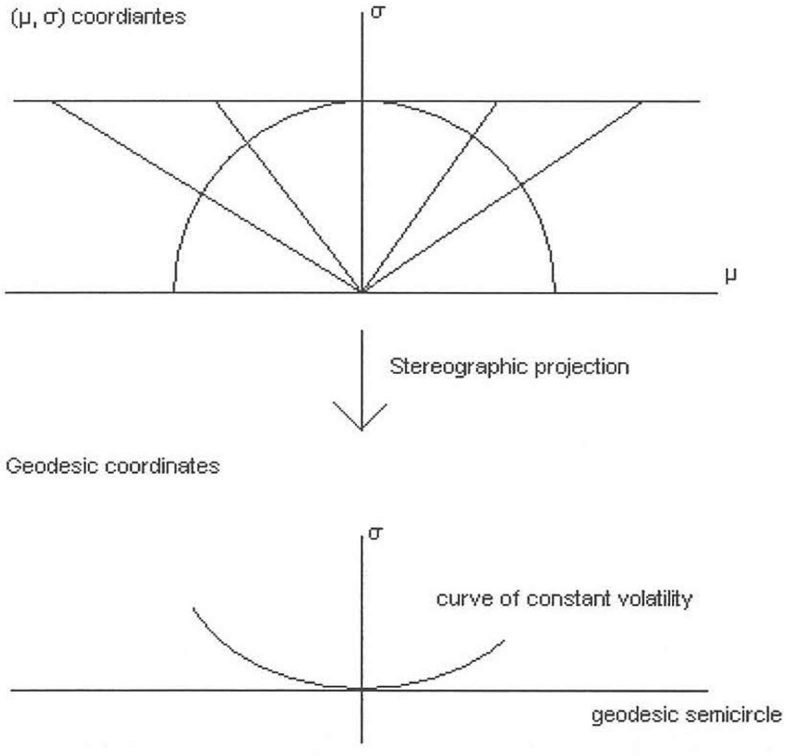

Fig. 1. This figure shows how curves of constant volatility resembles the shape of a volatility smile when plotted on the geodesic polar coordinates.

At this stage we only make a heuristic argument that if the market prices risk with units of "information", then the correct choice of coordinates should be the geodesic normal coordinates generated by the Fisher Information metric. This explains precisely why we observe the phenomenon that in a Black-Scholes world, the volatility of an option whose strike is further away from the spot is high.

6.3 Derivative Pricing on H^2

The discussion in the previous section explained the volatility smile from an information theoretic point of view. To achieve compatibility between our asset dynamics and information flow, we should model any log-normal asset dynamics on the hyperbolic plane H^2, where the dynamics of the underlying (or log-underlying) and its volatility are jointly considered to be one Brownian motion on H^2. More specifically, we will model the forward F_t as a log-normal process, we will consider $\xi_t = (W_t, \sigma_t)$ as a H^2-valued Brownian motion, possibly with correlation $\rho(W_t, \sigma_t)$. The dynamics of the forward F_t can then be written as

$$dF_t = \sigma_t b(F_t) dW_t.$$

where b is a function from $\mathbb{R}$ to $\mathbb{R}$. When $b(f) = f^\alpha$, the resulting model becomes the well known Constant Elasticity of Variance model (CEV), and the explicit solution to the case when $\alpha = 0$ and $\alpha = 1$ are known. For purposes of demonstration, we will for now assume that $\alpha = 0$. Let f denote the joint density function of $\zeta_t = (W_t, \sigma_t)$ with $\zeta_0 = (x, y)$, then to determine f we solve the heat equation on H^2:

$$\frac{\partial f}{\partial t} = \frac{1}{2} y^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right)$$

with initial condition $f_0(x, y) = \delta_x \delta_y$. Here, we are solving the above PDE on the upper half plane $\{z \in \mathbb{C} : z = x + iy, y > 0\}$ and δ here is the Dirac Delta. Let $d(z, \zeta)$ denote the geodesic distance between two points $z, \zeta \in H^2$, with $z = x + iy$ and $\zeta = \xi + i\eta$. Then,

$$\cosh d(z, \zeta) = 1 + \frac{|z - \zeta|^2}{2y\eta},$$

where $|z - \zeta|$ is the Euclidean distance between z and ζ. Then, application of Theorem 5.4.2 allows us to solve the heat equation on the upper half plane in the following closed form:[3]

$$f(z,\zeta,t) = \frac{e^{-\frac{1}{2t}d(z,\zeta)^2}\sqrt{2}}{(4\pi t)^{3/2}} \int_{d(z,\zeta)}^{\infty} \frac{ue^{\frac{-u^2}{4t}}}{\sqrt{\cosh u - \cosh d(z,\zeta)}} du. \tag{6.3}$$

Remark 6.3.1. Note that since ζ_t can also be considered as a spherical Lévy process on H^2, it falls under the example computed in section 4.6. There, the spherical transform of ζ_t was explicitly computed, and the Harish-Chandra's $\mathbf{c}$ function is also explicitly known on H^2. It is left as an exercise to the reader to apply the inversion formula to obtain an expression of the density on the unit circle, and then apply Cayley transform to reproduce (6.3) above.

In the general case (with drift), the δ-spherical functions on H^2 (unit disc model) are given by (c.f. [29], p.p. 60)

$$\Phi_{\lambda,n}(r) = r^{|n|}\frac{\Gamma(|n|+\nu)}{\Gamma(\nu)|n|!} F\left(\nu, 1-\nu; |n|+1; \frac{r^2}{r^2-1}\right),$$

and $\Phi_{\lambda,n}(e^{i\theta}z) = e^{in\theta}\Phi_{\lambda,n}(z)$. Here, $r = |z|$, $\nu = \frac{1}{2}(i\lambda + 1)$ and $F(a,b;c;z)$ is the hypergeomtric function. This can also be used, together with Theorem 5.4.2, to compute the density of a general Lévy process on H^2.

Now, since F_t is an Itô process, we can write $F_t = F(\zeta) = F(W_t, \sigma_t)$ such that

$$dF(\sigma_t, W_t) = \sigma_t b(F_t) dW_t,$$

where $F : \mathbb{M} \to \mathbb{R}^2$ is a C^2 function. Further more, assume the value of the derivative has payoff function G, so that the derivative value $V(\zeta) = (G \circ F)(\zeta)$. In the case of a European call, $G(x,y) = (e^x - K)^+$ so that $(G \circ F)(\zeta_t) = (S_t - K)^+$. Here S_t is the exponential of the first component of $F(\zeta)$. Hence, knowledge of the joint density (6.2), in principle allows us to explicitly compute the derivative price at time t as $e^{T-t}\mathbb{E}_{\mathbb{Q}}(V(\zeta_T)|\mathcal{F}_t)$.

We make a final remark on Levy and jump-diffusion dynamics. This allows the movement of the underlying be exposed to sudden shocks, as well as random diffusion and the probability distribution induced by such a process is typically

[3]Remark 6.3.1 briefly describes the procedure that this formula can be obtained from.

non-Gaussian. The induced information manifold is thus more complicated than the hyperbolic plane. In that case, one can still apply Theorem 5.4.2 to obtain an explicit formula of the characteristic function of the underlying stochastic process, and then develop a Carr-Madan type of procedure to obtain the derivative price.

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