Witten index, spectral shift function and spectral flow

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Witten index, spectral shift function and spectral flow

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The notion of spectral flow has been introduced by Atiyah and Lustzig and is an important tool in geometry. In 1976 Atiyah, Patodi and Singer suggested that a path of elliptic operators on odd dimensional compact manifolds the spectral flow can be computed via the Fredholm index of so-called `suspension', which is a first order elliptic operator on a manifold of one higher dimension and the well-known ``Fredholm index=spectral flow" theorem has appeared for the first time. Later, Robbin and Salamon provided an abstract framework for ``Fredholm index=spectral flow" theorem with a crucial assumption that the operators in the path have discrete spectra and the endpoints are boundedly invertible, the assumption which is usually violated in the setting of differential operators coming from mathematical physics.

In 2008 Pushnitski added a new ingredient to this equality, the Krein spectral shift function. With this new ingredient the operators in the path are allowed to have some essential spectral away from zero. If one removes the assumption that the endpoints are boundedly invertible, then the suspension is not necessarily a Fredholm operator. The latter assumption was omitted in the works by Carey and Gesztesy and their collaborators, where the Fredholm index was replaced by Witten index. However, the framework of this new equality ``Witten index=spectral shift function" does not cover yet differential operators on locally compact manifolds even in dimension 1.

The present thesis provides a complete framework for the ``index=spectral shift function" theorem, which is suitable for differential operators on locally compact manifolds in all dimensions at once. When specialised to the classical situation (with discrete spectra) our result recovers classical results of Atiyah, Patodi and Singer. In addition, whenever the spectral flow for the path is well-defined we establish an extension of Robbin-Salamon type theorem which is suitable for differential operators with some essential spectrum away from zero in any dimension.

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Abstract

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In 2008 Pushnitski added a new ingredient to this equality, the Krein spectral shift function. With this new ingredient the operators in the path are allowed to have some essential spectral away from zero. If one removes the assumption that the endpoints are boundedly invertible, then the suspension is not necessarily a Fredholm operator. The latter assumption was omitted in the works by Carey and Gesztesy and their collaborators, where the Fredholm index was replaced by Witten index. However, the framework of this new equality "Witten index=spectral shift function" does not cover yet differential operators on locally compact manifolds even in dimension 1.

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Contents

Introduction

Suppose that $\{A(t)\}_{t\in\mathbb{R}}$ is a family of self-adjoint operators in a Hilbert space $\mathcal{H}$ and consider the operator

$$\boldsymbol{D_A} = \frac{d}{dt} + A(t),$$

in the Hilbert space $L_2(\mathbb{R},\mathcal{H})$. Operators of this form were studied by Atiyah, Patodi and Singer [**5**], [**6**], [**7**] with $A(t),\quad t\in\mathbb{R}$, a first order elliptic differential operator on a compact odd-dimensional manifold with the asymptotes (in a suitable topology)

$$A_\pm = \lim_{t\to\pm\infty} A(t)$$

boundedly invertible and purely discrete spectra of $A_\pm, A(t), t\in\mathbb{R}$. The assumption that $A_\pm$ are boundedly invertible guarantees that the operator $\boldsymbol{D_A}$ is Fredholm, and therefore the Fredholm index, $\mathrm{index}(\boldsymbol{D_A})$, of the operator $\boldsymbol{D_A}$ is well-defined. Atiyah, Patodi and Singer showed that $\mathrm{index}(\boldsymbol{D_A})$ is equal to the spectral flow $\mathrm{sf}\{A(t)\}_{t=-\infty}^{\infty}$ of the path $\{A(t)\}_{t\in\mathbb{R}}$. The spectral flow here is intuitively understood as the net number of net number of eigenvalues (counting multiplicities) of $A(t)$ which pass through zero as t runs from $-\infty$ to $+\infty$.

An abstract framework for this "Fredholm index=spectral flow" theorem was established by Robbin and Salamon in [**70**]. In that paper the authors proved that the equality

$$\mathrm{index}(\boldsymbol{D_A}) = \mathrm{sf}\{A(t)\}_{t=-\infty}^{\infty} \tag{0.1}$$

holds under the assumption that the self-adjoint operators $A(t)$ and $A_\pm$ with common domain have purely discrete spectra and the asymptotes $A_\pm$ are boundedly invertible. Equality (0.1) has become an important result, with many applications including Morse theory, Floer homology, Morse and Maslov indices etc. It has been also extended in various directions, including Banach setting [**68**], a noncommutative analogue for C^*-algebras [**55**] and examples where equality (0.1) fails and the Fredholm index of $\boldsymbol{D_A}$ depends on the path $\{A(t)\}_{t\in\mathbb{R}}$ and not only on the endpoints has been discussed in [**1**].

However, the bulk of the literature on the "index-spectral flow" theorem focuses on the operators with purely discrete spectra with examples arising from geometrically defined operators on compact manifolds. If one is interested in the operators coming from physics, then the natural setting is locally compact manifolds and operators with at least some essential spectra.

One way to include operators with some essential spectra into consideration is to replace the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$, with which the operators $A(t)$ are affiliated, by a general semifinite von Neumann algebra $\mathcal{M}$ with faithful normal semifinite trace τ and assume that the path $\{A(t)\}$ consists of operators affiliated with $\mathcal{M}$ with resolvents compact with respect to the trace τ. In this case, Phillips definition of spectral flow (see Section 1.3 for precise details) can be used to define the right-hand side of (0.1), while the Fredholm index on the left-hand side of (0.1) is replaced by Breuer-Fredholm index. It is shown in [**11**] that with this adjustments equality (0.1) holds in this more general setting. Nevertheless, this setting is not suitable for the classical Dirac and Schroedinger operators in $\mathbb{R}^d$ and other important physical operators.

The first extension of "index-spectral flow" equality in the classical setting of the algebra $\mathcal{B}(\mathcal{H})$ for operators with some essential spectrum was obtained by Pushnitski [**67**]. There the author showed that the "index-spectral flow" equality is preserved if one replaces the Robbin-Salamon assumption of discrete spectra by the integrability condition with respect to the trace-class norm $\|\cdot\|_1$ on the trace-class ideal $\mathcal{L}_1(\mathcal{H})$ on $\mathcal{H}$, that is

$$\int_{\mathbb{R}} \|A'(t)\|_1 dt < \infty, \tag{0.2}$$

with $A'(t) = \frac{dA(t)}{dt}$ in a suitable topology.

Furthermore, motivated by [**67**], the result of Theorem (0.1) has been extended for a larger class of operators in [**47**]. The main assumption in [**47**] is that the family $\{A'(t)\}_{t\in\mathbb{R}}$ consists of relatively trace class perturbations of the operators A_-, that is

$$A'(t)(A_- + i)^{-1} \in \mathcal{L}_1(\mathcal{H}), \quad \int_{\mathbb{R}} \|A'(t)(A_- + i)^{-1}\|_1 dt < \infty. \tag{0.3}$$

A particular importance of [**67**] and [**47**] is the introduction of a new ingredient in the equality (0.1), the spectral shift function from scattering theory (see Section 1.2 for the detailed description).

THEOREM 0.1. [**67**], [**47**] *Assume that the family* $\{A(t)\}_{t\in\mathbb{R}}$ *satisfies* (0.2) *or* (0.3) *with boundedly invertible asymptotes* $A_\pm$. *Then*

$$\operatorname{index}(\boldsymbol{D_A}) = \operatorname{sf}\{A(t)\}_{t=-\infty}^{+\infty} = \xi(0; A_+, A_-), \tag{0.4}$$

where $\xi(\cdot; A_+, A_-)$ *denotes the spectral shift function for the pair* (A_+, A_-).

The addition of spectral shift function to the equality "index=spectral flow" and the fact that the assumption on spectra of operators is not essential for existence of spectral shift function yield that the right-hand side of (0.4) can be well-defined even if the assumption invertibility of $A_\pm$ is dropped. However, omitting the assumption of invertibility of $A_\pm$ implies that the operator $\boldsymbol{D_A}$ is no longer Fredholm, in general. This obstacle has been bypassed in [**31**] by replacing Fredholm index by the Witten index (see Section 1.4 for detailed discussion). The Witten index (in its resolvent regularisation) is defined as the limit

$$W(\boldsymbol{D_A}) = \lim_{\lambda\uparrow 0} (-\lambda) \operatorname{tr}\big((\boldsymbol{D_A}\boldsymbol{D_A^*} - \lambda)^{-1} - (\boldsymbol{D_A^*}\boldsymbol{D_A} - \lambda)^{-1}\big),$$

whenever this limit exists. Here tr denotes the classical trace on the Hilbert space $L_2(\mathbb{R}, \mathcal{H})$. The Witten index of $\boldsymbol{D_A}$ in the theorem below is expressed in terms on value of the spectral shift function $\xi(\,\cdot\,; A_+, A_-)$ in Lebesgue sense (see precise definition in Definition 6.2.1).

THEOREM 0.2. [**31**] *Assume* (0.3) *and suppose that* 0 *is a left and a right Lebesgue point of* $\xi(\,\cdot\,; A_+, A_-)$ *(denoted by* $\xi_L(0_-; A_+, A_-)$ *and* $\xi_L(0_+; A_+, A_-)$

respectively). Then the Witten index of the operator $\boldsymbol{D_A}$ exists and equals

$$W(\boldsymbol{D_A}) = \frac{1}{2}\big(\xi_L(0_+; A_+, A_-) + \xi_L(0_-; A_+, A_-)\big).$$

Thus, with omission of the assumption $A_\pm^{-1} \in \mathcal{B}(\mathcal{H})$, one can still have "index=spectral flow" type formula, if the Fredholm index is replaced by the Witten index and spectral flow is replaced by the (value of) spectral shift function, that is we now have

"index=spectral shift function"

theorem.

The major drawback of assumptions in [**67**] and [**47**], [**31**] is that they are suitable for certain pseudo-differential perturbations of a fixed differential operators or higher-order differential operators in low dimensions. If one considers even the simplest example of first order differential operator $A_- = \frac{d}{idx}$ on the one dimensional locally compact manifold $\mathbb{R}$ with perturbation given by multiplication operator M_f by a sufficiently nice function $f \neq 0$, then

$$M_f(A_- + i)^{-1}$$

belongs to Schatten ideals $\mathcal{L}_{1+\varepsilon}(L_2(\mathbb{R}))$ for any $\varepsilon > 0$, but not $\varepsilon = 0$ [**73**, Chapter 4]. Thus, first order differential operators can be treated neither by [**67**] nor by [**47**].

The first advancements for the "index=spectral shift function" equality applicable for differential operators on locally compact manifold were made in [**30**], [**26**], [**29**]. In particular, it was proved that for the operators $A_- = \frac{d}{idx}$ and $A_+ = \frac{d}{idx} + M_f$ on $L_2(\mathbb{R})$, with sufficently good f, the spectral shift function $\xi(\cdot; \frac{d}{idx} + M_f, \frac{d}{idx})$ is continuous at zero, the Witten index for the corresponding suspension $\boldsymbol{D_A}$ exists and

$$W(\boldsymbol{D_A}) = \frac{1}{2}\xi(0; \frac{d}{idx} + M_f, \frac{d}{idx}) = \frac{1}{2\pi}\int_{\mathbb{R}} f(s)ds. \tag{0.5}$$

However, the technique used in [**30**], [**26**] [**29**] can not be adapted to differential operators in higher-dimensions as the crucial assumption $(A_+ - A_-)(A_- + i)^{-2} \in \mathcal{L}_1(\mathcal{H})$ for many auxiliary results in [**30**], [**26**], [**29**] is not satisfied, in general, in higher dimensions.

The primary aim of the present thesis is to provide a general framework for the equality "index=spectral shift function", which is applicable for differential operators in any dimension at once and without any restrictions on their spectra. In addition, if we impose the assumption that the asymptotes $A_\pm$ have discrete spectra at 0 (without any restriction for the spectra outside 0) then we show that the "index=spectral flow" equality of Atiyah-Patodi-Singer and Robbin-Salamon holds in this more general setting too.

For our framework we assume that A_- is a self-adjoint operator on $\mathcal{H}$ and $\{B(t)\}_{t\in\mathbb{R}}$ is a family of bounded self-adjoint operators such that $B(t)$ is a *p-relative trace class perturbation* of A_-, that is there exists $p \in \mathbb{N} \cup \{0\}$ such that

$$B'(t)(A_- + i)^{-p-1} \in \mathcal{L}_1(\mathcal{H}), \quad \int_{\mathbb{R}} \|B'(t)(A_- + i)^{-p-1}\|_1 dt < \infty.$$

In this case, the family $\{A(t)\}_{t\in\mathbb{R}}$ is defined as

$$A(t) = A_- + B(t), \quad t \in \mathbb{R}.$$

The assumptions in [**47**], [**31**] and [**30**] are all particular cases on this assumption with $p = 0$ and $p = 1$, respectively. In addition, this is an assumption which is typically satisfied by differential operators on locally compact (and compact) manifolds for sufficiently large p, which depends on the dimension of the manifold and the order of differential operator A_-.

Assuming in addition some regularity of the path $\{B(t)\}_{t\in\mathbb{R}}$ (see Hypothesis 3.5.1 for the precise assumption) we prove the following (see Theorem 6.2.3).

THEOREM 0.3. *Assume Hypothesis 3.5.1. If* 0 *is a left and a right Lebesgue point of* $\xi(\,\cdot\,; A_+, A_-)$ *(denoted by* $\xi_L(0_-; A_+, A_-)$ *and* $\xi_L(0_+; A_+, A_-)$ *respectively), then the Witten index of the operator* $\boldsymbol{D_A}$ *exists and equals*

$$W(\boldsymbol{D_A}) = \frac{1}{2}\big(\xi_L(0_+; A_+, A_-) + \xi_L(0_-; A_+, A_-)\big).$$

Thus, our results establish "index=spectral shift function" equality in the framework which is suitable for differential operators in any dimension, which covers [**7**], [**70**] and [**67**], [**47**], [**31**],[**30**] at once. We discuss our approach in the proof of Theorem 0.3 in Section 1.6.

With an additional assumption that the operators $A_\pm$ are Fredholm we prove the following result (see Theorem 6.3.9).

THEOREM 0.4. *In addition to Hypothesis 3.5.1, assume that the asymptotes* $A_\pm$ *are Fredholm. Then the Witten index of the operator* $\boldsymbol{D_A}$ *exists, the spectral flow* $\mathrm{sf}(\{A(t)\}_{t=-\infty}^{\infty})$ *is well-defined and*

$$W(\boldsymbol{D_A}) = \frac{1}{2}\big(\xi(0+; A_+, A_-) + \xi(0-; A_+, A_-)\big) \tag{0.6}$$

$$= \mathrm{sf}(\{A(t)\}_{t=-\infty}^{\infty}) - \frac{1}{2}[\dim(\ker(A_+)) - \dim(\ker(A_-))]. \tag{0.7}$$

The result of Theorem 0.4 is new in several ways. Firstly, because we do not have to assume trivial kernels for $A_\pm$ or that the end points are unitarily equivalent the operators $\boldsymbol{D_A}$ need not be Fredholm. Replacing the Fredholm index on the left-hand side of equality (0.1) by the Witten index and adding correction term on the right-hand side, we establish an analogue of the "index=spectral flow" equality of Robbin-Salamon in the setting when the asymptotes $A_\pm$ are not invertible.

Secondly, if we impose the condition of trivial kernels for $A_\pm$, then we obtain exactly equations (0.1) and (0.4) for the operators with some essential spectra outside 0, which also hold for differential operators on locally compact manifolds in any dimension.

Thirdly, under the assumption that the operator A_- has compact resolvent and the perturbation $A_+ - A_-$ is bounded, equation (0.7) has been established in [**10**]. Therefore, our result provides a generalisation of [**10**] for the operators with essential spectra.

We also consider several examples of one dimensional differential operators, where we compute explicitly the spectral shift function $\xi(\cdot; A_+, A_-)$ in terms of the perturbation $A_+ - A_-$ (see Chapter 7). Therefore, taking its value at 0 we compute the Witten index $W(\boldsymbol{D_A})$ as well as spectral flow for examples where it is well-defined.

Structure of the thesis

The thesis is organised in the following way. In Chapter 1 we give a brief overview of the topic at hand. Firstly, we discuss the defintion of spectral shift function, spectral flow and Witten index as well as their basic properties. Then, in Section 1.5 we specify the exact assumptions made in [**67**] and [**47**], [**31**] as well as discuss the approach in the proof of Theorem 0.1. In Section 1.6 we give the outline of the proof of Theorems 0.3 and 0.4 and explain how we employ the results from [**31**].

In Chapter 2 we introduce the key technical tool used in our approach, the theory of Double Operator Integrals. We recall the definition of double operator integrals and their properties in Section 2.1. Then we present the details of construction from [**83**] of double operator integrals build over spectral measures of operators, such that difference of high power of resolvents falls into some Schatten ideal, and in Section 2.3 we show that these double operator integrals converge with respect to spectral measures.

Chapter 3 starts with explicit definitions of the operators involved in Theorems 0.3 and 0.4 as well as their basic properties. In this Chapter we introduce the approximation scheme, which is employed throughout the thesis. We also state our main assumption, Hypothesis 3.5.1, and prove its immediate corollaries.

In Chapter 4 we firstly recall the construction of spectral shift function from [**83**] and prove that it is continuous with respect to the operator parameter in Section 4.1. Then, in Section 4.2 we use these results to introduce uniquely the spectral shift function $\xi(\cdot; A_+, A_-)$. In Section 4.3 we also explain why we introduce the unique $\xi(\cdot; A_+, A_-)$ employing the continuity result from Section 4.1 rather than using the standard methonds from scattering theory.

Chapter 5 contains a crucial result in our approach, so-called principle trace formula. We explain in details the importance of this formula in Sections 1.5 and 1.6. Then, in Chapter 6 we establish our main results, Theorems 0.3 and 0.4.

We conclude with a chapter dedicated for examples for our framework. Firstly, in Section 7.1, we show that our main Hypothesis 3.5.1 is indeed satisfied for differential operators on locally compact manifolds. As an example we consider Dirac operator on $\mathbb{R}^d$. This chapter also contains several one dimensional examples, where the spectral shift function $\xi(\cdot; A_+, A_-)$ is computed explicitly.

The results of this thesis are published in the following papers and presented at the talks on various international and local conferences listed below.

(i) A. Carey, F. Gesztesy, G. Levitina, D. Potapov, F. Sukochev, *On the relationship of spectral flow to the Fredholm index and its extension to non-Fredholm operators,* to appear.
(ii) A. Carey, F. Gesztesy, G. Levitina, F. Sukochev, *On the index of a non-Fredholm model operator.* Oper. Matrices **10** (2016), no. 4, 881–914.
(iii) A. Carey, F. Gesztesy, G. Levitina, R. Nichols, D. Potapov, F. Sukochev, *Double operator integral methods applied to continuity of spectral shift functions.* J. Spectr. Theory **6** (2016), no. 4, 747–779.
(iv) A. Carey, F. Gesztesy, H. Grosse, G. Levitina, D. Potapov, F. Sukochev, D. Zanin, *Trace formulas for a class of non-Fredholm operators: a review.* Rev. Math. Phys. **28** (2016), no. 10, 1630002, 55 pp.
(v) A. Carey, F. Gesztesy, G. Levitina, D. Potapov, F. Sukochev, D. Zanin, *On index theory for non-Fredholm operators: a* $(1+1)$*-dimensional example.* Math. Nachr. **289** (2016), no. 5–6, 575–609.
(vi) A. Carey, F. Gesztesy, G. Levitina, F. Sukochev, *The spectral shift function and the Witten index.* Spectral theory and mathematical physics, 71105, Oper. Theory Adv. Appl., 254, Birkhäuser/Springer, [Cham], 2016.
(vii) G. Levitina, *The principal trace formula and its applications to index theory, spectral shift function and spectral flow*, 60th Annual AustMS Meeting, December 2016, Australian National University.
(viii) G. Levitina, *Witten index and spectral shift function,* International Workshop on Operator Theory and Applications, IWOTA, July, 2016, Washington University in St. Louis, USA.
(ix) G. Levitina, *Continuity of spectral shift function with respect to operator parameter*, plenary talk at Annual School Postgraduate Conference, June 2016, UNSW.
(x) G. Levitina *The spectral shift function and the Witten index*, Analysis and Partial Differential Equations workshop, July 2015, University of Wollongong.
(xi) G. Levitina *The spectral shift function and the Witten index*, plenary talk at Annual School Postgraduate Conference, June 2015, UNSW.

CHAPTER 1

Overview

The principal aim of this chapter is to introduce the central objectives of the present thesis, the Witten index, spectral shift function and spectral flow. We also briefly discuss the original Pushnitski's approach for the proof of Theorem 0.1 as well as give the outline of our approach.

1.1. Notations

In this section we collect the notations we employ throughout the thesis.

For a Banach space $\mathcal{X}$ we denote by $\mathcal{B}(\mathcal{X})$ the algebra of all linear bounded operators on $\mathcal{X}$.

In case when $\mathcal{X} = \mathcal{H}$ is a separable complex Hilbert space $\mathcal{H}$, we use notation $\|\cdot\|$ for the uniform norm. The corresponding ℓ_p-based Schatten–von Neumann ideals on $\mathcal{H}$ are denoted by $\mathcal{L}_p(\mathcal{H})$, with associated norm abbreviated by $\|\cdot\|_p$, $p \geq 1$. Moreover, $\operatorname{tr}_{\mathcal{H}}(A)$ denotes the trace of a trace class operator $A \in \mathcal{L}_1(\mathcal{H})$.

We note that for $A \in \mathcal{L}_p(\mathcal{H})$, $B \in \mathcal{L}_{p'}(\mathcal{H})$ with $\frac{1}{p}+\frac{1}{p'} = \frac{1}{q}$, the noncommutative Hölder inequality implies that $AB \in \mathcal{L}_q(\mathcal{H})$ and

$$\|AB\|_q \leq \|A\|_p \|B\|_{p'}. \tag{1.1.1}$$

We use symbols n-lim and s-lim to denote the operator norm limit (i.e., convergence in the topology of $\mathcal{B}(\mathcal{H})$), and the operator strong limit.

If T is a linear operator mapping (a subspace of) a Hilbert space into another, then $\operatorname{dom}(T)$ and $\ker(T)$ denote the domain and kernel (i.e., null space) of T. The closure of a closable operator S is denoted by $\overline{S}$.

The spectrum and resolvent set of a closed linear operator in $\mathcal{H}$ will be denoted by $\sigma(\cdot)$, and $\rho(\cdot)$, respectively. The spectral projections of a self-adjoint operator S in $\mathcal{H}$ we denote by $E_S(\cdot)$.

For a Fredholm operator T, we denote by $\operatorname{index}(T)$ its Fredholm index.

By $L_2(\mathbb{R}, \mathcal{H})$ we denote the Hilbert space off all $\mathcal{H}$-valued Bochner square integrable function on $\mathbb{R}$.

We define an auxiliary functions g and g_z, $z \in \mathbb{C} \setminus [0, \infty)$ by setting

$$g(t) = \frac{t}{(t^2+1)^{1/2}}, \quad t \in \mathbb{R}. \tag{1.1.2}$$

$$g_z(t) = \frac{t}{(t^2-z)^{1/2}}, \quad t \in \mathbb{R}, z \in \mathbb{C} \setminus [0, \infty). \tag{1.1.3}$$

The notation $[\cdot, \cdot]$ stands for commutator of two operators, that is

$$[A, B] = AB - BA.$$

We also note that for operators A, B, with A invertible, we have

$$[A^{-1}, B] = -A^{-1}[A, B]A^{-1}. \tag{1.1.4}$$

Throughout the thesis we use the convention that constants C_d, c_d, const etc. are strictly positive constants whose value depends only on their subscripts and can change from line to line.

The space of all Schwartz function on $\mathbb{R}^d$ is denoted by $S(\mathbb{R}^d)$ and the Sobolev spaces are denoted by $W^{p,q}(\mathbb{R}^d)$. Unless explicitly stated otherwise, whenever we write $L_p(\mathbb{R}^d)$ ($L_p(0,\infty)$ etc.) we assume the classical Lebesgue measure on $\mathbb{R}^d$ ($(0,\infty)$ etc.). The space of all functions with continuous derivative up to order n is denoted by $C^n(\mathbb{R})$ (or $C^n(a,b), a<b$). If all derivatives up to order n are also bounded functions then, the space is denoted by $C^n_b(\mathbb{R})$ ($C^n_b(a,b)$, respectively).

1.2. Spectral shift function in the classical setting

In this section we discuss the notion of spectral shift function and its properties. The material presented here can be found in [**82**, Chapter 8].

In 1947, a well-known physicist I. M. Lifshitz considered perturbations of an operator H_0 (arising as the Hamiltonian of a lattice model in quantum mechanics) by a finite-rank perturbation V and found some formulae and quantitative relations for the size of the shift of the eigenvalues. In one of his papers the spectral shift function (SSF), $\xi(\,\cdot\,; H_0+V, H_0)$, appeared for the first time, and formulae for it in the case of a finite-rank perturbation were obtained.

Lifshitz later continued these investigations and applied them to the problem of computing the trace of the operator $f(H_0+V)-f(H_0)$, where H_0 is the unperturbed self-adjoint operator, V is a self-adjoint, finite-dimensional perturbation, and f is an appropriate function (belonging to a fairly broad class). He obtained (or, rather, surmised) the remarkable relation

$$\operatorname{tr}(f(H_0+V)-f(H_0)) = \int_{\mathbb{R}} f'(\lambda)\xi(\lambda; H_0+V, H_0)\,d\lambda, \tag{1.2.1}$$

where the function $\xi(\,\cdot\,; H_0+V, H_0)$ depends on operators H_0 and V only.

In [**53**], M. G. Krein established the proper mathematical framework for spectral shift function in terms of trace-class perturbations V. He also described the broad class of functions f for which (1.2.1) holds.

THEOREM 1.2.1. *Suppose that H and H_0 are self-adjoint operators such that $H-H_0 \in \mathcal{L}_1(\mathcal{H})$ and assume that $f \in C^1(\mathbb{R})$ and its derivative admits the representation*

$$f'(\lambda) = \int_{\mathbb{R}} \exp(-i\lambda t)\,dm(t), \quad |m|(\mathbb{R}) < \infty,$$

for a finite (complex) measure m. Then $[f(H)-f(H_0)] \in \mathcal{L}_1(\mathcal{H})$, and there exists unique *function $\xi(\cdot; H, H_0) \in L_1(\mathbb{R})$ such the following trace formula holds*

$$\operatorname{tr}(f(H)-f(H_0)) = \int_{\mathbb{R}} f'(\lambda)\xi(\lambda; H, H_0)\,d\lambda. \tag{1.2.2}$$

The trace formula 1.2.2 is customarily referred to a the the (Lifshitz-)Krein trace formula.

REMARK 1.2.2. (i) The proof of existence of $\xi(\cdot; H, H_0)$ relies on so-called perturbation determinant $\Delta_{H/H_0}(z)$ (see [**82**]), which is an analytic function in the open upper and lower half planes of $\mathbb{C}$ and the integral representation

$$\ln(\Delta_{H/H_0}(z)) = \int_{\mathbb{R}} \frac{\xi(\lambda; H, H_0)\, d\lambda}{\lambda - z}, \quad \operatorname{Im}(z) \neq 0.$$

In particular, Krein's proof heavily relies on complex analysis. The full purely real-analytic proof has been recently provided in [**66**].

(ii) The function $\xi(\,\cdot\,; H, H_0)$ is an element of $L_1(\mathbb{R})$, that is, it represents an equivalence class of Lebesgue measurable functions. Therefore, generally speaking, the notation $\xi(\lambda; H, H_0)$ is meaningless for a fixed $\lambda \in \mathbb{R}$. This represents a particular technical difficulty for equality (0.3) as we wish to equate the Witten index with a particular value of the spectral shift function $\xi(\cdot; A_+, A_-)$. To overcome this difficulty we take the value of $\xi(\cdot; A_+, A_-)$ in Lebesgue sense (see Definition 6.2.1).

Next we discuss properties of the spectral shift function. For complete proofs we refer to [**82**, Chapter 8].

Let H_0 and H be such that $(H - H_0) \in \mathcal{L}_1(\mathcal{H})$ and let δ be an interval on the real line (possibly unbounded) such that $\delta \subset \rho(H_0) \cap \rho(H)$. Then $\xi(\,\cdot\,; H, H_0)$ takes a constant integer value on δ, that is,

$$\xi(\lambda; H, H_0) = n, \quad n \in \mathbb{Z},\ \lambda \in \delta.$$

If the interval δ contains a half-line, then the L_1-condition on ξ implies that $n = 0$.

Let μ be an isolated eigenvalue of multiplicity $\alpha_0 < \infty$ of H_0 and multiplicity α for H. Then

$$\xi(\mu_+; H, H_0) - \xi(\mu_-; H, H_0) = \alpha_0 - \alpha. \tag{1.2.3}$$

Equation (1.2.3) can be generalized as follows. Suppose that in some interval (a_0, b_0) the spectrum of H_0 is discrete. Then, by Weyl's theorem on the invariance of essential spectra (see, e.g., [**52**, Theorem 5.35]), H has discrete spectrum in (a_0, b_0) as well.

Let $\delta = (a, b)$, $a_0 < a < b < b_0$. Introduce the eigenvalue counting functions $N_0(\delta)$ and $N(\delta)$ of the operators H_0 and H, respectively, in the interval δ as the sum of the multiplicities of the eigenvalues in δ of the operator H_0, respectively, H. Since the interval δ is finite and both operators H_0, H have discrete spectrum, $N_0(\delta)$ and $N(\delta)$ are finite. In this case one has the equality,

$$\xi(b_-; H, H_0) - \xi(a_+; H, H_0) = N_0(\delta) - N(\delta). \tag{1.2.4}$$

The equation (1.2.4) can give some insight on the reason why $\xi(\cdot; H, H_0)$ is called a *spectral shift* function. However, one of the most important properties of spectral shift function is its connection with determinant of so-called scattering matrix via Birman-Krein formula. Namely, denote by $S(\lambda) = S(\lambda; H, H_0)$ the scattering matrix (see [**82**] for precise definition). Then, for almost every $\lambda \in \mathbb{R}$ we have [**13**], [**14**] (see also [**82**, Section 8.4])

$$\det(S(\lambda)) = \exp(-2\pi i \xi(\lambda; H, H_0)).$$

This identity is often used as definition of spectral shift function and has some deep applications in scattering theory. Here, we do not intend to use this equation and hence we do not go into details.

Of course the requirement that $H - H_1$ is a trace-class operator is very strict and rules our classical differential operators. The first result, generalising the class of operators H_0, H is due to M.G.Krein [**54**].

THEOREM 1.2.3 (Resolvent comparable case). *Let H_0, H be self-adjoint operators such that*

$$(H-z)^{-1}-(H_0-z)^{-1}\in\mathcal{L}_1(\mathcal{H}),\quad z\in\rho(H_0)\cap\rho(H).$$

Suppose that a function f on $\mathbb{R}$ has two bounded derivatives and

$$\frac{d^l}{d\lambda^l}\big(f(\lambda)-f_0\lambda^{-1}\big)=O(|\lambda|^{-l-1-\varepsilon}),\ as\ \lambda\to\pm\infty,\quad l=0,1,2$$

where the constant f_0 is the same for $\lambda\to\pm\infty$. Then $f(H)-f(H_0)\in\mathcal{L}_1(\mathcal{H})$ and there exists a spectral shift function $\xi(\,\cdot\,;H,H_0)$, satisfying the weighted integrability condition

$$\xi(\lambda;H,H_0)\in L_1\big(\mathbb{R};(1+\lambda^2)^{-1}d\lambda\big)$$

and

$$\operatorname{tr}\big(f(H)-f(H_0)\big)=\int_{\mathbb{R}}f'(\lambda)\xi(\lambda;H,H_0)d\lambda. \tag{1.2.5}$$

We emphasize that in the resolvent comparable case the spectral shift function is defined only up to an (integer-valued) additive constant (see [**82**, Section 8.7]). So, in general, there is a class of functions from $L_1\big(\mathbb{R};(1+\lambda^2)^{-1}d\lambda\big)$, which differ by an additive constant and satisfy (1.2.5).

Just as in the case of a trace class perturbation, the SSF for resolvent comparable operators H_0, H possesses the following property.

If in some interval (a_0,b_0) the spectrum of H_0 is discrete and let $\delta=(a,b)$, $a_0<a<b<b_0$. Then the analogue of (1.2.4) holds, that is,

$$\xi(b_-;H,H_0)-\xi(a_+;H,H_0)=N_0(\delta)-N(\delta), \tag{1.2.6}$$

where $N_0(\delta)$ (respectively, $N(\delta)$) are the sum of the multiplicities of the eigenvalues of H_0 (respectively, H) in δ.

In the particular case of lower semibounded operators H_0 and H equality (1.2.6) allows us to naturally fix the additive constant in the following way. To the left of the spectra of H_0 and H, the eigenvalue counting functions $N_0(\cdot)$ and $N(\cdot)$ are zero. Therefore, by equality (1.2.6) the SSF $\xi(\,\cdot\,;H,H_0)$ is a constant to the left of the spectra of H_0 and H, and it is custom to set this constant equal to zero,

$$\xi(\lambda;H,H_0)=0,\quad \lambda<\inf(\sigma(H_0)\cup\sigma(H)). \tag{1.2.7}$$

In the following we describe a particular way to introduce the SSF for the pair (H,H_0) by what is usually called the *invariance principle* and which is often used to fix an additive constant for the spectral shift function $\xi(\cdot;H,H_0)$ for resolvent comparable case.

Let Ω be an interval containing the spectra of H_0 and H, and let f be an arbitrary bounded monotone "sufficiently" smooth function on Ω. Suppose that

$$f(H)-f(H_0)\in\mathcal{L}_1(\mathcal{H}) \tag{1.2.8}$$

then, the SSF $\xi(\,\cdot\,;H,H_0)$ can be fixed as follows:

$$\xi(\lambda;H,H_0)=\operatorname{sgn}\big(f'(\lambda)\big)\,\xi(f(\lambda);f(H),f(H_0)). \tag{1.2.9}$$

For the function $\xi(\,\cdot\,;H,H_0)$ the Lifshitz–Krein trace formula (1.2.2) holds for some class of admissible functions. The latter class depends on f. The assumption that the function f is monotone is crucial in invariance principle, since it allows to take the inverse of f in the Krein trace formula. Namely, for sufficiently nice function h we have

$$\begin{aligned}\operatorname{tr}\big(h(H)-h(H_0)\big) &= \operatorname{tr}\big((h\circ f^{-1})(f(H))-(h\circ f^{-1})(f(H_0))\big)\\ &= \int_{\mathbb{R}}(h\circ f^{-1})'(\mu)\,\xi(\mu;f(H),f(H_0))d\mu.\end{aligned}$$

Using the substitution $\mu=f(\lambda)$, one can obtain that

$$\begin{aligned}\operatorname{tr}\big(h(H)-h(H_0)\big) &= \int_{\mathbb{R}} h'(\lambda)\,\xi(f(\lambda);f(H),f(H_0))d\lambda\\ &= \int_{\mathbb{R}} h'(\lambda)\,\xi(\lambda;H,H_0)d\lambda,\end{aligned}$$

where the last equation follows from (1.2.9). In Section 4.3 we give an example of pair (H,H_0), such that inclusion (1.2.8) does not hold no matter how smooth the function f is, which yields an example of operators for which the invariance principle is not applicable.

1.3. The Phillips definition of spectral flow and its analytic formulas

In this section we recall Phillips definition of the spectral flow as well as its integral formula. For a survey of the notion of spectral flow and its analytic formulas we recommend [**12**].

Given a continuous path of bounded Fredholm operators $\{F_t : t\in[0,1]\}$, J. Phillips [**61**] introduced an analytic definition of spectral flow along this path. This definition is more useful than the original topological approach of Atiyah-Patodi-Singer.

Let χ be the characteristic function of the interval $[0,\infty)$ and let $\{F_t\}_{t\in[0,1]}$ be a norm continuous path of bounded self-adjoint Fredholm operator on $\mathcal{H}$. Denote by π the projection onto the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Then one may show that $\pi(\chi(F_t)) = \chi(\pi(F(t)))$. Since the spectra of $\pi(F(t))$ are bounded away from 0, this latter path is continuous. By compactness we can choose a partition $0=t_0<t_1<\cdots<t_k=1$ so that for each $i=1,2,...,k$

$$\|\pi(\chi(F(t)))-\pi(\chi(F(s)))\|<1/2 \quad \text{for all } t,s\in[t_{i-1},t_i].$$

Letting $P_i=\chi(F_{t_i})$ for $i=0,1,...,k$, then by the previous inequality is equivalent to

$$\|\pi(P_i)-\pi(P_{i-1})\|<1/2.$$

By [**8**, Propisition 3.1] implies that the operator $P_{i-1}P_i : P_i\mathcal{H}\to P_{i-1}\mathcal{H}$ is Fredholm. Then we define the spectral flow of the path $\{F_t\}_{t\in[0,1]}$ to be the number:

$$\operatorname{sf}(\{F_t\}_{t\in[0,1]}) = \sum_{i=1}^{k}\operatorname{index}(P_{i-1}P_i),$$

where $\operatorname{index}(P_{i-1}P_i)$ is Fredholm index of $P_{i-1}P_i$ as an operator from $P_i\mathcal{H}$ to $P_{i-1}\mathcal{H}$.

The main results of [**61**] show that this analytic notion is well defined being independent of the partition into ‘small’ intervals and that it reproduces the usual topological point of view. In addition, this analytic point of view recovers the intersection number approach to spectral flow when the operators in question have discrete spectrum.

The standard approach (see e.g. [**34**], [**12**]) in the analytic definition of spectral flow for a path $\{D(t)\}_{t\in[0,1]}$ of unbounded Fredholm operators is to reduce it to the path of bounded Fredholm operators $\{g(D(t))\}_{t\in[0,1]}$, where g is the Riesz mapping

$$g(t) = \frac{t}{(1+t^2)^{1/2}}.$$

We firstly recall the following notion of continuity for a path of unbounded operators.

Definition 1.3.1. *(i) A path $\{D(t)\}_{t\in[0,1]}$ is called Γ-differentiable at the point $t = t_0$ if and only if there is a bounded linear operator G such that*

$$\lim_{t\to t_0} \left\| \frac{D(t)-D(t_0)}{t}(1+D(t_0)^2)^{-1/2} - G \right\|_{\mathcal{B}(\mathcal{H})} = 0.$$

In this case, we set $\dot{D}(t_0) = G(1+D(t_0)^2)^{1/2}$. By [**36**, Lemma 25] *the operator $\dot{D}(t)$ is a symmetric linear operator with the domain* $\mathrm{dom}(D(t))$.

(ii) If the mapping $t \mapsto \dot{D}(t)(1+D(t)^2)^{-1/2}$ is defined and continuous with respect to the operator norm, then the path $\{D(t)\}_{t\in[0,1]}$ is called continuously Γ-differentiable or a C^1_Γ-path

One of the crucial results in [**36**] is the following

Theorem 1.3.2. [**36**, Theorem 22] *If $\{D_t\}_{t\in[0,1]}$ is a C^1_Γ-path of self- adjoint linear operators, then the path $\{g(D_t)\}_{t\in[0,1]}$ is a C^1-path with respect to the operator norm.*

Theorem 1.3.2 implies that the following definition makes sense.

Definition 1.3.3. *Suppose that $\{D_t\}_{t\in[0,1]}$ is a C^1_Γ-path of self-adjoint Fredholm operators. We introduce the spectral flow as follows*

$$\mathrm{sf}(\{D_t\}_{t\in[0,1]}) := \mathrm{sf}(\{g(D_t)\}_{t\in[0,1]}). \tag{1.3.1}$$

In our approach to the proof of Theorem 0.4 we employ so-called integral formulas for spectral flow. The idea that spectral flow is given by integrating a one form suggested by Singer [**74**] and the first paper which presents a systematic approach to this idea is that of Getzler [**50**].

A different approach to finding an analytic formula for spectral flow may be found in [**34, 35**] motivated by the study of spectral flow in semifinite von Neumann algebras. Though [**34, 35**] drew inspiration from [**50**] their approach is based on Phillips' analytic viewpoint. In [**34, 35**] formulas for spectral flow are given for paths of both bounded and unbounded Fredholm operators. We note that [**34**], [**35**] and [**50**] prove the integral formulas for spectral flow for operators with purely discrete spectra (and some additional summability condition). We refer the reader to [**12**] and [**45**] for more extensive discussion of integral formulas for spectral flow.

Substantial refinements of this early work cited in the preceding paragraphs are found in [**10**] and [**36**]. In particular, [**36**] fully resolves Singer's conjecture for integral formulas of spectral flow. Moreover, [**36**] establishes an analytic formula for spectral flow for operators with some essential spectrum .

To be more specific, we state the theorem of [**36**] for one particular function we use in our proof. Below, the notation erf stands for the error function

$$\operatorname{erf}(x) = \frac{2}{\pi^{1/2}} \int_0^x e^{-y^2}\, dy, \quad x \in \mathbb{R}. \tag{1.3.2}$$

THEOREM 1.3.4. [**36**, Theorem 9] *Let $\{D_t\}_{t\in[0,1]}$ be a C^1_Γ-path of (unbounded) self adjoint Fredholm operators joining endpoints D_0, D_1. Suppose that*

(i) $\int_0^1 \|\dot{D}_t e^{-\lambda D_t^2}\|_1 dt < \infty$, $\lambda > 0$;

(ii) The operator $[\frac{1}{2}\operatorname{erf}(\lambda^{1/2}D_1) - \frac{1}{2}\operatorname{erf}(\lambda^{1/2}D_0)] - [\chi_{[0,\infty)}(D_1) - \chi_{[0,\infty)}(D_0)]$ *is a trace-class operator.*

Then

$$\begin{aligned}\operatorname{sf}(\{D_t\}_{t\in[0,1]}) &= \int_0^1 \operatorname{tr}(\dot{D}_t e^{-\lambda D_t^2}) dt \\ &\quad + \operatorname{tr}\Big([\frac{1}{2}\operatorname{erf}(\lambda^{1/2}D_1) - \frac{1}{2}\operatorname{erf}(\lambda^{1/2}D_0)] - [\chi_{[0,\infty)}(D_1) - \chi_{[0,\infty)}(D_0)]\Big).\end{aligned}$$

For completeness we present a short explanation how Theorem 1.3.4 can be obtained from [**36**, Theorem 9].

PROOF. To use [**36**, Theorem 9] set $g(x) = \sqrt{\frac{\lambda}{\pi}} e^{-\lambda x^2}$, $x \in \mathbb{R}$, $\lambda > 0$. Then clearly

$$\int_{\mathbb{R}} g(x) dx = 1$$

and hence assumptions (i) and (ii) of [**36**, Theorem 9] are satisfied. In addition, the antiderivative $G(\cdot)$ of $g(\cdot)$ satisfying $G(\pm\infty) = \pm\frac{1}{2}$ is given by (see [**36**, Theorem 11])

$$G(x) = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^x e^{-\lambda s^2} ds - \frac{1}{2} = \sqrt{\frac{\lambda}{\pi}} \int_0^x e^{-\lambda s^2} ds = \frac{1}{2}\operatorname{erf}(\sqrt{\lambda} x), \quad x \in \mathbb{R}, \lambda > 0.$$

Hence, the assumption (iii) of [**36**, Theorem 9] is precisely what is assumed in (ii). Hence, [**36**, Theorem 9] applies and gives the required formula. □

REMARK 1.3.5. We note that [**36**, Theorem 11] states exactly the same analytic formula for the spectral flow as we presented above. However, one of the assumptions of [**36**, Theorem 11] requires that $e^{-\lambda D_t^2}$ is a trace-class operator for every $t \in [0,1]$ and $\lambda > 0$. In our setting we do not have this assumption in general.

To conclude section we discuss connection of spectral shift function and the spectral flow. Recall that one particular property of spectral shift function for a pair (D, D_0) of operators with discrete spectra is that

$$\xi(b_-; D, D_0) - \xi(a_+; D, D_0) = N_0(\delta) - N(\delta),$$

where $N_0(\cdot)$ and $N(\delta)$ denotes the eigenvalue counting function of D_0 and D respectively on the interval $a_0 < a < b < b_0$. In addition, the naive definition of the spectral flow for a path $\{D(t)\}_{t\in[0,1]}$ connecting D_0 and D is the net number eigenvalues (counting multiplicities) of $D(t)$ which pass through zero as t runs from 0 to 1. Thus, on intuitive level we have the equality

$$\operatorname{sf}(\{D(t)\}_{t\in[0,1]}) = \frac{1}{2}\big(\xi(0+; D, D_0) + \xi(0-; D, D_0)\big).$$

In fact, this equality indeed holds for a path with unitarily equivalent endpoints D, D_0. This result was proved rigorously in [**10**, Section 3.5] for a general setting of operators affiliated with a semifinite von Neumann algebra with faithful normal semifinite trace. Here, we recall the result for a special case when $\mathcal{M} = \mathcal{B}(\mathcal{H})$.

THEOREM 1.3.6. *Suppose that D_0 is a self-adjoint operator with purely discrete spectra and let $D - D_0$ be a bounded operator. Then the spectral shift function $\xi(\cdot; D, D_0)$ and the spectral flow* $\mathrm{sf}(\{D(t)\}_{t\in[0,1]})$ *exist and*

$$\begin{aligned}\mathrm{sf}(\{D(t)\}_{t\in[0,1]}) = &\frac{1}{2}\big(\xi(0+; D, D_0) + \xi(0-; D, D_0)\big) \\ &+ \frac{1}{2}\big(\dim(\ker(D)) - \dim(\ker(D_0))\big).\end{aligned} \tag{1.3.3}$$

Comparing the latter equality with equality (0.7) (see also Theorem 6.3.8) we conclude that our technique provides an extension of equality (1.3.3) for the p-relative perturbations of a self-adjoint operator, such that D and D_0 do not have essential spectra at 0. Thus equality (1.3.3) holds also for operators with some essential spectra.

1.4. The Witten Index

In this section we recall the definition of Witten index and its basic properties.

In his paper [**81**], Witten introduced a number, which counts the difference in the number of bosonic and fermionic zero-energy modes of a Hamiltonian. This quantity, called the Witten index, became popular in connection with a variety of examples in supersymmetric quantum mechanics in the 1980's and in [**20**], [**21**], [**49**] has been put in mathematical framework using two different regularisation, which we recall next.

We start with the following facts on trace class properties of resolvent and semigroup differences.

PROPOSITION 1.4.1. *(see e.g.* [**31**, Lemma 3.1]*) Suppose that $0 \le S_j$, $j = 1, 2$, are nonnegative, self-adjoint operators in $\mathcal{H}$.*

(i) [**79**, p. 178] *If $\big[(S_2 - z_0)^{-1} - (S_1 - z_0)^{-1}\big] \in \mathcal{L}_1(\mathcal{H})$ for some $z_0 \in \rho(S_1) \cap \rho(S_2)$, then*

$$\big[(S_2 - z)^{-1} - (S_1 - z)^{-1}\big] \in \mathcal{L}_1(\mathcal{H}) \text{ for all } z \in \rho(S_1) \cap \rho(S_2).$$

(ii) If $\big[e^{-t_0 S_2} - e^{-t_0 S_1}\big] \in \mathcal{L}_1(\mathcal{H})$ for some $t_0 > 0$, then

$$\big[e^{-tS_2} - e^{-tS_1}\big] \in \mathcal{L}_1(\mathcal{H}) \text{ for all } t \ge t_0.$$

The preceding fact allows one to consider the following two definitions.

Let T be a closed, linear, densely defined operator in $\mathcal{H}$. Suppose that for some (and hence for all) $z \in \mathbb{C}\backslash[0,\infty) \subseteq [\rho(T^*T) \cap \rho(TT^*)]$,

$$\big[(T^*T - z)^{-1} - (TT^* - z)^{-1}\big] \in \mathcal{L}_1(\mathcal{H}).$$

Then one introduces the resolvent regularization

$$\Delta_r(T, \lambda) = (-\lambda)\,\mathrm{tr}_{\mathcal{H}}\big((T^*T - \lambda)^{-1} - (TT^* - \lambda)^{-1}\big), \quad \lambda < 0. \tag{1.4.1}$$

DEFINITION 1.4.2. The resolvent regularized Witten index $W_r(T)$ *of T is defined by*

$$W_r(T) = \lim_{\lambda\uparrow 0} \Delta_r(T, \lambda),$$

whenever this limit exists.

Similarly, suppose that for some $t_0 > 0$

$$\left[e^{-t_0 T^*T} - e^{-t_0 TT^*}\right] \in \mathcal{L}_1(\mathcal{H}).$$

Then $\left(e^{-tT^*T} - e^{-tTT^*}\right) \in \mathcal{L}_1(H)$ for all $t > t_0$ and one introduces the semigroup regularization

$$\Delta_s(T,t) = \operatorname{tr}_{\mathcal{H}}\left(e^{-tT^*T} - e^{-tTT^*}\right), \quad t > 0. \tag{1.4.2}$$

DEFINITION 1.4.3. *The semigroup regularized Witten index $W(T)$ of T is defined by*

$$W(T) = \lim_{t\uparrow\infty} \Delta_s(T,t),$$

whenever this limit exists.

As proved in [**49**], [**20**], the Witten index of an operator T in $\mathcal{H}$ is a natural substitute for the Fredholm index of T in cases where the operator T ceases to have the Fredholm property. Namely, the following result states that both (resolvent and semigroup) regularized Witten indices coincide with the Fredholm index in the special case of Fredholm operators.

THEOREM 1.4.4. [**49, 20**] *Let T be an (unbounded) Fredholm operator in H. Suppose that $\left[(T^*T - z)^{-1} - (TT^* - z)^{-1}\right]$, $\left[e^{-t_0 T^*T} - e^{-t_0 TT^*}\right] \in \mathcal{L}_1(\mathcal{H})$ for some $z \in \mathbb{C}\backslash[0,\infty)$, and $t_0 > 0$. Then*

$$\operatorname{index}(T) = W_r(T) = W(T).$$

We note that the regularisations (1.4.1) and (1.4.2) has been used before [**49, 20**] to compute the Fredholm index of an operator (see e.g. [**23**]).

In general (i.e., if T is not Fredholm), $W_r(T)$ (respectively, $W(T)$) is not necessarily integer-valued; in fact, it can be any real number. As a concrete example, we mention the two-dimensional magnetic field system discussed by Aharonov and Casher [**3**] which demonstrates that the resolvent and semigroup regularized Witten indices have the meaning of (non-quantized) magnetic flux $F \in \mathbb{R}$ which indeed can be any prescribed real number.

Expressing the Witten index $W(T)$ (respectively, $W_r(T)$) of an operator T in terms of the spectral shift function $\xi(\,\cdot\,; T^*T, TT^*)$ requires of course the choice of a concrete representative of the SSF:

THEOREM 1.4.5. [**20, 49**] *(i) Suppose that $\left[e^{-t_0 T^*T} - e^{-t_0 TT^*}\right] \in \mathcal{L}_1(\mathcal{H})$ for some $t_0 > 0$ and the spectral shift function $\xi(\,\cdot\,; T^*T, TT^*)$, uniquely defined by the requirement $\xi(\lambda; T^*T, TT^*) = 0$, $\lambda < 0$, is continuous from above at $\lambda = 0$. Then the semigroup regularized Witten index $W(T)$ of T exists and*

$$W(T) = -\xi(0_+; T^*T, TT^*).$$

(ii) Suppose that $\left[(T^*T - z)^{-1} - (TT^* - z)^{-1}\right] \in \mathcal{L}_1(\mathcal{H})$, $z \in \mathbb{C}\backslash[0,\infty)$ and $\xi(\,\cdot\,; T^*T, TT^*)$, uniquely defined by the requirement $\xi(\lambda; T^*T, TT^*) = 0$, $\lambda < 0$, is bounded and piecewise continuous on $\mathbb{R}$. Then the resolvent regularized Witten index $W(T)$ of T exists and

$$W_r(T) = -\xi(0_+; T^*T, TT^*).$$

The first relations between index theory for not necessarily Fredholm operators and the Lifshitz–Krein spectral shift function were established in [**20**], [**46**], [**49**], and independently in [**37**]. In fact, inspired by index calculations of Callias [**23**] in connection with noncompact manifolds, the more general notion of the Witten index was studied and identified with the value of an appropriate spectral shift function at zero in [**20**] and [**49**] (see also [**46**], [**78**, Ch. 5]). Similiar investigations in search of an index theory for non-Fredholm operators were untertaken in [**37**] in a slightly different direction, based on principal functions and their connection to Krein's spectral shift function.

While [**20**] and [**46**] focused on index theorems for concrete one and two-dimensional supersymmetric systems, [**49**] treated abstract Fredholm and Witten indices in terms of the spectral shift function and proved their invariance with respect to appropriate classes of perturbations. Soon after, a general abstract approach to supersymmetric scattering theory involving the spectral shift function was developed in [**22**] and applied to relative index theorems in the context of manifolds Euclidean at infinity.

The intrinsic value of the Witten index $W(T)$ lies in its stability properties with respect to additive perturbations, analogous to stability properties of the Fredholm index. As shown in [**49**], [**20**], the Witten index possesses stability properties with respect to additive perturbations, however, necessarily under considerably stronger hypotheses (very roughly speaking, relative trace class type perturbations) than in the case of Fredholm indices (where relatively compact perturbations can be handled). In this relation we refer also to a recent paper [**33**] where a new approach for topological invariance has been suggested.

1.5. Pushnitski's setting

In this section we give the precise assumptions made in [**67**], [**47**], [**31**] and the discuss the crucial steps in the proof of Theorems 0.1 and 0.2.

We start with the assumption of Pushnitski [**67**].

HYPOTHESIS 1.5.1 (The Pushnitski Assumptions).

(i) Assume $A_- \in \mathcal{B}(\mathcal{H})$ is self-adjoint in $\mathcal{H}$.

(ii) Suppose there exists a family of bounded self-adjoint operators $\{B(t)\}_{t\in\mathbb{R}}$ (the allowed perturbations of A_-) in $\mathcal{H}$ with $B(\cdot)$ weakly locally absolutely continuous on $\mathbb{R}$, implying the existence of a family of bounded self-adjoint operators $\{B'(t)\}_{t\in\mathbb{R}}$ in $\mathcal{H}$ such that for a.e. $t \in \mathbb{R}$,

$$\frac{d}{dt}(g, B(t)h)_{\mathcal{H}} = (g, B'(t)h)_{\mathcal{H}}, \quad g, h \in \mathcal{H}.$$

(iii) Assume that $B'(t) \in \mathcal{L}_1(\mathcal{H})$, $t \in \mathbb{R}$, and

$$\int_{\mathbb{R}} \big\|B'(t)\big\|_{\mathcal{L}_1(\mathcal{H})}\, dt < \infty.$$

For comparison we state the key assumption of [**47**] on these perturbations that replaces item *(iii)* of the Pushnitski assumptions.

HYPOTHESIS 1.5.2 (The assumptions in [**47**] and [**31**]). *Instead of assumption (iii) in Hypothesis 1.5.1 assume that* *(iii′) Assume the relatively trace class perturbation assumption*

$$\big\|B'(t)(A_-^2+1)^{-1/2}\big\|_{\mathcal{L}_1(\mathcal{H})}<\infty \text{ and } \int_{\mathbb{R}}\big\|B'(t)(A_-^2+1)^{-1/2}\big\|_{\mathcal{L}_1(\mathcal{H})}\,dt<\infty.$$

The Pushnitski's assumption 1.5.1 guarantees, in particular, that the limit

$$B_+:=\lim_{t\to+\infty}B(t),$$

exists in the uniform norm (see (3.1.4)), in addition,

$$B_+\in\mathcal{L}_1(\mathcal{H}).$$

Introduce the family of self-adjoint operators $A(t)$, $t\in\mathbb{R}$, in $\mathcal{H}$, by

$$A(t)=A_-+B(t),\quad \operatorname{dom}(A(t))=\operatorname{dom}(A_-),\quad t\in\mathbb{R}$$

and the self-adjoint asymptote

$$A_+=A_-+B_+.$$

For the family $\{A(t)\}_{t\in\mathbb{R}}$ in $\mathcal{H}$ we denote by $\boldsymbol{A}$ the operator acting in the Hilbert space $L_2(\mathbb{R},\mathcal{H})$ defined by

$$\begin{aligned}&(\boldsymbol{A}f)(t)=A(t)f(t)\ \text{ for a.e. } t\in\mathbb{R},\\ &f\in\operatorname{dom}(\boldsymbol{A})=\bigg\{g\in L_2(\mathbb{R},\mathcal{H})\,\bigg|\, g(t)\in\operatorname{dom}(A(t)) \text{ for a.e. } t\in\mathbb{R};\\ &\quad t\mapsto A(t)g(t) \text{ is (weakly) measurable; } \int_{\mathbb{R}}dt\,\|A(t)g(t)\|_{\mathcal{H}}^2<\infty\bigg\}.\end{aligned}$$

In addition, we introduce the operator $\boldsymbol{D_A}$ in $L_2(\mathbb{R},\mathcal{H})$ by

$$\boldsymbol{D_A}=\frac{d}{dt}+\boldsymbol{A},\quad \operatorname{dom}(\boldsymbol{D_A})=W^{1,2}(\mathbb{R},\mathcal{H})\cap\operatorname{dom}(\boldsymbol{A}_-).$$

Here the operator d/dt in $L_2(\mathbb{R},\mathcal{H})$ is defined by

$$\begin{aligned}&\left(\frac{d}{dt}f\right)(t)=f'(t)\ \text{ for a.e. } t\in\mathbb{R},\\ &f\in\operatorname{dom}(d/dt)=\big\{g\in L_2(\mathbb{R},\mathcal{H})\,\big|\, g\in AC_{\mathrm{loc}}\big(\mathbb{R},\mathcal{H}\big),\ g'\in L_2(\mathbb{R},\mathcal{H})\big\}\\ &\qquad\qquad\qquad =W^{1,2}\big(\mathbb{R},\mathcal{H}\big).\end{aligned}$$

For simplicity we also introduce the nonnegative, self-adjoint operators $\boldsymbol{H}_j$, $j=1,2$, in $L_2(\mathbb{R},\mathcal{H})$ by setting

$$\boldsymbol{H}_1=\boldsymbol{D_A^*}\boldsymbol{D_A},\quad \boldsymbol{H}_2=\boldsymbol{D_A}\boldsymbol{D_A^*}.$$

The crucial step in the proof of Theorem 0.1 (as well as Theorem 0.2) is the (resolvent version of) so-called principal trace formula which we state next.

Theorem 1.5.3. [**67**] *(see also* [**47**] *for relative trace-class setting) Assume that Pushnitski's assumption 1.5.1 holds. Then*

$$\big(g_z(A_+)-g_z(A_-)\big)\in\mathcal{L}_1(\mathcal{H}),\quad \big((\boldsymbol{H}_2-z)^{-1}-(\boldsymbol{H}_1-z)^{-1}\big)\in\mathcal{L}_1(L_2(\mathbb{R},\mathcal{H}))$$

and

$$(-z)\operatorname{tr}\big((\boldsymbol{H}_2-z)^{-1}-(\boldsymbol{H}_1-z)^{-1}\big)=-\frac{1}{2}\operatorname{tr}\big(g_z(A_+)-g_z(A_-)\big)\qquad(1.5.1)$$

for all $z\in\mathbb{C}\setminus[0,\infty)$.

In the setting of a finite-dimensional Hilbert space $\mathcal{H}$, this trace formula was proved for the first time by Callias [**23**] and for $\dim(\mathcal{H}) = 1$ in [**20**].

With assumption that $\boldsymbol{D_A}$ is a Fredholm operator, the left-hand side of (1.5.1) computes index$(\boldsymbol{D_A})$. In addition, the fact that B_+ is a trace-class operator implies that there exists unique spectral shift function $\xi(\cdot; A_+, A_-)$ for the pair (A_+, A_-). On the other hand, since $\big((\boldsymbol{H}_2 - z)^{-1} - (\boldsymbol{H}_1 - z)^{-1}\big) \in \mathcal{L}_1(L_2(\mathbb{R}, \mathcal{H}))$ there exists spectral shift function $\xi(\cdot; \boldsymbol{H}_2, \boldsymbol{H}_1)$ for the pair $(\boldsymbol{H}_2, \boldsymbol{H}_1)$. In addition, since $\boldsymbol{H}_2$ and $\boldsymbol{H}_1$ are nonnegative, the spectral shift function $\xi(\cdot; \boldsymbol{H}_2, \boldsymbol{H}_1)$ can be fixed (see (1.2.7)) by the requirement

$$\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) = 0, \quad \lambda < 0.$$

Using the Krein trace formula (see (1.2.2) and (1.2.5)) for both sides of the principal trace formula (1.5.1) one can write

$$\begin{aligned}
&\int_{[0,\infty)} \frac{\xi\big(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1\big)\, d\lambda}{(\lambda - z)^{-2}} \\
&\qquad = -\operatorname{tr}_{L_2(\mathbb{R},\mathcal{H})} \Big(\big(\boldsymbol{H}_2 - z\big)^{-1} - \big(\boldsymbol{H}_1 - z\big)^{-1}\Big) \\
&\qquad = -\frac{1}{2z} \operatorname{tr}_{\mathcal{H}} \big(g_z(A_+) - g_z(A_-)\big) \\
&\qquad = \frac{1}{2} \int_{\mathbb{R}} \frac{\xi(\nu; A_+, A_-)\, d\nu}{(\nu^2 - z)^{3/2}}, \quad z \in \mathbb{C}\backslash[0, \infty).
\end{aligned}$$

The latter equality implies the following result [**67**] (see also [**47**] for relative trace-class setting).

Theorem 1.5.4 (Pushnitski's formula). *Assume Hypothesis 1.5.1. We have*

$$\xi\big(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1\big) = \frac{1}{\pi} \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_+, A_-)\, d\nu}{(\lambda - \nu^2)^{1/2}}, \tag{1.5.2}$$

Pushnitski recognised that "index=spectral flow" can be interpreted as a particular limiting form of the general equality (1.5.2).

For our approach in the proof of our main result, Theorem 0.3, we aim for the same Pushnitski's formula (1.5.2), however, the employ quite different methods. We explain the outline of the proof in the next section.

1.6. The outline of the proof

As discussed in the previous section, the main step in the proof of Theorems 0.1 and 0.2 is the principal trace formula (1.5.1). However, as we show in Section 4.3 (see Theorem 4.3.10) the operator

$$g(A_+) - g(A_-)$$

is not necessarily trace-class operator in our setting. Therefore, the right-hand side of (1.5.1) is not well-defined in our case.

Furthermore, the difference of resolvents

$$\Big(\big(\boldsymbol{H}_2 - z\big)^{-1} - \big(\boldsymbol{H}_1 - z\big)^{-1}\Big)$$

is not necessarily trace-class operators for examples of higher-dimensional differential operators. Therefore, we can not consider the resolvent regularisation of the Witten index (1.4.1) for the operator $\boldsymbol{D_A}$. Hence, in the thesis we consider only the semigroup regularised Witten index. For that index we need to study the limit of

$$\operatorname{tr}\left(e^{-t\boldsymbol{H}_2} - e^{-t\boldsymbol{H}_1}\right)$$

as $t \to \infty$.

Since we aim to express this limit in terms of spectral shift function $\xi(\cdot; A_+, A_-)$ for the pair (A_+, A_-), we need to relate $\operatorname{tr}\left(e^{-t\boldsymbol{H}_2} - e^{-t\boldsymbol{H}_1}\right)$ with some expression containing A_+ and A_-. To this end we prove principle trace formula in its heat kernel version, namely, for all $t > 0$, we have

$$\operatorname{tr}\left(e^{-t\boldsymbol{H}_2} - e^{-t\boldsymbol{H}_1}\right) = -\Big(\frac{t}{\pi}\Big)^{1/2} \int_0^1 \operatorname{tr}\left(e^{-tA_s^2}(A_+ - A_-)\right) ds, \tag{1.6.1}$$

where $A_s = A_- + s(A_+ - A_-)$, $s \in [0, 1]$ is the straight line connecting A_+ and A_-. As in [**67**], [**47**] and [**31**], this version of the principal trace formula is the crucial step in the proof of Theorems 0.3 and 0.4.

This formula can be found in [**12**] under the assumption that the resolvent of operators $A_\pm$ are compact (and some additional 'summability' assumption). Here, we prove the formula without any assumptions on the spectra of the operators A_+ and A_-.

The key idea in the proof of (1.6.1) is an approximation argument and employment of double operator integrals (see Section 2.1). The first step is to introduce a spectral 'cut-off'

$$P_n = \chi_{[-n,n]}(A_-)$$

and the path $\{B_n(t)\}_{t\in\mathbb{R}}$ of reduced operators by setting

$$B_n(t) = P_n B(t) P_n, \quad n \in \mathbb{N},\ t \in \mathbb{R}.$$

The p-relative trace-class assumption on $B(t)$ guarantees that $\{B_n(t)\}_{t\in\mathbb{R}}$ is path of trace-class operators satisfying Pushnitski's assumption 1.5.1 (and therefore assumption of [**31**] too, see Hypothesis 1.5.2).

Employing results of [**31**] we write

$$\operatorname{tr}\left(e^{-t\boldsymbol{H}_{2,n}} - e^{-t\boldsymbol{H}_{1,n}}\right) = -\frac{1}{2}\operatorname{tr}\left(\operatorname{erf}(t^{1/2}A_{+,n}) - \operatorname{erf}(t^{1/2}A_-)\right), \tag{1.6.2}$$

where the erf stands for the error function (see (1.3.2)) and operators $\boldsymbol{H}_{j,n}$, $j = 1, 2$ denote the operators corresponding to the path $\{A_- + B_n(t)\}_{t\in\mathbb{R}}$.

One can think that we can approximate now both sides of equation (1.6.2) and obtain a more general version of the formula obtained in [**31**]. However, this is not true in general. Consider the case, when the operator A_- is the two-dimensional Dirac operator (see (4.3.1)) and the perturbed operator A_+ is given by

$$A_+ = A_- + 1 \otimes M_\varphi, \quad \varphi \in S(\mathbb{R}^2).$$

Clearly, $\operatorname{erf}' > 0$ is a Schwartz function. Hence, Theorem 4.3.14 below we have that the operator

$$\operatorname{erf}(t^{1/2}A_+) - \operatorname{erf}(t^{1/2}A_-)$$

is not a trace-class operator. Thus, on the right-hand side of (1.6.2) we can not pass to the limit, in general.

Using then Daletski-Krein formula (from the theory of double operator integrals) we then write

$$\frac{1}{2}\,\mathrm{tr}\,\Big(\,\mathrm{erf}(t^{1/2}A_{+,n}) - \mathrm{erf}(t^{1/2}A_{-})\Big) = \Big(\frac{t}{\pi}\Big)^{1/2}\int_0^1 \mathrm{tr}\,\Big(e^{-tA_{s,n}^2}(A_{+,n}-A_{-})\Big)ds,$$

with the convergent integral of the right-hand side. Thus, we obtain the required form of the principal trace formula for reduced operators.

$$\mathrm{tr}\,\Big(e^{-t\boldsymbol{H}_{2,n}} - e^{-t\boldsymbol{H}_{1,n}}\Big) = -\Big(\frac{t}{\pi}\Big)^{1/2}\int_0^1 \mathrm{tr}\,\Big(e^{-tA_{s,n}^2}(A_{+,n}-A_{-})\Big)ds.$$

Using again double operator integrals we show that both, left-hand and right-hand side yield the required limit as $n \to \infty$, thus proving that (see Section 5.1)

$$\mathrm{tr}\,\Big(e^{-t\boldsymbol{H}_{2}} - e^{-t\boldsymbol{H}_{1}}\Big) = -\Big(\frac{t}{\pi}\Big)^{1/2}\int_0^1 \mathrm{tr}\,\Big(e^{-tA_{s}^2}(A_{+}-A_{-})\Big)ds.$$

Having established the principal trace formula in its heat kernel version we use Laplace transform to show (see Section 6.1) that Pushnitski's formula (1.5.2) holds in our general framework. As an application of Pushnitski's formula we obtain Theorem 0.3.

To prove Theorem 0.4 we use again the approximation technique combined with the integral formula for spectral flow from Theorem 1.3.4.

CHAPTER 2

Double operator integrals

In this chapter we present the foundations of the theory of double operator integrals. We start with a detailed exposition of definition of double operator integrals given by Birman and Solomyak in [**15**]. We also discuss several sufficient conditions for a double operator integral to be a bounded mapping on the Schatten class $\mathcal{L}_p(\mathcal{H})$, $1 \leq p < \infty$ and on $\mathcal{B}(\mathcal{H})$.

Next, we give details of construction from [**83**] of double operator integrals build over spectral measures of self-adjoint operators such that the difference of high enough power of their resolvents fall into Schatten class. This detailed exposition is essential for the result of Section 2.3 on the limiting process for this types of double operator integrals. The main result of this section, Theorem 2.3.9, plays a crucial rule in the approximation of the left-hand side of the principal trace formula (1.6.2) (see discussion in Section 1.6). The results of Section 2.3 are presented in [**27**].

2.1. Definition of double operator integrals and their properties

In this section we give the definition of double operator integrals and present their basic properties. The theory of double operator integrals originated in [**15**], [**16**], [**17**] and has become an important tool in many areas of mathematics, most notably perturbation theory. There are different approaches to double operator integrals (see e.g. [**40, 60, 62, 9**]) and the choice of a particular approach depends on the question studied. In the present thesis we use double operator integrals as operators on $\mathcal{L}_1(H)$ only, and therefore, we recall here the classical definition of DOI due to Birman and Solomyak.

Firstly, we recall the definition of double operator integral as a mapping (transformator) on the Hilbert-Schmidt class $\mathcal{L}_2(\mathcal{H})$. Below we present the results for self-adjoint operators, but note that the same construction works for a pair of unitary operator U, V with the replacement of $\mathbb{R}$ by $\mathbb{T}$.

Suppose that A, B are self-adjoint operators with common dense domain. Denote by E and F the ($\mathcal{B}(H)$-valued) spectral measures on $\mathbb{R}$ of A and B, respectively. Consider the $\mathcal{B}(\mathcal{L}_2(H))$-valued measures on $\mathbb{R}$ defined by

$$\mathcal{E}(\sigma_1) : X \to E(\sigma_1)X,$$
$$\mathcal{F}(\sigma_2) : X \to XF(\sigma_2),$$

where σ_1, σ_2 are Borel sets in $\mathbb{R}$. It is clear that the $\mathcal{E}$ and $\mathcal{F}$ are commuting spectral measure on $\mathbb{R}$.

Define the product of two measures $\mathcal{E}$ and $\mathcal{F}$

$$\nu(\sigma_1 \times \sigma_2) = \mathcal{E}(\sigma_1)\mathcal{F}(\sigma_2),$$

that is

$$\nu(\sigma_1 \times \sigma_2)(X) = E(\sigma_1)XF(\sigma_2).$$

It is proved by Birman and Solomyak [**15**] that this is a countably additive (in the strong operator topology) projection-valued measure on $\mathbb{R}^2$, and therefore the following definition makes sense.

DEFINITION 2.1.1. *For a function $\psi \in L_\infty(\mathbb{R}^2, \nu)$ the double operator integral $T_\psi^{A,B} : \mathcal{L}_2(H) \to \mathcal{L}_2(H)$ is defined as the integral of the symbol ψ with respect to the spectral measure ν, that is*

$$T_\psi^{A,B}(X) := \int_{\mathbb{R}^2} \psi(\omega) d\nu(\omega)(X), \quad X \in \mathcal{L}_2(H).$$

The other frequently used notation is

$$T_\psi^{A,B}(X) := \int_{\mathbb{R}\times\mathbb{R}} \psi(\lambda, \mu) dE(\lambda) X dF(\mu), \quad X \in \mathcal{L}_2(H).$$

REMARK 2.1.2. For a DOI operator integral $T_\psi^{A,B}$ with symbol ψ the values of ψ outside some Borel subset $\mathcal{B} \in \mathbb{R}^2$ containing $\sigma(A) \cup \sigma(B)$ are inessential. Namely, if $\sigma(A) \cup \sigma(B) \subset \mathcal{B}$ and $\psi|_\mathcal{B}$ denotes the restriction of ψ onto $\mathcal{B}$, then

$$T_\psi^{A,B} = T_{\psi|_\mathcal{B}}^{A,B}.$$

In the special case, when the the operators A and B have discrete spectra, the definition of double operator integral becomes substantially easier. Indeed, suppose that A, B are self-adjoint operators with common dense domain and discrete spectra. Let $\{\lambda_i\}_{i\in\mathbb{N}}$ and $\{\mu_j\}_{j\in\mathbb{N}}$ denote the sequence of eigenvalues of A and B, respectively, with corresponding orthonormal bases of eigenvectors $\{p_i\}_{i\in\mathbb{N}}$ and $\{q_j\}_{j\in\mathbb{N}}$.

In this case, the product measure ν is supported on a discrete set $\{(\lambda_i, \mu_j),\, i, j \in \mathbb{N}\} \subset \mathbb{R}^2$ and for $\psi \in L_\infty(\mathbb{R}^2, \nu)$ we have

$$T_\psi^{A,B}(X) = \sum_{i,j} \psi(\lambda_i, \mu_j) P_{p_i}\, X\, P_{q_j},$$

where P_{p_i} and P_{q_j} denote the projections on the vector space spanned by p_i and q_j, respectively.

In particular,

$$\langle T_\psi^{A,B}(X) q_j, p_i\rangle = \psi(\lambda_i, \mu_j) x_{ij},$$

where

$$x_{ij} = \langle X(q_j), p_i\rangle,$$

is the representation of the operator $X \in \mathcal{L}_2(H)$ as an infinite matrix with respect to $\{p_i\}_{i\in\mathbb{N}}$ and $\{q_j\}_{j\in\mathbb{N}}$.

Thus, in the discrete case, the double operator integral $T_\psi^{A,B}(X)$ is simply the Schur product $\{\psi(\lambda_i, \mu_j)\}_{i,j} * X$ of the matrices $\{\psi(\lambda_i, \mu_j)\}_{i,j}$ and $X = \{x_{ij}\}_{i,j}$. Therefore, double operator integrals are considered as continuous version of Schur multipliers. In fact, using the decomposing the Hilbert space $\mathcal{H}$ into direct integral (with respect to $\mathcal{E}$ and $\mathcal{F}$) one can write the double operator integral as a multiplier transformation of the kernel on integral operators (see [**19**]).

The following proposition gathers the elementary properties of double operator integrals, which easily follow from the spectral theorem.

PROPOSITION 2.1.3. *Let A, B be arbitrary self-adjoint operators on H with common dense domain. Suppose that $\psi, \psi_1, \psi_2 \in L_\infty(\mathbb{R}^2, \nu)$. We have* *(i)* $T^{A,B}_{\psi_1+\psi_2} = T^{A,B}_{\psi_1} + T^{A,B}_{\psi_2}$.

(ii) $T^{A,B}_{\psi_1\psi_2} = T^{A,B}_{\psi_1} \circ T^{A,B}_{\psi_2}$.

(iii) if $\psi(t_1,t_2) = h_1(t_1)$, *then* $T^{A,B}_{\psi_1}(X) = h_1(A)X$.

(iv) if $\psi(t_1,t_2) = h_2(t_2)$, *then* $T^{A,B}_{\psi}(X) = Xh_2(B)$.

Next, we discuss double operator integrals on $\mathcal{B}(\mathcal{H})$. Recall that $\mathcal{B}(\mathcal{H})$ is adjoint to $\mathcal{L}_1(\mathcal{H})$ via trace duality given by

$$\langle T, S\rangle = \mathrm{tr}(TS^*), \quad T \in \mathcal{L}_1(\mathcal{H}), S \in \mathcal{B}(\mathcal{H}).$$

Since $\mathcal{L}_1(\mathcal{H}) \subset \mathcal{L}_2(\mathcal{H})$, one can easily conclude that $T^{A,B}_{\psi}(X) \in \mathcal{L}_2(\mathcal{H})$ for any $X \in \mathcal{L}_1(\mathcal{H})$ and $\psi \in L_\infty(\mathbb{R}^2,\nu)$. If, in addition, ψ is such that $T^{A,B}_{\psi}$ is a bounded operator on $\mathcal{L}_1(\mathcal{H})$, then $T^{A,B}_{\bar\psi}$ is also a bounded operator on $\mathcal{L}_1(\mathcal{H})$. Therefore, one can define the double operator integral $T^{A,B}_{\psi}$ on $\mathcal{B}(\mathcal{H})$ by duality

$$T^{A,B}_{\psi}(T) = (T^{A,B}_{\bar\psi})^*(T), \quad T \in \mathcal{B}(\mathcal{H}). \tag{2.1.1}$$

Thus, the definition of the double operator integral $T^{A,B}_{\bar\psi}$ on $\mathcal{B}(\mathcal{H})$ heavily relies on the fact that $T^{A,B}_{\bar\psi}$ is bounded operator on $\mathcal{L}_1(\mathcal{H})$. However, in contrast to the double operator integrals on $\mathcal{L}_2(\mathcal{H})$, the condition that $\psi \in L_\infty(\mathbb{R}^2,\nu)$ does not guarantee that $T^{A,B}_{\psi} \in \mathcal{B}(\mathcal{L}_1(\mathcal{H}))$. Below we will recall a result describing the class of functions ψ such that $T^{A,B}_{\psi} \in \mathcal{B}(\mathcal{L}_1(\mathcal{H}))$. We introduce

$$\begin{aligned} \mathfrak{M}_1 &:= \big\{\psi \in L_\infty(\mathbb{R}^2;\nu) \,\big|\, T^{A,B}_{\psi} \in \mathcal{B}(\mathcal{L}_1(\mathcal{H}))\big\},\\ \mathfrak{M}_\infty &:= \big\{\psi \in L_\infty(\mathbb{R}^2;\nu) \,\big|\, T^{A,B}_{\psi} \in \mathcal{B}(\mathcal{B}(\mathcal{H}))\big\}. \end{aligned} \tag{2.1.2}$$

In addition, we set

$$\|\psi\|_{\mathfrak{M}_1} := \|T^{A,B}_{\psi}\|_{\mathcal{B}(\mathcal{L}_1(\mathcal{H}))}, \quad \|\psi\|_{\mathfrak{M}_\infty} := \|T^{A,B}_{\psi}\|_{\mathcal{B}(\mathcal{B}(\mathcal{H}))}.$$

It follows from the definition that

$$\mathfrak{M} := \mathfrak{M}_1 = \mathfrak{M}_\infty,$$

and

$$\|\psi\|_{\mathfrak{M}} := \|\psi\|_{\mathfrak{M}_1} = \|\psi\|_{\mathfrak{M}_\infty}, \quad \psi \in \mathfrak{M}.$$

We recall the following result.

THEOREM 2.1.4. [**15, 17, 60**] *(see also* [**19**, Theorem 4.1]*) The following conditions are equivalent:*

(i) $\psi \in \mathfrak{M}$*;*

(ii) The function $\psi(\,\cdot\,,\,\cdot\,)$ *admits a representation of the form*

$$\psi(\lambda,\mu) = \int_\Omega \alpha(\lambda,t)\beta(\mu,t)\,d\eta(t), \quad (\lambda,\mu) \in \mathbb{R}^2,$$

where $(\Omega, d\eta(t))$ *is an auxiliary measure space and*

$$C_\alpha^2 := \sup_{\lambda\in\mathbb{R}} \int_\Omega |\alpha(\lambda,t)|^2\,d\eta(t) < \infty, \quad C_\beta^2 := \sup_{\mu\in\mathbb{R}} \int_\Omega |\beta(\mu,t)|^2\,d\eta(t) < \infty.$$

In this case,

$$\|\psi\|_{\mathfrak{M}} \le C_\alpha C_\beta.$$

Next we recall an important equality, which is one of the main reasons why the theory of double operator integrals is so fruitful.

Suppose that f is a differential function on $\mathbb{R}$. Define the divided difference

$$f^{[1]}(\lambda,\mu) := \begin{cases} \frac{f(\lambda)-f(\mu)}{\lambda-\mu}, & \text{if } \lambda \neq \mu \\ f'(\lambda), & \text{if } \lambda = \mu, \quad \lambda,\mu \in \mathbb{R}. \end{cases} \tag{2.1.3}$$

We note that particular value of $f^{[1]}$ on the diagonal is inessential.

By the Mean Value Theorem, the function $f^{[1]}$ is bounded, and so the DOI $T^{A,B}_{f^{[1]}}$ is a bounded operator on $\mathcal{L}_2(\mathcal{H})$.

THEOREM 2.1.5. [**17**, Theorem 4.5] *Let A, B be self-adjoint operators on $\mathcal{H}$ with common domain such that $A - B \in \mathcal{L}_1(\mathcal{H})$ (respectively, $A - B \in \mathcal{B}(\mathcal{H})$). Suppose also that function f on $\mathbb{R}$ is such that $f^{[1]} \in \mathfrak{M}$. Then*

$$f(B) - f(A) = T^{A,B}_{f^{[1]}}(B - A).$$

In particular, $f(B) - f(A) \in \mathcal{L}_1(\mathcal{H})$ (respectively, $f(B) - f(A) \in \mathcal{B}(\mathcal{H})$) with

$$\|f(B) - f(A)\|_1 \leq \|f^{[1]}\|_{\mathfrak{M}}\|B - A\|_1$$

(and

$$\|f(B) - f(A)\| \leq \|f^{[1]}\|_{\mathfrak{M}}\|B - A\|,$$

respectively).

Next, we recall a sufficient condition on a function f, so that the double operator integral with symbol $f^{[1]}$ is bounded on $\mathcal{L}_1(\mathcal{H})$ and on $\mathcal{B}(\mathcal{H})$.

Recall that we say that a function f on $\mathbb{R}$ is of Hölder class α, $0 \leq \alpha \leq 1$ if

$$\|f\|_{\Lambda_\alpha} = \sup_{t_1,t_2} \frac{|f(t_1) - f(t_2)|}{|t_1 - t_2|^\alpha} < \infty.$$

THEOREM 2.1.6. [**64**, Theorem 4 and Corollary 2] *Let $f : \mathbb{R} \to \mathbb{C}$. Assume that for some $0 \leq \theta < 1$ and $0 < \varepsilon \leq 1$ we have $\|f\|_{\Lambda_\theta}, \|f'\|_\infty, \|f'\|_{\Lambda_\varepsilon} < \infty$. Then the double operator integral $T^{A,B}_{f^{[1]}}$ is bounded on $\mathcal{L}_1(\mathcal{H})$ and on $\mathcal{B}(\mathcal{H})$.*

We also recall (see e.g. [**63**]) that if $X \in \mathcal{L}_1(\mathcal{H})$, $V \in \mathcal{B}(\mathcal{H})$ and $\psi \in \mathfrak{M}$ then

$$\operatorname{tr}(T^{A,B}_\psi(X) \cdot V) = \operatorname{tr}(X \cdot T^{A,B}_\psi(V)). \tag{2.1.4}$$

In addition, if $f^{[1]} \in \mathfrak{M}$, then

$$T^{A,A}_{f^{[1]}}(1) = f'(A). \tag{2.1.5}$$

Introduce the class

$$\mathfrak{M}_p := \{\psi \in L_\infty(\mathbb{R}^2;\nu) \,|\, T^{A,B}_\psi \in \mathcal{B}(\mathcal{L}_p(\mathcal{H}))\}, \quad p \in (1,\infty),$$

with

$$\|\psi\|_{\mathfrak{M}_p} := \|T^{A,B}_\psi\|_{\mathcal{B}(\mathcal{L}_p(\mathcal{H}))},\ p \in (1,\infty).$$

REMARK 2.1.7. By interpolation, the inclusion $\psi \in \mathfrak{M}$ implies that $\psi \in \mathfrak{M}_p$ for any $p \in (1,\infty)$, and $\|\psi\|_{\mathfrak{M}_p} \leq \|\psi\|_{\mathfrak{M}}$, $p \in (1,\infty)$.

However, for $p \in (1,\infty)$ the class $\mathfrak{M}_p$ is strictly larger that the class $\mathfrak{M}$. Indeed, it is proved in [**65**] that for any Lipschitz function f, its divided difference $f^{[1]}$ belongs to the class $\mathfrak{M}_p$ for any $1 < p < \infty$. However, as shown in [**43**] there exists a Lipschitz function f with $f^{[1]} \notin \mathfrak{M}$.

For future purposes we also recall the following result from [**47**]. Define the function

$$\varphi(\lambda,\mu) := \frac{\lambda(\lambda^2+1)^{-1/2} - \mu(\mu^2+1)^{-1/2}}{(\lambda^2+1)^{-1/4}\,(\lambda-\mu)\,(\mu^2+1)^{-1/4}}, \quad (\lambda,\mu)\in\mathbb{R}^2. \tag{2.1.6}$$

Note that

$$\varphi(\lambda,\mu) = (\lambda^2+1)^{1/4}\cdot g^{[1]}(\lambda,\mu)\cdot(\mu^2+1)^{1/4},$$

where g is defined by (1.1.2).

THEOREM 2.1.8. [**47**, Lemma 6.6] *Suppose that A, B are self-adjoint operators such that $A-B\in\mathcal{L}_p(\mathcal{H})$ $1\le p<\infty$. The double operator integral $T_\varphi^{A,B}$ with function φ defined by* (2.1.6) *is bounded on $\mathcal{L}_p(\mathcal{H})$, $1\le p<\infty$ and on $\mathcal{B}(\mathcal{H})$ and*

$$g(A)-g(B) = T_\varphi^{A,B}\big((A^2+1)^{-1/4}(A-B)(B^2+1)^{1/4}\big).$$

To conclude this section we also recall a result for boundedness of double operator integral $T_\psi^{U,V}$ built over two unitary operators U, V.

THEOREM 2.1.9. [**15**, Theorem 11] *Let U, V be unitary operator on $\mathcal{H}$ and let g be a function on $\mathbb{T}$ such that g' satisfies Hölder condition with exponent $\varepsilon>0$. Then the double operator integral $T_{g^{[1]}}^{U,V}$ is bounded on $\mathcal{L}_p(\mathcal{H})$, $p\in[1,\infty)$ and on $\mathcal{B}(\mathcal{H})$.*

2.2. Double operator integrals for resolvent comparable operators

The result of Theorem 2.1.5 says that for sufficiently nice function f, a bounded perturbation $A-B$ for self-adjoint operators A, B with common domain, give bounded $f(A)-f(B)$. In this section we consider double operator integrals such that bounded perturbation $A-B$ with some additional resolvent comparability condition is mapped to $f(A)-f(B)$ belonging to some Schatten class. The results of this section are proved in [**83**]. However, for the proof of Theorem 2.3.9 we require the details of the construction from [**83**], and therefore, we present it in full details. The results of this section are also presented in [**27**].

In the proof of the main theorem of this section, we need two results from [**83**] and [**19**]. Since these results were stated without proof in those papers, we now present a proof for convenience of the reader.

THEOREM 2.2.1. [**19**, Theorem 5.2] *Suppose that there exist $0\le m_1<1$ and $1<m_2$ such that*

$$\sup_{\mu\in\mathbb{R}}\int_{\mathbb{R}}\big(|\xi|^{m_1}+|\xi|^{m_2}\big)\big|\widehat{\psi}(\xi,\mu)\big|^2\,d\xi = C_0^2<\infty, \tag{2.2.1}$$

where $\widehat{\psi}(\xi,\mu)$ stands for the partial Fourier transform of ψ with respect to the first variable,

$$\widehat{\psi}(\xi,\mu) = (2\pi)^{-1}\int_{\mathbb{R}}\psi(\lambda,\mu)e^{-i\xi\lambda}\,d\lambda, \quad (\xi,\mu)\in\mathbb{R}^2.$$

Then $\psi\in\mathfrak{M}$ and

$$\|\psi\|_{\mathfrak{M}}\le \operatorname{const} C_0,$$

where the constant depends on m_1 and m_2 only.

PROOF. In view of
$$m_1 < 1 < m_2,$$
one obtains
$$\int_{\mathbb{R}} \big(|\xi|^{m_1} + |\xi|^{m_2}\big)^{-1} d\xi = 2\int_0^{+\infty} \frac{dr}{|r|^{m_1} + |r|^{m_2}} =: C^2 \in (0,\infty). \tag{2.2.2}$$
That is, $f_{m_1,m_2}(\xi) = (|\xi|^{m_1} + |\xi|^{m_2})^{-1/2}$, $m_1 < 1 < m_2$, satisfies $f_{m_1,m_2} \in L_2(\mathbb{R})$. Therefore, by (2.2.1) and Hölder's inequality, one obtains
$$\begin{aligned}\int_{\mathbb{R}} \big|\widehat{\psi}(\xi,\mu)\big|\, d\xi &= \int_{\mathbb{R}} \Big[\big(|\xi|^{m_1} + |\xi|^{m_2}\big)^{\frac12} \big|\widehat{\psi}(\xi,\mu)\big|\Big] \big(|\xi|^{m_1} + |\xi|^{m_2}\big)^{-\frac12} d\xi \\ &\le \bigg(\int_{\mathbb{R}} \Big[\big(|\xi|^{m_1} + |\xi|^{m_2}\big)^{\frac12} \big|\widehat{\psi}(\xi,\mu)\big|\Big]^2 d\xi\bigg)^{1/2} \bigg(\int_{\mathbb{R}} \big(|\xi|^{m_1} + |\xi|^{m_2}\big)^{-1} d\xi\bigg)^{1/2} \\ &\le C_0 \bigg(\int_{\mathbb{R}} \big(|\xi|^{m_1} + |\xi|^{m_2}\big)^{-1} d\xi\bigg)^{1/2} \overset{(2.2.2)}{=} C_0\, C \end{aligned} \tag{2.2.3}$$
uniformly for $\mu \in \mathbb{R}$. Hence,
$$\widehat{\psi}(\,\cdot\,,\mu) \in L_1(\mathbb{R}), \tag{2.2.4}$$
and
$$\sup_{\mu\in\mathbb{R}} \big\|\widehat{\psi}(\,\cdot\,,\mu)\big\|_{L_1(\mathbb{R})} < \infty. \tag{2.2.5}$$
By the inverse Fourier transform theorem
$$\begin{aligned}\psi(\lambda,\mu) &= \int_{\mathbb{R}} \widehat{\psi}(\xi,\mu)\, e^{i\xi\lambda}\, d\xi \\ &= \int_{\mathbb{R}} e^{i\xi\lambda} \big(|\xi|^{m_1} + |\xi|^{m_2}\big)^{-1/2} \cdot \Big[\big(|\xi|^{m_1} + |\xi|^{m_2}\big)^{1/2} \widehat{\psi}(\xi,\mu)\Big]\, d\xi.\end{aligned} \tag{2.2.6}$$
Thus, introducing the functions
$$\alpha(\lambda,\xi) = e^{i\lambda\xi}\big(|\xi|^{m_1} + |\xi|^{m_2}\big)^{-\frac12}, \quad \beta(\mu,\xi) = \big(|\xi|^{m_1} + |\xi|^{m_2}\big)^{\frac12}\, \widehat{\psi}(\xi,\mu). \tag{2.2.7}$$
we have
$$\psi(\lambda,\mu) = \int_{\mathbb{R}} \alpha(\lambda,\xi)\beta(\mu,\xi) d\xi.$$
Moreover, by (2.2.1) and (2.2.2), the functions α and β satisfy the condition of Theorem 2.1.4 with respect to the measure space $(\Omega, d\eta(t)) = (\mathbb{R}, d\xi)$. Hence, by Theorem 2.1.4, $\psi \in \mathfrak{M}$ and $\|\psi\|_{\mathfrak{M}} \le CC_0$, where the constant C depends on m_1 and m_2 only. □

PROPOSITION 2.2.2. [**83**, Proposition 3.1] *Assume that A and B are self-adjoint operators in $\mathcal{H}$. Suppose that function $K(\cdot,\cdot)$ on $\mathbb{R}^2$ satisfies*
$$|K(\lambda,\mu)| \le C_K < \infty, \quad (\lambda,\mu) \in \mathbb{R}^2, \tag{2.2.8}$$
and is differentiable with respect to λ with
$$\left|\frac{\partial K(\lambda,\mu)}{\partial\lambda}\right| \le \widetilde{C}_K\big(1+\lambda^2\big)^{-1}, \quad (\lambda,\mu) \in \mathbb{R}^2, \tag{2.2.9}$$
where the constant $\widetilde{C}_K$ is independent of μ. Assume, in addition, that for every fixed $\mu \in \mathbb{R}$
$$\lim_{\lambda\to-\infty} K(\lambda,\mu) = \lim_{\lambda\to+\infty} K(\lambda,\mu), \tag{2.2.10}$$

where the limits exist by (2.2.9). *Then* $T_K^{A,B} \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ *and* $T_K^{A,B} \in \mathcal{B}(\mathcal{L}_p(\mathcal{H}))$, $p \in [1, \infty)$.

Proof. By (2.2.8) and (2.2.10) the function

$$k(\mu) := \lim_{\lambda \to \pm\infty} K(\lambda, \mu), \quad \mu \in \mathbb{R}, \tag{2.2.11}$$

is well-defined and bounded on $\mathbb{R}$.

We set

$$K_0(\lambda, \mu) := K(\lambda, \mu) - k(\mu), \quad (\lambda, \mu) \in \mathbb{R}^2, \tag{2.2.12}$$

and claim that this function satisfies the conditions of Theorem 2.2.1. Indeed, since

$$\frac{\partial K_0}{\partial \lambda} = \frac{\partial K}{\partial \lambda}, \tag{2.2.13}$$

one infers from (2.2.9) that

$$\frac{\partial K_0}{\partial \lambda}(\,\cdot\,, \mu) \in L_2(\mathbb{R}), \quad \mu \in \mathbb{R}, \text{ with } \sup_{\mu \in \mathbb{R}} \left\| \frac{\partial K_0}{\partial \lambda}(\,\cdot\,, \mu) \right\|_{L_2(\mathbb{R})} < \infty. \tag{2.2.14}$$

Furthermore, by the definition of the function K_0,

$$\lim_{\lambda \to \pm\infty} K_0(\lambda, \mu) = 0,$$

and therefore,

$$K_0(\lambda, \mu) = \begin{cases} -\displaystyle\int_\lambda^{+\infty} \frac{\partial K_0}{\partial \lambda}(t, \mu)\, dt, & \lambda > 0, \\ \displaystyle\int_{-\infty}^{\lambda} \frac{\partial K_0}{\partial \lambda}(t, \mu)\, dt, & \lambda < 0. \end{cases}$$

Hence, by (2.2.9) for $\lambda > 0$,

$$|K_0(\lambda, \mu)| \leq \int_\lambda^{+\infty} \left| \frac{\partial K_0}{\partial \lambda}(t, \mu) \right| dt \leq C \int_\lambda^{+\infty} (1 + t^2)^{-1}\, dt,$$

for an appropriate constant $C > 0$. A similar estimate for $\lambda < 0$ yields

$$K_0(\lambda, \mu) = O\big(|\lambda|^{-1}\big) \text{ as } \lambda \to \pm\infty,$$

uniformly for $\mu \in \mathbb{R}$. Hence, $K_0(\cdot, \mu) \in L_2(\mathbb{R})$ and by Parseval's identity, one obtains

$$\sup_{\mu \in \mathbb{R}} \int_{\mathbb{R}} |\xi|^2 \big| \widehat{K}_0(\xi, \mu) \big|^2\, d\xi < \infty. \tag{2.2.15}$$

That is, the function $K_0(\cdot, \cdot)$ satisfies the condition in Theorem 2.2.1 with

$$m_1 = 0 \text{ and } m_2 = 2. \tag{2.2.16}$$

Hence, Theorem 2.2.1 implies that the operator $T_{K_0}^{A,B} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is bounded. Furthermore, since $K(\lambda, \mu) = K_0(\lambda, \mu) + k(\mu)$, $(\lambda, \mu) \in \mathbb{R}^2$, Proposition 2.1.3 implies

$$T_K^{A,B}(X) = T_{K_0}^{A,B}(X) + Xk(B), \quad X \in \mathcal{B}(\mathcal{H}). \tag{2.2.17}$$

Since the function k is bounded (see (2.2.11)) one infers that the operator $T_K^{A,B}$ is bounded on $\mathcal{B}(\mathcal{H})$. Finally, Remark 2.1.7 implies that the operator $T_K^{A,B}$ is also bounded on any $\mathcal{L}_p(\mathcal{H})$, $p \in [1, \infty)$. $\square$

Corollary 2.2.3. *The norms* $\|T_K^{A,B}\|_{\mathcal{B}(\mathcal{B}(\mathcal{H}))}$, $\|T_K^{A,B}\|_{\mathcal{B}(\mathcal{L}_p(\mathcal{H}))}$, $p \in [1, \infty)$, *do not depend on the spectral measures* E_A *and* E_B.

PROOF. This follows from the proof of Proposition 2.2.2 and Theorem 2.2.1. □

To prove the norm bounds required for the proof of Proposition 4.1.6, we now introduce the following notion. It should be understood by reference to the classical 'resolvent comparability' in [**82**].

DEFINITION 2.2.4. *Let $m \in \mathbb{N}$ and let $p \in [1, \infty)$. Assume that A and B are self-adjoint operators in the Hilbert space $\mathcal{H}$. We say that A, B are m-resolvent comparable in $\mathcal{L}_p(\mathcal{H})$ if for all $a \in \mathbb{R}\backslash\{0\}$, we have*

$$\big[(B - ai)^{-m} - (A - ai)^{-m}\big] \in \mathcal{L}_p\big(\mathcal{H}\big). \tag{2.2.18}$$

If S and T are self-adjoint operators in $\mathcal{H}$ and for some $z_0 \in \mathbb{C}\backslash\mathbb{R}$,

$$[(S - z_0)^{-1} - (T - z_0)^{-1}] \in \mathcal{L}_1(\mathcal{H}), \tag{2.2.19}$$

then actually (see Proposition 1.4.1)

$$[(S - z)^{-1} - (T - z)^{-1}] \in \mathcal{L}_1(\mathcal{H}), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

However, an analogous result does not hold for higher powers of the resolvent as the following remarkably simple example illustrates.

EXAMPLE 2.2.5. [**27**] *Suppose $\mathcal{H}$ is an infinite-dimensional Hilbert space, and let $P_j \in \mathcal{B}(\mathcal{H})$, $j \in \{1, 2\}$, be infinite-dimensional orthogonal projections satisfying*

$$P_1 P_2 = 0 \quad \text{and} \quad P_1 + P_2 = 1.$$

Set

$$A = \sqrt{3}(P_1 + P_2), \quad B = \sqrt{3}(P_1 - P_2).$$

Evidently, $A^2 = B^2 = 3$, and

$$(A - i)^3 = A^3 - 3iA^2 + 3(-i)^2 A - i^3 = -8i.$$

Similarly, one obtains $(B - i)^3 = -8i$, and consequently,

$$(A - i)^{-3} - (B - i)^{-3} = 0 \in \mathcal{L}_1(\mathcal{H}).$$

However, if $z \in \mathbb{C}\backslash\{i\}$, then

$$(A + z)^3 = A^3 + 3zA^2 + 3z^2 A + z^3 \tag{2.2.20}$$

Taking, for example, $z = 3i$ in (2.2.20), one computes

$$(A + z)^3 = A(A^2 + 3z^2) + z(3A^2 + z^2) = -24A,$$

and similarly,

$$(B + 3i)^3 = -24B.$$

Computing inverses, one infers

$$(A + 3i)^{-3} = -\frac{1}{24}A^{-1} = -\frac{1}{24\sqrt{3}}(P_1 + P_2),$$
$$(B + 3i)^{-3} = -\frac{1}{24}B^{-1} = -\frac{1}{24\sqrt{3}}(P_1 - P_2),$$

so that

$$(A + 3i)^{-3} - (B + 3i)^{-3} = -\frac{1}{12\sqrt{3}}P_2 \notin \mathcal{L}_1(\mathcal{H}),$$

due to the fact that P_2 is an infinite-dimensional projection in $\mathcal{H}$.

Due to these reasons in the definition of higher power resolvent comparable operators we assume inclusion (2.2.18) for all $z = ai$, $a \neq 0$, whenever $m \geq 2$.

REMARK 2.2.6. In connection with Definition 2.2.4, we recall that a Cauchy-type formula implies the following elementary fact (cf. [**82**, p. 210])): Let A, B, be self-adjoint operators in some complex, separable Hilbert space $\mathcal{H}$. If

$$\big[(A-z)^{-m} - (B-z)^{-m}\big] \in \mathcal{L}_p(\mathcal{H}), \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{2.2.21}$$

for some $p \in [1, \infty)$ and some $m \in \mathbb{N}$, then

$$\big[(A-z)^{-n} - (B-z)^{-n}\big] \in \mathcal{L}_p(\mathcal{H}), \quad z \in \mathbb{C}\backslash\mathbb{R},\ n \geq m. \tag{2.2.22}$$

In the case where A and B are bounded from below, see also [**82**, Proposition 8.9.2]. Hence, if the operators A and B are m-resolvent com[arable in $\mathcal{L}_p(\mathcal{H})$, then A and B are n-resolvent comparable for any $n \geq m$.

For the rest of this section we assume that A and B are m-resolvent comparable operators in $\mathcal{L}_p(\mathcal{H})$ for some *odd* $m \in \mathbb{N}$.

The following construction is taken from [**83**]. Fix a bijection $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying for some $c > 0$ and $r > 0$,

$$\varphi \in C^2(\mathbb{R}), \quad \varphi(\lambda) = \lambda^m,\ |\lambda| \geq r, \quad \varphi'(\lambda) \geq c,\ \lambda \in \mathbb{R}. \tag{2.2.23}$$

Let $r > 0$ be such that $\varphi(\lambda) = \lambda^m$ for $|\lambda| \geq r$. We choose a function $\theta \in C^2(\mathbb{R})$ such that $\theta(\lambda) = 0$ for $|\lambda| \leq r/2$, $\theta(\lambda) = 1$ for $|\lambda| \geq r$ and

$$\frac{1}{\varphi(\lambda) - i} = \theta(\lambda)\frac{1}{\lambda^m - i} + (1 - \theta(\lambda))\frac{1}{\varphi(\lambda) - i} =: g_1(\lambda) + g_2(\lambda), \quad \lambda \in \mathbb{R}. \tag{2.2.24}$$

We note that $g_2 \in C^2(\mathbb{R})$ has compact support.

Thus,

$$(\varphi(A) - i)^{-1} - (\varphi(B) - i)^{-1} = g_1(A) - g_1(B) + g_2(A) - g_2(B). \tag{2.2.25}$$

Next, we denote

$$\begin{aligned} G_{1,a}(\lambda, \mu) &= \frac{g_1(\lambda) - g_1(\mu)}{(\lambda - ia)^{-m} - (\mu - ia)^{-m}}, \\ G_{2,a}(\lambda, \mu) &= \frac{g_2(\lambda) - g_2(\mu)}{(\lambda - ia)^{-m} - (\mu - ia)^{-m}}, \quad \lambda, \mu \in \mathbb{R}, \end{aligned} \tag{2.2.26}$$

where $a \in \mathbb{R}\backslash\{0\}$. In [**83**, Proposition 3.3] it is proved that there exists a (sufficiently small) $a_1 \in \mathbb{R}\backslash\{0\}$, such that the function G_{1,a_1} satisfies the assumption of Proposition 2.2.2. Therefore, Proposition 2.2.2 implies that

$$g_1(A) - g_1(B) = T^{A,B}_{G_{1,a_1}}\big((A - a_1 i)^{-m} - (B - a_1 i)^{-m}\big) \tag{2.2.27}$$

and

$$\|g_1(A) - g_1(B)\|_p \leq C_1 \big\|(A - a_1 i)^{-m} - (B - a_1 i)^{-m}\big\|_p, \tag{2.2.28}$$

for some constant $C_1 = C_1(a_1, m) \in (0, \infty)$ (and a corresponding estimate for the $\mathcal{B}(\mathcal{H})$-norm). Moreover, in [**83**, Proposition 3.2] it is proved that there exists a (sufficiently large) $a_2 \in \mathbb{R}\backslash\{0\}$, such that the function G_{2,a_2} satisfies the assumption of Proposition 2.2.2. Therefore,

$$g_2(A) - g_2(B) = T^{A,B}_{G_{2,a_2}}\big((A - a_2 i)^{-m} - (B - a_2 i)^{-m}\big) \tag{2.2.29}$$

and

$$\|g_2(A) - g_2(B)\|_p \le C_2 \big\|(A - a_2 i)^{-m} - (B - a_2 i)^{-m}\big\|_p \tag{2.2.30}$$

for some constant $C_2 = C_2(a_2, m) \in (0, \infty)$ (and a corresponding estimate for the $\mathcal{B}(\mathcal{H})$-norm).

Combining this with (2.2.25) one arrives at the following result.

Proposition 2.2.7. [**83**] *Suppose that A, B are m-resolvent comparable operators in $\mathcal{L}_p(\mathcal{H})$ for some odd $m \in \mathbb{N}$ and let functions $\varphi, G_{1,a}, G_{2,a}$ be as in (2.2.23) and (2.2.26), respectively. Then there exist $a_1, a_2 \in \mathbb{R}\backslash\{0\}$ such that the double operator integrals $T^{A,B}_{G_{1,a_1}}$ and $T^{A,B}_{G_{2,a_2}}$ are bounded on $\mathcal{L}_p(\mathcal{H})$ and on $\mathcal{B}(\mathcal{H})$ and*

$$\begin{aligned}(\varphi(A) - i)^{-1} &- (\varphi(B) - i)^{-1}\\ &= T^{A,B}_{G_{1,a_1}}\big((A - a_1 i)^{-m} - (B - a_1 i)^{-m}\big)\\ &\quad + T^{A,B}_{G_{2,a_2}}\big((A - a_2 i)^{-m} - (B - a_2 i)^{-m}\big)\end{aligned} \tag{2.2.31}$$

Proposition 2.2.7 implies that there exists $C = C(a_1, a_2, m) \in (0, \infty)$ such that

$$\begin{aligned}\big\|(\varphi(A) &- i)^{-1} - (\varphi(B) - i)^{-1}\big\|_p\\ &\le C\Big(\big\|(A - a_1 i)^{-m} - (B - a_1 i)^{-m}\big\|_p\\ &\quad + \big\|(A - a_2 i)^{-m} - (B - a_2 i)^{-m}\big\|_p\Big).\end{aligned} \tag{2.2.32}$$

In what follows we fix $a_1, a_2 \in \mathbb{R} \setminus \{0\}$ from Proposition 2.2.7.

Next, we introduce the class of functions for which we the main results of this and the next section hold.

Definition 2.2.8. [**83**] *Let $m \in \mathbb{N}$. Define the class of functions $\mathfrak{F}_m(\mathbb{R})$ by*

$$\begin{aligned}\mathfrak{F}_m(\mathbb{R}) := \big\{f \in C^2(\mathbb{R}) \,\big|\, f^{(\ell)} \in L_\infty(\mathbb{R});\ &\text{there exists } \varepsilon > 0 \text{ and } f_0 \in \mathbb{C}\\ \text{such that } \big(d^\ell/d\lambda^\ell\big)\big[f(\lambda) - f_0\lambda^{-m}\big] &\underset{|\lambda|\to\infty}{=} O\big(|\lambda|^{-\ell-m-\varepsilon}\big),\ \ell = 0, 1, 2\big\}.\end{aligned} \tag{2.2.33}$$

(*It is implied that f_0 is the same as $\lambda \to \pm\infty$.*)

In particular, one notes that for all $m \in \mathbb{N}$,

$$S(\mathbb{R}) \subset \mathfrak{F}_m(\mathbb{R}), \tag{2.2.34}$$

and

$$f(\lambda) \underset{|\lambda|\to\infty}{=} f_0\lambda^{-m} + O\big(|\lambda|^{-m-\epsilon}\big), \quad f \in \mathfrak{F}_m(\mathbb{R}). \tag{2.2.35}$$

Let $f \in \mathfrak{F}_m(\mathbb{R})$ and let φ be as before (see (2.2.23)). The assumptions on the functions φ and f imply that $f_0 := f \circ \varphi^{-1} \in \mathfrak{F}_1(\mathbb{R})$ (see [**83**]). It follows from the discussion before [**82**, Theorem 8.7.1] that there is a continuously differentiable function h on $\mathbb{T}$, with h' satisfying the Hölder condition with exponent $\varepsilon > 0$, such that

$$f_0(\lambda) = h(\gamma(\lambda)), \tag{2.2.36}$$

where $\gamma(\lambda) = \frac{\lambda+i}{\lambda-i}$, $\lambda \in \mathbb{R}$, denotes the Cayley transform.

We denote

$$U_A = \gamma(\varphi(A)), \quad U_B = \gamma(\varphi(B)). \tag{2.2.37}$$

By Proposition 2.2.7, there exist $a_1, a_2 \in \mathbb{R}\backslash\{0\}$ and a constant $C = C(a_1, a_2, m) > 0$ such that

$$\begin{aligned} U_A - U_B &= 2i\big((\varphi(A) - i)^{-1} - (\varphi(B) - i)^{-1}\big) \\ &= 2iT^{A,B}_{G_{1,a_1}}\left((A - a_1 i)^{-m} - (B - a_1 i)^{-m}\right) \\ &\quad + 2iT^{A,B}_{G_{2,a_2}}\left((A - a_2 i)^{-m} - (B - a_2 i)^{-m}\right) \end{aligned} \tag{2.2.38}$$

with $T^{A,B}_{G_{j,a_j}} \in \mathcal{B}(\mathcal{L}_p(\mathcal{H}))$, $j = 1, 2$, and

$$\begin{aligned} \|U_A - U_B\|_p \le 2C\Big(&\big\|(A - a_1 i)^{-m} - (B - a_1 i)^{-m}\big\|_p \\ &+ \big\|(A - a_2 i)^{-m} - (B - a_2 i)^{-m}\big\|_p\Big). \end{aligned} \tag{2.2.39}$$

Since h' satisfies the Hölder condition with exponent $\varepsilon > 0$, Theorem 2.1.9 implies that the double operator integral $T^{U,V}_{h^{[1]}}$ is bounded on $\mathcal{L}_p(\mathcal{H})$, $p \in [1, \infty)$. Thus,

$$\begin{aligned} f(A) - f(B) = f_0(\varphi(A)) - f_0(\varphi(B)) &= h(U_A) - h(U_B) \\ &= T^{U_A,U_B}_{h^{[1]}}(U_A - U_B). \end{aligned}$$

Therefore, recalling (2.2.38)

$$\begin{aligned} f(A) - f(B) &= T^{U_A,U_B}_{h^{[1]}}(U_A - U_B) \\ &= 2i(T^{U_A,U_B}_{h^{[1]}} \circ T^{A,B}_{G_{1,a_1}})\big((A - a_1 i)^{-m} - (B - a_1 i)^{-m}\big) \\ &\quad + 2i(T^{U,V}_{h^{[1]}} \circ T^{A,B}_{G_{2,a_2}})\big((A - a_2 i)^{-m} - (B - a_2 i)^{-m}\big). \end{aligned} \tag{2.2.40}$$

In particular, $[f(A) - f(B)] \in \mathcal{L}_p(\mathcal{H})$ and

$$\begin{aligned} \big\|f(A) - f(B)\|_p &\le \|T^{U_A,U_B}_{h^{[1]}}\|_{\mathcal{B}(\mathcal{L}_p(\mathcal{H}))}\|U_A - U_B\|_p \\ &\le \text{const}\,\Big(\big\|(A - a_1 i)^{-m} - (B - a_1 i)^{-m}\big\|_p \\ &\quad + \big\|(A - a_2 i)^{-m} - (B - a_2 i)^{-m}\big\|_p\Big), \quad f \in \mathfrak{F}_m(\mathbb{R}). \end{aligned} \tag{2.2.41}$$

Here the constant $C = C(f, a_1, a_2, m) \in (0, \infty)$ is independent of $p \in [1, \infty)$.

We summarise the construction in the following

Definition 2.2.9. *Assume that self-adjoint operators A, B are m-resolvent comparable in $\mathcal{L}_p(\mathcal{H})$, $1 \le p < \infty$ for some odd $m \in \mathbb{N}$ and let $f \in \mathfrak{F}_m(\mathbb{R})$. Let $a_1, a_2 \in \mathbb{R} \setminus \{0\}$ and the functions φ, G_{1,a_1}, G_{2,a_2} be as in Proposition 2.2.7. Let h be the function on the circle satisfying*

$$f \circ \varphi^{-1} = h \circ \gamma,$$

where γ is the Cayley transform. Introduce also the operators $U_A = \gamma(\varphi(A)), U_B = \gamma(\varphi(B))$. Then, the double operator integral $T^{A,B}_{f,a_j}$, $j = 1, 2$, is defined by setting

$$T^{A,B}_{f,a_j} = 2iT^{U_A,U_B}_{h^{[1]}} \circ T^{A,B}_{G_{j,a_j}}, \quad j = 1, 2. \tag{2.2.42}$$

Using the notation of Definition 2.2.9 and recalling Proposition 2.2.7, (2.2.40) and (2.2.41) we conclude the following result, which is proved in [**83**].

Proposition 2.2.10. *Assume that for some odd $m \in \mathbb{N}$ the operators A, B are m-resolvent comparable in $\mathcal{L}_p(\mathcal{H})$ for some $p \in [1, \infty)$ and let $f \in \mathfrak{F}_m(\mathbb{R})$. Then there exist (sufficiently small) $a_1 \in \mathbb{R} \setminus \{0\}$ and (sufficiently large) $a_2 \in \mathbb{R} \setminus \{0\}$,* *such that the double operator integral* $T^{A,B}_{f,a_j}$, $j = 1, 2$, *introduced in* (2.2.42) *is bounded on* $\mathcal{L}_p(\mathcal{H})$ *and*

$$f(A) - f(B) = \sum_{j=1,2} T^{A,B}_{f,a_j}\big((A - a_j i)^{-m} - (B - a_j i)^{-m}\big) \in \mathcal{L}_p(\mathcal{H}).$$

REMARK 2.2.11. Assume that for some odd $m \in \mathbb{N}$ the operators A, B are m-resolvent comparable in $\mathcal{L}_p(\mathcal{H})$ for some $p \in (1, \infty)$. Then Proposition 2.2.10 holds for a wider class of functions f, and the constant C in (2.2.41) can be sharpened. Indeed, assume that function f on $\mathbb{R}$ is such that the function h on $\mathbb{T}$ defined by (1.1.2) is a Lipschitz function on $\mathbb{T}$. Then combining [**10**, Theorem 2] and [**38**, Corollary 5.5] one obtains $[f(A) - f(B)] \in \mathcal{L}_p(\mathcal{H})$ and

$$\begin{aligned} \|f(A) - f(B)\|_p &\leq 32\Big(C_1\frac{p^2}{p-1} + 9\Big)\|U_A - U_B\|_p \\ &\overset{(2.2.39)}{\leq} 64C_2\left(C_1\frac{p^2}{p-1} + 9\right)\Big(\big\|(A - a_1 i)^{-m} - (B - a_1 i)^{-m}\big\|_p \\ &\qquad\qquad + \big\|(A - a_2 i)^{-m} - (B - a_2 i)^{-m}\big\|_p\Big), \end{aligned} \tag{2.2.43}$$

where the constants $C_1 = C_1(f) \in (0, \infty)$ and $C_2 = C_2(a_1, a_2, m) \in (0, \infty)$ are independent of $p \in (1, \infty)$.

2.3. Limiting Process for Double Operator Integrals

In this section we firstly recall the classical results of limits of double operator integrals with respect to a varying spectral measure. Then we present limiting results for double operators build over spectral measures of operators which are m-resolvent comparable in $\mathcal{L}_p(\mathcal{H})$ (see Theorem 2.3.9). The latter result is published in [**27**].

We firstly recall the following result, which follows from [**17**, Proposition 7.8, Theorem 5.7] and [**17**, Proposition 5.6 (3)].

PROPOSITION 2.3.1. *Let* $f \in \mathbb{C}^2_b(\mathbb{R})$ *be such that* $f' \in L_p(\mathbb{R})$ *for some* $p \geq 1$ *and* f' *satisfies Hölder condition for some* $\varepsilon > 0$. *Let* A_n, B_n *and* A, B *be self-adjoint operators on* $\mathcal{H}$, *such that* $A_n \to A$, $B_n \to B$ *in the strong resolvent sense. Then*

$$T^{A_n,B_n}_{f^{[1]}} \to T^{A,B}_{f^{[1]}} \tag{2.3.1}$$

pointwise on $\mathcal{L}_1(\mathcal{H})$.

The following proposition is an easy corollary of Daletski-Krein formula (see e.g. [**17**]). We refer also to [**72**]. For completeness we present the full proof.

PROPOSITION 2.3.2. *Let* A *be a self-adjoint operator acting in a separable Hilbert space* $\mathcal{H}$, $B \in \mathcal{L}_1(\mathcal{H})$ *and let* $f \in C^2_b(\mathbb{R})$ *be such that* $f' \in L_p(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$. *Then, letting* $A_s = A + sB$, $s \in [0, 1]$, *we have that*

$$\mathrm{tr}(f(A_1) - f(A_0)) = \int_0^1 \mathrm{tr}\,\Big(f'(A_s)B\Big)ds. \tag{2.3.2}$$

PROOF. Let us show firstly that the function

$$s \mapsto \mathrm{tr}(f(A_s) - f(A_0))$$

is a $C^1[0, 1]$-function.

Let $s, t \in [0,1]$. We have

$$\begin{aligned}
|\operatorname{tr}(f(A_s) - f(A_0)) - \operatorname{tr}(f(A_t) - f(A_0))| &= |\operatorname{tr}(f(A_s) - f(A_t))| \\
&= \left|\operatorname{tr}(T^{A_s,A_t}_{f^{[1]}}(A_s - A_t))\right| \le \|T^{A_s,A_t}_{f^{[1]}}\|_{\mathcal{L}_1\to\mathcal{L}_1}\|A_s - A_t\|_1 \\
&\le \mathrm{const}\,|s-t|\|B\|_1 \to 0,
\end{aligned}$$

as $t \to s$. Hence, we infer that $|\operatorname{tr}(f(A_s) - f(A_0)) - \operatorname{tr}(f(A_t) - f(A_0))| \to 0$, as $t \to s$.

To prove that this function is continuously differentiable, we write

$$\begin{aligned}
&\frac{\operatorname{tr}(f(A_s) - f(A_0)) - \operatorname{tr}(f(A_t) - f(A_0))}{s-t} \\
&= \frac{\operatorname{tr}(f(A_s) - f(A_t))}{s-t} = \operatorname{tr}\Big(T^{A_s,A_t}_{f^{[1]}}(\frac{A_s - A_t}{s-t})\Big) \\
&= \operatorname{tr}\Big(T^{A_s,A_t}_{f^{[1]}}(B)\Big).
\end{aligned}$$

Since the function f satisfies the assumptions in Proposition 2.3.1, we further infer that

$$T^{A_s,A_t}_{f^{[1]}}(B) \to T^{A_t,A_t}_{f^{[1]}}(B), \quad s \to t,$$

in $\|\cdot\|_1$-norm. Hence,

$$\frac{d}{ds}\operatorname{tr}(f(A_s) - f(A_0))|_{s=t} = \operatorname{tr}\Big(T^{A_t,A_t}_{f^{[1]}}(B)\Big). \tag{2.3.3}$$

Moreover, we infer that the function $s \mapsto \operatorname{tr}(f(A_s) - f(A_0))$ is a $C^1[0,1]$-function.

Hence, by the fundamental theorem of calculus we have

$$\begin{aligned}
\operatorname{tr}(f(A_1) - f(A_0)) &= \int_0^1 \frac{d}{ds}\operatorname{tr}(f(A_s) - f(A_0))ds \\
&= \int_0^1 \operatorname{tr}\Big(T^{A_s,A_s}_{f^{[1]}}(B)\Big)ds \\
&\overset{(2.1.4)}{=} \int_0^1 \operatorname{tr}\Big(T^{A_s,A_s}_{f^{[1]}}(1)\cdot B\Big)ds \\
&= \int_0^1 \operatorname{tr}\Big(f'(A_s)\cdot B\Big)ds,
\end{aligned}$$

as required. □

Next, we recall several results, which are necessary for the proof of Theorem 2.3.8.

Let A_n, B_n, A, B be self-adjoint operators in the Hilbert space $\mathcal{H}$. We recall the definition of the classes $\mathfrak{A}^s_r(E_A)$ and $\mathfrak{A}^s_l(E_B)$ [**17**, p. 40]. Suppose $\varphi(\,\cdot\,,\,\cdot\,)$ admits a representation of the form

$$\varphi(\lambda,\mu) = \int_\Omega \alpha(\lambda,t)\beta(\mu,t)\,d\eta(t), \quad (\lambda,\mu)\in\mathbb{R}^2, \tag{2.3.4}$$

where $(\Omega, d\eta(t))$ is an auxiliary measure space and

$$C^2_\alpha := \sup_{\lambda\in\mathbb{R}}\int_\Omega |\alpha(\lambda,t)|^2\,d\eta(t) < \infty, \quad C^2_\beta := \sup_{\mu\in\mathbb{R}}\int_\Omega |\beta(\mu,t)|^2\,d\eta(t) < \infty. \tag{2.3.5}$$

Set

$$\begin{aligned} a(t) &:= \int_{\mathbb{R}} \alpha(\lambda,t)\,dE_A(\lambda), \quad b(t) := \int_{\mathbb{R}} \beta(\mu,t)\,dE_B(\mu), \\ a_n(t) &:= \int_{\mathbb{R}} \alpha(\lambda,t)\,dE_{A_n}(\lambda), \quad b_n(t) := \int_{\mathbb{R}} \beta(\mu,t)\,dE_{B_n}(\mu), \quad n \in \mathbb{N}, \end{aligned} \tag{2.3.6}$$

and introduce

$$\begin{aligned} \varepsilon_n(v,\alpha) &= \left[\int_{\Omega} \|a_n(t)v - a(t)v\|^2\,d\eta(t)\right]^{1/2}, \\ \delta_n(v,\beta) &= \left[\int_{\Omega} \|b_n(t)v - b(t)v\|^2\,d\eta(t)\right]^{1/2}, \quad n \in \mathbb{N},\ v \in \mathcal{H}, \end{aligned} \tag{2.3.7}$$

and

$$\mathfrak{A}_r^s(E_A) := \left\{\varphi \text{ in } (2.3.4) \;\middle|\; \lim_{n\to\infty} \varepsilon_n(v,\alpha) = 0,\ v \in \mathcal{H}\right\}, \tag{2.3.8}$$

$$\mathfrak{A}_l^s(E_B) := \left\{\varphi \text{ in } (2.3.4) \;\middle|\; \lim_{n\to\infty} \delta_n(v,\alpha) = 0,\ v \in \mathcal{H}\right\}. \tag{2.3.9}$$

If A_n, B_n, A, B are unitary operators on $\mathcal{H}$, the classes $\mathfrak{A}_r^s(E_A), \mathfrak{A}_l^s(E_A)$ are introduced similarly by taking the corresponding spectral measures and functions over the circle $\mathbb{T}$.

We note that the definitions of the classes $\mathfrak{A}_r^s(E_A), \mathfrak{A}_l^s(E_A)$ impose certain restrictions on convergences $A_n \to A$ and $B_n \to B$ as well as on the properties of the function φ, given in (2.3.4).

PROPOSITION 2.3.3. *If $\varphi, \psi \in \mathfrak{A}_r^s(E_A)$ (respectively, $\varphi, \psi \in \mathfrak{A}_l^s(E_B)$), then $(\varphi + \psi) \in \mathfrak{A}_r^s(E_A)$ (respectively, $(\varphi + \psi) \in \mathfrak{A}_l^s(E_B)$).*

PROOF. We prove the assertion only for the set $\mathfrak{A}_r^s(E_A)$, since for the set $\mathfrak{A}_l^s(E_B)$ the proof is similar.

Let the functions φ and ψ have the representations

$$\varphi(\lambda,\mu) = \int_{\Omega_1} \alpha_1(\lambda,t)\beta_1(\mu,t)\,d\eta_1(t), \quad \psi(\lambda,\mu) = \int_{\Omega_2} \alpha_2(\lambda,t)\beta_2(\mu,t)\,d\eta_2(t),$$

for some measure spaces $(\Omega_i, d\eta_j(t))$, and functions α_j, β_j, $j \in \{1,2\}$.

Let $(\Omega, \Sigma, d\eta(t))$ be the direct sum of the measure spaces $(\Omega_1, d\eta_1(t))$ and $(\Omega_2, d\eta_2(t))$ (so $\Omega = \Omega_1 \sqcup \Omega_2$, the disjoint union of Ω_1 and Ω_2, etc.). Define the function

$$\alpha(\lambda,t) = \begin{cases} \alpha_1(\lambda,t), & t \in \Omega_1, \\ \alpha_2(\lambda,t), & t \in \Omega_2. \end{cases}$$

Evidently, the function α satisfies condition (2.3.5). In addition,

$$a_n(t) = \begin{cases} \displaystyle\int_{\mathbb{R}} \alpha_1(\lambda,t)\,dE_{A_n}(t) = a_n^{(1)}(t), & t \in \Omega_1, \\ \displaystyle\int_{\mathbb{R}} \alpha_2(\lambda,t)\,dE_{A_n}(t) = a_n^{(2)}(t), & t \in \Omega_2, \end{cases}$$

and

$$a(t) = \begin{cases} \displaystyle\int_{\mathbb{R}} \alpha_1(\lambda,t)\,dE_A(t) = a^{(1)}(t), & t \in \Omega_1, \\ \displaystyle\int_{\mathbb{R}} \alpha_2(\lambda,t)\,dE_A(t) = a^{(2)}(t), & t \in \Omega_2, \end{cases}$$

where $a_n^{(j)}(\cdot)$ and $a^{(j)}(\cdot)$ denote the operators defined by (2.3.6) with respect to the functions α_j, $j \in \{1, 2\}$. Hence, for every fixed $v \in \mathcal{H}$,

$$\begin{aligned}
\varepsilon_n(v, \alpha) &= \Big(\int_{\Omega} \|a_n(t)v - a(t)v\|^2 d\eta(t) \Big)^2 \\
&\leq \Big(\int_{\Omega_1} \big\|a_n^{(1)}(t)v - a^{(1)}(t)v\big\|^2 d\eta_1(t) \Big)^2 \\
&\quad + \Big(\int_{\Omega_2} \big\|a_n^{(2)}(t)v - a^{(2)}(t)v\big\|^2 d\eta_2(t) \Big)^2 \\
&= \varepsilon_n(v, \alpha_1) + \varepsilon_n(v, \alpha_2) \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Thus, $(\varphi + \psi) \in \mathfrak{A}_r^s(E_A)$. □

Our proof of Theorem 2.3.8 is based on the following result in [**17**].

PROPOSITION 2.3.4. [**17**, Proposition 5.6] *Let $\psi \in \mathfrak{A}_r^s(E_A) \cap \mathfrak{A}_l^s(E_B)$. Then for any $T \in \mathcal{L}_p(\mathcal{H})$, $p \in [1, \infty)$,*

$$\lim_{n\to\infty} \big\|T_\psi^{A_n,B_n}(T) - T_\psi^{A,B}(T)\big\|_p = 0, \quad p \in [1, \infty). \tag{2.3.10}$$

Next, we provide a sufficient condition for inclusion how additional assumption $\varphi \in \mathfrak{A}_r^s(E_A)$, in the case, when we impose additional condition that $A_n \to A$ in the strong resolvent sense.

LEMMA 2.3.5. *Assume that A, A_n, $n \in \mathbb{N}$, are self-adjoint operators such that $A_n \to A$ as $n \to \infty$ in the strong resolvent sense. If a function $\varphi(\cdot, \cdot)$ satisfies the condition of Theorem 2.2.1, then $\varphi \in \mathfrak{A}_r^s(E_A)$.*

PROOF. This argument is based on the proof of Theorem 2.2.1.

Let $(\Omega, d\eta(t)) = (\mathbb{R}, dt)$ and let $\alpha(\lambda, t) = e^{i\lambda t}\big(|t|^{m_1} + |t|^{m_2}\big)^{-1/2}$. If $v \in \mathcal{H}$, then

$$\varepsilon_n(v, \alpha) = \Big[\int_{\mathbb{R}} \Big(|t|^{m_1} + |t|^{m_2}\Big)^{-1} \big\|e^{itA_n}v - e^{itA}v\big\|_{\mathcal{H}}^2 \, dt \Big]^{\frac{1}{2}}, \quad n \in \mathbb{N}. \tag{2.3.11}$$

Fix $\delta > 0$. Since $\int_{\mathbb{R}} \big(|t|^{m_1} + |t|^{m_2}\big)^{-1} dt < \infty$ (cf., eq. (2.2.2)), there exists $R > 0$ such that

$$\int_{|t|>R} \big(|t|^{m_1} + |t|^{m_2}\big)^{-1} dt < \delta. \tag{2.3.12}$$

On the other hand, since the family of functions $\big\{e^{i\lambda t}\big\}_{t\in[-R,R]}$ is uniformly continuous, [**69**, Theorem VIII.21] and the comment following its proof guarantees for each $v \in \mathcal{H}$,

$$\lim_{n\to 0} \big\|e^{itA_n}v - e^{itA}v\big\|_{\mathcal{H}} = 0, \tag{2.3.13}$$

uniformly in $t \in [-R, R]$. Therefore, for each $v \in \mathcal{H}$, there exists $N \in \mathbb{N}$ such that

$$\big\|e^{itA_n}v - e^{itA}v\big\|_{\mathcal{H}} < \delta, \quad n \geq N, \ t \in [-R, R]. \tag{2.3.14}$$

Hence, for every $v \in \mathcal{H}$,

$$\begin{aligned}\lim_{n\to\infty} \varepsilon_n(v,\alpha) &\le \lim_{n\to\infty} \left[\int_{|t|\le R} \left\| e^{itA_n}v - e^{itA}v \right\|_{\mathcal{H}}^2 dt \right]^{\frac{1}{2}} \\ &\quad + \lim_{n\to\infty} \left[\int_{|t|>R} \left\| e^{itA_n}v - e^{itA}v \right\|_{\mathcal{H}}^2 dt \right]^{\frac{1}{2}} \\ &\le 2\delta \|v\|_{\mathcal{H}}.\end{aligned} \tag{2.3.15}$$

Since $\delta > 0$ was arbitrary, one concludes

$$\lim_{n\to\infty} \varepsilon_n(v,\alpha) = 0, \quad v \in \mathcal{H}. \tag{2.3.16}$$

□

The next corollary is an immediate consequence of Lemma 2.3.5 and Proposition 2.2.2.

Corollary 2.3.6. *Assume that A, A_n, $n \in \mathbb{N}$, are self-adjoint operators such that $A_n \to A$ as $n \to \infty$ in the strong resolvent sense. If a function K on $\mathbb{R}^2$ satisfies the assumption of Proposition 2.2.2, then $K \in \mathfrak{A}_r^s(E_A)$.*

Proof. As in the proof of Proposition 2.2.2 (see (2.2.11) and (2.2.12)), we set

$$k(\mu) = \lim_{\lambda\to\pm\infty} K(\lambda,\mu), \quad K_0(\lambda,\mu) = K(\lambda,\mu) - k(\mu), \quad \lambda,\mu \in \mathbb{R},$$

and write

$$K(\lambda,\mu) = K_0(\lambda,\mu) - k(\mu). \tag{2.3.17}$$

As established in the course of the proof of Proposition 2.2.2, the function K_0 satisfies the assumption of Theorem 2.2.1. Therefore, by Lemma 2.3.5 we have $K_0 \in \mathfrak{A}_r^s(E_A)$. In addition, for the function $\varphi(\lambda,\mu) := k(\mu)$ we can write

$$\varphi(\lambda,\mu) = \int_{\mathbb{R}} \alpha(\lambda,t)\beta(\mu,t)\, dm(t),$$

where $\alpha(\lambda,t) = 1$, $\beta(\mu,t) = k(\mu)$, and m is the measure defined on the σ-algebra $2^{\mathbb{R}}$ by setting

$$m(A) = \begin{cases} 1, & 0 \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Since for the function $\alpha(\lambda,t) = 1$, the corresponding operators $a(t)$ and $a_n(t)$, defined in (2.3.6) are just the identity operator, it is clear that the function φ belongs to the class $\mathfrak{A}_r^s(E_A)$. Hence, equality (2.3.17) combined with Proposition 2.3.3 implies that $K \in \mathfrak{A}_r^s(E_A)$. □

To proceed further, we now strengthen the assumptions on the operators A_n, A and B_n, B, $n \in \mathbb{N}$, as follows.

Hypothesis 2.3.7. *Assume that A, A_n, B, B_n, $n \in \mathbb{N}$, are self-adjoint operators such that $A_n \to A$ and $B_n \to B$ as $n \to \infty$ in the strong resolvent sense. In addition we assume that for some $m \in \mathbb{N}$, m odd, $p \in [1,\infty)$, and every $a \in \mathbb{R}\backslash\{0\}$,*

$$\begin{aligned} R(a) &:= \left[(A+ia)^{-m} - (B+ia)^{-m}\right] \in \mathcal{L}_p(\mathcal{H}), \\ R_n(a) &:= \left[(A_n+ia)^{-m} - (B_n+ia)^{-m}\right] \in \mathcal{L}_p(\mathcal{H}), \end{aligned} \tag{2.3.18}$$

and

$$\lim_{n\to\infty}\|R_n(a)-R(a)\|_p=0. \tag{2.3.19}$$

With this hypothesis at hand, the following theorem is the main result of this section.

THEOREM 2.3.8. [**27**] *Assume Hypothesis 2.3.7. Let $f\in\mathfrak{F}_m(\mathbb{R})$ and let the double operator integrals $T^{A,B}_{f,a_j}$ and T^{A_n,B_n}_{f,a_j} be as in Definition 2.2.9 (with A,B replaced by A_n,B_n respectively for T^{A_n,B_n}_{f,a_j}). Then*

$$T^{A_n,B_n}_{f,a_j}\to T^{A,B}_{f,a_j},\quad j=1,2,$$

pointwise on $\mathcal{L}_p(\mathcal{H})$, $p\in[1,\infty)$.

PROOF. The proof for $j=1$ and $j=2$ are identical, and therefore we prove it only for $j=1$.

By definition we have that

$$T^{A,B}_{f,a_1}=2iT^{U_A,U_B}_{g^{[1]}}\circ T^{A,B}_{G_{1,a_1}},$$

and

$$T^{A_n,B_n}_{f,a_1}=2iT^{U_{A_n},U_{B_n}}_{g^{[1]}}\circ T^{A_n,B_n}_{G_{1,a_1}},$$

and therefore, we divide our proof into two steps.

Step 1. In this step we prove that

$$T^{A_n,B_n}_{G_{1,a_1}}\to T^{A,B}_{G_{1,a_1}}$$

pointwise on $\mathcal{L}_p(\mathcal{H})$.

By Proposition 2.3.4 it is sufficient to show that $G_{1,a_1}\in\mathfrak{A}^s_r(E_A)\cap\mathfrak{A}^s_l(E_B)$. Since by definition of G_{1,a_1}, we have $G_{1,a_1}(\lambda,\mu)=G_{1,a_1}(\mu,\lambda)$, it suffices to show that $G_{1,a_1}\in\mathfrak{A}^s_r(E_A)$. The latter inclusion follows from the fact that the function G_{1,a_1} satisfies the assumptions of Proposition 2.2.2 and hence also of Corollary 2.3.6, that is, $G_{1,a_1}\in\mathfrak{A}^s_r(E_A)\cap\mathfrak{A}^s_l(E_B)$, as required.

Step 2. Fix $X\in\mathcal{L}_p(\mathcal{H})$. By the previous step we have that

$$T^{A_n,B_n}_{G_{1,a_1}}(X)\to T^{A,B}_{G_{1,a_1}}(X)$$

in $\mathcal{L}_p(\mathcal{H})$. For simplicity we denote,

$$Y_n=T^{A_n,B_n}_{G_{1,a_1}}(X),\quad Y=T^{A,B}_{G_{1,a_1}}(X).$$

To conclude the proof it is sufficient to show that

$$T^{U_{A_n},U_{B_n}}_{h^{[1]}}(Y_n)\to T^{U_A,U_B}_{h^{[1]}}(Y),$$

in $\mathcal{L}_p(\mathcal{H})$.

We write

$$\begin{aligned}&T^{U_{A_n},U_{B_n}}_{h^{[1]}}(Y_n)-T^{U_A,U_B}_{h^{[1]}}(Y)\\&\qquad=T^{U_{A_n},U_{B_n}}_{h^{[1]}}(Y_n-Y)+\big(T^{U_{A_n},U_{B_n}}_{h^{[1]}}-T^{U_A,U_B}_{h^{[1]}}\big)(Y).\end{aligned} \tag{2.3.20}$$

Since $Y_n\to Y$ in $\mathcal{L}_p(\mathcal{H})$ and the sequence $\{\|T^{U_{A_n},U_{B_n}}_{h^{[1]}}\|_{p\to p}\}_{n\in\mathbb{N}}$ is uniformly bounded by $\|h^{[1]}\|_{\mathfrak{M}}$, we conclude that the first term converges to 0 in $\mathcal{L}_p(\mathcal{H})$.

Recall, that the function h (see (2.2.36)) is such that h' satisfies the Hölder condition with exponent $\varepsilon>0$. Therefore, a combination of [**17**, Proposition 7.5] and [**17**, Theorem 5.9], as well as the discussion following the latter theorem, implies that $h^{[1]}$ belongs to the class $\mathfrak{A}_l^s(E_V)\cap\mathfrak{A}_r^s(E_U)$ and therefore, by Proposition 2.3.4, one infers

$$T_{h^{[1]}}^{U_{A_n},U_{B_n}} \to T_{h^{[1]}}^{U_A,U_B}$$

pointwise on $\mathcal{L}_p(\mathcal{H})$, which suffices to conclude that the second term on the right hand-side of (2.3.20) converges to zero in $\mathcal{L}_p(\mathcal{H})$. Thus, the proof is complete. $\square$

The following result is an immediate corollary of Theorem 2.3.8.

THEOREM 2.3.9. [**27**] *Assume Hypothesis 2.3.7. Then for any function $f \in \mathfrak{F}_m(\mathbb{R})$,*

$$\lim_{n\to\infty} \big\|[f(A_n) - f(B_n)] - [f(A) - f(B)]\big\|_p = 0. \tag{2.3.21}$$

PROOF. By Proposition 2.2.10 we have that

$$f(A) - f(B) = \sum_{j=1,2} T_{f,a_j}^{A,B}(R(a_j)),$$

$$f(A_n) - f(B_n) = \sum_{j=1,2} T_{f,a_j}^{A_n,B_n}(R_n(a_j)),$$

where $R(a_j)$ and $R_n(a_j)$ are defined in (2.3.18).

Hence, we can write

$$[f(A_n) - f(B_n)] - [f(A) - f(B)] = \sum_{j=1,2}\Big(T_{f,a_j}^{A_n,B_n}(R_n(a_j)) - T_{f,a_j}^{A,B}(R(a_j))\Big)$$

$$= \sum_{j=1,2}\Big(T_{f,a_j}^{A_n,B_n}(R_n(a_j) - R(a_j))\Big) + \sum_{j=1,2}\Big(\big(T_{f,a_j}^{A_n,B_n} - T_{f,a_j}^{A,B}\big)(R(a_j))\Big).$$

By assumption of Hypothesis 2.3.7 we have that $R_n(a_j) \to R(a_j)$ in $\mathcal{L}_p(\mathcal{H})$. Since, in addition, the sequence $\{\|T_{f,a_j}^{A_n,B_n}\|_{p\to p}\}_{n\in\mathbb{N}}$ is uniformly bounded, we conclude that the first term on the right hand side above converges to 0 in $\mathcal{L}_p(\mathcal{H})$. The second term converges to 0 in $\mathcal{L}_p(\mathcal{H})$ since $R(a_j) \in \mathcal{L}_p(\mathcal{H})$ and by Theorem 2.3.8 we have that $T_{f,a_j}^{A_n,B_n} \to T_{f,a_j}^{A,B}$ pointwise on $\mathcal{L}_p(\mathcal{H})$. This completes the proof. $\square$

CHAPTER 3

The main setting and some preliminaries

The present chapter collects all the definitions and necessary preliminaries, which will be used in the rest of the thesis. The result presented in this chapter are taken from [**28**].

In the first section of this chapter we give precise definitions of the operators involved and immediate implications of the p-relative trace-class assumption. In Section 3.2 we introduce the approximation scheme we employ in our approach. The idea is to introduce spectral 'cut-off' $\{B_n(t)\}_{t\in\mathbb{R}}$, which satisfy the Pushnitski's assumptions (Hypothesis 1.5.1) and well as assumptions of [**47**], [**31**] (see Hypothesis 1.5.2). Hence, the results of [**67**], [**47**] and [**31**] are applicable to $\{B_n(t)\}_{t\in\mathbb{R}}$.

In Section 3.3 we prove our first key result (see Theorem 3.3.2), which establishes that the operators A_+ and A_- are p-resolvent comparable in $\mathcal{L}_1(\mathcal{H})$ (in the sense of Definition 2.2.4). Moreover, we show that

$$(A_+ - z)^{-p} - (A_- - z)^{-p}, \quad z \in \mathbb{C}\setminus\mathbb{R}$$

can be approximated in the $\mathcal{L}_1(\mathcal{H})$-norm by

$$(A_{+,n} - z)^{-p} - (A_- - z)^{-p},$$

where $A_{+,n}$ denotes the asymptote for the reduced family $\{B_n(t)\}_{t\in\mathbb{R}}$. We also establish several important immediate corollaries of Theorem 3.3.2, which we use in the sequel.

To establish similar results for the operators $\boldsymbol{H}_j, j = 1, 2$, we firstly prove uniform norm estimates in Section 3.4. The proof of these estimates uses techniques, which came from noncommutative geometry (see e.g. [**24**, Corollary 2.7]) and are the reason why the final Hypothesis 3.5.1 contains an additional 'smoothness' assumption (v).

We introduce our main Hypothesis 3.5.1 in a separate Section 3.5 and also discuss its version for a special case, when the path $\{B(t)\}_{t\in\mathbb{R}}$ is a path with 'separable' variables, that is $B(t) = \theta(t)B_+$, for some function θ on $\mathbb{R}$ and a bounded perturbation B_+ of A_-.

Finally, the uniform norm estimates from Section 3.4 we prove (Theorem 3.6.1) that the operators $\boldsymbol{H}_j, j = 1, 2$, are m-resolvent comparable with $m = \lceil \frac{p}{2} \rceil$ and in fact the difference of powers of resolvents of $\boldsymbol{H}_j, j = 1, 2$, can be approximated in trace-norm by the difference of powers of resolvents of the reduced operators $\boldsymbol{H}_{j,n}, j = 1, 2$.

3.1. The basic setup

In this section we introduce precisely the operators we work with. Our paths are restricted by the final Hypothesis 3.5.1. In this introductory discussion however we will work under the less restrictive Hypothesis 3.1.1.

Hypothesis 3.1.1. *Suppose $\mathcal{H}$ is a complex, separable Hilbert space.*
(i) Assume A_- is self-adjoint on $\mathrm{dom}(A_-) \subseteq \mathcal{H}$.
(ii) Suppose we have a family of bounded self-adjoint operators $\{B(t)\}_{t\in\mathbb{R}} \subset \mathcal{B}(\mathcal{H})$, continuously differentiable with respect to t in the uniform operator norm, such that

$$\|B'(\cdot)\|_\infty \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R}). \tag{3.1.1}$$

(iii) Suppose that the family $\{B(t)\}$ is p**-relative trace-class operators** *with respect to A_- for some $p \in \mathbb{N} \cup \{0\}$, that is*

$$B'(t)(A_- + i)^{-p-1} \in \mathcal{L}_1(\mathcal{H}), \quad \int_{\mathbb{R}} \|B'(t)(A_- + i)^{-p-1}\|_1 dt < \infty. \tag{3.1.2}$$

In what follows, we always choose the smallest $p \in \mathbb{N}\cup\{0\}$ which satisfies (3.1.2).

Remark 3.1.2. The Hypothesis 1.5.2 used in [**47**] corresponds to the p-relative trace-class assumption with $p = 0$. The hypothesis in [**30**] corresponds to a 1-relative trace-class assumption.

Remark 3.1.3.

(i) The fact that the function $f(t) = \frac{(t+i)^q}{(t^2+1)^{q/2}}$, $q \in \mathbb{R}$, is bounded together with its inverse, implies that for any $C \in \mathcal{B}(\mathcal{H})$ the operators

$$C(A_- + i)^{-q} \quad \text{and } C(A_-^2 + 1)^{-q/2}$$

belong to the same ideal of $\mathcal{B}(\mathcal{H})$. In what follows we use this fact repeatedly without additional explanations.

(ii) By three lines theorem [**51**, Chapter III, Theorem 16.1]) (see also [**48**, Theorem 3.2]), Hypothesis 3.1.1(iii) above implies that

$$(A_- + i)^{-k} B'(t)(A_- + i)^{-\ell} \in \mathcal{L}_1(\mathcal{H}), \quad \int_{\mathbb{R}} \|(A_- + i)^{-k} B'(t)(A_- + i)^{-\ell}\|_1 dt < \infty,$$

for all $k, \ell \in \mathbb{N} \cup \{0\}$ with $k + \ell = 1 + p$.

Given Hypothesis 3.1.1 we introduce the family of self-adjoint operators $A(t)$, $t \in \mathbb{R}$, in $\mathcal{H}$, by

$$A(t) = A_- + B(t), \quad \mathrm{dom}(A(t)) = \mathrm{dom}(A_-), \quad t \in \mathbb{R}.$$

Writing

$$B(t) = B(t_0) + \int_{t_0}^{t} B'(s)\, ds, \quad t, t_0 \in \mathbb{R}, \tag{3.1.3}$$

with the convergent Bochner integral on the right-hand side, we conclude that the self-adjoint asymptotes

$$\operatorname*{n-lim}_{t\to\pm\infty} B(t) := B_\pm \in \mathcal{B}(\mathcal{H}) \tag{3.1.4}$$

exist, where the limit is taken in the uniform norm. In particular, purely for convenience of notations, we will make the choice

$$B_- = 0$$

in the following and also introduce the asymptote,

$$A_+ = A_- + B_+, \quad \mathrm{dom}(A_+) = \mathrm{dom}(A_-). \tag{3.1.5}$$

Assumption (3.1.1) and equality (3.1.3) also yield,

$$\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{B}(\mathcal{H})} \leq \int_{\mathbb{R}} \|B'(t)\|_{\mathcal{B}(\mathcal{H})} dt < \infty. \tag{3.1.6}$$

A simple application of the resolvent identity yields (with $t \in \mathbb{R}$, $z \in \mathbb{C}\backslash\mathbb{R}$)

$$(A(t) - zI)^{-1} = (A_\pm - zI)^{-1} - (A(t) - zI)^{-1}[B(t) - B_\pm](A_\pm - zI)^{-1},$$
$$\big\|(A(t) - zI)^{-1} - (A_\pm - zI)^{-1}\big\|_{\mathcal{B}(\mathcal{H})} \leq |\mathrm{Im}(z)|^{-2}\|B(t) - B_\pm\|_{\mathcal{B}(\mathcal{H})},$$

and hence proves that

$$\operatorname*{n-lim}_{t\to\pm\infty}(A(t) - zI)^{-1} = (A_\pm - zI)^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

This is relevant to whether spectral flow between A_- and A_+ along the path $\{A(t)\}$, exists [**56**].

Repeating the argument of [**47**, (3.49)] one can prove that

$$B_+(A_- + i)^{-1-p}, \quad B(t)(A_- + i)^{-1-p} \in \mathcal{L}_1(\mathcal{H}). \tag{3.1.7}$$

REMARK 3.1.4. (i) The inclusion (3.1.7) together with the fact that B_+ is a bounded operator implies that

$$B_+(A_- + i)^{-j} \in \mathcal{L}_{\frac{p+1}{j}}(\mathcal{H}), \quad j = 1, \ldots, p + 1, \tag{3.1.8}$$

and

$$\|B_+(A_- + i)^{-j}\|_{\frac{p+1}{j}} \leq \|B_+\|^{\frac{j}{p+1}} \cdot \|B_+(A_- + i)^{-p+1}\|_1^{\frac{p+1-j}{p+1}}. \tag{3.1.9}$$

Indeed, an application of three-line theorem (see e.g. [**73**], [**51**]) the function $T(z) = B_+(A_-^2 + 1)^{\frac{(p+1)(z-1)}{2}}$, $0 \leq \mathrm{Re}(z) \leq 1$ with $T(0) = B_+(A_-^2 + 1)^{-p-1} \in \mathcal{L}_1(\mathcal{H})$ and $T(1) = B_+ \in \mathcal{B}(\mathcal{H})$ yields the required result with $\mathrm{Re}(z) = \frac{p+1-j}{p+1}$.

(ii) Combining (3.1.8) together with the three lines theorem [**51**, Chapter III, Theorem 16.1]) (see also [**48**, Theorem 3.2]), we obtain that for any fixed $j = 1, \ldots, p + 1$ and any $k, l \in \mathbb{N}_0$ such that $k + l = j$, we have

$$(A_- + i)^{-k}B_+(A_- + i)^{-l} \in \mathcal{L}_{\frac{p+1}{j}}(\mathcal{H}).$$

and

$$\|(A_- + i)^{-k}B_+(A_- + i)^{-l}\|_{\frac{p+1}{j}} \leq \|B_+(A_- + i)^{-j}\|_{\frac{p+1}{j}}. \tag{3.1.10}$$

Inclusion (3.1.8) combined with Weyl's theorem (see e.g. [**79**, Theorem 9.13]) implies the following

PROPOSITION 3.1.5. *For all $t \in \mathbb{R}$ we have that*

$$\sigma_{ess}(A(t)) = \sigma_{ess}(A_-) = \sigma_{ess}(A_+).$$

Next, we turn to the pair $(\boldsymbol{H}_2, \boldsymbol{H}_1)$.

For a family of operators $\{T(t)\}_{t\in\mathbb{R}}$ in $\mathcal{H}$ we denote by $\boldsymbol{T}$ the operator acting in the Hilbert space $L_2(\mathbb{R},\mathcal{H})$ defined by

$$\begin{aligned}
&(\boldsymbol{T}f)(t) = T(t)f(t) \ \text{ for a.e. } t\in\mathbb{R},\\
&f\in \operatorname{dom}(\boldsymbol{T}) = \bigg\{ g\in L_2(\mathbb{R},\mathcal{H}) \,\bigg|\, g(t)\in\operatorname{dom}(T(t)) \text{ for a.e. } t\in\mathbb{R};\\
&\qquad t\mapsto T(t)g(t) \text{ is (weakly) measurable; } \int_{\mathbb{R}} \|T(t)g(t)\|_{\mathcal{H}}^2\,dt<\infty\bigg\}.
\end{aligned} \tag{3.1.11}$$

REMARK 3.1.6. If the family of operators $\{T(t)\}_{t\in\mathbb{R}}$ in $\mathcal{H}$ is given by $T(t) = \theta(t)T$ for some fixed operator T in $\mathcal{H}$ and a suitable function θ on $\mathbb{R}$, then with the identification of the Hilbert space $L_2(\mathbb{R},\mathcal{H})$ and $L_2(\mathbb{R})\otimes\mathcal{H}$ we have

$$\boldsymbol{T} = M_\theta\otimes T.$$

Let $\boldsymbol{A}_-$ be the operator acting in $L_2(\mathbb{R},\mathcal{H})$ defined by

$$\begin{aligned}
&(\boldsymbol{A}_-f)(t) = A_-f(t),\\
&f\in \operatorname{dom}(\boldsymbol{A}_-) = \bigg\{ g\in L_2(\mathbb{R},\mathcal{H}) \,\bigg|\, g(t)\in\operatorname{dom}(A_-) \text{ for a.e. } t\in\mathbb{R};\\
&\qquad t\mapsto A_-g(t) \text{ is (weakly) measurable; } \int_{\mathbb{R}} \|A_-g(t)\|_{\mathcal{H}}^2dt<\infty\bigg\}.
\end{aligned} \tag{3.1.12}$$

Identifying the Hilbert spaces $L_2(\mathbb{R},\mathcal{H})$ and $L_2(\mathbb{R})\otimes\mathcal{H}$ we have that

$$\boldsymbol{A}_- = 1\otimes A_-.$$

Let the operators $\boldsymbol{A}$, $\boldsymbol{B}, \boldsymbol{A}'=\boldsymbol{B}'$, be defined in terms of the families $A(t)$, $B(t)$, and $B'(t)$, $t\in\mathbb{R}$. Since $B(t), B'(t)$ are bounded operators for every $t\in\mathbb{R}$ and $\|B(\cdot)\|, \|B'(\cdot)\|\in L_\infty(\mathbb{R})$ (see (3.1.6) and Hypothesis 3.1.1 (ii)) we have that

$$\boldsymbol{B},\ \boldsymbol{B}'\in\mathcal{B}(L_2(\mathbb{R},\mathcal{H})).$$

Since, in addition, $A(t) = A_- + B(t)$, we infer that

$$\boldsymbol{A} = \boldsymbol{A}_- + \boldsymbol{B},\quad \operatorname{dom}(\boldsymbol{A}) = \operatorname{dom}(\boldsymbol{A}_-).$$

Now we introduce the operator $\boldsymbol{D_A}$ in $L_2(\mathbb{R},\mathcal{H})$ by

$$\boldsymbol{D_A} = \frac{d}{dt} + \boldsymbol{A},\quad \operatorname{dom}(\boldsymbol{D_A}) = W^{1,2}(\mathbb{R},\mathcal{H})\cap\operatorname{dom}(\boldsymbol{A}_-). \tag{3.1.13}$$

Recall that the operator d/dt in $L_2(\mathbb{R},\mathcal{H})$ is defined by

$$\begin{aligned}
&\left(\frac{d}{dt}f\right)(t) = f'(t) \ \text{ for a.e. } t\in\mathbb{R},\\
&f\in\operatorname{dom}(d/dt) = \big\{g\in L_2(\mathbb{R},\mathcal{H}) \,\big|\, g\in AC_{\rm loc}\big(\mathbb{R},\mathcal{H}\big),\ g'\in L_2(\mathbb{R},\mathcal{H})\big\}\\
&\qquad\qquad = W^{1,2}\big(\mathbb{R},\mathcal{H}\big).
\end{aligned} \tag{3.1.14}$$

Assuming Hypothesis 3.1.1 and repeating the proof of [**47**, Lemma 4.4] one can show that the operator $\boldsymbol{D_A}$ is densely defined and closed in $L_2(\mathbb{R},\mathcal{H})$. Furthermore, the adjoint operator $\boldsymbol{D_A^*}$ of $\boldsymbol{D_A}$ in $L_2(\mathbb{R},\mathcal{H})$ is then given by (cf. [**47**])

$$\boldsymbol{D_A^*} = -\frac{d}{dt} + \boldsymbol{A},\quad \operatorname{dom}(\boldsymbol{D_A^*}) = W^{1,2}(\mathbb{R},\mathcal{H})\cap\operatorname{dom}(\boldsymbol{A}_-).$$

This enables us to introduce the nonnegative, self-adjoint operators $\boldsymbol{H}_j$, $j = 1, 2$, in $L_2(\mathbb{R}, \mathcal{H})$ by

$$\boldsymbol{H}_1 = \boldsymbol{D}_{\boldsymbol{A}}^* \boldsymbol{D}_{\boldsymbol{A}}, \quad \boldsymbol{H}_2 = \boldsymbol{D}_{\boldsymbol{A}} \boldsymbol{D}_{\boldsymbol{A}}^*. \tag{3.1.15}$$

The following result is proved in [**31**, Theorem 2.6] under a relatively trace class perturbation assumption. However, as noted in [**31**, Remark 2.7] the result holds without the assumption of relative trace-class perturbation. Thus, in our more general setting the following theorem holds.

THEOREM 3.1.7. *Assume Hypothesis 3.1.1. Then the operator $\boldsymbol{D}_{\boldsymbol{A}}$ is Fredholm if and only if $0 \in \rho(A_+) \cap \rho(A_-)$.*

For future purposes we also introduce $\boldsymbol{H}_0$ in $L_2(\mathbb{R}, \mathcal{H})$ by

$$\boldsymbol{H}_0 = -\frac{d^2}{dt^2} + \boldsymbol{A}_-^2, \quad \operatorname{dom}(\boldsymbol{H}_0) = W^{2,2}(\mathbb{R}, \mathcal{H}) \cap \operatorname{dom}\left(\boldsymbol{A}_-^2\right). \tag{3.1.16}$$

By [**69**, Theorem VIII.33], the operator $\boldsymbol{H}_0$ is self-adjoint and positive. We note, that the operators $\boldsymbol{A}_-$ and $\boldsymbol{H}_0$ commute and

$$\operatorname{dom} \boldsymbol{H}_0^{1/2} = \operatorname{dom}(d/dt) \cap \operatorname{dom} \boldsymbol{A}_-. \tag{3.1.17}$$

The proof of the following result can be found in [**47**, Lemma 4.7]. Observe, that the proof given there does not require the full strength of the assumptions made in that paper. The statement is formulated using Hypothesis 3.1.1. In fact, it requires only Hypothesis 3.1.1 (i).

LEMMA 3.1.8. [**47**, Lemma 4.7] *The operator $\boldsymbol{A}_-(\boldsymbol{H}_0 - z)^{-1/2}, z < 0$ is bounded and*

$$\|\boldsymbol{A}_-(\boldsymbol{H}_0 - z)^{-1/2}\|_\infty \le 1, \quad z < 0.$$

In what follows, we need to strengthen Hypothesis 3.1.1 as follows.

HYPOTHESIS 3.1.9. *In addition to Hypothesis 3.1.1, assume that* $\operatorname{dom}(\boldsymbol{H}_0^{1/2})$ *is invariant with respect to the operator $\boldsymbol{B}$.*

Assuming Hypothesis 3.1.9 in the following, we have that $\boldsymbol{A}_-\boldsymbol{B}$ is an operator well defined on $\operatorname{dom} \boldsymbol{H}_0^{1/2}$, since $\operatorname{dom} \boldsymbol{H}_0^{1/2} \subset \operatorname{dom} \boldsymbol{A}_-$ (see (3.1.17)). Therefore, recalling that $\boldsymbol{A} = \boldsymbol{A}_- + \boldsymbol{B}$ one can decompose $\boldsymbol{H}_j$, $j = 1, 2$, as

$$\begin{aligned}
\boldsymbol{H}_j &= -\frac{d^2}{dt^2} + \boldsymbol{A}^2 + (-1)^j \boldsymbol{A}' \\
&= -\frac{d^2}{dt^2} + \boldsymbol{A}_-^2 + \boldsymbol{B}\boldsymbol{A}_- + \boldsymbol{A}_-\boldsymbol{B} + \boldsymbol{B}^2 + (-1)^j \boldsymbol{B}' \\
&= \boldsymbol{H}_0 + \boldsymbol{B}\boldsymbol{A}_- + \boldsymbol{A}_-\boldsymbol{B} + \boldsymbol{B}^2 + (-1)^j \boldsymbol{B}', \\
&\operatorname{dom}(\boldsymbol{H}_j) = \operatorname{dom}(\boldsymbol{H}_0), \quad j = 1, 2.
\end{aligned} \tag{3.1.18}$$

Using the standard resolvent identity, we obtain

$$(\boldsymbol{H}_j - z\boldsymbol{I})^{-1} - (\boldsymbol{H}_0 - z\boldsymbol{I})^{-1} = -(\boldsymbol{H}_j - z\boldsymbol{I})^{-1}(\boldsymbol{H}_j - \boldsymbol{H}_0)(\boldsymbol{H}_0 - z\boldsymbol{I})^{-1}$$

$$= -(\boldsymbol{H}_j - z\boldsymbol{I})^{-1}(\boldsymbol{B}\boldsymbol{A}_- + \boldsymbol{A}_-\boldsymbol{B} + \boldsymbol{B}^2 + (-1)^j \boldsymbol{B}')(\boldsymbol{H}_0 - z\boldsymbol{I})^{-1}, \tag{3.1.19}$$

for $j = 1, 2$ and $z \in \mathbb{C} \setminus \mathbb{R}_+$.

3.2. The approximation scheme

In this section we introduce the key technical ideas that enable us to use the old results of [**47**] in an approximation scheme.

Throughout the text we will constantly use the following result, which can be found in [**73**] and [**47**, Lemma 3.4].

LEMMA 3.2.1. *Let $p \in [1, \infty)$ and assume that $R, R_n, T, T_n \in \mathcal{B}(\mathcal{H})$, $n \in \mathbb{N}$, satisfy*

$$\operatorname*{s-lim}_{n\to\infty} R_n = R, \ \operatorname*{s-lim}_{n\to\infty} T_n = T$$

and that $S, S_n \in \mathcal{L}_p(\mathcal{H})$, $n \in \mathbb{N}$, satisfy $\lim_{n\to\infty} \|S_n - S\|_p = 0$. Then

$$\lim_{n\to\infty} \|R_n S_n T_n^* - RST^*\|_p = 0.$$

Next we introduce a spectral 'cut-off'

$$P_n = \chi_{[-n,n]}(A_-). \tag{3.2.1}$$

It follows from the spectral theory that

$$\operatorname*{s-lim}_{n\to\infty} P_n = 1. \tag{3.2.2}$$

REMARK 3.2.2. The precise form of the cut-offs P_n is of course immaterial. We just need two facts: that $\operatorname{s-lim}_{n\to\infty} P_n = 1$, $\sup_{n\in\mathbb{N}} \|P_n\| < \infty$ and that $P_n B_+ P_n \in \mathcal{L}_1(\mathcal{H})$ (see (3.2.7) below).

Let A_- and $\{B(t)\}_{t\in\mathbb{R}}$ satisfy Hypothesis 3.1.9. We introduce the family $\{B_n(t)\}_{t\in\mathbb{R}}$, $n \in \mathbb{N}$, of reduced operators by setting

$$B_n(t) := P_n B(t) P_n, \quad t \in \mathbb{R}, n \in \mathbb{N}. \tag{3.2.3}$$

In this case,

$$A_n(t) := A_- + B_n(t), \quad \operatorname{dom}(A_n(t)) = \operatorname{dom}(A_-), \quad n \in \mathbb{N}, \ t \in \mathbb{R}. \tag{3.2.4}$$

In particular, one concludes that

$$B_{+,n} := \operatorname*{n-lim}_{t\to+\infty} B_n(t) = P_n B_+ P_n, \tag{3.2.5}$$

and therefore for the reduces asymptotes $A_{+,n}$ we obtain

$$A_{+,n} := A_- + B_{+,n} = A_- + P_n B_+ P_n, \quad \operatorname{dom}(A_{+,n}) = \operatorname{dom}(A_-). \tag{3.2.6}$$

The following proposition shows that the family $\{B_n(t)\}_{t\in\mathbb{R}}$ of 'approximants' consists of trace-class operators, and so for this family the results of [**67, 47, 31**] hold.

PROPOSITION 3.2.3. *The family $\{B_n(t)\}_{t\in\mathbb{R}}$ consists of trace-class perturbations of A_- and satisfies the Pushnitski's assumption 1.5.1.*

PROOF. The proof easily follows from the fact that the function $x \mapsto (x + i)^{p+1}\chi_{[n,n]}(x)$, $x \in \mathbb{R}$, is bounded for every fixed $n \in \mathbb{N}$, and therefore the definition of P_n (see (3.2.1)) implies that

$$B_n'(t) = P_n B(t)' P_n = P_n \cdot B(t)'(A_- + i)^{-p-1} \cdot (A_- + i)^{p+1} P_n \in \mathcal{L}_1(\mathcal{H}).$$

□

REMARK 3.2.4. We note that the equality (3.1.7) together with the definition of the projections P_n implies that

$$B_{+,n} = P_n B_+ P_n \in \mathcal{L}_1(\mathcal{H}). \tag{3.2.7}$$

LEMMA 3.2.5. *Assume Hypothesis 3.1.1. We have that*

(i) $A_{+,n} \to A_+$ *in the strong resolvent sense.*

(ii) Let $j \in \mathbb{N}$. *For any* $k, l \in \mathbb{N}$ *such that* $k + l \geq j$ *and* $z \in \mathbb{C} \setminus \mathbb{R}$, *we have*

$$\|\cdot\|_{\frac{p+1}{j}} - \lim_{n\to\infty} (A_- - z)^{-k} B_{+,n} (A_- - z)^{-l} = (A_- - z)^{-k} B_+ (A_- - z)^{-l}.$$

PROOF. (i). Since $A_{+,n} = A_- + P_n B_+ P_n$ and $A_+ = A_- + B_+$ and the operator B_+ is bounded, the operators $A_{+,n}$ and A_+ have common core $\mathrm{dom}(A_-)$. Therefore, by [**69**, Theorem VIII.25 (a)] it is sufficient to show that $A_{+,n}\xi \to A_+\xi$ for all $\xi \in \mathrm{dom}(A_-)$. Let $\xi \in \mathrm{dom}(A_-)$. Since $P_n \to 1$ in the strong operator topology, we have that $B_{+,n}\xi = P_n B_+ P_n \xi \to B_+\xi$. Hence

$$\|A_{+,n}\xi - A_+\xi\| = \|B_{+,n}\xi - B_+\xi\| \to 0.$$

Thus, $A_{+,n} \to A_+$ in the strong resolvent sense.

(ii). Since $k + l \geq j$, Remark 3.1.4 implies that

$$(A_- - z)^{-k} B_+ (A_- - z)^{-l} \in \mathcal{L}_{\frac{p+1}{j}}(\mathcal{H}).$$

Therefore, since

$$(A_- - z)^{-k} B_{+,n} (A_- - z)^{-l} = P_n (A_- - z)^{-k} B_+ (A_- - z)^{-l} P_n,$$

and $P_n \to 1$ in the strong operator topology, it follows from Lemma 3.2.1 that

$$(A_- - z)^{-k} B_{+,n} (A_- - z)^{-l} \to (A_- - z)^{-k} B_+ (A_- - z)^{-l}$$

in $\mathcal{L}_{\frac{p+1}{j}}(\mathcal{H})$.

□

Next, we turn to the operators $\boldsymbol{H}_j, j = 1, 2$ and their reduced counterparts $\boldsymbol{H}_{j,n}, j = 1, 2, n \in \mathbb{N}$. Recall that the family $\{B_n(t)\}_{t\in\mathbb{R}}, n \in \mathbb{N}$ is defined by (see (3.2.5))

$$B_n(t) = P_n B(t) P_n, \quad P_n = \chi_{[-n,n]}(A_-).$$

In this case, the corresponding operator $\boldsymbol{A}_n$ is defined as

$$\boldsymbol{A}_n = \boldsymbol{A}_- + \boldsymbol{B}_n,$$

where $\boldsymbol{B}_n$ is defined by (3.1.11) with $\{T(t)\}_{t\in\mathbb{R}} = \{B_n(t)\}_{t\in\mathbb{R}}$.

Denote by $\boldsymbol{H}_{j,n}$, $j = 1, 2$, the operator defined by (3.1.15) with $\boldsymbol{D}_{\boldsymbol{A}}$ replaced by the corresponding operator $\boldsymbol{D}_{\boldsymbol{A}_n} = \frac{d}{dt} + \boldsymbol{A}_n$. Similarly to (3.1.18), assuming Hypothesis 3.1.9, one obtains the decompositions for the operators $\boldsymbol{H}_{j,n}$, $j = 1, 2$, of the following form

$$\begin{aligned}
\boldsymbol{H}_{j,n} &= \frac{d^2}{dt^2} + \boldsymbol{A}_n^2 + (-1)^j \boldsymbol{A}_n' \\
&= \boldsymbol{H}_0 + \boldsymbol{B}_n \boldsymbol{A}_- + \boldsymbol{A}_- \boldsymbol{B}_n + \boldsymbol{B}_n^2 + (-1)^j \boldsymbol{B}_n',
\end{aligned} \tag{3.2.8}$$
$$\mathrm{dom}(\boldsymbol{H}_{j,n}) = \mathrm{dom}(\boldsymbol{H}_0) = W^{2,2}(\mathbb{R}) \cap \mathrm{dom}\left(\boldsymbol{A}_-^2\right), \quad n \in \mathbb{N},\ j = 1, 2,$$

with

$$\boldsymbol{B}_n = \boldsymbol{P}_n \boldsymbol{B} \boldsymbol{P}_n, \quad \boldsymbol{B}_n' = \boldsymbol{P}_n \boldsymbol{B}' \boldsymbol{P}_n, \quad n \in \mathbb{N},$$

where $\boldsymbol{P}_n = \chi_{[-n,n]}(\boldsymbol{A}_-) = 1 \otimes P_n$.

The following result can be found in [**30**, Lemma 3.12 (i)] as well as [**28**].

LEMMA 3.2.6. *Assume Hypothesis 3.1.9. The operators* $\boldsymbol{H}_{j,n}$ *converge to* $\boldsymbol{H}_j$, $j = 1, 2$, *in the strong resolvent sense.*

Proof. Since the operators $\boldsymbol{H}_j$ and $\boldsymbol{H}_{j,n}$ have the common core $\operatorname{dom}(\boldsymbol{H}_0)$, by [**69**, Theorem VIII.25] (see also [**79**, Theorem 9.16]) it is sufficient to show that

$$\boldsymbol{H}_{j,n} f \underset{n\to\infty}{\longrightarrow} \boldsymbol{H}_j f, \quad f \in \operatorname{dom}(\boldsymbol{H}_0).$$

Using the decompositions (3.1.18) and (3.2.8) it is sufficient to show the convergence of every term separately. First, rewriting

$$\boldsymbol{B}' - \boldsymbol{B}'_n = \boldsymbol{B}' - \boldsymbol{P}_n \boldsymbol{B}' \boldsymbol{P}_n = (\boldsymbol{I} - \boldsymbol{P}_n)\boldsymbol{B}' + \boldsymbol{P}_n \boldsymbol{B}'(\boldsymbol{I} - \boldsymbol{P}_n),$$

the convergence $\operatorname{s-lim}_{n\to\infty} \boldsymbol{B}'_n = \boldsymbol{B}'$ follows since the operator $\boldsymbol{B}'$, is a bounded operator, and $\boldsymbol{P}_n \underset{n\to\infty}{\longrightarrow} \boldsymbol{I}$ in the strong operator topology. Arguing similarly, one infers that $\operatorname{s-lim}_{n\to\infty} \boldsymbol{B}_n = \boldsymbol{B}$. Next, one notes that

$$\begin{aligned}\boldsymbol{B}^2 - \boldsymbol{B}_n^2 &= \boldsymbol{B}^2 - \boldsymbol{P}_n \boldsymbol{B} \boldsymbol{P}_n \boldsymbol{B} \boldsymbol{P}_n \\ &= (\boldsymbol{I} - \boldsymbol{P}_n)\boldsymbol{B}^2 + \boldsymbol{P}_n \boldsymbol{B}\Big(\boldsymbol{B}(\boldsymbol{I} - \boldsymbol{P}_n) + (\boldsymbol{I} - \boldsymbol{P}_n)\boldsymbol{B}\boldsymbol{P}_n\Big),\end{aligned}$$

implying, $\operatorname{s-lim}_{n\to\infty} \boldsymbol{B}_n^2 = \boldsymbol{B}^2$. Thus, appealing to (3.1.18) and (3.2.8), it remains to show that $\operatorname{s-lim}_{n\to\infty} \boldsymbol{B}_n \boldsymbol{A}_- f = \boldsymbol{B}\boldsymbol{A}_- f$ and $\operatorname{s-lim}_{n\to\infty} \boldsymbol{A}_- \boldsymbol{B}_n f = \boldsymbol{A}_- \boldsymbol{B} f$ for all $f \in \operatorname{dom}(\boldsymbol{H}_0)$. The fact that,

$$\begin{aligned}\boldsymbol{B}\boldsymbol{A}_- f - \boldsymbol{B}_n \boldsymbol{A}_- f &= \boldsymbol{B}\boldsymbol{A}_- f - \boldsymbol{P}_n \boldsymbol{B}\boldsymbol{A}_- \boldsymbol{P}_n f \\ &= (\boldsymbol{I} - \boldsymbol{P}_n)\boldsymbol{B}\boldsymbol{A}_- f + \boldsymbol{P}_n \boldsymbol{B}(\boldsymbol{I} - \boldsymbol{P}_n)\boldsymbol{A}_- f, \quad f \in \operatorname{dom} \boldsymbol{H}_0\end{aligned}$$

and a similar equality $\boldsymbol{A}_- \boldsymbol{B}_n f = \boldsymbol{A}_- \boldsymbol{B} f$, $f \in \operatorname{dom} \boldsymbol{H}_0$ implies the required convergences. Consequently,

$$\lim_{n\to\infty} \boldsymbol{H}_{j,n} f = \boldsymbol{H}_j f, \quad f \in \operatorname{dom}(\boldsymbol{H}_0),$$

which completes the proof. □

3.3. Approximation results for the pair (A_+, A_-)

In this section we prove Theorem 3.3.2, which is our first key result. It can be found in [**28**].

Suppose that $j = 1, \dots, p$ and let $i \in \mathbb{N}$. For $z \in \mathbb{C}\setminus\mathbb{R}$ and $X_1, \dots, X_i \in \mathcal{B}(\mathcal{H})$ we introduce the following mappings

$$\begin{aligned}T_1^{(j)}(X_1) &= \sum_{k_0+k_1=j-1} (A_- - z)^{-k_0-1} X_1 (A_- - z)^{-k_1-1}, \\ T_2^{(j)}(X_1, X_2) &= \sum_{k_0+k_1+k_2=j-1} (A_- - z)^{-k_0-1} X_1 (A_- - z)^{-k_1-1} X_2 (A_- - z)^{-k_2-1}, \\ &\dots \\ T_i^{(j)}(X_1, \dots, X_i) &= \sum_{k_0+\dots+k_i=j-1} (A_- - z)^{-k_0-1} X_1 (A_- - z)^{-k_1-1} \dots X_i (A_- - z)^{-k_i-1},\end{aligned} \tag{3.3.1}$$

and for a self-adjoint $B \in \mathcal{B}(\mathcal{H})$ we denote

$$\begin{aligned}&\widetilde{T}_i^{(j)}(B; X_1, \dots, X_i) \\ &\quad = \sum_{k_0+\dots+k_i=j-1} (A_- + B - z)^{-k_0-1} X_1 (A_- - z)^{-k_1-1} \dots X_i (A_- - z)^{-k_i-1}.\end{aligned}$$

PROPOSITION 3.3.1. *Fix $j = 1, \dots, p$ and assume that B is a p-relative trace-class perturbation of A_-, that is, $B(A_- + 1)^{-p-1} \in \mathcal{L}_1(\mathcal{H})$. Set $P_n = \chi_{[-n,n]}(A_-)$ and $B_n := P_n B P_n$. The following assertions hold:*

(i) For every $i \in \mathbb{N}$ we have $T_i^{(j)}(B, \dots, B) \in \mathcal{L}_{\frac{p+1}{j+i}}(\mathcal{H})$.

(ii) For every $i \in \mathbb{N}$ we have the convergence

$$\|\cdot\|_{\frac{p+1}{j+i}} - \lim_{n\to\infty} T_i^{(j)}(B_n, \dots, B_n) = T_i^{(j)}(B, \dots, B).$$

(iii) For every $i \in \mathbb{N}$ we have $\widetilde{T}_i^{(j)}(B; B, \dots, B) \in \mathcal{L}_{\frac{p+1}{i}}(\mathcal{H})$.

(iv) For every $i \in \mathbb{N}$ we have the convergence

$$\|\cdot\|_{\frac{p+1}{i}} - \lim_{n\to\infty} \widetilde{T}_i^{(j)}(B_n; B_n, \dots, B_n) = \widetilde{T}_i^{(j)}(B; B, \dots, B).$$

PROOF. (i). Since the operator B is p-relative trace-class perturbation of A_-, Remark 3.1.4 implies that

$$(A_- - z)^{-k_0-1} B (A_- - z)^{-k_1-1} \in \mathcal{L}_{\frac{p+1}{k_0+k_1+2}}(\mathcal{H}) \tag{3.3.2}$$

and

$$B(A_- - z)^{-k_l-1} \in \mathcal{L}_{\frac{p+1}{k_l+1}}, \quad l = 2, \dots i. \tag{3.3.3}$$

Hence, by Hölder inequality we have that

$$(A_- - z)^{-k_0-1} B (A_- - z)^{-k_1-1} \dots B(A_- - z)^{-k_i-1} \in \mathcal{L}_{\frac{p+1}{k_0+k_1+\dots+k_i+i+1}}(\mathcal{H}).$$

Since $k_0 + k_1 + \dots + k_i = j - 1$, we obtain that

$$\begin{aligned} T_i^{(j)}(B, \dots, B) = &\sum_{k_0+\dots+k_i=j-1} (A_- - z)^{-k_0-1} B (A_- - z)^{-k_1-1} \dots B(A_- - z)^{-k_i-1} \\ &\in \mathcal{L}_{\frac{p+1}{j+i}}(\mathcal{H}), \end{aligned}$$

as required.

(ii). Since $P_n \to 1$ in the strong operator topology, Lemma 3.2.1 and inclusions (3.3.2), (3.3.3) imply that

$$\begin{aligned} (A_- - z)^{-k_0-1} B_n (A_- - z)^{-k_1-1} &= P_n (A_- - z)^{-k_0-1} B (A_- - z)^{-k_1-1} P_n \\ &\to (A_- - z)^{-k_0-1} B (A_- - z)^{-k_1-1} \end{aligned} \tag{3.3.4}$$

in $\mathcal{L}_{\frac{p+1}{k_0+k_1+2}}(\mathcal{H})$ and

$$B_n(A_- - z)^{-k_l-1} = P_n B (A_- - z)^{-k_l-1} P_n \to B(A_- - z)^{-k_l-1} \tag{3.3.5}$$

in $\mathcal{L}_{\frac{p+1}{k_l+1}}(\mathcal{H})$ for every $l = 2, \dots i$. Hence, using again Hölder inequality we obtain that

$$\|\cdot\|_{\frac{p+1}{j+i}} - \lim_{n\to\infty} T_i^{(j)}(B_n, \dots, B_n) = T_i^{(j)}(B, \dots, B).$$

(iii). By Remark 3.1.4 we have that

$$B(A_- - z)^{-1} \in \mathcal{L}_{p+1}(\mathcal{H}),$$

which implies that

$$B(A_- - z)^{-k_l-1} \in \mathcal{L}_{p+1}(\mathcal{H}),$$

for any $k_l \in \mathbb{Z}_+$. Therefore, by the Hölder inequality

$$B(A_- - z)^{-k_1-1} \dots B(A_- - z)^{-k_i-1} \in \mathcal{L}_{\frac{p+1}{i}}(\mathcal{H}),$$

which implies that

$$\widetilde{T}_i^{(j)}(B; B, \dots, B) \in \mathcal{L}_{\frac{p+1}{i}}(\mathcal{H}).$$

The proof of part (iv) can be obtained similarly to (ii) taking into account also that

$$(A_- + B_n - z)^{-l} \to (A_- + B - z)^{-l}, \quad l \in \mathbb{N}$$

with respect to the strong operator topology. □

The following theorem establishes, in particular, that the p-relative trace-class assumption (Hypothesis 3.1.1 (iii)) implies that A_+ and A_- are p-resolvent comparable. The theorem below is our first key result.

THEOREM 3.3.2. [**28**] *Assume Hypothesis 3.1.1 and let $z \in \mathbb{C}\backslash\mathbb{R}$, $j = 1, \dots, p$. Then*

$$(A_+ - z)^{-j} - (A_- - z)^{-j}, \quad (A_{+,n} - z)^{-j} - (A_- - z)^{-j} \in \mathcal{L}_{\frac{p+1}{j+1}}(\mathcal{H})$$

and

$$\lim_{n\to\infty} \Big\| \Big[(A_{+,n} - z)^{-j} - (A_- - z)^{-j} \Big] - \Big[(A_+ - z)^{-j} - (A_- - z)^{-j} \Big] \Big\|_{\frac{p+1}{j+1}} = 0.$$

PROOF. Using the elementary identity

$$A^j - B^j = \sum_{k_0+k_1=j-1} A^{k_0}[A - B]B^{k_1}, \quad A, B \in \mathcal{B}(\mathcal{H}),\ j \in \mathbb{N},$$

and the resolvent identity we can write

$$\begin{aligned}
&(A_+ - z)^{-j} - (A_- - z)^{-j} \\
&\qquad = \sum_{k_0+k_1=j-1} (A_+ - z)^{-k_0}\Big((A_+ - z)^{-1} - (A_- - z)^{-1} \Big)(A_- - z)^{-k_1} \\
&\qquad = - \sum_{k_0+k_1=j-1} (A_+ - z)^{-k_0-1} B_+ (A_- - z)^{k_1-1}.
\end{aligned}$$

Writing

$$(A_+ - z)^{-k_0-1} = (A_- - z)^{-k_0-1} + \big((A_+ - z)^{-k_0-1} - (A_- - z)^{-k_0-1}\big)$$

and repeating the same argument for the second term on the right-hand side we obtain

$$\begin{aligned}
&(A_+ - z)^{-j} - (A_- - z)^{-j} \\
&\qquad = - \sum_{k_0+k_1=j-1} (A_- - z)^{-k_0-1} B_+ (A_- - z)^{k_1-1} \\
&\qquad\quad - \sum_{k_0+k_1=j-1} \Big((A_+ - z)^{-k_0-1} - (A_- - z)^{-k_0-1}\Big) B_+ (A_- - z)^{k_1-1} \\
&\qquad = - \sum_{k_0+k_1=j-1} (A_- - z)^{-k_0-1} B_+ (A_- - z)^{k_1-1} \\
&\qquad\quad - \sum_{k_0+k_1+k_2=j-1} (A_+ - z)^{-k_0-1} B_+ (A_- - z)^{-k_1-1} B_+ (A_- - z)^{k_1-1} \\
&\qquad = -T_1(B_+) - \widetilde{T}_2(B_+; B_+).
\end{aligned}$$

Repeating this process we can write

$$(A_+ - z)^{-j} - (A_- - z)^{-j}$$
$$= T_1^{(j)}(B_+) + \cdots + T_j^{(j)}(B_+, \ldots, B_+) + \widetilde{T}_{j+1}^{(j)}(B_+; B_+, \ldots, B_+). \tag{3.3.6}$$

By Proposition 3.3.1 we have that

$$T_i^{(j)}(B_+, \ldots, B_+) \in \mathcal{L}_{\frac{p+1}{j+i}}(\mathcal{H}) \subset \mathcal{L}_{\frac{p+1}{j+1}}(\mathcal{H}), \quad i = 1, \ldots, j$$

and

$$\widetilde{T}_{j+1}^{(j)}(B_+; B_+, \ldots, B_+) \in \mathcal{L}_{\frac{p+1}{j+1}}(\mathcal{H}).$$

Hence,

$$(A_+ - z)^{-j} - (A_- - z)^{-j} \in \mathcal{L}_{\frac{p+1}{j+1}}(\mathcal{H}).$$

To prove the convergence one can repeat the argument above to write

$$\begin{aligned}(A_{+,n} - z)^{-j} &- (A_- - z)^{-j} \\ &= T_1^{(j)}(B_{+,n}) + \cdots + T_j^{(j)}(B_{+,n}, \ldots, B_{+,n}) \\ &\quad + \widetilde{T}_{j+1}^{(j)}(B_{+,n}; B_{+,n}, \ldots, B_{+,n}).\end{aligned} \tag{3.3.7}$$

By Proposition 3.3.1 we have that

$$\| \cdot \|_{\frac{p+1}{j+i}} - \lim_{n\to\infty} T_i^{(j)}(B_{+,n}, \ldots, B_{+,n}) = T_i^{(j)}(B_+, \ldots, B_+)$$

and

$$\| \cdot \|_{\frac{p+1}{j+1}} - \lim_{n\to\infty} \widetilde{T}_{j+1}^{(j)}(B_{+,n}; B_{+,n}, \ldots, B_{+,n}) = \widetilde{T}_{j+1}^{(j)}(B_+; B_+, \ldots, B_+).$$

Thus, combining these convergences with equalities (3.3.6) and (3.3.7) we conclude the proof. $\square$

REMARK 3.3.3. Repeating the proof of Theorem 3.3.2 replacing A_+ and $A_{+,n}$ by the operators $A_s := A_- + sB_+$ and $A_{s,n} := A_- + sP_nB_+P_n$, $s \in (0, 1]$, respectively, and referring to (3.1.9) and (3.1.10), one can conclude that the functions

$$s \mapsto \| (A_s - z)^{-j} - (A_- - z)^{-j} \|_{\frac{p+1}{j+1}},$$
$$s \mapsto \| (A_{s,n} - z)^{-j} - (A_- - z)^{-j} \|_{\frac{p+1}{j+1}}$$

are continuous with respect to s and uniformly bounded with respect to $n \in \mathbb{N}$.

COROLLARY 3.3.4. *If a function h on $\mathbb{R}$ is such that $h \in L_\infty(\mathbb{R}, (\lambda^2 + 1)^{-\frac{p+1}{2}} d\lambda)$, then the sequence of functions*

$$s \mapsto \|h(A_{s,n})(A_{+,n} - A_-)\|_1, \quad s \in [0, 1]$$

is uniformly bounded (with respect to n) by a continuous function.

PROOF. By the assumption the function $t \mapsto h(t)(t + i)^{-p-1}$, $t \in \mathbb{R}$, is bounded, and therefore we can write

$$\begin{aligned}\|h(A_{s,n})(A_{+,n} - A_-)\|_1 &\le \|h(A_{s,n})(A_{s,n} + i)^{p+1}\|_\infty \cdot \|(A_{s,n} + i)^{p+1}(A_{+,n} - A_-)\|_1 \\ &\le \text{const} \,\|\big((A_{s,n} + i)^{p+1} - (A_- + i)^{p+1}\big)(A_{+,n} - A_-)\|_1 \\ &\quad + \text{const} \,\|(A_- + i)^{p+1}(A_{+,n} - A_-)\|_1,\end{aligned}$$

where the constant is independent of s and n.

By Remark 3.3.3 the first terms is a sequence of functions uniformly majorised by a continuous function. Since $A_{+,n} - A_- = P_n B_+ P_n$, the second terms is clearly uniformly majorised by the constant function

$$\text{const}\, \|(A_- + i)^{p+1} B_+\|_1.$$

□

3.4. Some uniform norm estimates

In this section we prove necessary uniform norm estimates (see Proposition 3.4.5), which will be used in the proof of one of the major results, Theorem 3.6.1 below. The technique used here comes from noncommutative geometry (see e.g. [**24**]).

For $k \in \mathbb{N}$ we introduce

$$\begin{gathered}\operatorname{dom}(\delta^k_{\boldsymbol{H}_0}) = \{\boldsymbol{T} \in \mathcal{B}(L_2(\mathbb{R}, \mathcal{H})) : \boldsymbol{T} \operatorname{dom}(\boldsymbol{H}_0^{j/2}) \subset \operatorname{dom}(\boldsymbol{H}_0^{j/2}), \forall j = 1, \dots, k, \\ \text{and the operator } [(1 + \boldsymbol{H}_0)^{1/2}, \boldsymbol{T}]^{(k)}, \text{ defined on } \operatorname{dom}(\boldsymbol{H}_0^{k/2}) \\ \text{extends to a bounded operator on } L_2(\mathbb{R}, \mathcal{H})\}.\end{gathered}$$

and set

$$\delta^k_{\boldsymbol{H}_0}(\boldsymbol{T}) = \overline{[(1 + \boldsymbol{H}_0)^{1/2}, \boldsymbol{T}]^{(k)}}, \quad \boldsymbol{T} \in \operatorname{dom} \delta^k_{\boldsymbol{H}_0}. \tag{3.4.1}$$

where the notation $[(1+\boldsymbol{H}_0)^{1/2}, \boldsymbol{T}]^{(k)}$ stands for k-th repeated commutator defined by

$$\begin{gathered}[(1 + \boldsymbol{H}_0)^{1/2}, T]^{(k)} = [(1 + \boldsymbol{H}_0)^{1/2}, \dots [(1 + \boldsymbol{H}_0)^{1/2}, [(1 + \boldsymbol{H}_0)^{1/2}, T]] \dots], \\ \operatorname{dom}([(1 + \boldsymbol{H}_0)^{1/2}, T]^{(k)}) = \operatorname{dom}(\boldsymbol{H}_0^{k/2}).\end{gathered}$$

For convenience, we set

$$[(1 + \boldsymbol{H}_0)^{1/2}, \boldsymbol{T}]^{(0)} = \boldsymbol{T}.$$

Remark 3.4.1. For all $k \in \mathbb{N}$, the set $\bigcap_{j=0}^k \operatorname{dom}(\delta^j_{\boldsymbol{H}_0})$ is a subalgebra in $\mathcal{B}(L_2(\mathbb{R}, \mathcal{H}))$.

We note that if $\boldsymbol{T} \in \operatorname{dom}(\delta_{\boldsymbol{H}_0})$, then for every $\xi \in \operatorname{dom}(\boldsymbol{H}_0^{1/2})$ we have

$$\boldsymbol{T}(\boldsymbol{H}_0 + 1)^{1/2}\xi = (\boldsymbol{H}_0 + 1)^{1/2}\boldsymbol{T}\xi - [(\boldsymbol{H}_0 + 1)^{1/2}, \boldsymbol{T}]\xi.$$

Hence, if $\boldsymbol{T} \in \bigcap_{j=1}^k \operatorname{dom}(\delta^j_{\boldsymbol{H}_0})$, for some $k \in \mathbb{N}$, then for every $\xi \in \operatorname{dom}(\boldsymbol{H}_0^{k/2})$, using this equality repeatedly we obtain

$$\begin{aligned}\boldsymbol{T}(\boldsymbol{H}_0 + 1)^{k/2}\xi &= \boldsymbol{T}(\boldsymbol{H}_0 + 1)^{1/2}(\boldsymbol{H}_0 + 1)^{\frac{k-1}{2}}\xi \\ &= (\boldsymbol{H}_0 + 1)^{1/2}\boldsymbol{T}(\boldsymbol{H}_0 + 1)^{\frac{k-1}{2}}\xi - [(\boldsymbol{H}_0 + 1)^{1/2}, \boldsymbol{T}](\boldsymbol{H}_0 + 1)^{\frac{k-1}{2}}\xi \\ &= \dots \\ &= \sum_{j=0}^{k}(-1)^j C_k^j (\boldsymbol{H}_0 + 1)^{j/2}[(\boldsymbol{H}_0 + 1)^{1/2}, \boldsymbol{T}]^{(k-j)}\xi,\end{aligned} \tag{3.4.2}$$

where C_k^j denotes the binomial coefficient.

LEMMA 3.4.2. *Assume* $\boldsymbol{B} \in \operatorname{dom}(\delta_{\boldsymbol{H}_0})$. *Then the operator* $(\boldsymbol{H}_i+1)^{-1/2}(\boldsymbol{H}_0+1)^{1/2}, i=1,2,$ *defined on* $\operatorname{dom}(\boldsymbol{H}_0^{1/2})$ *extends to a bounded operator on* $L_2(\mathbb{R},\mathcal{H})$ *and*

$$\left\|\overline{(\boldsymbol{H}_i+1)^{-1/2}(\boldsymbol{H}_0+1)^{1/2}}\right\| \le \operatorname{const}\cdot(\|\delta_{\boldsymbol{H}_0}(\boldsymbol{B})\|+\|\boldsymbol{B}\|+\|\boldsymbol{B}\|^2+\|\boldsymbol{B}'\|).$$

REMARK 3.4.3. Note that the first assertion in Lemma 3.4.2 follows immediately from the closed graph theorem and the fact that $\operatorname{dom}(\boldsymbol{H}_0)=\operatorname{dom}(\boldsymbol{H}_i), i=1,2$. However, we also need an estimate on the uniform norm of this operator.

PROOF. Since the operator $\boldsymbol{H}_1$ is positive we can write

$$(\boldsymbol{H}_1+1)^{-1/2}=\frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}(1+\lambda+\boldsymbol{H}_1)^{-1},$$

with the right-hand side being a convergent Bochner integral (see, e.g., [**52**, p. 282] for a more general result).

By the resolvent identity (3.1.19) we have

$$\begin{aligned}&(1+\lambda+\boldsymbol{H}_1)^{-1}=(1+\lambda+\boldsymbol{H}_0)^{-1}\\&\quad-(1+\lambda+\boldsymbol{H}_1)^{-1}(\boldsymbol{B}\boldsymbol{A}_-+\boldsymbol{A}_-\boldsymbol{B}+\boldsymbol{B}^2-\boldsymbol{B}')(1+\lambda+\boldsymbol{H}_0)^{-1}.\end{aligned}$$

Therefore, for all $\xi\in\operatorname{dom}(\boldsymbol{H}_0)^{1/2}$ we have

$$\begin{aligned}&(\boldsymbol{H}_1+1)^{-1/2}(\boldsymbol{H}_0+1)^{1/2}\xi\\&\quad=\xi+\frac{1}{\pi}\int_0^\infty\frac{d\lambda}{\lambda^{1/2}}(1+\lambda+\boldsymbol{H}_1)^{-1}\boldsymbol{A}_-\boldsymbol{B}\frac{(\boldsymbol{H}_0+1)^{1/2}}{1+\lambda+\boldsymbol{H}_0}\xi\\&\quad\quad+\frac{1}{\pi}\int_0^\infty\frac{d\lambda}{\lambda^{1/2}}(1+\lambda+\boldsymbol{H}_1)^{-1}\boldsymbol{B}\frac{\boldsymbol{A}_-}{(1+\lambda+\boldsymbol{H}_0)^{1/2}}\frac{(\boldsymbol{H}_0+1)^{1/2}}{(1+\lambda+\boldsymbol{H}_0)^{1/2}}\xi\\&\quad\quad+\frac{1}{\pi}\int_0^\infty\frac{d\lambda}{\lambda^{1/2}}(1+\lambda+\boldsymbol{H}_1)^{-1}(\boldsymbol{B}^2-\boldsymbol{B}')\frac{(\boldsymbol{H}_0+1)^{1/2}}{1+\lambda+\boldsymbol{H}_0}\xi\\&\quad=\xi+I_1\xi+I_2\xi+I_3\xi.\end{aligned}\tag{3.4.3}$$

Since

$$\left\|(1+\lambda+\boldsymbol{H}_0)^{-1}(\boldsymbol{H}_0+1)^{1/2}\right\|_\infty\le(1+\lambda)^{-1/2},$$

and

$$\left\|(1+\lambda+\boldsymbol{H}_1)^{-1}\right\|_\infty\le(1+\lambda)^{-1},$$

and the operators $\boldsymbol{B},\boldsymbol{B}'$ are bounded we obtain that the operator I_3 on the RHS of (3.4.3) converges in the uniform norm and

$$\|I_3\|\le\operatorname{const}(\|\boldsymbol{B}\|^2+\|\boldsymbol{B}'\|).$$

Similarly for I_2, using instead the estimate (see Lemma 3.1.8)

$$\left\|\boldsymbol{A}_-(1+\lambda+\boldsymbol{H}_0)^{-1/2}\right\|_\infty\le(1+\lambda)^{-1/2},$$

we obtain that I_2 is a bounded operator and

$$\|I_2\|\le\operatorname{const}\cdot\|\boldsymbol{B}\|.$$

Finally, for I_1 we write

$$\begin{aligned}
I_1\xi =& \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}(1+\lambda+\boldsymbol{H}_1)^{-1}\frac{\boldsymbol{A}_-}{(\boldsymbol{H}_0+1)^{1/2}}(\boldsymbol{H}_0+1)^{1/2}\boldsymbol{B}\,\frac{(\boldsymbol{H}_0+1)^{1/2}}{1+\lambda+\boldsymbol{H}_0}\xi \\
=& \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}(1+\lambda+\boldsymbol{H}_1)^{-1}\frac{\boldsymbol{A}_-}{(\boldsymbol{H}_0+1)^{1/2}}\boldsymbol{B}\frac{\boldsymbol{H}_0+1}{1+\lambda+\boldsymbol{H}_0}\xi \\
&+ \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}(1+\lambda+\boldsymbol{H}_1)^{-1}\frac{\boldsymbol{A}_-}{(\boldsymbol{H}_0+1)^{1/2}} \\
&\qquad\qquad \cdot [(\boldsymbol{H}_0+1)^{1/2},\boldsymbol{B}]\,\frac{(\boldsymbol{H}_0+1)^{1/2}}{1+\lambda+\boldsymbol{H}_0}\xi.
\end{aligned}$$

Since, $\boldsymbol{B}\in \mathrm{dom}(\delta_{\boldsymbol{H}_0})$, the operator $[(\boldsymbol{H}_0+1)^{1/2},\boldsymbol{B}]$ extends to a bounded operator on $L_2(\mathbb{R},\mathcal{H})$. Hence, repeating the argument above, we conclude that I_1 is a bounded operator with

$$\|I_1\| \le \mathrm{const}\cdot(\|\boldsymbol{B}\| + \|\delta_{\boldsymbol{H}_0}(\boldsymbol{B})\|).$$

Thus, by (3.4.3) we have that the operator $(\boldsymbol{H}_0+1)^{-1/2}(\boldsymbol{H}_1+1)^{1/2}$ extends to a bounded operator on $L_2(\mathbb{R},\mathcal{H})$ and

$$\left\|\overline{(\boldsymbol{H}_0+1)^{-1/2}(\boldsymbol{H}_1+1)^{1/2}}\right\|_\infty \le \mathrm{const}(1+\|\delta_{\boldsymbol{H}_0}(\boldsymbol{B})\| + \|\boldsymbol{B}\| + \|\boldsymbol{B}\|^2 + \|\boldsymbol{B}'\|).$$

□

The following result will be used later in the proof of the convergence of the left-hand side of the principal trace formula.

Proposition 3.4.4. *Assume* $\boldsymbol{B},\boldsymbol{B}'\in\bigcap_{j=1}^{k-1}\mathrm{dom}(\delta^j_{\boldsymbol{H}_0})$ *for some* $k\ge 2$. *Then the operator* $(\boldsymbol{H}_i+1)^{-k/2}(\boldsymbol{H}_0+1)^{k/2}, i=1,2,$ *defined on* $\mathrm{dom}(\boldsymbol{H}_0^{k/2})$ *extends to a bounded operator on* $L_2(\mathbb{R},\mathcal{H})$ *and*

$$\left\|\overline{(\boldsymbol{H}_i+1)^{-k/2}(\boldsymbol{H}_0+1)^{k/2}}\right\| \le \mathrm{const}\cdot Q(\|\delta^j_{\boldsymbol{H}_0}(\boldsymbol{B})\|, \|\delta^j_{\boldsymbol{H}_0}(\boldsymbol{B}')\|), \quad j=0,\dots,k-1,$$

for some polynomial Q *with positive coefficients.*

Proof. We prove the assertion only for $i=1$, since the proof for $i=2$ is identical.

We proceed by induction on k. For $k=1$ the assertion is proved in Lemma 3.4.2. Let $k=2$. By the resolvent identity (3.1.19) we have

$$\begin{aligned}
(\boldsymbol{H}_1&+1)^{-1}(\boldsymbol{H}_0+1)\xi \\
&= \xi - (\boldsymbol{H}_1+1)^{-1}\boldsymbol{B}\boldsymbol{A}_-\xi - (\boldsymbol{H}_1+1)^{-1}\boldsymbol{A}_-\boldsymbol{B}\xi \\
&\quad - (\boldsymbol{H}_1+1)^{-1}(\boldsymbol{B}^2-\boldsymbol{B}')\xi
\end{aligned} \tag{3.4.4}$$

for all $\xi\in\mathrm{dom}(\boldsymbol{H}_1)=\mathrm{dom}(\boldsymbol{H}_0)$. For the second term we write

$$\begin{aligned}
(\boldsymbol{H}_1+1)^{-1}\boldsymbol{B}\boldsymbol{A}_-\xi &= (\boldsymbol{H}_1+1)^{-1}\boldsymbol{B}(\boldsymbol{H}_0+1)^{1/2}\boldsymbol{A}_-(\boldsymbol{H}_0+1)^{-1/2}\xi \\
&= (\boldsymbol{H}_1+1)^{-1/2}\cdot(\boldsymbol{H}_1+1)^{-1/2}(\boldsymbol{H}_0+1)^{1/2}\cdot\boldsymbol{B}\boldsymbol{A}_-(\boldsymbol{H}_0+1)^{-1/2}\xi \\
&\quad - (\boldsymbol{H}_1+1)^{-1}[(\boldsymbol{H}_0+1)^{1/2},\boldsymbol{B}]\boldsymbol{A}_-(\boldsymbol{H}_0+1)^{-1/2}\xi.
\end{aligned}$$

By Lemma 3.4.2, the operator $(\boldsymbol{H}_1+1)^{-1/2}(\boldsymbol{H}_0+1)^{1/2}$ extends to a bounded operator, and by the assumption $[(\boldsymbol{H}_0+1)^{1/2},\boldsymbol{B}]$ also extends to a bounded operator. Hence, the operator $(\boldsymbol{H}_1+1)^{-1}\boldsymbol{B}\boldsymbol{A}_-$ extends to a bounded operator and

$$\begin{aligned}\left\|\overline{(\boldsymbol{H}_1+1)^{-1}\boldsymbol{B}\boldsymbol{A}_-}\right\| &\le \left\|\overline{(\boldsymbol{H}_1+1)^{1/2}(\boldsymbol{H}_0+1)^{-1/2}}\right\|\|\boldsymbol{B}\|+\|\delta_{\boldsymbol{H}_0}(\boldsymbol{B})\|\\ &\le \operatorname{const}\cdot(1+\|\delta_{\boldsymbol{H}_0}(\boldsymbol{B})\|+\|\boldsymbol{B}\|+\|\boldsymbol{B}\|^2+\|\boldsymbol{B}'\|)\cdot\|\boldsymbol{B}\|\\ &\qquad+\|\delta_{\boldsymbol{H}_0}(\boldsymbol{B})\|,\end{aligned}$$

where the latter inequality follows from Lemma 3.4.2.

For the third term on the right hand side (3.4.4), we write

$$(\boldsymbol{H}_1+1)^{-1}\boldsymbol{A}_-\boldsymbol{B}\xi=(\boldsymbol{H}_1+1)^{-1/2}\cdot(\boldsymbol{H}_1+1)^{-1/2}(\boldsymbol{H}_0+1)^{1/2}\cdot(\boldsymbol{H}_0+1)^{-1/2}\boldsymbol{A}_-\boldsymbol{B}\xi,$$

for $\xi\in\operatorname{dom}(\boldsymbol{H}_0)$, and therefore, by Lemma 3.4.2 we conclude that $\frac{1}{\boldsymbol{H}_1+1}\boldsymbol{A}_-\boldsymbol{B}$ also extends to a bounded operator.

Thus, $(\boldsymbol{H}_1+1)^{-1}(\boldsymbol{H}_0+1)$ extends to a bounded operator and by (3.4.4) we have

$$\begin{aligned}\left\|\overline{(\boldsymbol{H}_1+1)^{-1}(\boldsymbol{H}_0+1)}\right\| &\le 1+\left\|\overline{(\boldsymbol{H}_1+1)^{-1}\boldsymbol{B}\boldsymbol{A}_-}\right\|\\ &\qquad+\left\|\overline{(\boldsymbol{H}_1+1)^{-1}\boldsymbol{A}_-\boldsymbol{B}}\right\|+\left\|(\boldsymbol{H}_1+1)^{-1}(\boldsymbol{B}^2-\boldsymbol{B}')\right\|\\ &\le Q(\|\delta^j_{\boldsymbol{H}_0}(\boldsymbol{B})\|),\quad j=0,1.\end{aligned}$$

for some polynomial Q.

Suppose now that for some $k\ge 3$ the assertion holds for all $j\le k-1$. Let us prove it for $j=k$. For $\xi\in\operatorname{dom}(\boldsymbol{H}_0^{k/2})$, using the resolvent identity (3.1.19) we write

$$\begin{aligned}(\boldsymbol{H}_1+1)^{-k/2}(\boldsymbol{H}_0+1)^{k/2}\xi &= (\boldsymbol{H}_1+1)^{-(k-2)/2}(\boldsymbol{H}_1+1)^{-1}(\boldsymbol{H}_0+1)^{k/2}\xi\\ &= (\boldsymbol{H}_1+1)^{-(k-2)/2}(\boldsymbol{H}_0+1)^{(k-2)/2}\xi\\ &\quad+(\boldsymbol{H}_1+1)^{-k/2}(\boldsymbol{B}\boldsymbol{A}_-+\boldsymbol{A}_-\boldsymbol{B}+\boldsymbol{B}^2-\boldsymbol{B}')(\boldsymbol{H}_0+1)^{(k-2)/2}\xi.\end{aligned}\tag{3.4.5}$$

By the induction assumption, the operator $(\boldsymbol{H}_1+1)^{-(k-2)/2}(\boldsymbol{H}_0+1)^{(k-2)/2}$ extends to a bounded operator with the required estimate in the uniform norm.

For the second term on the right hand side of (3.4.5) equality (3.4.2) implies that

$$\begin{aligned}&(\boldsymbol{H}_1+1)^{-k/2}(\boldsymbol{B}\boldsymbol{A}_-+\boldsymbol{A}_-\boldsymbol{B}+\boldsymbol{B}^2-\boldsymbol{B}')(\boldsymbol{H}_0+1)^{(k-2)/2}\xi\\ &=\sum_{j=0}^{k-1}(-1)^jC_k^j(\boldsymbol{H}_1+1)^{-k/2}(\boldsymbol{H}_0+1)^{j/2}[(\boldsymbol{H}_0+1)^{1/2},\boldsymbol{B}]^{(j)}\boldsymbol{A}_-(\boldsymbol{H}_0+1)^{-1/2}\xi\\ &+\sum_{j=0}^{k-2}(-1)^jC_k^j(\boldsymbol{H}_1+1)^{-k/2}(\boldsymbol{H}_0+1)^{(j+1)/2}\boldsymbol{A}_-(\boldsymbol{H}_0+1)^{-1/2}[(\boldsymbol{H}_0+1)^{1/2},\boldsymbol{B}]^{(j)}\xi\\ &+\sum_{j=0}^{k-2}(-1)^jC_k^j(\boldsymbol{H}_1+1)^{-k/2}(\boldsymbol{H}_0+1)^{j/2}[(\boldsymbol{H}_0+1)^{1/2},\boldsymbol{B}^2-\boldsymbol{B}']^{(j)}\xi\end{aligned}$$

By the induction assumption, for every $j=0,\dots,k-1$, it follows that the operator

$$(\boldsymbol{H}_1+1)^{-k/2}(\boldsymbol{H}_0+1)^{j/2}$$

extends to a bounded operator. In addition, the operators

$$[(\boldsymbol{H}_0+1)^{1/2}, \boldsymbol{B}]^{(j)} \text{ and } [(\boldsymbol{H}_0+1)^{1/2}, \boldsymbol{B}']^{(j)}$$

also extend to bounded operators by the assumption of the proposition. Hence,

$$(\boldsymbol{H}_1+1)^{-k/2}(\boldsymbol{B}\boldsymbol{A}_- + \boldsymbol{A}_-\boldsymbol{B} + \boldsymbol{B}^2 - \boldsymbol{B}')(\boldsymbol{H}_0+1)^{(k-2)/2}$$

extends to a bounded operator and the required estimate in the uniform norm follows. □

PROPOSITION 3.4.5. *Let $\boldsymbol{B}, \boldsymbol{B}' \in \bigcap_{j=1}^{k-1} \operatorname{dom}(\delta^j_{\boldsymbol{H}_0})$ for some $k \in \mathbb{N}$. Then*

(i) *The operator $\overline{(\boldsymbol{H}_0-z)^{k/2}(\boldsymbol{H}_i-z)^{-k/2}}$, $i=1,2$ is bounded.*

(ii) *The operators $\overline{(\boldsymbol{H}_{i,n}-z)^{-k/2}(\boldsymbol{H}_0-z)^{k/2}}$, and $(\boldsymbol{H}_0-z)^{k/2}(\boldsymbol{H}_{i,n}-z)^{-k/2}$, $i=1,2$ are bounded.*

(iii) *The sequences*

$$\left\{\overline{(\boldsymbol{H}_{i,n}-z)^{-k/2}(\boldsymbol{H}_0-z)^{k/2}}\right\}_{n=1}^{\infty}, \quad \left\{(\boldsymbol{H}_0-z)^{k/2}(\boldsymbol{H}_{i,n}-z)^{-k/2}\right\}_{n=1}^{\infty}$$

are uniformly bounded.

PROOF. Without loss of generality we can assume that $z=-1$.
(i). As the operators $(\boldsymbol{H}_0+1)^{k/2}$ and $(\boldsymbol{H}_{i,n}+1)^{-k/2}$ are self-adjoint, both of the operators $(\boldsymbol{H}_{i,n}+1)^{-k/2}$ and $(\boldsymbol{H}_{i,n}+1)^{-k/2}(\boldsymbol{H}_0+1)^{k/2}$ are densely defined, [**79**, Theorem 4.19 (b)] implies that

$$\begin{aligned}(\boldsymbol{H}_0+1)^{k/2}(\boldsymbol{H}_i+1)^{-k/2} &= \left((\boldsymbol{H}_i+1)^{-k/2}(\boldsymbol{H}_0+1)^{k/2}\right)^* \\ &= \left(\overline{(\boldsymbol{H}_i+1)^{-k/2}(\boldsymbol{H}_0+1)^{k/2}}\right)^* \in \mathcal{B}(L_2(\mathbb{R},\mathcal{H})),\end{aligned}$$

where the last inclusion follows from Proposition 3.4.4.

(ii). Since $\boldsymbol{B}, \boldsymbol{B}' \in \bigcap_{j=1}^{k-1} \operatorname{dom}(\delta^j_{\boldsymbol{H}_0})$, $\boldsymbol{B}_n = \boldsymbol{P}_n\boldsymbol{B}\boldsymbol{P}_n$, $\boldsymbol{B}'_n = \boldsymbol{P}_n\boldsymbol{B}'\boldsymbol{P}_n$, and $\boldsymbol{P}_n$ commutes with $\boldsymbol{H}_0$, we infer that $\boldsymbol{B}_n, \boldsymbol{B}'_n \in \bigcap_{j=1}^{k-1} \operatorname{dom}(\delta^j_{\boldsymbol{H}_0})$. Therefore, applying Proposition 3.4.4 and part (i) to the operators $\boldsymbol{H}_{i,n}$ and $\boldsymbol{H}_0$, we obtain the assertion.

(iii). Note that for $j=1,\dots,k-1$, we have

$$\|\delta^j_{\boldsymbol{H}_0}(\boldsymbol{B}_n)\| \le \|\delta^j_{\boldsymbol{H}_0}(\boldsymbol{B})\|, \quad \|\delta^j_{\boldsymbol{H}_0}(\boldsymbol{B}'_n)\| \le \|\delta^j_{\boldsymbol{H}_0}(\boldsymbol{B}')\|.$$

Hence, Proposition 3.4.4 applied to the operators $\boldsymbol{H}_{i,n}$ and $\boldsymbol{H}_0$ implies that for some polynomial Q with positive coefficients, we have

$$\begin{aligned}\left\|\overline{(\boldsymbol{H}_{i,n}+1)^{-k/2}(\boldsymbol{H}_0+1)^{k/2}}\right\| &\le \operatorname{const} Q(\|\delta^j_{\boldsymbol{H}_0}(\boldsymbol{B}_n)\|, \|\delta^j_{\boldsymbol{H}_0}(\boldsymbol{B}'_n)\|) \\ &\le \operatorname{const} Q(\|\delta^j_{\boldsymbol{H}_0}(\boldsymbol{B})\|, \|\delta^j_{\boldsymbol{H}_0}(\boldsymbol{B}')\|), \quad j=0,\dots k-1,\end{aligned}$$

which together with the equality

$$\left\|(\boldsymbol{H}_0-z)^{k/2}(\boldsymbol{H}_{i,n}-z)^{-k/2}\right\| = \left\|\overline{(\boldsymbol{H}_{i,n}+1)^{-k/2}(\boldsymbol{H}_0+1)^{k/2}}\right\|,$$

concludes the proof. □

COROLLARY 3.4.6. *Let $\boldsymbol{B}, \boldsymbol{B}' \in \bigcap_{j=1}^{k-1} \operatorname{dom}(\delta^j_{\boldsymbol{H}_0})$ for some $k \in \mathbb{N}$. Then*

$$\operatorname{dom}(\boldsymbol{H}_{i,n}^{k/2}) = \operatorname{dom}(\boldsymbol{H}_i^{k/2}) \subset \operatorname{dom}(\boldsymbol{H}_0^{k/2}), \quad i=1,2, \quad n \in \mathbb{N}.$$

PROOF. We prove only the inclusion $\operatorname{dom}(\boldsymbol{H}_1^{k/2}) \subset \operatorname{dom}(\boldsymbol{H}_0^{k/2})$ since the others can be proved similarly.

Let $\xi \in \operatorname{dom}(\boldsymbol{H}_1^{k/2}) = \operatorname{dom}(\boldsymbol{H}_1+1)^{k/2} = \operatorname{ran}((\boldsymbol{H}_1+1)^{-k/2})$. Then there exists $\eta \in L_2(\mathbb{R},\mathcal{H})$ such that $\xi = (\boldsymbol{H}_1+1)^{-k/2}\eta$. Since $\eta \in L_2(\mathbb{R},\mathcal{H})$ and by Corollary 3.4.5 (i) the operator $(\boldsymbol{H}_0+1)^{k/2}(\boldsymbol{H}_1+1)^{-k/2}$ is bounded, we have that

$$(\boldsymbol{H}_0+1)^{k/2}\xi = (\boldsymbol{H}_0+1)^{k/2}(\boldsymbol{H}_1+1)^{-k/2}\eta \in \mathcal{H},$$

that is $\xi \in \operatorname{dom}(\boldsymbol{H}_0+1)^{k/2} = \operatorname{dom}(\boldsymbol{H}_0^{k/2})$. $\square$

By Propositions 3.4.4 and 3.4.5 the operators $\overline{(\boldsymbol{H}_{2,n}-z)^{\frac{-k}{2}}(\boldsymbol{H}_0-z)^{\frac{k}{2}}}$, and $\overline{(\boldsymbol{H}_2-z)^{\frac{-k}{2}}(\boldsymbol{H}_0-z)^{\frac{k}{2}}}$ are bounded. The following proposition establishes the strong-operator convergence of $\overline{(\boldsymbol{H}_{2,n}-z)^{\frac{-k}{2}}(\boldsymbol{H}_0-z)^{\frac{k}{2}}}$ to $\overline{(\boldsymbol{H}_2-z)^{\frac{-k}{2}}(\boldsymbol{H}_0-z)^{\frac{k}{2}}}$, which required for the proof of the principal trace formula. The result here should be compared with [**30**, Lemma 3.13 (ii)], where a much simpler case $k=2$ was treated.

PROPOSITION 3.4.7. *Assume that* $\boldsymbol{B}, \boldsymbol{B}' \in \bigcap_{j=1}^{k-1} \operatorname{dom}(\delta^j_{\boldsymbol{H}_0})$ *for some* $k \in \mathbb{N}$. *Then*

(i)
$$\overline{(\boldsymbol{H}_{2,n}-z)^{\frac{-k}{2}}(\boldsymbol{H}_0-z)^{\frac{k}{2}}} \to \overline{(\boldsymbol{H}_2-z)^{\frac{-k}{2}}(\boldsymbol{H}_0-z)^{\frac{k}{2}}}$$
in the strong operator topology.

(ii)
$$(\boldsymbol{H}_0-z)^{\frac{k}{2}}(\boldsymbol{H}_{1,n}-z)^{\frac{-k}{2}} \to (\boldsymbol{H}_0-z)^{\frac{k}{2}}(\boldsymbol{H}_1-z)^{\frac{-k}{2}}$$
in the strong operator topology.

PROOF. Without loss of generality we have $z=-1$.

(i). By Lemma 3.2.6 we have that $\boldsymbol{H}_{2,n} \to \boldsymbol{H}_2$ in the strong resolvent sense. Therefore, [**69**, Theorem VIII.20] implies that $(\boldsymbol{H}_{2,n}+1)^{\frac{-k}{2}} \to (\boldsymbol{H}_2+1)^{\frac{-k}{2}}$ in the strong operator topology. Hence, for every $\xi \in \operatorname{dom}(\boldsymbol{H}_0^{\frac{k}{2}})$ we have

$$(\boldsymbol{H}_{2,n}+1)^{\frac{-k}{2}}(\boldsymbol{H}_0+1)^{\frac{k}{2}}\xi \to (\boldsymbol{H}_2+1)^{\frac{-k}{2}}(\boldsymbol{H}_0+1)^{\frac{k}{2}}\xi.$$

Since $\operatorname{dom}(\boldsymbol{H}_0^{\frac{k}{2}})$ is a dense subset in $L_2(\mathbb{R},\mathcal{H})$ and by Proposition 3.4.5 (iii) the sequence $\{\overline{(\boldsymbol{H}_{2,n}+1)^{\frac{-k}{2}}(\boldsymbol{H}_0+1)^{\frac{k}{2}}}\}_{n\in\mathbb{N}}$ is uniformly bounded, we infer that

$$\overline{(\boldsymbol{H}_{2,n}+1)^{\frac{-k}{2}}(\boldsymbol{H}_0+1)^{\frac{k}{2}}} \to \overline{(\boldsymbol{H}_2+1)^{\frac{-k}{2}}(\boldsymbol{H}_0+1)^{\frac{k}{2}}}$$

in the strong operator topology.

(ii). By Corollary 3.4.6 we have that

$$\operatorname{dom}(\boldsymbol{H}_1+1)^{\frac{k}{2}} = \operatorname{dom}(\boldsymbol{H}_{1,n}+1)^{\frac{k}{2}} \subset \operatorname{dom}(\boldsymbol{H}_0+1)^{\frac{k}{2}},$$

and therefore both $(\boldsymbol{H}_{1,n}+1)^{\frac{-k}{2}}\xi$ and $(\boldsymbol{H}_1+1)^{\frac{-k}{2}}\xi$ lie in $\operatorname{dom}(\boldsymbol{H}_0+1)^{\frac{k}{2}}$ for every $\xi \in L_2(\mathbb{R},\mathcal{H})$. The strong resolvent convergence $\boldsymbol{H}_{1,n} \to \boldsymbol{H}_1$ and [**69**, Theorem VIII.20], imply that $(\boldsymbol{H}_{1,n}+1)^{\frac{-k}{2}} \to (\boldsymbol{H}_1+1)^{\frac{-k}{2}}$ in the strong operator topology. Hence,

$$(\boldsymbol{H}_0+1)^{\frac{k}{2}}(\boldsymbol{H}_{1,n}+1)^{\frac{-k}{2}}\xi \to (\boldsymbol{H}_0+1)^{\frac{k}{2}}(\boldsymbol{H}_1+1)^{\frac{-k}{2}}\xi$$

for every $\xi \in L_2(\mathbb{R},\mathcal{H})$, since the operator $(\boldsymbol{H}_0+1)^{\frac{k}{2}}$ is closed. $\square$

3.5. The main hypothesis

In this section we give the precise assumptions we impose for our results and discuss some details of this hypothesis.

HYPOTHESIS 3.5.1. *(i) Assume A_- is self-adjoint on* $\operatorname{dom}(A_-) \subseteq \mathcal{H}$ *with $\mathcal{H}$ a complex, separable Hilbert space.*

(ii) Suppose we have a family of bounded self-adjoint operators $\{B(t)\}_{t\in\mathbb{R}} \subset \mathcal{B}(\mathcal{H})$, continuously differentiable with respect to t in the uniform operator norm, such that $\|B'(\cdot)\|_{\mathcal{B}(\mathcal{H})} \in L_1(\mathbb{R}; dt) \cap L_\infty(\mathbb{R}; dt)$.

(iii) Suppose that the family $\{B(t)\}$ consists of p-relative trace-class perturbations with respect to A_- for some $p \in \mathbb{N} \cup \{0\}$, that is

$$B'(t)(A_- + i)^{-p-1} \in \mathcal{L}_1(\mathcal{H}), \quad \int_{\mathbb{R}} \|B'(t)(A_- + i)^{-p-1}\|_1 dt < \infty.$$

(iv) Let $m = \lceil \frac{p}{2} \rceil$. Assume that for all $z < 0$ we have that

$$\boldsymbol{B}'(\boldsymbol{H}_0 - z)^{-m-1} \in \mathcal{L}_1(L_2(\mathbb{R}, \mathcal{H})).$$

(v) $\boldsymbol{B}, \boldsymbol{B}' \in \bigcap_{j=1}^{2m-1} \operatorname{dom}(\delta^j_{\boldsymbol{H}_0})$.

In what follows we always take the smallest $p \in \mathbb{N} \cup \{0\}$ satisfying (iii).

Next, we discuss some details of our main assumption, Hypothesis 3.5.1, for the special type of the path $\{B(t)\}_{t\in\mathbb{R}}$. Suppose that a positive function θ on $\mathbb{R}$ satisfies

$$\begin{gathered} \theta \in C_b^\infty(\mathbb{R}), \quad \theta' \in L_1(\mathbb{R}), \\ \lim_{t\to-\infty} \theta(t) = 0, \quad \lim_{t\to+\infty} \theta(t) = 1. \end{gathered} \tag{3.5.1}$$

and assume that $B_+ \in \mathcal{B}(\mathcal{H})$.

Suppose that the family $\{B(t)\}_{t\in\mathbb{R}}$ is given by

$$B(t) = \theta(t) B_+. \tag{3.5.2}$$

PROPOSITION 3.5.2. *Suppose that B_+ is a p-relative trace class perturbation of A_- and let $\{B(t)\}_{t\in\mathbb{R}}$ be as in* (3.5.2) *with θ satisfying* (3.5.1)*. Then assumption of Hypothesis 3.1.1 (ii) and (iii) are satisfied.*

PROOF. By the definition of $B(t)$ it follows that $B'(t) = \theta'(t)B_+$. Hence, assumption (3.5.1) guarantee Hypothesis 3.1.1 (ii). Hypothesis 3.1.1 (iii) is also satisfied since

$$\int_{\mathbb{R}} \|B'(t)(A_- + i)^{-p-1}\|_1 dt = \int_{\mathbb{R}} |\theta'(t)| dt \cdot \|B_+(A_- + i)^{-p-1}\|_1 < \infty.$$

□

Moreover, an argument similar to the proof of [**32**, Proposition 2.2] guarantees the following

PROPOSITION 3.5.3. [**32**] *Suppose that B_+ is a p-relative trace class perturbation of A_- and let $\{B(t)\}_{t\in\mathbb{R}}$ be as in* (3.5.2) *with θ satisfying* (3.5.1)*. Then*

$$\boldsymbol{B}'(\boldsymbol{H}_0 - z)^{-m-1} \in \mathcal{L}_1(L_2(\mathbb{R}, \mathcal{H})),$$

that is assumption of Hypothesis 3.5.1 (iv) is satisfied.

Thus, for the special type of family $\{B(t)\}_{t\in\mathbb{R}} = \{\theta(t)B_+\}_{t\in\mathbb{R}}$ the assumptions Hypothesis 3.5.1 (ii), (iii) and (iv) are automatically guaranteed by the assumption (3.5.1) and the fact that B_+ is a p-relative trace class perturbation of A_-.

Next, we discuss Hypothesis 3.5.1 (v). By definition of $\delta_{\boldsymbol{H}_0}$ (see (3.4.1)) to check Hypothesis 3.5.1 (v) we have to consider repeated commutator with $(1+\boldsymbol{H}_0)^{1/2}$. However, in general, it is hard to work with these commutators. Therefore, we introduce below a different type of commutators, which are more manageable.

Similarly to [**25**, Section 1.3] we introduce the operator

$$\boldsymbol{L}^k_{\boldsymbol{H}_0}(\boldsymbol{T}) = \overline{(1+\boldsymbol{H}_0)^{-k/2}[\boldsymbol{H}_0, T]^{(k)}}$$

whose domain is

$$\begin{aligned}\operatorname{dom}(\boldsymbol{L}^k_{\boldsymbol{H}_0}) = \{\boldsymbol{T} \in \mathcal{B}(L_2(\mathbb{R},\mathcal{H})) : \boldsymbol{T}\operatorname{dom}(\boldsymbol{H}_0^j) \subset \operatorname{dom}(\boldsymbol{H}_0^j),\ j = 1,\dots,k\\ \text{and the operator } (1+\boldsymbol{H}_0)^{-k/2}[\boldsymbol{H}_0,\boldsymbol{T}]^{(k)} \text{ defined on } \operatorname{dom}(\boldsymbol{H}_0^k)\\ \text{extends to a bounded operator on } L_2(\mathbb{R},\mathcal{H})\}.\end{aligned}$$

The following result follows from the proof of [**25**, Lemma 1.29].

LEMMA 3.5.4. *If* $\boldsymbol{T} \in \bigcap_{j=1}^{2k}\operatorname{dom}(\boldsymbol{L}^j_{\boldsymbol{H}_0})$ *for some* $k\in\mathbb{N}$*, then* $\boldsymbol{T}\in\bigcap_{j=1}^{k}\operatorname{dom}(\delta^j_{\boldsymbol{H}_0})$.

PROOF. It follows from the proof of [**25**, Lemma 1.29] that for any $\xi \in \operatorname{dom}(\boldsymbol{H}_0^k)$ we have

$$\begin{aligned}&[(1+\boldsymbol{H}_0)^{1/2},\boldsymbol{T}]^{(k)}\xi\\ &= 2^{-n}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{2}{\pi}\right)^j\int_{\mathbb{R}_+^k}\prod_{i=1}^{j}\frac{\lambda_i^{1/2}(1+\boldsymbol{H}_0)}{(1+\lambda_i+\boldsymbol{H}_0)^2}\boldsymbol{L}^{k+j}_{\boldsymbol{H}_0}(\boldsymbol{T})\prod_{i=1}^{j}\frac{d\lambda_i}{1+\lambda_i+\boldsymbol{H}_0}\xi,\end{aligned}$$

where the right-hand side is well defined, since

$$\prod_{i=1}^{j}(1+\lambda_i+\boldsymbol{H}_0)^{-1} : \operatorname{dom}(\boldsymbol{H}_0^k)\to\operatorname{dom}(\boldsymbol{H}_0^{k+j})$$

and the operator $\boldsymbol{L}^{k+j}_{\boldsymbol{H}_0}(\boldsymbol{T})$ is initially defined on $\operatorname{dom}(\boldsymbol{H}_0^{k+j})$.

By the assumption, the operator $\boldsymbol{L}^{k+j}_{\boldsymbol{H}_0}(\boldsymbol{T})$ is bounded, $j = 1,\dots k$. In addition, the functional calculus yields

$$(1+\lambda+\boldsymbol{H}_0)^{-1} \le (1+\lambda)^{-1},\quad \lambda^{1/2}(1+\boldsymbol{H}_0)(1+\lambda+\boldsymbol{H}_0)^{-2} \le \lambda^{-1/2}/4,$$

we obtain that the operator $[(1+\boldsymbol{H}_0)^{1/2},\boldsymbol{T}]^{(k)}$ is bounded on $\operatorname{dom}(\boldsymbol{H}_0^k)$. Hence, there exists a unique bounded extension $\overline{[(1+\boldsymbol{H}_0)^{1/2},\boldsymbol{T}]^{(k)}} \in \mathcal{B}(L_2(\mathbb{R},\mathcal{H}))$ (see e.g. [**79**, Theorem 4.5]). $\square$

Next, we want to reduce the commutators with $\boldsymbol{H}_0$ to the commutators with $\boldsymbol{A}_-^2$. To this end, for a self-adjoint operator A on $\mathcal{H}$ we introduce the operator

$$L^k_{A^2}(T) = \overline{(1+A^2)^{-k/2}[A^2,T]^{(k)}} \tag{3.5.3}$$

with domain

$$\begin{aligned}\operatorname{dom}(L^k_{A^2}) = \{T \in \mathcal{B}(\mathcal{H}) : T \operatorname{dom}(A^j) \subset \operatorname{dom}(A^j),\ j = 1, \dots, 2k \\ \text{and the operator } (1 + A^2)^{-k/2}[A^2, T]^{(k)} \text{ defined on } \operatorname{dom}(A^{2k}) \\ \text{extends to a bounded operator on } \mathcal{H}\}.\end{aligned}$$

PROPOSITION 3.5.5. *Let $\{B(t)\}_{t\in\mathbb{R}}$ be as in* (3.5.2) *with θ satisfying* (3.5.1). *If $B_+ \in \bigcap_{j=1}^k \operatorname{dom}(L^j_{A^2_-})$, for some $k \in \mathbb{N}$, then $\boldsymbol{B}, \boldsymbol{B}' \in \bigcap_{j=1}^k \operatorname{dom}(\boldsymbol{L}^j_{\boldsymbol{H}_0})$.*

PROOF. We prove the assertion for $\boldsymbol{B}$ only.

Firstly we note that since

$$\boldsymbol{H}_0 = \frac{d^2}{dt^2} + \boldsymbol{A}^2_- = \frac{d^2}{dt^2} \otimes 1 + 1 \otimes A^2_-,$$

and $\theta \in C^\infty_b(\mathbb{R})$, it follows that the assumption $B_+ \operatorname{dom}(A^j_-) \subset \operatorname{dom}(A^j_-)$, $j = 1, \dots, 2k$, guarantees that the operator $\boldsymbol{B} = M_\theta \otimes B_+$ (see Remark 3.1.6) leaves $\operatorname{dom}(\boldsymbol{H}^j_0), j = 1, \dots, k$, invariant.

Furthermore, on $\operatorname{dom}(\boldsymbol{H}_0)$ we have

$$\begin{aligned}[\boldsymbol{H}_0, \boldsymbol{B}] &= [\frac{d^2}{dt^2} \otimes 1, M_\theta \otimes B_+] + [1 \otimes A^2_-, M_\theta \otimes B_+] \\ &= [\frac{d^2}{dt^2}, M_\theta] \otimes B_+ + M_\theta \otimes [A^2_-, \otimes B_+].\end{aligned}$$

By Lemma 3.1.8 we have that the operator

$$\boldsymbol{C} := (1 - \frac{d^2}{dt^2})^{\frac{l}{2}} (1 + \boldsymbol{A}^2_-)^{\frac{k-l}{2}} (1 + \boldsymbol{H}_0)^{-k/2}$$

is bounded for any $l = 0, \dots, k$.

Hence, on $\operatorname{dom}(\boldsymbol{H}^k_0)$ we have

$$\begin{aligned}\boldsymbol{L}^k_{\boldsymbol{H}_0}(\boldsymbol{B}) &= (1 + \boldsymbol{H}_0)^{-k/2}[\boldsymbol{H}_0, M_\theta \otimes B_+]^{(k)} \\ &= (1 + \boldsymbol{H}_0)^{-k/2} \sum_{l=0}^k [\frac{d^2}{dt^2}, M_\theta]^{(l)} \otimes [A^2_-, B_+]^{(k-l)} \\ &= \sum_{l=0}^k \boldsymbol{C} \cdot (1 - \frac{d^2}{dt^2})^{-\frac{l}{2}} [\frac{d^2}{dt^2}, M_\theta]^{(l)} \otimes (1 + A^2_-)^{-k-l/2}[A^2_-, B_+]^{(k-l)}.\end{aligned}$$

It follows from inclusion (7.1.4) below (for $n = 1$) that $(1 - \frac{d^2}{dt^2})^{-\frac{l}{2}}[\frac{d^2}{dt^2}, M_\theta]^{(l)}$ extends to a bounded operator on $L_2(\mathbb{R})$ for any $l = 0, \dots, k$. By assumption the operator $(1 + A^2_-)^{-k-l/2}[A^2_-, B_+]^{(k-l)}$ extends to a bounded operator on $\mathcal{H}$. Therefore, $\boldsymbol{L}^k_{\boldsymbol{H}_0}(\boldsymbol{B})$ also extends to a bounded operator on $L_2(\mathbb{R}, \mathcal{H})$, as required. □

We now formulate the Hypothesis 3.5.1 for the special case when $\{B(t)\}_{t\in\mathbb{R}} = \{\theta(t)B_+\}_{t\in\mathbb{R}}$.

HYPOTHESIS 3.5.6. *(i) Assume that A_- is self-adjoint on* $\operatorname{dom}(A_-) \subseteq$ *$\mathcal{H}$ with $\mathcal{H}$ a complex, separable Hilbert space and let θ satisfies* (3.5.1).

(ii) Suppose that an operator B_+ is p-relative trace-class perturbations with respect to A_- for some $p \in \mathbb{N}$, that is

$$B_+(A_- + i)^{-p-1} \in \mathcal{L}_1(\mathcal{H}).$$

(iii) *Assume also that* $B_+ \in \bigcap_{j=1}^{2p} \operatorname{dom}(L^j_{A^2_-})$, *where the mapping* $L^j_{A^2_-}$ *is defined by* (3.5.3).

The proof of the following proposition follows from combination of Propositions 3.5.2, 3.5.3 and 3.5.5.

PROPOSITION 3.5.7. *For the special case when* $\{B(t)\}_{t\in\mathbb{R}} = \{\theta(t)B_+\}_{t\in\mathbb{R}}$, *Hypothesis 3.5.6 guarantees that Hypothesis 3.5.1 is satisfied.*

3.6. Approximation results for the pair $(\boldsymbol{H}_2, \boldsymbol{H}_1)$.

In this section we develop further our approximation scheme that is essential for the proof of the principal trace formula in Section 5.1 below. The following theorem is our second key result. The proof of this result crucially uses the results obtained in Section 3.4. We note firstly that by [**48**, Theorem 3.2], Hypothesis 3.5.1 (iv) implies that

$$(\boldsymbol{H}_0 - z)^{-m+j-1}\boldsymbol{B}'(\boldsymbol{H}_0 - z)^{-j} \in \mathcal{L}_1(L_2(\mathbb{R}, \mathcal{H})) \tag{3.6.1}$$

for all $j = 1, \dots, m$.

THEOREM 3.6.1. *Assume Hypothesis 3.5.1. Let* $z \in \mathbb{C} \setminus \mathbb{R}_+$.

(i) *Both* $(\boldsymbol{H}_2 - z)^{-m} - (\boldsymbol{H}_1 - z)^{-m}$ *and* $(\boldsymbol{H}_{2,n} - z)^{-m} - (\boldsymbol{H}_{1,n} - z)^{-m}$ *are trace class.*

(ii) *We have*

$$\|\cdot\|_1 - \lim_{n\to\infty} \Big((\boldsymbol{H}_{2,n} - z)^{-m} - (\boldsymbol{H}_{1,n} - z)^{-m}\Big) = (\boldsymbol{H}_2 - z)^{-m} - (\boldsymbol{H}_1 - z)^{-m}.$$

PROOF. (i). Using again the resolvent identity and the elementary relation

$$A^k - B^k = \sum_{j=1}^{k} A^{k-j}[A-B]B^{j-1}, \quad A, B \in \mathcal{B}(\mathcal{H}),\ k \in \mathbb{N},$$

we write

$$\begin{aligned}
&(\boldsymbol{H}_2 - z)^{-m} - (\boldsymbol{H}_1 - z)^{-m} \\
&\qquad = \sum_{j=1}^{m} (\boldsymbol{H}_2 - z)^{-m+j}((\boldsymbol{H}_2 - z)^{-1} - (\boldsymbol{H}_1 - z)^{-1})(\boldsymbol{H}_1 - z)^{-j+1} \\
&\qquad = -2\sum_{j=1}^{m} (\boldsymbol{H}_2 - z)^{-m+j-1}\boldsymbol{B}'(\boldsymbol{H}_1 - z)^{-j}
\end{aligned}$$

Thus

$$\begin{aligned}
(\boldsymbol{H}_2 - z)^{-m} - (\boldsymbol{H}_1 - z)^{-m} &= -2\sum_{j=1}^{m} \overline{(\boldsymbol{H}_2 - z)^{-m+j-1}(\boldsymbol{H}_0 - z)^{m-j+1}} \\
&\times (\boldsymbol{H}_0 - z)^{-m+j-1}\boldsymbol{B}'(\boldsymbol{H}_0 - z)^{-j} \times (\boldsymbol{H}_0 - z)^{j}(\boldsymbol{H}_1 - z)^{-j}
\end{aligned} \tag{3.6.2}$$

By (3.6.1), the operators $(\boldsymbol{H}_0 - z)^{-m+j-1}\boldsymbol{B}'(\boldsymbol{H}_0 - z)^{-j}$ are trace-class operators for all $j = 1, \dots, m$. Since, in addition, by Proposition 3.4.5 (i) and (ii), the operators $\overline{(\boldsymbol{H}_2 - z)^{-m+j-1}(\boldsymbol{H}_0 - z)^{m-j+1}}$ and $(\boldsymbol{H}_0 - z)^{j}(\boldsymbol{H}_1 - z)^{-j}$ are bounded, we infer that $(\boldsymbol{H}_2 - z)^{-m} - (\boldsymbol{H}_1 - z)^{-m} \in \mathcal{L}_1(L_2(\mathbb{R}, \mathcal{H}))$.

Arguing similarly, one can obtain that

$$
\begin{aligned}
&(\boldsymbol{H}_{2,n}-z)^{-m}-(\boldsymbol{H}_{1,n}-z)^{-m} \\
&= -2\sum_{j=1}^{m}(\boldsymbol{H}_{2,n}-z)^{-m+j-1}\boldsymbol{P}_n\boldsymbol{B}'\boldsymbol{P}_n(\boldsymbol{H}_{1,n}-z)^{-j-1} \\
&= -2\sum_{j=1}^{m}\overline{(\boldsymbol{H}_{2,n}-z)^{-m+j-1}(\boldsymbol{H}_0-z)^{m-j+1}} \qquad (3.6.3)\\
&\qquad \times \boldsymbol{P}_n(\boldsymbol{H}_0-z)^{-m+j-1}\boldsymbol{B}'\frac{1}{(\boldsymbol{H}_0-z)^j}\boldsymbol{P}_n\times(\boldsymbol{H}_0-z)^j(\boldsymbol{H}_{1,n}-z)^{-j}.
\end{aligned}
$$

Referring to Proposition 3.4.4 and 3.4.5 and hence that $(\boldsymbol{H}_{2,n}-z)^{-m}-(\boldsymbol{H}_{1,n}-z)^{-m}\in\mathcal{L}_1(L_2(\mathbb{R},\mathcal{H}))$.

(ii). Using decompositions (3.6.2) and (3.6.3) we see that it is sufficient to prove the convergence of each term separately.

By (3.6.1), the operator $(\boldsymbol{H}_0-z)^{-m+j-1}\boldsymbol{B}'(\boldsymbol{H}_0-z)^{-j}\in\mathcal{L}_1(L_2(\mathbb{R},\mathcal{H}))$ for all $j=1,\dots,m$, and therefore, by Lemma 3.2.1 we have that

$$
\boldsymbol{P}_n(\boldsymbol{H}_0-z)^{-m+j-1}\boldsymbol{B}'(\boldsymbol{H}_0-z)^{-j}\boldsymbol{P}_n\xrightarrow{\|\cdot\|_1}(\boldsymbol{H}_0-z)^{-m+j-1}\boldsymbol{B}'(\boldsymbol{H}_0-z)^{-j}.
$$

In addition, by Proposition 3.4.7 we have

$$
\overline{(\boldsymbol{H}_{2,n}-z)^{-m+j-1}(\boldsymbol{H}_0-z)^{m-j+1}}\to\overline{(\boldsymbol{H}_2-z)^{-m+j-1}(\boldsymbol{H}_0-z)^{m-j+1}}
$$

and

$$
(\boldsymbol{H}_0-z)^j(\boldsymbol{H}_{1,n}-z)^{-j}\to(\boldsymbol{H}_0-z)^j(\boldsymbol{H}_1-z)^{-j},\quad j=1,\dots,m,
$$

in the strong operator topology. Thus, appealing again to Lemma 3.2.1 we complete the proof. $\square$

CHAPTER 4

Spectral shift function

In this chapter we discuss the spectral shift functions for the pairs (A_+, A_-) and $(\boldsymbol{H}_2, \boldsymbol{H}_1)$. Since the difference of resolvents of these operators is not necessarily trace class and only the difference of higher power of resolvents gives a trace class operator, we define spectral shift functions $\xi(\cdot, A_+, A_-)$, $\xi(\cdot; \boldsymbol{H}_2, \boldsymbol{H}_1)$ using Yafaev's construction from [**83**]. We firstly recall this construction in Section 4.1. In this section we also prove that spectral shift function constructed in this way is continuous (in a certain topology) with respect to operator parameter. This result is published in [**27**]. One particularly useful application of this continuity is the fact that it allows us to fix the additive constant in the spectral shift function $\xi(\cdot; A_+, A_-)$.

In Section 4.2 we introduce spectral shift functions $\xi(\cdot, A_+, A_-)$, $\xi(\cdot; \boldsymbol{H}_2, \boldsymbol{H}_1)$ and fix the additive constants in their definitions.

Finally, in Section 4.3 we give an example of a self-adjoint H and its bounded self-adjoint perturbation V, such that

$$f(H+V) - f(H) \notin \mathcal{L}_1(\mathcal{H})$$

for a sufficiently nice monotone function f on $\sigma(H) \cup \sigma(H+V)$, which implies that the invariance principle for the spectral shift function (see Section 1.2) is not applicable for the pair $(H+V, H)$. This is the reason why we fix the spectral shift function $\xi(\cdot; A_+, A_-)$ using the continuity result from Section 4.1. The results of Section 4.3 are taken from [**57**].

4.1. Continuity of spectral shift function with respect to the operator parameter

In this section we firstly recall the construction of spectral shift function for m-resolvent comparable (in $\mathcal{L}_1(\mathcal{H})$) operators, $m \in \mathbb{N}$ is odd, due to D. Yafaev [**83**] and then prove that this spectral shift function is continuous with respect to the operator parameter.

Suppose that A_0 and B_0 are fixed self-adjoint operators in the Hilbert space $\mathcal{H}$, which are m-resolvent comparable in $\mathcal{L}_1(\mathcal{H})$ (see Definition 2.2.4) for some odd $m \in \mathbb{N}$. That is for all $a \in \mathbb{R} \setminus \{0\}$ we have

$$\big[(B_0 - ai)^{-m} - (A_0 - ai)^{-m}\big] \in \mathcal{L}_1\big(\mathcal{H}\big). \tag{4.1.1}$$

As in Section 2.2 we denote by $\varphi : \mathbb{R} \to \mathbb{R}$ a bijection satisfying for some $c > 0$,

$$\varphi \in C^2(\mathbb{R}), \quad \varphi(\lambda) = \lambda^m,\ |\lambda| \geq 1, \quad \varphi'(\lambda) \geq c. \tag{4.1.2}$$

By (2.2.32) we have that

$$\big[(\varphi(B_0) - i)^{-1} - (\varphi(A_0) - i)^{-1}\big] \in \mathcal{L}_1(\mathcal{H}). \tag{4.1.3}$$

Therefore, by Theorem 1.2.3 there exists the (class) of spectral shift functions $\xi(\cdot;\varphi(B_0),\varphi(A_0))$ for the pair $(\varphi(B_0),\varphi(A_0))$ satisfying

$$\xi(\,\cdot\,;\varphi(B_0),\varphi(A_0)) \in L_1\big(\mathbb{R};(\mu^2+1)^{-1}d\mu\big). \tag{4.1.4}$$

Therefore, one can introduce the (class) of spectral shift functions $\xi(\cdot;B_0,A_0)$ for the pair (B_0,A_0) by setting

$$\xi(\nu;B_0,A_0) := \xi(\varphi(\nu);\varphi(B_0),\varphi(A_0)), \quad \nu\in\mathbb{R}. \tag{4.1.5}$$

In particular, the condition $\varphi'(\lambda)\geq c>0$ implies that the inverse function φ^{-1} is differentiable. Therefore, with the simple substitution

$$\mu=\varphi(\nu)\in\mathbb{R}, \quad \nu\in\mathbb{R}, \tag{4.1.6}$$

in (4.1.4) implies that

$$\xi(\,\cdot\,;B_0,A_0) \in L_1\big(\mathbb{R};(|\nu|^{m+1}+1)^{-1}d\nu\big). \tag{4.1.7}$$

Furthermore, for $f\in\mathfrak{F}_m(\mathbb{R})$, the fact that $\varphi(\lambda)=\lambda^m$ for sufficiently large (in absolute value) λ (see (2.2.23)) implies that $f\circ\varphi^{-1}\in\mathfrak{F}_1(\mathbb{R})$. Hence, using Theorem 1.2.3 and the change of variables (4.1.6), the corresponding trace formula is of the form

$$\begin{aligned}
\operatorname{tr}(f(B_0)-f(A_0)) &= \operatorname{tr}\big((f\circ\varphi^{-1})(\varphi(B_0))-(f\circ\varphi^{-1})(\varphi(A_0))\big)\\
&= \int_{\mathbb{R}} (f\circ\varphi^{-1})'(\mu)\,\xi(\mu;\varphi(B_0),\varphi(A_0))d\mu\\
&= \int_{\mathbb{R}} f'(\nu)\,\xi(\varphi(\nu);\varphi(B_0),\varphi(A_0))d\nu\\
&= \int_{\mathbb{R}} f'(\nu)\,\xi(\nu;B_0,A_0)d\nu, \quad f\in\mathfrak{F}_m(\mathbb{R}),
\end{aligned}$$

where the last equality follows from (4.1.5).

Thus, we have the following result.

THEOREM 4.1.1. [**83**] *Suppose that operators A_0 and B_0 are m-resolvent comparable with $m\in\mathbb{N}$ odd. Then there exits spectral shift function $\xi(\cdot;B_0,A_0)$, satisfying*

$$\xi(\,\cdot\,;B_0,A_0) \in L_1\big(\mathbb{R};(|\nu|^{m+1}+1)^{-1}d\nu\big)$$

and

$$\operatorname{tr}(f(B_0)-f(A_0)) = \int_{\mathbb{R}} f'(\nu)\,\xi(\nu;B_0,A_0)d\nu, \quad f\in\mathfrak{F}_m(\mathbb{R}). \tag{4.1.8}$$

REMARK 4.1.2. Assume that A_0 and B_0 are m-resolvent comparable with $m\in\mathbb{N}$ odd. We note that, the definition of spectral shift function $\xi(\cdot;B_0,A_0)$ and the properties of spectral shift function for resolvent comparable operators $\varphi(A_0),\varphi(B_0)$ immediately imply two following observations.

(i) If in some interval (a_0,b_0) the spectra of A_0 and B_0 are discrete and let $\delta=(a,b)$, $a_0<a<b<b_0$. Then, similarly to (1.2.6), we have

$$\xi(b_-;B_0,A_0)-\xi(a_+;B_0,A_0) = N_{A_0}(\delta)-N_{B_0}(\delta), \tag{4.1.9}$$

where $N_{A_0}(\delta)$ (respectively, $N_{B_0}(\delta)$) are the sum of the multiplicities of the eigenvalues of A_0 (respectively, B_0) in δ.

(ii) There is a class of spectral shift functions satisfying (4.1.7) and (4.1.8), which differ by an additive constant.

Next, we discuss the continuity of spectral shift function with respect to the operator parameter.

Let T be some fixed self-adjoint operator on $\mathcal{H}$. Denote by $\Gamma(T)$ the space of all self-adjoint operators, which are resolvent comparable with T, that is S on $\mathcal{H}$ such that

$$(T-z)^{-1}-(S-z)^{-1} \in \mathcal{L}_1(\mathcal{H})$$

for some $z \in \mathbb{C} \setminus \mathbb{R}$. For every $z \in \mathbb{C}$, $\operatorname{Im}(z) > 0$, one defines a metric on $\Gamma(T)$ by setting

$$d_z(S_1, S_2) = 2\operatorname{Im}(z)\big\|(S_2-z)^{-1}-(S_1-z)^{-1}\big\|_1. \tag{4.1.10}$$

By a standard resolvent identity, the metrics d_{z_1} and d_{z_2} are equivalent for different values of z_1 and z_2 in $\mathbb{C} \setminus \mathbb{R}$.

The following result due to Yafaev establishes continuity of spectral shift function with respect to the operator parameter.

Proposition 4.1.3. [**82**, Lemma 8.7.5] *Let A_0, B_0, and B_1 denote self-adjoint operators in $\mathcal{H}$ with $B_0, B_1 \in \Gamma(A_0)$, and let $\{B(s)\}_s \subset \Gamma(B_0)$ be a continuous (with respect to s) path from B_0 to B_1 in $\Gamma(B_0)$. Assume also that the spectral shift function $\xi_0(\cdot; B_0, A_0)$ is fixed representative from the class $\xi(\cdot; B_0, A_0)$. Then, there exists a unique representative $\xi(\cdot; B(s), A_0)$, continuous in s with respect to the norm in $L_1(\mathbb{R}, (\lambda^2+1)^{-1}d\lambda)$, such that*

$$\xi(\cdot; B(0), A_0) = \xi_0(\cdot; B_0, A_0).$$

In the rest of this section we prove an analogue of Proposition 4.1.3 for m-resolvent comparable operators.

Definition 4.1.4. *Let T be self-adjoint in $\mathcal{H}$ and $m \in \mathbb{N}$ odd. Then $\Gamma_m(T)$ denotes the set of all self-adjoint operators S in $\mathcal{H}$ for which the inclusion*

$$\big[(S-ai)^{-m}-(T-ai)^{-m}\big] \in \mathcal{L}_1(\mathcal{H}), \quad a \in \mathbb{R}\backslash\{0\}, \tag{4.1.11}$$

holds. In particular, $\Gamma_1(T) = \Gamma(T)$.

We note that for each $m \in \mathbb{N}$, $\Gamma_m(T)$ can be equipped with the family $\mathcal{D} = \{d_{m,a}\}_{a\in\mathbb{R}\backslash\{0\}}$ of pseudometrics (see [**41**, Definition IX.10.1] for a precise definition) defined by

$$d_{m,a}(S_1, S_2) = \big\|(S_2-ai)^{-m}-(S_1-ai)^{-m}\big\|_1, \quad S_1, S_2 \in \Gamma_m(T). \tag{4.1.12}$$

For each fixed $\varepsilon > 0$, $a \in \mathbb{R}\backslash\{0\}$, and $S \in \Gamma_m(T)$, define

$$B(S; d_{m,a}, \varepsilon) = \{S' \in \Gamma_m(T) \,|\, d_{m,a}(S, S') < \varepsilon\},$$

to be the ε-ball centered at S with respect to the pseudometric $d_{m,a}$.

Definition 4.1.5. *$\mathcal{T}_m(T)$ is the topology on $\Gamma_m(T)$ with the subbasis*

$$\mathfrak{B}_m(T) = \{B(S; d_{m,a}, \varepsilon) \,|\, S \in \Gamma_m(T),\, a \in \mathbb{R}\backslash\{0\},\, \varepsilon > 0\}.$$

That is, $\mathcal{T}_m(T)$ is the smallest topology on $\Gamma_m(T)$ which contains $\mathfrak{B}_m(T)$.

Proposition 4.1.6. *Suppose that $\{B(s)\}_{s\in[0,1]} \in \Gamma_m(A_0)$ is a path continuous with respect to the topology $\mathcal{T}_m(A_0)$. Then the path $\{\varphi(B(s))\}_{s\in[0,1]} \in \Gamma(\varphi(A_0))$ is continuous in $\Gamma(\varphi(A_0))$ with respect to the metric $d_z(\cdot,\cdot)$.*

PROOF. The assertion $\{\varphi(B(s))\}_{s\in[0,1]} \subset \Gamma_1(\varphi(B_0))$ follow immediately from the inequality (2.2.32). Similarly, by the definition of the metric $d_i(\cdot,\cdot)$ and pseudometric $d_{m,a}(\cdot,\cdot)$ we have

$$\begin{aligned} d_i\big(\varphi(B(s_1)),\varphi(B(s_2))\big) &= 2\Big\|(\varphi(B(s_1))-i)^{-1}-\varphi(B(s_2)-i)^{-1}\Big\|_1 \\ &\overset{(2.2.32)}{\leq} \text{const}\,\Big(\big\|(B(s_1)-a_1 i)^{-m}-(B(s_2)-a_1 i)^{-m}\big\|_1 \\ &\qquad + \big\|(B(s_1)-a_2 i)^{-m}-(B(s_2)-a_2 i)^{-m}\big\|_1\Big) \\ &= \text{const}\,\big(d_{m,a_1}(B(s_1),B(s_2))+d_{m,a_2}(B(s_1),B(s_2))\big), \end{aligned}$$

which suffices to conclude the proof. □

The following theorem is the principal result of this section.

THEOREM 4.1.7. [**27**] *Let A_0, B_0, and B_1 denote self-adjoint operators in $\mathcal{H}$ with $B_0, B_1 \in \Gamma_m(A_0)$, and let $\{B(s)\}_s \subset \Gamma(B_0)$ be a continuous (with respect to s) path from B_0 to B_1 in the topology $\mathcal{T}_m(B_0)$. Assume also that the spectral shift function $\xi_0(\cdot; B_0, A_0)$, defined by* (4.1.5)*, is a fixed representative from the class $\xi(\cdot; B_0, A_0)$. Then, there there exists a unique representative $\xi(\cdot; B(s), A_0)$, continuous is s with respect to the norm in $L_1(\mathbb{R}, (|\nu|^{m+1}+1)^{-1}d\nu)$, such that*

$$\xi(\cdot; B(0), A_0) = \xi_0(\cdot; B_0, A_0).$$

PROOF. By Proposition 4.1.6 we have that the path $\{\varphi(B(s))\}_{s\in[0,1]} \subset \Gamma(\varphi(B_0))$ is a continuous path with respect to $d_{1,i}(\,\cdot\,,\,\cdot\,)$. In addition, since $\xi_0(\cdot; B_0, A_0) = \xi_0(\varphi(\cdot); \varphi(B_0), \varphi(A_0))$ is fixed, Proposition 4.1.3 implies that there exists a unique spectral shift function $\xi(\,\cdot\,; \varphi(B(s)), \varphi(A_0)) \in L_1(\mathbb{R}; (\lambda^2+1)^{-1}d\lambda)$, depending continuously on $s \in [0,1]$ in the $L_1(\mathbb{R}; (\lambda^2+1)^{-1}d\lambda)$-norm and such that

$$\xi(\cdot, \varphi(B(0)), \varphi(A_0)) = \xi_0(\cdot, \varphi(B_0), \varphi(A_0)). \tag{4.1.13}$$

For each $s \in [0,1]$, let $\xi(\,\cdot\,; B(s), A_0)$ denote the spectral shift function for the pair $(B(s), A_0)$ defined by (4.1.5). Equality (4.1.13) then implies that

$$\xi(\,\cdot\,; B(0), A_0) = \xi(\cdot, \varphi(B(0)), \varphi(A_0)) = \xi_0(\cdot, \varphi(B_0), \varphi(A_0)) = \xi_0(\cdot, B_0, A_0),$$

as required.

It only remains to establish continuity of $\xi(\,\cdot\,; B(s), A_0)$ with respect to the $L_1(\mathbb{R}; (|\nu|^{m+1}+1)^{-1}d\nu)$-norm. Using the substitution (4.1.6) we have

$$\begin{aligned} &\int_{\mathbb{R}} \big|\xi(\nu; B(s_1), A_0) - \xi(\nu; B(s_2), A_0)\big| \big(|\nu|^{m+1}+1\big)^{-1}\, d\nu \\ &\quad = \int_{\mathbb{R}} \frac{\big|\xi(\mu; \varphi(B_\tau), \varphi(A_0)) - \xi(\mu; \varphi(B_{\tau'}), \varphi(A_0))\big|}{(|\varphi^{-1}(\mu)|^{m+1}+1)\varphi'(\varphi^{-1}(\mu))}\, d\mu. \end{aligned} \tag{4.1.14}$$

By the properties of φ (see (4.1.2)) we have that

$$\frac{1}{(|\varphi^{-1}(\mu)|^{m+1}+1)\varphi'(\varphi^{-1}(\mu))} \leq \text{const}\,\mu^2+1, \quad \mu \in \mathbb{R}. \tag{4.1.15}$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}} &\frac{\left|\xi(\nu; B(s_1), A_0) - \xi(\nu; B(s_2), A_0)\right|}{|\nu|^{m+1}+1}\, d\nu \\ &\leq \text{const} \int_{\mathbb{R}} \frac{\left|\xi(\mu; \varphi(B(s_1)), \varphi(A_0)) - \xi(\mu; \varphi(B(s_2)), \varphi(A_0))\right|}{\mu^2+1}\, d\mu \qquad (4.1.16)\end{aligned}$$

for $s_1, s_2 \in [0,1]$. Thus, continuity of $\xi(\,\cdot\,; B(s), A_0)$ in $L_1(\mathbb{R}; (|\nu|^{m+1}+1)^{-1}d\nu)$ follows from continuity of $\xi(\,\cdot\,; \varphi(B(s)), \varphi(A_0))$ in $L_1(\mathbb{R}; (\mu^2+1)^{-1}d\mu)$. □

We now apply Theorem 4.1.7 as a tool to fix the spectral shift function, in the case when the perturbation is given by $(m-1)$-relative trace class perturbation.

Let A_0 be a self-adjoint operator and let

$$P_n = \chi_{[-n,n]}(A_0), \quad A_0 \in \mathbb{N}.$$

THEOREM 4.1.8. *Suppose that A_0 is a self-adjoint operator on $\mathcal{H}$ and let $B \in \mathcal{B}(\mathcal{H})$ be an $(m-1)$-relative trace class perturbation of A_0, $m \in \mathbb{N}$ is odd. Then there exists unique spectral shift function $\xi(\cdot; A_0+B, A_0)$ such that*

$$\xi(\cdot; A_0+B, A_0) = \lim_{n\to\infty} \xi(\cdot; A_0 + P_nBP_n, A_0)$$

in $L_1(\mathbb{R}; (|\nu|^{m+1}+1)^{-1}d\nu)$.

PROOF. Introduce the path $\{B(s)\}_{s\in[0,1]}$, by setting

$$\begin{aligned}&B(s) = A_0 + \hat{P}_s B \hat{P}_s, \quad \operatorname{dom}(B(s)) = \operatorname{dom}(A_0), \quad s \in [0,1], \\ &\hat{P}_s = \chi_{[-\frac{1}{1-s}, \frac{1}{1-s}]}(A_0), s \in [0,1), \quad \hat{P}_1 = 1.\end{aligned} \qquad (4.1.17)$$

We note that

$$B(0) = A_0 + P_1BP_1, \quad B(1) = A_0 + B. \qquad (4.1.18)$$

It follows from Remark 3.2.4 that $P_nBP_n \in \mathcal{L}_1(\mathcal{H})$, and therefore $\hat{P}_sB\hat{P}_s \in \mathcal{L}_1(\mathcal{H})$ for any $s<1$. Hence, there exists a *unique* spectral shift function $\xi(\cdot; A_0 + \hat{P}_sB\hat{P}_s, A_0)$, $s<1$ satisfying

$$\xi(\cdot; A_0 + \hat{P}_sB\hat{P}_s, A_0) \in L_1(\mathbb{R}). \qquad (4.1.19)$$

Moreover, in complete analogy to Theorem 3.3.2, the family $B(s)$ depends continuously on $s \in [0,1]$ with respect to the family of pseudometric $d_{m,a}(\cdot,\cdot)$ defined by (4.1.12). Thus, the hypotheses of Theorem 4.1.7 are satisfied and hence there exists a unique spectral shift function $\xi(\,\cdot\,; B(s), A_0)$ for the pair $(B(s), A_0)$ depending continuously on $s \in [0,1]$ in the space $L_1\big(\mathbb{R}; (|\nu|^{m+1} + 1)^{-1}d\nu\big)$, satisfying $\xi(\,\cdot\,; B(0), A_0) = \xi(\,\cdot\,; A_0 + P_1BP_1, A_0)$. Taking $s = \frac{n-1}{n}$ we obtain that

$$\begin{aligned}\xi(\cdot; A_0+B, A_0) &= \lim_{s\to 1} \xi(\cdot; B(s), A_0) = \lim_{n\to\infty} \xi(\cdot; B(\frac{n-1}{n}), A_0) \\ &= \lim_{n\to\infty} \xi(\cdot; A_0 + P_nBP_n, A_0)\end{aligned}$$

in $L_1\big(\mathbb{R}; (|\nu|^{m+1}+1)^{-1}d\nu\big)$, as required. □

We conclude with an elementary consequence of Theorem 4.1.8.

COROLLARY 4.1.9. *Let A_0, B and P_n be as in Theorem 4.1.8. If $f \in L_\infty(\mathbb{R})$, then*

$$\lim_{n\to\infty} \|\xi(\,\cdot\,; A_0+P_nBP_n, A_0)f - \xi(\,\cdot\,; A_0+B, A_0)f\|_{L_1(\mathbb{R};(|\nu|^{m+1}+1)^{-1}d\nu)} = 0. \quad (4.1.20)$$

In particular, for $h \in L_\infty(\mathbb{R})$ such that $\sup_{\nu\in\mathbb{R}} \big|(|\nu|^{m+1}+1)h(\nu)\big| < \infty$ *we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}} \xi(\nu; A_0 + P_nBP_n, A_0)h(\nu)d\nu = \int_{\mathbb{R}} \xi(\nu; A_0 + B, A_0)h(\nu)d\nu \qquad (4.1.21)$$

for all $g \in L_\infty(\mathbb{R})$ such that $\operatorname{esssup}_{\nu\in\mathbb{R}}\big|(|\nu|^{\mathrm{m}+1}+1)\mathrm{g}(\nu)\big| < \infty$.

4.2. The spectral shift functions for the pairs (A_+, A_-) and $(\boldsymbol{H}_2, \boldsymbol{H}_1)$

In this section we introduce the spectral shift functions for the pairs (A_-, A_+) and $(\boldsymbol{H}_1, \boldsymbol{H}_2)$. From the point of view of computing Witten index and spectral flow the spectral shift function is our main tool. Throughout this section we assume Hypothesis 3.5.1.

We want to introduce the spectral shift function $\xi(\cdot; \boldsymbol{H}_2, \boldsymbol{H}_1)$ following Theorem 4.1.1. Combining Theorem 3.6.1 and Remark 2.2.6 we obtain the following

PROPOSITION 4.2.1. *Let m be as in Hypothesis 3.5.1 (iv). Then for all $z \in \mathbb{C} \setminus [0,\infty)$ we have that both $(\boldsymbol{H}_2 - z\,\boldsymbol{I})^{-m} - (\boldsymbol{H}_1 - z\,\boldsymbol{I})^{-m}$ and $(\boldsymbol{H}_2 - z\,\boldsymbol{I})^{-m-1} - (\boldsymbol{H}_1 - z\,\boldsymbol{I})^{-m-1}$ are trace class operators.*

For simplicity, for $k \in \mathbb{N}$ we use the notation

$$\widehat{k} = \begin{cases} k, & \text{if } k \text{ is odd},\\ k+1, & \text{if } k \text{ is even}.\end{cases} \qquad (4.2.1)$$

By Proposition 4.2.1 $(\boldsymbol{H}_2 - z\,\boldsymbol{I})^{-\widehat{m}} - (\boldsymbol{H}_1 - z\,\boldsymbol{I})^{-\widehat{m}} \in \mathcal{L}_1\big(L_2(\mathbb{R},\mathcal{H})\big)$, for all $z \in \mathbb{C}\backslash\mathbb{R}$. Therefore, Theorem 4.1.1 implies that there is a spectral shift function $\xi(\,\cdot\,; \boldsymbol{H}_2, \boldsymbol{H}_1)$ for the pair $(\boldsymbol{H}_2, \boldsymbol{H}_1)$ that satisfies

$$\xi(\,\cdot\,; \boldsymbol{H}_2, \boldsymbol{H}_1) \in L_1\big(\mathbb{R}; (|\lambda|^{\widehat{m}+1}+1)^{-1}d\lambda\big).$$

Since $\boldsymbol{H}_j \geq 0$, $j = 1, 2$, $\xi(\,\cdot\,; \boldsymbol{H}_2, \boldsymbol{H}_1)$ may be specified uniquely by requiring that

$$\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) = 0, \quad \lambda < 0. \qquad (4.2.2)$$

In addition, Theorem 4.1.1 implies also the trace formula

$$\operatorname{tr}\big(f(\boldsymbol{H}_2) - f(\boldsymbol{H})\big) = \int_{[0,\infty)} f'(\lambda)\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1)\, d\lambda, \quad f \in \mathfrak{F}_{\widehat{m}}(\mathbb{R}). \qquad (4.2.3)$$

We introduce now the spectral shift function for the pair (A_+, A_-) using again Theorem 4.1.1. Firstly, we state the following proposition, which follows by the same argument as in Theorem 3.3.2.

PROPOSITION 4.2.2. *Assume Hypothesis 3.5.1. Then*

$$(A_+ - z)^{-p} - (A_- - z)^{-p}, \quad (A_+ - z)^{-p-1} - (A_- - z)^{-p-1} \in \mathcal{L}_1(\mathcal{H})$$

for every $z \in \mathbb{C} \setminus \mathbb{R}$. Furthermore,

$$\|\cdot\|_1 - \lim_{n\to\infty} \Big((A_{+,n} - z)^{-p-1} - (A_- - z)^{-p-1} \Big) = (A_+ - z)^{-p-1} - (A_- - z)^{-p-1}$$

Combining now Proposition 4.2.2 and Theorem 4.1.1, we infer that there exists a function

$$\xi(\cdot; A_+, A_-) \in L_1(\mathbb{R}, (1+|\lambda|)^{-\widehat{p}-1}\, d\lambda) \tag{4.2.4}$$

such that

$$\operatorname{tr}\,(f(A_+) - f(A_-)) = \int_{\mathbb{R}} f'(\lambda) \cdot \xi(\lambda; A_+, A_-)\, d\lambda, \quad f \in \mathfrak{F}_{\widehat{p}}(\mathbb{R}). \tag{4.2.5}$$

However, the spectral shift function $\xi(\cdot; A_+, A_-)$ introduced above is not unique, in general, and therefore we have to fix one particular SSF which satisfies (4.2.4) and (4.2.5). It order to fix this constant we apply Theorem 4.1.8 for the operators $A_0 = A_-$ and $B = B_+$ and obtain the following

Theorem 4.2.3. *Assume Hypothesis 3.5.1. There exists unique spectral shift function $\xi(\cdot, A_+, A_-)$ such that*

$$\xi(\cdot, A_+, A_-) = \lim_{n\to\infty} \xi(\cdot, A_- + B_{+,n}, A_-) \tag{4.2.6}$$

in $L_1(\mathbb{R}; (|\nu|^{\widehat{p}+1} + 1)^{-1} d\nu)$.

Since every $\xi(\,\cdot\,; A_{+,n}, A_-), n \in \mathbb{N}$, is uniquely defined, Theorem 4.2.3 implies that we can fix uniquely the spectral shift function $\xi(\,\cdot\,; A_+, A_-)$ satisfying conditions (4.2.6). In what follows, we adopt this fixation for the remainder of the thesis.

For convenience we state Corollary 4.1.9 for the pair (A_+, A_-)

Corollary 4.2.4. *Assume Hypothesis 3.5.1 and suppose that $f \in L_\infty(\mathbb{R})$. Then*

$$\lim_{n\to\infty} \int_{\mathbb{R}} \xi(\nu; A_{+,n}, A_-) h(\nu) d\nu = \int_{\mathbb{R}} \xi(\nu; A_+, A_-) h(\nu) d\nu \tag{4.2.7}$$

for all $h \in L_\infty(\mathbb{R})$ such that $\sup_{\nu\in\mathbb{R}} \left|(|\nu|^{\widehat{p}+1} + 1)^{-1} h(\nu)\right| < \infty$.

4.3. On invariance principle for spectral shift function

In this section we present an example of operators $H, H+V$ such that the inclusion

$$f(H+V) - f(H) \in \mathcal{L}_1(\mathcal{H})$$

(see (1.2.8)) does not hold for a very wide class of functions f. The results of the present section can be found in the joint paper [**57**].

The example is based on the two-dimensional Dirac operator $\mathcal{D}$ and its electric potential $1 \otimes M_\varphi$. We note that the result holds for Dirac operator with electromagnetic potential for any dimension greater than 2 [**57**].

Throughout this section we assume that

$$\mathcal{H} = \mathbb{C}^2 \otimes L_2(\mathbb{R}^2).$$

Let γ_1, γ_2 be the Pauli matrices, that is

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Define a self-adjoint operator ∂_k, $k = 1, 2$, in $L_2(\mathbb{R}^2)$ by

$$\partial_k = -i\frac{\partial}{\partial t_k},$$

which corresponds to partial differentiation with respect to the k-th argument, and whose domain is the Sobolev space $W^{1,2}(\mathbb{R}^2)$.

Define the two-dimensional Dirac operator as a self-adjoint operator acting in the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes L_2(\mathbb{R}^2)$ by

$$\mathcal{D} = \gamma_1 \otimes \partial_1 + \gamma_2 \otimes \partial_2, \tag{4.3.1}$$

with domain $\mathrm{dom}(\mathcal{D}) = \mathbb{C}^2 \otimes W^{1,2}(\mathbb{R}^2)$.

For a bounded function $\varphi \in L_\infty(\mathbb{R}^2)$, we denote by M_φ the multiplication operator of φ, which is the bounded operator on $L_2(\mathbb{R}^2)$ defined by pointwise multiplication by φ. For simplicity we consider the case when the function $\varphi \neq 0$ is a real-valued Schwartz function on $\mathbb{R}^2$. For a more general result we refer to [**57**].

Suppose now that we have a *monotone* function f on $\mathbb{R} = \sigma(\mathcal{D}) \cup \sigma(\mathcal{D}+1\otimes M_\varphi)$, such that f' is a Schwartz function (in particular, $f' > 0$). The main result of this section is the that

$$f(\mathcal{D} + 1 \otimes M_\varphi) - f(\mathcal{D}) \notin \mathcal{L}_1(\mathcal{H}). \tag{4.3.2}$$

We note that the assumption on f can be weakened significantly, however, here we assume that $f' \in S(\mathbb{R})$ for simplicity.

The strategy of the proof of (4.3.2) is the following:

(i) Consider the auxiliary function g defined by (1.1.2) and show that

$$g(\mathcal{D} + 1 \otimes M_\varphi) - g(\mathcal{D}) \notin \mathcal{L}_1(\mathcal{H}).$$

(ii) Using double operator integrals to reduce the question for the operator $f(\mathcal{D} + 1 \otimes M_\varphi) - f(\mathcal{D})$ to the operator $g(\mathcal{D} + 1 \otimes M_\varphi) - g(\mathcal{D})$.

The first step of the proof is divided into two parts, where we firstly find a nice decomposition for $g(\mathcal{D} + 1 \otimes M_\varphi) - g(\mathcal{D})$ modulo $\mathcal{L}_1(\mathcal{H})$, and then show that the remaining operator is not in $\mathcal{L}_1(\mathcal{H})$. We proof each step in a separate subsection.

Before we proceed, we recall some preliminary results.

Firstly, observe that $\mathcal{D}^2 = 1 \otimes (-\Delta)$, where Δ is the Laplace operator $\Delta = \partial_1^2 + \partial_2^2$.

We have the following commutation relations. Suppose that φ is a Schwartz function on $\mathbb{R}^2$. Then, for $k = 1, 2$, we have that $M_\varphi(\mathrm{dom}\,\partial_k) \subset \mathrm{dom}(\partial_k)$, so the operator $[\partial_k, M_\varphi]$ is well-defined on $\mathrm{dom}(\partial_k) = W^{1,2}(\mathbb{R}^2)$, and moreover,

$$[\partial_k, M_\varphi] = M_{\partial_k \varphi}, \tag{4.3.3}$$

on the domain $W^{1,2}(\mathbb{R}^2)$. In particular, $[\partial_k, M_\varphi]$ extends to a bounded operator on $L_2(\mathbb{R}^2)$.

Throughout this section, we shall make use of the following notations

$$R_{0,\lambda} := \big(\mathcal{D} + i(1+\lambda)^{1/2}\big)^{-1}, \quad R_{1,\lambda} := \big(\mathcal{D} + 1 \otimes M_\varphi + i(1+\lambda)^{1/2}\big)^{-1}, \tag{4.3.4}$$

where $\lambda > 0$. We make several immediate observations. Note that

$$|R_{0,\lambda}| = (1 + \lambda + \mathcal{D}^2)^{-1/2} = 1 \otimes (1 + \lambda - \Delta)^{-1/2}. \tag{4.3.5}$$

Furthermore, using the resolvent identity repeatedly, we obtain

$$\begin{aligned} R_{1,\lambda} - R_{0,\lambda} &= -R_{1,\lambda}(1 \otimes M_\varphi)R_{0,\lambda} \\ &= R_{1,\lambda}((1 \otimes M_\varphi)R_{0,\lambda})^2 - R_{0,\lambda}(1 \otimes M_\varphi)R_{0,\lambda} \qquad (4.3.6) \\ &= -R_{1,\lambda}((1 \otimes M_\varphi)R_{0,\lambda})^3 + R_{0,\lambda}((1 \otimes M_\varphi)R_{0,\lambda})^2 \qquad (4.3.7) \\ &\quad - R_{0,\lambda}(1 \otimes M_\varphi)R_{0,\lambda}. \end{aligned}$$

We also recall so-called Cwikel estimates (see [**73**, Chapter 4]). Since we refer to these estimates in the later section (see Chapter 7) we present the estimates in general dimension.

Firstly, we define function spaces due to Birman and Solomyak [**18**] (see also [**73**]). Let $d \in \mathbb{N}$ and let $Q = [-\frac{1}{2}, \frac{1}{2}]^d \subset \mathbb{R}^d$ be the unit cube centred at the origin. For $n \in \mathbb{Z}^d$, let $Q + n$ denote the unit cube centred at n. For $1 \le p < 2$ define the space

$$l_p(L_2)(\mathbb{R}^d) := \Big\{ f \in L_0(\mathbb{R}^d) \, : \, \sum_{n \in \mathbb{Z}^d} \|f\chi_{Q+n}\|_2^p < \infty \Big\},$$

with the corresponding norm

$$\|f\|_{l_p(L_2)(\mathbb{R}^d)} := \Big(\sum_{n \in \mathbb{Z}^d} \|f\chi_{Q+n}\|_2^p \Big)^{1/p}, \qquad f \in l_p(L_2)(\mathbb{R}^d).$$

It is clear that $S(\mathbb{R}^d) \subset l_p(L_2)(\mathbb{R}^d)$, for any $d \in \mathbb{N}$.

We now state a particular case of Cwikel estimates. For general Cwikel estimates we refer to [**73**, Chapter 4] (see also [**58**]).

THEOREM 4.3.1. *Let* $\lambda \ge 0$.

(i) [**73**, Theorem 4.1] *Suppose* $2 \le p < \infty$, *and suppose* $\delta > d/2p$. *If* $\varphi \in L_p(\mathbb{R}^d)$, *then* $M_\varphi(1 + \lambda - \Delta)^{-\delta} \in \mathcal{L}_p(L_2(\mathbb{R}^d))$, *and*

$$\big\|M_\varphi(1 + \lambda - \Delta)^{-\delta}\big\|_p \le \mathrm{const} \cdot (1+\lambda)^{d/2p-\delta} < \infty.$$

(ii) [**73**, Theorem 4.5] *Suppose* $1 \le p < 2$, *and suppose* $\delta > d/2p$. *If* $\varphi \in l_p(L_2)(\mathbb{R}^d)$, *then* $M_\varphi(1 + \lambda - \Delta)^{-\delta} \in \mathcal{L}_p(L_2(\mathbb{R}^d))$ *and*

$$\big\|M_\varphi(1 + \lambda - \Delta)^{-\delta}\big\|_p \le \mathrm{const} \cdot (1+\lambda)^{d/4-\delta} < \infty.$$

For the convenience we also rewrite Cwikel estimates in the form we use them in this section.

COROLLARY 4.3.2. *Suppose* $\varphi \in S(\mathbb{R}^2)$. *The following assertions hold:*

(i) $(1 \otimes M_\varphi)R_{0,\lambda} \in \mathcal{L}_3(\mathcal{H})$ *and*

$$\big\|(1 \otimes M_\varphi)R_{0,\lambda}\big\|_3 \le \mathrm{const} \cdot (1+\lambda)^{-1/6}.$$

(ii) $(1 \otimes M_\varphi)R_{0,\lambda}^3 \in \mathcal{L}_{3/2}(\mathcal{H})$ *and*

$$\big\|(1 \otimes M_\varphi)R_{0,\lambda}^3\big\|_{3/2} \le \mathrm{const} \cdot (1+\lambda)^{-5/6}.$$

(iii) $M_\varphi(1 - \Delta)^{-2} \in \mathcal{L}_1(\mathbb{R}^2)$.

PROOF. We present the proof for part (i) only, since other parts can be proved similarly.

(i). Writing $R_{0,\lambda} = (1 + \lambda + \mathcal{D}^2)^{-1/2} \operatorname{sgn}(R_{0,\lambda})$ and recalling (4.3.5), it is sufficient to prove the assertion for the operator $M_\varphi(1 + \lambda - \Delta)^{-1/2}$. Since the

functions φ is in $L_3(\mathbb{R}^2)$, the required inclusion and the estimate follow from Theorem 4.3.1.

□

4.3.1. Decomposition for the operator $g(\mathcal{D}+1\otimes M_\varphi)-g(\mathcal{D})$. In this section we show that

$$g(\mathcal{D}+1\otimes M_\varphi)-g(\mathcal{D})\in\frac{1}{2}\gamma_2\gamma_1\otimes(M_{\partial_2\varphi}\partial_1-M_{\partial_1\varphi}\partial_2)(1-\Delta)^{-3/2}+\mathcal{L}_1(\mathcal{H}).$$

Firstly, we present a suitable integral decomposition for the operator $g(\mathcal{D}+1\otimes M_\varphi)-g(\mathcal{D})$. Recall that the operator $R_{1,\lambda}, -R_{0,\lambda}$ are defined in (4.3.4).

LEMMA 4.3.3. [**26**] *We have*

$$g(\mathcal{D}+1\otimes M_\varphi)-g(\mathcal{D})=\pi^{-1}\mathrm{Re}\Big(\int_0^\infty\lambda^{-1/2}[R_{1,\lambda}-R_{0,\lambda}]\,d\lambda\Big),$$

with the convergent Bochner integral on the right-hand side.

PROOF. We recall the fact that for any self-adjoint operator T in $\mathcal{H}$,

$$\big(T^2+1\big)^{-1/2}=\pi^{-1}\int_0^\infty\lambda^{-1/2}\big(T^2+1+\lambda\big)^{-1}\,d\lambda$$

(see, e.g., [**52**, p. 282] for a more general result). Thus,

$$\begin{aligned}&g(\mathcal{D}+1\otimes M_\varphi)-g(\mathcal{D})\\&\quad=\frac{1}{\pi}\int_0^\infty\lambda^{1/2}\Big[\frac{\mathcal{D}+1\otimes M_\varphi}{(\mathcal{D}+1\otimes M_\varphi)^2+1+\lambda}-\frac{\mathcal{D}}{\mathcal{D}^2+1+\lambda}\Big]\,d\lambda.\end{aligned}$$

Since

$$\frac{t}{t^2+1+\lambda}=\mathrm{Re}\big(\frac{1}{t+i(1+\lambda)^{1/2}}\big),\quad t\in\mathbb{R}$$

we have that

$$\frac{\mathcal{D}+1\otimes M_\varphi}{(\mathcal{D}+1\otimes M_\varphi)^2+1+\lambda}-\frac{\mathcal{D}}{\mathcal{D}^2+1+\lambda}=\mathrm{Re}\big(R_{1,\lambda}-R_{0,\lambda}\big),$$

which concludes the proof. □

Combining Lemma 4.3.3 with (4.3.7), we may represent the difference $g(\mathcal{D}+1\otimes M_\varphi)-g(\mathcal{D})$ as the Bochner integral

$$\begin{aligned}g(\mathcal{D}+1\otimes M_\varphi)-g(\mathcal{D})&=\frac{1}{\pi}\mathrm{Re}\Big(\int_0^\infty\frac{1}{\lambda^{1/2}}(R_{1,\lambda}-R_{0,\lambda})\,d\lambda\Big)\\&\overset{(4.3.7)}{=}\frac{1}{\pi}\mathrm{Re}\Big(-\int_0^\infty\frac{d\lambda}{\lambda^{1/2}}R_{1,\lambda}\big((1\otimes M_\varphi)R_{0,\lambda}\big)^3\\&\qquad+\int_0^\infty\frac{d\lambda}{\lambda^{1/2}}R_{0,\lambda}\big((1\otimes M_\varphi)R_{0,\lambda}\big)^2\\&\qquad\qquad-\int_0^\infty\frac{d\lambda}{\lambda^{1/2}}R_{0,\lambda}(1\otimes M_\varphi)R_{0,\lambda}\Big).\end{aligned}\tag{4.3.8}$$

We shall be interested in showing that the first two terms of our decomposition lie in $\mathcal{L}_1(\mathcal{H})$, and can therefore be neglected for our purposes. We begin with the first term of (4.3.8).

LEMMA 4.3.4. *Suppose $\varphi \in S(\mathbb{R}^2)$. Then*

$$\int_0^\infty \frac{d\lambda}{\lambda^{1/2}} R_{1,\lambda}\big((1\otimes M_\varphi)R_{0,\lambda}\big)^3 \in \mathcal{L}_1(\mathcal{H}).$$

PROOF. Appealing to (4.3.4), we have that $\|R_{1,\lambda}\|_\infty = (1+\lambda)^{-1/2}$, and by Corollary 4.3.2 (i), we have $\big\|(1\otimes M_\varphi)R_{0,\lambda}\big\|_3 \le \mathrm{const}\cdot(1+\lambda)^{-1/6}$. Hence, by the noncommutative Hölder inequality (1.1.1), we have that

$$\begin{aligned}\bigg\|\int_0^\infty \frac{d\lambda}{\lambda^{1/2}} R_{1,\lambda}\big((1\otimes M_\varphi)R_{0,\lambda}\big)^3\bigg\|_1 &\le \int_0^\infty \frac{d\lambda}{\lambda^{1/2}}\big\|R_{1,\lambda}((1\otimes M_\varphi)R_{0,\lambda})^3\big\|_1\\ &\le \int_0^\infty \frac{d\lambda}{\lambda^{1/2}}\|R_{1,\lambda}\|_\infty\big\|(1\otimes M_\varphi)R_{0,\lambda}\big\|_3^3\\ &\le \mathrm{const}\cdot\int_0^\infty \frac{d\lambda}{\lambda^{1/2}(1+\lambda)}<\infty,\end{aligned}$$

as required. □

Before moving on to the second term of (4.3.8), we state the following easy corollary of functional calculus and simple computations. We supplement the proof in Appendix A (see Lemma A.1 (ii)).

LEMMA 4.3.5. *We have*

$$\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}\mathrm{Re}(R_{0,\lambda}^3) = -\frac{3\pi}{2}\mathcal{D}(1+\mathcal{D}^2)^{-5/2}.$$

Before we proceed to the following lemma we note that since $R_{0,\lambda}+(\mathcal{D}+i(1+\lambda)^{1/2})^{-1}$, it follows from (1.1.4) and (4.3.3) that

$$[R_{0,\lambda},1\otimes M_\varphi] \overset{(1.1.4)}{=} -R_{0,\lambda}[\mathcal{D},1\otimes M_\varphi]R_{0,\lambda} \overset{(4.3.3)}{=} -\sum_{k=1}^d R_{0,\lambda}(\gamma_k\otimes M_{\partial_k\varphi})R_{0,\lambda}. \tag{4.3.9}$$

LEMMA 4.3.6. *Suppose $\varphi \in S(\mathbb{R}^2)$. Then*

$$\mathrm{Re}\Big(\int_0^\infty \frac{d\lambda}{\lambda^{1/2}} R_{0,\lambda}\big((1\otimes M_\varphi)R_{0,\lambda}\big)^2\Big) \in \mathcal{L}_1(\mathcal{H}).$$

PROOF. Applying (4.3.9) twice to $R_{0,\lambda}\big((1\otimes M_\varphi)R_{0,\lambda}\big)^2$ yields

$$\begin{aligned}&R_{0,\lambda}(1\otimes M_\varphi)R_{0,\lambda}(1\otimes M_\varphi)R_{0,\lambda}\\ &\quad= R_{0,\lambda}(1\otimes M_\varphi)R_{0,\lambda}^2(1\otimes M_\varphi)\\ &\qquad+\sum_{k=1}^2 R_{0,\lambda}(1\otimes M_\varphi)R_{0,\lambda}^2(\gamma_k\otimes M_{\partial_k\varphi})R_{0,\lambda}\\ &\quad= (1\otimes M_\varphi)R_{0,\lambda}^3(1\otimes M_\varphi) - \sum_{k=1}^2 R_{0,\lambda}(\gamma_k\otimes M_{\partial_k\varphi})R_{0,\lambda}^3(1\otimes M_\varphi)\\ &\qquad+\sum_{k=1}^d R_{0,\lambda}(1\otimes M_\varphi)R_{0,\lambda}^2(\gamma_k\otimes M_{\partial_k\varphi})R_{0,\lambda}.\end{aligned} \tag{4.3.10}$$

We consider each term on the right-hand side of (4.3.10) individually.

For each $k = 1, 2$, Hölder's inequality (1.1.1) and Corollary 4.3.2 imply

$$\begin{aligned}&\big\|R_{0,\lambda}(1\otimes M_\varphi)R_{0,\lambda}^2(\gamma_k\otimes M_{\partial_k\varphi})R_{0,\lambda}\big\|_1\\&\quad\le \big\|(1\otimes M_{|\varphi|^{1/2}})R_{0,\lambda}\big\|_3\big\|(1\otimes M_{|\varphi|^{1/2}})R_{0,\lambda}^2\big\|_3\big\|(\gamma_k\otimes M_{\partial_k\varphi})R_{0,\lambda}\big\|_3\\&\quad\le \text{const}\cdot(1+\lambda)^{-1}.\end{aligned} \tag{4.3.11}$$

Similarly, we have that

$$\begin{aligned}&\big\|R_{0,\lambda}(\gamma_k\otimes M_{\partial_k\varphi})R_{0,\lambda}^3(1\otimes M_\varphi)\big\|_1\\&\quad\le \big\|(\gamma_k\otimes M_{\partial_k\varphi})R_{0,\lambda}\big\|_3\big\|(1\otimes M_\varphi)R_{0,\lambda}^3\big\|_{3/2}\le \text{const}\cdot(1+\lambda)^{-1}.\end{aligned} \tag{4.3.12}$$

Hence, the second and third terms on the right-hand side of (4.3.10) generate an operator in $\mathcal{L}_1(\mathcal{H})$.

Now, we consider the first term on the right-hand side of (4.3.10), and show that it also belongs to $\mathcal{L}_1(\mathcal{H})$. By Lemma 4.3.5, we have that

$$\begin{aligned}\mathrm{Re}\Big((1\otimes M_\varphi)\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}(R_{0,\lambda}^3)(1\otimes M_\varphi)\Big) &= (1\otimes M_\varphi)\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}\mathrm{Re}(R_{0,\lambda}^3)(1\otimes M_\varphi)\\&= \frac{3\pi}{2}(1\otimes M_\varphi)(1+\mathcal{D}^2)^{-2}\cdot\mathcal{D}(1+\mathcal{D}^2)^{-1/2}(1\otimes M_\varphi).\end{aligned}$$

Therefore, by Corollary 4.3.2 we conclude that the first term on the right-hand side of (4.3.10) is a trace-class operator. Thus, combining the obtained results with (4.3.10), we conclude the proof. $\square$

So far, we have established that the first two integrals on the right-hand side of (4.3.8) belong to $\mathcal{L}_1(\mathcal{H})$. This leaves only the third integral term of (4.3.8). Since this is the last remaining term in the expression, we claim that this term is not in $\mathcal{L}_1(\mathcal{H})$. First, we need the following auxiliary lemma, whose proof can be found in Appendix A (see Lemma A.2 and Lemma A.1 (i), respectively).

LEMMA 4.3.7. *(i) Suppose $k = 1, 2$. Then*

$$\begin{aligned}&\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}\big(R_{0,\lambda}(\gamma_k\otimes 1)R_{0,\lambda}^2 + R_{0,\lambda}^*(\gamma_k\otimes 1)(R_{0,\lambda}^*)^2\big)\\&\quad= \frac{\pi}{2}[\mathcal{D},\gamma_k\otimes 1](1+\mathcal{D}^2)^{-3/2} - \frac{3\pi}{2}\{\mathcal{D},\gamma_k\otimes 1\}(1+\mathcal{D}^2)^{-5/2},\end{aligned} \tag{4.3.13}$$

where $\{\cdot,\cdot\}$ denotes the anticommutator.

(ii) We have

$$\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}\mathrm{Re}(R_{0,\lambda}^2) = -2\pi(1+\mathcal{D}^2)^{-3/2}.$$

LEMMA 4.3.8. *Suppose $\varphi\in S(\mathbb{R}^2)$. Then*

$$\begin{aligned}&\mathrm{Re}\Big(\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}R_{0,\lambda}(1\otimes M_\varphi)R_{0,\lambda}\Big)\\&\quad\in \frac{\pi}{2}\gamma_2\gamma_1\otimes(M_{\partial_2\varphi}\partial_1 - M_{\partial_1\varphi}\partial_2)(1-\Delta)^{-3/2} + \mathcal{L}_1(\mathcal{H}).\end{aligned}$$

Proof. By applying (4.3.9) twice, we get that

$$\begin{aligned}
R_{0,\lambda}(1\otimes M_\varphi)R_{0,\lambda} &= (1\otimes M_\varphi)R_{0,\lambda}^2 - \sum_{k=1}^{2}\big(R_{0,\lambda}(1\otimes M_{\partial_k\varphi})\big)(\gamma_k\otimes 1)R_{0,\lambda}^2\\
&= (1\otimes M_\varphi)R_{0,\lambda}^2 - \sum_{k=1}^{2}(1\otimes M_{\partial_k\varphi})R_{0,\lambda}(\gamma_k\otimes 1)R_{0,\lambda}^2+\\
&\quad + \sum_{j,k=1}^{2} R_{0,\lambda}(\gamma_j\otimes M_{\partial_j\partial_k\varphi})R_{0,\lambda}(\gamma_k\otimes 1)R_{0,\lambda}^2. \qquad (4.3.14)
\end{aligned}$$

By Hölder's inequality (1.1.1) and Corollary 4.3.2, we have that

$$\begin{aligned}
&\big\|R_{0,\lambda}(\gamma_j\otimes M_{\partial_j\partial_k\varphi})R_{0,\lambda}(\gamma_k\otimes 1)R_{0,\lambda}^2\big\|_1\\
&\qquad\overset{(4.3.5)}{\leq} \big\|(1+\lambda+\mathcal{D}^2)^{-1/2}(1\otimes M_{\partial_j\partial_k\varphi})(1+\lambda+\mathcal{D}^2)^{-3/2}\big\|_1\\
&\qquad\leq \text{const}\cdot(1+\lambda)^{-1}. \qquad (4.3.15)
\end{aligned}$$

We treat $(R_{0,\lambda}(1\otimes M_\varphi)R_{0,\lambda})^*$ similarly. Since $M_{\partial_k\varphi}^* = -M_{\partial_k\varphi}$ and $M_{\partial_j\partial_k\varphi}^* = M_{\partial_j\partial_k\varphi}$, by shifting $(1\otimes M_\varphi)$ to the right with (4.3.9) instead of the left before taking the adjoint, we observe that

$$\begin{aligned}
(R_{0,\lambda}(1\otimes M_\varphi)R_{0,\lambda})^* &= (1\otimes M_\varphi)(R_{0,\lambda}^*)^2 - \sum_{k=1}^{2}(1\otimes M_{\partial_k\varphi})R_{0,\lambda}^*(\gamma_k\otimes 1)(R_{0,\lambda}^*)^2+\\
&\quad + \sum_{j,k=1}^{2}\mathcal{D}_{0,\lambda}^*(\gamma_j\otimes M_{\partial_j\partial_k\varphi})R_{0,\lambda}^*(\gamma_k\otimes 1)(R_{0,\lambda}^*)^2, \quad (4.3.16)
\end{aligned}$$

and by a similar argument to that of (4.3.15), we arrive at

$$\big\|R_{0,\lambda}^*(\gamma_j\otimes M_{\partial_j\partial_k\varphi})R_{0,\lambda}^*(\gamma_k\otimes 1)(R_{0,\lambda}^*)^2\big\|_1 \leq \text{const}\cdot(1+\lambda)^{-1}. \qquad (4.3.17)$$

Hence, the third terms on the right-hand sides of (4.3.14) and (4.3.16) generate an operator from $\mathcal{L}_1(\mathcal{H})$. Therefore, by (4.3.14) and (4.3.16) we have

$$\begin{aligned}
&\operatorname{Re}\Big(\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}R_{0,\lambda}(1\otimes M_\varphi)R_{0,\lambda}\Big)\\
&\quad\in \frac{1}{2}\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}\big((1\otimes M_\varphi)R_{0,\lambda}^2 + (1\otimes M_\varphi)(R_{0,\lambda}^*)^2\big)\\
&\qquad - \frac{1}{2}\sum_{k=1}^{2}\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}(1\otimes M_{\partial_k\varphi})\big(R_{0,\lambda}(\gamma_k\otimes 1)R_{0,\lambda}^2 + R_{0,\lambda}^*(\gamma_k\otimes 1)(R_{0,\lambda}^*)^2\big)\\
&\qquad + \mathcal{L}_1(\mathcal{H})\\
&\quad= \frac{(1\otimes M_\varphi)}{2}\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}\operatorname{Re}\big(R_{0,\lambda}^2\big)\\
&\qquad - \frac{1}{2}\sum_{k=1}^{2}(1\otimes M_{\partial_k\varphi})\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}\big(R_{0,\lambda}(\gamma_k\otimes 1)R_{0,\lambda}^2 + R_{0,\lambda}^*(\gamma_k\otimes 1)(R_{0,\lambda}^*)^2\big)\\
&\qquad + \mathcal{L}_1(\mathcal{H}).
\end{aligned}$$

(4.3.18)

By Lemma 4.3.7 (ii) we have that

$$\frac{(1\otimes M_\varphi)}{2}\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}\mathrm{Re}\big(R_{0,\lambda}^2\big) = -\pi(1\otimes M_\varphi)(1+\mathcal{D}^2)^{-3/2}\in\mathcal{L}_1(\mathcal{H}), \tag{4.3.19}$$

where the inclusion into $\mathcal{L}_1(\mathcal{H})$ follows from Corollary 4.3.2.

Combining this inclusion with Lemma 4.3.7 (ii), we conclude that (4.3.18) can be written as

$$\begin{aligned}
&\mathrm{Re}\Big(\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}R_{0,\lambda}(1\otimes M_\varphi)R_{0,\lambda}\Big)\\
&\in -\frac12\sum_{k=1}^2(1\otimes M_{\partial_k\varphi})\int_0^\infty\frac{d\lambda}{\lambda^{1/2}}\big(R_{0,\lambda}(\gamma_k\otimes 1)R_{0,\lambda}^2+R_{0,\lambda}^*(\gamma_k\otimes 1)(R_{0,\lambda}^*)^2\big)\\
&\qquad +\mathcal{L}_1(\mathcal{H}) \qquad\qquad (4.3.20)\\
&\overset{(4.3.13)}{=} -\frac{\pi}{4}\sum_{k=1}^2(1\otimes M_{\partial_k\varphi})[\mathcal{D},\gamma_k\otimes 1](1+\mathcal{D}^2)^{-3/2}\\
&\qquad -\frac{3\pi}{4}\sum_{k=1}^2(1\otimes M_{\partial_k\varphi})\{\mathcal{D},\gamma_k\otimes 1\}(1+\mathcal{D}^2)^{-5/2}+\mathcal{L}_1(\mathcal{H}).
\end{aligned}$$

Observe that, for the second term on the right-hand side of (4.3.18), Corollary 4.3.2 implies that

$$\Big\|(1\otimes M_{\partial_k\varphi})\{\mathcal{D},\gamma_k\otimes 1\}(1+\mathcal{D}^2)^{-5/2}\Big\|_1\le 2\big\|(1\otimes M_{\partial_k\varphi})(1+\mathcal{D}^2)^{-2}\big\|_1$$

so the third term of (4.3.18) lies in $\mathcal{L}_1(\mathcal{H})$.

Thus, by rearranging this term, we conclude that

$$\begin{aligned}
&\mathrm{Re}\Big(\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}R_{0,\lambda}(1\otimes M_\varphi)R_{0,\lambda}\Big)\\
&\overset{(4.3.18)}{\in} \frac{\pi}{4}\sum_{k=1}^2(1\otimes M_{\partial_k\varphi})[\gamma_k\otimes 1,\mathcal{D}](1+\mathcal{D}^2)^{-3/2}+\mathcal{L}_1(\mathcal{H})\\
&= \frac{\pi}{4}\sum_{k,j=1,2}\big((\gamma_k\gamma_j-\gamma_j\gamma_k)\otimes M_{\partial_k\varphi}\partial_j\big)(1+\mathcal{D}^2)^{-3/2}+\mathcal{L}_1(\mathcal{H})\\
&= \frac{\pi}{2}\big(\gamma_2\gamma_1\otimes(M_{\partial_2\varphi}\partial_1-M_{\partial_1\varphi}\partial_2)\big)(1+\mathcal{D}^2)^{-3/2}+\mathcal{L}_1(\mathcal{H}),
\end{aligned}$$

as required. □

Recalling (4.3.8), the following expression immediately follows from Lemmas 4.3.4, 4.3.6 and 4.3.8, which is the main result of this subsection.

Corollary 4.3.9. *Suppose $\varphi\in S(\mathbb{R}^2)$. Then*

$$g(\mathcal{D}+(1\otimes M_\varphi))-g(\mathcal{D})\in\frac12\gamma_2\gamma_1\otimes(M_{\partial_2\varphi}\partial_1-M_{\partial_1\varphi}\partial_2)(1-\Delta)^{-3/2}+\mathcal{L}_1(\mathcal{H}). \tag{4.3.21}$$

In particular,

$$g(\mathcal{D}+(1\otimes M_\varphi))-g(\mathcal{D})\in\mathcal{L}_1(\mathcal{H}) \textit{ if and only if } \tag{4.3.22}$$

$$\big(M_{\varphi_1}\partial_2-M_{\varphi_2}\partial_1\big)\frac{1}{(1-\Delta)^{3/2}}\in\mathcal{L}_1(\mathcal{H}).$$

4.3.2. The auxiliary operator $g(\mathcal{D}+1\otimes M_\varphi)-g(\mathcal{D})$ is not trace-class. In this subsection we show that $g(\mathcal{D}+1\otimes M_\varphi)-g(\mathcal{D})$ is not a trace-class operator for $0\neq\varphi\in S(\mathbb{R}^2)$. We note that this results holds for higher-dimensional examples. As we show in Section 7.3 the operator $g(\mathcal{D}+1\otimes M_\varphi)-g(\mathcal{D})$ is a trace-class operator if $\mathcal{D}$ is one-dimensional Dirac operator.

The key tools in the proof presented in [**57**, Theorem 5.3] are singular traces and Connes' trace theorem in the form proved in [**77**, Lemma 16]. Here we present a different approach which uses only simple geometrical properties of the trace ideal $\mathcal{L}_1(\mathcal{H})$ and elementary computations.

THEOREM 4.3.10. *Let $\varphi\in S(\mathbb{R}^2), \varphi\neq 0$. Then $g(\mathcal{D}+1\otimes M_\varphi)-g(\mathcal{D})$ is not a trace-class operator.*

PROOF. To shorten notation we denote $\varphi_1=\partial_1\varphi$ and $\varphi_2=\partial_2\varphi$. By Corollary 4.3.9 it is sufficient to show, that the operator $K=\big(M_{\varphi_1}\partial_2-M_{\varphi_2}\partial_1\big)\frac{1}{(1-\Delta)^{3/2}}$ is not trace-class. Suppose the contrary. We note that $\varphi_1,\varphi_2\neq 0$, since otherwise the Schwartz function φ equals zero.

For all $n,m\in\mathbb{Z}$ we set

$$\psi:=\chi_{[-\frac12,\frac12]\times[-\frac12,\frac12]},\quad \psi_{n,m}(\cdot,\cdot):=\psi(\cdot-n,\cdot-m).$$

By [**39**, Proposition 3.3] we have that

$$\sum_{n,m}\psi_{n,m}(\partial_1,\partial_2)K\psi_{n,m}(\partial_1,\partial_2)\prec\prec K$$

in the sense of Hardy-Littlewood-Polya submajorization (see e.g. [**59**, Section 3.3]) Since $\mathcal{L}_1(\mathcal{H})$ is fully symmetric and by assumption $K\in\mathcal{L}_1(\mathcal{H})$, it follows that

$$\begin{aligned}\sum_{n,m\in\mathbb{Z}}\|\psi_{n,m}(\partial_1,\partial_2)K\psi_{n,m}(\partial_1,\partial_2)\|_1&=\Big\|\sum_{n,m\in\mathbb{Z}}\psi_{n,m}(\partial_1,\partial_2)K\psi_{n,m}(\partial_1,\partial_2)\Big\|_1\\&\leq\|K\|_1<\infty.\end{aligned}$$

We claim that the series above on the left-hand side does not converge, and this would apply that the operator K is not trace-class. To this end, it is sufficient to show that the series

$$\sum_{n,m\geq 0}\|\psi_{n,m}(\partial_1,\partial_2)K\psi_{n,m}(\partial_1,\partial_2)\|_1,\quad m\geq c(\varphi)n, \tag{4.3.23}$$

does not converge for sufficiently large $c(\varphi)$.

By the definition of the operator K we have

$$\begin{aligned}&\|\psi_{n,m}(\partial_1,\partial_2)K\psi_{n,m}(\partial_1,\partial_2)\|_1\\&\qquad\geq\|\psi_{n,m}(\partial_1,\partial_2)M_{\varphi_1}\partial_2(1-\Delta)^{-3/2}\psi_{n,m}(\partial_1,\partial_2)\|_1\\&\qquad\quad-\|\psi_{n,m}(\partial_1,\partial_2)M_{\varphi_2}\partial_1(1-\Delta)^{-3/2}\psi_{n,m}(\partial_1,\partial_2)\|_1.\end{aligned}\tag{4.3.24}$$

Note, that the right-hand side is well-defined since both operators are trace-class (see Theorem 4.3.1). We estimate two terms on the right-hand side above separately.

For $(s,t)\in[n-\frac12,n+\frac12]\times[m-\frac12,m+\frac12]$ we have the estimate

$$\frac{t}{(1+s^2+t^2)^{3/2}}\geq\frac{m-\frac12}{(1+(m+\frac12)^2+(n+\frac12)^2)^{3/2}}.$$

Therefore,

$$\psi_{n,m}(s,t)\cdot\frac{m-\frac{1}{2}}{(1+(m+\frac{1}{2})^2+(n+\frac{1}{2})^2)^{3/2}}=\theta(s,t)\psi_{n,m}(s,t)\frac{t}{(1+s^2+t^2)^{3/2}} \tag{4.3.25}$$

for some θ with $0\le\theta(s,t)\le 1$. Therefore, for the first term in (4.3.24) we have

$$\begin{aligned}
&\|\psi_{n,m}(\partial_1,\partial_2)M_{\varphi_1}\partial_2(1-\Delta)^{-3/2}\psi_{n,m}(\partial_1,\partial_2)\|_1\\
&\qquad\ge\|\psi_{n,m}(\partial_1,\partial_2)M_{\varphi_1}\partial_2(1-\Delta)^{-3/2}\psi_{n,m}(\partial_1,\partial_2)\|_1\|\theta(\partial_1,\partial_2)\|_\infty\\
&\qquad\ge\|\psi_{n,m}(\partial_1,\partial_2)M_{\varphi_1}\partial_2(1-\Delta)^{-3/2}\psi_{n,m}(\partial_1,\partial_2)\theta(\partial_1,\partial_2)\|_1\\
&\qquad\overset{(4.3.25)}{=}\|\psi_{n,m}(\partial_1,\partial_2)M_{\varphi_1}\psi_{n,m}(\partial_1,\partial_2)\|_1\frac{m-\frac{1}{2}}{(1+(m+\frac{1}{2})^2+(n+\frac{1}{2})^2)^{3/2}}.
\end{aligned} \tag{4.3.26}$$

Using the fact, that ∂_1 (respectively, ∂_2) is generator of translation, we have that $\partial_1+n=M_{h_n}\partial_1M_{h_n}$ (respectively, $\partial_2+m=M_{h_m}\partial_2M_{h_m}$), where $h_n=e^{-ins}$ (respectively, $h_m=e^{-imt}$). Hence, we obtain that

$$\psi_{n,m}(\partial_1,\partial_2)=\psi(\partial_1+n,\partial_2+m)=M_{h_nh_m}\psi(\partial_1,\partial_2)M_{h_n^{-1}h_m^{-1}}, \tag{4.3.27}$$

and therefore combining this equality with (4.3.26) we obtain

$$\begin{aligned}
&\|\psi_{n,m}(\partial_1,\partial_2)M_{\varphi_1}\partial_2(1-\Delta)^{-3/2}\psi_{n,m}(\partial_1,\partial_2)\|_1\\
&\qquad\ge\|\psi(\partial_1,\partial_2)M_{\varphi_1}\psi(\partial_1,\partial_2)\|_1\frac{m-\frac{1}{2}}{(1+(m+\frac{1}{2})^2+(n+\frac{1}{2})^2)^{3/2}},
\end{aligned} \tag{4.3.28}$$

for all $n,m\in\mathbb{Z}$.

On the other hand, for $(s,t)\in[n-\frac{1}{2},n+\frac{1}{2}]\times[m-\frac{1}{2},m+\frac{1}{2}]$ we have the estimate

$$\frac{s}{(1+s^2+t^2)^{3/2}}\le\frac{n+\frac{1}{2}}{(1+(m-\frac{1}{2})^2+(n-\frac{1}{2})^2)^{3/2}}.$$

Hence, using this estimate and the unitary equivalence (4.3.27) for the second term in (4.3.24) we have

$$\begin{aligned}
&\|\psi_{n,m}(\partial_1,\partial_2)M_{\varphi_2}\partial_1(1-\Delta)^{-3/2}\psi_{n,m}(\partial_1,\partial_2)\|_1\\
&\qquad=\|\psi_{n,m}(\partial_1,\partial_2)M_{\varphi_2}\psi_{n,m}(\partial_1,\partial_2)\partial_1(1-\Delta)^{-3/2}\|_1\\
&\qquad\le\frac{n+\frac{1}{2}}{(1+(m-\frac{1}{2})^2+(n-\frac{1}{2})^2)^{3/2}}\|\psi_{n,m}(\partial_1,\partial_2)M_{\varphi_2}\psi_{n,m}(\partial_1,\partial_2)\|_1\\
&\qquad\le\frac{n+\frac{1}{2}}{(1+(m-\frac{1}{2})^2+(n-\frac{1}{2})^2)^{3/2}}\|\psi(\partial_1,\partial_2)M_{\varphi_2}\psi(\partial_1,\partial_2)\|_1
\end{aligned}$$

Combining this estimate with (4.3.28) we infer from (4.3.24) that

$$\begin{aligned}
&\|\psi_{n,m}(\partial_1,\partial_2)K\psi_{n,m}(\partial_1,\partial_2)\|_1\\
&\qquad\ge\|\psi(\partial_1,\partial_2)M_{\varphi_1}\psi(\partial_1,\partial_2)\|_1\frac{m-\frac{1}{2}}{(1+(m+\frac{1}{2})^2+(n+\frac{1}{2})^2)^{3/2}}\\
&\qquad\quad-\frac{n+\frac{1}{2}}{(1+(m-\frac{1}{2})^2+(n-\frac{1}{2})^2)^{3/2}}\|\psi(\partial_1,\partial_2)M_{\varphi_2}\psi(\partial_1,\partial_2)\|_1
\end{aligned} \tag{4.3.29}$$

Next, we claim that

$$\|\psi(\partial_1,\partial_2)M_{\varphi_1}\psi(\partial_1,\partial_2)\|_1, \quad \|\psi(\partial_1,\partial_2)M_{\varphi_2}\psi(\partial_1,\partial_2)\|_1 \neq 0. \tag{4.3.30}$$

We concentrate firstly on the operator $\psi(\partial_1,\partial_2)M_{\varphi_1}\psi(\partial_1,\partial_2)$.

If $\psi(\partial_1,\partial_2)M_{\varphi_1}\psi(\partial_1,\partial_2) = 0$, then repeating the argument for the function $\psi_N := \chi_{[-\frac{N}{2},\frac{N}{2}]\times[-\frac{N}{2},\frac{N}{2}]}$, $N \in \mathbb{N}$, we again arrive to estimate similar to (4.3.29). If there exists no $N \in \mathbb{N}$ such that $\psi_N(\partial_1,\partial_2)M_{\varphi_1}\psi_N(\partial_1,\partial_2) \neq 0$, then since $\psi_N \to I$ in the strong operator topology, we infer that $0 = \psi_N(\partial_1,\partial_2)M_{\varphi_1}\psi_N(\partial_1,\partial_2) \xrightarrow{(so)} M_{\varphi_1}$, that is $\varphi_1 = 0$, which is not the case. Therefore, there exists $N_1 \in \mathbb{N}$, such that

$$\psi_{N_1}(\partial_1,\partial_2)M_{\varphi_1}\psi_{N_1}(\partial_1,\partial_2) \neq 0.$$

Similarly, there exists $N_2 \in \mathbb{N}$, such that

$$\psi_{N_2}(\partial_1,\partial_2)M_{\varphi_1}\psi_{N_2}(\partial_1,\partial_2) \neq 0.$$

Taking $N_0 = \max\{N_1, N_2\}$ we have that

$$\psi_{N_0}(\partial_1,\partial_2)M_{\varphi_1}\psi_{N_0}(\partial_1,\partial_2), \quad \psi_{N_0}(\partial_1,\partial_2)M_{\varphi_2}\psi_{N_0}(\partial_1,\partial_2) \neq 0.$$

Thus, without loss of generality we can assume (4.3.30).

Hence, using (4.3.29) and (4.3.30) for the series in (4.3.23) we infer

$$\begin{aligned}
&\sum_{n,m\geq 0, m\geq c(\varphi)n} \|\psi_{n,m}(\partial_1,\partial_2)K\psi_{n,m}(\partial_1,\partial_2)\|_1 \\
&\geq \sum_{n,m\geq 0, m\geq c(\varphi)n} \Big(\|\psi(\partial_1,\partial_2)M_{\varphi_1}\psi(\partial_1,\partial_2)\|_1 \frac{m-\frac{1}{2}}{(1+(m+\frac{1}{2})^2+(n+\frac{1}{2})^2)^{3/2}} \\
&\quad - \frac{n+\frac{1}{2}}{(1+(m-\frac{1}{2})^2+(n-\frac{1}{2})^2)^{3/2}} \|\psi(\partial_1,\partial_2)M_{\varphi_2}\psi(\partial_1,\partial_2)\|_1 \Big) \\
&\geq \operatorname{cosnt}(\varphi) \sum_{n,m\geq 0, m\geq c(\varphi)n} \frac{m-(n+1)}{(1+(m+\frac{1}{2})^2+(n+\frac{1}{2})^2)^{3/2}}.
\end{aligned}$$

Since $m \geq c(\varphi)n$ for $c(\varphi)$ sufficiently large, we have that

$$\begin{aligned}
&\sum_{n,m\geq 0, m\geq c(\varphi)n} \|\psi_{n,m}(\partial_1,\partial_2)K\psi_{n,m}(\partial_1,\partial_2)\|_1 \\
&\geq \text{const} \sum_{n,m\geq 0, m\geq c(\varphi)n} \frac{m}{(1+(m+\frac{1}{2})^2+(n+\frac{1}{2})^2)^{3/2}}.
\end{aligned}$$

Since the series on the right-hand side diverges, we infer that the series

$$\sum_{n,m\geq 0, m\geq c(\varphi)n} \|\psi_{n,m}(\partial_1,\partial_2)K\psi_{n,m}(\partial_1,\partial_2)\|_1$$

is also divergent.

The obtained contradiction implies that the operator $\big(M_{\varphi_1}\partial_2 - M_{\varphi_2}\partial_1\big)\frac{1}{(1-\Delta)^{3/2}}$ is not trace-class, which suffices to conclude the proof. $\square$

4.3.3. The proof of the main result. Now, having proven Theorem 4.3.10, we prove our main result of this section, Theorem 4.3.14, for "smoother" functions f. Firstly, we work with the square g^2 of the auxiliary function g (see (1.1.2)).

LEMMA 4.3.11. *Suppose $\varphi \in S(\mathbb{R}^2)$. Then*

$$g^2(\mathcal{D} + 1 \otimes M_\varphi) - g^2(\mathcal{D}) \in \mathcal{L}_1(\mathcal{H}).$$

PROOF. First, observe that $g^2(t) = t^2(1+t^2)^{-1} = 1-(1+t^2)^{-1}$, so by applying the second resolvent identity, we see that

$$\begin{aligned} g^2(\mathcal{D}+1 \otimes M_\varphi) - g^2(\mathcal{D}) &= (1+\mathcal{D}^2)^{-1} - \big(1 + (\mathcal{D} + 1 \otimes M_\varphi)^2\big)^{-1} \\ &= (1+\mathcal{D}^2)^{-1}\big((\mathcal{D} + 1 \otimes M_\varphi)^2 - \mathcal{D}^2\big)\big(1 + (\mathcal{D} + (1 \otimes M_\varphi))^2\big)^{-1} \\ &= (1+\mathcal{D}^2)^{-1}\Big(\mathcal{D}(1 \otimes M_\varphi) + (1 \otimes M_\varphi)\mathcal{D} + (1 \otimes M_\varphi)^2\Big)(1+\mathcal{D}^2)^{-1} \\ &\quad \times \frac{1+\mathcal{D}^2}{1+(\mathcal{D}+(1 \otimes M_\varphi))^2}. \end{aligned}$$

Since $(1 \otimes M_\varphi)$ is bounded, [**34**, Lemma B.6] implies that $(1+\mathcal{D}^2)\big(1+(\mathcal{D}+(1 \otimes M_\varphi))^2\big)^{-1}$ is bounded, and therefore it suffices to show that

$$\begin{aligned} (1+\mathcal{D}^2)^{-1}\mathcal{D}(1 \otimes M_\varphi)(1+\mathcal{D}^2)^{-1}, \quad (1+\mathcal{D}^2)^{-1}(1 \otimes M_\varphi)\mathcal{D}(1+\mathcal{D}^2)^{-1}, \\ (1+\mathcal{D}^2)^{-1}(1 \otimes M_\varphi)^2(1+\mathcal{D}^2)^{-1} \in \mathcal{L}_1(\mathcal{H}). \end{aligned}$$

We show this only for the first operator, since the others can be treated similarly. By the noncommutative Hölder inequality (1.1.1) and Corollary 4.3.2, we have that

$$\begin{aligned} &\big\|(1+\mathcal{D}^2)^{-1}\mathcal{D}(1 \otimes M_\varphi)(1+\mathcal{D}^2)^{-1}\big\|_1 \\ &\qquad \leq \big\|\mathcal{D}(1+\mathcal{D}^2)^{-1/2}\big\|_\infty \big\|(1+\mathcal{D}^2)^{-1/2}(1 \otimes M_{|\varphi|^{1/2}})\big\|_3 \\ &\qquad\quad \times \big\||(1 \otimes M_{|\varphi|^{1/2}})(1+\mathcal{D}^2)^{-1}\big\|_{3/2} < \infty. \end{aligned}$$

□

LEMMA 4.3.12. *Suppose $\varphi \in S(\mathbb{R}^2)$. If $f_0 \in C_b^2(\mathbb{R})$ is an even function, then*

$$f_0(\mathcal{D} + 1 \otimes M_\varphi) - f_0(\mathcal{D}) \in \mathcal{L}_1(\mathcal{H}).$$

PROOF. Suppose $f_0 \in C_b^2(\mathbb{R})$ is even. Since g^2 is an even function, and since $g^2 : [0,\infty) \to [0,1)$ is injective, we may write $f_0 = h \circ g^2$, where $h = f_0 \circ g^{-2} : [0,1] \to \mathbb{R}$ is a C^2-function. Hence, by Theorem 2.1.6 we have that $T_{h^{[1]}}^{g^2(\mathcal{D}+1\otimes M_\varphi), g^2(\mathcal{D})} \in \mathcal{B}\big(\mathcal{L}_1(\mathcal{H})\big)$. Therefore, by Lemma 4.3.11 we conclude that

$$\begin{aligned} f_0(\mathcal{D} + 1 \otimes M_\varphi) - f_0(\mathcal{D}) &= h\big(g^2(\mathcal{D} + 1 \otimes M_\varphi)\big) - h\big(g^2(\mathcal{D})\big) \\ &= T_{h^{[1]}}^{g^2(\mathcal{D}+1\otimes M_\varphi), g^2(\mathcal{D})}\big(g^2(\mathcal{D} + 1 \otimes M_\varphi) - g^2(\mathcal{D})\big) \in \mathcal{L}_1(\mathcal{H}), \end{aligned}$$

as required. □

LEMMA 4.3.13. *Suppose that $\varphi \in S(\mathbb{R}^2)$. Then*

$$\big(g(\mathcal{D} + 1 \otimes M_\varphi) - g(\mathcal{D})\big)(1+\mathcal{D}^2)^{-1} \in \mathcal{L}_1(\mathcal{H}).$$

PROOF. By Theorem 4.3.1, $M_{\partial_j\varphi}(1+\mathcal{D}^2)^{-2} \in \mathcal{L}_1(\mathcal{H})$. Hence, by Corollary 4.3.9, we have

$$\begin{aligned}&\big(g(\mathcal{D}+1\otimes M_\varphi)-g(\mathcal{D})\big)(1+\mathcal{D}^2)^{-1}\\&\quad\in \frac{1}{2}\big(\gamma_2\gamma_1\otimes(M_{\partial_2\varphi}\partial_1 - M_{\partial_1\varphi}\partial_2)\big)(1+\mathcal{D}^2)^{-5/2}+\mathcal{L}_1(\mathcal{H}) = \mathcal{L}_1(\mathcal{H}).\end{aligned}$$

□

Now, in the following, let f be a real-valued function on $\mathbb{R}$ such that $0 < f' \in S(\mathbb{R})$. We denote limits at infinity by $f(+\infty) := \lim_{t\to\infty} f(t)$ and $f(-\infty) := \lim_{t\to-\infty} f(t)$. Since $f(+\infty) \neq f(-\infty)$ we can define the functions $f_0, f_1, f_{0,m}, f_{1,m}$ by setting

$$f_0(t) := \frac{f(t)+f(-t)}{2}, \qquad f_1(t) := \frac{f(t)-f(-t)}{2},$$

$$f_{0,m}(t) := f_0(t) - f_0(+\infty), \qquad f_{1,m}(t) := \begin{cases} \dfrac{f_1(t)}{g(t)f_1(+\infty)} - 1, & \text{if } t \neq 0,\\ -1, & \text{if } t = 0.\end{cases}$$

One can check that $f_{0,m} \in S(\mathbb{R})$, $f_{1,m} \in C_b^2(\mathbb{R})$. We are now ready to prove the second main result of the present section.

THEOREM 4.3.14. *Suppose $\varphi \in S(\mathbb{R}^2)$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that $0 < f' \in S(\mathbb{R})$. Then*

$$f(\mathcal{D}+1\otimes M_\varphi) - f(\mathcal{D}) \notin \mathcal{L}_1(\mathcal{H}).$$

PROOF. Firstly, for $t \in \mathbb{R}$, observe that we may write

$$\begin{aligned}f(t) = f_0(t) + f_1(t) &= f_0(+\infty) + \big(f_0(t) - f_0(+\infty)\big) + g(t)\Big(\frac{f_1(t)}{g(t)f_1(+\infty)}\Big)f_1(+\infty)\\ &= f_1(+\infty) + f_{0,m}(t) + g(t)\big(1+f_{1,m}(t)\big)f_1(+\infty),\end{aligned}$$

Now, since $f_{0,m} \in C_b^2(\mathbb{R})$ is even, Lemma 4.3.12 implies that

$$f_{0,m}(\mathcal{D}+1\otimes M_\varphi) - f_{0,m}(\mathcal{D}) \in \mathcal{L}_1(\mathcal{H}).$$

Furthermore,

$$\begin{aligned}&g(\mathcal{D}+1\otimes M_\varphi)\big(1+f_{1,m}(\mathcal{D}+1\otimes M_\varphi)\big) - g(\mathcal{D})\big(1+f_{1,m}(\mathcal{D})\big)\\&\quad= g(\mathcal{D}+1\otimes M_\varphi)\Big[f_{1,m}(\mathcal{D}+1\otimes M_\varphi) - f_{1,m}(\mathcal{D})\Big]\\&\qquad+\Big[g(\mathcal{D}+1\otimes M_\varphi) - g(\mathcal{D})\Big]\big(1+f_{1,m}(\mathcal{D})\big),\end{aligned}$$

and, since $f_{1,m} \in C_b^2(\mathbb{R})$ and is even, again using Lemma 4.3.12, we have that

$$g(\mathcal{D}+1\otimes M_\varphi)\Big[f_{1,m}(\mathcal{D}+1\otimes M_\varphi) - f_{1,m}(\mathcal{D})\Big] \in \mathcal{L}_1(\mathcal{H}).$$

Hence,

$$\begin{aligned}&f(\mathcal{D}+1\otimes M_\varphi) - f(\mathcal{D})\\&\quad\in f_1(+\infty)\Big[g(\mathcal{D}+1\otimes M_\varphi) - g(\mathcal{D})\Big]\big(1+f_{1,m}(\mathcal{D})\big) + \mathcal{L}_1(\mathcal{H})\\&\quad= f_1(+\infty)\big[g(\mathcal{D}+1\otimes M_\varphi) - g(\mathcal{D})\big]\\&\qquad+ f_1(+\infty)\big[g(\mathcal{D}+1\otimes M_\varphi) - g(\mathcal{D})\big]\cdot f_{1,m}(\mathcal{D}) + \mathcal{L}_1(\mathcal{H})\end{aligned} \tag{4.3.31}$$

For the second term on the right-hand side of (4.3.31) we note that since $\theta_f(t) = f_{1,m}(t)(1+t^2)$ is bounded, Lemma 4.3.13, implies that

$$\begin{aligned}&\big[g(\mathcal{D}+1\otimes M_\varphi) - g(\mathcal{D})\big] f_{1,m}(\mathcal{D})\\ &\qquad = \big[g(\mathcal{D}+1\otimes M_\varphi) - g(\mathcal{D})\big](1+\mathcal{D}^2)^{-1}\cdot\theta_f(\mathcal{D}) \in \mathcal{L}_1(\mathcal{H}).\end{aligned}$$

Thus, we conclude that

$$f(\mathcal{D}+1\otimes M_\varphi) - f(\mathcal{D}) \in f_1(+\infty)\big[g(\mathcal{D}+1\otimes M_\varphi) - g(\mathcal{D})\big] + \mathcal{L}_1(\mathcal{H}).$$

Finally, by Theorem 4.3.10, we have that $g(\mathcal{D}+1\otimes M_\varphi) - g(\mathcal{D}) \notin \mathcal{L}_1(\mathcal{H})$, so

$$f(\mathcal{D}+1\otimes M_\varphi) - f(\mathcal{D}) \notin \mathcal{L}_1(\mathcal{H}).$$

$\square$

CHAPTER 5

The principal trace formula

In this chapter we prove the fundamental result of the present thesis, the principal trace formula, which states that

$$\operatorname{tr}\Big(e^{-t\boldsymbol{H}_2} - e^{-t\boldsymbol{H}_1}\Big) = -\Big(\frac{t}{\pi}\Big)^{1/2} \int_0^1 \operatorname{tr}\Big(e^{-tA_s^2}(A_+ - A_-)\Big) ds, \quad t > 0, \qquad (5.0.1)$$

where $A_s = A_- + s(A_+ - A_-), s \in [0, 1]$ is the straight line path joining A_- and A_+.

As mentioned in Section 1.6, this principal trace formula is the most important result, which allows us to establish the further results for the Witten index (see Chapter 6 below). One can say that the results of Chapter 6 are (almost immediate) corollaries of this principal trace formula.

In the first section of the present chapter we establish the principal trace formula in its heat kernel version.

To demonstrate that our technique is applicable for many other versions of trace formulas of the type (5.0.1), we we also prove version of the principal trace formula in its resolvent difference form. In this case, the formula is

$$\operatorname{tr}\Big((\boldsymbol{H}_2-z)^{-m}-(\boldsymbol{H}_1-z)^{-m}\Big) = -\frac{(2m-1)!!}{2^m(m-1)!}\int_0^1 \operatorname{tr}\big((A_s^2-z)^{-\frac{1}{2}-m}(A_+-A_-)\big)ds,$$

where $z < 0$ and $A_s = A_- + s(A_+ - A_-), s \in [0, 1]$, as before. The latter result provides an alternative proof of [**32**, Theorem 1.1].

The results of this chapter are presented in [**28**].

5.1. The heat kernel version of the PTF

As we already mentioned in Section 1.6, our approach in the proof of the principal trace formula relies on approximation of results already known for the path $\{A_n(t)\}_{t\in\mathbb{R}}$ of reduced operators

$$A_n(t) = A_- + P_nB(t)P_n,$$

where, as before, $P_n = \chi_{[-n,n]}(A_-)$. Hence, we firstly recall the result from [**31**], which is used here. The notation erf stands for the error function

$$\operatorname{erf}(x) = \frac{2}{\pi^{1/2}}\int_0^x e^{-y^2}\,dy, \quad x \in \mathbb{R}. \qquad (5.1.1)$$

Proposition 5.1.1. [**31**, Example B.6 (ii) and Theorem B.5] *For the the path $\{A_n(t)\}_{t\in\mathbb{R}}$ of reduced operators we have that*

$$e^{-t\boldsymbol{H}_{2,n}} - e^{-t\boldsymbol{H}_{1,n}} \in \mathcal{L}_1(L_2(\mathbb{R},\mathcal{H})), \quad \operatorname{erf}(t^{1/2}A_{+,n}) - \operatorname{erf}(t^{1/2}A_-) \in \mathcal{L}_1(\mathcal{H})$$

and the equation

$$\operatorname{tr}\Big(e^{-t\boldsymbol{H}_{2,n}} - e^{-t\boldsymbol{H}_{1,n}}\Big) = -\frac{1}{2}\operatorname{tr}\Big(\operatorname{erf}(t^{1/2}A_{+,n}) - \operatorname{erf}(t^{1/2}A_-)\Big), \qquad (5.1.2)$$

holds.

PROOF. By Proposition 3.2.3 the family $\{A_n(t)\}$ satisfies the Pushnitski assumptions (see Hypothesis 1.5.1), and hence, obviously, it satisfies Hypothesis 1.5.2, which guarantees that the results of [**31**] are applicable.

Take the Schwartz function

$$f(\lambda) = e^{-t\lambda}, \quad \lambda \in [0,\infty),\ t \in (0,\infty).$$

Then for the function F on $\mathbb{R}$, defined by (cf. [**31**, (B.57)])

$$F(\nu) = \frac{\nu}{2\pi} \int_{[\nu^2,\infty)} \lambda^{-1}(\lambda-\nu^2)^{-1/2}[f(\lambda)-f(0)]d\lambda,$$

we have [**31**, (B.76)]

$$F(\nu) = -\frac{1}{2}\operatorname{erf}\left(t^{1/2}\nu\right).$$

Hence, by [**31**, Theorem B.5] we obtain that

$$\begin{aligned}\operatorname{tr}\left(e^{-t\boldsymbol{H}_{2,n}} - e^{-t\boldsymbol{H}_{1,n}}\right) &= \operatorname{tr}\left(f(\boldsymbol{H}_{2,n}) - f(\boldsymbol{H}_{1,n})\right)\\ &= \operatorname{tr}(F(A_{+,n}) - F(A_-))\\ &= -\frac{1}{2}\operatorname{tr}\Big(\operatorname{erf}(t^{1/2}A_{+,n}) - \operatorname{erf}(t^{1/2}A_-)\Big),\end{aligned}$$

as required. □

Next, applying the (noncommutative) Fundamental Theorem of Calculus obtained in Proposition 2.3.2 to the right-hand side of (5.1.2), we aim to rewrite the principal trace formula obtained in Proposition 5.1.1 for the reduced operators.

LEMMA 5.1.2. *For the path $\{A_n(t)\}_{t\in\mathbb{R}}$ of reduced operators we have*

$$\operatorname{tr}\left(e^{-t\boldsymbol{H}_{2,n}} - e^{-t\boldsymbol{H}_{1,n}}\right) = -\left(\frac{t}{\pi}\right)^{1/2}\int_0^1 \operatorname{tr}\left(e^{-tA_{s,n}^2}(A_{+,n}-A_-)\right)ds, \tag{5.1.3}$$

where $A_{s,n} = A_- + sP_nB_+P_n$, $s \in [0,1]$.

PROOF. By Proposition 5.1.1 we have

$$\operatorname{tr}\left(e^{-t\boldsymbol{H}_{2,n}} - e^{-t\boldsymbol{H}_{1,n}}\right) = -\frac{1}{2}\operatorname{tr}\Big(\operatorname{erf}(t^{1/2}A_{+,n}) - \operatorname{erf}(t^{1/2}A_-)\Big).$$

Since the operator $B_{+,n} = A_{+,n} - A_-$ is a trace-class operator (see (3.2.7)), it follows that the path $B_{s,n} = s(A_{+,n} - A_-)$ is a C^1-path of trace-class operators. Applying now Proposition 2.3.2 for this path (with $f = \operatorname{erf}$, which clearly satisfies the assumption of this proposition since f' is a Schwartz function) we obtain that

$$\frac{1}{2}\operatorname{tr}\Big(\operatorname{erf}(t^{1/2}A_{+,n}) - \operatorname{erf}(t^{1/2}A_-)\Big) = \left(\frac{t}{\pi}\right)^{1/2}\int_0^1 \operatorname{tr}\left(e^{-tA_{s,n}^2}(A_{+,n}-A_-)\right)ds. \tag{5.1.4}$$

Hence,

$$\operatorname{tr}\left(e^{-t\boldsymbol{H}_{2,n}} - e^{-t\boldsymbol{H}_{1,n}}\right) = -\left(\frac{t}{\pi}\right)^{1/2}\int_0^1 \operatorname{tr}\left(e^{-tA_{s,n}^2}(A_{+,n}-A_-)\right)ds,$$

as required. □

Thus, by Lemma 5.1.2 it is sufficient to pass to the limit as $n \to \infty$ to prove the desired principal trace formula (5.0.1). We firstly prove the integral of the right-hand side of (5.0.1) is well-defined.

PROPOSITION 5.1.3. *Assume Hypothesis 3.5.1. The function*

$$s \mapsto \operatorname{tr}\Big(e^{-tA_s^2}(A_+ - A_-)\Big), \quad s \in [0,1]$$

is continuous, and hence, the integral

$$\int_0^1 \operatorname{tr}\Big(e^{-tA_s^2}(A_+ - A_-)\Big) ds$$

is well-defined.

PROOF. Firstly we show that the operator $e^{-tA_s^2}(A_+ - A_-)$ is a trace class operator for any fixed $s \in [0,1]$. Since the operator

$$(A_s + i)^{p+1} e^{-tA_s^2}$$

is bounded, it is sufficient to show that

$$(A_s + i)^{-p-1}(A_+ - A_-) = (A_s + i)^{-p-1} B_+ \in \mathcal{L}_1(\mathcal{H}).$$

We can write

$$(A_s + i)^{-p-1} B_+ = \Big((A_s + i)^{-p-1} - (A_- + i)^{-p-1}\Big) B_+ + (A_- + i)^{-p-1} B_+.$$

By (3.1.7) we have that the second term on the right-hand side is a trace-class operator. On the other hand, Theorem 3.3.2 implies the operator $(A_s + i)^{-p-1} - (A_- + i)^{-p-1}$ is a trace-class operator. Hence, $(A_s + i)^{-p-1} B_+ \in \mathcal{L}_1(\mathcal{H})$ for any $s \in [0,1]$.

Now, let $s_1, s_2 \in [0,1]$. By Proposition 2.2.10 we have

$$\begin{aligned} e^{-tA_{s_1}^2} B_+ - e^{-tA_{s_2}^2} B_+ &= (e^{-tA_{s_1}^2} - e^{-tA_{s_2}^2}) B_+ \\ &= \sum_{j=1,2} T_{f,a_j}^{A_{s_1}, A_{s_2}} \Big((A_{s_1} - a_j i)^{-p-1} - (A_{s_2} - a_j i)^{-p-1}\Big) \cdot B_+, \end{aligned} \tag{5.1.5}$$

where $f(x) = e^{-tx^2}, x \in \mathbb{R}, t > 0$.

By Remark 3.3.3 we have that

$$\Big\|(A_{s_1} - a_j i)^{-p-1} - (A_{s_2} - a_j i)^{-p-1}\Big\|_1 \to 0, \quad \text{as } s_1 - s_2 \to 0.$$

Furthermore, by Theorem 2.3.8 the double operator integral $T_{f,a_j}^{A_{s_1}, A_{s_2}}$, $j = 1, 2$, converges pointwise on $\mathcal{L}_1(\mathcal{H})$ to $T_{f,a_j}^{A_{s_1}, A_{s_1}}$, as $s_2 \to s_1$. Therefore,

$$\begin{aligned} \|\cdot\|_1 - \lim_{s_2 \to s_1} T_{f,a_j}^{A_{s_1}, A_{s_2}} &\Big((A_{s_1} - a_j i)^{-p-1} - (A_{s_2} - a_j i)^{-p-1}\Big) \\ &= T_{f,a_j}^{A_{s_1}, A_{s_1}}(0) = 0. \quad j = 1, 2. \end{aligned}$$

Thus, equality (5.1.5) implies that

$$\|e^{-tA_{s_1}^2} B_+ - e^{-tA_{s_2}^2} B_+\|_1 \to 0, \quad s_1 - s_2 \to 0,$$

as required. □

We now ready to prove the principal trace formula in its heat kernel version, which is the main result of this chapter.

THEOREM 5.1.4 (The principal trace formula). *Assume Hypothesis 3.5.1. Let* $A_s = A_- + s(A_+ - A_-), s \in [0,1]$, *be the straight line path joining* A_- *and* A_+. *Then for all* $t > 0$, *we have*

$$\operatorname{tr}\Big(e^{-t\boldsymbol{H}_2} - e^{-t\boldsymbol{H}_1}\Big) = -\Big(\frac{t}{\pi}\Big)^{1/2} \int_0^1 \operatorname{tr}\Big(e^{-tA_s^2}(A_+ - A_-)\Big)ds.$$

PROOF. By Lemma 5.1.2 we have

$$\operatorname{tr}\Big(e^{-t\boldsymbol{H}_{2,n}} - e^{-t\boldsymbol{H}_{1,n}}\Big) = -\Big(\frac{t}{\pi}\Big)^{1/2} \int_0^1 \operatorname{tr}\Big(e^{-tA_{s,n}^2}(A_{+,n} - A_-)\Big)ds. \tag{5.1.6}$$

We now pass to the limit as $n \to \infty$.

For the left hand side of (5.1.6) we firstly note that since the function $f(\lambda) = e^{-t\lambda^2}$, $t > 0$, is a Schwartz function, it follows from (2.2.34) that it belongs to the class $\mathfrak{F}_m$ (see Definition 2.2.8). Therefore, Theorem 3.6.1 (ii) and Theorem 2.3.9 (with $A_n = \boldsymbol{H}_{2,n}$, $A = \boldsymbol{H}_2$, $B_n = \boldsymbol{H}_{1,n}$, $B = \boldsymbol{H}_1$) imply that

$$\lim_{n\to\infty} \operatorname{tr}\Big(e^{-t\boldsymbol{H}_{2,n}} - e^{-t\boldsymbol{H}_{1,n}}\Big) = \operatorname{tr}\Big(e^{-t\boldsymbol{H}_2} - e^{-t\boldsymbol{H}_1}\Big). \tag{5.1.7}$$

For the right hand side of (5.1.6), we firstly write

$$e^{-tA_{s,n}^2}(A_{+,n} - A_-) = (A_{s,n} + i)^{p+1} e^{-tA_{s,n}^2} \cdot (A_{s,n} + i)^{-p-1} B_{+,n}.$$

Since $A_{s,n} \to A_s$ in the strong resolvent sense (see Lemma 3.2.5) and the function $x \mapsto e^{-tx^2}(x+i)^{p+1}$ is continuous and bounded, [**69**, Theorem VIII.23] implies that $(A_{s,n} + i)^{p+1} e^{-tA_{s,n}^2} \to (A_s + i)^{p+1} e^{-tA_s^2}$ strongly. Hence, by Lemma 3.2.1, the convergence

$$\|\cdot\|_1 - \lim_{n\to\infty} e^{-tA_{s,n}^2}(A_{+,n} - A_-) = e^{-tA_s^2}(A_+ - A_-) \tag{5.1.8}$$

will follow from the convergence

$$\|\cdot\|_1 - \lim_{n\to\infty} (A_{s,n} + i)^{-p-1} B_{+,n} = (A_s + i)^{-p-1} B_+. \tag{5.1.9}$$

To prove convergence (5.1.9) we write

$$\begin{aligned}
&(A_{s,n} + i)^{-p-1} B_{+,n} \\
&\quad= \Big((A_{s,n} + i)^{-p-1} - (A_- + i)^{-p-1}\Big) B_{+,n} + (A_- + i)^{-p-1} B_{+,n} \\
&\quad= \Big((A_{s,n} + i)^{-p-1} - (A_- + i)^{-p-1}\Big) B_{+,n} + P_n (A_- + i)^{-p-1} B_+ P_n.
\end{aligned}$$

Theorem 3.3.2 (see also Remark 3.3.3) implies that $\Big((A_{s,n}+i)^{-p-1} - (A_-+i)^{-p-1}\Big)$ converges to $\Big((A_s + i)^{-p-1} - (A_- + i)^{-p-1}\Big)$ in $\mathcal{L}_1(\mathcal{H})$. Moreover, the assumption that $(A_- + i)^{-p-1} B_+ \in \mathcal{L}_1(\mathcal{H})$ (see (3.1.7)), the strong convergence $P_n \to 1$ combined with Lemma 3.2.1 imply that

$$P_n (A_- + i)^{-p-1} B_+ P_n \to (A_- + i)^{-p-1} B_+$$

in $\mathcal{L}_1(\mathcal{H})$. Hence,

$$\begin{aligned}
&\|\cdot\|_1 - \lim_{n\to\infty} (A_{s,n} + i)^{-p-1} B_{+,n} \\
&\qquad= \Big((A_s + i)^{-p-1} - (A_- + i)^{-p-1}\Big) B_+ + (A_- + i)^{-p-1} B_+ \\
&\qquad= (A_s + i)^{-p-1} B_+
\end{aligned}$$

for every fixed $s \in [0, 1]$, which suffices to prove (5.1.8).

By Corollary 3.3.4 we have that the sequence of functions

$$\{s \mapsto \|e^{-tA_{s,n}^2}(A_{+,n} - A_-)\|_1\}_{n\in\mathbb{N}} \tag{5.1.10}$$

is uniformly bounded with respect to $n \in \mathbb{N}$. Hence, using (5.1.8) with (5.1.10) and employing the dominated convergence theorem we infer that

$$\lim_{n\to\infty} \int_0^1 \operatorname{tr}\Big(e^{-tA_{s,n}^2}(A_{+,n} - A_-)\Big)ds = \int_0^1 \operatorname{tr}\Big(e^{-tA_s^2}(A_+ - A_-)\Big)ds. \tag{5.1.11}$$

Thus, (5.1.6) and (5.1.7) imply that

$$\operatorname{tr}\Big(e^{-t\boldsymbol{H}_2} - e^{-t\boldsymbol{H}_1}\Big) = -\Big(\frac{t}{\pi}\Big)^{1/2} \int_0^1 \operatorname{tr}\Big(e^{-tA_s^2}(A_+ - A_-)\Big)ds,$$

which concludes the proof. □

5.2. Resolvent version of the principal trace formula

In this section we show how our technique yields an alternative proof of the resolvent version of the principal trace formula proved in [**32**]. Our argument is based on the approximation technique described previously and differentiation of the original principal trace formula (1.5.1) from [**67, 47**] with respect to the parameter z. We firstly supply the necessary technical details of our differentiation procedure.

Recall that, the reduced family $\{B_n(t)\}_{t\in\mathbb{R}}, n \in \mathbb{N}$, satisfies the Pushnitski's assumption (see Proposition 3.2.3). Therefore, by Theorem 1.5.3 for the operators $\boldsymbol{H}_{j,n}$, $j = 1, 2$, $n \in \mathbb{N}$ we have that

$$(\boldsymbol{H}_{2,n} - z)^{-1} - (\boldsymbol{H}_{1,n} - z)^{-1} \in \mathcal{L}_1(L_2(\mathbb{R}, \mathcal{H})). \tag{5.2.1}$$

In particular, the equality

$$\begin{aligned}(\boldsymbol{H}_{2,n} - z)^{-k} &- (\boldsymbol{H}_{1,n} - z)^{-k} \\ &= \sum_{j=1}^{k} (\boldsymbol{H}_{2,n} - z)^{j-k}\Big((\boldsymbol{H}_{2,n} - z)^{-1} - (\boldsymbol{H}_{1,n} - z)^{-1}\Big)(\boldsymbol{H}_{1,n} - z)^{1-j},\end{aligned}$$

implies that

$$(\boldsymbol{H}_{2,n} - z)^{-k} - (\boldsymbol{H}_{1,n} - z)^{-k} \in \mathcal{L}_1(L_2(\mathbb{R}, \mathcal{H})). \tag{5.2.2}$$

for any $k \in \mathbb{N}$.

Moreover, inclusion (5.2.1) combined with Theorem 1.2.3 guarantees that there exists spectral shift function $\xi(\cdot; \boldsymbol{H}_{2,n}, \boldsymbol{H}_{1,n})$ for the pair $(\boldsymbol{H}_{2,n}, \boldsymbol{H}_{1,n})$, satisfying

$$\xi(\cdot, \boldsymbol{H}_{2,n}, \boldsymbol{H}_{1,n}) \in L_1(\mathbb{R}; (\lambda^2 + 1)^{-1}d\lambda). \tag{5.2.3}$$

The following lemma gives the well-known formula for derivative (with respect to z) of (5.2.1). The proof, which is based on the Krein trace formula (1.2.5), is supplied for completeness. We note, that the formula below can be proved using simple algebraic tools, however Krein trace formula provides a shorter proof.

Lemma 5.2.1. *Let $m \in \mathbb{N}$. We have*

$$\frac{d^m}{dz^m}\operatorname{tr}\Big((\boldsymbol{H}_{2,n} - z)^{-1} - (\boldsymbol{H}_{1,n} - z)^{-1}\Big) = m!\operatorname{tr}\Big((\boldsymbol{H}_{2,n} - z)^{-m-1} - (\boldsymbol{H}_{1,n} - z)^{-m-1}\Big)$$

for all $z \in \mathbb{C} \setminus [0, \infty)$.

Proof. Throughout this proof we fix $z \in \mathbb{C} \setminus [0, \infty)$ and choose $\Delta z \in \mathbb{C}$ satisfying $|\Delta z| < \varepsilon$ with $\varepsilon > 0$ sufficiently small such that $z + \Delta z \in \mathbb{C} \setminus [0, \infty)$.

It is sufficient to prove, that

$$\frac{d}{dz} \operatorname{tr}\Big((\boldsymbol{H}_{2,n} - z)^{-m} - (\boldsymbol{H}_{1,n} - z)^{-m}\Big) = m \operatorname{tr}\Big((\boldsymbol{H}_{2,n} - z)^{-m-1} - (\boldsymbol{H}_{1,n} - z)^{-m-1}\Big).$$

By Krein's trace formula we have

$$\begin{aligned}
\frac{1}{\Delta z}\Big[&\operatorname{tr}\big((\boldsymbol{H}_{2,n} - z - \Delta z)^{-m} - (\boldsymbol{H}_{1,n} - z - \Delta z)^{-m}\big) \\
&\qquad - \operatorname{tr}\big((\boldsymbol{H}_{2,n} - z)^{-m} - (\boldsymbol{H}_{1,n} - z)^{-m}\big)\Big] \\
&= -m\frac{1}{\Delta z}\int_0^\infty \xi(\lambda, \boldsymbol{H}_{2,n}, \boldsymbol{H}_{1,n})(\lambda - z - \Delta z)^{-m-1} d\lambda \qquad (5.2.4)\\
&\qquad + m\frac{1}{\Delta z}\int_0^\infty \xi(\lambda, \boldsymbol{H}_{2,n}, \boldsymbol{H}_{1,n})(\lambda - z)^{-m-1} d\lambda \\
&= -m\int_0^\infty \xi(\lambda, \boldsymbol{H}_{2,n}, \boldsymbol{H}_{1,n})\Big(\frac{1}{\Delta z}\big((\lambda - z - \Delta z)^{-m-1} - (\lambda - z)^{-m-1}\big)\Big) d\lambda.
\end{aligned}$$

We have that

$$\lim_{\Delta z \to 0} \frac{1}{\Delta z}\big((\lambda - z - \Delta z)^{-m-1} - (\lambda - z)^{-m-1}\big) = (m+1)(\lambda - z)^{-m-2}. \qquad (5.2.5)$$

Furthermore,

$$\begin{aligned}
&\Big|\frac{1}{\Delta z}\big((\lambda - z - \Delta z)^{-m-1} - (\lambda - z)^{-m-1}\big)\Big| \\
&\qquad = \Big|\frac{1}{(\lambda - z - \Delta z)(\lambda - z)}\sum_{j=0}^m (\lambda - z - \Delta z)^{-m+j}(\lambda - z)^{-j}\Big| \\
&\qquad \le \text{const}\,|(\lambda - z)^{-2}|
\end{aligned}$$

uniformly with respect to Δz. Since the function $\xi(\cdot, \boldsymbol{H}_{2,n}, \boldsymbol{H}_{1,n})(\lambda - z)^{-2}$ is integrable (see (5.2.3)), we obtain that the integrand on the right-hand side of (5.2.4) is uniformly majorised by an integrable function. Therefore, by (5.2.5) and the dominated convergence theorem we obtain that (5.2.4) converges to

$$-m(m+1)\int_0^\infty \xi(\lambda, \boldsymbol{H}_{2,n}, \boldsymbol{H}_{1,n})(\lambda - z)^{-m-2} d\lambda$$

as $|\Delta z| \to 0$. By the Krein trace formula the latter term is equal to

$$m \operatorname{tr}\Big((\boldsymbol{H}_{2,n} - z)^{-m-1} - (\boldsymbol{H}_{1,n} - z)^{-m-1}\Big).$$

Thus,

$$\begin{aligned}
&\frac{d}{dz}\operatorname{tr}\Big((\boldsymbol{H}_{2,n} - z)^{-m} - (\boldsymbol{H}_{1,n} - z)^{-m}\Big) \\
&\qquad = -\frac{d}{dz} m\int_0^\infty \xi(\lambda, \boldsymbol{H}_{2,n}, \boldsymbol{H}_{1,n})(\lambda - z)^{-m-1} d\lambda \\
&\qquad = -m(m+1)\int_0^\infty \xi(\lambda, \boldsymbol{H}_{2,n}, \boldsymbol{H}_{1,n})(\lambda - z)^{-m-2} d\lambda \\
&\qquad = m \operatorname{tr}\Big((\boldsymbol{H}_{2,n} - z)^{-m-1} - (\boldsymbol{H}_{1,n} - z)^{-m-1}\Big),
\end{aligned}$$

as required.

□

LEMMA 5.2.2. *Let $m \in \mathbb{N}$, let D_0 be a self-adjoint operator on $\mathcal{H}$ and let B be self-adjoint trace-class operator, $D_1 = D_0 + B$. For every $z < 0$ we have in trace norm*

$$\frac{d^m}{dz^m}(D_1^2 - z)^{-\frac{3}{2}}B = C_m(D_1^2 - z)^{-\frac{3}{2}-m}B,$$

where $C_m = \frac{(2m+1)!!}{2^m}$.

PROOF. Since, the operator B is trace-class, by Lemma 3.2.1 it is sufficient to show that

$$(so) - \frac{d^m}{dz^m}(A^2 - z)^{-\frac{3}{2}} = C_m(A^2 - z)^{-\frac{3}{2}-m},$$

which in its turn follows from

$$(so) - \frac{d}{dz}(A^2 - z)^{-\frac{3}{2}-m} = (\frac{3}{2} + m)(A^2 - z)^{-\frac{3}{2}-m-1}.$$

By Leibniz's rule we have

$$\begin{aligned}(so) - \frac{d}{dz}(A^2 - z)^{-\frac{3}{2}-m} \qquad\qquad (5.2.6)\\ = \Big((so) - \frac{d}{dz}(A^2 - z)^{-1/2}\Big) \times (A^2 - z)^{-1-m}\\ + (A^2 - z)^{-1/2} \times \Big((so) - \frac{d}{dz}(A^2 - z)^{-1-m}\Big).\end{aligned}$$

The equality $(so) - \frac{d}{dz}(A^2 - z)^{-1-m} = (1 + m)(A^2 - z)^{-2-m}, m \in \mathbb{N} \cup \{0\}$ can be easily proved by induction. Thus, we need only to show that

$$(so) - \frac{d}{dz}(A^2 - z)^{-1/2} = \frac{1}{2}(A^2 - z)^{-\frac{3}{2}}.$$

Let $z, z_0 < 0$ and let $f(t) = (t^2 - z)^{-1/2}$, we have

$$\frac{(A^2 - z)^{-1/2} - (A^2 - z_0)^{-1/2}}{z - z_0} = \frac{1}{z - z_0}T_{f^{[1]}}^{A^2-z,A^2-z_0}(z - z_0) = T_{f^{[1]}}^{A^2-z,A^2-z_0}(I).$$

Since the function $f \in C_b^2[z', \infty)$, where $z' = \min\{-z, -z_0\} > 0$, we have that (see e.g. [**9**, Proposition 4.9 (ii)])

$$T_{f^{[1]}}^{(A^2-z,A^2-z_0)}(1) \to T_{f^{[1]}}^{A^2-z_0,A^2-z_0}(1), \text{ as } z \to z_0$$

in the strong operator topology. Hence, by (2.1.5) we conclude

$$(so) - \frac{d}{dz}(A^2 - z)^{-1/2}\Big|_{z=z_0} = T_{f^{[1]}}^{A^2-z_0,A^2-z_0}(1) = f'(A^2 - z_0) = \frac{1}{2}(A^2 - z_0)^{-\frac{3}{2}}.$$

□

The next theorem gives an alternative proof of [**32**, Theorem 1.1].

THEOREM 5.2.3. *Assume Hypothesis 3.5.1. Let $A_s = A_- + s(A_+ - A_-), s \in [0, 1]$ by the straight line joining A_- and A_+. Then for all $z < 0$, we have that*

$$\operatorname{tr}\Big((\boldsymbol{H}_2 - z)^{-m} - (\boldsymbol{H}_1 - z)^{-m}\Big) = -\frac{(2m-1)!!}{2^m(m-1)!}\int_0^1 \operatorname{tr}\big((A_s^2 - z)^{-1/2-m}(A_+ - A_-)\big)ds.$$

Proof. By Proposition 3.2.3 the family $\{A_n(t)\}$ satisfies the Pushnitski assumption 1.5.1. Hence, by Theorem 1.5.3 we have the principal trace formula (1.5.1) for the reduced operators,

$$\operatorname{tr}\left[(\boldsymbol{H}_{1,n}-z)^{-1}-(\boldsymbol{H}_{2,n}-z)^{-1}\right]=\frac{1}{2z}\operatorname{tr}\left[g_z\left(A_{+,n}\right)-g_z\left(A_-\right)\right],\quad n\in\mathbb{N},$$

where g_z is defined by (1.1.3).

Recall that $B_{s,n}$ is a C^1-path of trace-class operators. In addition, for the function $g_z\in C^\infty(\mathbb{R})$ we have $g_z'(t)=-z(t^2-z)^{-3/2}$, that is $g_z'\in L_1(\mathbb{R})$. Hence, applying Proposition 2.3.2 we infer

$$\frac{1}{2z}\operatorname{tr}\left[g_z\left(A_{+,n}\right)-g_z\left(A_-\right)\right]=-\frac{1}{2}\int_0^1\operatorname{tr}\Big((A_{s,n}^2-z)^{-\frac{3}{2}}(A_{+,n}-A_-)\Big)ds.$$

Thus, for every $n\in\mathbb{N}$ we have

$$\begin{aligned}&\operatorname{tr}\left[(\boldsymbol{H}_{1,n}-z)^{-1}-(\boldsymbol{H}_{2,n}-z)^{-1}\right]\\&=-\frac{1}{2}\int_0^1\operatorname{tr}\Big((A_{s,n}^2-z)^{-\frac{3}{2}}(A_{+,n}-A_-)\Big)ds,\quad z\in\mathbb{C}\setminus[0,\infty).\end{aligned}\tag{5.2.7}$$

On the left hand side of (5.2.7), by Lemma 5.2.1, we have that

$$\begin{aligned}&\frac{d^{m-1}}{dz^{m-1}}\operatorname{tr}\left[(\boldsymbol{H}_{1,n}-z)^{-1}-(\boldsymbol{H}_{2,n}-z)^{-1}\right]\\&\qquad=(m-1)!\operatorname{tr}\left[(\boldsymbol{H}_{1,n}-z)^{-m}-(\boldsymbol{H}_{2,n}-z)^{-m}\right].\end{aligned}$$

On the other hand by Lemma 5.2.2 for the right-hand side of (5.2.7) we have

$$\begin{aligned}&\frac{d^{m-1}}{dz^{m-1}}\int_0^1\operatorname{tr}\Big((A_{s,n}^2-z)^{-\frac{3}{2}}(A_{+,n}-A_-)\Big)ds\\&=\int_0^1\frac{d^{m-1}}{dz^{m-1}}\operatorname{tr}\Big((A_{s,n}^2-z)^{-\frac{3}{2}}(A_{+,n}-A_-)\Big)ds\\&=\frac{(2m-1)!!}{2^{m-1}}\int_0^1\operatorname{tr}\Big((A_{s,n}^2-z)^{-m-1/2}(A_{+,n}-A_-)\Big)ds.\end{aligned}$$

Therefore, for every $n\in\mathbb{N}$

$$\begin{aligned}&\operatorname{tr}\left[(\boldsymbol{H}_{1,n}-z)^{-m}-(\boldsymbol{H}_{2,n}-z)^{-m}\right]\\&=-\frac{(2m-1)!!}{(m-1)!2^m}\int_0^1\operatorname{tr}\Big((A_{s,n}^2-z)^{-m-1/2}(A_{+,n}-A_-)\Big)ds.\end{aligned}\tag{5.2.8}$$

By Theorem 3.6.1 the left hand side converges to $\operatorname{tr}\Big((\boldsymbol{H}_2-z)^{-m}-(\boldsymbol{H}_1-z)^{-m}\Big)$ as $n\to\infty$. We claim that the right-hand side converges to $\frac{(2m-1)!!}{(m-1)!2^m}\int_0^1\operatorname{tr}\big((A_s^2-z)^{-1/2-m}(A_+-A_-)\big)ds$. By Corollary 3.3.4 we have that the functions

$$[0,1]\ni s\mapsto\left\|(A_{s,n}^2-z)^{-m-1/2}(A_{+,n}-A_-)\right\|_1\tag{5.2.9}$$

are uniformly bounded with respect to $n\in\mathbb{N}$.

It follows from (5.1.9) that

$$\|\cdot\|_1-\lim_{n\to\infty}(A_{s,n}+i)^{-2m-1}B_{+,n}=(A_s+i)^{-2m-1}B_+.$$

In addition, the function $x \mapsto (x^2 - z)^{-m-1/2}(x+i)^{2m+1}$, $x \in \mathbb{R}$, is continuous and bounded, and therefore [**69**, Theorem VIII.23] implies that $(A_{s,n}+i)^{p+1}e^{-tA_{s,n}^2} \to (A_s+i)^{p+1}e^{-tA_s^2}$ strongly. Hence, by Lemma 3.2.1, the convergence

$$\|\cdot\|_1 - \lim_{n\to\infty} (A_{s,n}^2 - z)^{-m-1/2}(A_{+,n} - A_{-,n}) = (A_s^2 - z)^{-m-1/2}(A_+ - A_-) \quad (5.2.10)$$

holds. Combining this fact with (5.2.9), by the dominated convergence theorem we infer that the sequence

$$\frac{(2m-1)!!}{(m-1)!2^m}\int_0^1 \operatorname{tr}\Big((A_{s,n}^2 - z)^{-m-1/2}(A_{+,n} - A_{-,n})\Big)ds$$

converges to

$$\frac{(2m-1)!!}{(m-1)!2^m}\int_0^1 \operatorname{tr}\big((A_s^2 - z)^{-1/2-m}(A_+ - A_-)\big)ds$$

as $n \to \infty$.

Thus, by (5.2.8) we obtain

$$\operatorname{tr}\Big((\boldsymbol{H}_2 - z)^{-m} - (\boldsymbol{H}_1 - z)^{-m}\Big) = -\frac{(2m-1)!!}{(m-1)!2^m}\int_0^1 \operatorname{tr}\big((A_s^2 - z)^{-1/2-m}(A_+ - A_-)\big)ds.$$

□

Remark 5.2.4. (i) Here we presented the proof, which uses differentiation with respect to z. We note that this step is redundant and, in fact, can be replaced by the reference to [**31**, Theorem B.5]. Indeed, consider the function

$$f(\lambda) = (\lambda - z)^{-m}, \quad \lambda \in [0,\infty), m \in \mathbb{N}.$$

Then for the function F on $\mathbb{R}$, defined by (cf. [**31**, (B.57)])

$$F(\nu) = \frac{\nu}{2\pi}\int_{[\nu^2,\infty)} \lambda^{-1}(\lambda - \nu^2)^{-1/2}[f(\lambda) - f(0)]d\lambda,$$

we have (see [**31**, (B.51)])

$$F'(\nu) = \frac{1}{\pi}\int_{[0,\infty)} \lambda^{-1/2} f'(\lambda + \nu^2)d\lambda, \quad \nu \in \mathbb{R}.$$

For this choice of f, one can compute that

$$\begin{aligned} F'(\nu) &= -\frac{m}{\pi}\int_{[0,\infty)} \frac{1}{\lambda^{1/2}(\lambda - z + \nu^2)^{m+1}}d\lambda \\ &= \frac{m}{\pi}\frac{\Gamma(m+1/2)\sqrt{\pi}}{\Gamma(m+1)}\frac{1}{(\nu^2 - z)^{m+1/2}} \\ &= -\frac{(2m-1)!!}{2^m(m-1)!}\frac{1}{(\nu^2 - z)^{m+1/2}}, \end{aligned}$$

where the second equality follows from the definition of Beta function and the last equality one from properties of Gamma function.

Hence, by [**31**, Theorem B.5] and Proposition 2.3.2 we have

$$\begin{aligned}
&\operatorname{tr}\Big((\boldsymbol{H}_{2,n}-z)^{-m}-(\boldsymbol{H}_{1,n}-z)^{-m}\Big)\\
&\quad=\operatorname{tr}\big(F(A_{+,n})-F(A_{-})\big)\\
&\quad=\int_0^1\operatorname{tr}\big(F'(A_{s,n})(A_{+,n}-A_{-})\big)ds\\
&\quad=-\frac{(2m-1)!!}{2^m(m-1)!}\int_0^1\operatorname{tr}\big((A_{s,n}^2-z)^{-m-1/2}(A_{+,n}-A_{-})\big)ds,
\end{aligned}$$

which is precisely the same as (5.2.8). Passing to the limit as $n\to\infty$ as before, we infer the principal trace formula in the resolvent form.

(ii) If fact, our technique can be used to prove trace formulas for a wide class of functions. However, since we do not aim to prove principle trace formula for a general class of functions, we skip the proof.

CHAPTER 6

Witten index in terms of spectral shift function and spectral flow

In this chapter we prove the main results of the present thesis. The results presented in htis chapter are taken from [**28**].

The first main result here (Section 6.1) is the Pushnitski's formula

$$\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) = \frac{1}{\pi} \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_+, A_-)\, d\nu}{(\lambda - \nu^2)^{1/2}}.$$

Thus, we generalise the formula, obtained in [**67**], [**47**] and [**30**], for the setting of p-relative trace-class perturbations of a self-adjoint operator A_-.

We note that our approach to the proof of Pushnitski's formula is completely different from that of [**67**], [**47**] and [**30**]. The fundamental step in our approach is the principal trace formula (in its heat kernel version) and the Laplace transform.

As an application of Pushnitski's formula we prove (see Theorem 6.2.3) that

$$W(\boldsymbol{D_A}) = [\xi_L(0_+; A_+, A_-) + \xi_L(0_-; A_+, A_-)]/2,$$

provided that the right-hand side is well-defined. Here our proof is based on [**31**]. Thus, if the values (in Lebesgue sense) of the spectral shift function $\xi(\cdot; A_+, A_-)$ are well defined from the left and the right of zero, the Witten index of the operator $\boldsymbol{D_A}$ can be computed via $\xi(\cdot; A_+, A_-)$.

In Section 6.3 we impose an additional assumption that the asymptotes $A_\pm$ are Fredholm. This assumption guarantees that the spectral flow $\mathrm{sf}(\{A(t)\}_{t\in\mathbb{R}})$ for the family $\{A(t)\}_{t\in\mathbb{R}}$ is well-defined. Moreover, it also yields that spectral shift function $\xi(\cdot; A_+, A_-)$ is left and right continuous at zero. Here, we prove that

$$\begin{aligned}\frac{1}{2}\big(\xi(0+; A_+, A_-) + \xi(0-; A_+, A_-)\big)\\ = \mathrm{sf}\{A(t)\}_{t=-\infty}^{\infty} - \frac{1}{2}[\dim(\ker(A_+)) - \dim(\ker(A_-))],\end{aligned}$$

thus showing that an intuitive equality that the spectral shift function on discrete spectra computes the spectral flow holds in this general setting.

In addition, combining this with our formula for the Witten index we conclude that

$$\begin{aligned}W(\boldsymbol{D_A}) &= \frac{1}{2}\big(\xi(0+; A_+, A_-) + \xi(0-; A_+, A_-)\big)\\ &= \mathrm{sf}\{A(t)\}_{t=-\infty}^{\infty} - \frac{1}{2}[\dim(\ker(A_+)) - \dim(\ker(A_-))].\end{aligned}$$

Thus, we prove the 'index=spectral flow' type theorem of Atiyah-Patodi-Singer, Robbin-Salamon etc., which is now suitable for paths $\{A(t)\}_{t\in\mathbb{R}}$ with not necessarily invertible asymptotes $A_\pm$. Furthermore, this result also holds for paths

$\{A(t)\}_{t\in\mathbb{R}}$ of differential operator on locally compact manifolds, where some essential spectra outside 0 is typically present.

6.1. Pushnitski's formula

The following is generalisation of Pushnitski's formula (1.5.2) from [**67**] (see also, [**47**], [**30**]).

THEOREM 6.1.1. *Assume Hypothesis 3.5.1. Let* $\xi(\cdot; A_+, A_-)$ *be the spectral shift function for the pair* (A_+, A_-) *fixed in* (4.2.6) *and let* $\xi(\cdot; \boldsymbol{H}_2, \boldsymbol{H}_1)$ *be the spectral shift function for the pair* $(\boldsymbol{H}_2, \boldsymbol{H}_1)$ *fixed by equality* (4.2.2). *Then for a.e.* $\lambda > 0$ *we have*

$$\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) = \frac{1}{\pi}\int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_+, A_-)\, d\nu}{(\lambda-\nu^2)^{1/2}} \tag{6.1.1}$$

with a convergent Lebesgue integral on the right-hand side of (6.1.1).

PROOF. By the principle trace formula for semigroup difference (see Theorem 5.1.4) we have that

$$\mathrm{tr}(e^{-t\boldsymbol{H}_2} - e^{-t\boldsymbol{H}_1}) = -\Big(\frac{t}{\pi}\Big)^{1/2}\int_0^1 \mathrm{tr}\left(e^{-tA_s^2}(A_+ - A_-)\right) ds$$

for all $t > 0$. For the right hand side of this formula (5.1.11) and (5.1.4) imply that

$$\begin{aligned}
&\Big(\frac{t}{\pi}\Big)^{1/2}\int_0^1 \mathrm{tr}(e^{-tA_s^2}(A_+ - A_-))ds\\
&\qquad \overset{(5.1.11)}{=} \lim_{n\to\infty}\Big(\frac{t}{\pi}\Big)^{1/2}\int_0^1 \mathrm{tr}(e^{-tA_{s,n}^2}(A_{+,n} - A_-))ds\\
&\qquad \overset{(5.1.4)}{=} \lim_{n\to\infty}\frac{1}{2}\,\mathrm{tr}(\mathrm{erf}(t^{1/2}A_{+,n}) - \mathrm{erf}(t^{1/2}A_-))
\end{aligned} \tag{6.1.2}$$

By the Krein's trace formula (1.2.2) and the definition of error function (5.1.1) we have that

$$\frac{1}{2}\,\mathrm{tr}(\mathrm{erf}(t^{1/2}A_{+,n}) - \mathrm{erf}(t^{1/2}A_-)) = \Big(\frac{t}{\pi}\Big)^{1/2}\int_{\mathbb{R}} e^{-ts^2}\xi(s, A_{+,n}, A_-)ds. \tag{6.1.3}$$

It is clear that the function $s \mapsto e^{-ts^2}, s\in\mathbb{R}, t>0$, satisfies the assumption of Corollary 4.2.4, and therefore, we obtain that

$$\lim_{n\to\infty}\Big(\frac{t}{\pi}\Big)^{1/2}\int_{\mathbb{R}} e^{-ts^2}\xi(s, A_{+,n}, A_-)ds = \Big(\frac{t}{\pi}\Big)^{1/2}\int_{\mathbb{R}} e^{-ts^2}\xi(s, A_+, A_-)ds. \tag{6.1.4}$$

Thus, combining (6.1.2), (6.1.3) and (6.1.4) we conclude that the right-hand side of the principal trace formula can be written as

$$\Big(\frac{t}{\pi}\Big)^{1/2}\int_0^1 \mathrm{tr}(e^{-tA_s^2}(A_+ - A_-))ds = \Big(\frac{t}{\pi}\Big)^{1/2}\int_{\mathbb{R}} e^{-ts^2}\xi(s, A_+, A_-)ds.$$

Since the function $s \mapsto e^{-ts}, s\in\mathbb{R}, t>0$, is a Schwartz function, it belongs to the class $\mathfrak{F}_{\widehat{m}}$ (see (2.2.34)). Hence, by the Krein's trace formula (4.2.3) for the left hand side of the principal trace formula we have that

$$\mathrm{tr}(e^{-t\boldsymbol{H}_2} - e^{-t\boldsymbol{H}_1}) = -t\int_0^\infty \xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1)e^{-t\lambda}\, d\lambda.$$

Thus,

$$\begin{aligned}\int_0^\infty \xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) e^{-t\lambda}\, d\lambda &= \Big(\frac{1}{\pi \cdot t}\Big)^{1/2} \int_{\mathbb{R}} \xi(s, A_+, A_-) e^{-ts^2} ds \\ &= \Big(\frac{1}{\pi \cdot t}\Big)^{1/2} \int_0^\infty \frac{\xi(\sqrt{s}, A_+, A_-) + \xi(-\sqrt{s}, A_+, A_-)}{\sqrt{s}} e^{-ts} ds, \qquad (6.1.5)\end{aligned}$$

where for the last integral we used the substitutions $s \mapsto \sqrt{s}$ and $s \mapsto -\sqrt{s}$ for the integrals on $(0, \infty)$ and on $(-\infty, 0)$, respectively.

Let us denote by L the Laplace transform on $L^1_{loc}(\mathbb{R})$. It is well-known that $L(\frac{1}{\pi\sqrt{s}})(t) = \frac{1}{\sqrt{\pi t}}$ (see e.g. [**2**, 29.3.4]). Therefore, introducing

$$\xi_0(s) := \frac{\xi(\sqrt{s}, A_+, A_-) + \xi(-\sqrt{s}, A_+, A_-)}{\sqrt{s}}, \ s \in [0, \infty),$$

equality (6.1.5) can be rewritten as

$$L\big(\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1)\big)(t) = L\big(\frac{1}{\pi\sqrt{s}}\big)(t) \cdot L\big(\xi_0(s)\big)(t).$$

By [**4**, Proposition 1.6.4] the right-hand side of the above equality is equal to $L\Big(\frac{1}{\pi\sqrt{s}} * \xi_0(s)\Big)(t)$. Therefore, by the Uniqueness theorem for the Laplace transform (see e.g. [**4**, Theorem 1.7.3]) we have $\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) = \big(\frac{1}{\pi\sqrt{s}} * \xi_0(s)\big)(\lambda)$ for a.e. $\lambda \in [0, \infty)$. Thus, for a.e. $\lambda \in [0, \infty)$ we have

$$\begin{aligned}\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) &= \frac{1}{\pi} \int_0^\lambda \frac{1}{\sqrt{\lambda - s}} \xi_0(s) ds \\ &= \frac{1}{\pi} \int_0^\lambda \frac{1}{\sqrt{\lambda - s}} \frac{\xi(\sqrt{s}, A_+, A_-) + \xi(-\sqrt{s}, A_+, A_-)}{\sqrt{s}} ds \\ &= \frac{1}{\pi} \int_0^\lambda \frac{\xi(\sqrt{s}, A_+, A_-)}{\sqrt{s}\sqrt{\lambda - s}} ds + \frac{1}{\pi} \int_0^\lambda \frac{\xi(-\sqrt{s}, A_+, A_-)}{\sqrt{s}\sqrt{\lambda - s}} ds \\ &= \frac{1}{\pi} \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{\xi(s; A_+, A_-) ds}{\sqrt{\lambda - s^2}}.\end{aligned}$$

□

6.2. Witten index in terms of spectral shift function

In this section, employing the Pushnitski's formula (6.1.1) in the general setting of p-relative trace-class perturbations, we prove the formula relating the Witten index and spectral shift function, which follows the detailed treatment in [**31**]. Our results enable us to weaken the "relatively trace class perturbation assumption" of Hypothesis 1.5.2 from [**31**] as well as "relatively Hilbert-Schmidt class perturbation assumption" from [**30**]. In Section 7.1 we show that our Hypothesis 3.5.1 permits consideration of differential operators (in particular, Dirac operators) in any dimension uniformly.

We firstly recall necessary definitions.

Definition 6.2.1. *Let* $f \in L_{1,\mathrm{loc}}(\mathbb{R})$ *and* $h > 0$.

(i) *The point* $x \in \mathbb{R}$ *is called a right Lebesgue point of* f *if there exists an* $\alpha_+ \in \mathbb{C}$ *such that*

$$\lim_{h \downarrow 0} \frac{1}{h} \int_x^{x+h} |f(y) - \alpha_+| dy = 0. \tag{6.2.1}$$

One then denotes $\alpha_+ = f_L(x_+)$.

(ii) *The point* $x \in \mathbb{R}$ *is called a left Lebesgue point of* f *if there exists an* $\alpha_- \in \mathbb{C}$ *such that*

$$\lim_{h \downarrow 0} \frac{1}{h} \int_{x-h}^{x} |f(y) - \alpha_-| dy = 0. \tag{6.2.2}$$

One then denotes $\alpha_- = f_L(x_-)$.

For convenience we also recall the following result from [**31**].

Lemma 6.2.2. [**31**, Lemma 4.1] *Introduce the linear operators* $S : L_{1,\mathrm{loc}}(\mathbb{R}) \to L_{1,\mathrm{loc}}(0,\infty)$ *defined by*

$$(Sf)(\lambda) = \frac{1}{\pi} \int_0^{\lambda^{1/2}} (\lambda - \nu^2)^{-1/2} f(\nu) d\nu, \quad \lambda > 0.$$

If 0 *is a right Lebesgue point for* $f \in L_{1,\mathrm{loc}}(\mathbb{R})$, *then it is also right Lebesgue point for* Sf *and*

$$(Sf)_L(0_+) = \frac{1}{2} f_L(0_+). \tag{6.2.3}$$

Now we state our main result of the present chapter. Its proof closely follows the argument as used in [**31**].

Theorem 6.2.3. *Assume Hypothesis 3.5.1 and assume that* 0 *is a right and a left Lebesgue point of* $\xi(\cdot; A_+, A_-)$. *Then* 0 *is a right Lebesgue point of* $\xi(\cdot; \boldsymbol{H}_2, \boldsymbol{H}_1)$ *and* $W(\boldsymbol{D}_{\boldsymbol{A}})$ *exists and equals*

$$W(\boldsymbol{D}_{\boldsymbol{A}}) = \xi_L(0_+; \boldsymbol{H}_2, \boldsymbol{H}_1) = [\xi_L(0_+; A_+, A_-) + \xi_L(0_-; A_+, A_-)]/2. \tag{6.2.4}$$

Proof. First, one rewrites (6.1.1) in the form,

$$\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) = \frac{1}{\pi} \int_0^{\lambda^{1/2}} \frac{d\nu\, [\xi(\nu; A_+, A_-) + \xi(-\nu; A_+, A_-)]}{(\lambda - \nu^2)^{1/2}}, \quad \lambda > 0. \tag{6.2.5}$$

Define the function $f(\nu) = [\xi(\nu, A_+, A_-) + \xi(-\nu, A_+, A_-)]$. Equality (6.2.5) implies that

$$\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) = (Sf)(\lambda), \quad \lambda > 0,$$

where S is defined in Lemma 6.2.2. By assumption, 0 is a right and a left Lebesgue point of $\xi(\,\cdot\,; A_+, A_-)$, and therefore, 0 is a right Lebesgue point of f. Hence, by Lemma 6.2.2 we obtain that 0 is a right Lebesgue point of $\xi(\,\cdot\,; \boldsymbol{H}_2, \boldsymbol{H}_1)$ and

$$\xi_L(0_+; \boldsymbol{H}_2, \boldsymbol{H}_1) = \frac{1}{2} f_L(0_+) = \frac{1}{2}(\xi_L(0_+; A_+, A_-) + \xi_L(0_-; A_+, A_-)).$$

Next, to prove the first equality we introduce the function

$$\Xi(r; \boldsymbol{H}_2, \boldsymbol{H}_1) = \int_0^r \xi(s; \boldsymbol{H}_2, \boldsymbol{H}_1)\, ds, \quad r > 0.$$

By Krein's trace formula (4.1.8) we have that

$$\begin{aligned}\frac{1}{t}\operatorname{tr}\big(e^{-t\boldsymbol{H}_2}-e^{-t\boldsymbol{H}_1}\big) &= -\int_0^\infty \xi(s;\boldsymbol{H}_2,\boldsymbol{H}_1)\,e^{-ts}\,ds\\ &= -\int_0^\infty e^{-ts}d\,\Xi(s;\boldsymbol{H}_2,\boldsymbol{H}_1).\end{aligned}$$

We have already established, that 0 is a right Lebesgue point of $\xi(\,\cdot\,;\boldsymbol{H}_2,\boldsymbol{H}_1)$. Hence, one obtains that

$$\lim_{r\downarrow 0+}\frac{\Xi(r;\boldsymbol{H}_2,\boldsymbol{H}_1)}{r} = \Xi'(0_+;\boldsymbol{H}_2,\boldsymbol{H}_1) = \xi_L(0_+;\boldsymbol{H}_2,\boldsymbol{H}_1)$$

exists. Hence, an Abelian theorem for Laplace transforms [**80**, Theorem 1, p. 181] (with $\gamma=1$) implies that

$$-\lim_{t\to\infty}\operatorname{tr}_{\mathcal{H}}\big(e^{-t\boldsymbol{H}_2}-e^{-t\boldsymbol{H}_1}\big) = \lim_{r\downarrow 0+}\frac{\Xi(r;\boldsymbol{H}_2,\boldsymbol{H}_1)}{r} = \xi_L(0_+;\boldsymbol{H}_2,\boldsymbol{H}_1).$$

□

6.3. Connection to spectral flow

In this section we establish the connection of the Witten index of the operator $\boldsymbol{D_A}$ to the spectral flow along the path $\{A(t)\}_{t\in\mathbb{R}}$ in the special case, when $A_\pm$ are Fredholm operators, and so the spectral flow is well-defined. This provides an extension of the Robbin-Salamon result to the situation where the endpoints of the path are not invertible so that $\boldsymbol{D_A}$ is not Fredholm.

The key tool in our proof is the analytic formula for the spectral flow of Theorem 1.3.4 applied for the path of approximants $\{A_n(t)\}_{t\in\mathbb{R}}$ (see the proof of Proposition 6.3.5 below).

In this section we assume the following:

Hypothesis 6.3.1. *In addition to Hypothesis 3.1.1 we assume that*

(i) the operators $A_\pm$ are Fredholm;
(ii) For all $t\in\mathbb{R}$ and some $k\in\mathbb{N}$ we have

$$B'(t)(A_-+i)^{-1}\in\mathcal{L}_{p+1}(\mathcal{H}),\qquad \int_{\mathbb{R}}\|B'(t)(A_-+i)^{-1}\|_{p+1}dt<\infty.$$

Remark 6.3.2. Note that using the three line theorem, one can show that the assumption $B'(t)(A_-+i)^{-p-1}\in\mathcal{L}_1(\mathcal{H})$ implies that $B'(t)(A_-+i)^{-1}\in\mathcal{L}_{p+1}(\mathcal{H})$, however, we also need integrability of the norm.

In order to relate the Witten index to spectral flow we will again use our approximation method, that is, as before, we introduce the family

$$A_n(t)=A_-+P_nB(t)P_n,\quad t\in\mathbb{R},\quad A_{-,n}=A_-,\quad A_{+,n}=A_-+P_nB_+P_n,$$

where $P_n=\chi_{[-n,n]}(A_-)$.

Repeating the proof of [**47**, Remark 3.3] and referring to Remark 3.1.4 we obtain that

$$B(t)(|A_-|+I)^{-1}=\int_{-\infty}^t B'(s)(|A_-|+I)^{-1}\,ds\in\mathcal{L}_{p+1}(\mathcal{H}),\quad t\in\mathbb{R}\tag{6.3.1}$$

and

$$B_+(|A_-|+I)^{-1}=\int_{-\infty}^\infty B'(s)(|A_-|+I)^{-1}\,ds\in\mathcal{L}_{p+1}(\mathcal{H}).\tag{6.3.2}$$

As the key ingredient of our argument is Theorem 1.3.4 and that results defines the spectral flow for a path $\{S(t)\}$, $t \in [0,1]$ (see Section 1.3), we re-parametrise our path $\{A(t)\}_{t=-\infty}^{\infty}$ to avoid confusion. Let $r : [0,1] \to \mathbb{R}$ be a continuously differentiable strictly increasing function. Introduce the path $\{S(t)\}_{t=0}^{1}$ by letting

$$S(0) = A_-, \quad S(t) = A(r(t)), t \in (0,1), \quad S(1) = A_+, \tag{6.3.3}$$

and the corresponding path of 'cut-off' operators $\{S_n(t)\}_{t=0}^{1}$ by

$$S_n(0) = A_-, \quad S_n(t) = A_n(r(t)), t \in (0,1), \quad S_n(1) = A_{+,n}, \tag{6.3.4}$$

Recall that Theorem 1.3.4 is established for C^1_Γ-path of unbounded Fredholm operators (see Definition 1.3.1 for the precise definition of a C^1_Γ-path). Therefore, we begin by showing that both the paths $\{A(t)\}_{t=-\infty}^{+\infty}$, $\{A_n(t)\}_{t=-\infty}^{+\infty}$ (and equivalently, $\{S(t)\}_{t=0}^{1}$ and $\{S_n(t)\}_{t=0}^{1}$) are continuous paths of (self-adjoint) Fredholm operators.

Lemma 6.3.3. *The paths $\{S(t)\}_{t\in[0,1]}$ and $\{S_n(t)\}_{t\in[0,1]}$ are C^1_Γ-paths of Fredholm operators.*

Proof. We prove the assertion only for the path $\{S(t)\}_{t\in[0,1]}$, as the argument for $\{S_n(t)\}_{t\in[0,1]}$ is similar.

To prove that the path consists of Fredholm operators, we note that (6.3.1) states that $B(t), t \in \mathbb{R}$ is an A_--relatively compact operators. Hence by Weyl theorem (see e.g. [**52**, Theorem 5.35]) we obtain that $A(t) = A_- + B(t), t \in \mathbb{R}$ has the same essential spectra as A_-, which by Hypothesis 6.3.1 implies that $A(t), t \in \mathbb{R}$, are Fredholm operators.

Next, we show that $\{S(t)\}_{t\in[0,1]}$ is a C^1_Γ-path. By Hypothesis 3.5.1 we have that $\{S(t)\}_{t\in[0,1]}$ is Γ-differentiable at any point and $\dot S(t) = A'(r(t)) \cdot r'(t) = B'(r(t)) \cdot r'(t)$. Next for arbitrary $t_1, t_2 \in [0,1]$ we have

$$\begin{aligned}
&\Big\| \dot S(t_1)(1+S(t_1)^2)^{-1/2} - \dot S(t_2)(1+S(t_2)^2)^{-1/2} \Big\| \\
&\qquad\le \|B'(r(t_1)) - B'(r(t_2))\| \|(1+S(t_1)^2)^{-1/2}\| \\
&\qquad\quad + \|B'(r(t_2))\| \Big\| (1+S(t_1)^2)^{-1/2} - (1+S(t_2)^2)^{-1/2} \Big\|.
\end{aligned}$$

Since the family $\{B(t)\}_{t=-\infty}^{\infty}$ is continuously differentiable with respect to uniform norm and the function r is continuous, we obtain that, as $t_1 - t_2 \to 0$ we have $\|B'(r(t_1)) - B'(r(t_2))\| \to 0$. In addition, we have

$$\begin{aligned}
&(1+S(t_1)^2)^{-1/2} - (1+S(t_2)^2)^{-1/2} \\
&\qquad = \frac{1}{\pi}\int_0^\infty d\lambda\, \lambda^{-1/2}((1+\lambda+S(t_1)^2)^{-1} - (1+\lambda+S(t_2)^2)^{-1}).
\end{aligned}$$

Using the resolvent identity and continuity of the path $\{B(t)\}_{t=-\infty}^{\infty}$ one can conclude, that $\|(1+S(t_1)^2)^{-1/2} - (1+S(t_2)^2)^{-1/2}\| \to 0$ as $t_1 - t_2 \to 0$.

Thus, $\|\dot S(t_1)(1+S(t_1)^2)^{-1/2} - \dot S(t_2)(1+S(t_2)^2)^{-1/2}\|_{\mathcal{B}(\mathcal{H})} \to 0$ as $t_1 - t_2 \to 0$, which proves that the mapping $t \mapsto \dot S(t)(1+S(t)^2)^{-1/2}$ is continuous, and hence, concludes the proof.

□

Thus, by Lemma 6.3.3 both $\{A(t)\}_{t=-\infty}^{\infty}$ and $\{A_n(t)\}_{t=-\infty}^{\infty}$ are C^1_Γ-paths of Fredholm operators. Therefore, by Definition 1.3.3 we can define the spectral flow for both paths $\{A(t)\}_{t=-\infty}^{\infty}$ and $\{A_n(t)\}_{t=-\infty}^{\infty}$ by setting

$$\mathrm{sf}(\{A(t)\}_{t=-\infty}^{\infty}) := \mathrm{sf}(\{S(t)\}_{t=0}^{1}) = \mathrm{sf}(\{g(S(t))\}_{t=0}^{1}) \tag{6.3.5}$$

and

$$\mathrm{sf}(\{A_n(t)\}_{t=-\infty}^{\infty}) := \mathrm{sf}(\{S_n(t)\}_{t=0}^{1}) = \mathrm{sf}(\{g(S_n(t))\}_{t=0}^{1}). \tag{6.3.6}$$

In the next lemma we show that the path $\{S_n(t)\}_{t=0}^{1}$ satisfies the assumptions of Theorem 1.3.4.

LEMMA 6.3.4. *The path $\{S_n(t)\}_{t=0}^{1}$ satisfies the assumptions of Theorem 1.3.4.*

PROOF. By Lemma 6.3.3 the path $\{S_n(t)\}_{t\in[0,1]}$ is a C^1_Γ-path. Moreover, $\dot{S}(t) = P_n B'(r(t)) P_n$, which implies that $\dot{S}(t)$ is a trace-class operator for any $t \in [0,1]$. Hence assumption (i) of Theorem 1.3.4is satisfied.

Next, by Proposition 5.1.1we have that

$$\frac{1}{2}\,\mathrm{erf}(t^{1/2}A_{+,n}) - \frac{1}{2}\,\mathrm{erf}(t^{1/2}A_-) \in \mathcal{L}_1(\mathcal{H}).$$

Thus, we only need to show that

$$\chi_{[0,\infty)}(A_{+,n}) - \chi_{[0,\infty)}(A_-) \in \mathcal{L}_1(\mathcal{H}).$$

Since 0 is an isolated eigenvalue of $\sigma(A_{+,n})$ and $\sigma(A_-)$, there exists $\varepsilon > 0$, such that $P_{(0,\varepsilon)}(A_{+,n}) = P_{(0,\varepsilon)}(A_-) = 0$ and $\varepsilon \notin \sigma(A_{+,n}), \sigma(A_-)$. Therefore,

$$\begin{aligned}\chi_{[0,\infty)}(A_{+,n}) - \chi_{[0,\infty)}(A_-) &= -[\chi_{(-\infty,0)}(A_{+,n}) - \chi_{(-\infty,0)}(A_-)]\\ &= -[\chi_{(-\infty,\varepsilon)}(A_{+,n}) - \chi_{(-\infty,\varepsilon)}(A_-)]\\ &= -[\chi_{(-\infty,0)}(A_{+,n}-\varepsilon) - \chi_{(-\infty,0)}(A_- - \varepsilon)].\end{aligned} \tag{6.3.7}$$

Introduce a smooth cut-off function $\varphi \in C^\infty(\mathbb{R})$ satisfying

$$\varphi(\nu) = \begin{cases} 1, & \nu \le -\nu_0,\\ 0, & \nu \ge \nu_0,\end{cases} \quad \text{and} \quad \int_{-\nu_0}^{\nu_0} \varphi'(\nu)\,d\nu = -1. \tag{6.3.8}$$

The choice of the function φ guarantees that φ satisfies the assumptions on Theorem 1.2.1. Hence, the fact that $(A_{+,n} - \varepsilon) - (A_- - \varepsilon) = A_{+,n} - A_- \in \mathcal{L}_1(\mathcal{H})$ combined with Theorem 1.2.1 implies that

$$\varphi(A_{+,n} - \varepsilon) - \varphi(A_- - \varepsilon) \in \mathcal{L}_1(\mathcal{H}). \tag{6.3.9}$$

Note that φ coincides with the characteristic function of $(-\infty, 0)$ on the spectra of $A_{+,n} - \varepsilon$ and $A_- - \varepsilon$, and therefore

$$\chi_{(-\infty,0)}(A_{+,n} - \varepsilon) = \varphi(A_{+,n} - \varepsilon), \quad \chi_{(-\infty,0)}(A_- - \varepsilon) = \varphi(A_- - \varepsilon).$$

Hence, combining (6.3.7) and (6.3.9) we conclude that

$$\begin{aligned}\chi_{[0,\infty)}(A_{+,n}) - \chi_{[0,\infty)}(A_-) &= [\chi_{(-\infty,0)}(A_{+,n}-\varepsilon) - \chi_{(-\infty,0)}(A_- - \varepsilon)]\\ &= -[\varphi(A_{+,n}-\varepsilon) - \varphi(A_- - \varepsilon)] \in \mathcal{L}_1(\mathcal{H}),\end{aligned}$$

which concludes the proof. □

The next step of our approach is to give a formula, similar to that of Theorem 1.3.6, relating the spectral flow along the path $\{A_n(t)\}_{t=-\infty}^{\infty}$ and the spectral shift function $\xi(\cdot; A_{+,n}, A_-)$. To this end, we recall (see (4.1.19) and (1.2.2)) that for each $n \in \mathbb{N}$ there exists a (unique) spectral shift function $\xi(\cdot; A_{+,n}, A_-)$ for the pair $(A_{+,n}, A_-)$. By Proposition 6.3.3 the operators $A_{+,n}$ and A_- are Fredholm, and therefore, it follows from the properties of the spectral shift function (see (1.2.4)) that for every $n \in \mathbb{N}$, the function $\xi(\cdot; A_{+,n}, A_-)$ is left and right continuous at zero and

$$\xi(0+; A_{+,n}, A_-) - \xi(0-; A_{+,n}, A_-) = \dim\ker(A_-) - \dim\ker(A_{+,n}). \qquad (6.3.10)$$

The next proposition is the first step in the proof of our formula for the Witten index in terms of spectral flow. This formula connects the spectral flow $\mathrm{sf}(\{A_n(t)\}_{t=-\infty}^{\infty})$ to the spectral shift function $\xi(\cdot, A_{+,n}, A_-)$. The main ingredient of our argument is Theorem 1.3.4.

Proposition 6.3.5. *Let $A_{+,n}, \{A_n(t)\}_{t=-\infty}^{\infty}$ and $\{S_n(t)\}_{t\in[0,1]}$ be as before. Then for $\mathrm{sf}(\{A_n(t)\}_{t=-\infty}^{\infty})$, defined by (6.3.6) we have*

$$\begin{aligned}\mathrm{sf}(\{A_n(t)\}_{t=-\infty}^{\infty}) = \frac{1}{2}\big[\xi(0_+, A_{+,n}, A_-) + \xi(0_-, A_{+,n}, A_-)\big]\\ + \frac{1}{2}\big[\dim(\ker(A_{+,n})) - \dim(\ker(A_-))\big].\end{aligned}$$

Proof. By Lemma 6.3.4 the path $\{S_n(t)\}_{t\in[0,1]}$ satisfies the assumption of Theorem 1.3.4. Hence,

$$\begin{aligned}&\mathrm{sf}(\{A_n(t)\}_{t=-\infty}^{\infty}) = \mathrm{sf}(\{S_n(t)\}_{t\in[0,1]})\\ &= \int_0^1 \mathrm{tr}\,\big((A_{+,n} - A_-)e^{-\lambda A_n^2(r(t))}\big)dt + \frac{1}{2}\,\mathrm{tr}[\mathrm{erf}(\lambda^{1/2}A_{+,n}) - \mathrm{erf}(\lambda^{1/2}A_-)]\\ &\quad - \mathrm{tr}[\chi_{[0,\infty)}(A_{+,n}) - \chi_{[0,\infty)}(A_-)].\end{aligned}$$

By Proposition 2.3.2 we have that

$$\int_0^1 \mathrm{tr}\,\big((A_{+,n} - A_-)e^{-\lambda A_n^2(r(t))}\big)dt = \frac{1}{2}\,\mathrm{tr}[\mathrm{erf}(\lambda^{1/2}A_{+,n}) - \mathrm{erf}(\lambda^{1/2}A_-)],$$

and therefore

$$\begin{aligned}\mathrm{sf}(\{A_n(t)\}_{t=-\infty}^{\infty}) = \mathrm{tr}[\mathrm{erf}(\lambda^{1/2}A_{+,n}) - \mathrm{erf}(\lambda^{1/2}A_-)]\\ - \mathrm{tr}[\chi_{[0,\infty)}(A_{+,n}) - \chi_{[0,\infty)}(A_-)].\end{aligned}$$

We now compute $\mathrm{tr}[\chi_{[0,\infty)}(A_{+,n}) - \chi_{[0,\infty)}(A_-)]$. Fix $\varepsilon > 0$ as in the proof of Lemma 6.3.4. By (6.3.7) we have that

$$\begin{aligned}&\mathrm{tr}\,\big(\chi_{[0,\infty)}(A_{+,n}) - \chi_{[0,\infty)}(A_-)\big)\\ &\qquad = -\,\mathrm{tr}\,\big([\chi_{(-\infty,0)}(A_{+,n} - \varepsilon) - \chi_{(-\infty,0)}(A_- - \varepsilon)]\big).\end{aligned} \qquad (6.3.11)$$

Introducing the family $\tilde{B}_n(t) = B_n(t) - \varepsilon$, we have that $\tilde{B}_n'(t) = B_n'(t)$ for all $t \in \mathbb{R}$, and hence the family $\{\tilde{B}_n(t)\}_{t\in\mathbb{R}}$ satisfies the Pushnitski assumptions 1.5.1 relative to $A_- - \varepsilon$. In addition, for the corresponding asymptotes $A_{+,n} - \varepsilon, A_- - \varepsilon$, 0 is not in their spectra and $(A_{+,n} - \varepsilon) - (A_- - \varepsilon) = A_{+,n} - A_- \in \mathcal{L}_1(\mathcal{H})$.

Hence, by [**47**, Lemma 8.7.5] we have that

$$\mathrm{tr}[\chi_{(-\infty,0)}(A_{+,n} - \varepsilon) - \chi_{(-\infty,0)}(A_- - \varepsilon)] = -\xi(0, A_{+,n} - \varepsilon, A_- - \varepsilon),$$

which implies that

$$\mathrm{tr}[\chi_{[0,\infty)}(A_{+,n}) - \chi_{[0,\infty)}(A_-)] = \xi(0, A_{+,n} - \varepsilon, A_- - \varepsilon) = \xi(\varepsilon, A_{+,n}, A_-). \quad (6.3.12)$$

Thus, combining (6.3.11) with (6.3.12) we conclude that

$$\mathrm{sf}(\{A_n(t)\}_{t=-\infty}^{\infty}) = \mathrm{tr}[\mathrm{erf}(\lambda^{1/2}A_{+,n}) - \mathrm{erf}(\lambda^{1/2}A_-)] + \xi(\varepsilon, A_{+,n}, A_-).$$

Next, we take the limit as $\lambda \to \infty$. Firstly, applying Proposition 2.3.2 and (5.1.3) we have that

$$\begin{aligned}\mathrm{sf}(\{A_n(t)\}_{t=-\infty}^{\infty}) &= 2\,\mathrm{tr}[\mathrm{erf}(\lambda^{1/2}A_{+,n}) - \mathrm{erf}(\lambda^{1/2}A_-)] + \xi(\varepsilon, A_{+,n}, A_-)\\ &\overset{(5.1.3)}{=} -2\,\mathrm{tr}(e^{-t\boldsymbol{H}_{2,n}} - e^{-t\boldsymbol{H}_{1,n}}) - \xi(\varepsilon, A_{+,n}, A_-).\end{aligned}$$

Now, by Theorem 0.2 we have that

$$-2\lim_{t\to\infty}\mathrm{tr}(e^{-t\boldsymbol{H}_{2,n}} - e^{-t\boldsymbol{H}_{1,n}}) = 2W(\boldsymbol{D}_{\boldsymbol{A}_n}) = [\xi(0_+, A_{+,n}, A_-) + \xi(0_-, A_{+,n}, A_-)].$$

Therefore,

$$\begin{aligned}\mathrm{sf}(\{A_n(t)\}_{t=-\infty}^{\infty}) &= [\xi(0_+, A_{+,n}, A_-) + \xi(0_-, A_{+,n}, A_-)] - \xi(\varepsilon, A_{+,n}, A_-)\\ &= \frac{1}{2}[\xi(0_+, A_{+,n}, A_-) + \xi(0_-, A_{+,n}, A_-)]\\ &\quad - \frac{1}{2}[\xi(0_+, A_{+,n}, A_-) - \xi(0_-, A_{+,n}, A_-)].\end{aligned}$$

Referring to (6.3.10) we have that

$$\begin{aligned}\mathrm{sf}(\{A_n(t)\}_{t=-\infty}^{\infty}) &= \frac{1}{2}[\xi(0_+, A_{+,n}, A_-) + \xi(0_-, A_{+,n}, A_-)]\\ &\quad + \frac{1}{2}[\dim(\ker(A_{+,n})) - \dim(\ker(A_-))],\end{aligned}$$

which concludes the proof. □

Having established the desired formula for the reduced operators we now want to pass to the limit as $n \to \infty$. We state some of the necessary aproximation results in separate lemmas.

To handle the kernel dimensions we prove that the kernel of the operator $A_{+,n}$ has the same dimension as the kernel of A_+ for sufficiently large $n \in \mathbb{N}$.

Lemma 6.3.6. *For sufficiently large $n \in \mathbb{N}$ we have that* $\dim(\ker(A_{+,n})) = \dim(\ker(A_+))$.

Proof. By Theorem 3.3.2 we have

$$\begin{aligned}\|(A_{+,n} - i)^{-1} - (A_+ - i)^{-1}\|_\infty &\le \|(A_{+,n} - i)^{-1} - (A_+ - i)^{-1}\|_{p+1}\\ &= \Big\|\big((A_{+,n} - i)^{-1} - (A_- - i)^{-1}\big) - \big((A_+ - i)^{-1} - (A_- - i)^{-1}\big)\Big\|_{p+1}\\ &\to 0\end{aligned}$$

as $n \to \infty$. That is $A_{+,n} \to A_+$ in the norm resolvent sense. Therefore, by [**69**, Theorem VIII.23 (i)] we obtain that 0 is an isolated eigenvalue of $\sigma(A_{+,n})$ for sufficiently large $n \in \mathbb{N}$. In addition, by [**69**, Theorem VIII.23 (ii)] for sufficiently small $\varepsilon > 0$ we have that $\|P_{(-\varepsilon,\varepsilon)}(A_{+,n}) - P_{(-\varepsilon,\varepsilon)}(A_+)\|_\infty \to 0$. Therefore, for sufficiently large $n \in \mathbb{N}$, the rank of $P_{(-\varepsilon,\varepsilon)}(A_{+,n})$ equals the rank of $P_{(-\varepsilon,\varepsilon)}(A_+)$, that is for sufficiently large $n \in \mathbb{N}$ the multiplicity of 0 for $A_{+,n}$ is the same as multiplicity for A_+. □

Next, we handle the approximation of spectral flow.

Lemma 6.3.7. *For n sufficiently large*

$$\mathrm{sf}\{A(t)\}_{t=-\infty}^{\infty} = \mathrm{sf}\{A_n(t)\}_{t=-\infty}^{\infty}. \tag{6.3.13}$$

PROOF. By (6.3.5) we have that $\mathrm{sf}\{A(t)\}_{t=-\infty}^{\infty} = \mathrm{sf}\{g(S(t))\}_{t\in[0,1]}$, and, similarly, by (6.3.6) $\mathrm{sf}\{A_n(t)\}_{t=-\infty}^{\infty} = \mathrm{sf}\{g(S_n(t))\}_{t\in[0,1]}$. Recall that $S(0) = S_n(0) = A_-$, $S(1) = A_+$ and $S_n(1) = A_{+,n}$ (see (6.3.3) and (6.3.4)). We form the following loop

$$g(A_-) \longrightarrow g(A_+) \xrightarrow{\ell} g(A_{+,n}) \longrightarrow g(A_-), \tag{6.3.14}$$

where the operators $g(A_+)$ and $g(A_{+,n})$ are joined by the straight line ℓ. We claim that this loop is contractible, and therefore there is no spectral flow around this loop. To this end, it is sufficient to show that all operators in the loop are compact perturbations of a fixed operator, say, $g(A_-)$.

Firstly, we show that the difference $g(A(t)) - g(A_-)$ is compact for all $-\infty \leq t \leq \infty$. By Theorem 2.1.8 we have

$$g(A(t)) - g(A_-) = T_\varphi^{A(t),A_-}\Big((A(t)^2+1)^{-1/4}\big(A(t)-A_-\big)(A_-^2+1)^{-1/4}\Big), \tag{6.3.15}$$

where φ is defined by setting

$$\varphi(\lambda,\mu) := \frac{\lambda(\lambda^2+1)^{-1/2} - \mu(\mu^2+1)^{-1/2}}{(\lambda^2+1)^{-1/4}\,(\lambda-\mu)\,(\mu^2+1)^{-1/4}}, \quad (\lambda,\mu)\in\mathbb{R}^2$$

and the operator $T_\varphi^{A(t),A_-}$ is bounded on $\mathcal{L}_p(\mathcal{H})$ for any $p \geq 1$. Hence, by equality (6.3.15) it is sufficient to show that $(A(t)^2+1)^{-1/4}\big(A(t)-A_-\big)(A_-^2+1)^{-1/4} \in \mathcal{L}_p(\mathcal{H})$ for some $p \geq 1$.

We have

$$(A(t)^2+1)^{-1/4}\big(A(t)-A_-\big)(A_-^2+1)^{-1/4}$$
$$= -\overline{(A(t)^2+1)^{-1/4}(A_-^2+I)^{1/4}} \times (A_-^2+I)^{-1/4}B(t)(A_-^2+I)^{-1/4}.$$

Repeating the argument in [**47**, Remark 3.9] one can prove that the operator

$$\overline{(A(t)^2+1)^{-1/4}(A_-^2+I)^{1/4}}$$

is bounded. Using the fact that

$$B(t)(A_-^2+I)^{-1/2} \in \mathcal{L}_{p+1}(\mathcal{H}), \quad -\infty \leq t \leq \infty$$

(see (6.3.1) for $t < \infty$ and (6.3.2) for $t = \infty$) and the three line theorem (see also [**47**, Lemma 6.6]), we infer that $(A_-^2+I)^{-1/4}B(t)(A_-^2+I)^{-1/4} \in \mathcal{L}_{p+1}(\mathcal{H})$, which implies that

$$g(A(t)) - g(A_-) \in \mathcal{L}_{p+1}(\mathcal{H}), \quad -\infty \leq t \leq \infty.$$

Repeating the same argument, one can obtain that

$$g(A_n(t)) - g(A_-) \in \mathcal{L}_{p+1}(\mathcal{H}), \quad -\infty \leq t \leq \infty.$$

Hence, the loop (6.3.14) consists of compact perturbations of the operator operators $g(A_-)$, that is, it is contractible. Thus, there is no spectral flow around this loop, which means that

$$\mathrm{sf}\{g(S(t))\}_{t\in[0,1]} + \mathrm{sf}\{g(A_+), g(A_{+,n})\} + \mathrm{sf}\{g(S_n(t))\}_{t\in[1,0]} = 0. \tag{6.3.16}$$

Finally, by Lemma 6.3.6 we have $\mathrm{sf}\{g(A_+), g(A_{+,n})\} = 0$ for sufficiently large $n \in \mathbb{N}$. Hence, equality (6.3.16) implies that

$$\mathrm{sf}\{g(S(t))\}_{t\in[0,1]} = -\,\mathrm{sf}\{g(S_n(t))\}_{t\in[1,0]} = \mathrm{sf}\{g(S_n(t))\}_{t\in[0,1]},$$

which completes the proof. $\square$

Prior to proving the next result of this section, we recall that the spectral shift function $\xi(\cdot; A_+, A_-)$ for the pair (A_+, A_-) is defined in (4.2.4), (4.2.3) and fixed via the requirement (4.2.3). In addition, the operators $A_\pm$ are Fredholm, and therefore, by (4.1.9) we have that $\xi(\cdot; A_+, A_-)$ is left and right-continuous at zero. In particular, 0 is a left and right Lebesgue point of $\xi(\cdot; A_+, A_-)$.

Theorem 6.3.8. *Assume Hypothesis 6.3.1. We have*

$$\begin{aligned}\frac{1}{2}\big(\xi(0+; A_+, A_-) + \xi(0-; A_+, A_-)\big)& \\ = \mathrm{sf}\{A(t)\}_{t=-\infty}^{\infty} - \frac{1}{2}[\dim(\ker(A_+)) - \dim(\ker(A_-))].&\end{aligned} \tag{6.3.17}$$

Proof. By Proposition 6.3.5 we have that

$$\begin{aligned}\frac{1}{2}\big[\xi(0_+, A_{+,n}, A_-)+\xi(0_-, A_{+,n}, A_-)\big]& \\ = \mathrm{sf}(\{A_n(t)\}_{t=-\infty}^{\infty}) - \frac{1}{2}\big[\dim(\ker(A_{+,n})) - \dim(\ker(A_-))\big].&\end{aligned} \tag{6.3.18}$$

By Lemma 6.3.6 we have that $\dim(\ker(A_{+,n})) = \dim(\ker(A_+))$ for sufficiently large $n \in \mathbb{N}$. In addition, since $A_\pm$ and $A_{+,n}$ have discrete spectra at 0, the spectral shift functions $\xi(\cdot, A_{+,n}, A_-)$ and $\xi(\cdot, A_+, A_-)$ are step functions on sufficiently small interval containing 0 (see (1.2.4) and (4.1.9) respectively). Hence, Corollary 4.2.6 implies that $\xi(0_+, A_{+,n}, A_-) = \xi(0_+, A_+, A_-)$ and $\xi(0_-, A_{+,n}, A_-) = \xi(0_-, A_+, A_-)$ for sufficiently large $n \in \mathbb{N}$. Thus, for sufficiently large $n \in \mathbb{N}$ we have

$$\begin{aligned}\frac{1}{2}\big[\xi(0_+, A_+, A_-)+\xi(0_-, A_+, A_-)\big]& \\ = \mathrm{sf}(\{A_n(t)\}_{t=-\infty}^{\infty}) - \frac{1}{2}\big[\dim(\ker(A_+)) - \dim(\ker(A_-))\big].&\end{aligned} \tag{6.3.19}$$

Referring to Lemma 6.3.7 we conclude that

$$\begin{aligned}\frac{1}{2}\big[\xi(0_+, A_+, A_-)+\xi(0_-, A_+, A_-)\big]& \\ = \mathrm{sf}(\{A(t)\}_{t=-\infty}^{\infty}) - \frac{1}{2}\big[\dim(\ker(A_+)) - \dim(\ker(A_-))\big],&\end{aligned}$$

as required.

□

As a corollary of Theorem 6.2.3 and 6.3.8 we obtain the following theorem, which is the main result of this section. This result is an extension of Robbin-Salamon theorem (see equation (0.1)) for the operators with some essential spectra outside 0 without the assumption that that asymptotes $A_\pm$ are boundedly invertible. As we will show in Section 7.1 below our framework is suitable for differential operators on locally compact manifolds in any dimension.

THEOREM 6.3.9. *Assume Hypothesis 3.5.1 and 6.3.1. Then the Witten index of the operator $\boldsymbol{D_A}$ exists and equals*

$$W(\boldsymbol{D_A}) = \frac{1}{2}\big(\xi(0+; A_+, A_-) + \xi(0-; A_+, A_-)\big) \tag{6.3.20}$$

$$= \mathrm{sf}\{A(t)\}_{t=-\infty}^{\infty} - \frac{1}{2}[\dim(\ker(A_+)) - \dim(\ker(A_-))]. \tag{6.3.21}$$

PROOF. As we discussed above (see also (4.1.9)), 0 is a left and right Lebesgue point of $\xi(\cdot; A_+, A_-)$. Hence, by Theorem 6.2.3 we have that the Witten index of the operator $\boldsymbol{D_A}$ exists and equals $W(\boldsymbol{D_A}) = \frac{1}{2}\big(\xi(0+; A_+, A_-)+\xi(0-; A_+, A_-)\big)$. On the other hand, Theorem 6.3.8 implies (6.3.21). $\square$

CHAPTER 7

Examples

In this chapter we supplement several examples, for which our general assumption are satisfied and hence the results of Chapter 6 hold.

Firstly, in Section 7.1, we prove that our primary example, multidimensional Dirac operator on $\mathbb{R}^d$ and its perturbation given by multiplication by a sufficiently nice functions, satisfies Hypothesis 3.5.1. Thus, our framework is indeed suitable for differential operators on locally-compact manifolds.

In the rest of the chapter we consider one-dimensional differential operators and compute explicitly the spectral shift function $\xi(\cdot; A_+, A_-)$. In all our computation we use the Krein trace formula (see (4.2.5))

$$\operatorname{tr}(f(A_+) - f(A_-)) = \int_{\mathbb{R}} \xi(\lambda; A_+, A_-) d\lambda, \quad f \in \mathfrak{F}_{\widehat{p}}(\mathbb{R}), \tag{7.0.1}$$

where the spectral shift function $\xi(\cdot; A_+, A_-)$ is fixed by (4.2.6).

However, for our examples the class $f \in \mathfrak{F}_{\widehat{p}}(\mathbb{R})$ is not large enough to compute $\xi(\cdot; A_+, A_-)$ by computing the left-hand side of (7.0.1). Therefore, in Section 7.2 we firstly enlarge the class of admissible function f so that the trace formula (7.0.1) holds for our specific choice of the additive constant in $\xi(\cdot; A_+, A_-)$.

In Section 7.3 we consider the example of differential operators on $L_2(\mathbb{R})$, namely

$$A_- = -i\frac{d}{dx}, \quad A_+ = \frac{d}{idx} + M_\varphi, \quad \operatorname{dom}(A_-) = \operatorname{dom}(A_+) = W^{1,2}(\mathbb{R}),$$

where $\varphi \in W^{1,1}(\mathbb{R}) \cap C_b(\mathbb{R})$. In this case the spectra of operators $A_\pm$ is purely absolutely continuous. We show that the spectral shift function $\xi(\cdot; \frac{d}{idx} + M_\varphi, \frac{d}{idx})$ is a constant function and

$$\xi(\nu; \frac{d}{idx} + M_\varphi, \frac{d}{idx}) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(x)\, dx \text{ for a.e. } \nu \in \mathbb{R}.$$

The result of this chapter are taken from [**26**]. We note that [**29**] presents an alternative approach for the computation of $\xi(\cdot; \frac{d}{idx} + M_\varphi, \frac{d}{idx})$ which uses tools from scattering theory.

In Section 7.4 we consider the example, when $A_- = \frac{d}{idx}$ is differentiation operator on $L_2[0, 2\pi]$ with twisted periodic boundary conditions and $A_+ = \frac{d}{idx} + M_\varphi$. This example is taken from [**28**]. Since this operator has purely discrete spectra, by computing the spectral shift function $\xi(\cdot; \frac{d}{idx} + M_\varphi, \frac{d}{idx})$ we also compute the spectral flow.

Finally, in Section 7.5 we consider the 'discrete differentiation' operator on $\ell_2(\mathbb{Z})$.

7.1. The Dirac operators

In this section we show that the main Hypothesis 3.5.1 is satisfied for the multidimensional Dirac operator on $\mathbb{R}^d$ and its perturbation given by multiplication operator.

Through this section we fix $d \in \mathbb{N}$. For each $k = 1, \dots, d$, define a self-adjoint operator in $L_2(\mathbb{R}^d)$ by

$$\partial_k = -i\frac{\partial}{\partial t_k}, \quad \mathrm{dom}(\partial_k) = W^{1,2}(\mathbb{R}^d).$$

Let $n(d) = 2^{\lceil \frac{d}{2} \rceil}$. Let $\gamma_k \in M_{n(d)}(\mathbb{C})$, $1 \le k \le d$, be Clifford algebra generators, that is,

(i) $\gamma_k = \gamma_k^*$ and $\gamma_k^2 = 1$ for $1 \le k \le d$.
(ii) $\gamma_{k_1}\gamma_{k_2} = -\gamma_{k_2}\gamma_{k_1}$ for $1 \le k_1, k_2 \le d$, such that $k_1 \ne k_2$.

Definition 7.1.1. *Define the Dirac operator as an unbounded operator $\mathcal{D}$ acting in the Hilbert space $\mathbb{C}^{n(d)} \otimes L_2(\mathbb{R}^d)$ with domain $\mathrm{dom}(\mathcal{D}) = \mathbb{C}^{n(d)} \otimes W^{1,2}(\mathbb{R}^d)$ by the formula*

$$\mathcal{D} = \sum_{k=1}^{d} \gamma_k \otimes \partial_k. \tag{7.1.1}$$

We have

$$\mathcal{D}^2 = -1 \otimes \Delta,$$

where $\Delta : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ is the Laplace operator (i.e. $\Delta = \sum_{k=1}^d \Delta_k$ with $\Delta_k = \frac{\partial^2}{\partial^2 x_k}$).

Suppose that $\Phi = \{\varphi_{ij}\}_{i,j=1}^{n(d)}$ is a matrix of function, such that $\varphi_{ij} \in L_\infty(\mathbb{R}^d)$. We denote by M_Φ the multiplication operator by Φ on the Hilbert space $\mathbb{C}^{n(d)} \otimes L_2(\mathbb{R}^d)$.

For this example we set

$$A_- = \mathcal{D}, \quad B_+ = M_\Phi$$

and

$$B(t) = \theta B_+,$$

where *theta* satisfies (3.5.1). In the rest of this section gradually impose assumptions on the matrix $\Phi = \{\varphi_{ij}\}$, such that the family $\{\theta(t)M_\Phi\}$ satisfies Hypothesis 3.5.6 with $p = d$. By Proposition 3.5.7 this ensures that Hypothesis 3.5.1 is also satisfied.

Recall (see Section 4.3) that the space $l_1(L_2)(\mathbb{R}^d)$ is defined as

$$l_1(L_2)(\mathbb{R}^d) := \left\{ f \in L_0(\mathbb{R}^d) \, : \, \sum_{n \in \mathbb{Z}^d} \|f\chi_{Q+n}\|_2 < \infty \right\},$$

with the corresponding norm

$$\|f\|_{l_1(L_2)(\mathbb{R}^d)} := \sum_{n \in \mathbb{Z}^d} \|f\chi_{Q+n}\|_2, \qquad f \in l_1(L_2)(\mathbb{R}^d).$$

Proposition 7.1.2. *Assume that $\Phi = \{\varphi_{ij}\}_{i,j=1}^{n(d)}$ is such that $\varphi_{ij} \in l_1(L_2)(\mathbb{R}^d)$. Then the operator M_Φ is a d-relative trace-class operators with respect to $\mathcal{D}$, that is Hypothesis 3.5.6 (ii) is satisfied.*

Proof. Since $\varphi_{ij} \in l_1(L_2)(\mathbb{R}^d)$, Theorem 4.3.1 implies that the matrix elements $M_{\varphi_{ij}}(-\Delta+1)^{-\frac{d+1}{2}}$ are trace class operators (on $L_2(\mathbb{R}^d)$) for all $1 \le i, j \le n(d)$. Hence, the operator

$$M_\Phi(-1\otimes\Delta+1)^{\frac{-d-1}{2}} = \Big(M_{\varphi_{ij}}(-\Delta+1)^{\frac{-d-1}{2}}\Big)_{i,j=1}^{n(d)}$$

is a trace class operator (on $\mathbb{C}^{n(d)} \otimes L_2(\mathbb{R}^d)$). Therefore, $M_\Phi(\mathcal{D}+i)^{-1-d} \in \mathcal{L}_1(\mathbb{C}^{n(d)} \otimes L_2(\mathbb{R}^d))$. □

In the next proposition we establish sufficient condition on the matrix $\Phi = \{\varphi_{ij}\}$ for Hypothesis 3.5.6.

Recall that (see (3.5.3)) the operator $L^j_{\mathcal{D}^2}, j \in \mathbb{N}$ is defined by

$$L^k_{\mathcal{D}^2}(T) = \overline{(1+A_-^2)^{-k/2}[A_-^2, T]^{(k)}}$$

with domain

$$\begin{aligned}\operatorname{dom}(L^k_{\mathcal{D}^2}) = \{T \in \mathcal{B}(\mathcal{H}) : T\operatorname{dom}(\mathcal{D}^j) \subset \operatorname{dom}(\mathcal{D}^j),\ j = 1,\dots,2k\\ \text{and the operator } (1+\mathcal{D}^2)^{-k/2}[\mathcal{D}^2, T]^{(k)} \text{ defined on } \operatorname{dom}(\mathcal{D}^{2k})\\ \text{extends to a bounded operator on } \mathcal{H}\}.\end{aligned}$$

Since $\mathcal{D}^2 = -1\otimes\Delta$, we have that

$$[\mathcal{D}^2, M_\Phi] = \Big([-\Delta, M_{\varphi_{ij}}]\Big)_{i,j=1}^{n(d)},$$

whenever the commutators $[-\Delta, M_{\varphi_{ij}}]$ are well-defined. Therefore, introducing $L^j_{-Delta}, j \in \mathbb{N}$ as in (3.5.3), we obtain that $M_\Phi \in \operatorname{dom}(L^k_{\mathcal{D}^2})$ for some $k \in \mathbb{N}$, provided that $M_{\varphi_{ij}} \in \operatorname{dom}(L^k_{-\Delta})$ for any $i, j = 1,\dots,n(d)$. In this case,

$$L^k_{\mathcal{D}^2}(M_\Phi) = \Big(L^k_{-\Delta}(M_{\varphi_{ij}})\Big)_{i,j=1}^{n(d)}. \tag{7.1.2}$$

Next, we give a condition on φ, which is sufficient for the inclusion $M_\varphi \in \bigcap_{j=1}^k \operatorname{dom}(L^j_{-\Delta})$.

Proposition 7.1.3. *Let $k \in \mathbb{N}$ be fixed. If $\varphi \in W^{2k,\infty}(\mathbb{R}^n)$, then $M_\varphi \in \bigcap_{j=1}^k \operatorname{dom}(L^j_{-\Delta})$.*

Proof. Let $k \in \mathbb{N}$ be fixed. Since $\varphi \in W^{2k,\infty}(\mathbb{R}^n)$, we have that $(\Delta)^j(\varphi\xi) \in L_2(\mathbb{R}^n)$ for every $\xi \in \operatorname{dom}(\Delta)^j$, $j = 1,\dots k$. That is $M_\varphi \operatorname{dom}(\Delta)^j \subset \operatorname{dom}(\Delta)^j$ for all $j = 1,\dots,2k$.

Recall that $\partial_k = \frac{\partial}{i\partial x_j}$ and if $\varphi \in L_\infty(\mathbb{R}^d)$ with $\frac{\partial\varphi}{\partial x_k} \in L_\infty(\mathbb{R}^d)$, $k = 1,\dots,d$, then $\varphi\operatorname{dom}(\partial_k) \subset \operatorname{dom}(\partial_k)$ and for all $\xi \in \operatorname{dom}(\partial_k)$ we have

$$[\partial_k, M_\varphi]\xi = \frac{1}{i}\frac{\partial\varphi}{\partial x_k}\xi, \quad k = 1,\dots,d. \tag{7.1.3}$$

By (7.1.3) we have

$$\begin{aligned}
[\Delta, M_\varphi] &= \sum_{j=1}^n [\partial_j^2, M_\varphi] = \sum_{j=1}^n \partial_j [\partial_j, M_\varphi] + \sum_{j=1}^n [\partial_j, M_\varphi]\partial_j \\
&= \frac{1}{i}\sum_{j=1}^n \Big(\partial_j M_{\frac{\partial\varphi}{\partial x_j}} + M_{\frac{\partial\varphi}{\partial x_j}}\partial_j\Big) = \frac{1}{i}\sum_{j=1}^n \Big(2\partial_j M_{\frac{\partial\varphi}{\partial x_j}} - [\partial_j, M_{\frac{\partial\varphi}{\partial x_j}}]\Big) \\
&= \frac{2}{i}\sum_{j=1}^n \partial_j M_{\frac{\partial\varphi}{\partial x_j}} + \sum_{j,\ell=1}^n M_{\frac{\partial^2\varphi}{\partial x_j \partial x_\ell}}.
\end{aligned}$$

Therefore,

$$(1-\Delta)^{-1/2}[\Delta, M_\varphi] = \frac{2}{i}\sum_{j=1}^n \partial_j (1-\Delta)^{-1/2} M_{\frac{\partial\varphi}{\partial x_j}} + \sum_{j,\ell=1}^n (1-\Delta)^{-1/2} M_{\frac{\partial^2\varphi}{\partial x_j \partial x_\ell}}.$$

Since $\varphi \in W^{2k,\infty}(\mathbb{R}^n)$, the operators $M_{\frac{\partial^2\varphi}{\partial x_j \partial x_\ell}}$ and $M_{\frac{\partial\varphi}{\partial x_j}}, j, \ell = 1, \dots, n$, are bounded. Since the operator $\partial_j(1-\Delta)^{-1/2}$ is also bounded, we infer that

$$\overline{(1-\Delta)^{-1/2}[\Delta, M_\varphi]} \in \mathcal{B}(L_2(\mathbb{R}^n)).$$

Continuing this process, we obtain that

$$\overline{(1-\Delta)^{-j}[\Delta, M_\varphi]^{(j)}} \in \mathcal{B}(L_2(\mathbb{R}^n)), \quad j = 1, \dots, k, \tag{7.1.4}$$

that is $M_\varphi \in \bigcap_{j=1}^{2k} \operatorname{dom}(L_{-\Delta}^j)$. □

Combining now Proposition 7.1.2 and 7.1.3 we arrive at the following

THEOREM 7.1.4. *Let $\mathcal{D}$ be the Dirac operator on $C^{n(d)} \otimes L_2(\mathbb{R}^d)$ defined by (7.1.1), $d \in \mathbb{N}$, and let $m = \lceil \frac{d}{2} \rceil$. Assume that $\Phi = \{\varphi_{ij}\}_{i,j=1}^{n(d)}$ is such that*

$$\varphi_{ij} \in l_1(L_2)(\mathbb{R}^d) \cap W^{4p,\infty}(\mathbb{R}^d), \quad i, j = 1, \dots, n(d).$$

Then the operator $A_- = \mathcal{D}$ and the perturbation $B_+ = M_\Phi$ satisfy Hypothesis 3.5.6 (and hence also Hypothesis 3.5.1) with $p = d$.

7.2. An auxiliary result

Recall that (see (4.2.5))

$$\operatorname{tr}\left(f(A_+) - f(A_-)\right) = \int_{\mathbb{R}} f'(\lambda) \cdot \xi(\lambda; A_+, A_-)\, d\lambda, \quad f \in \mathfrak{F}_{\widehat{p}}(\mathbb{R}), \tag{7.2.1}$$

and the spectral shift function $\xi(\cdot; A_+, A_-)$ is fixed by (4.2.3).

The examples we consider below are examples of one-dimensional differential operators and for these type of operators the formula above holds for a significantly larger class of functions f. We aim to use this fact to compute the spectral shift function, since the class of admissible function is large enough to compute $\xi(\cdot; A_+, A_-)$ by computing the left-hands side of (7.2.1).

The purpose of this section is to show that under some additional conditions suitable for one-dimensional differential operators we can extend the trace formula (7.2.1) to the required class of functions f keeping the choice of the spectral shift function $\xi(\cdot; A_+, A_-)$ as in (4.2.3).

PROPOSITION 7.2.1. *Let $A_\pm$, P_n and $A_{+,n}$, $n \in \mathbb{N}$, be as before (see* (3.1.5), (3.2.1) *and* (3.2.6)*). Assume that $g(A_+) - g(A_-) \in \mathcal{L}_1(\mathcal{H})$ and*

$$\|\cdot\|_1 - \lim_{n\to\infty} (g(A_{+,n}) - g(A_-)) = g(A_+) - g(A_-).$$

Then for any $F \in C_b^\infty(\mathbb{R})$ such that $F'(\lambda) \leq \mathrm{const}(1 + \lambda^{\widehat{p}+1})^{-1}, \lambda \in \mathbb{R}$, we have $F(A_+) - F(A_-) \in \mathcal{L}_1(\mathcal{H})$ and

$$\operatorname{tr}\big(F(A_+) - F(A_-)\big) = \int_{\mathbb{R}} F'(\lambda)\xi(\lambda; A_+, A_-)d\lambda,$$

where the spectral shift function $\xi(\cdot; A_+, A_-)$ is fixed by (4.2.3)*.*

PROOF. Since $F'(\lambda) \leq \mathrm{const}(1 + \lambda^{\widehat{p}+1})^{-1}, \lambda \in \mathbb{R}$, Corollary 4.2.4 implies that

$$\int_{\mathbb{R}} F'(\lambda)\xi(\lambda; A_+, A_-)d\lambda = \lim_{n\to\infty} \int_{\mathbb{R}} F'(\lambda)\xi(\lambda; A_{+,n}, A_-)d\lambda.$$

Since $B_{+,n} = A_{+,n} - A_-$ is a trace-class operator (see (3.2.7)) and $F \in C_b^\infty(\mathbb{R})$, the Krein trace formula (1.2.2) implies that

$$\int_{\mathbb{R}} F'(\lambda)\xi(\lambda; A_{+,n}, A_-)d\lambda = \operatorname{tr}\big(F(A_{+,n}) - F(A_-)\big).$$

Set $\psi = F \circ g^{-1}$. By the assumption on F, we have that ψ satisfies the assumptions of Theorem 2.1.6. Hence, the double operator integral $T_{\psi^{[1]}}^{g(A_{+,n}),g(A_-)}$ is bounded on $\mathcal{L}_1(\mathcal{H})$. Therefore,

$$\begin{aligned} F(A_{+,n}) - F(A_-) &= \psi(g(A_{+,n})) - \psi(g(A_-)) \\ &= T_{\psi^{[1]}}^{g(A_{+,n}),g(A_-)}\big(g(A_{+,n}) - g(A_-)\big) \end{aligned}$$

and similarly

$$F(A_+) - F(A_-) = T_{\psi^{[1]}}^{g(A_+),g(A_-)}\big(g(A_+) - g(A_-)\big)$$

In particular, we have that $F(A_+) - F(A_-) \in \mathcal{L}_1(\mathcal{H})$.

By the assumption on F, the function ψ satisfies the assumption of Proposition 2.3.1. Therefore, combining Proposition 2.3.1 implies that

$$\begin{aligned} &\|\cdot\|_1 - \lim_{n\to\infty} \big(F(A_{+,n}) - F(A_-)\big) \\ &\qquad = \|\cdot\|_1 - \lim_{n\to\infty} T_{\psi^{[1]}}^{g(A_{+,n}),g(A_-)}\big(g(A_{+,n}) - g(A_-)\big) \\ &\qquad = T_{\psi^{[1]}}^{g(A_+),g(A_-)}\big(g(A_+) - g(A_-)\big) \\ &\qquad = \psi(g(A_+)) - \psi(g(A_-)) = \big(F(A_+) - F(A_-)\big). \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \operatorname{tr}\big(F(A_{+,n}) - F(A_-)\big) = \operatorname{tr}\big(F(A_+) - F(A_-)\big).$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}} F'(\lambda)\xi(\lambda; A_+, A_-)d\lambda &= \lim_{n\to\infty} \int_{\mathbb{R}} F'(\lambda)\xi(\lambda; A_{+,n}, A_-)d\lambda \\ &= \lim_{n\to\infty} \operatorname{tr}\big(F(A_{+,n}) - F(A_-)\big) \\ &= \operatorname{tr}\big(F(A_+) - F(A_-)\big) \end{aligned}$$

as required. $\square$

In addition to Proposition 7.2.1 we prove here a lemma where a useful integral decomposition for the difference $g(A_+) - g(A_-)$ is given. This decomposition will be used whenever we show that the assumptions of Proposition 7.2.1 are satisfied for a specific choice of $A_\pm$. We used a similar integral decomposition ito prove Corollary 4.3.9 for the two-dimensional Dirac operator.

For brevity, we introduce (cf. (4.3.4)) the notations $R_{+,\lambda}(z)$, $R_{-,\lambda}(z)$ for appropriate resolvents of the operators A_+ and A_-, respectively, that is,

$$R_{+,\lambda} = \big(A_+ + i(1+\lambda)^{1/2}\big)^{-1}, \; R_{-,\lambda} = \big(A_- + i(1+\lambda)^{1/2}\big)^{-1}, \; \lambda > 0. \tag{7.2.2}$$

We also introduce

$$U_\lambda = (A_+ - A_-)R_{-,\lambda} = B_+ R_{-,\lambda}. \tag{7.2.3}$$

LEMMA 7.2.2. *Suppose that B_+ leaves the domain of A_- invariant. We have*

$$\begin{aligned} g(A_+) - g(A_-) &= -B_+(A_-^2 + 1)^{-3/2} \\ &+ \frac{1}{2\pi}\int_0^\infty \lambda^{-1/2}\big(R_{-,\lambda}[A_-, B_+]R_{-,\lambda}^2 + R_{-,\lambda}^*[A_-, B](R_{-,\lambda}^*)^2\big)d\lambda \\ &+ \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\mathrm{Re}\Big(R_{+,\lambda}U_\lambda^2\Big)d\lambda. \end{aligned}$$

PROOF. By Lemma 4.3.3 we have that

$$g(A_+) - g(A_-) = \frac{1}{\pi}\mathrm{Re}\Big(\int_0^\infty \lambda^{-1/2}[R_{+,\lambda} - R_{-,\lambda}]d\lambda\Big).$$

Using the resolvent identity twice one can write

$$\begin{aligned} R_{+,\lambda} - R_{-,\lambda} &= -R_{+,\lambda}B_+R_{-,\lambda} \\ &= -R_{-,\lambda}B_+R_{-,\lambda} + R_{+,\lambda}B_+R_{-,\lambda}B_+R_{-,\lambda} \\ &= -R_{-,\lambda}B_+R_{-,\lambda} + R_{+,\lambda}U_\lambda^2. \end{aligned}$$

Therefore, we have

$$R_{+,\lambda} - R_{-,\lambda} = -B_+R_{-,\lambda}^2 - [R_{-,\lambda}, B_+]R_{-,\lambda} + R_{+,\lambda}U_\lambda^2.$$

Applying the formula $[C^{-1}, B] = -C^{-1}[C, B]C^{-1}$ to the second term we obtain

$$R_{+,\lambda} - R_{-,\lambda} = -B_+R_{-,\lambda}^2 + R_{-,\lambda}[A_-, B_+]R_{-,\lambda}^2 + R_{+,\lambda}U_\lambda^2.$$

Similarly

$$\begin{aligned} \big(R_{+,\lambda} - R_{-,\lambda}\big)^* &= -R_{-,\lambda}^* + R_{+,\lambda}^* = -R_{-,\lambda}^*B_+R_{+,\lambda}^* \\ &= -R_{-,\lambda}^*B_+R_{-,\lambda}^* + R_{-,\lambda}^*B_+R_{-,\lambda}^*B_+R_{+,\lambda}^* \\ &= -R_{-,\lambda}^*B_+R_{-,\lambda}^* + (R_{+,\lambda}U_\lambda^2)^* \\ &= -B_+(R_{-,\lambda}^*)^2 - [B_+, R_{-,\lambda}^*]R_{-,\lambda}^* + (R_{+,\lambda}U_\lambda^2)^* \\ &= -B_+(R_{-,\lambda}^*)^2 + R_{-,\lambda}^*[A_-, B](R_{-,\lambda}^*)^2 + (R_{+,\lambda}U_\lambda^2)^*. \end{aligned}$$

Therefore, we have

$$\begin{aligned} g(A_+) - g(A_-) &= \frac{1}{2\pi}\int_0^\infty \lambda^{-1/2} B_+\big((R^*_{-,\lambda})^2 + (R_{-,\lambda})^2\big)d\lambda \\ &+ \frac{1}{2\pi}\int_0^\infty \lambda^{-1/2}\big(R_{-,\lambda}[A_-,B_+]R^2_{-,\lambda} + R^*_{-,\lambda}[A_-,B](R^*_{-,\lambda})^2\big)d\lambda \\ &+ \frac{1}{2\pi}\int_0^\infty \lambda^{-1/2}\big(R_{+,\lambda}U^2_\lambda + (R_{+,\lambda}U^2_\lambda)^*\big)d\lambda \\ &= \frac{1}{\pi}B_+\mathrm{Re}\Big(\int_0^\infty \lambda^{-1/2}(R_{-,\lambda})^2 d\lambda\Big) \\ &+ \frac{1}{2\pi}\int_0^\infty \lambda^{-1/2}\big(R_{-,\lambda}[A_-,B_+]R^2_{-,\lambda} + R^*_{-,\lambda}[A_-,B](R^*_{-,\lambda})^2\big)d\lambda \\ &+ \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\mathrm{Re}\Big(R_{+,\lambda}U^2_\lambda\Big)d\lambda. \end{aligned}$$

It follows from Lemma A.1 (i), that

$$\frac{1}{\pi}B_+\mathrm{Re}\Big(\int_0^\infty \lambda^{-1/2}(R_{-,\lambda})^2 d\lambda\Big) = -B_+(A_-^2+1)^{-3/2},$$

which suffices to complete the proof. □

7.3. Locally compact one dimensional example

In this section we compute the spectral shift function for the one dimensional operator $A_- = -i\frac{d}{dx}$ on $L_2(\mathbb{R})$ and its perturbation by multiplication operator. This example illustrates Theorem 6.2.4 in the case, when the operators $A_\pm$ has purely absolutely continuous spectra coinciding with the whole real line.

Throughout this section we assume that A_-, acting in the Hilbert space $L_2(\mathbb{R})$, is the self-adjoint operator

$$A_- = D = -i\frac{d}{dx}, \quad \mathrm{dom}(A_-) = W^{1,2}(\mathbb{R})$$

and its perturbation B_+ is given by the self-adjoint operator

$$B_+ = M_\varphi,$$

where, as before, M_φ denotes the multiplication operator by a bounded real-valued function on $\mathbb{R}$.

Subsequently, we will exploit the unitary equivalence of the operators $A_- = D$ and $A_+ = D + M_\varphi$. The following lemma establishes this fact.

Assuming that the function φ is locally integrable. We define the function

$$\psi(x) = \exp\big(-i\int_0^x \varphi(y)\,dy\big), \quad x\in\mathbb{R}. \tag{7.3.1}$$

LEMMA 7.3.1. *Assume that $\varphi \in C_b(\mathbb{R})$ and let ψ be defined by* (7.3.1). *We have*

$$M^*_\psi(D+M_\varphi)M_\psi = D. \tag{7.3.2}$$

PROOF. By the definition of the function ψ we have that

$$\psi' = -i\varphi\psi.$$

Therefore, we have

$$\begin{aligned}&\big([-i(d/dx) + M_\varphi]M_\psi\xi)(x)\\&\quad= -i\psi'(x)\xi(x) - i\psi\xi'(x) + i\varphi(x)\psi(x)\xi(x)\\&\quad= -i\psi\xi'(x) = (M_\psi D\xi)(x), \quad \xi \in C_0^\infty(\mathbb{R}),\end{aligned}$$

as required. □

COROLLARY 7.3.2. *Since the operators A_- and A_+ are self-adjoint, Lemma 7.3.1 and the functional calculus imply that*

$$h(D + M_\varphi) = M_\psi h(D) M_\psi^* \tag{7.3.3}$$

for any locally bounded Borel function $h\colon \mathbb{R} \to \mathbb{C}$.

It follows from Theorem 7.1.4 that Hypothesis 3.5.6 is satisfied (with $p = 1$) if we assume that $\varphi \in l_1(L_2)(\mathbb{R}) \cap W^{4,\infty}(\mathbb{R})$. Therefore, if we show that 0 is a right and left Lebesgue point for $\xi(\cdot; D+M_\varphi, D)$, the Witten index of the corresponding operator $\boldsymbol{D_A}$ can be computed via $\xi(\cdot; D + M_\varphi, D)$.

For convenience, we divide the exposition into several subsections.

7.3.1. On the difference $g(D + M_\varphi) - g(D)$. .

In this subsection, we show that the assumptions of Proposition 7.2.1 are satisfied for our choice of the operators $A_- = D$, $A_+ = D + M_\varphi$.

Recall (see (7.2.2)) that the operators $R_{+,\lambda}(z)$, $R_{-,\lambda}(z)$ are defined for the operators $A_+ = D + M_\varphi$ and $A_- = D$ by setting

$$R_{+,\lambda} = \big(D + M_\varphi + i(1+\lambda)^{1/2}\big)^{-1}, \ R_{-,\lambda} = \big(D + i(1+\lambda)^{1/2}\big)^{-1}, \ \lambda > 0.$$

Recall also (see (7.2.3))

$$U_\lambda = (A_+ - A_-)R_{-,\lambda} = M_\varphi R_{-,\lambda}.$$

The next result yields the first claim in Theorem 1.5.3 in the present setting. Here we do not resort to the double operator integration technique as in [**47**], but instead apply more elementary means.

PROPOSITION 7.3.3. *Suppose that $A_- = D$, $A_+ = D + M_\varphi$ with $\varphi \in W^{1,1}(\mathbb{R}) \cap C_b(\mathbb{R})$. We have that $g(A_+) - g(A_-) \in \mathcal{L}_1(L_2(\mathbb{R}))$ and*

$$\|g(D + M_\varphi) - g(D)\|_1 \le \|\varphi\|_{1,1}.$$

PROOF. To prove the first statement we use Lemma 7.2.2 to write

$$\begin{aligned}g(D + M_\varphi) - g(D) &= -M_\varphi(D^2+1)^{-3/2}\\&\quad+ \frac{1}{2\pi}\int_0^\infty \lambda^{-1/2}\big(R_{-,\lambda}[D, M_\varphi]R_{-,\lambda}^2 + R_{-,\lambda}^*[D, M_\varphi](R_{-,\lambda}^*)^2\big)d\lambda\\&\quad+ \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\mathrm{Re}\Big(R_{+,\lambda}U_\lambda^2\Big)d\lambda.\end{aligned} \tag{7.3.4}$$

First, we show that

$$\int_0^\infty \lambda^{-1/2} R_{+,\lambda} U_\lambda^2\, d\lambda \in \mathcal{L}_1(L_2(\mathbb{R})). \tag{7.3.5}$$

Employing the noncommutative Hölder inequality

$$\Big\| \int_0^\infty \lambda^{-1/2} R_{+,\lambda} U_\lambda^2 \, d\lambda \Big\|_1 \le \int_0^\infty \lambda^{-1/2} \big\| R_{+,\lambda} U_\lambda^2 \big\|_1 \, d\lambda$$
$$\le \int_0^\infty \lambda^{-1/2} \| R_{+,\lambda} \|_\infty \| U_\lambda \|_2^2 \, d\lambda. \tag{7.3.6}$$

Thus, applying Theorem 4.3.1 (i),

$$\| U_\lambda \|_2 = \big\| M_\varphi \big(D + i(\lambda+1)^{1/2} I \big)^{-1} \big\|_2$$
$$\le \text{const } \|\varphi\|_2 \|h_1\|_2 \le \text{ const } \|\varphi\|_1 \|\varphi\|_\infty \|h_1\|_2,$$

where $h_1(t) = (t + i(\lambda+1)^{1/2})^{-1}$. Since $\|h_1\|_2 = \text{const}(\lambda+1)^{-1/4}$, one infers that

$$\|U_\lambda\|_2 \le \text{ const } \|\varphi\|_1 \|\varphi\|_\infty (\lambda+1)^{-1/4}. \tag{7.3.7}$$

In addition, (see (7.2.2)) one has

$$\|R_{+,\lambda}\|_\infty \le \sup_{t \in \mathbb{R}} (|t + i(\lambda+1)^{1/2}|)^{-1} = (\lambda+1)^{-1/2}.$$

Hence, combining this estimate with (7.3.6) and (7.3.7), one obtains

$$\Big\| \int_0^\infty \lambda^{-1/2} R_{+,\lambda} U_\lambda^2 \, d\lambda \Big\|_1 \le \text{const } \|\varphi\|_1 \|\varphi\|_\infty \int_0^\infty \lambda^{-1/2} (\lambda+1)^{-1} \, d\lambda. \tag{7.3.8}$$

Since the integral on the right-hand side converges, the claim (7.3.5) follows.

Next, we show that

$$\int_0^\infty \lambda^{-1/2} R_{-,\lambda} [D, M_\varphi] R_{-,\lambda}^2 \, d\lambda \in \mathcal{L}_1\big(L_2(\mathbb{R})\big). \tag{7.3.9}$$

Since $[D, M_\varphi] = \frac{1}{i} M_{\varphi'}$ we have

$$\Big\| \int_0^\infty \lambda^{-1/2} R_{-,\lambda} [D, M_\varphi] R_{-,\lambda}^2 \, d\lambda \Big\|_1$$
$$\le \int_0^\infty \lambda^{-1/2} \big\| R_{-,\lambda} M_{\varphi'} R_{-,\lambda}^2 \big\|_1 \, d\lambda$$
$$\le \int_0^\infty \lambda^{-1/2} \big\| R_{-,\lambda} M_{|\varphi'|^{1/2}} \big\|_2 \big\| M_{|\varphi'|^{1/2}} R_{-,\lambda}^2 \big\|_2 \, d\lambda. \tag{7.3.10}$$

Since by assumption, $\varphi' \in L_1(\mathbb{R})$, Theorem 4.3.1 (i) implies that

$$\big\| R_{-,\lambda} M_{|\varphi'|^{1/2}} \big\|_2 \le \text{ const } \big\| |\varphi'|^{1/2} \big\|_2 \|h_1\|_2.$$

Arguing similarly, one obtains that

$$\big\| M_{|\varphi'|^{1/2}} R_{-,\lambda}^2 \big\|_2 \le \text{ const } \big\| |\varphi'|^{1/2} \big\|_2 \|h_2\|_2,$$

where $h_2(t) = (t + i(\lambda+1)^{1/2})^{-2}$. It is easy to check that $\|h_2\|_2 = C(\lambda+1)^{-3/4}$. Therefore, (7.3.9) is proved by estimating the right-hand side of (7.3.10) as follows

$$\Big\| \int_0^\infty \lambda^{-1/2} R_{-,\lambda} [D, M_\varphi] R_{-,\lambda}^2 \, d\lambda \Big\|_1 \le \text{const } \|\varphi'\|_1 \int_0^\infty \lambda^{-1/2} (\lambda+1)^{-1} \, d\lambda < \infty. \tag{7.3.11}$$

Using the same argument one can show that

$$\int_0^\infty \lambda^{-1/2} R_{-,\lambda}^* [D, M_\varphi] (R_{-,\lambda}^2)^* \, d\lambda \in \mathcal{L}_1\big(L_2(\mathbb{R})\big). \tag{7.3.12}$$

and

$$\Big\| \int_0^\infty \lambda^{-1/2} R_{-,\lambda}[D, M_\varphi](R^2_{-,\lambda})^* \, d\lambda \Big\|_1 \leq \text{const} \, \|\varphi'\|_1 \int_0^\infty \lambda^{-1/2}(\lambda+1)^{-1} \, d\lambda < \infty. \tag{7.3.13}$$

Finally, by Theorem 4.3.1 and Lemma 7.3.10 we have

$$\begin{aligned} \big\|M_\varphi\big(A_-^2+1\big)^{-3/2}\big\|_1 \leq & \ \text{const} \, \|\varphi\|_{\ell^1(L_2)(\mathbb{R})} \big\|((\cdot)^2 - z)^{-3/2}\big\|_{\ell^1(L^2(\mathbb{R}))} \\ \leq & \ \text{const} \, \|\varphi'\|_{1,1}. \end{aligned} \tag{7.3.14}$$

Thus, combining equality (7.3.4) with the estimates obtained in (7.3.8), (7.3.11), (7.3.13) and (7.3.14) imply that $g(A_+) - g(A_-)$ is a trace-class operator and

$$\|g(A_+) - g(A_-)\|_1 \leq \ C\,[\|\varphi\|_1\|\varphi\|_\infty + \|\varphi\|_{1,1}]. \tag{7.3.15}$$

To remove the term $\|\varphi\|_1\|\varphi\|_\infty$ from the estimate (7.3.15) we fix $l \in \mathbb{N}$ and write

$$\begin{aligned} &\|g(A_+) - g(A_-)\|_1 \\ &\quad = \Big\| \sum_{k=0}^{l-1} \big(g\big(A_- + \frac{k+1}{l}M_\varphi\big) - g\big(A_- + \frac{k}{l}M_\varphi\big)\big) \Big\|_1 \\ &\quad \leq \sum_{k=1}^{n-1} \big\|g\big(A_- + \frac{k+1}{l}M_\varphi\big) - g\big(A_- + \frac{k}{l}M_\varphi\big)\big\|_1. \end{aligned} \tag{7.3.16}$$

Applying Lemma 7.3.1 one obtains for fixed $k \in \mathbb{N}$ the existence of a sequence of unimodular functions $\psi_{k,l}$ such that $A_- + \frac{k}{l}M_\varphi = M_{\psi_{k,l}} A_- M^*_{\psi_{k,l}}$. Hence we have

$$\begin{aligned} A_- + \frac{k+1}{l}M_\varphi &= A_- + \frac{k}{l}M_\varphi + \frac{1}{l}M_\varphi \\ &= M_{\psi_{k,l}} A_- M^*_{\psi_{k,l}} + \frac{1}{l}M_\varphi \\ &= M_{\psi_{k,l}}\big(A_- + \frac{1}{l}M^*_{\psi_{k,l}} M_\varphi M_{\psi_{k,l}}\big) M^*_{\psi_{k,l}}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\Big\|g(A_- + \frac{k+1}{l}M_\varphi) - g(A_- + \frac{k}{l}M_\varphi)\Big\|_1 \\ &\quad = \Big\|g(A_-) - g(A_- + \frac{1}{l}M_{\psi^*_{k,l}\varphi\psi_{k,l}})\Big\|_1. \end{aligned}$$

Combining this with (7.3.16) and using that every $\psi_{k,l}$ is a unimodular function yields

$$\begin{aligned} &\|g(A_+) - g(A_-)\|_1 \\ &\quad \leq \sum_{k=0}^{l-1} \big\|g(A_-) - g\big(A_- + \frac{1}{l}M_{\psi^*_{k,l}\varphi\psi_{k,l}}\big)\big\|_1 \\ &\quad \leq \ \text{const} \sum_{k=0}^{l-1} \Big(\|\frac{1}{l}\psi^*_{k,l}\varphi\psi_{k,l}\|_\infty \|\frac{1}{l}\psi^*_{k,l}\varphi\psi_{k,l}\|_1 + \|\frac{1}{l}\psi^*_{k,l}\varphi\psi_{k,l}\|_{1,1}\Big) \\ &\quad \leq l \, \text{const}[\|\frac{1}{l}\varphi\|_\infty\|\frac{1}{l}\varphi\|_1 + \|\frac{1}{l}\varphi\|_{1,1}]. \end{aligned}$$

Hence,

$$\begin{aligned}
&\|g(A_+) - g(A_-)\|_1 \\
&\leq \ \text{const} \lim_{l\to\infty} l[\|\frac{1}{l}\varphi\|_\infty \|\frac{1}{l}\varphi\|_1 + \|\frac{1}{l}\varphi\|_{1,1}] \\
&\leq \text{const} \lim_{l\to\infty} l[l^{-2}\|\varphi\|_\infty\|\varphi\|_1 + l^{-1}\|\varphi\|_{1,1}] = \text{const}\,\|\varphi\|_{1,1},
\end{aligned}$$

as required. □

The following theorem is the main result of this subsection; it yields a trace norm approximation of the operator $[g(A_+) - g(A_-)]$ as required for Proposition 7.2.1.

Firstly, for the reduced operators we also define

$$R^{(n)}_{+,\lambda} = \big(A_{+,n} + i(\lambda - z)^{1/2} I\big)^{-1}, \quad U^{(n)}_\lambda = P_n U_\lambda P_n, \quad n \in \mathbb{N}.$$

THEOREM 7.3.4. *Suppose that $\varphi \in W^{1,1}(\mathbb{R}) \cap C_b(\mathbb{R})$. Then,*

$$\lim_{n\to\infty} \big\|[g(A_{+,n}) - g(A_-)] - [g(A_+) - g(A_-)]\big\|_1 = 0.$$

PROOF. Recall (see (7.3.4)) that

$$\begin{aligned}
g(D + M_\varphi) - g(D) = &-M_\varphi(D^2+1)^{-3/2} \\
&+ \frac{1}{2\pi}\int_0^\infty \lambda^{-1/2}\big(R_{-,\lambda}[D, M_\varphi]R^2_{-,\lambda} + R^*_{-,\lambda}[D, M_\varphi](R^*_{-,\lambda})^2\big)d\lambda \\
&+ \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\mathrm{Re}\Big(R_{+,\lambda}U^2_\lambda\Big)d\lambda.
\end{aligned}$$

Using similar argument we can write

$$\begin{aligned}
g(D + P_n M_\varphi P_n) - g(D) = &-P_n M_\varphi P_n (D^2+1)^{-3/2} \\
&+ \frac{1}{2\pi}\int_0^\infty \lambda^{-1/2}\big(P_n R_{-,\lambda}[D, M_\varphi]R^2_{-,\lambda}P_n + P_n R^*_{-,\lambda}[D, M_\varphi](R^*_{-,\lambda})^2 P_n\big)d\lambda \\
&+ \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\mathrm{Re}\Big(R^{(n)}_{+,\lambda}(U^{(n)}_\lambda)^2\Big)d\lambda.
\end{aligned}$$

To prove the stated convergence it suffices to show that we have trace norm convergence:

$$\begin{aligned}
&P_n M_\varphi\big(A^2_- + 1\big)^{-3/2}P_n \underset{n\to\infty}{\longrightarrow} M_\varphi\big(A^2_- + 1\big)^{-3/2}, \\
&P_n\int_0^\infty \lambda^{-1/2}R_{-,\lambda}[D, M_\varphi]R^2_{-,\lambda}\,d\lambda P_n \underset{n\to\infty}{\longrightarrow} \int_0^\infty \lambda^{-1/2}R_{-,\lambda}[D, M_\varphi]R^2_{-,\lambda}\,d\lambda, \\
&P_n\int_0^\infty \lambda^{-1/2}R^*_{-,\lambda}[D, M_\varphi](R^2_{-,\lambda})^*\,d\lambda P_n \underset{n\to\infty}{\longrightarrow} \int_0^\infty \lambda^{-1/2}R^*_{-,\lambda}[D, M_\varphi](R^2_{-,\lambda})^*\,d\lambda, \\
&\int_0^\infty \lambda^{-1/2}R^{(n)}_{+,\lambda}(U^{(n)}_\lambda)^2\,d\lambda \underset{n\to\infty}{\longrightarrow} \int_0^\infty \lambda^{-1/2}R_{+,\lambda}U^2_\lambda\,d\lambda.
\end{aligned} \tag{7.3.17}$$

By (7.3.11) the operator $M_\varphi(A^2_- + 1)^{-3/2}$ is a trace-class operator. Since $P_n \underset{n\to\infty}{\longrightarrow} 1$ in the strong operator topology, Lemma 3.2.1 implies convergence of the first term in (7.3.17). Similarly, using (7.3.13) and (7.3.14) instead of (7.3.11), we obtain the convergence of the second and third term in (7.3.17).

For the fourth term one obtains

$$\begin{aligned}\Big\| \int_0^\infty \lambda^{-1/2} R^{(n)}_{+,\lambda} (U^{(n)}_\lambda)^2 \, d\lambda - \int_0^\infty \lambda^{-1/2} R_{+,\lambda} U^2_\lambda \, d\lambda \Big\|_1 \\ \leq \int_0^\infty \lambda^{-1/2} \big\| R^{(n)}_{+,\lambda} (U^{(n)}_\lambda)^2 - R_{+,\lambda} U^2_\lambda \big\|_1 \, d\lambda,\end{aligned} \tag{7.3.18}$$

Since $U^{(n)}_\lambda = P_n U_\lambda P_n$ and $P_n \to 1$ in the strong operator topology, $U_\lambda \in \mathcal{L}_2(L_2(\mathbb{R}))$, Lemma 3.2.1 implies that

$$\|U^{(n)}_\lambda - U_\lambda\|_2 \to 0, \quad n \to \infty.$$

Therefore, $\lim_{n\to\infty} \big\|(U^{(n)}_\lambda)^2 - U^2_\lambda\big\|_1 = 0$.

Combining strong resolvent convergence of $A_{+,n}$ to A_+ as $n \to \infty$ (see Lemma 3.2.5), and trace norm convergence of $(U^{(n)}_\lambda)^2$ to U^2_λ as $n \to \infty$, Lemma 3.2.1 implies that the integrands on the right-hand side of (7.3.18) converge to zero. Since, in addition,

$$\big\|R^{(n)}_{+,\lambda}\big\|_\infty, \quad \|R_{+,\lambda}\|_\infty \leq (\lambda+1)^{-1/2},$$

and

$$\|U_\lambda\|_2^2 \leq (\lambda+1)^{-1/2},$$

one infers that $\lambda^{-1/2}\big\|R^{(n)}_{+,\lambda}(U^{(n)}_\lambda)^2 - R_{+,\lambda}U^2_\lambda\big\|_1$ is dominated by the integrable function $\lambda^{-1/2}(\lambda+1)^{-1}$. Thus, the dominated convergence theorem implies that

$$\int_0^\infty \lambda^{-1/2} \big\| R^{(n)}_{+,\lambda} (U^{(n)}_\lambda)^2 - R_{+,\lambda} U^2_\lambda \big\|_1 \, d\lambda \underset{n\to\infty}{\longrightarrow} 0,$$

as required. □

7.3.2. Computation of spectral shift function $\xi(\cdot; D + M_\varphi, D)$. In this subsection we compute explicitly the spectral shift function $\xi(\cdot; D + M_\varphi, D)$.

Firstly, we prove that the spectral shift function for the pair $(D + M_\varphi, D)$ is a constant function.

Proposition 7.3.5. *Let $\varphi \in W^{1,1}(\mathbb{R}) \cap C_b(\mathbb{R})$, $A_- = D$, $A_+ = A_- + M_\varphi$. Then*

$$\xi(\nu; A_+, A_-) = \text{const.} \ \textit{for a.e.}\ \nu \in \mathbb{R}.$$

Proof. Let h be such that h' is a Schwartz function. Then $h \circ g^{-1} \in C^2[-1,1]$, and hence by Proposition 7.2.1, $[h(A_+) - h(A_-)] \in \mathcal{L}_1\big(L_2(\mathbb{R})\big)$. We claim that

$$\operatorname{tr}(h(A_+) - h(A_-)) = \operatorname{tr}(h(A_+ + \alpha) - h(A_- + \alpha)), \quad \alpha \in \mathbb{R}. \tag{7.3.19}$$

As $D = -id/dx$ on $\operatorname{dom}(D) = W^{1,2}(\mathbb{R})$ is the generator of translations in $L_2(\mathbb{R})$, introducing $\psi_0 = e^{-i\alpha x}$, $\alpha \in \mathbb{R}$, yields $D + \alpha = M_{\psi_0} D M^*_{\psi_0}$ and hence,

$$h(D+\alpha) = M_{\psi_0} h(D) M^*_{\psi_0}, \quad h(D + M_\varphi + \alpha) = M_{\psi_0} h(D + M_\varphi) D M^*_{\psi_0}.$$

Consequently, by the unitary invariance of tr

$$\begin{aligned}\operatorname{tr}\big(h(D + M_\varphi + \alpha) - h(D+\alpha)\big) &= \operatorname{tr}\big(M_{\psi_0}(h(D+M_\varphi) - h(D))M^*_{\psi_0}\big) \\ &= \operatorname{tr}(h(D+M_\varphi) - h(D)),\end{aligned}$$

which proves (7.3.19). By Proposition (7.2.1) we have

$$\begin{aligned}\operatorname{tr}\big(h(D+M_\varphi+\alpha)-h(D+\alpha)\big) &= \int_{\mathbb{R}} h'(\nu+\alpha)\xi(\nu;D+M_\varphi,D)\,d\nu\\ &= \int_{\mathbb{R}} h'(\nu)\xi(\nu-\alpha;D+M_\varphi,D)\,d\nu.\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}} h'(\nu)[\xi(\nu-\alpha;D+M_\varphi,D)-\xi(\nu;D+M_\varphi,D)]\,d\nu = 0.$$

Since h' is an arbitrary Schwartz function, it follows by the Lemma of Du Bois-Reymond that

$$\xi(\nu-\alpha;D+M_\varphi,D)-\xi(\nu;D+M_\varphi,D)=0 \text{ for a.e. } \nu\in\mathbb{R}.$$

Since $\alpha\in\mathbb{R}$ was arbitrary, $\xi(\,\cdot\,;A_+,A_-)$ is constant a.e. on $\mathbb{R}$. □

By Proposition 7.3.5, the spectral shift function $\xi(\,\cdot\,;A_+,A_-)$ is constant a.e. on $\mathbb{R}$. In particular, we obtain that 0 is a right and a left Lebesgue point of $\xi(\,\cdot\,;A_+,A_-)$. In the rest of this subsection we achieve our principal goal, which is to compute the actual constant value of $\xi(\,\cdot\,;A_+,A_-)$.

In order to calculate the precise value of the constant $\xi(\,\cdot\,;A_+,A_-)$ we consider the auxiliary function $\arctan(\cdot)$. Since $\arctan'(t)=(1+t^2)^{-1}$, it follows from Proposition 7.2.1 that

$$[\arctan(A_+)-\arctan(A_-)]\in\mathcal{L}_1(L_2(\mathbb{R})), \tag{7.3.20}$$

and that for a.e. $\nu\in\mathbb{R}$,

$$\operatorname{tr}(\arctan(A_+)-\arctan(A_-)) = \int_{\mathbb{R}}\frac{\xi(\nu;A_+,A_-)}{\nu^2+1}\,d\nu = \pi\xi(\nu;A_+,A_-).$$

Equivalently,

$$\xi(\nu;A_+,A_-) = \frac{1}{\pi}\operatorname{tr}(\arctan(A_+)-\arctan(A_-)) \text{ for a.e. } \nu\in\mathbb{R}. \tag{7.3.21}$$

Thus, our task is the computing the value of the right-hand side in (7.3.21).

Given $\varphi\in W^{1,1}(\mathbb{R})\cap C_b(\mathbb{R})$, our aim is to represent the operator $[\arctan(A_+)-\arctan(A_-)]$ as an integral operator on $L_2(\mathbb{R})$ (cf. (7.3.30)). The unitary equivalence in (7.3.2) implies

$$\begin{aligned}\arctan(A_+)-\arctan(A_-) &= M_\psi\arctan(A_-)M_\psi^* - \arctan A_-\\ &= M_\psi\mathcal{F}^{-1}M_{\arctan}\mathcal{F}M_\psi^* - \mathcal{F}^{-1}M_{\arctan}\mathcal{F},\end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform on $L_2(\mathbb{R})$. Fix $\eta\in L_2(\mathbb{R})\cap L_1(\mathbb{R})$, then

$$(\mathcal{F}^{-1}\arctan(\cdot)\mathcal{F}\eta)(x) = (2\pi)^{-1}\int_{\mathbb{R}^2}\eta(x_1)\arctan(s_0)e^{-is_0(x_1-x)}\,ds_0dx_1. \tag{7.3.22}$$

We would like to identify the quantity on the right-hand side of (7.3.22) with the integral

$$(2\pi)^{-1/2}\int_{\mathbb{R}}\eta(s_1)(\mathcal{F}\arctan)(s_1-s)\,ds_1. \tag{7.3.23}$$

However, this identification is not possible straight away due to the fact that

$$(\mathcal{F}\arctan)(s) = \frac{1}{is}\mathcal{F}\Big(\frac{1}{1+x^2}\Big)(s) = \Big(\frac{\pi}{2}\Big)^{1/2}\frac{1}{is}e^{-|s|},$$

that is, the function $(\mathcal{F}\arctan)(s_1 - s)$ is discontinuous at the point $s_1 - s = 0$. Thus, we have to replace (7.3.23) by the principal value

$$\frac{1}{2i}\lim_{\varepsilon\to 0}\int_{|s_1-s|>\varepsilon}\frac{e^{-|s_1-s|}\eta(s_1)}{s_1-s}ds_1. \tag{7.3.24}$$

The identification of the right-hand sides of (7.3.22) and (7.3.24) will be done in Lemma 7.3.6 below.

Prior to proving that lemma we show that the principle value of $\frac{e^{-|x|}}{x}$ (in the sense of distributions), abbreviated by p.v.$\frac{e^{-|x|}}{x}$, is a tempered distribution, and hence the convolution on right-hand side of the equality in Lemma 7.3.6 is well-defined.

Since $\arctan(\cdot)$ is bounded, we may regard it as a tempered distribution (see, e.g., [**75**, Section I.3]). We consider the principle value of $\frac{e^{-|x|}}{x}$ introduced by the equality

$$\text{p.v.}\frac{e^{-|x|}}{x}(\eta) = \lim_{\varepsilon\downarrow 0}\int_{|x|>\varepsilon}\frac{e^{-|x|}\eta(x)}{x}\,dx, \quad \eta\in S(\mathbb{R}). \tag{7.3.25}$$

This is a tempered distribution since for arbitrary $\eta \in S(\mathbb{R})$,

$$\begin{aligned}\text{p.v.}\frac{e^{-|x|}}{x}(\eta) &= \lim_{\varepsilon\downarrow 0}\int_{\varepsilon<|x|<1}\frac{e^{-|x|}\eta(x)}{x}\,dx + \int_{|x|>1}\frac{e^{-|x|}\eta(x)}{x}\,dx \\ &= \lim_{\varepsilon\downarrow 0}\int_{\varepsilon<|x|<1}\frac{e^{-|x|}(\eta(x)-\eta(0))}{x}\,dx + \eta(0)\lim_{\varepsilon\downarrow 0}\int_{\varepsilon<|x|<1}\frac{e^{-|x|}}{x}\,dx \\ &\quad + \int_{|x|>1}\frac{e^{-|x|}\eta(x)}{x}\,dx,\end{aligned}$$

and since the next to last integral equals zero,

$$\left|\text{p.v.}\frac{e^{-|x|}}{x}(\eta)\right| \le \ \text{const.}\,[\|\eta'\|_\infty + \|\eta\|_\infty].$$

Thus, by [**75**, Section 1.3, Theorem 3.11], p.v.$\frac{e^{-|x|}}{x}$ is a tempered distribution.

The next lemma is crucial for our representation of the operator $\arctan(A_+) - \arctan(A_-)$ as an integral operator.

LEMMA 7.3.6. *Let* $D = -i\frac{d}{dx}$. *Then,*

$$(\arctan D)(\eta) = -\frac{1}{2i}\text{p.v.}\frac{e^{-|x|}}{x} * \eta, \quad \eta\in S(\mathbb{R}).$$

PROOF. For every $t > 0$, consider the function $Q_t : \mathbb{R} \to \mathbb{R}$ defined by

$$Q_t(x) = \frac{x}{t^2+x^2}e^{-|x|}, \quad x\in\mathbb{R}.$$

It is clear that $Q_t \in L_2(\mathbb{R})$, $t > 0$, and hence the Fourier transform of Q_t, $t > 0$, is also square-integrable.

One can consider the function Q_t as a tempered distribution [**75**, Section 3.3]. Next, we claim that

$$\lim_{t\downarrow 0} Q_t = \text{p.v.}\frac{e^{-|x|}}{x} \tag{7.3.26}$$

in the sense of tempered distributions, that is,

$$\lim_{t\downarrow 0} Q_t(\eta) = \text{p.v.}\frac{e^{-|x|}}{x}(\eta), \quad \eta \in S(\mathbb{R})$$

(see also a similar, but slightly different result in [**42**, Proposition 3.1]). Indeed, one can write $Q_t = \frac{1}{2}\big(\frac{1}{x+it} + \frac{1}{x-it}\big)e^{-|x|}$, and by the Sokhotski–Plemelj formulas (see, e.g., [**44**, p. 33–34]) obtain for every $\eta \in S(\mathbb{R})$,

$$\lim_{t\downarrow 0}\int_{\mathbb{R}} \frac{1}{x+it} e^{-|x|}\eta(x)\,dx = -i\pi\eta(0) + \text{p.v.}\int_{\mathbb{R}} \frac{e^{-|x|}\eta(x)}{x}\,dx,$$

and

$$\lim_{t\downarrow 0}\int_{\mathbb{R}} \frac{1}{x-it} e^{-|x|}\eta(x)\,dx = +i\pi\eta(0) + \text{p.v.}\int_{\mathbb{R}} \frac{e^{-|x|}\eta(x)}{x}\,dx,$$

that is,

$$\begin{aligned}\lim_{t\downarrow 0} Q_t(\eta) &= \lim_{t\downarrow 0}\int_{\mathbb{R}} \frac{1}{x+it} e^{-|x|}\eta(x)\,dx + \lim_{t\downarrow 0}\int_{\mathbb{R}} \frac{1}{x-it} e^{-|x|}\eta(x)\,dx\\ &= \text{p.v.}\frac{e^{-|x|}}{x}(\eta), \quad \eta \in S(\mathbb{R}).\end{aligned}$$

Next, standard properties of the Fourier transform imply

$$\begin{aligned}\mathcal{F}(Q_t)(s) &= \mathcal{F}\bigg(\frac{x}{t^2+x^2}e^{-|x|}\bigg)(s) = \frac{1}{(2\pi)^{1/2}}\bigg(\mathcal{F}\bigg(\frac{x}{t^2+x^2}\bigg) * \mathcal{F}(e^{-|x|})\bigg)(s)\\ &= i\frac{1}{(2\pi)^{1/2}}\bigg(\bigg(\mathcal{F}(\frac{1}{t^2+x^2})\bigg)' * \mathcal{F}(e^{-|x|})\bigg)(s)\\ &= -i\frac{1}{(2\pi)^{1/2}}\bigg((e^{-t|x|}\operatorname{sgn}(x)) * \frac{1}{1+x^2}\bigg)(s).\end{aligned}$$

Lebesgue's dominated convergence theorem implies

$$\begin{aligned}\lim_{t\downarrow 0}\mathcal{F}(Q_t)(s) &= -i\frac{1}{(2\pi)^{1/2}}\lim_{t\downarrow 0}\int_{\mathbb{R}} e^{-t|x|}\operatorname{sgn}(x)\frac{1}{1+(x-s)^2}\,dx\\ &= -i\frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}}\operatorname{sgn}(x)\frac{1}{1+(x-s)^2}\,dx = -\frac{2i}{(2\pi)^{1/2}}\arctan(s).\end{aligned} \tag{7.3.27}$$

In addition, since the Fourier transform is a continuous map of $S'(\mathbb{R})$ onto itself (see, e.g., [**71**, Theorem 7.15]),

$$\mathcal{F}\bigg(\text{p.v.}\frac{e^{-|x|}}{x}\bigg) = \mathcal{F}(\lim_{t\downarrow 0} Q_t) = \lim_{t\to 0}\mathcal{F}(Q_t),$$

in $S'(\mathbb{R})$, or equivalently,

$$\mathcal{F}\bigg(\text{p.v.}\frac{e^{-|x|}}{x}\bigg)(\eta) = -i\frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}}\big((e^{-t|x|}\operatorname{sgn}(x)) * (1+x^2)^{-1}\big)(s)\eta(s)\,ds,$$
$$\eta \in S(\mathbb{R}).$$

Since

$$\big\|(e^{-t|\cdot|}\operatorname{sgn}(\cdot)) * (1+|\cdot|^2)^{-1}\big\|_\infty \le \big\|(1+|\cdot|^2)^{-1}\big\|_1\big\|e^{-t|\cdot|}\operatorname{sgn}(\cdot)\big\|_\infty \le \pi$$

(see, e.g., [**75**, Section 1.1, Theorem 1.3]), and $\eta \in S(\mathbb{R})$, one infers that the integrand $((e^{-t|x|}\operatorname{sgn}(x)) * (1+x^2)^{-1})(\cdot)\eta(\cdot)$ is dominated by the integrable function

$\pi\eta(\cdot)$. Hence, by (7.3.27), applying once again Lebesgue's dominated convergence theorem, one arrives at

$$\mathcal{F}\bigg(\text{p.v.}\frac{e^{-|x|}}{x}\bigg)(\eta) = -\frac{2i}{(2\pi)^{1/2}}\int_{\mathbb{R}} \arctan(s)\eta(s)\,ds,$$

that is, the distribution $\mathcal{F}\big(\text{p.v.}\frac{e^{-|x|}}{x}\big)$ is, in fact, the function $-\frac{2i}{(2\pi)^{1/2}}\arctan(\cdot)$. Thus,

$$\mathcal{F}^{-1}\arctan(\cdot) = -\frac{(2\pi)^{1/2}}{2i}\text{p.v.}\frac{e^{-|x|}}{x}. \tag{7.3.28}$$

Finally, for an arbitrary $\eta \in S(\mathbb{R})$ by [**71**, Theorem 7.19] one obtains

$$\begin{aligned}(\arctan(D)\eta)(s) &= (\mathcal{F}^{-1}\arctan(\cdot)\mathcal{F}\eta)(s) = \mathcal{F}^{-1}(\arctan\cdot\mathcal{F}\eta)(s)\\ &= \frac{1}{(2\pi)^{1/2}}\big(\eta * \mathcal{F}^{-1}\arctan\big)(s) \overset{(7.3.28)}{=} -\frac{1}{2i}\bigg(\eta * \text{p.v.}\frac{e^{-|x|}}{x}\bigg)(s).\end{aligned}$$

□

For the special case where the operator D is perturbed by a Schwartz function $\varphi \in S(\mathbb{R})$, we also state the following result:

Corollary 7.3.7. *Let $\varphi \in S(\mathbb{R})$. Then the operator $A_+ = D + M_\varphi$ with $\operatorname{dom}(A_+) = \operatorname{dom}(D) = W^{1,2}(\mathbb{R})$, in $L_2(\mathbb{R})$ satisfies*

$$(\arctan A_+)\eta = -\frac{1}{2i}\psi\ \text{p.v.}\frac{e^{-|x|}}{x} * (\overline{\psi}\eta), \quad \eta \in S(\mathbb{R}).$$

Proof. Since φ is a Schwartz test function, $\psi(x) = \exp(-i\int_0^x \varphi(x')\,dx')$ is infinitely differentiable and $\overline{\psi}\eta \in S(\mathbb{R})$ for every $\eta \in S(\mathbb{R})$. Hence, one can write

$$(\arctan A_+)\eta = \psi\arctan(D)\psi\eta = \psi\cdot\arctan(D)(\overline{\psi}\eta),$$

and Lemma 7.3.6 completes the proof. □

Proposition 7.3.8. *Let $\varphi \in S(\mathbb{R})$ and introduce $A_- = D$, $A_+ = A_- + M_\varphi$, $\operatorname{dom}(A_\pm) = W^{1,2}(\mathbb{R})$, in $L_2(\mathbb{R})$. Then,*

$$\operatorname{tr}(\arctan(A_+) - \arctan(A_-)) = \frac{1}{2}\int_{\mathbb{R}}\varphi(x)\,dx. \tag{7.3.29}$$

Proof. To prove (7.3.29), let $\eta \in S(\mathbb{R})$. Combining Lemma 7.3.6 and Corollary 7.3.7 one infers,

$$\begin{aligned}&((\arctan(A_+) - \arctan(A_-))\eta)(y)\\ &\quad = -\frac{1}{2i}\bigg(\psi\ \text{p.v.}\frac{e^{-|x|}}{x} * (\overline{\psi}\eta)(y) - \text{p.v.}\frac{e^{-|x|}}{x} * (\eta)(y)\bigg)\\ &\quad = -\frac{1}{2i}\lim_{\varepsilon\downarrow 0}\int_{|x|>\varepsilon}\big(\psi(y)\overline{\psi(y-x)} - 1\big)\frac{e^{-|x|}}{x}\eta(y-x)\,dx\\ &\quad = -\frac{1}{2i}\lim_{\varepsilon\downarrow 0}\int_{|y-x|>\varepsilon}\big(\psi(y) - \psi(x)\big)\overline{\psi(x)}\,\frac{e^{-|y-x|}}{y-x}\eta(x)\,dx\\ &\quad = -\frac{1}{2i}\int_{\mathbb{R}}\overline{\psi(x)}\,\frac{\psi(y)-\psi(x)}{y-x}e^{-|y-x|}\eta(x)\,dx,\end{aligned}$$

where the last equality is due to continuity of $\overline{\psi(x)}\,\frac{\psi(y)-\psi(x)}{y-x}e^{-|y-x|}\eta(x)$ for all $x \in \mathbb{R}$ (given $y \in \mathbb{R}$).

Next, we will show that the preceding equality can be extended to arbitrary $\eta \in L_2(\mathbb{R})$ and thus

$$\big((\arctan(A_+) - \arctan(A_-))\eta\big)(y) = -\frac{1}{2i}\int_{\mathbb{R}} \overline{\psi(x)}\,\frac{\psi(y)-\psi(x)}{y-x}e^{-|y-x|}\eta(x)\,dx \tag{7.3.30}$$

holds. Since $S(\mathbb{R})$ is dense in $L_2(\mathbb{R})$, for every $\eta \in L_2(\mathbb{R})$ there exists a sequence $\{\eta_n\}_{n=1}^{\infty} \subset S(\mathbb{R})$, such that $\|\eta_n - \eta\|_2 \underset{n\to\infty}{\longrightarrow} 0$. On one hand,

$$\|(\arctan(A_+) - \arctan(A_-))(\eta_n - \eta)\|_2 \underset{n\to\infty}{\longrightarrow} 0,$$

since $[\arctan(A_+) - \arctan(A_-)] \in \mathcal{B}\big(L_2(\mathbb{R})\big)$. On the other hand, we claim that the integral operator K in $L_2(\mathbb{R})$ with integral kernel

$$K(x,y) = -\frac{1}{2i}\overline{\psi(x)}\,\frac{\psi(y)-\psi(x)}{y-x}e^{-|x-y|}$$

is a bounded operator on $L_2(\mathbb{R})$. By [**18**, Equation (2.2)] this will follow from the estimates

$$\|K(\,\cdot\,,\,\cdot\,)\|_{L_\infty(\mathbb{R};dx;L_1(\mathbb{R};dy))} < \infty, \quad \|K(\,\cdot\,,\,\cdot\,)\|_{L_\infty(\mathbb{R};dy,L_1(\mathbb{R};dx))} < \infty \tag{7.3.31}$$

(Bochner norms are used in this context). Since $|K(x,y)| = \frac{1}{2}\Big|\frac{\psi(y)-\psi(x)}{y-x}\Big|e^{-|y-x|}$, it is sufficient to estimate one of the two norms in (7.3.31). We estimate the norm of $\|K(\,\cdot\,,\,\cdot\,)\|_{L_\infty(\mathbb{R};dx;L_1(\mathbb{R};dy))}$ next:

$$\begin{aligned}\|K(\,\cdot\,,\,\cdot\,)\|_{L_\infty(\mathbb{R};dx;L_1(\mathbb{R};dy))} &= \frac{1}{2}\sup_{x\in\mathbb{R}}\int_{\mathbb{R}}\left|\frac{\psi(y)-\psi(x)}{y-x}\right|e^{-|y-x|}\,dy\\ &\le \frac{1}{2}\sup_{x\in\mathbb{R}}\int_{\mathbb{R}}\sup_{(x,y)\in\mathbb{R}^2}\left|\frac{\psi(y)-\psi(x)}{y-x}\right|e^{-|y-x|}\,dy\\ &\le \frac{1}{2}\|\psi'\|_\infty \sup_{x\in\mathbb{R}}\int_{\mathbb{R}} e^{-|y-x|}\,dy \le \|\psi'\|_\infty < \infty.\end{aligned}$$

Hence indeed, $K \in \mathcal{B}\big(L_2(\mathbb{R})\big)$ and $K\eta_n \underset{n\to\infty}{\longrightarrow} K\eta$ in $L_2(\mathbb{R})$. Thus, equality (7.3.30) holds for all $\eta \in L_2(\mathbb{R})$. Moreover, since the integral kernel $K(\,\cdot\,,\,\cdot\,)$ is continuous, an application of (7.3.30) yields

$$\begin{aligned}\operatorname{tr}(\arctan(A_+) - \arctan(A_-)) &= \int_{\mathbb{R}} K(x,x)\,dx\\ &= -\frac{1}{2i}\int_{\mathbb{R}}\psi'(x)\overline{\psi(x)}\,dx = \frac{1}{2}\int_{\mathbb{R}}\varphi(x)\,dx.\end{aligned}$$

□

By Proposition 7.3.8 and equality (7.3.21) we have

$$\xi(\nu; D + M_\varphi, D) = \frac{1}{2\pi}\int_{\mathbb{R}}\varphi(x)\,dx \ \text{ for a.e. } \nu \in \mathbb{R}, \tag{7.3.32}$$

provided that $\varphi \in S(\mathbb{R})$. The following theorem extends this result to an arbitrary $\varphi \in W^{1,1}(\mathbb{R}) \cap C_b(\mathbb{R})$.

THEOREM 7.3.9. *Assume that $\varphi \in W^{1,1}(\mathbb{R}) \cap C_b(\mathbb{R})$, $m \in \mathbb{N}$. Then,*

$$\xi(\nu; D + M_\varphi, D) = \frac{1}{2\pi}\int_{\mathbb{R}}\varphi(x)\,dx \ \textit{ for a.e. } \nu \in \mathbb{R}.$$

PROOF. Since $\varphi \in W^{1,1}(\mathbb{R})$, one concludes the existence of a sequence of Schwartz functions $\{\varphi_n\}_{n=1}^{\infty}$ such that $\|\varphi_n - \varphi\|_{1,1} \underset{n\to\infty}{\longrightarrow} 0$. By Lemma 7.3.1, $D + M_{\varphi_n} = M_{\psi_n} D M_{\psi_n}^*$, and

$$\begin{aligned} D + M_\varphi &= D + M_{\varphi_n} + M_{\varphi-\varphi_n} \\ &= M_{\psi_n}\big(D + M_{\psi_n}^*(M_{\varphi-\varphi_n})M_{\psi_n}\big)M_{\psi_n}^*, \end{aligned}$$

that is, $D+M_\varphi$ is unitarily equivalent to $D+M_{\psi_n}^*(M_{\varphi-\varphi_n})M_{\psi_n}$. Hence, applying Proposition 7.3.3,

$$\begin{aligned} &\big|\operatorname{tr}\big(g(D+M_\varphi)-g(D)\big)-\operatorname{tr}\big(g(D+M_{\varphi_n})-g(D)\big)\big| \\ &\quad\le \|g(D+M_\varphi)-g(D+M_{\varphi_n})\|_1 \\ &\quad= \big\|g\big(D+M_{\psi_n}^*(M_{\varphi-\varphi_n})M_{\psi_n}\big)-g(D)\big\|_1 \\ &\quad\le \text{ const.}\, \|\psi_n^*(\varphi-\varphi_n)\psi_n\|_{1,1} \le \text{ const.}\, \|\varphi-\varphi_n\|_{1,1} \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

that is,

$$\operatorname{tr}\big(g(D+M_\varphi)-g(D)\big) = \lim_{n\to\infty}\operatorname{tr}\big(g(D+M_{\varphi_n})-g(D)\big).$$

Since $\varphi_n \in S(\mathbb{R})$, $n\in\mathbb{N}$, Proposition 7.2.1 and equality (7.3.32) imply that

$$\begin{aligned} \operatorname{tr}\big(g(D+M_\varphi)-g(D)\big) &= \lim_{n\to\infty}\xi(\,\cdot\,;D+M_\varphi,D)\big(g(+\infty)-g(-\infty)\big) \\ &= \lim_{n\to\infty}\frac{1}{\pi}\int_{\mathbb{R}}\varphi_n(x)\,dx, \end{aligned}$$

Moreover, the convergence $\|\varphi_n-\varphi\|_{1,1} \underset{n\to\infty}{\longrightarrow} 0$ implies that $\int_{\mathbb{R}}\varphi_n(x)\,dx \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}}\varphi(x)\,dx$, that is,

$$\operatorname{tr}\big(g(D+M_\varphi)-g(D)\big) = \frac{1}{\pi}\int_{\mathbb{R}}\varphi(x)\,dx.$$

On the other hand, by Proposition 7.2.1 we have

$$\begin{aligned} \operatorname{tr}\big(g(D+M_\varphi)-g(D)\big) &= \int_{\mathbb{R}} g'(\nu)\xi(\nu;D+M_\varphi,D)d\nu \\ &= \xi(\nu;D+M_\varphi,D)\int_{\mathbb{R}} g'(\nu)d\nu = 2\xi(\nu;D+M_\varphi,D). \end{aligned}$$

Thus,

$$\xi(\nu;D+M_\varphi,D) = \frac{1}{2\pi}\int_{\mathbb{R}}\varphi(x)\,dx \;\text{ for a.e. } \nu\in\mathbb{R}.$$

□

7.3.3. Main result for the pair $(D+M_\varphi, D)$**.** Thus, we obtain that if $\varphi \in l_1(L_2)(\mathbb{R})\cap W^{4,\infty}(\mathbb{R})$, then by Theorem 6.2.3, the Witten index of the operator $\boldsymbol{D_A}$ can be computed via the spectral shift function $\xi(\cdot;D+M_\varphi,D)$. On the other hand, by Theorem 7.3.9 the spectral shift function $\xi(\cdot;D+M_\varphi,D)$ is a.e. equal to the constant $\frac{1}{2\pi}\int_{\mathbb{R}}\varphi(x)\,dx$ if $\varphi\in W^{1,1}(\mathbb{R})\cap C_b(\mathbb{R})$. Before we formulate the main result of this section we prove that the latter condition implies that $\varphi\in l_1(L_2)(\mathbb{R})$.

LEMMA 7.3.10. *If $f\in W^{1,p}(\mathbb{R})\cap C_b(\mathbb{R}), 1\le p<\infty$, then $f\in \ell^p(L_2)(\mathbb{R})$ and*

$$\|f\|_{\ell^p(L_2)(\mathbb{R})} \le C\,\|f\|_{1,p}.$$

PROOF. Since f is continuous, for every $n \in \mathbb{Z}$, there exists $x_n \in [n, n+1]$, such that

$$|f(x_n)| = \left(\int_n^{n+1} |f(x)|^p \, dx\right)^{1/p}.$$

For every $x \in [n, n+1]$, one has

$$\begin{aligned} |f(x)| &= \left| f(x_n) + \int_{x_n}^{x} f'(s) \, ds \right| \leq |f(x_n)| + \int_{x_n}^{x} |f'(s)| \, ds \\ &\leq \left(\int_n^{n+1} |f(s)|^p \, ds\right)^{1/p} + \int_n^{n+1} |f'(s)| \, ds \\ &\leq \left(\int_n^{n+1} |f(s)|^p \, ds\right)^{1/p} + \left(\int_n^{n+1} |f'(s)|^p \, ds\right)^{1/p} \\ &\leq 2^{1-1/p}\left(\int_n^{n+1} |f(s)|^p \, ds + \int_n^{n+1} |f'(s)|^p \, ds\right)^{1/p}. \end{aligned}$$

Thus,

$$\begin{aligned} \|f\|_{\ell^p(L_2)(\mathbb{R})} \leq \|f\|_{\ell^p(L_\infty)(\mathbb{R})} &= \sum_{n \in \mathbb{Z}} \sup_{x \in [n, n+1)} |f(x)|^p \\ &\leq \ C \sum_{n \in \mathbb{Z}} \left(\int_n^{n+1} |f(x)|^p \, dx + \int_n^{n+1} |f'(x)|^p \, dx\right)^{1/p} \\ &\leq \ C \, (\|f\|_p + \|f'\|_p), \end{aligned}$$

as required. □

Combining now Theorems 7.3.9 and 6.2.3 with Lemma 7.3.10 we obtain the following result.

THEOREM 7.3.11. *Suppose that $\varphi \in W^{1,1}(\mathbb{R}) \cap C_b(\mathbb{R}) \cap W^{4,\infty}(\mathbb{R})$. Let $A_- = D = -id/dx$ and let $B_+ = M_\varphi$. Then for the corresponding operator $\boldsymbol{D_A}$ the Witten index exists and equals*

$$W(\boldsymbol{D_A}) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(x) \, dx.$$

PROOF. Since $\varphi \in W^{1,1}(\mathbb{R}) \cap C_b(\mathbb{R})$, Lemma 7.3.10 implies that $\varphi \in l_1(L_2)(\mathbb{R})$. Therefore, $\varphi \in l_1(L_2)(\mathbb{R}) \cap W^{4,\infty}(\mathbb{R})$, which means that Theorem 6.2.3 hold for the operators $A_- = D$ and $B_+ = M_\varphi$. Hence,

$$W(\boldsymbol{D_A}) = \frac{1}{2}\big(\xi_L(0_+; D + M_\varphi, D) + \xi_L(0_-; D + M_\varphi, D)\big) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(x) \, dx,$$

where the last equality follows from Theorem 7.3.9. □

7.4. Compact one-dimensional example

In this section we consider compact one-dimensional example with $A_- = -i\frac{d}{dx}$ on $L_2[0, 2\pi]$ with twisted periodic boundary conditions and its perturbation B_+ given by multiplication operator.

Let $\alpha \in [0, 1)$ and let $D_\alpha = \frac{d}{idx}$ be the differentiation operator on $L_2[0, 2\pi]$ with twisted periodic boundary conditions, that is

$$\mathrm{dom}(D_\alpha) = \{\xi \in L_2[0, 2\pi] : \xi \in AC[0, 2\pi], \quad \xi(0) = e^{i2\pi\alpha}\xi(2\pi)\}.$$

Recall, the operator D_α has eigenvalues $\lambda_{\alpha,n}$ and eigenfunctions $e_{\alpha,n}$ given by

$$\lambda_{\alpha,n} = n - \alpha, \quad e_{\alpha,n}(t) = (2\pi)^{-1/2} e^{i\lambda_{\alpha,n}t}, \quad n \in \mathbb{Z}, \tag{7.4.1}$$

where every eigenvalue has multiplicity 1 (see e.g. [**76**, Section X.2]). In particular,

$$(D_\alpha + i)^{-1} \in \mathcal{L}_{1+\varepsilon}(L_2[0, 2\pi]), \quad \varepsilon > 0.$$

Therefore, for any bounded perturbation M_φ, $f \in L_\infty[0, 2\pi]$, we have that

$$M_\varphi(D_\alpha + i)^{-2} \in \mathcal{L}_1(L_2[0, 2\pi]),$$

which implies that the operator M_φ is 1-relative trace-class perturbation of D_α.

Recall also (see e.g. [**76**, Section X.2]), that the operator D_α is unitary equivalent to the operator $D_0 - \alpha$, namely

$$U_\alpha^* D_\alpha U_\alpha = D_0 - \alpha, \tag{7.4.2}$$

where the unitary operator U_α is given by multiplication on the function $s \mapsto e^{-i\alpha s}$, $s \in [0, 2\pi]$.

In contrast to the locally compact case (see Lemma 7.3.1), the operators D_α and $D_\alpha + M_\varphi$ are unitary equivalent only under an additional condition on φ.

Lemma 7.4.1. *If $\int_0^{2\pi} h(s)ds \in 2\pi\mathbb{Z}$, then the operators $D_\alpha + M_h$ and D_α are unitary equivalent. That is, for the unimodular function*

$$\psi(t) = \exp(-i \int_0^t h(s)ds), \quad t \in [0, 2\pi],$$

we have

$$M_\psi D_\alpha M_{\bar{\psi}} = D_\alpha + M_h.$$

Proof. We firstly note that since $\int_0^{2\pi} h(s)ds = 2\pi\mathbb{Z}$, we have

$$\psi(0) = \psi(2\pi).$$

Since, in addition, there exists $\psi' = -i\psi h$, we have that $\psi\xi \in \mathrm{dom}(D_\alpha)$ for all $\xi \in \mathrm{dom}(D_\alpha)$.

Now, for an arbitrary $\xi \in \mathrm{dom}(D_\alpha)$ we have,

$$[M_\psi, D_\alpha]\xi = M_\psi D_\alpha \xi - D_\alpha(\psi\xi) = \frac{1}{i}(\psi\xi' - (\xi\psi)') = \frac{-\psi'}{i}\xi = \psi h\xi = M_{\psi h}\xi.$$

Thus,

$$M_\psi D_\alpha - D_\alpha M_\psi = M_h M_\psi \text{ or, equivalently, } (D_\alpha + M_h)M_\psi = M_\psi D_\alpha$$

and the claim follows. □

Let $\varphi \in C^1[0, 2\pi]$ be an arbitrary function. We set

$$c = \frac{1}{2\pi}\int_0^{2\pi} \varphi(s)ds \quad \textit{and} \quad \varphi_0 := \varphi - c. \tag{7.4.3}$$

Then

$$\varphi = \varphi_0 + c, \quad \int_0^{2\pi} \varphi_0(s)ds = 0.$$

REMARK 7.4.2. It follows from Lemma 7.4.1 and (7.4.1) that for any $\varphi \in L_\infty[0,2\pi]$ we have

$$\begin{aligned}\sigma(D_\alpha + M_\varphi) &= \sigma(D_\alpha + M_{\varphi_0} + c) = \sigma(D_\alpha + c) = \{\lambda_{\alpha,n} + c, n \in \mathbb{Z}\} \\ &= \Big\{n - \alpha + c, n \in \mathbb{Z}\Big\},\end{aligned}$$

where every eigenvalue has multiplicity one.

REMARK 7.4.3. Of course, the full description of the spectra of D_α and $D_\alpha + M_\varphi$ is sufficient to know the jumps of spectral shift function $\xi(\cdot; D_\alpha + M_\varphi, D_\alpha)$. However, to compute the Witten index, the additive constant in $\xi(\cdot; D_\alpha + M_\varphi, D_\alpha)$ (which is fixed by (4.2.3)) has to be computed too. We do this using Proposition 7.2.1.

As we intend to use Proposition 7.2.1, we need to ensure that the assumptions of this proposition are satisfied.

PROPOSITION 7.4.4. *Let $\varphi \in C^1[0,2\pi]$ with $\varphi(0) = \varphi(2\pi)$. Then the operator $g(D_\alpha + M_\varphi) - g(D_\alpha)$ is trace-class. In addition, if $P_n = \chi_{[-n,n]}(D_\alpha)$ then*

$$\|\cdot\|_1 - \lim_{n\to\infty} \Big(g(D_\alpha + P_n M_\varphi P_n) - g(D_\alpha)\Big) = g(D_\alpha + M_\varphi) - g(D_\alpha).$$

PROOF. Using the decomposition obtained in Lemma 7.2.2 we can write

$$\begin{aligned}g(D_\alpha + M_\varphi) - g(D_\alpha) &= -M_\varphi(D_\alpha^2 + 1)^{-3/2} \\ &+ \frac{1}{2\pi}\int_0^\infty \lambda^{-1/2}\big(R_{-,\lambda}[D_\alpha, M_\varphi]R_{-,\lambda}^2 + R_{-,\lambda}^*[D_\alpha, M_\varphi](R_{-,\lambda}^*)^2\big)d\lambda \\ &+ \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\mathrm{Re}\Big(R_{+,\lambda}U_\lambda^2\Big)d\lambda.\end{aligned} \tag{7.4.4}$$

where, as before,

$$R_{+,\lambda} = \frac{1}{D_\alpha + M_\varphi + i(\lambda+1)^{1/2}}, \quad R_{-,\lambda} = \frac{1}{D_\alpha + i(\lambda+1)^{1/2}},$$
$$U_\lambda = M_\varphi R_{-,\lambda}, \quad \lambda > 0.$$

We note that both integrals on the right-hand side converges in the uniform norm.

The first term on the right hand side of (7.4.4) is a trace-class operator, since

$$\Big\|M_\varphi \frac{1}{(D_\alpha^2+1)^{3/2}}\Big\|_1 \le \|f\|_\infty \Big\|\frac{1}{(D_\alpha^2+1)^{3/2}}\Big\|_1 = \|f\|_\infty \sum_{n\in\mathbb{Z}} \frac{1}{((n-\alpha)^2+1)^{3/2}} < \infty.$$

For the second term in (7.4.4) we note that since the function f is differentiable and $\varphi(0) = \varphi(2\pi)$ we have that the commutator $[D_\alpha, M_\varphi]$ extends to a bounded operator on $L_2[0,2\pi]$ (which we still denote by $[D_\alpha, M_\varphi]$). Hence, we can estimate

$$\begin{aligned}
&\Big\| \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} R_{-,\lambda}[D_\alpha, M_\varphi] R_{-,\lambda}^2 \Big\|_1 \\
&\qquad \le \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} \|R_{-,\lambda}[D_\alpha, M_\varphi]\|_3 \|R_{-,\lambda}^2\|_{3/2} \\
&\qquad \le \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} \|[D_\alpha, M_\varphi]\|_\infty \Big\| \frac{1}{D_\alpha + i(\lambda+1)^{1/2}} \Big\|_3^3 \\
&\qquad \le \text{const} \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} \sum_{n\in\mathbb{Z}} \frac{1}{((n-\alpha)^2 + (1+\lambda))^{3/2}} \\
&\qquad \le \text{const} \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} (1+\lambda)^{-1} < \infty,
\end{aligned}$$

and therefore, $\int_0^\infty \frac{d\lambda}{\lambda^{1/2}} R_{-,\lambda}[D_\alpha, M_\varphi] R_{-,\lambda}^2 \in \mathcal{L}_1(L_2[0,2\pi])$. A similar argument shows that $\int_0^\infty \frac{d\lambda}{\lambda^{1/2}} R_{-,\lambda}^*[D_\alpha, M_\varphi](R_{-,\lambda}^2)^* \in \mathcal{L}_1(L_2[0,2\pi])$. That is, the second term in (7.4.4) is also a trace-class operator.

For the third term in (7.4.4) we have

$$\Big\| \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} R_{+,\lambda} U_\lambda^2 \Big\|_1 \le \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} \|R_{+,\lambda}\|_\infty \|U_\lambda\|_2^2$$

and since $\|R_{+,\lambda}\|_\infty \le \text{const}(1+\lambda)^{-1/2}$ and

$$\begin{aligned}
\|U_\lambda\|_2^2 &= \Big\| M_\varphi \frac{1}{D_\alpha + i(\lambda+1)^{1/2}} \Big\|_2^2 \le \|\varphi\|_\infty^2 \Big\| \frac{1}{D_\alpha + i(\lambda+1)^{1/2}} \Big\|_2^2 \\
&\le \text{const} \sum_{n\in\mathbb{Z}} \frac{1}{(n-\alpha)^2 + 1 + \lambda} \le \text{const}(1+\lambda)^{-1/2}.
\end{aligned}$$

we infer that

$$\Big\| \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} R_{+,\lambda} U_\lambda^2 \Big\|_1 \le \text{const} \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} (1+\lambda)^{-1} < \infty.$$

That is, the second term in (7.4.4) is also trace-class operator.

The proof of the convergence follows an argument similar to that of Proposition 7.3.3, and therefore is omitted. ☐

Remark 7.4.5. (i) We note that by equality (7.4.4) for a small $t \in \mathbb{R}$ we have that

$$g(D_\alpha + tM_\varphi) - g(D_\alpha) = tV + o^{\mathcal{L}_1}(t),$$

where

$$V = M_\varphi \frac{1}{(D_\alpha^2 + 1)^{3/2}} - \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} \big(R_{-,\lambda}[D_\alpha, M_\varphi] R_{-,\lambda}^2 + R_{-,\lambda}^*[D_\alpha, M_\varphi](R_{-,\lambda}^2)^* \big).$$

(ii) We note that we need that assumption that $f(0) = f(2\pi)$, since in the proof of Proposition 7.4.4 we use the commutator $[D_\alpha, M_\varphi]$.

By the assumption of Proposition 7.2.1 are satisfied, and therefore, we have that

$$\operatorname{tr}\big(F(D_\alpha + M_\varphi) - F(D_\alpha)\big) = \int_{\mathbb{R}} F'(\lambda) \xi(\lambda; D_\alpha + M_\varphi, D_\alpha) d\lambda \tag{7.4.5}$$

provided that F is such that $F' \in S(\mathbb{R})$. To compute the spectral shift function $\xi(\cdot, D_\alpha + M_\varphi, D_\alpha)$ we now compute the left hand of the equality above for F with $F' \in S(\mathbb{R})$.

Let $\varphi \in C^1[0, 2\pi]$. Recall that

$$c := \frac{1}{2\pi}\int_0^{2\pi} \varphi(s)ds \quad \textit{and} \quad \varphi_0 := \varphi - c$$

and

$$\varphi = \varphi_0 + c, \qquad \int_0^{2\pi} \varphi_0(s)ds = 0. \tag{7.4.6}$$

It is clear, that $\varphi_0 \in C^1[0, 2\pi]$ and $\varphi_0(0) = \varphi_0(2\pi)$.

Taking into account that $\varphi = \varphi_0 + c$, we write

$$\begin{aligned}\operatorname{tr}(F(D_\alpha + M_\varphi) - F(D_\alpha))& \\ &= \operatorname{tr}(F(D_\alpha + M_{\varphi_0} + c) - F(D_\alpha + M_{\varphi_0})) \\ &\quad + \operatorname{tr}(F(D_\alpha + M_{\varphi_0}) - F(D_\alpha)).\end{aligned}$$

Since $\int_0^{2\pi} \varphi_0(s)ds = 0$, Lemma 7.4.1 implies that the operator $D_\alpha + M_{\varphi_0}$ is unitary equivalent to the operator D_α. Therefore, we have

$$\begin{aligned}&\operatorname{tr}(F(D_\alpha + M_\varphi) - F(D_\alpha)) \\ &\quad = \operatorname{tr}(F(D_\alpha + c) - F(D_\alpha)) + \operatorname{tr}(F(D_\alpha + M_{\varphi_0}) - F(D_\alpha)).\end{aligned} \tag{7.4.7}$$

We claim that $\operatorname{tr}(F(D_\alpha + M_{\varphi_0}) - F(D_\alpha)) = 0$. Prior to prove this equality, we establish the following auxiliary result.

Lemma 7.4.6. *If $\Psi \in S(\mathbb{R})$, then $\Psi(D_\alpha) \in \mathcal{L}_1(L_2[0, 2\pi])$. Moreover, if $\varphi_0 \in C^1[0, 2\pi]$ is such that $\varphi_0(0) = \varphi_0(2\pi)$ and $\int_0^{2\pi} \varphi_0(s)ds = 0$, then $\operatorname{tr}(M_{\varphi_0}\Psi(D_\alpha)) = 0$.*

Proof. Since $\Psi \in S(\mathbb{R})$, there exists $C \geq 0$, such that $|\Psi(s)| \leq C(1 + s^2)^{-1}$, $s \in \mathbb{R}$. Therefore,

$$\|\Psi(D_\alpha)\|_1 = \sum_{n\in\mathbb{Z}} |\Psi(\lambda_n)| \leq C \sum_{n\in\mathbb{Z}} \frac{1}{1 + (n-\alpha)^2} < \infty,$$

that is $\Psi(D_\alpha) \in \mathcal{L}_1$.

By the unitary equivalence given in (7.4.2) we have

$$\operatorname{tr}(M_{\varphi_0}\Psi(D_\alpha)) = \operatorname{tr}\big(U_\alpha M_{\varphi_0}\Psi\big(D_0 - \alpha\big)U_\alpha^*\big) = \operatorname{tr}\big(M_{\varphi_0}\Psi\big(D_0 - \alpha\big)\big). \tag{7.4.8}$$

Denoting $\lambda_n := \lambda_{0,n}$ and $e_n := e_{0,n}$ (here, $\lambda_{0,n}$ and $e_{0,n}$ are given by (7.4.1) with $\alpha = 0$) for an arbitrary $\xi \in L_2[0, 2\pi]$ we can write

$$\begin{aligned}\Big(M_{\varphi_0}\Psi\big(D_0 - \alpha\big)\xi\Big)(t) &= \sum_{n_1\in\mathbb{Z}} \hat{\varphi}_0(n_1)e_{n_1} \cdot \sum_{n_2\in\mathbb{Z}} \Psi\big(\lambda_n - \alpha\big)\hat{\xi}(n_2)e_{n_2} \\ &= \sum_{n\in\mathbb{Z}} e_n \sum_{n_1+n_2=n} \Psi\big(\lambda_n - \alpha\big)\hat{\xi}(n_2)\hat{\varphi}_0(n_1),\end{aligned}$$

where $\hat{\eta}(n)$ denotes the n-th Fourier coefficient of a function $\eta \in L_2[0, 2\pi], \eta(0) = \eta(2\pi)$ and the series converge in $L_2[0, 2\pi]$ norm. That is

$$\Big(\widehat{M_{\varphi_0}\Psi(D_0 - \alpha)\xi}\Big)(n) = \sum_{n_2\in\mathbb{Z}} \hat{\xi}(n_2)\Psi\big(\lambda_n - \alpha\big)\hat{\varphi}_0(n - n_2).$$

Hence, the matrix elements $K(n, n_2)$ of the operator $M_{\varphi_0}\Psi(D-\alpha)$ are given by $K(n, n_2) = \Psi(\lambda_n - \alpha)\hat{\varphi}_0(n-n_2)$, and therefore,

$$\begin{aligned}\operatorname{tr}\Big(M_{\varphi_0}\Psi\big(D_0-\alpha\big)\Big) &= \sum_{n\in\mathbb{Z}} K(n,n) = \sum_{n\in\mathbb{Z}} \Psi\big(\lambda_n-\alpha\big)\hat{\varphi}_0(0)\\ &= \sum_{n\in\mathbb{Z}} \Psi\big(\lambda_n-\alpha\big)\int_0^{2\pi}\varphi_0(s)ds \overset{(7.4.6)}{=} 0,\end{aligned}$$

as required. $\square$

Lemma 7.4.7. *Let F be such that $F' \in S(\mathbb{R})$ and let $\varphi_0 \in C^1[0,2\pi]$, $\varphi_0(0) = \varphi_0(2\pi)$ with $\int_0^{2\pi}\varphi_0(s)ds = 0$. Then* $\operatorname{tr}(F(D_\alpha + M_{\varphi_0}) - F(D_\alpha)) = 0$.

Proof. We set

$$H(t) = F(D_\alpha + tM_{\varphi_0}) - F(D_\alpha), \quad t \in \mathbb{R}.$$

Since $t\varphi_0 \in C^1[0,2\pi], t\varphi_0(0) = t\varphi_0(2\pi)$ for all $t \in \mathbb{R}$, Proposition 7.4.4 implies that $H(t) \in \mathcal{L}_1(L_2[0,2\pi])$ for all $t \in \mathbb{R}$. We claim that the $\mathcal{L}_1$-valued function $H(t)$ is differentiable in $\mathcal{L}_1$-norm.

Let $t, t_0 \in \mathbb{R}$. Since $\int_0^{2\pi}\varphi_0(s)ds = 0$, it follows that $\int_0^{2\pi} t_0\varphi_0(s)ds = 0$ for all $t_0 \in \mathbb{R}$, in particular, the operator $D_\alpha + t_0M_{\varphi_0}$ is unitarily equivalent to the operator D_α via the operator $M_{\psi_{t_0}}$, where $\psi_{t_0}(\nu) = \exp(-i\int_0^s t_0\varphi_0(s)ds)$. In addition, since $M_{\psi_{t_0}}$ commutes with M_{φ_0}, we also have that $D_\alpha + t_0M_{\varphi_0} + (t-t_0)M_{\varphi_0} = M_{\psi_{t_0}}(D_\alpha + (t-t_0)M_{\varphi_0})M_{\bar{\psi}_{t_0}}$. Therefore,

$$\begin{aligned}H(t) - H(t_0) &= F(D_\alpha + tM_h) - F(D_\alpha + t_0M_h)\\ &= F\big(D_\alpha + t_0M_h + (t-t_0)M_h\big) - F(D_\alpha + t_0M_h)\\ &= M_{\psi_{t_0}}\Big(F(D_\alpha + (t-t_0)M_h) - F(D_\alpha)\Big)M_{\bar{\psi}_{t_0}} = M_{\psi_{t_0}}H(t-t_0)M_{\bar{\psi}_{t_0}}.\end{aligned}$$

Thus, it is sufficient to prove differentiability of $H(t)$ at $t_0 = 0$ only.

Let t be small. Setting $G = F \circ g^{-1}$, we have that the function G is infinitely differentiable on $[-1,1]$, in particular, Theorem 2.1.6 implies that $G(o^{\mathcal{L}_1}(t)) = o^{\mathcal{L}_1}(t)$. Now, by Remark 7.4.5 we have

$$H(t) = G\big(g(D_\alpha + tM_{\varphi_0})\big) - G(g(D_\alpha)) = G(g(D_\alpha) + tV) - G(g(D_\alpha)) + o^{\mathcal{L}_1}(t).$$

Since $V \in \mathcal{L}_1(L_2[0,2\pi])$ (see Proposition 7.4.4) we can write

$$H(t) = T^{g(D_\alpha),g(D_\alpha)}_{G^{[1]}}(tV) + o^{\mathcal{L}_1}(t).$$

Hence,

$$\frac{d}{dt}H(t)\Big|_{t=0} = T^{g(D_\alpha),g(D_\alpha)}_{G^{[1]}}(V),$$

that is, the function $H(t)$ is differentiable in $\mathcal{L}_1$-norm.

Next, Proposition 2.3.2 we have

$$\begin{aligned}\operatorname{tr}(F(D_\alpha + M_h) - F(D_\alpha)) &= \operatorname{tr}(H(1)) = \operatorname{tr}\Big(\int_0^1 \frac{d}{dt}H(t)dt\Big)\\ &= \int_0^1 \operatorname{tr}(T^{g(D_\alpha),g(D_\alpha)}_{G^{[1]}}(V))dt\\ &= \operatorname{tr}(T^{g(D_\alpha),g(D_\alpha)}_{G^{[1]}}(V)) = \operatorname{tr}(G'(g(D_\alpha))V) \qquad (7.4.9)\end{aligned}$$

where the last equality follows from (2.1.4) and (2.1.5).

It remains now to compute the trace $\mathrm{tr}(G'(g(D_\alpha))V)$. Recall that

$$V = M_\varphi \frac{1}{(D_\alpha^2+1)^{3/2}} - \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} \big(R_{-,\lambda}[D_\alpha, M_\varphi]R_{-,\lambda}^2 + R_{-,\lambda}^*[D_\alpha, M_\varphi](R_{-,\lambda}^2)^*\big),$$

where every separate term is a trace-class operator. Since the function $G' \circ g$ is a Schwartz function, it follows from Lemma 7.4.6 that $G'(g(D_\alpha)) \in \mathcal{L}_1(L_2[0, 2\pi])$. Therefore, again using Lemma 7.4.6 we have

$$\begin{aligned}\mathrm{tr}(G'(g(D_\alpha))V) &= \mathrm{tr}\,\big(G'(g(D_\alpha))M_{\varphi_0}\frac{1}{(D_\alpha^2+1)^{3/2}}\big) \\ &\quad - \int_0^\infty \frac{d\lambda}{\lambda^{1/2}}\,\mathrm{tr}\,\Big(G'(g(D_\alpha))R_{-,\lambda}[D_\alpha, M_{\varphi_0}]R_{-,\lambda}^2\Big) \qquad (7.4.10)\\ &\quad - \int_0^\infty \frac{d\lambda}{\lambda^{1/2}}\,\mathrm{tr}\,\Big(G'(g(D_\alpha))R_{-,\lambda}^*[D_\alpha, M_{\varphi_0}](R_{-,\lambda}^2)^*\Big).\end{aligned}$$

Since $G'(g(D_\alpha)) \in \mathcal{L}_1(L_2[0, 2\pi])$, we have that

$$tr\big(G'(g(D_\alpha))M_{\varphi_0}\frac{1}{(D_\alpha^2+1)^{3/2}}\big) = tr\big(M_{\varphi_0}G'(g(D_\alpha))\frac{1}{(D_\alpha^2+1)^{3/2}}\big) = 0,$$

where the last equality follows from Lemma 7.4.6. Using similar argument (while opening the commutator) one can show that the second and third terms in (7.4.10) are also 0. Hence, we infer from (7.4.9) that

$$\mathrm{tr}(F(D_\alpha + M_{\varphi_0}) - F(D_\alpha)) = 0.$$

□

Now, we are ready to prove compute explicitly the spectral shift function $\xi(\cdot; D_\alpha + M_\varphi, D_\alpha)$.

THEOREM 7.4.8. *Let* $D_\alpha = \frac{d}{idx}$ *with* α*-twisted periodic boundary conditions on* $[0, 2\pi]$, $\alpha \in [0, 2\pi)$, *and let* $\varphi \in C^1[0, 2\pi]$, $\varphi(0) = \varphi(2\pi)$, $c = \frac{\int_0^{2\pi}\varphi(s)ds}{2\pi}$. *Then*

$$\xi(\cdot; D_\alpha + M_\varphi, D_\alpha) = \mathrm{sgn}(c)\sum_{n\in\mathbb{Z}} \chi_{(\lambda_{\alpha,n}, \lambda_{\alpha,n}+c)} \ a.e.,$$

where notation $\chi_{(\lambda_{\alpha,n}, \lambda_{\alpha,n}+c)}$ *stands for characteristic function of the set* $(\lambda_{\alpha,n} + c, \lambda_{\alpha,n})$ *if* $c < 0$.

PROOF. Let F be an arbitrary function with $F' \in S(\mathbb{R})$. By (7.4.5) we have

$$\mathrm{tr}(F(D_\alpha + M_\varphi) - F(D_\alpha)) = \int_{\mathbb{R}} F'(s)\xi(s; D_\alpha + M_\varphi, D_\alpha)ds. \qquad (7.4.11)$$

By equality (7.4.7) and Lemma 7.4.7 we have that $\mathrm{tr}(F(D_\alpha + M_\varphi) - F(D_\alpha)) = \mathrm{tr}(F(D_\alpha + c) - F(D_\alpha))$, and therefore, it is sufficient to compute the trace $\mathrm{tr}(F(D_\alpha + c) - F(D_\alpha))$. We have

$$\begin{aligned}\mathrm{tr}(F(D_\alpha + c) - F(D_\alpha)) &= \sum_{n\in\mathbb{Z}}\Big(F(\lambda_{\alpha,n} + c) - F(\lambda_{\alpha,n})\Big) \\ &= \sum_{n\in\mathbb{Z}}\int_{\lambda_{\alpha,n}}^{\lambda_{\alpha,n}+c} F'(s)ds.\end{aligned}$$

Therefore, for an arbitrary F such that $F' \in S(\mathbb{R})$ we have

$$\int_{\mathbb{R}} F'(s)\xi(s, D_\alpha + M_\varphi, D_\alpha)ds = \sum_{n\in\mathbb{Z}} \int_{\lambda_{\alpha,n}}^{\lambda_{\alpha,n}+c} F'(s)ds \tag{7.4.12}$$
$$= \operatorname{sgn}(c) \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \chi_{(\lambda_{\alpha,n},\lambda_{\alpha,n}+c)}(s)F'(s)ds.$$

Since F is an arbitrary function with $F' \in S(\mathbb{R})$, it follows from (7.4.12) that

$$\xi(\cdot, D_\alpha + M_\varphi, D_\alpha) = \operatorname{sgn}(c) \sum_{n\in\mathbb{Z}} \chi_{(\lambda_{\alpha,n},\lambda_{\alpha,n}+c)} \quad \text{a.e..}$$

□

Having computed explicitly the spectral shift function $\xi(\cdot, D_\alpha + M_\varphi, D_\alpha)$ we now compute the Witten index and spectral flow.

Everywhere below we denote by $\lfloor\cdot\rfloor$ the floor function (that is, $\lfloor x\rfloor$ is the largest integer which less than or equal $x \in \mathbb{R}$) and $\{x\} = x - \lfloor x\rfloor, x \in \mathbb{R}$.

THEOREM 7.4.9. *Let $\varphi \in C^1[0, 2\pi]$, $\varphi(0) = \varphi(2\pi)$, let θ satisfies* (3.5.1) *and let $D_\alpha = \frac{d}{idx}$ on $L^2[0, 2\pi]$ with α-twisted periodic boundary conditions, $\alpha \in [0, 2\pi)$. Introduce*

$$c = \frac{1}{2\pi}\int_0^{2\pi} \varphi(s)ds.$$

Then the Witten index $W(\boldsymbol{D_A})$ exists and we have the following

(i) *If $\alpha = 0$, then the operator $\boldsymbol{D_A}$ is not Fredholm for any φ and*

$$W(\boldsymbol{D_A}) = \begin{cases} \lfloor c\rfloor, & \text{if } c \in \mathbb{Z} \\ \lfloor c\rfloor + \frac{1}{2}\operatorname{sgn}(c), & \text{otherwise} \end{cases}.$$

(ii) *If $\alpha \neq 0$, then the operator $\boldsymbol{D_A}$ is Fredholm if and only if $c \notin \alpha + \mathbb{Z}$. In this case*

$$W(\boldsymbol{D_A}) = \operatorname{index}(\boldsymbol{D_A}) = \begin{cases} \lfloor c\rfloor + \operatorname{sgn}(c), & \{c\} > \alpha \\ \lfloor c\rfloor, & \{c\} < \alpha. \end{cases}.$$

If $c \in \alpha + \mathbb{Z}$, then the operator $\boldsymbol{D_A}$ is not Fredholm and

$$W(\boldsymbol{D_A}) = \lfloor c\rfloor + \frac{1}{2}\operatorname{sgn}(c).$$

PROOF. If $\alpha = 0$, then $0 \in \sigma(D)$, hence, by Theorem 3.1.7 the operator $\boldsymbol{D_A}$ is not Fredholm. If $\alpha \neq 0$, then $0 \notin \sigma(D_\alpha)$, and therefore $\boldsymbol{D_A}$ is Fredholm if and only if $0 \notin \sigma(D_\alpha + M_\varphi)$. By Remark 7.4.2 we have that

$$\sigma(D_\alpha + M_\varphi) = \Big\{n - \alpha + c, n \in \mathbb{Z}\Big\}.$$

Hence $0 \in \sigma(D_\alpha + M_\varphi)$ if and only if $c \in \alpha + \mathbb{Z}$. Thus, if $\alpha \neq 0$, the operator $\boldsymbol{D_A}$ is Fredholm if and only if $c \notin \alpha + \mathbb{Z}$.

Now, we turn to computing the index of the operator $\boldsymbol{D_A}$. It follows from Theorem 7.4.8 that the spectral shift function $\xi(\,\cdot\,; D_\alpha + M_\varphi, D_\alpha)$ is piecewise constant, and therefore, 0 is a right and a left Lebesgue point of $\xi(\cdot\,; D_\alpha + M_\varphi, D_\alpha)$. By Theorem 6.2.3 the Witten index $W(\boldsymbol{D_A})$ exists and equals

$$\begin{aligned} W(\boldsymbol{D_A}) &= [\xi_L(0_+; D_\alpha + M_\varphi, D_\alpha) + \xi_L(0_-; D_\alpha + M_\varphi, D_\alpha)]/2 \\ &= [\xi(0_+; D_\alpha + M_\varphi, D_\alpha) + \xi(0_-; D_\alpha + M_\varphi, D_\alpha)]/2. \end{aligned} \tag{7.4.13}$$

By Theorem 7.4.8 we have

$$\xi(\cdot; D_\alpha + M_\varphi, D_\alpha) = \lfloor c \rfloor + \operatorname{sgn}(c) \sum_{n\in\mathbb{Z}} \chi_{(\lambda_{\alpha,n}, \lambda_{\alpha,n}+\{c\})},$$

and therefore

$$\xi(0_\pm; D_\alpha + M_\varphi, D_\alpha) = \lfloor c \rfloor + \operatorname{sgn}(c)\chi_{(\lambda_{\alpha,0}, \lambda_{\alpha,0}+\{c\})}(0_\pm). \tag{7.4.14}$$

Now we consider the cases $\alpha = 0$ and $\alpha \neq 0$ separately. Assume first that $\alpha = 0$, then by (7.4.14) we have

$$\xi(0_\pm; D_\alpha + M_\varphi, D_\alpha) = \lfloor c \rfloor + \operatorname{sgn}(c)\chi_{(0,\{c\})}(0_\pm).$$

Thus,

$$\xi(0_+; D_\alpha + M_\varphi, D_\alpha) = \begin{cases} \lfloor c \rfloor + \operatorname{sgn}(c), & \{c\} \neq 0, \\ \lfloor c \rfloor, & \text{otherwise} \end{cases}.$$

and

$$\xi(0_-; D_\alpha + M_\varphi, D_\alpha) = \lfloor c \rfloor.$$

Hence, for the case, when $\alpha = 0$ we infer from (7.4.13) the following

$$W(\boldsymbol{D_A}) = \begin{cases} \lfloor c \rfloor, & \text{if } c \in \mathbb{Z} \\ \lfloor c \rfloor + \dfrac{1}{2}\operatorname{sgn}(c), & \text{otherwise} \end{cases}.$$

Now, let $\alpha \neq 0$. Then by (7.4.14) we have

$$\xi(0_\pm, D_\alpha + M_\varphi, D_\alpha) = \lfloor c \rfloor + \operatorname{sgn}(c)\chi_{(-\alpha, -\alpha+\{c\})}(0\pm).$$

Thus,

$$\xi(0_+, D_\alpha + M_\varphi, D_\alpha) = \begin{cases} \lfloor c \rfloor, & \{c\} \leq \alpha \\ \lfloor c \rfloor + \operatorname{sgn}(c), & \{c\} > \alpha \end{cases},$$

and

$$\xi(0_-, D_\alpha + M_\varphi, D_\alpha) = \begin{cases} \lfloor c \rfloor, & \{c\} < \alpha \\ \lfloor c \rfloor + \operatorname{sgn}(c), & \{c\} \geq \alpha \end{cases}.$$

Combining these two equalities with equality (7.4.13) we obtain the following precise value of the Witten index of the operator $\boldsymbol{D_A}$

$$W(\boldsymbol{D_A}) = \begin{cases} \lfloor c \rfloor, & \{c\} < \alpha \\ \lfloor c \rfloor + \dfrac{1}{2}\operatorname{sgn}(c), & \{c\} = \alpha \\ \lfloor c \rfloor + \operatorname{sgn}(c), & \{c\} > \alpha \end{cases}$$

□

Remark 7.4.10. (i) It follows from Theorem 7.4.9, that if $\alpha \neq 0$ and $c \notin \alpha+\mathbb{Z}$, then we are in the purely Fredholm situation (i.e. the operator $D_\alpha + M_\varphi, D_\alpha$ and $\boldsymbol{D_A}$ are Fredholm) with discrete spectra as in [**70**].

(ii) It is worth noting that in this special compact case the Witten index $W(\boldsymbol{D_A})$ can take only half-integer values, while for the locally compact case (i.e. the operator $D = \frac{d}{idx}$ acts on $L^2(\mathbb{R})$) the Witten index $W(\boldsymbol{D_A})$ could be any real number (see [**29**], [**26**]). $\diamond$

Finally, we would like to discuss connections with spectral flow. By Theorem 6.3.9 we have that

$$\mathrm{sf}(D_\alpha, D_\alpha + M_\varphi) = \xi(0, D_\alpha + M_\varphi, D_\alpha) + \frac{1}{2}\big(\operatorname{tr}(\ker(D_\alpha + M_\varphi)) - \operatorname{tr}(\ker(D_\alpha))\big),$$

where the value of the spectral shift function $\xi(\cdot, D_\alpha + M_\varphi, D_\alpha)$ at discontinuity points is defined as a half-sum of the left and the right limits.

We again consider the cases $\alpha = 0$ and $\alpha \neq 0$ separately. First let $\alpha = 0$. It is clear that $\dim(\ker(D_\alpha)) = 1$. By Remark 7.4.2 we have

$$\dim(\ker(D_\alpha + M_\varphi)) = \dim(\ker(D_\alpha + c)) = \begin{cases} 1, & \text{if } c \in \mathbb{Z} \\ 0, & \text{otherwise.} \end{cases}$$

Thus, combining these equalities with Theorem 7.4.9 we obtain

$$\mathrm{sf}(D_\alpha, D_\alpha + M_\varphi) = \begin{cases} \lfloor c \rfloor, & c \in \mathbb{Z} \\ \lfloor c \rfloor + \frac{1}{2}\operatorname{sgn}(c) - \frac{1}{2} & \text{otherwise.} \end{cases}$$

Now, let $\alpha \neq 0$. If $c \neq \alpha + \mathbb{Z}$, then $0 \in \rho(D_\alpha) \cap \rho(D_\alpha + M_\varphi)$, and therefore, $\operatorname{tr}(\ker(D_\alpha + M_\varphi)) = \operatorname{tr}(\ker(D_\alpha)) = 0$. In addition, by Theorem 7.4.9 we are in the Fredholm situation and have the equality

$$W(\boldsymbol{D_A}) = \operatorname{index}(\boldsymbol{D_A}) = \mathrm{sf}(D_\alpha, D_\alpha + M_\varphi),$$

which is consistent with the result of [**47**, Theorem 9.13]. If $c = \alpha + \mathbb{Z}$, then $\operatorname{tr}(\ker(D_\alpha + M_\varphi)) = 1$, and by Theorem 7.4.9 we obtain

$$\mathrm{sf}(D_\alpha, D_\alpha + M_\varphi) = \lfloor c \rfloor + \frac{1}{2}\operatorname{sgn}(c) + 1/2.$$

Thus, we have the following

THEOREM 7.4.11. *Let $\varphi \in C^1[0, 2\pi]$, $\varphi(0) = \varphi(2\pi)$, let θ satisfies* (3.5.1) *and let $D_\alpha = \frac{d}{idx}$ on $L^2[0, 2\pi]$ with α-twisted periodic boundary conditions, $\alpha \in [0, 2\pi)$. Introduce*

$$c = \frac{1}{2\pi}\int_0^{2\pi} \varphi(s)ds.$$

Then

(i) If $\alpha = 0$, then

$$\mathrm{sf}(D_\alpha, D_\alpha + M_\varphi) = \begin{cases} \lfloor c \rfloor, & c \in \mathbb{Z} \\ \lfloor c \rfloor + \frac{1}{2}\operatorname{sgn}(c) - \frac{1}{2} & \textit{otherwise.} \end{cases}.$$

(ii) If $\alpha \neq 0$, then

$$\mathrm{sf}(D_\alpha, D_\alpha + M_\varphi) = \begin{cases} \lfloor c \rfloor, & \{c\} < \alpha \\ \lfloor c \rfloor + \frac{1}{2}\operatorname{sgn}(c) + \frac{1}{2}, & \{c\} = \alpha \\ \lfloor c \rfloor + \operatorname{sgn}(c), & \{c\} > \alpha \end{cases}$$

7.5. Discrete differentiation operator

In this section we consider the easiest example of discrete (one-dimensional) differentiation operator. This example is essentially commutative and therefore, the argument is substantially simpler.

Let d be 'differentiation' operator on $l_2(\mathbb{Z})$ given by multiplication operator by the sequence $\{n\}_{n\in\mathbb{Z}}$, that is

$$d(y) = \{ny_n\}_{n\in\mathbb{Z}}, \quad y = \{y_n\}_{n\in\mathbb{Z}} \in \mathrm{dom}(d)$$
$$\mathrm{dom}(d) = \{x = \{x_n\}_{n\in\mathbb{Z}} \in l_2(\mathbb{Z}) : \{nx_n\}_{n\in\mathbb{Z}} \in l_2(\mathbb{Z})\}.$$

It is clear that

$$\sigma(d) = \{n : n \in \mathbb{Z}\}, \tag{7.5.1}$$

where every eigenvalue has multiplicity one.

Since d acts by multiplication by the sequence $\{n\}_{n\in\mathbb{Z}}$, it follows that d commutes with any operator M_x given by multiplication by a bounded sequence $x \in l_\infty(\mathbb{Z})$, that is

$$[d, M_x] = 0. \tag{7.5.2}$$

As before, in order to compute the spectral shift function $\xi(\cdot; d + M_x, d)$, we verify firstly that Proposition 7.2.1 holds in this setting. The proof of this fact can be proved by an agriment similar to Proposition 7.4.4. We omit the proof.

PROPOSITION 7.5.1. *Let $x \in l_\infty(\mathbb{Z})$. Then the operator $g(d + M_x) - g(d)$ is trace-class and*

$$\|\cdot\|_1 - \lim_{n\to\infty} (g(d + P_n M_x Pn) - g(d)) = g(d + M_x) - g(d).$$

Hence, by Proposition 7.2.1 we infer that

$$\mathrm{tr}(F(d + M_x) - F(d)) = \int_{\mathbb{R}} \xi(s; d + M_x, d)F'(s)d(s) \tag{7.5.3}$$

for any F with $F' \in S(\mathbb{R})$. The following proposition gives explicit formula for the spectral shift function $\xi(\cdot; d + M_x, d)$.

PROPOSITION 7.5.2. *Let d be 'differentiation' operator on $l_2(\mathbb{Z})$ acting by multiplication on the sequence $\{n\}_{n\in\mathbb{Z}}$ and let $x = \{x_n\} \in l_\infty(\mathbb{Z})$. Then*

$$\xi(\cdot; d + M_x, d) = \sum_{n\in\mathbb{Z}} \mathrm{sgn}(x_n)\chi_{(n,n+x_n)} \ a.e.,$$

where the notation $\chi_{(n,n+x_n)}, n \in \mathbb{Z}$ stands for characteristic function of the set $(n + x_n, n)$ if $x_n \le 0$.

PROOF. As usual, in order to compute the spectral shift function $\xi(\cdot, d + M_x, d)$ we compute the trace $\mathrm{tr}(F(d+M_x)-F(d))$ on the left-hand side of (7.5.3) for an arbitrary admissible function F (that is, F is such that $F' \in S(\mathbb{R})$). We have

$$\mathrm{tr}(F(d + M_x) - F(d)) = \sum_{n\in\mathbb{Z}} \big(F(n + x_n) - F(n)\big) = \sum_{n\in\mathbb{Z}} \int_n^{n+x_n} F'(s)ds.$$

Therefore, for an arbitrary F such that $F' \in S(\mathbb{R})$ we have

$$\int_{\mathbb{R}} F'(s)\xi(s; d+M_x, d)ds \overset{(7.5.3)}{=} \sum_{n\in\mathbb{Z}} \int_n^{n+x_n} F'(s)ds \tag{7.5.4}$$
$$= \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \operatorname{sgn}(x_n)\chi_{(n,n+x_n)}(s)F'(s)ds.$$

Since F is an arbitrary function with $F' \in S(\mathbb{R})$, it follows from (7.4.12) that

$$\xi(\cdot; d+M_x, d) = \sum_{n\in\mathbb{Z}} \operatorname{sgn}(x_n)\chi_{(n,n+x_n)} \quad \text{a.e..}$$

□

Proposition 7.5.3. *Let $x \in l_\infty(\mathbb{Z})$, let θ satisfies* (3.5.1) *and let d be the operator on $l_2(\mathbb{Z})$ be the operator acting by multiplication on the sequence $\{n\}_{n\in\mathbb{Z}}$. Then the operator $\boldsymbol{D_A}$ is not Fredholm, the Witten index $W(\boldsymbol{D_A})$ exists and*

$$W(\boldsymbol{D_A}) = \sum_{\substack{x_n:|x_n|>-n\\ n<0}} \operatorname{sgn}(x_n) + \frac{1}{2}\sum_{\substack{x_n:|x_n|=-n\\ n<0}} \operatorname{sgn}(x_n) + \frac{1}{2}\operatorname{sgn}(x_0).$$

Proof. Since $0 \in \sigma(d)$, Theorem 3.1.7 implies that the operator $\boldsymbol{D_A}$ is not Fredholm.

Now, we turn to computing the index of the operator $\boldsymbol{D_A}$. It follows from Proposition 7.5.2 that the spectral shift function $\xi(\,\cdot\,; d+M_x, d)$ is piecewise constant, and therefore, 0 is a right and a left Lebesgue point of $\xi(\,\cdot\,; d+M_x, d)$. By Theorem 6.2.3 the Witten index $W(\boldsymbol{D_A})$ exists and equals

$$W(\boldsymbol{D_A}) = [\xi_L(0_+; d+M_x, d) + \xi_L(0_-; d+M_x, d)]/2$$
$$= [\xi(0_+; d+M_x, d) + \xi(0_-; d+M_x, d)]/2. \tag{7.5.5}$$

For the right-limit at zero we have that

$$\xi(0_+; d+M_x, d) = \operatorname{card}\{x_n, n<0 : x_n > -n\} - \operatorname{card}\{x_n, n>0 : x_n \le -n\} + \delta_+, \tag{7.5.6}$$

where $\delta_+ = 1$ if $x_0 > 0$ and $\delta_+ = 0$ if $x_0 \le 0$. For the left-limit at zero we have

$$\xi(0_-; d+M_x, d) = \operatorname{card}\{x_n, n<0 : x_n \ge -n\} - \operatorname{card}\{x_n, n>0 : x_n < -n\} - \delta_-, \tag{7.5.7}$$

where $\delta_- = 1$ if $x_0 < 0$ and $\delta_- = 0$ if $x_0 \ge 0$. Since the sequence $x = \{x_n\}_{n\in\mathbb{Z}}$ is bounded, it follows that cardinality of all sets above are finite.

Therefore,

$$W(\boldsymbol{D_A}) = \operatorname{card}\{x_n, n<0 : x_n > -n\} - \operatorname{card}\{x_n, n>0 : x_n < -n\}$$
$$+ \frac{1}{2}\Big(\operatorname{card}\{x_n, n<0 : x_n = -n\} - \operatorname{card}\{x_n, n>0 : x_n = -n\}\Big)$$
$$\frac{1}{2}\operatorname{sgn}(x_0).$$

□

Finally, we would like to discuss connections with spectral flow.

By Theorem 6.3.9 we have

$$\operatorname{sf}(d, d+M_x) = \xi(0, d+M_x, d) + \frac{1}{2}\big(\operatorname{tr}(\ker(d+M_x)) - \operatorname{tr}(\ker(d))\big),$$

where the value of spectral shift function $\xi(\cdot, d+M_x, d)$ at discontinuity points is defined as a half-sum of the left and the right limits.

It is clear that $\dim(\ker(d)) = 1$ and

$$\dim(\ker(d+M_x)) = \dim(\ker(M_{\{n+x_n\}_{n\in\mathbb{Z}}})) = \operatorname{card}\{x_n : x_n = -n, n \in \mathbb{Z}\}.$$

Since the sequence $x = \{x_n\}_{n\in\mathbb{Z}}$ is bounded, it follows that $\dim(\ker(d+M_x)) < \infty$.

Thus, combining these equalities with (7.5.6) and (7.5.7) we obtain

$$\begin{aligned}\operatorname{sf}(d, d+M_x) &= \operatorname{card}\{x_n, n<0 : x_n > -n\} - \operatorname{card}\{x_n, n>0 : x_n < -n\} \\ &\quad + \frac{1}{2}\Big(\operatorname{card}\{x_n, n<0 : x_n = -n\} - \operatorname{card}\{x_n, n>0 : x_n = -n\}\Big) \\ &\quad + \frac{1}{2}\operatorname{sgn}(x_0) + \frac{1}{2}(\operatorname{card}\{x_n : x_n = -n, n\in\mathbb{Z}\} - 1) \\ &= \operatorname{card}\{x_n, n<0 : x_n > -n\} - \operatorname{card}\{x_n, n>0 : x_n < -n\} \\ &\quad + \operatorname{card}\{x_n, n<0 : x_n = -n\} + \frac{1}{2}(\delta_{x_0} + \operatorname{sgn}(x_0) - 1).\end{aligned}$$

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APPENDIX A

Explicit computation of some integrals

In the appendix we explicitly compute the Bochner integrals used in Section 4.3 and in 7.3.

Let $\mathcal{D}$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$. Recall that, when $\alpha > -1$, $\beta > \alpha + 1$, we have

$$\int_0^\infty \lambda^\alpha (1 + \lambda + \mathcal{D}^2)^{-\beta}\, d\lambda = B(\alpha + 1; \beta - \alpha - 1)(1 + \mathcal{D}^2)^{\alpha - \beta + 1}, \tag{A.1}$$

where $B(\cdot\,;\cdot)$ denotes the Beta function

We denote (cf. (4.3.4))

$$R_{0,\lambda} = (\mathcal{D} + i(1 + \lambda)^{1/2})^{-1}, \quad \lambda > 0.$$

A simple computation shows that

$$\begin{aligned} R_{0,\lambda} &= \frac{\mathcal{D}}{1 + \lambda + \mathcal{D}^2} - i\frac{(1 + \lambda)^{1/2}}{1 + \lambda + \mathcal{D}^2}, \\ R_{0,\lambda}^2 &= \frac{\mathcal{D}^2 - (1 + \lambda)}{(1 + \lambda + \mathcal{D}^2)^2} - 2i\frac{(1 + \lambda)^{1/2}\mathcal{D}}{(1 + \lambda + \mathcal{D}^2)^2}, \end{aligned} \tag{A.2}$$

LEMMA A.1. *We have*

(i)

$$\frac{1}{2}\int_0^\infty \frac{d\lambda}{\lambda^{1/2}} \mathrm{Re}(R_{0,\lambda}^2) = -\pi(1 + \mathcal{D}^2)^{-3/2}$$

(ii)

$$\int_0^\infty \frac{d\lambda}{\lambda^{1/2}} \mathrm{Re}(R_{0,\lambda}^3) = -\frac{3\pi}{2}\mathcal{D}(1 + \mathcal{D}^2)^{-5/2}.$$

PROOF. To prove the first equality using (A.2) we obtain

$$\begin{aligned} \frac{1}{2}\int_0^\infty \frac{d\lambda}{\lambda^{1/2}} \mathrm{Re}(R_{0,\lambda}^2) &= \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} \Big(\frac{\mathcal{D}^2 - (1 + \lambda)}{(1 + \lambda + \mathcal{D}^2)^2}\Big) \\ &= (\mathcal{D}^2 - 1)\int_0^\infty \lambda^{-1/2}(1 + \lambda + \mathcal{D}^2)^{-2}\, d\lambda - \int_0^\infty \lambda^{1/2}(1 + \lambda + \mathcal{D}^2)^{-2}\, d\lambda \\ &\overset{(A.1)}{=} \frac{\pi}{2}(\mathcal{D}^2 - 1)(1 + \mathcal{D}^2)^{-3/2} - \frac{\pi}{2}(1 + \mathcal{D}^2)^{-1/2} = -\pi(1 + \mathcal{D}^2)^{-3/2}. \end{aligned}$$

For the second equality, appealing to the definition of $R_{0,\lambda}$ (see (4.3.4)), and by computing of the real and imaginary parts of the complex number $\big(t + i(1 + \lambda)^{1/2}\big)^{-3}$, we obtain that

$$\mathrm{Re}(R_{0,\lambda}^3) = \big(\mathcal{D}^3 - 3\mathcal{D}(1 + \lambda)\big)(1 + \lambda + \mathcal{D}^2)^{-3}.$$

Consequently, applying (A.1), we obtain the identity

$$\begin{aligned}
&\Big(\int_0^\infty \frac{d\lambda}{\lambda^{1/2}} \mathrm{Re}(R_{0,\lambda}^3)\Big) \\
&\quad = (\mathcal{D}^3 - 3\mathcal{D}) \int_0^\infty \lambda^{-1/2}(1+\lambda+\mathcal{D}^2)^{-3}\,d\lambda - 3\mathcal{D}\int_0^\infty \lambda^{1/2}(1+\lambda+\mathcal{D}^2)^{-3}\,d\lambda \\
&\quad = \frac{3\pi}{8}(\mathcal{D}^3-3\mathcal{D})(1+\mathcal{D}^2)^{-5/2} - \frac{3\pi}{8}\mathcal{D}(1+\mathcal{D}^2)^{-3/2} = -\frac{3\pi}{2}\mathcal{D}(1+\mathcal{D}^2)^{-5/2}.
\end{aligned}$$

□

Now, let $\mathcal{D}$ denote the two-dimensional Dirac operator on $\mathbb{R}^2$, see (7.1.1).

Lemma A.2. *We have*

$$\begin{aligned}
&\int_0^\infty \frac{d\lambda}{\lambda^{1/2}} \big(R_{0,\lambda}(\gamma_k\otimes 1)R_{0,\lambda}^2 + R_{0,\lambda}^*(\gamma_k\otimes 1)(R_{0,\lambda}^*)^2\big) \\
&\quad = \frac{\pi}{2}[\mathcal{D},\gamma_k\otimes 1](1+\mathcal{D}^2)^{-3/2} - \frac{3\pi}{2}\{\mathcal{D},\gamma_k\otimes 1\}(1+\mathcal{D}^2)^{-5/2},
\end{aligned}$$

where $\{\cdot,\cdot\}$ denotes the anticommutator.

Proof. Fixing $k = 1, 2$, by expanding and cancelling similar terms, we obtain

$$\begin{aligned}
&R_{0,\lambda}(\gamma_k\otimes 1)R_{0,\lambda}^2 + R_{0,\lambda}^*(\gamma_k\otimes 1)(R_{0,\lambda}^*)^2 \\
&\quad \overset{(A.2)}{=} \Big(\frac{\mathcal{D}}{1+\lambda+\mathcal{D}^2} - i\frac{(1+\lambda)^{1/2}}{1+\lambda+\mathcal{D}^2}\Big)(\gamma_k\otimes 1)\times \\
&\qquad\qquad \times\Big(\frac{\mathcal{D}^2-(1+\lambda)}{(1+\lambda+\mathcal{D}^2)^2} - 2i\frac{(1+\lambda)^{1/2}\mathcal{D}}{(1+\lambda+\mathcal{D}^2)^2}\Big) \\
&\qquad + \Big(\frac{\mathcal{D}}{1+\lambda+\mathcal{D}^2} + i\frac{(1+\lambda)^{1/2}}{1+\lambda+\mathcal{D}^2}\Big)(\gamma_k\otimes 1)\times \\
&\qquad\qquad \times\Big(\frac{\mathcal{D}^2-(1+\lambda)}{(1+\lambda+\mathcal{D}^2)^2} + 2i\frac{(1+\lambda)^{1/2}\mathcal{D}}{(1+\lambda+\mathcal{D}^2)^2}\Big) \\
&\quad = 2\frac{\mathcal{D}}{1+\lambda+\mathcal{D}^2}(\gamma_k\otimes 1)\frac{\mathcal{D}^2-(1+\lambda)}{(1+\lambda+\mathcal{D}^2)^2} \\
&\qquad - 4(1+\lambda)\frac{1}{1+\lambda+\mathcal{D}^2}(\gamma_k\otimes 1)\frac{\mathcal{D}}{(1+\lambda+\mathcal{D}^2)^2} \\
&\quad = \frac{2\mathcal{D}(\gamma_k\otimes 1)(\mathcal{D}^2-1) - 2\mathcal{D}(\gamma_k\otimes 1)\lambda - 4(\gamma_k\otimes 1)\mathcal{D}(1+\lambda)}{(1+\lambda+\mathcal{D}^2)^3},
\end{aligned}$$

where in the last line we used the fact that $\mathcal{D}^2$ commutes with $\gamma_k \otimes 1$. Hence, for $k = 1, 2$, by (A.1), we get that

$$
\begin{aligned}
&\frac{1}{2}\int_0^\infty \frac{d\lambda}{\lambda^{1/2}}\big(R_{0,\lambda}(\gamma_k \otimes 1)R_{0,\lambda}^2 + R_{0,\lambda}^*(\gamma_k \otimes 1)(R_{0,\lambda}^*)^2\big)\\
&= \big((1+\mathcal{D}^2) - 2\big)\Big(\int_0^\infty \lambda^{-1/2}(1+\lambda+\mathcal{D}^2)^{-3}\, d\lambda\Big)\mathcal{D}(\gamma_k \otimes 1) -\\
&\qquad - \Big(\int_0^\infty \lambda^{1/2}(1+\lambda+\mathcal{D}^2)^{-3}\, d\lambda\Big)\mathcal{D}(\gamma_k \otimes 1) -\\
&\qquad - 2(\gamma_k \otimes 1)\mathcal{D}\Big(\int_0^\infty \lambda^{-1/2}(1+\lambda+\mathcal{D}^2)^{-3}\, d\lambda\Big)\\
&\qquad - 2(\gamma_k \otimes 1)\mathcal{D}\Big(\int_0^\infty \lambda^{1/2}(1+\lambda+\mathcal{D}^2)^{-3}\, d\lambda\Big)\\
&\overset{(A.1)}{=} \frac{3\pi}{8}\big((1+\mathcal{D}^2) - 2\big)(1+\mathcal{D}^2)^{-5/2}\mathcal{D}(\gamma_k \otimes 1)\\
&\qquad - \frac{\pi}{8}(1+\mathcal{D}^2)^{-3/2}\mathcal{D}(\gamma_k \otimes 1) - -\frac{3\pi}{4}(\gamma_k \otimes 1)\mathcal{D}(1+\mathcal{D}^2)^{-5/2}\\
&\qquad\qquad - \frac{\pi}{4}(\gamma_k \otimes 1)\mathcal{D}(1+\mathcal{D}^2)^{-3/2}\\
&= \frac{\pi}{4}\big(\mathcal{D}(\gamma_k \otimes 1) - (\gamma_k \otimes 1)\mathcal{D}\big)(1+\mathcal{D}^2)^{-3/2}\\
&\qquad - \frac{3\pi}{4}\big(\mathcal{D}(\gamma_k \otimes 1) + (\gamma_k \otimes 1)\mathcal{D}\big)(1+\mathcal{D}^2)^{-5/2}.
\end{aligned}
$$

$\square$