Retirement savings and housing

Author:
Xu, Mengyi

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Retirement savings and housing

Mengyi XU

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In addition to occupational pension, housing is another important source of retirement savings. For young people entering the work force, home property purchase can significantly affect their savings for retirement. Literature on housing tenure choice largely focuses on determinants of home ownership, while its impact on retirement planning is rarely explored. This thesis investigates the question of when to become a homeowner from the perspective of financing retirement and conducts a welfare analysis. The presence of home equity can also affect individual demand for retirement products, such as annuities and long-term care insurance (LTCI). Despite high home ownership rates among retirees and significance of housing wealth in retired homeowners' portfolios, housing is often excluded from the literature of optimal consumption and portfolio choice. This thesis explains how homeowners' demand for annuities and LTCI differ from that of non-homeowners.

Individuals have heterogeneous levels of risk aversion and willingness to substitute consumption over time, as measured by elasticity of intertemporal substitution (EIS). LTCI insures against uncertain healthcare costs, so retirees of higher risk aversion will demand more coverage. Annuities smooth consumption over time while providing longevity insurance, so EIS is potentially the chief determinant of optimal annuitisation rate. Existing research exploring demand for retirement products mostly uses a power utility function that imposes risk aversion is the reciprocal of EIS, reflecting preference of only a small group of retirees. This thesis extends the literature by separating these two factors and showing their different impact on demand for annuities and LTCI.

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Abstract

The transition from defined benefit to defined contribution (DC) pension schemes has raised concern about whether retirees will have adequate income for their retirement. Prior literature has studied the optimal investment strategies for DC funds that provide minimum guarantees. But far less attention is paid to portfolio insurance strategies, despite their suitability to pension funds and their optimality to certain investors. This thesis evaluates the performance of option-based and constant proportion portfolio insurance strategies. They are implemented to a DC fund that targets an inflation- and longevity-protected annuity at retirement. The results show that both strategies provide strong protection against downside risk.

In addition to occupational pension, housing is another important source of retirement savings. For young people entering the work force, home property purchase can significantly affect their savings for retirement. Literature on housing tenure choice largely focuses on determinants of home ownership, while its impact on retirement planning is rarely explored. This thesis investigates the question of when to become a homeowner from the perspective of financing retirement and conducts a welfare analysis. The presence of home equity can also affect individual demand for retirement products, such as annuities and long-term care insurance (LTCI). Despite high home ownership rates among retirees and significance of housing wealth in retired homeowners' portfolios, housing is often excluded from the literature of optimal consumption and portfolio choice. This thesis explains how homeowners' demand for annuities and LTCI differ from that of non-homeowners.

Individuals have heterogeneous levels of risk aversion and willingness to substitute consumption over time, as measured by elasticity of intertemporal substitution (EIS). LTCI insures against uncertain healthcare costs, so retirees of higher risk aversion will demand more coverage. Annuities smooth consumption over time while providing longevity insurance, so EIS is potentially the chief determinant of optimal annuitisation rate. Existing research exploring demand for retirement products mostly uses a power utility function that imposes risk aversion is the reciprocal of EIS, reflecting preference of only a small group of retirees. This thesis extends the literature by separating these two factors and showing their different impact on demand for annuities and LTCI.

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List of Abbreviations

ABS	Australian Bureau of Statistics
ASFA	Association of Superannuation Funds of Australia
CAPM	capital asset pricing model
CEV	constant elasticity of variance
CI	confidence interval
CIR	Cox-Ingersoll-Ross
CPI	consumer price index
CPPI	constant proportion portfolio insurance
CRRA	constant relative risk aversion
CV	coefficient of variation
DB	defined benefit
DC	defined contribution
EIS	elasticity of intertemporal substitution
GBM	geometric Brownian motion
GDP	gross domestic product
GLM	generalised linear model
HARA	hyperbolic absolute risk aversion
HILDA	Household, Income and Labour Dynamics in Australia Survey
HRS	Health and Retirement Study
i.i.d.	independent and identically distributed
LTCI	long-term care insurance
OBPI	option-based portfolio insurance
VAR	vector autoregressive
VaR	value at risk
w.r.t.	with respect to

List of Symbols

Asset returns in discrete time

μ_H	Drift and volatility parameters of the log-normal distribution
σ_H	of house price growth (used in Chapter 6)
R_f	Gross real return on cash
$R_{H,t}$	Real property capital growth in period t
$R_{S,t}$	Gross real return on stocks in period t

Continuous time asset pricing models

$\alpha(\cdot,\cdot)$ $\beta(\cdot,\cdot)$	Deterministic process in pricing a zero-coupon bond
κ	Degree of mean reversion in the Vasicek model
$\boldsymbol{\lambda}$	Defined as $\begin{pmatrix}\lambda_S & \lambda_r\end{pmatrix}'$
λ_r	Market price of real interest rate
λ_S	Market price of equity fund price risk
Ω	Sample space
$\boldsymbol{\sigma}$	Defined as $\begin{pmatrix}\sigma_S & \sigma_r\end{pmatrix}'$
σ_P	Zero-coupon bond (denominated in the equity fund price) volatility
σ_r	Real interest rate volatility
σ_S	Equity fund's own volatility
σ_{Sr}	Equity fund volatility associated with interest rate
$\sigma_{\overline{T}}$	Bond fund volatility
$D(t)$	Discount process defined as $1/M_t$
$d_{1,t}$	Argument of the $N(\cdot)$ used in option pricing
$d_{2,t}$	(similar to the d_1 and d_2 in the Black-Scholes model)
$\mathbb{E}$	Expectation operator under the real world probability measure
$\widetilde{\mathbb{E}}$	Expectation operator under the risk neutral probability measure
$\mathcal{F}$	Information structure generated by $W_r(t)$ and $W_S(t)$
$K_j^{(S)}(t)$	Strike price (denominated in the equity fund price) in valuing the option price Q_t at time t
M_t	Price (in real term) of the cash fund at time t
n	Number of options
$\mathbb{P}$	Real world probability measure
$\widetilde{\mathbb{P}}$	Risk neutral probability measure for the cash fund numéraire
$\widetilde{\mathbb{P}}^{(S)}$	Risk neutral probability measure for the equity fund numéraire
$P(t,T)$	Price (in real term) of a zero-coupon bond at time t that matures at time T
$P^{(S)}(\cdot,\cdot)$	Price (denominated in the equity fund price) of a zero-coupon bond
$P_t^{\overline{T}}$	Price (in real term) of the bond fund with a constant maturity $\overline{T}$ at time t

Q_t	Value of a single option at time t
$\overline{r}$	Long-run mean of the real interest rate
r_t	Real interest rate at time t
S_t	Price (in real term) of the equity fund at time t
$\boldsymbol{W}(t)$	Defined as $\begin{pmatrix} W_S(t) & W_r(t) \end{pmatrix}'$
$\widetilde{W}_P^{(S)}(t)$	Standard Brownian motions w.r.t. the risk neutral measure $\widetilde{\mathbb{P}}^{(S)}$
$W_r(t), W_S(t)$	Independent standard Brownian motions w.r.t. the real world measure $\mathbb{P}$
$\widetilde{W}_r^{(S)}(t), \widetilde{W}_S^{(S)}(t)$	Independent standard Brownian motions w.r.t. the risk neutral measure $\widetilde{\mathbb{P}}^{(S)}$

Consumption and wealth

$\omega_{S,t}$	Proportion of liquid wealth invested in the risky asset in period t
φ	Maintenance and depreciation cost as a proportion of home value
$\mathcal{B}_t$	Amount of liquid wealth invested in cash in period t
B	Endowment of liquid wealth at retirement
B_t	Liquid wealth at the beginning of period t (used in Chapter 6[1])
C_t	Consumption of non-durable goods in period t
C^f	Consumption floor
$D_{H,t}$	Amount of rental payment in period t
H_t	Housing asset at the beginning of period t
LW_t	Liquid wealth at the beginning of period t (used in Chapter 5)
$\mathcal{S}_t$	Amount of liquid wealth invested in stocks in period t
W_t	Net worth at the beginning of period t
W_t^H	Home equity at the beginning of period t
W^{H}	Endowment of home equity at retirement
W_t^R	Pension balance at the beginning of period t

Health dynamics and costs

η_k	Coefficient of the polynomial used for graduating transition rates
$\pi(k\|j)$	Transition probability from health state j to health state k
σ_{jk}	Transition intensity from health state j to health state k
e_x	Central exposure to risk for x-year-old individuals
$h(s_t, t)$	Out-of-pocket health expenditure in health state s_t at time t
K	Degree of the polynomial used for graduating transition rates
m_x	Expected number of transitions between two health states for x-year-old individuals
q	Health expenditure inflation in excess of consumer price index inflation
s_t	Health state at time t

Insurance

α	Proportion of liquid assets annuitised
$\mathsf{\lambda}$	Percentage coverage of long-term care insurance to purchase
P	Actuarially fair premium for a full coverage long-term care insurance policy
Y	Annual income from the life annuity

Labour income

[1] Chapter 6 abstracts from the equity market, so the liquid wealth consists of cash only.

π	Employment-based pension contribution rate
K_t	Amount of labour income received in period t
$\underline{K_t^H}$	Lower bound in labour income in period t

Longevity index and target annuitisation level

δ	Probability that an x-year-old individual will live $T^d_{x,t}$ years
π_t	Directly observable inflation at time t
$\widehat{\pi}_t$	Unobservable core inflation at time t
A_t	Target annuitisation level at time t
f_t	Inflation rate at time t
g	Annual post-retirement consumption target
g_0	Parameter used in the exponential smooth
H	Inflation forecast horizon
$I_{x,t}$	Longevity index for an individual aged x at time t
i_t	Nominal interest rate at time t
J	Annuity term used to calculate the target annuitisation level
$T^d_{x,t}$	Term used to calculate the longevity index for an individual aged x at time t

Mortgage

$\underline{\lambda}$	Minimum deposit for the mortgage
DP_t	Down payment made in period t
$\mathbb{I}_t^{\text{own}}$	Home owner indicator in period t
$\overline{LVR}$	Maximum loan-to-value ratio
$\mathcal{M}_t$	Mortgage balance in period t before the repayment has been made
P_t^H	House price in period t
$\underline{P_t^H}$	Lower bound in house prices in period t
RP_t	Mortgage loan repayment made in period t

Preference

β	Subjective discount factor
γ	Coefficient of relative risk aversion
ψ	Elasticity of intertemporal substitution
ρ	Inverse of the elasticity of intertemporal substitution
θ	Defined as $(1-\gamma)/(1-\rho)$
b	Strength of retirement saving motive (used in Chapter 5[2]) Strength of bequest motive (used in Chapter 6)
O_t	Set of choice variables at time t
$u(\cdot)$	Constant relative risk aversion utility function
U_t	Expected present value of lifetime utility at time t
V_t	Indirect utility value at time t
$v(\cdot)$	Utility of savings for retirement

Portfolio of the target annuitisation fund

$\Omega^L_T(t)$	Bond fund that replicates the loan, L_t

[2]Chapter 5 models the pre-retirement phase in which the terminal period is the time of retirement.

$\Omega_M^L(t)$	Cash fund that replicates the loan, L_t
$\Omega_{\overline{T}}(t)$	Wealth invested in the bond fund for the portfolio X_t
$\Omega_M(t)$	Wealth invested in the cash fund for the portfolio X_t
$\Omega_S(t)$	Wealth invested in the equity fund for the portfolio X_t
$\Omega_{\overline{T}}^Y(t)$	Wealth invested in the bond fund for the portfolio Y_t
$\Omega_M^Y(t)$	Wealth invested in the cash fund for the portfolio Y_t
$\Omega_S^Y(t)$	Wealth invested in the equity fund for the portfolio Y_t
$\mathcal{C}_t$	Cushion of the constant proportion portfolio insurance strategy at time t
$c(t)$	Contribution amount to the pension fund at time t
$\mathcal{E}_t$	Exposure to the equity fund at time t
L_t	Loan that corresponds to the future contributions to the fund at time t
m	Multiplier of the constant proportion portfolio insurance strategy
X_t	Value of the target annuitisation fund at time t
Y_t	Value of the self-financing portfolio at time t

Probability and statistics

$\mathcal{N}(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2
$N(\cdot)$	Cumulative distribution function of the standard normal distribution
${}_np_{x,s_1}$	Probability that an x-year-old individual in health state s_1 will survive for the next n years

Vector autoregressive process

$\mathbf{\Phi}_0$	Vector of intercepts
$\mathbf{\Phi}_1$	Matrix of slope coefficients
$\mathbf{v}_t$	Shocks to the state variables at time t
$\mathbf{\Sigma}_v$	Variance of $\mathbf{v}_t$
$\mathbf{x}_t$	Vector of state variables at time t, $\mathbf{x}_t = \left(r_{S,t}, r_{H,t}, r_{L,t}, y_{S,t}, y_{H,t}, r_{G,t}\right)'$
$r_{G,t}$	Real GDP growth rate at time t
$r_{H,t}$	Real property capital growth rate at time t
$r_{S,t}$	Log real return on stocks at time t
$Y_{H,t}$	Rental yield at time t
$y_{H,t}$	Log rental yield at time t
$y_{S,t}$	Log dividend yield at time t
$r_{L,t}$	Real labour income growth rate at time t

Chapter 1

Introduction

Occupational pension and housing wealth constitute the major components of retirement savings at the individual or the household level (see e.g. Doling and Ronald, 2010; Knoef et al., 2016). The employment-based pension has been shifting from defined benefit (DB) to defined contribution (DC) plans, raising concern about financial security of future retirees. In addition, housing is typically the single largest asset for retired homeowners, but little research has been done on how housing tenure choice affects retirement planning, and on how housing wealth influences consumption and portfolio choices of retirees. Furthermore, individuals have heterogeneous levels of risk aversion and willingness to substitute consumption over time, as measured by elasticity of intertemporal substitution (EIS). These two factors are likely to have different impact on retirement products providing insurance and consumption smoothing, but the widely-used power utility function is unable to distinguish between these two factors.

This thesis addresses the question of how to improve financial security for DC pension plans, investigates the role of housing in retirement planning for both pre- and post-retirement phases, and analyses how risk aversion differs from EIS in influencing demand for retirement products. In particular, this thesis studies the following research questions.

- How to quantify a target annuity at retirement? How does the target annuity level evolve over time in four countries: Australia, Japan, the U.K. and the U.S.? These questions will be addressed in Chapter 3.

- What are the asset allocation strategies to target an inflation- and longevity-protected annuity in a DC pension plan? How effective are the portfolio insurance strategies in managing downside risk for such a DC pension fund? These questions will be addressed in Chapter 4.
- How does housing tenure choice affect pre-retirement consumption level and retirement savings? At what age before retirement should one become a homeowner? Chapter 5 will answer these questions.
- How does home equity affect demand for prominent retirement financial products, namely life annuities and long-term care insurance (LTCI)? How does risk aversion differ from EIS in determining demand for annuities and LTCI? These questions will be answered in Chapter 6.

The remainder of the chapter is organised as follows. Section 1.1 discusses the research motivation for the thesis. Section 1.2 presents the contributions made by the thesis to the existing literature while giving an overview of the results. Section 1.3 outlines the structure of the remainder of the thesis.

1.1 Motivation

The provision for retirement benefits around the world is undergoing two dramatic transitions: *1*) shifts from unfunded public pension to private funding, and *2*) shifts from DB to DC plans within the private funding. Take Australia for example. The Superannuation Guarantee system was introduced in 1992 to generate private savings for retirement. Australia is seen at the forefront of shifts from DB to DC plans. Back to 1982/83, 82% of fund members belonged to DB plans. By 2005/06, 97% of members were in funds that provided DC or a hybrid of DC and DB plans (Australian Prudential Regulation Authority, 2007). The U.K. and the U.S. have experienced similar declining trends in the membership of DB schemes. In the U.K., the decline of DB schemes was most prominent between

2000 and 2005, with the membership in the private sector falling by almost 50% (Turner and Hughes, 2008). In the U.S., the number of active participants in DB schemes peaked in mid 1980s and steadily declined afterwards. By contrast, the number of participants in DC schemes have increased dramatically, from less than 30% in 1975 to more than 70% in 2014.[1] The transition from DB to DC plans transfers the risk from plan sponsors to individual members. This significantly increases fund members' responsibility in retirement planning, and places a huge burden of complex decision making on individuals, who usually lack the skills to make complicated investment choices (Lusardi and Mitchell, 2014).

The operation of DC pension funds can generally be divided into two phases *1*) pre-retirement accumulation, and *2*) post-retirement drawdown. During accumulation phase, fund managers mostly aim to maximise fund value at retirement, while the connection with decumulation phase has not received enough consideration. The inadequate design of current DC plans leads to the discussion around providing sustainable income flows as the investment objective (Blake et al., 2008; Financial System Inquiry, 2014). In particular, the accumulated wealth at retirement should finance a desired consumption profile during retirement within a confidence interval. Such investment products can be labelled as "target annuitisation funds" (Impavido et al., 2012). The target is only probabilistic, so the pension fund managers have no liability. This feature is compatible with the DC nature of the pension plans. Such investment products are attracting increasing attention as they offer a possible solution to linking the accumulation and retirement phases in DC pension funds (Impavido et al., 2012). The increased attention motivates the study to construct and compare longevity index, a possible way to model the investment objective of the target annuitisation fund. It also motives the study on pension design for target annuitisation funds during the accumulation phase.

Apart from savings through occupational pension schemes, individuals have investment outside of their pension. Housing is typically the most significant part. In 2011-12, an

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[1] Source: Form 5500 filings with the U.S. Department of Labor available at `www.dol.gov/ebsa/publications/form5500excelhistorical.xls` [accessed November 21, 2014].

average Australian household owned about $370,000 worth of owner-occupied property, which accounted for 43% of household assets (Australian Bureau of Statistics, 2013a). Owner-occupied housing is not only an investment asset, but also a durable consumption good. It insurers homeowners against rental fluctuations since homeowners effectively pay for the housing-service consumption up front when they purchase the property. The dual purpose of housing implies its crucial role in retirement planning. Home ownership can protect against poverty among retirees, while renting in old age is often associated with poverty due to lower non-housing wealth and higher housing costs (Yates and Bradbury, 2010). Housing is also a potential source of retirement income. Housing wealth can be unlocked through equity release products to improve retirement living standard or to fund healthcare and long-term care. Furthermore, older people tend to develop attachment to their family home. A recent survey indicates that the majority of Australians aged 60 and over prefer "age in place" due to their desire to stay in the local community (Productivity Commission, 2015). The dual purpose of owner-occupied housing and its importance in retirement planning motivate the study that investigates when to buy a residential home from the perspective of financing retirement.

Retirees tend to have high home ownership rates. In both the U.S. and Australia, households headed by people aged 65 and over have had one of the highest home ownership rates among all age groups over the past few decades (U.S. Census Bureau, 2017; Reserve Bank of Australia, 2015). Retired homeowners have a large fraction of household portfolio held in the form of home equity. The median ratio of home equity to all assets is estimated to be 0.56 among the elderly homeowners in the U.S. (Davidoff, 2009). Home equity is generally not reduced among people who continue to own their homes (Venti and Wise, 1990; Venti and Wise, 1991; Venti and Wise, 2004). The preserved home equity will be left to heirs. Selling the house is often associated with losing spouse or entering into a nursing home (Walker, 2004; Venti and Wise, 2004). This means home equity can serve as LTCI to pay for health expenses. Uncertain out-of-pocket healthcare costs represent a key source of risk during retirement, even in the presence of Medicare arrangements. Findings in McRae et al. (2012) suggest that around 570,000 Australians aged

55 and over spend more than 10% of their income on health, and around 250,000 spend more than 20%. In addition to health shocks, retirees also face the challenge of allocating their financial resources across time to avoid outliving their wealth. Life annuities are an effective instrument to combat the risk of outliving one's financial resources. The significance of housing combined with increased attention on retirement products motivates the study on the impact of housing wealth on demand for life annuities and LTCI.

Individuals differ in their risk aversion and EIS, implying different demands for retirement products such as annuities and LTCI. Annuities can smooth consumption over time while providing longevity insurance, so EIS is probably more important than risk aversion in determining the optimal annuitisation rate. LTCI insures against uncertain healthcare costs while transferring consumption from healthy states to unhealthy states, so retirees of higher risk aversion and/or higher EIS will demand more coverage. The widely-used power utility function imposes an inverse relationship between risk aversion and EIS. On the other hand, experimental studies have shown that risk tolerance and the EIS are essentially uncorrelated across individuals (Barsky et al., 1997), and that individuals have relative risk aversion greater than the reciprocal of the EIS (Brown and Kim, 2013). Therefore it is highly likely that power utility functions capture preference of only a small group of retirees. The drawback of the power utility function motivates the study that separates risk aversion and EIS to show their different impact on demand for annuities and LTCI.

1.2 Contributions to the literature

This section summarises the thesis contributions to the literature and gives an overview of the key results in each chapter.

Chapter 3 contributes to the literature of longevity index. The longevity index quantifies the impact of interest rate, inflation and mortality on affordability of retirement. This chapter constructs and compares longevity index for four countries starting from 1970s. Under the assumption that the levels of real interest rate and mortality at the time of calculation persist, the indices show volatilities as high as stock returns. This highlights the difficulty for an ordinary individual in accumulating sufficient wealth to provide longevity- and inflation-protected retirement income. The analysis in this chapter also helps to determine the term of annuity used in Chapter 4. The term annuity serves as an investment objective that provides adequate longevity protection for fund members.

Chapter 4 contributes to the literature of portfolio insurance strategies and DC pension designs. This chapter applies the option-based and constant proportion portfolio insurance strategies to managing target annuitisation funds. It investigates the portfolio weights in the cash fund, bond fund, and equity fund over the course of pre-retirement stage. It also compares the performance of the two portfolio insurance strategies in terms of downside risk protection and accumulated wealth at retirement.

The model takes into account interest rate risk and assumes pension funds receive deterministic contributions from their members. The investment objective at retirement is represented by a term annuity that ensures an inflation-linked and longevity-protected income stream with a high probability. By appropriate transformation, the minimum retirement benefit in the option-based strategy is similar to a call option on a coupon-bearing bond. Analytical solutions of portfolio weights are derived for both strategies.

The portfolio weights in the equity fund depend on the member's age upon joining the fund, displaying a downward trend for members joining the fund before mid-30s. The portfolio weights are highly volatile due to the volatility of the equity fund. Comparing the two strategies in terms of risk management, the constant proportion strategy performs better using the baseline parameter values. The option-based strategy provides more robust level of protection when equity market becomes more volatile or pension fund contributions are lowered. In addition, it often leads to higher accumulated savings at retirement.

Chapter 5 contributes to the literature of housing tenure choice. This chapter investigates how purchasing home property at different ages would affect an individual's pre-retirement consumption level, savings for retirement, and ultimately the lifetime utility level. The model takes into account practical issues that property purchase decision depends also on labour earnings, and that it affects non-housing consumption and portfolio choice.

This chapter uses a vector autoregressive (VAR) process to model the dynamics of state variables that determine asset prices and labour income, and performs Monte Carlo simulations. The parameters of consumption and asset allocation decisions are calibrated to the Household, Income and Labour Dynamics in Australia Survey (HILDA) data, reflecting an average Australian's decisions in his/her age group and home ownership status. The lifetime utility is then calculated based on the individual consumption and asset allocation decisions given the simulated state variables.

Purchasing the property earlier during the working life often leads to greater wealth at retirement due to higher home equity and more liquid assets. Purchasing the property, however, creates a dramatic consumption drop that lasts for a few years since the down payment transfers a significant amount of liquid assets to illiquid housing wealth. The consumption cut results in welfare loss, and the earlier the property is purchased, the higher its impact on lifetime utility. Renting throughout working life is unattractive both in terms of retirement wealth and welfare. Individuals who keep renting have to incur high rental costs. This not only constrains the spending on non-housing consumption, but also slows down the wealth accumulation.

Chapter 6 contributes to the literature of optimal consumption and portfolio choice at retirement. This chapter studies the impact of home equity as well as preference parameters on demand for life annuities and LTCI. An Epstein-Zin-Weil-type (Epstein and Zin, 1989; Epstein and Zin, 1991; Weil, 1989) utility function is used to represent individual preferences. It generalises the more commonly used power utility function by separately identifying the risk aversion and EIS.

This chapter builds a multi-period lifecycle model for a single retired homeowner who faces several sources of risk, namely uncertain lifespan, uncertain healthcare cost, and uncertain house price. At the point of retirement, individuals have access to fairly priced ordinary life annuities and LTCI. Housing wealth will either be bequeathed or liquidated upon moving into a long-term care facility. Health transition probabilities are estimated from the U.S. Health and Retirement Study (HRS), and the other parameters take commonly used values in the literature.

The impact of home equity on demand for life annuities depends on the availability of LTCI. When retirees have no access to LTCI, the presence of home equity generally increases the optimal annuitisation rate. For retired homeowners who tend to sell the property at the time of moving into a nursing home, home equity can serve as a bequest and precautionary savings. Prior research has shown that bequest motive (Lockwood, 2012) or precautionary savings for healthcare costs (Sinclair and Smetters, 2004; Turra and Mitchell, 2008) can weaken demand for life annuities. The presence of home equity enhances demand for annuities by lowering the barrier to annuitisation. When LTCI is also available, the presence of home equity can make life annuities more attractive if there are sufficient liquid assets. If the amount of liquid assets is low, the spending on purchasing LTCI can impair demand for life annuities. Given retirees tend to liquidate housing wealth in the event of moving to a long-term care facility, home equity typically crowds out demand for LTCI regardless of the availability of life annuities.

The sensitivity analysis on preference parameters shows the importance of separately identifying risk aversion and EIS. An individual with a higher degree of risk aversion wants more LTCI coverage and less annuities (to hold more precautionary savings in liquid assets). A lower level of EIS has the opposite effect. The power utility model imposes an inverse relationship on risk aversion and EIS, meaning a more risk averse individual will inevitably has a lower EIS. Such a rigid structure is unlikely to represent the preference of a large majority of retirees, so the Epstein-Zin model is more suitable to determine demand for life annuities and LTCI to accommodate individuals with various levels of risk aversion and EIS.

1.3 Outline of thesis

The rest of the thesis is structured as follows. Chapter 2 reviews the relevant literature and identifies the gaps to be addressed in the thesis. Chapter 3 provides an empirical study on longevity index, serving as a background for Chapter 4, which compares and contrasts two prominent portfolio insurance strategies for managing target annuitisation funds. Chapter 5 investigates the optimal time to become a homeowner during pre-retirement. Chapter 6 analyses the impact of home equity and preference parameters on demand for life annuities and LTCI. Chapter 7 concludes.

Chapter 2

Literature review

2.1 Introduction

This thesis studies retirement savings and their interaction with housing. Chapter 4 proposes to use portfolio insurance strategies to manage defined contribution (DC) pension plans that provide a minimum income stream during retirement. This requires an understanding of DC pension designs in the accumulation phase and portfolio insurance strategies. The relevant papers are reviewed in Section 2.2 and Section 2.3, respectively. The two bodies of literature can be linked through the expected utility theory, which is discussed towards the end of Section 2.3. Not much attention, however, has been paid to applying portfolio insurance strategies to DC pension funds, although research has shown the suitability of portfolio insurance strategies to pension funds and their optimality to certain investors.

Chapter 5 investigates the optimal time to become a homeowner during one's working life. This chapter builds on the papers of housing tenure decisions (Section 2.4). The existing literature largely focuses on determinants of home ownership, such as housing expenditure, labour income and borrowing constraints. But the impact of housing tenure choice on retirement planning is rarely explored despite the importance of housing wealth to financing retirement.

Chapter 6 discusses the impact of home equity on demand for life annuities and long-term care insurance (LTCI) in a lifecycle framework. This chapter is related to an extensive stream of literature that studies the optimal consumption and portfolio choice decisions. Since the seminal work of Samuelson (1969) and Merton (1969), the model has been greatly extended to allow for more realistic features. These papers are reviewed in Section 2.5. In spite of the large amount of literature in this area, a very limited number of papers consider the interaction between housing wealth and healthcare expenditure. In addition, most studies use a power utility function that imposes a rigid structure on the relationship between risk aversion and elasticity of intertemporal substitution (EIS). Section 2.6 discusses how this thesis will address the above-mentioned gaps in the existing literature.

2.2 Defined contribution pension plan designs in the accumulation phase

One of the most important decisions in DC pension designs is about portfolio allocation. There are two broad categories of the literature that studies portfolio choice from the perspective of a DC pension fund manager: *1*) using some form of asset allocation strategies as an input and investigating their performance, *2*) using the desired fund performance as an input and deriving the asset allocation strategies.

Among the papers in the first category, the so-called lifestyle investment strategies have gained much attention. The strategy involves automatically switching away from risky assets as fund members approach retirement. Although the strategy has some theoretical soundness (Viceira, 2009), a number of studies have found its drawbacks and proposed improvements. The relevant papers are reviewed in Section 2.2.1.

The papers belonging to the second category are usually based on utility maximisation. Since one of the earliest papers by Boulier et al. (2001), the literature has been extended in several directions by, for example, including additional sources of risks, considering various asset return models, and using different forms of the utility function. Some studies also consider the optimal contribution to the fund and drawdown behaviour after retirement. These papers are reviewed in Section 2.2.2.

Papers in the second strand of literature often find it optimal to adopt stochastic lifestyling strategies that generally decrease equity exposure as members approach retirement. To differentiate, the lifestyle strategies in the first strand of literature are often referred to as deterministic lifestyling strategies. The main difference between the deterministic and stochastic ones is the feedback system. Deterministic lifestyling decreases allocation in equities in a predetermined manner, whereas stochastic lifestyling makes the switch based on human capital and portfolio performance.

2.2.1 Lifestyle investment strategies

Lifestyle investment strategies involve switching from risky assets (e.g. stocks) when plan participants are young to safe assets (e.g. cash band bond) when they approach retirement through a predetermined glide path. The funds adopting such strategies are also known as lifecycle funds, or target-date funds.

Lifecycle funds offer a possible solution to address behaviour barriers to retirement planning. Many pension fund participants are known to exhibit inertia (Madrian and Shea, 2001). They tend not to adjust their retirement portfolios over time to suit their personal situations. In addition, for those who actively choose their investment portfolios, they tend to possess naïve diversification, overlooking the asset classes in each investment option (Benartzi and Thaler, 2001). Lifecycle funds offer a one-off solution that automatically adjusts portfolio weights. Viceira (2009) reviews several theoretical foundations for lifecycle funds[1], including mean-reversion in stock returns and human capital. While mean reversion only lends partial support to lifecycle funds and there is ongoing debate

[1]Note that Viceira (2009) differentiates between the terms “life-style” funds and “life-cycle” funds. “Life-style” funds are referred to those which adopt risk-based investment strategy (i.e. the investment horizon is irrelevant), whereas “life-cycle” funds adopt an “age-based” asset allocation strategy (i.e. a lifestyle investment strategies). To avoid confusion with “lifestyle investment strategies”, the thesis uses the terms “lifecycle” and “lifestyle” interchangeably when it comes to the investment strategy.

on its existence, the literature on optimal portfolio choice with uncertain labour income (see e.g. Viceira, 2001) does show investors should tilt their portfolios towards stocks when they are young.

Despite the growing popularity and theoretical support from the literature, a number of studies have criticised the conventional lifestyle strategies. Basu and Drew (2009) argue that adjusting portfolio weights using predetermined rules ignores the portfolio size effect. In a follow up paper, Basu et al. (2011) consider dynamic rebalancing strategies whereby the portfolio is switching to bond and cash only if the value of the portfolio exceeds some target. They find the dynamic lifecycle strategy dominate the conventional one if the terminal wealth exceeds \$500,000[2].

Several studies have challenged the existence of lifecycle funds on the ground that balanced funds (i.e. with age-invariant asset allocation strategies) perform similarly to the lifestyle strategies for some performance measures. Poterba et al. (2009) study the distribution of retirement wealth by looking at its 1st, 10th, 50th, 90th percentiles and mean. The age-invariant asset allocation strategies set the portfolio weight in equities as the average equity share in the lifestyle investment strategies. Guillén et al. (2013) propose to compare the fund performance with respect to an equivalent reference product. The equivalent product has a constant stock proportion, and its retirement wealth has the same level of riskiness as the fund under comparison. The risk is measured by value at risk (VaR) or conditional tail expectation. Guillén et al. (2013) find that compared to the commercial lifecycle products considered in the paper, the equivalent product performs better in terms of average (or median) retirement wealth. Graf (2016) assesses the probability distribution of returns at the end of the investment horizon by comparing the full probability densities. The balanced fund is found by matching the first two moments of retirement wealth.

[2]The magnitude of this threshold is not large considering the following assumptions: *1*) 9% of labour earnings contributed to the fund, *2*) a total of 40 annual contributions, *3*) salary starting at \$25,000 and growing linearly at a rate of 4% per annum. The average annual rate of return required to achieve \$500,000 is about 4.5%, which is far below the 10% per annum target rate of return set in the paper.

DC pension plans take regular contributions, which are usually a proportion of members' labour income. The labour income profile can therefore affect the attractiveness of lifecycle funds. Bodie and Treussard (2007) and Gomes et al. (2008) conduct welfare analyses on target-date funds taking into account different aspects of labour income. Bodie and Treussard (2007) focus on the impact of human capital risk, along with that of risk aversion. Human capital risk is measured as the ratio of the volatility of human capital to the volatility of stocks. Gomes et al. (2008) incorporate flexible labour supply, which acts like self-insurance against unfavourable market movements. Bodie and Treussard (2007) find that target-date funds are far from optimal for members who have high relative risk aversion and high human capital risk. They subsequently propose a safe fund to improve welfare of members with high relative risk aversion or high human capital risk. The safe fund guarantees an inflation-indexed annuity starting at the target retirement date. Guarantees of similar types are used in other papers, but it is questionable whether such a guarantee is compatible with the nature of DC plans. Gomes et al. (2008) find that a lifecycle fund that follows the average optimal asset allocation path implied by the lifecycle model leads to negligible welfare loss. The contrasting results in Bodie and Treussard (2007) and Gomes et al. (2008) can be contributed to different assumptions about labour income and the investment strategies adopted by target-date funds. Gomes et al. (2008) calibrate the wage income profile to empirical data, whereas Bodie and Treussard (2007) examine a range of human capital risk. The target-date fund investigated in Gomes et al. (2008) reduces the exposure to stocks from early 30s, whereas most target-date funds (including the one tested in Bodie and Treussard (2007)) reduce the exposure to risky assets at much older ages.

2.2.2 Optimal portfolio choice

The traditional way of comparing fund performance based on quantile and mean does not take into account the risk of a very poor outcome, albeit with a small probability. Introducing the expected utility to assessing the fund performance takes into account individual's risk aversion to adverse outcome. In addition, pension funds could make the products more attractive by providing downside risk protection for retirement. The floor is usually modelled as a minimum guarantee in terms of a life annuity (or a term annuity) purchased at retirement.

This stream of literature employs the utility maximisation (or disutility minimisation) framework to find the optimal investment strategies, which usually mimic some form of lifestyle strategies discussed in the previous section (Section 2.2.1). The studies often use the dynamic programming to solve the optimal investment strategies. The solution technique of dynamic programming is developed by the literature in optimal asset allocation (to be reviewed in Section 2.5).

2.2.2.1 Interest rate risk

The accumulation phase of a pension plan involves a long time horizon, typically 30 to 40 years. It is therefore important to consider stochastic interest rates. The Ornstein-Uhlenbeck process of Vasicek (1977) is widely used in the context of optimal asset allocation, because it usually gives closed-form solutions.

Boulier et al. (2001) propose an optimal pension fund strategy in which the stochastic interest rate follows the Vasicek model. The fund provides a minimum guarantee that is a life annuity whose value is determined by the short interest rate at retirement. The model is based on a few simplified assumptions. The contribution, the annuity rate and the time of death are assumed to be deterministic. The contribution rate and the annuity rate increase at the same constant rate, which can be thought of as the inflation rate. Boulier et al. (2001) introduce the idea of rolling bond, which is widely used in later studies (see e.g. Battocchio and Menoncin, 2004; Guan and Liang, 2014). A rolling bond has a constant maturity. It is preferred over zero coupon bonds because it is unrealistic to assume the existence of infinite zero coupon bonds in the market. In fact a rolling bond can be linked to zero coupon bonds via the riskless asset. Boulier et al. (2001) find that the optimal pension fund can be divided into three components: *1*) a loan corresponding to the present value of all the future contributions, 2) a portfolio that replicates the guarantee, and 3) a hedge fund. The interest rate risk is managed by investing into cash and bonds.

Some extensions include the Cox-Ingersoll-Ross (CIR) process suggested by Cox et al. (1985) and the affine process proposed by Duffie and Kan (1996). The CIR process overcomes the deficiency of negative interest rates in the Vasicek process. The affine process is a more general framework that includes the Vasicek and CIR models as special cases. Some of the following papers adopt the affine framework.

2.2.2.2 Labour income risk

One defining feature of the DC pension fund is the fact that employees (or employers on behalf of employees) contribute a certain proportion of their salary to the fund. The model framework is extended to consider the labour income risk and the stochastic interest rate simultaneously. There are generally two ways to model the labour income risk, depending on whether the risk can be hedged using the financial assets in the market.

Deelstra et al. (2003) build on Boulier et al. (2001) to include labour income risk. The contribution follows a general stochastic process that can be perfectly hedged (i.e. the market is complete). They also use a more general affine process to model the term structure of interest rates. The minimum guarantee is in terms of a deterministic minimum return on the pension fund. From a practical point of view, life annuity guarantee is more meaningful than investment return guarantee, since it provides a minimum income stream instead of a minimum capital.

Cairns et al. (2006) study stochastic lifestyling asset allocation strategies, taking into account the non-hedgeable salary risk and interest rate risk. Various cases are considered in the study, including those with or without contribution, with or without non-hedgeable salary risk. The case with zero contribution is particularly relevant to plan members who have accumulated some wealth and have chosen not to contribute more. Compared to previous studies, a key difference in their approach is using final salary as a numeraire, so the utility function depends on the surplus (i.e. the terminal wealth over a minimum guarantee) as a proportion of final salary rather than the surplus itself. This implies that the plan member's objective focuses on the replacement ratio. The minimum guarantee is given by a life annuity in retirement. Cairns et al. (2006) show that the optimal strategy is represented by two or three efficient mutual funds, depending on whether interest rate is stochastic. The first fund is heavily dominated with equities to satisfy the plan member's risk appetite. The second fund is heavily dominated with cash to hedge the salary risk. The third fund is heavily dominated with bond to hedge the interest rate risk. Cairns et al. (2006) argue that it is worthwhile to implement the stochastic lifestyling strategy in practice, since it significantly improves the expected terminal utility compared to some other popular strategies.

2.2.2.3 Inflation risk

Inflation is another key source of risk given the long investment horizon in the accumulation phase. Inflation risk can be hedged using a portfolio dominated by cash (Battocchio and Menoncin, 2004) or inflation-indexed bond (Han and Hung, 2012).

Battocchio and Menoncin (2004) consider two background risks, the salary risk and the inflation risk in optimal pension management. In particular, the labour income process is generated by the market (i.e. the stock and the interest rate) as well as the inflation that does not belong to the financial market. A constant proportion of the wages is contributed to the pension fund. The consumer price index follows an Itô process. In addition, the nominal risk-free interest rate satisfies the Ornstein-Uhlenbeck process. The fund manager can invest in three assets: a riskless asset, a stock, and a rolling bond. The dynamics of these three assets are expressed in the nominal amount. The financial market remains complete after the introduction of the inflation risk, since the riskless asset becomes risky in the real value. The utility function is in an exponential form, so the value function used in solving the dynamic problem can be separated into the real wealth and all the other state variables. Battocchio and Menoncin (2004) show that the optimal portfolio consists of three components: *1*) a pure technical hedging component, *2*) a speculative component that is proportional to the portfolio Sharpe ratio and the inverse of the coefficient of absolute risk aversion, and *3*) a hedging component that depends on the parameters of the state variables. Battocchio and Menoncin (2004) extend the framework of optimal management of pension fund by explicitly modelling the salary risk and the inflation risk, but they ignore the minimum guarantee, which is a common feature in related studies. The paper has another major drawback, as pointed out in Ma (2011), that the optimal weights in each of the three assets are determined independently. In fact only two of them can be determined independently.

Han and Hung (2012) extend the work of Boulier et al. (2001), Deelstra et al. (2003), and Battocchio and Menoncin (2004) by considering the inflation risk, the labour income risk and the downside protection simultaneously. Battocchio and Menoncin (2004) use a portfolio dominated by cash to hedge against the inflation risk. By contrast, Han and Hung (2012) introduce the inflation-indexed bond, which is a real risk-free asset for long-term investors (Campbell and Viceira, 2001). Investors can allocate their assets among cash, nominal bond, stock and the inflation-index bond. This addresses the problem that investors cannot arbitrarily adjust their asset portfolios based on their risk preference, due to the non-existence of real risk-free assets. Indeed Han and Hung (2012) highlight the benefits of issuing inflation-indexed bond to pension industry. The guarantee on the terminal benefits is an indexed annuity purchased at the time of retirement, which can be replicated by inflation-indexed bonds. Such a guarantee is attractive in that it helps to maintain a constant purchasing power for retirees. However, it has no longevity insurance feature since the time of death is assumed to be deterministic. The optimal asset allocation can be represented by three components: *1*) a mean-variance efficient portfolio, *2*) a replicating portfolio of the present value of all the future contributions, and *3*) a portfolio that replicates the guarantee.

2.2.2.4 Alternative asset return models: Stochastic volatility

Most of the papers reviewed so far assume the stock market follows a geometric Brownian motion (GBM). Several papers extend the GBM to include stochastic volatility. The resulting optimal investment strategies usually have a term that hedges the volatility risk.

Heston model Guan and Liang (2014) consider stochastic volatility in the Heston model (Heston, 1993). Since the stochastic volatility cannot be hedged, the market is incomplete. In addition, the interest rate follows an affine model, which is also used in Deelstra et al. (2003). The stochastic contribution rate is closely related to the stock index, so it can be replicated by the existing assets. They avoid non-hedgeable labour income to get an explicit solution. The minimum guarantee is a life annuity purchased at retirement, as in Deelstra et al. (2003). Guan and Liang (2014) extend the form of guarantee to allow for the random time of death with deterministic force of mortality. Unlike Battocchio and Menoncin (2004) and Han and Hung (2012), the inflation risk is not considered in the model. Given the long investment horizon for the pension fund and the compounding effect of the inflation, it is necessary to incorporate inflation risk in optimal pension fund management. Compared to the complete market case in Deelstra et al. (2003), there are additional terms in the optimal investment strategy to hedge stochastic volatility.

Constant elasticity of variance model Gao (2009) uses the constant elasticity of variance (CEV) model to allow for stochastic volatility in the stock market. Compared to the GBM model, the CEV model has one additional parameter to capture the correlation between asset price and asset volatility. The main contribution of the paper is to derive explicit solution of the optimal investment strategies for the power and exponential utility functions, but this is at the cost of assuming a constant interest rate and a constant contribution to the pension fund.

2.2.2.5 Alternative objective functions: Loss aversion

A common approach to the optimal asset allocation problem is to maximise the expected utility from the terminal wealth over a minimum guarantee. The study on behavioural finance, however, argues that loss aversion is better than expected utility theory in describing individuals' risk attitudes (see e.g. Rabin and Thaler, 2001). The concept of loss aversion is first introduced by Kahneman and Tversky (1979). The theory suggests that individuals perceive gains and losses relative to some reference point. This gives rise to an alternative approach, whereby the investment strategy is driven by a final target and a series of interim targets. Vigna and Haberman (2001) and Haberman and Vigna (2002) use quadratic loss function to penalise deviations from the target, whereas Blake et al. (2013) adopt the prospect utility function.

Quadratic cost function Vigna and Haberman (2001) analyse investment risk and annuity risk borne by members of a DC pension scheme. They choose a quadratic disutility function, which penalises the differences above and below the target to the same extent. Deviations from the interim targets and the final one are penalised differently to reflect the higher importance of achieving the final target. The target is set in terms of a required return on the pension fund. A number of assumptions are made on grounds of simplicity. The fund can be invested in the two uncorrelated assets, a low-risk one and a high-risk one. The contribution to the fund is a fixed percentage of salary, which has no real increase. The simulation results confirm the validity of the lifestyle strategy to reduce the investment risk. Members of the pension fund seem to bear considerable annuity risk, as the level of annuity that can be purchased at retirement shows large variability in the case of variable conversion rate.

Haberman and Vigna (2002) extend the model in Vigna and Haberman (2001) to a total of n correlated assets, and generalise the disutility function such that the deviations above the target are rewarded to a certain level, after which they are penalised. This disutility function is more realistic as it incentivises reasonable gains from high market returns. The optimal investment strategy for a risk-averse member is again lifestyle strategy. The risk of failing to attain the target replacement ratio is measured by the probability of failing the target, the mean shortfall, and a VaR measure. The simulation results on two-asset base give contradictory indications as to whether more cautious strategies can reduce the downside risk. This can be explained by the fact that more cautious strategies increase the number of failures but limit the magnitude of losses when a failure occurs.

Prospect theory Blake et al. (2013) adopt the loss aversion framework of Tversky and Kahneman (1992). It assumes that an individual is risk seeking when the fund value is below the pre-defined target, and risk averse when the fund value is above the target. Besides, individuals are more sensitive to a unit loss than to a corresponding unit gain. Blake et al. (2013) argue that the prospect theory utility function is more appropriate than the quadratic cost function used in Vigna and Haberman (2001) and Haberman and Vigna (2002) as it more closely reflects the behaviour exhibited by investors. The paper considers stochastic labour income, and imposes no-borrowing and no-short-selling constraints. Members of the pension fund target a replacement ratio of two-thirds of income at retirement, and receive a real life annuity during retirement. Since the labour income is stochastic, the final target and the interim target are path dependent. The optimal investment strategy is a target-driven threshold strategy. If the fund value is below the target, the proportion invested in equity should increase, and vice versa. If the fund level is well above the target, the optimal investment strategy is portfolio insurance. Compared with strategies associated with maximising expected utility, the proposed optimal strategy significantly reduces the risk of failing to attain the target replacement ratio.

2.2.2.6 Alternative objective functions: Recursive utility

Blake et al. (2014) uses an Epstein and Zin (1989) type utility function, which separates relative risk aversion and EIS. Another difference to the previous studies is that instead of setting a certain contribution rate, Blake et al. (2014) make the funding decision endogenous. In other words, contribute rates vary over time to reflect consumption adjustment.

During pre-retirement phase, fund assets are allocated between a riskless asset (bond fund) and a risky asset (equity fund). Members earn stochastic labour income, which is allocated between consumption and contribution. During post-retirement phase, members can invest between life annuities and equity fund, since bond fund is dominated by life annuity. Blake et al. (2014) assumes constant risk-free interest rate, which is unlike to hold in the long time horizon. This assumption, however, greatly simplifies the optimisation problem.

Blake et al. (2014) show that the optimal funding strategy involves an age-dependent contribution rate. In particular, it remains zero until age 35, increases steadily to around 30-35% around age 55, and then decreases slightly before retirement. The optimal pre-retirement investment strategy is stochastic lifestyling, which is extensively analysed in Cairns et al. (2006). The optimal post-retirement investment strategy is phased annuitisation. It involves exchanging the bond fund for life annuities at retirement, and gradually selling equities for more annuities.

2.3 Portfolio insurance strategies

Portfolio insurance strategies provide investors with the potential to limit downside risk and to participate on the upside. Strategies that buy stocks when the market rises and sell stocks as the market falls (i.e. buy high and sell low) represent the purchase of portfolio insurance (Perold and Sharpe, 1988). Two most prominent examples are option-based portfolio insurance and constant proportion portfolio insurance. The two strategies are discussed in detail in Section 2.3.1. Section 2.3.2 reviews some of the literature that compares the two portfolio insurance strategies. Section 2.3.3 and Section 2.3.4 discuss the literature that links portfolio insurance strategies to optimal portfolio choice and DC pension management, respectively.

2.3.1 Theory

Option-based portfolio insurance (OBPI) are introduced in Brennan and Schwartz (1976) and Rubinstein and Leland (1981). The strategy involves holding the reference portfolio and purchasing a put option on the portfolio with strike price equal to the desired floor value at the investment horizon. It is usually difficult, if not impossible, to find the put option with suitable maturity and strike price on the market. As a result, a synthetic put option is often used, and the overall exposure is managed by delta hedging.

OBPI strategies are usually costly to implement and not easy to communicate. Black and Jones (1987) propose a simplified dynamic hedging approach called constant proportion portfolio insurance (CPPI), which is arguably more robust than OBPI. The strategy takes a simple form

$$\mathcal{E} = m\mathcal{C}, \tag{2.1}$$

where $\mathcal{E}$ is the exposure to risky assets, m is a fixed multiplier, and $\mathcal{C}$ is the cushion that represents the different between the portfolio value and a protected floor value.

Black and Perold (1992) further develop the theory of CPPI. They provide a two-asset framework to study the CPPI strategies. Wealth is allocated among "active" and "reserve" assets. The reserve asset is a safe asset. Although most of the later studies interpret "safe" as risk-free, it may mean that the safe asset closely tracks a liability stream. Black and Perold (1992) show that in a frictionless market with no borrowing constraints, the portfolio payoff at time t depends only on the values of the active and reserve assets at that time and on the number of trades along the way. This implies a weak form of path independence.

2.3.2 Comparison

Many studies are devoted to comparing OBPI and CPPI from different perspectives under different assumptions. The results are inconclusive as to which strategy is superior.

This implies that it is worthwhile to investigate both methods for pre-retirement investment strategies in Chapter 4. Pézier and Scheller (2013) give an overview of six papers that compare OBPI and CPPI. The key features and results are summarised in Table 1 of their paper.

The terminal payoffs of OBPI and CPPI can by compared by means of stochastic dominance criteria. Most studies draw the conclusion that is it hard to discriminate between these two approaches using such criteria. Bertrand and Prigent (2005) find no stochastic dominance at first order for either type of portfolio insurance strategy. Since the first order stochastic dominance criterion is not sufficient to capture the behaviour of risk-averse investors (Levy, 1992), Zagst and Kraus (2009) extend the analysis of stochastic dominance up to third order. They give sets of conditions for second and third order stochastic dominance of CPPI over OBPI. Both Bertrand and Prigent (2005) and Zagst and Kraus (2009) derive the results theoretically. Annaert et al. (2009) take a different approach, using block-bootstrap simulation from an empirical distribution. They also find no stochastic dominance between these two portfolio insurance strategies.

Some studies show that CPPI strategies are superior to OBPI strategies using alternative measures or in a more realistic setting. Bertrand and Prigent (2011) use the Kappa performance measure, and in particular the omega measure. The Kappa measure takes into account the whole return distribution, which makes it suitable to measure the convex portfolio insurance payoffs. They show that CPPI generally performs better than OBPI for Kappa measures. Pézier and Scheller (2013) use a certainty equivalence return based on a hyperbolic absolute risk aversion (HARA) utility function. When portfolios cannot be rebalanced continuously and asset prices processes are discontinuous, they show that CPPI strategies are superior to OBPI strategies.

2.3.3 Optimality

OBPI and CPPI can be linked to the optimal investment strategies in the framework of expected utility. It is well known that constant mix strategy (which is a special case of the

CPPI strategy) is the optimal investment strategy in a Black-Scholes setup for an investor with constant relative risk aversion (CRRA) utility function (see e.g. Merton, 1971).

Kingston (1989) studies the optimal consumption and portfolio decision for an investor whose utility is derived from the total consumption of a family. The family consists of infinitely lived individuals with identical preferences represented by a HARA utility function. The main feature of the utility function is that it considers the growth of the investor's family. Kingston (1989) concludes that the CPPI strategy is optimal if and only if the investor has decreasing absolute and relative risk aversion. By contrast, Black and Perold (1992) show that the CPPI strategy is optimal for an investor with piecewise utility function that consists of a power function (i.e. of the CRRA form) and a linear function. The investor also faces a minimum consumption constraint.

Both Kingston (1989) and Black and Perold (1992) assume lifetime utility is derived from a consumption stream of infinite horizon. Alternatively, utility can be derived from terminal portfolio value. Under such an assumption, Balder and Mahayni (2010) show that CPPI is optimal when the utility is of HARA family, i.e. the utility is measured as the portfolio value less the guarantee, and that OBPI is optimal when the terminal portfolio value is constrained to be above the guarantee. The payoffs of both approaches consist of the payoff of a constant mix strategy plus an additional term from the guarantee.

Bernard and Kwak (2016) provide further evidence to the optimality of the CPPI strategy. Prior studies tend to specify a parametric form of the utility function before solving the optimisation problem. By contrast, Bernard and Kwak (2016) infer a class of utility function for which a given investment strategy is optimal. They find a CPPI strategy with an adapted multiplier is optimal for an investor with HARA utility function. The multiplier is time varying to accommodate possibly stochastic equity risk premium and equity volatility.

2.3.4 Application to defined contribution pension plans

Unlike traditional insurance, where everyone can benefit from risk pooling, portfolio insurance involves hedging against a common risk, which is market risk. For every investor purchasing portfolio insurance, there must be some other investor(s) selling it. Leland (1980) analyses the type of investors who will benefit from purchasing portfolio insurance. He concludes that pension fund managers who want to provide a minimum fund value, and can take some risks thereafter will find portfolio insurance attractive. With the expansion of DC pension schemes and the intention to provide downside risk protection in DC plans, portfolio insurance strategies have been increasingly applied to managing DC pension plans.

Blake et al. (2001) investigate the impact of asset allocation strategies on the performance of DC pension plans in the accumulation phase. They consider five investment strategies, including both static and dynamic ones. The dynamic ones contain a threshold strategy and a CPPI strategy. The threshold strategy rebalances portfolio in the opposite direction of the portfolio insurance strategies. When the fund is performing well, it invests more in low-risk portfolios to protect its value; when the fund is performing poorly, it allocates more assets to high-risk portfolios to benefit from higher expected returns. Comparing these two dynamic strategies, Blake et al. (2001) find that the CPPI strategy gives a higher median pension at retirement than the threshold strategy. The CPPI strategy is also less expensive to fund in that the required contribution rate to achieve the same level of pension outcome is lower.

Pézier and Scheller (2011) apply a CPPI strategy for performance sharing rules between pension fund sponsors and fund members. They propose a cumulative performance sharing rule to improve the welfare of both fund managers and members. The proposed cumulative rule would, however, expose future pensioners to credit risk on fund managers, i.e. the risk that the final fund value is below the guaranteed minimum one. They show that the risk can be greatly reduced by following investment strategies of CPPI style.

While Blake et al. (2001) and Pézier and Scheller (2011) explicitly consider CPPI strategies, Blake et al. (2013) find that the optimal investment strategy is in line with the portfolio insurance strategy when the fund is sufficiently above its target. Otherwise, the fund should follow a threshold strategy.

2.4 Housing tenure decisions

Housing tenure decisions can be studied from a number of perspectives, such as demographic, social or psychological. Since Chapter 5 analyses the impact of housing tenure choice on consumption and retirement savings, the present section reviews the relevant papers adopting an economic perspective.

The model presented in Henderson and Ioannides (1983) and Fu (1991) lays the foundation for studying housing tenure choice. A key feature of the model is to recognise dual purpose of housing and separately identify consumption demand and investment demand for housing. If an individual's consumption demand is less than the investment demand, Henderson and Ioannides (1983) show that they should owner occupy the housing they consume in the absence of capital market imperfections. If the consumption demand for housing exceeds the investment demand, Henderson and Ioannides (1983) make an assumption that an individual cannot consume housing services from both owner-occupied and rented housing. Under such an assumption, one may still choose to own to avoid rental externality[3].

Fu (1995) extends the consumption-investment framework of Henderson and Ioannides (1983) to include a liquidity constraint, prohibiting borrowing against future income and future gains of housing investment. The presence of liquidity constraint leads to competing effects on housing tenure decision. On the one hand, an increase in the certainty-equivalent future wealth reduces the marginal utility of consumption, increasing current

[3]As pointed out in Section I of Henderson and Ioannides (1983), the rental externality arises from the maintenance problem. Since a landlord is unable to collect all the costs of utilisation from a tenant, e.g. wear and tear, tenants pay less than owners for maintenance costs. This presents an externality problem in the rental market.

consumption and decreasing investment. On the other hand, it decreases the risk aversion to uncertain house price appreciation, resulting in more investment and less current consumption.

The impact of house price uncertainty on housing tenure choice derived from theoretical models are later tested empirically, using both national- and micro-level data. These papers are reviewed in Section 2.4.1. Potential homeowners who take out a mortgage face the borrowing constraints that impose limits on the loan-to-value ratio and the mortgage-payment-to-income ratio. Two separate streams of literature considers the impact of uncertain labour income and borrowing constraints on home ownership. The relevant papers are reviewed in Sections 2.4.2 and 2.4.3, respectively.

2.4.1 Uncertain housing expenditure

Rosen et al. (1984), using the annual U.S. data, find that housing price uncertainty significantly reduces the aggregate proportion of homeowners. Turner (2003), using micro-level survey data, finds a similar result. In addition, Turner (2003) shows that conditional on home ownership, households demand less housing when expected house-price volatility is relatively high. Rosen et al. (1984) assume renting and owning are mutually exclusive and that house prices and rent are uncorrelated. Turner (2003) excludes anticipated volatility in rent when estimating the propensity to own. This may overestimate the riskiness of home ownership, since houses provide a hedge against rent risk (Sinai and Souleles, 2005).

Sinai and Souleles (2005) consider asset price risk together with rent risk to capture the entire risk position in housing tenure choice. They explicitly allow future house prices to endogenously fluctuate with rent shocks. Sinai and Souleles (2005) find an increase in housing market volatility can increase the demand for owning due to the desire to hedge rental expenditure risk.

2.4.2 Uncertain labour income

In analysing factors that affect the likelihood of home ownership, another factor found to be significant is income variability. The variability of income can be measured in several ways. The coefficient of variation (CV) is used by Haurin (1991) to measure the intertemporal variability of income for a period of eight years. Haurin (1991) shows that the likelihood of home ownership is negatively correlated with income risk, but the relationship between income risk and housing demand is not statistically significant. Robst et al. (1999) refine the measure to remove income variability due to increasing work experience and job tenure, and draw a similar conclusion to that in Haurin (1991).

More recent papers start considering the income uncertainty in conjunction with uncertain housing expenditure. The covariance between income and housing costs determines the extent to which uncertain housing costs affect fluctuations in non-housing consumption. Ortalo-Magné and Rady (2002) show analytically that households with higher covariance between labour earnings and rent are more likely to rent. The high covariance means rental payment can greatly offset the fluctuations of income to generate a smooth non-housing consumption path.

Davidoff (2006) provides empirical evidence to support the finding in Ortalo-Magné and Rady (2002), although the negative effect on the probability of ownership is of marginal significance. He shows that the covariance has a much larger impact on housing demand conditional on home ownership. A one standard deviation increase in covariance between labour income and housing prices, on average, reduces housing investment by approximately $7,500.

2.4.3 Borrowing constraints

When becoming homeowners, households often need to take out mortgages. The maximum allowable loan is typically constrained by the requirements on down payment and mortgage repayment to income. Such constraints would certainly affect housing tenure choice. Brueckner (1986) explores the trade-off induced by the down payment. In a two-period model, one needs to sacrifice initial consumption in order to benefit from tax-exempt mortgage payment associated with home ownership. Note that, however, Australian tax rules do not allow tax deductibles on mortgage interest if the residential property is owner occupied.

The empirical studies of the impact of borrowing constraints on home ownership are initiated by Linneman and Wachter (1989) and Zorn (1989). A common approach used in these and later studies consists of three steps: *1*) stratify the sample into constrained and unconstrained households; *2*) estimate the desired amount of housing services from the unconstrained group; *3*) examine how the constrained group would behave if constraints were relaxed. Linneman and Wachter (1989) use a regression method to determine the optimal services, while Zorn (1989) uses a utility maximisation approach. Subsequent studies, such as Duca and Rosenthal (1994) and Haurin et al. (1997), use more sophisticated methods to split the sample into two groups depending on whether the housing tenure choice is affected by borrowing constraints. Most of these studies find that wealth constraints reduce home ownership propensities more than income constraints.

With the increased use of automatic underwriting in mortgage market, which are largely based on credit scores, researchers include credit quality into the borrowing constraints along with income and wealth. Rosenthal (2002) shows that credit constraint presents a barrier to home ownership, especially for Hispanic and lower-income households. Barakova et al. (2003) extend the work of Rosenthal (2002) to compare the impact of income, wealth, and credit quality constraints on home ownership over a decade. They find income- and wealth-based constraints have been less binding, while the importance of credit-based constraints might have been increased.

2.5 Household portfolio choices

The literature on the optimal household portfolio choices can be dated back to the seminal papers of Samuelson (1969) and Merton (1969). Merton (1971) and Merton (1973) further develop the model framework. These papers are reviewed in Section 2.5.1. Early studies tend to assume that asset returns are independent and identically distributed (i.i.d.) over time, whereas in reality investment opportunities usually vary over time. Section 2.5.2 discusses optimal asset allocation under stochastic investment opportunities. The model framework is further extended to allow for labour income (Section 2.5.3) and housing (Section 2.5.4).

Optimal consumption and asset allocation towards the end of lifecycle are complicated by uncertain future lifetime (Section 2.5.5). Since the seminal paper of Yaari (1965), there have been extensive studies on the role of life annuities in retirement. The theoretical attractiveness of lifetime annuities accompanied with the low voluntary annuitisation rate across the globe gives rise to extensive research to explain the annuity puzzle. One of the reasons is liquidity to cover sizeable out-of-pocket health expenditure (Section 2.5.6). Uncertain health expenses create a need for precautionary savings, which can be in the form of liquid assets and home equity. Section 2.5.7 reviews the papers that consider housing wealth in the post-retirement stage.

2.5.1 Foundations

Modern portfolio theory is often thought to have started with Markowitz (1952) mean-variance analysis. The model assumes investors care only about expected return and variance of return over a single period. Markowitz (1952) shows investors should choose "mean-variance-efficient" portfolios, which minimise the variance of portfolio return given expected return, and maximise the expected return given variance.

One theoretical shortcoming of mean-variance analysis is the single-period assumption, so that investors cannot rebalance portfolios repeatedly over time. A multi-period generalisation that corresponds to lifetime planning of consumption and investment decisions is more realistic. Samuelson (1969) and Merton (1969) extend the portfolio selection problem into multiple periods. They assume an individual can invest in two assets: a risk-free asset with a constant rate of return, and a risky asset with a constant expected rate of return. The individual has only financial wealth and no labour income. The results give two sets of conditions under which the optimal portfolio decisions are independent of time. In other words, investors behaves myopically, as if the current period is the last one. The first set of condition is that investors have CRRA utilities and asset returns are i.i.d. over time. The second condition is that investors have log utility.

Merton (1971) extends the results in Merton (1969) in a few directions: *1*) considering a more general utility function (the HARA family), *2*) introducing wage income, and *3*) exploring alternative price behaviour other than the GBM. One important result of Merton (1971) is the "mutual fund" theorem for a multi-period investor. In particular, given n assets whose prices are stationary and log-normally distributed, all investors (i.e. with different preferences, wealth distribution, and time horizon) allocate between the same pair of mutual funds (each constructed from a linear combination of these assets). In fact it is possible to choose one fund to be the risk-free asset and the other fund to hold only risky assets, in which case the "mutual fund" theorem in Merton (1971) is more similar to that developed in Tobin (1958). The result implies that one can work with two-asset case without loss of generality provided log-normality of prices is assumed.

A further extension in Merton (1971) is to incorporate wage income. Under the simplified conditions that future wages can be borrowed against, and wage income is constant, the optimal consumption and portfolio rules are to capitalise the wage income at the risk-free rate and treat the capitalised value as an additional to the current portfolio. The case of a stochastic wage income is briefly discussed in an example where the event of a wage increase follows a Poisson process. It is often impossible to borrow against future labour income due to the moral hazard problem. An employee is unlikely to continue working when he has spent future wages. A more realistic setting is to assume non-tradable labour income. The relevant literature is reviewed in Section 2.5.3.

Merton (1971) also considers three alternative price mechanisms to recognise time-varying investment opportunities in the real world. These assumptions are in contrast to the constant interest rate and constant risk premium assumed in Samuelson (1969) and Merton (1969). The first alternative for the price mechanism (which is called the "asymptotic 'normal' price-level") is of particular interest, since the result gives rise to an intertemporal hedging demand. The mechanism assumes the existence of "normal" price function, and that the investor expects the "long-run" price to proceed towards the normal price. If the investor has an infinite time horizon and a constant absolute risk aversion preference, he will allocate a constant proportion of his wealth to the risky asset. This proportion is higher than the one who believes GBM hypothesis would hold, even though both investors have the same utility function and identical expectations about the instantaneous mean and variance. Merton (1971) identifies the intertemporal hedging demand, and this feature is left for formal discussion in Merton (1973).

Merton (1973) extends the capital asset pricing model (CAPM) model[4] to the intertemporal capital asset pricing model. Merton (1973) shows that the demand functions for assets are different from those derived from one-period model, since investors in multi-periods would desire to hedge against future changes in investment opportunities. For instance, an investor would hold long-term bonds to hedge against declines in interest rates. When interest rates fall, the value of the long-term bonds increase to compensate for the loss of income generated by the portfolio. Merton (1973) generalises the "mutual fund" theorem developed in Merton (1971) to the "three fund" theorem. In particular, given n risky assets and a riskless asset with time-varying interest rate, all risk-averse investors allocate among three portfolios ("mutual funds") constructed from these assets. Two of the funds provide an efficient frontier, while the rest plays the role of intertemporal hedge against unfavourable shifts in the frontier. Similar to the "mutual fund" theorem, the construction of each fund is purely technological, and the investor's allocation among

[4] Treynor (1961), Sharpe (1964), Lintner (1965), and Mossin (1966) introduce the CAPM, which builds on the Markowitz (1952) mean-variance analysis. The CAPM predicts the relation between risk and expected return under the market equilibrium conditions.

the three funds depends only on the individual preference. Merton (1973) assumes that the variation in investment opportunities comes from time variation in the interest, but there are other sources of changing investment opportunities, e.g. time-varying risk premia and volatility. This issue is further explored in Section 2.5.2.

2.5.2 Stochastic investment opportunities

Empirical evidence has shown time variation or predictability in asset returns (see e.g. Keim and Stambaugh, 1986; Campbell and Shiller, 1988; Fama and French, 1988; Bekaert and Hodrick, 1992). This motivates further work on dynamic optimal portfolio choice under stochastic investment opportunities following the seminal paper of Merton (1973). One common objective function of the optimal investment strategy is to maximise an investor's expected utility from his terminal wealth, as opposed to the intertemporal consumption. Most of these studies show that the optimal portfolio choice consists of some hedging portfolio(s) with a mean-variance efficient portfolio.

Sørensen (1999) and Brennan and Xia (2000) study the optimal cash-bond-stock mix under stochastic interest rates and constant risk premia. Sørensen (1999) assumes the short interest rate follows Vasicek (1977) one factor model, whereas Brennan and Xia (2000) assume the two-factor Hull and White (1990) model. The results are similar in that the investor will require extra components in his optimal portfolio to hedge against changes in the investment opportunity set, alongside a mean-variance efficient portfolio. The investment horizon is important in determining the optimal portfolio mix. Sørensen (1999) shows that the hedging portfolio is the zero-coupon bond that expires at the investment horizon. Brennan and Xia (2000) show that, in general, the investor needs two portfolios to hedge against changes in the instantaneous interest rate and the central tendency of the rate. If one of the two bonds matures at the investment horizon, the hedging portfolio degenerates to one, which is the case in Sørensen (1999).

Kim and Omberg (1996) examine the optimal portfolio strategy for an HARA investor who can trade a risk-free asset and a risky asset, assuming a constant risk-free rate and a stochastic risk premium[5] that follows the Ornstein-Uhlenbeck process. The optimal asset allocation strategies are of considerable complexity and variety. They can depend on investors' risk preference, the correlation between the risk premium and the risky asset return, and investment horizon. One generalised result is that all investors except for those with log utility function always hedge against or speculate risk premium uncertainty.

Brennan et al. (1997) set up a more complicated investment problem in which time variation in opportunity set is driven by three state variables: the short-term interest rate, the yield on a long-term bond, and the dividend yield on a stock portfolio. The reason for choosing the first two variables is their power in predicting stock returns. The last one is chosen because of its relationship with expected changes in the short rate. The number of state variables is limited to three due to computational constraint. Brennan et al. (1997) show long-term investors tend to allocate a larger fraction to risky assets compared to myopic investors, because the mean reversion in stock and bond returns make them less risky in the long run.

Liu (2007) derives analytical results of optimal portfolio problem when the asset returns are quadratic and investor has a CRRA utility function. His solutions include as special cases the results obtained in many previous studies. Liu (2007) contributes to the literature through three applications. The first application is a bond portfolio selection problem where the term structure model is quadratic; the second application is a stock portfolio selection problem where the stochastic volatility follows the Heston (1993) model; the third application is a bond-stock portfolio selection problem where the short rate follows the CIR model (Cox et al., 1985) and the stock volatility follows the Heston (1993) model. He proves that when the state variables that govern interest rates and stock returns are independent, the optimal bond-stock mix problem can be decomposed into bond-only and stock-only portfolio selection problems.

Long-term bonds can hedge against the risk that interest rates will decline. This hedging effect is, however, weakened by the sensitivity of prices to changes in expected inflation

[5]Kim and Omberg (1996) define the risk premium on the risky asset as the Sharpe ratio.

rates. None of the papers discussed so far consider stochastic inflation. Brennan and Xia (2002) develop a framework for analysing dynamic asset allocation problems under inflation in a continuous-time setting. They assume the investor has a finite horizon and can only invest in nominal assets. The expected rate of inflation and the real interest rate follow correlated Ornstein-Uhlenbeck processes, and the risk premia are constant.

2.5.3 Labour income

It is reasonable to assume investors' wealth consists only of financial assets for institutional investors and retired individual investors. It is, however, an unrealistic assumption for individuals earning labour income. Those individuals own a valuable asset called human capital, which refers to the present value of expected future labour earnings.

Bodie et al. (1992) study the impact of labour flexibility on consumption, saving, and portfolio investment decisions. Under flexible labour supply, an individual can continuously vary the quantity of labour to supply and the amount of leisure to consume. Under fixed labour supply, the individual cannot change his leisure and labour once they are determined. Bodie et al. (1992) show that if future wages are certain, an employed investor should tilt his financial portfolio towards stocks relative to a retired investor. The proportion of wealth invested in risky asset decreases as he approaches retirement. These results hold for both flexible and fixed labour supply. If labour supply is flexible, employed individuals will take greater risk in financial investments.

In practice future wages are uncertain for most investors. Bodie et al. (1992) study the case where stochastic wage income can be perfectly hedged using the risky asset. This is a rather simplified assumption. Markets may be incomplete such that income risk cannot be traded away. Viceira (2001) examines optimal portfolio choice with risky labour income, which is not perfectly correlated with stock returns. When labour income risk is independent of stock return risk, an employed investor should still hold a larger proportion of financial portfolio in the risky asset than a retired investor. A small positive correlation between labour income and risky asset returns can significantly decrease the optimal allocation to stocks. Apart from correlation with stock returns, Viceira (2001) considers another aspect of labour income risk: the variance of labour income. Labour income needs to be highly volatile to significantly decrease the tilt towards the risky asset.

2.5.4 Housing

Housing is the single most important asset in most investors' portfolios. Nevertheless, none of the models discussed so far include housing as an asset. Housing differs from other financial assets in several aspects. It serves a dual purpose, both an asset in a portfolio and a consumption good. Housing investment is often leveraged through mortgages and relatively undiversified. Housing is illiquid in that transaction costs must be paid to adjust the housing asset. In addition, there is a parallel market for housing such that a household can transit between renting and home ownership.

Grossman and Laroque (1990) analyse optimal consumption and investment decisions in which consumption is derived from a single illiquid durable consumption good (such as a house). The durable good is indivisible in the sense that the only way to adjust the consumption level (beyond appreciation) is to sell the current durable good for a new one. It is assumed that the transaction costs are a fraction of its value. The consumer should adjust his consumption only when the consumption to wealth ratio deviates substantially from his target level. The adjustment in consumption is deferred due to the transaction cost. Damgaard et al. (2003) carry over the approach and the conclusions in Grossman and Laroque (1990) to a more general framework with a perishable and a durable consumption good. More specifically, an individual with an infinite time horizon can allocate his wealth among one riskless and n risky financial assets, and two consumption goods. There is no stochastic labour income stream.

Cocco (2004) studies the impact of housing investment on the portfolio choice in a richer framework. The model allows for the stochastic labour income during pre-retirement phase and assumes deterministic income during post-retirement phase. The investor has a certain lifespan and a given retirement age. The model also allows for the house price to be correlated with labour income shocks and stock returns. The investor needs to pay a one-time fixed cost to participate in the equity market. The model does not allow for rental market. Cocco (2004) concludes that housing investment reduces the benefit of equity market participation for the young and the poor, because their financial assets are largely dominated by real estate. In addition, house price risks crowd out stock holdings, especially for investors with low financial net worth.

Yao and Zhang (2005) introduce a house rental market and survival probabilities to optimal portfolio choice model, but they omit the fixed cost of participating in the equity market, as in Cocco (2004). Yao and Zhang (2005) show that wealth to labour income ratio and age are two key factors in making the renting versus owning decisions. The wealth to labour income ratio is important because it determines the level of liquidity constraint. An investor with a high wealth to labour income ratio is more able to make the down payment. The wealth to labour income ratio that triggers home ownership generally decreases with age, reaching the trough at the age of retirement, and then monotonically increases. The decreasing pattern in the pre-retirement phase reflects the investor's anticipated wage increases. The investor wants to own a larger house relative to his labour income to minimise the cost of moving when he is younger. The increasing pattern in the post-retirement phase is due to the bequest motive. The investor needs to have a high wealth to labour income[6] ratio to justify the house liquidation cost at the time of death. When indifferent between renting and owning, homeowners hold a lower proportion of wealth in equity. This substitution effect is also found in Cocco (2004). Yao and Zhang (2005) find the proportion of equity in liquid assets is higher for homeowners. This result can be explained by the low correlation between stock returns and housing returns, so the investor can benefit from the diversification effect.

Hu (2005) is similar to Yao and Zhang (2005) in that renting versus buying decision is endogenous, but Hu's model abstracts from survival probability, bequest motive, and housing endowment. In particular, the model assumes no mortality prior to the final

[6]The labour income after retirement is essentially non-financial income, e.g. pension, social security payments, and distributions from retirement account. Yao and Zhang (2005) stick with the term "labour income" to avoid multiple definitions for the state variable.

period, no bequest, and no homeowners in the first period. But Hu (2005) explicitly models the cost of increasing the level of mortgage. This feature is absent from Yao and Zhang (2005).

Flavin and Yamashita (2002) examine the optimal portfolio problem in the presence of owner-occupied housing in a different way. Instead of using a dynamic lifecycle model, they use a static mean-variance framework, while the lifecycle effects are captured by the exogenously given ratio of house value to wealth, or the "housing constraint". The returns on financial assets are adjusted by the inflation and income tax. Mortgage borrowing is limited to the housing value, and no short position is allowed in other financial assets. Flavin and Yamashita (2002) find that housing constraint plays a significant role in optimal portfolios. The results are similar to Cocco (2004). Young households who are typically highly leveraged and are therefore in a highly risky position tend to pay down their mortgage or buy bonds. Older households with a lower ratio of housing to wealth hold more fraction of assets in shares.

2.5.5 Uncertain lifespan

Optimal consumption and asset allocation towards the end of lifecycle is complicated by uncertain future lifetime. In theory, life annuities play a key role in retirement planning. Yaari (1965) develops a basic lifecycle model with one single interest bearing asset and no uncertainty other than future life time. He concludes that an individual without bequest motive should annuitise all his wealth, provided that the annuity price is actuarially fair. The reason for this is that annuities provide a higher return relative to risk-free assets due to mortality credits, thus allowing for a higher consumption level. Besides, annuities provide lifetime income streams that can hedge against the risk of outliving one's financial resources. Richard (1975) obtains the results in Yaari (1965) using the Merton (1971) framework.

Davidoff et al. (2005) relax many of the restrictive assumptions imposed in Yaari (1965). In particular, annuities need not be actuarially fair, and the utility can be intertemporally dependent. The full annuitisation is still optimal for a consumer without a bequest motive, so long as the market for annuities is complete in that the consumer can find an annuity income stream to match his desired consumption path. Even when the market is incomplete, a risk averse individual should annuitise the majority of his wealth.

Despite the theoretical attractiveness of annuities, there is a lack of voluntary annuitisation around the world. This gives rise to an extensive research on explaining the so-called "annuity puzzle". Brown (2007) reviews an extensive list of the rational and behavioural reasons for limited voluntary annuitisation. Among those rational reasons, uncertain out-of-pocket health expenditure is of particular interest as it relates to Chapter **??**. The related literature is reviewed in the next section (Section 2.5.6).

2.5.6 Uncertain health expenditure

The literature offers inconclusive findings regarding the impact of uncertain health expenses on annuitisation. On the one hand, precautionary saving motives to cover uncertain health costs can reduce demand for annuities. For example, Sinclair and Smetters (2004) and Turra and Mitchell (2008) show that health expenditure shocks tend to reduce the annuity demand, due to the motivation for precautionary savings. In addition, health shocks can shorten the life expectancy, which impairs the value of annuities. On the other hand, the need to hedge future increased health expenditure can increase annuity demand. Davidoff et al. (2005) show, in a two-period model, that the net effect depends on the timing of health expenditure. If it tends to occur early in life, the need for liquidity might reduce the value of illiquid annuities. By contrast, health shocks occurring later in life can enhance annuity purchases. The mortality premium in annuities makes them superior to bonds in substituting for LTCI.

Peijnenburg et al. (2016) and Ai et al. (2017) extend the literature in different directions. Peijnenburg et al. (2016) consider the more general background risk. They find full annuitisation at retirement remains optimal with background risk, since annuitants can save out of annuity income to insure against the background risk. Nevertheless, the result is based on the assumption that the background risk follows a serially uncorrelated log-normal distribution, which may not be appropriate to model uncertain healthcare cost. Ai et al. (2017) consider systematic improvement of longevity, which can lead to morbidity expansion or compression. They show that except for retirees with low wealth who can get government subsidies, morbidity expansion or slight compression leads to higher annuity values. Severe morbidity compression, on the other hand, might reduce demand for annuities. The results are consistent with the conclusion in Davidoff et al. (2005). Morbidity expansion associated with increased longevity essentially means the timing of health shocks are delayed, enhancing demand for annuities.

Apart from the timing of health shocks, the assumption about the timing of purchasing annuities plays a role in explaining the impact of health expenditure on annuity demand. Most papers reviewed in this section assume annuitisation is a one-off decision made at retirement. By contrast, Pang and Warshawsky (2010) and Peijnenburg et al. (2015) consider the possibility of additional annuitisation after retirement. When retirees have the flexibility to rebalance their portfolio among annuities[7], bonds, and stocks every year, the presence of uninsured health expenses can make annuities more appealing, especially at more advanced ages. As uninsured health expenses generally result in precautionary savings, asset allocation is shifted away from risky assets to riskless assets. Annuities dominate bonds in hedging against life-contingent health expenses and longevity risk, and the superiority increases with age.

The papers reviewed so far in this section assume exogenous health expenditure. While empirical studies find little evidence that higher healthcare utilisation can lengthen life expectancy (see e.g. Finkelstein and McKnight, 2008), individuals might be able to choose healthcare cost to improve their welfare. Yogo (2016) consider endogenous health expenditure. The retiree can improve her health but cannot reduce it through negative expenditure. He finds retirees in poorer health tend to spend a higher proportion of income in healthcare. Another main contribution of Yogo (2016) is the inclusion of housing. He shows that portfolio share in housing is negatively correlated with health status and that

[7] Unlike bonds or stocks, however, retirees cannot sell annuities.

it declines sharply with age. The literature that consider housing over the course of retirement is discussed in more detail in the next section (Section 2.5.7).

2.5.7 Housing revisited

An emerging stream of literature studies the role of housing in the post-retirement stage when uncertain healthcare expenditure represents a key risk. While a large number of papers (reviewed in Section 2.5.4) examine the optimal portfolio choice in the presence of housing when households labour income risk, relatively few research includes both housing wealth and healthcare expenditure (De Nardi et al., 2016). Yet the significance of home equity in elderly homeowners makes it worthwhile to examine its role during retirement.

The elderly usually hold substantial wealth in the form of home equity (see e.g. Venti and Wise, 2004; Poterba et al., 2011). They rarely drawdown their housing assets unless late in life or when in long-term care (see e.g. Walker, 2004). Under these assumptions, Davidoff (2009) shows, in a two-period model, that illiquid home equity can crowd out demand for annuities and LTCI separately, and can reverse the complementarity between annuities and LTCI. Davidoff (2010) uses a one-period model to show that home equity can weaken demand for LTCI since home equity provides an alternative source to finance long-term care costs.

For asset-rich-cash-poor retirees, home equity release products offer a way to increase liquid wealth while still staying at home. Hanewald et al. (2016) compare two such products, namely reverse mortgage and home reversion plan, and conclude that reverse mortgage gives higher utility gains. Shao et al. (2017) build on the work of Davidoff (2010) and Hanewald et al. (2016) to investigate the complementarity between LTCI and reverse mortgage in a multi-period setting. They find the presence of reverse mortgage greatly enhances demand for LTCI, improving retirees' welfare.

In addition to health risk, another key component featuring the post-retirement stage is public pension. Andréasson et al. (2017) model optimal housing at retirement, optimal consumption and risky asset allocation over the course of retirement for an individual receiving means-tested Age Pension (i.e. the public pension in Australia). Housing allocation creates a trade-off for future consumption. Since home equity is exempted from the means-tested Age Pension, a higher proportion of wealth in housing means retirees are entitled to more Age Pension income. On the other hand, higher home equity as a proportion of total wealth leaves less liquid wealth to be consumed. The optimisation results show that households should allocate a high proportion of total wealth to housing to benefit from Age Pension.

2.6 Conclusions

A number of papers derive the optimal asset allocation strategies for a DC pension fund that provides a minimum guarantee in the form of an annuity or a cumulative rate of return. In addition, research has shown the suitability of adopting portfolio insurance strategies in pension funds that provide minimum guarantee, and proved the optimality of portfolio insurance strategies under certain conditions. But no study so far has examined the performance of portfolio insurance strategies in a DC pension plan that targets an annuity at retirement. Chapter 4, building on some of the results in Chapter 3, will fill in this gap by implementing option-based and constant proportion strategies in a DC fund that aims to provide longevity- and inflation-protected income streams in retirement with a high probability.

There has been growing discussion on using home equity to fund retirement (see e.g. Mitchell and Piggott, 2004), but the decision to purchase a home property is not often considered in conjunction with retirement planning in the literature. The theoretical work of housing tenure choice is typically confined to a two-period model that yields analytical results to predict the probability of home ownership; the empirical work usually develops econometric models to regress home ownership against a number of social and economic factors. Chapter 5 explicitly considers the impact of home ownership on retirement savings in a multi-period model, and conducts a welfare analysis.

The role of housing in portfolio choice has been explored widely to explain the household portfolio compositions over the lifecycle. These papers typically focus on the sources of risk in the pre-retirement stage, e.g. labour income risk, while risks faced by retirees are less of a concern. Among the studies that discuss the impact of uncertain lifespan and health shocks on annuitisation, most of them exclude home equity from household portfolios. Yet the role of housing wealth among the elderly can hardly be overlooked. As discussed in Section 1.1, retirees have high home ownership rates and home equity constitutes a significant proportion of wealth among retired homeowners. Chapter 6 will fill in the gap by studying whether retired homeowners have different demand for annuities and LTCI from that of retired non-homeowners. Both groups of retirees face idiosyncratic mortality risk and uncertain health expenditure.

Among a handful of papers considering home equity when studying optimal consumption and portfolio choice at retirement, most of them use a power utility function, which imposes that the coefficient of risk aversion is the inverse of the EIS. However, as discussed in Section 1.1, such a rigid structure may fail to capture the preference of a large majority of retirees. Chapter 6 will use an Epstein-Zin-Weil (Epstein and Zin, 1989; Epstein and Zin, 1991; Weil, 1989) type utility function that breaks the link between risk aversion and the EIS.

Chapter 3

Longevity index: A cross-country study

3.1 Introduction

This chapter conducts an empirical study on the longevity index, providing a background for Chapter 4. The longevity index can be used to model the investment objective of the target annuitisation fund. It refers to the amount of wealth required at retirement to meet a minimum level of inflation- and longevity-protected retirement income stream with a high probability (e.g. 95%). Ideas similar to the longevity index are also seen in the industry. BlackRock publishes CoRI^{TM} Retirement Indexes, which reflect the cost of a lifetime annuity for an individual with an average life expectancy. The difference between the longevity index and the CoRI^{TM} index is that the former assumes longevity risk is borne by individuals and hence it is partially insured to reflect the thin annuity market, whereas the latter assumes the full insurance of longevity risk by annuity providers.

The rest of the chapter is organised as follows. Section 3.2 defines the longevity index. Section 3.3 constructs and compares the longevity indices across four countries: Australia, Japan, the U.K. and the U.S. Section 3.4 concludes.

3.2 Index definition

The methodology outlined in Sherris (2009) is used to construct the longevity index for the four countries. The index uses fixed interest rates and inflation rates that reflect the current levels of these economic factors.

Denote $I_{x,t}$ as the index for an individual aged x at time t. It is defined as the present value of an income stream that allows an individual to consume 1 per annum in real term for the rest of his or her life with probability greater than or equal to $1-\delta$. Suppose consumption occurs at the start of each year. The index is given by

$$I_{x,t} = \frac{(1+r_t) - \left(\frac{1}{1+r_t}\right)^{T^d_{x,t}}}{r_t}, \tag{3.1}$$

where r_t is the real interest rate at time t, and $T^d_{x,t}$ is the smallest integer such that the probability of dying after age $x + T^d_{x,t}$ last birthday is less than or equal to δ. The horizon $T^d_{x,t}$ changes over time to reflect systematic mortality improvement. Since the real interest rate cannot be directly observed, it is derived from

$$r_t = \frac{1+i_t}{1+f_t} - 1,$$

where i_t is the nominal interest rate and f_t is the inflation rate at time t.

3.3 Index construction

3.3.1 Data

Table 3.1 lists the data used to construct the longevity index for each of the four countries. The ten-year government bond yield is chosen as the nominal interest rate since it reflects the long-term nature of the cash flows and avoids the impact of credit risk. The inflation rate should also reflect the long time horizon to make it consistent with the interest rate.

The core inflation can be used to measure the long-run inflation. The fundamental concept of core inflation is that it should well indicate the underlying inflation trend. It tracks the component of overall price change that is expected to persist for several years (Clark, 2001).

Table 3.1. Data used to construct the longevity index and data sources.

Data item	Source
Australia	
Consumer price index (CPI)	Australian Bureau of Statistics
Capital market yield - Government bonds	Reserve Bank of Australia
Japan	
CPI	Statistics Bureau, Ministry of Internal Affiars & Communications
Government guaranteed bond yield - 10 year	Bank of Japan
U.K.	
Retail price index	Office for National Statistics
U.K. gross redemption yield on 10-year gilt edged stocks	Bank of England
U.S.	
CPI - All urban consumers	Bureau of Labor Statistics
Market yield on U.S. Treasury securities at 10-year constant maturity	Federal Reserve

Note: The mortality data for all four countries is retrieved from Human Mortality Database.

A number of core inflation measures have been proposed to remove transient noise. The most widely used ones are inflation excluding food and energy published by the U.S. Bureau of Labor Statistics, inflation excluding energy (Clark, 2001), trimmed mean and median inflation (Bryan and Cecchetti, 1994). These measures are easy to calculate, but they contain a substantial amount of high frequency variation and do not forecast medium- to long-term inflation very well.

Cogley (2002) proposes an adaptive measure of core inflation. The essence of this measure is to exponentially smooth the aggregate inflation. Compared to other measures mentioned above, it filters out transient noise more effectively and better predicts the changes in inflation. It is also simple to implement compared to other statistical models such as the one proposed in Ball et al. (1990). As a result, the measure introduced in Cogley (2002) is adopted to calculate the long-run inflation. The exponential smooth method is given by

$$\widehat{\pi}_t = \begin{cases} g_0\pi_t + (1-g_0)\widehat{\pi}_{t-1} & \text{for } t > 1 \\ \pi_1 & \text{for } t = 1 \end{cases}, \tag{3.2}$$

where π_t is the actual inflation that is directly observable, $\widehat{\pi}_t$ the core inflation that is not directly observable, and g_0 a "gain" parameter between 0 and 1. The parameter g_0 is calibrated to minimise the forecast error given by

$$\sum_t \left(\pi_{t+\mathsf{H}} - \widehat{\pi}_t\right)^2, \tag{3.3}$$

where H is the forecast horizon in quarters. The forecast errors and the corresponding values of g_0 that minimise the forecast errors are displayed in Table 3.3. The smallest error(s) in each row is in bold. Table 3.2 shows the values of g_0 chosen for each country.

Table 3.2. The g_0 chosen for each country. g_0 is a "gain" parameter to smooth the inflation.

Country	Australia	Japan	U.K.	U.S.
g_0	0.275	0.175	0.125	0.125

Table 3.3. Forecast errors of quarterly inflation rates with different forecast horizons.

	g_0							
H	0.025	0.075	0.125	0.175	0.225	0.275	0.325	0.375
Australia								
1	0.0257	0.0164	0.0140	0.0130	0.0125	**0.0123**	**0.0123**	0.0124
2	0.0264	0.0173	0.0149	0.0139	0.0134	**0.0131**	**0.0131**	**0.0131**
3	0.0271	0.0183	0.0159	0.0149	0.0144	0.0141	0.0140	**0.0139**
4	0.0278	0.0194	0.0171	0.0162	0.0158	0.0156	0.0155	**0.0154**
5	0.0283	0.0203	0.0183	0.0176	**0.0175**	**0.0175**	0.0177	0.0180
6	0.0289	0.0210	0.0189	0.0182	0.0180	**0.0179**	0.0180	0.0182
7	0.0296	0.0219	0.0199	0.0192	**0.0190**	**0.0190**	0.0192	0.0194
8	0.0295	0.0219	0.0199	0.0191	0.0189	**0.0188**	0.0189	0.0191
9	0.0299	0.0225	0.0205	0.0198	**0.0195**	**0.0195**	0.0196	0.0198
10	0.0305	0.0233	0.0212	0.0204	0.0201	**0.0200**	**0.0200**	**0.0200**
11	0.0312	0.0243	0.0222	0.0215	**0.0212**	**0.0212**	**0.0212**	0.0213
12	0.0316	0.0249	0.0229	0.0221	0.0218	**0.0217**	**0.0217**	0.0218
Japan								
1	0.0219	0.0169	0.0149	0.0139	0.0133	0.0130	**0.0128**	**0.0128**
2	0.0226	0.0181	0.0164	0.0156	0.0151	0.0148	**0.0147**	**0.0147**
3	0.0228	0.0188	0.0173	0.0165	0.0161	0.0158	0.0156	**0.0154**
4	0.0235	0.0200	0.0189	0.0184	0.0181	**0.0180**	**0.0180**	**0.0180**
5	0.0238	0.0210	0.0204	**0.0203**	0.0205	0.0208	0.0212	0.0216
6	0.0241	0.0215	0.0210	**0.0209**	0.0212	0.0215	0.0220	0.0224
7	0.0243	0.0219	0.0212	**0.0211**	**0.0211**	0.0213	0.0215	0.0218
8	0.0246	0.0224	0.0218	0.0216	**0.0215**	0.0216	0.0217	0.0218
9	0.0250	0.0233	**0.0230**	**0.0230**	0.0231	0.0233	0.0236	0.0239
10	0.0254	0.0239	**0.0236**	**0.0236**	0.0237	0.0239	0.0241	0.0244
11	0.0257	0.0244	0.0242	**0.0241**	0.0242	0.0242	0.0243	0.0244
12	0.0248	0.0238	**0.0237**	0.0238	0.0238	0.0239	0.0239	0.0239
U.K.								
1	0.0509	0.0407	0.0380	0.0370	**0.0366**	**0.0366**	0.0369	0.0374
2	0.0520	0.0423	0.0397	0.0387	0.0383	**0.0381**	0.0382	0.0384
3	0.0527	0.0439	0.0420	**0.0416**	0.0418	0.0421	0.0426	0.0432
4	0.0530	0.0443	0.0423	0.0417	0.0415	0.0413	0.0412	**0.0411**
5	0.0523	0.0450	**0.0443**	0.0449	0.0460	0.0471	0.0483	0.0495
6	0.0519	0.0447	**0.0438**	0.0443	0.0451	0.0461	0.0471	0.0480
7	0.0527	0.0458	**0.0451**	0.0458	0.0470	0.0482	0.0495	0.0507
8	0.0523	0.0452	**0.0441**	0.0443	0.0450	0.0457	0.0464	0.0469
9	0.0526	0.0467	**0.0465**	0.0478	0.0497	0.0518	0.0539	0.0559
10	0.0529	0.0464	**0.0455**	0.0461	0.0473	0.0488	0.0503	0.0517
11	0.0533	0.0470	**0.0460**	0.0466	0.0479	0.0495	0.0513	0.0530
12	0.0537	0.0469	0.0450	**0.0447**	0.0451	0.0458	0.0466	0.0473
U.S.								
1	0.0175	0.0144	0.0137	**0.0134**	**0.0134**	0.0135	0.0137	0.0140
2	0.0178	0.0149	0.0142	**0.0140**	**0.0140**	**0.0140**	0.0142	0.0145
3	0.0181	0.0151	0.0144	0.0141	0.0139	0.0138	**0.0137**	**0.0137**
4	0.0184	0.0157	0.0151	0.0148	0.0146	0.0144	0.0142	**0.0140**
5	0.0189	0.0166	**0.0165**	0.0168	0.0171	0.0173	0.0176	0.0179
6	0.0191	**0.0170**	**0.0170**	0.0173	0.0177	0.0181	0.0185	0.0188
7	0.0193	**0.0172**	**0.0172**	0.0175	0.0179	0.0183	0.0186	0.0189
8	0.0195	0.0174	**0.0173**	0.0176	0.0179	0.0182	0.0184	0.0185
9	0.0199	**0.0181**	0.0183	0.0190	0.0197	0.0204	0.0210	0.0217
10	0.0200	**0.0181**	0.0183	0.0189	0.0196	0.0203	0.0209	0.0216
11	0.0201	0.0180	**0.0179**	0.0182	0.0187	0.0192	0.0196	0.0201
12	0.0201	0.0179	**0.0176**	0.0178	0.0182	0.0185	0.0187	0.0189

Figure 3.1 plots the smoothed inflation rates, ten-year government bond yields, and real interest rates for the four countries. It is noticeable from the graph that all the countries, except for the U.S., experienced extremely high inflation rates, and accordingly negative real interest rates during 1970s.

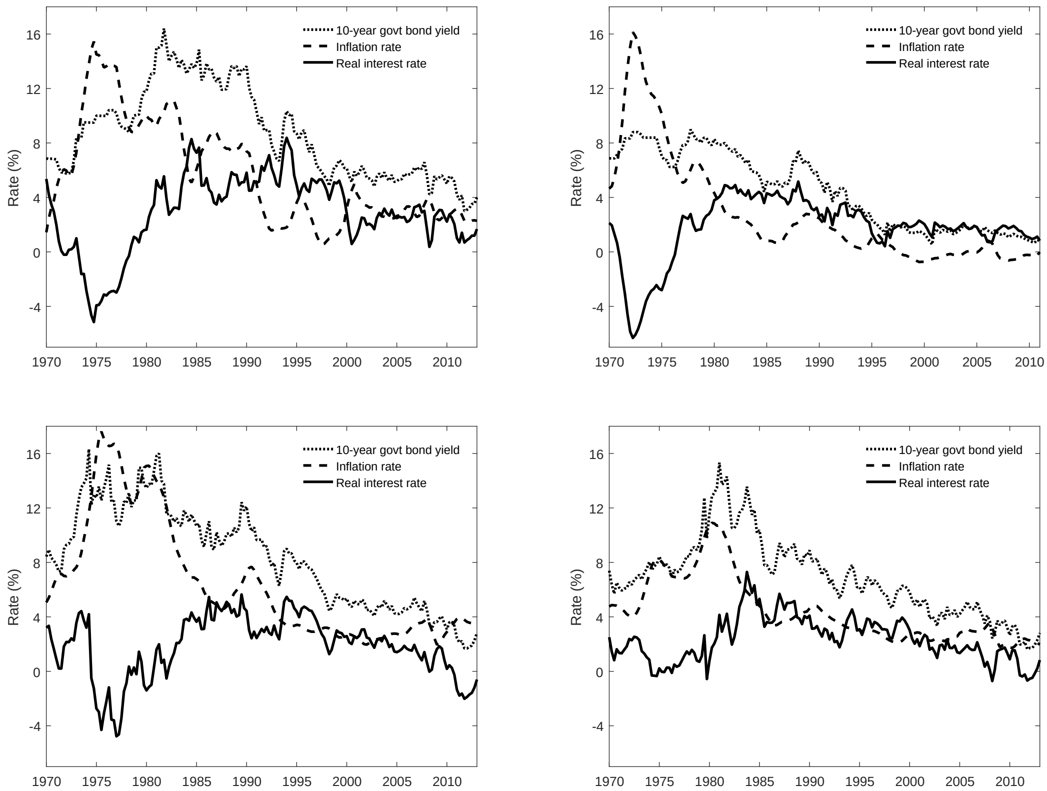


Figure 3.1. Economic series used to construct the longevity index: (Top Left) Australia; (Top Right) Japan; (Bottom Left) the U.K.; (Bottom Right) the U.S.

3.3.2 Comparison of index

The longevity index is constructed for both males and females aged 55, 60, 65 and 70 every quarter. Index values are calculated using the latest available life table at each point of calculation without incorporating future systematic mortality improvements. The probability of confidence, $1 - \delta$, takes the value of 95%. The indices for each country are plotted in Figure 3.2 to Figure 3.5. The evolutions of the indices of the four countries are similar in that they are highly volatile and show upward trends due to a combined impact of historical mortality improvement and decreasing real interest rate. The general patterns of the indices are similar for Australia, Japan, and the U.K. Japan has the highest overall index level for its well-known long life expectancy. The spikes in 1970s were caused by the high inflation rates over that period. As a result, the real returns became negative. This suggests that inflation can have a huge impact on the amount of wealth required to fund the post-retirement life. Inflation-linked bonds (e.g. NSW Waratah Bond issued by the New South Wales Treasury Corporation) are valuable instruments to hedge against inflation risk. The indices for the U.S. have a different pattern compared to the other countries. The overall level of longevity index is low due to its relatively low life expectancy. It did not experience a sudden increase in inflation rates over 1970s, so the spikes observed in other countries are less evident in Figure 3.5.

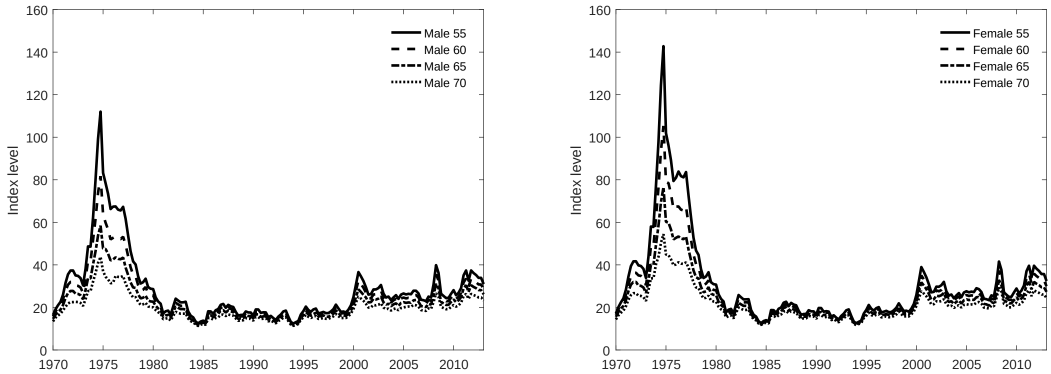


Figure 3.2. Longevity index for Australia: (Left Panel) males; (Right Panel) females.

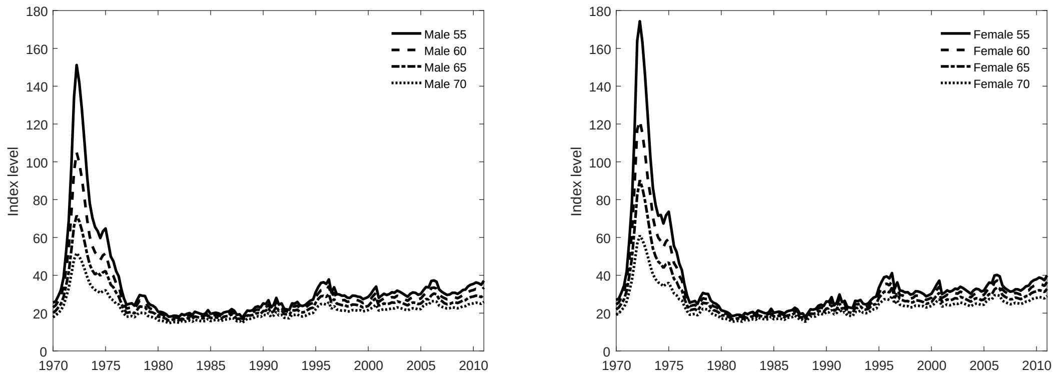


Figure 3.3. Longevity index for Japan: (Left Panel) males; (Right Panel) females.

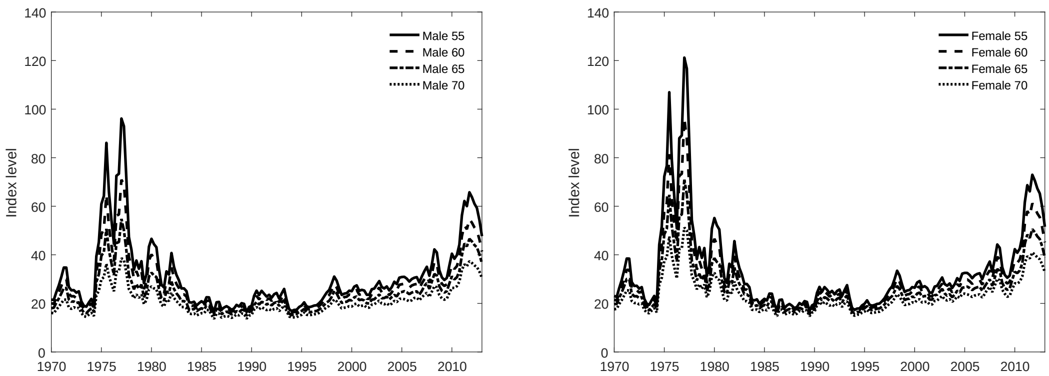


Figure 3.4. Longevity index for the U.K.: (Left Panel) males; (Right Panel) females.

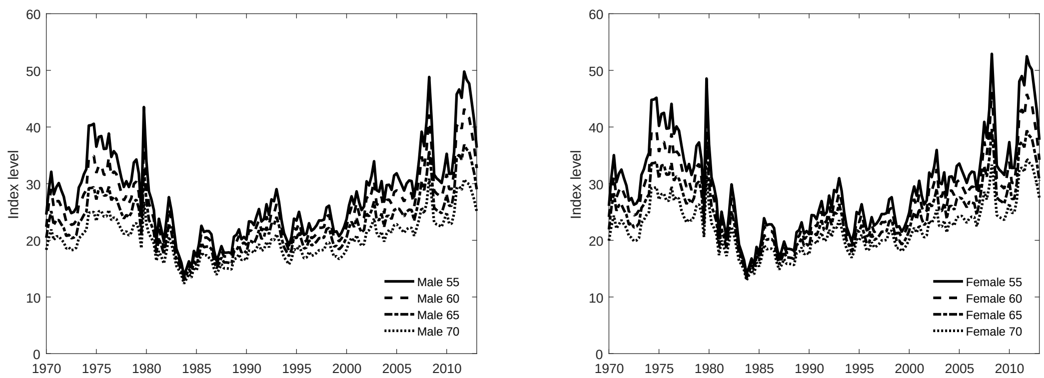


Figure 3.5. Longevity index for the U.S.: (Left Panel) males; (Right Panel) females.

The patterns observed in Figures 3.2 to 3.5 are confirmed in Table 3.4, which compares the mean, standard deviation and volatility across four countries for two different periods. The first period includes the whole sample period. The second period starts from the first quarter of 1980, removing the impact of the great inflation of the 1970s. The volatility (Vol) is calculated as the standard deviation (Std) of the log difference of the index across time, i.e.

$$\mathrm{Vol} = \mathrm{Std}\left(\ln \frac{I_{x,t}}{I_{x,t-1}}\right).$$

The standard deviations in the top panel are significantly higher for Australia, Japan, the U.K. compared to those in the bottom panel, whereas those for the U.S. remain virtually unchanged. The table also reveals the high volatility of the longevity index, which can be as high as the volatility of a single stock price.

3.4 Conclusions

I use the concept of longevity index to analyse the investment objective of the target annuitisation fund. In particular, I construct and compare longevity indices across four countries to quantify the volatility and trend in the cost of providing a unit of inflation- and longevity-protected income streams at different retirement ages. Under the assumption that the current levels of real interest rate and mortality persist, the indices can be as volatile as stock returns. This highlights the difficulty for an ordinary individual in accumulating sufficient fund to provide sustainable income post-retirement life.

Table 3.4. The mean, standard deviation (Std) and volatility (Vol) of the longevity index for the period starting in the early 1970s, and the period starting in the first quarter of 1980.

	M55	M60	M65	M70	F55	F60	F65	F70
Australia								
1970 Q3 - 2013 Q3								
Mean	28.38	25.24	22.37	19.58	31.03	27.80	24.66	21.66
Std	16.42	11.95	8.63	6.12	21.09	15.86	11.46	8.04
Vol	0.27	0.23	0.20	0.18	0.29	0.26	0.23	0.20
1980 Q1 - 2013 Q3								
Mean	22.44	20.96	19.37	17.56	23.29	21.97	20.47	18.76
Std	6.34	5.44	4.65	3.77	6.82	5.92	5.05	4.15
Vol	0.23	0.20	0.18	0.16	0.24	0.22	0.20	0.18
Japan								
1972 Q1 - 2013 Q3								
Mean	33.69	29.44	25.57	22.02	36.37	32.03	28.10	24.20
Std	21.98	14.87	9.77	6.62	25.97	18.02	12.60	8.05
Vol	0.26	0.22	0.19	0.15	0.28	0.24	0.21	0.17
1980 Q1 - 2013 Q3								
Mean	26.80	24.77	22.55	20.07	28.26	26.34	24.24	21.90
Std	5.53	4.74	4.00	3.27	6.27	5.50	4.75	4.02
Vol	0.12	0.10	0.10	0.08	0.13	0.11	0.10	0.09
U.K.								
1970 Q3 - 2013 Q3								
Mean	31.24	27.59	24.20	20.78	34.15	30.41	26.86	23.30
Std	14.94	10.99	8.28	5.79	18.62	14.15	10.45	7.44
Vol	0.29	0.25	0.22	0.18	0.32	0.28	0.25	0.21
1980 Q1 - 2013 Q3								
Mean	28.40	25.70	22.94	20.11	30.31	27.61	24.88	22.02
Std	10.69	8.55	6.89	5.25	12.21	9.82	7.69	5.87
Vol	0.19	0.17	0.15	0.13	0.21	0.19	0.17	0.15
U.S.								
1970 Q3 - 2013 Q3								
Mean	27.83	25.46	22.88	20.25	29.53	27.17	24.71	22.18
Std	7.53	6.14	4.87	3.77	8.31	6.88	5.56	4.44
Vol	0.18	0.17	0.15	0.13	0.20	0.18	0.16	0.14
1980 Q1 - 2013 Q3								
Mean	26.66	24.58	22.26	19.81	28.02	25.97	23.78	21.49
Std	7.78	6.41	5.14	4.00	8.32	6.96	5.66	4.56
Vol	0.19	0.17	0.15	0.13	0.21	0.19	0.17	0.15

Chapter 4

Portfolio insurance strategies for target annuitisation funds

4.1 Introduction

Chapter 3 gives an empirical study on longevity index, guiding the setup of investment objective in the present chapter. In particular, the analysis in the previous chapter indicates the term of an annuity, starting at the time of retirement, required to provide adequate self-protection against longevity risk. This chapter studies the management of the fund from the perspective of a fund manager. A number of papers have derived the optimal investment strategies with minimum guarantee for defined contribution (DC) pension funds, considering different sources of risk, e.g. interest rate risk and salary risk. These studies (reviewed in Section 2.2.2 in detail) focus on optimising investment strategies, whereby the expected utility is maximised in the presence of a minimum guarantee. On the flip side, prior research has shown the optimality of portfolio insurance strategies under certain conditions (see Section 2.3.3 for more details). In addition, the downside risk protection feature makes the strategy attractive to DC fund managers. Therefore, I opt to use portfolio insurance strategies to limit the downside risk. In particular, I use option-based portfolio insurance strategies (Brennan and Schwartz, 1976; Rubinstein and Leland, 1981) and constant proportion portfolio insurance strategies (Black and Jones, 1987;

Black and Perold, 1992) to limit the risk that the portfolio value is below an annuity-based target at retirement.

I analyse asset allocation strategies, and assess the portfolio values against the target for different cohorts. The trend of the portfolio weights in the equity fund depends on the member's age when joining the fund. It shows a downward (upward) trend for members joining the fund before (after) mid-30s. The difference is mainly because older cohorts have lower amounts of future contributions, which are in the form of safe assets. In addition, the portfolio weights are highly volatile, reflecting the volatility of the equity fund. In terms of the risk management performance, the constant proportion strategy provides a better downside risk protection in the base case, but the degree of protection is sensitive to the equity market volatility and the amount of contributions to the fund. The option-based strategy often leads to a higher average portfolio value at retirement, and its ability to ensure a minimum level of payoff is more robust to unfavourable external factors. I also analyse the portfolio weights for a one-off pension fund that is closed to new members after its inception. The asset allocation strategies depend on the cohort mix of the fund.

The rest of the chapter is organised as follows. Section 4.2 introduces the model framework. Section 4.3 discusses the two portfolio insurance strategies. Section 4.4 presents numerical illustrations to compare the two portfolio insurance strategies. Section 4.5 concludes.

4.2 The model

This section presents

1. the financial market, including the assets available and their dynamics,
2. the dynamics of the fund value, which can be split into a loan that represents the present value of future contributions and a self-financing portfolio, and
3. the target, i.e. the pre-retirement investment objective.

4.2.1 The financial market

Consider a complete and frictionless financial market. The market is continuously open, has no transaction costs, borrowing constraint, taxes or margin requirements, and does not provide any arbitrage opportunities. The randomness is described by two independent Brownian motions $W_r(t)$ and $W_S(t)$, where $t \geq 0$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Ω denotes the sample space. The filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ represents the information structure generated by the Brownian motions, and $\mathbb{P}$ denotes the real world probability measure. The independence assumption about $W_r(t)$ and $W_S(t)$ implies no loss of generality since correlated Wiener processes can be created from uncorrelated ones via the Cholesky decomposition of the correlation matrix.

The pre-retirement investment horizon typically involves 30 to 40 years, so it is important to allow for stochastic interest rate. Assume the real interest rate r_t follows an Ornstein-Uhlenbeck process

$$\mathrm{d}r_t = \kappa(\overline{r} - r_t)\mathrm{d}t + \sigma_r \mathrm{d}W_r(t), \tag{4.1}$$

where $\overline{r}$ describes the long-run mean of the real interest rate, κ describes the degree of mean reversion, and σ_r is the real interest rate volatility. Given a constant market price of real interest rate risk, λ_r, the real price of a zero-coupon bond that matures at the time of retirement T ($T > t$) is given by

$$P(t,T) = \alpha(t,T)e^{-\beta(t,T)r_t},$$

where

$$\begin{aligned}
\alpha(t,T) &= \exp\left\{\left(\overline{r} - \frac{\sigma_r\lambda_r}{\kappa} - \frac{\sigma_r^2}{2\kappa^2}\right)\Big[\beta(t,T) - T + t\Big] - \frac{\sigma_r^2}{4\kappa}\big(\beta(t,T)\big)^2\right\}, \\
\beta(t,T) &= \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right).
\end{aligned}$$

The stochastic process for the zero-coupon bond price, under the $\mathbb{P}$ measure, is therefore given by

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = \left[r_t - \beta(t,T)\sigma_r\lambda_r\right]\mathrm{d}t - \sigma_r\beta(t,T)\mathrm{d}W_r(t). \tag{4.2}$$

The fund manager can invest in three assets defined by the following processes respectively. All variables are expressed in real terms.

1. A cash fund whose price M_t evolves according to

$$\frac{\mathrm{d}M_t}{M_t} = r_t\mathrm{d}t.$$

   Define the discount process as follows

$$D(t) = e^{-\int_0^t r_u \mathrm{d}u} = \frac{1}{M_t}.$$

2. A bond fund whose price $P_t^{\overline{T}}$ evolves according to

$$\frac{\mathrm{d}P_t^{\overline{T}}}{P_t^{\overline{T}}} = \left(r_t - \sigma_{\overline{T}}\lambda_r\right)\mathrm{d}t - \sigma_{\overline{T}}\mathrm{d}W_r(t),$$

   where $\overline{T}$ is the constant maturity of the bond, and

$$\sigma_{\overline{T}} = \frac{1 - e^{-\kappa\overline{T}}}{\kappa}\sigma_r.$$

   Investing in the bond fund can hedge interest rate risk. A bond fund with a constant maturity ($\overline{T}$ [1]) is introduced as it is unrealistic to assume the existence of zero-coupon bonds with any maturity (Boulier et al., 2001). In fact, the original zero-coupon bond price dynamics (4.2) can be obtained through a linear combination of the cash fund and the bond fund

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = \left(1 - \frac{\sigma_r\beta(t,T)}{\sigma_{\overline{T}}}\right)\frac{\mathrm{d}M_t}{M_t} + \frac{\sigma_r\beta(t,T)}{\sigma_{\overline{T}}}\frac{\mathrm{d}P_t^{\overline{T}}}{P_t^{\overline{T}}}. \tag{4.3}$$

[1] Note that $\overline{T}$ is different from T because T denotes the time of retirement.

3. An equity fund (with dividend reinvested) whose price S_t satisfies the following stochastic differential equation

$$\frac{\mathrm{d}S_t}{S_t} = (r_t + \sigma_S\lambda_S + \sigma_{Sr}\lambda_r)\,\mathrm{d}t + \sigma_S\mathrm{d}W_S(t) + \sigma_{Sr}\mathrm{d}W_r(t), \tag{4.4}$$

where σ_S is the equity fund's own volatility, σ_{Sr} equity fund volatility associated with interest rate, and λ_S the market price of equity fund risk. Using the notation

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_S \\ \sigma_{Sr} \end{pmatrix}, \quad \boldsymbol{\lambda} = \begin{pmatrix} \lambda_S \\ \lambda_r \end{pmatrix}, \quad \boldsymbol{W}(t) = \begin{pmatrix} W_S(t) \\ W_r(t) \end{pmatrix},$$

the diffusion process of S_t can also be written as

$$\frac{\mathrm{d}S_t}{S_t} = \big(r_t + \boldsymbol{\sigma}'\boldsymbol{\lambda}\big)\,\mathrm{d}t + \boldsymbol{\sigma}'\mathrm{d}\boldsymbol{W}(t). \tag{4.5}$$

4.2.2 Fund value dynamics

In a DC pension fund, a certain proportion of labour earnings are contributed to the fund in the pre-retirement phase. I assume the contribution amount, $c(t)$, is a continuous and deterministic process. Let $\Omega_M(t), \Omega_{\overline{T}}(t), \Omega_S(t)$ denote the wealth invested in the cash fund, the bond fund, and the equity fund, respectively. The fund value, X_t, must satisfy

$$X_t = \Omega_M(t) + \Omega_{\overline{T}}(t) + \Omega_S(t).$$

From the diffusion process of each asset and the budget constraint of the fund, the fund value satisfies the following stochastic differential equation

$$\begin{aligned} \mathrm{d}X_t = r_t X_t \mathrm{d}t - \Omega_{\overline{T}}(t)\Big[\sigma_{\overline{T}}\lambda_r \mathrm{d}t + \sigma_{\overline{T}}\mathrm{d}W_r(t)\Big] \\ + \Omega_S(t)\Big[\big(\sigma_S\lambda_S + \sigma_{Sr}\lambda_r\big)\mathrm{d}t + \sigma_S \mathrm{d}W_S(t) + \sigma_{Sr}\mathrm{d}W_r(t)\Big] + c(t)\mathrm{d}t. \end{aligned} \tag{4.6}$$

X_t is not a self-financing process in that the change of portfolio value is not entirely driven by the gains or losses of investment returns due to the continuous contribution to the fund. Therefore the fund is split into two parts to create a self-financing portfolio. The first part is a loan (L_t) that corresponds to all the future contributions from fund members. The loan will be paid back by these contributions. The value of the loan is given by

$$L_t = -\int_t^T c(u)P(t,u)\mathrm{d}u.$$

Using the Itô's formula, L_t can be replicated with the bond fund and the cash fund as follows

$$\Omega^L_{\overline{T}}(t) = -\int_t^T c(u)P(t,u)\frac{\sigma_r\beta(t,u)}{\sigma_{\overline{T}}}\mathrm{d}u, \tag{4.7}$$

$$\Omega^L_M(t) = L_t - \Omega^L_{\overline{T}}(t). \tag{4.8}$$

Proof. Since the loan corresponds to all the future contributions from the fund member and it will be paid back by these contributions, the diffusion equation of L_t is given by

$$\mathrm{d}L_t = c(t)\mathrm{d}t - \int_t^T c(u)\mathrm{d}P(t,u)\mathrm{d}u. \tag{4.9}$$

The dynamics of the zero-coupon bond are given by

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = \left[r_t - \beta(t,T)\sigma_r\lambda_r\right]\mathrm{d}t - \sigma_r\beta(t,T)\mathrm{d}W_r(t). \tag{4.2 revisited}$$

Substitute Equation (4.2) into Equation (4.9)

$$\begin{aligned}\mathrm{d}L_t &= c(t)\mathrm{d}t - \int_t^T c(u)P(t,u)\left[\left[r_t - \beta(t,u)\sigma_r\lambda_r\right]\mathrm{d}t - \sigma_r\beta(t,u)\mathrm{d}W_r(t)\right]\mathrm{d}u \\ &= c(t)\mathrm{d}t - r_t\left(\int_t^T c(u)P(t,u)\mathrm{d}u\right)\mathrm{d}t + \lambda_r\left(\int_t^T c(u)P(t,u)\sigma_r\beta(t,u)\mathrm{d}u\right)\mathrm{d}t \\ &\quad + \left(\int_t^T c(u)P(t,u)\sigma_r\beta(t,u)\mathrm{d}u\right)\mathrm{d}W_r(t).\end{aligned} \tag{4.10}$$

Since the randomness of L_t is driven by the Brownian motion $W_r(t)$ only, it can be replicated by the bond fund and the cash fund. Denote $\Omega^L_{\overline{T}}(t)$ as the wealth invested in the bond fund to replicate L_t. The diffusion term of $\mathrm{d}L_t$ and $\mathrm{d}\Omega^L_{\overline{T}}$ must be the same, so

$$\Omega^L_{\overline{T}}(t) \times (-\sigma_{\overline{T}}) = \int_t^T c(u)P(t,u)\sigma_r\beta(t,u)\mathrm{d}u. \tag{4.11}$$

Therefore,

$$\Omega^L_{\overline{T}}(t) = -\int_t^T c(u)P(t,u)\frac{\sigma_r\beta(t,u)}{\sigma_{\overline{T}}}\mathrm{d}u. \tag{4.7 revisited}$$

The result of $\mathrm{d}L_t$ will be used in proving Equation (4.13), so I continue to derive $\mathrm{d}L_t$ based on Equation (4.10). Note that $-\int_t^T c(u)P(t,u)\mathrm{d}u = L_t$, and substitute the left-hand side of Equation (4.11) into Equation (4.10)

$$\mathrm{d}L_t = \Big[c(t) + r_tL_t - \lambda_r\Omega^L_{\overline{T}}(t)\sigma_{\overline{T}}\Big]\mathrm{d}t - \Omega^L_{\overline{T}}(t)\sigma_{\overline{T}}\mathrm{d}W_r(t). \tag{4.12}$$

$\square$

The other part of the fund is given by

$$Y_t = X_t - L_t.$$

Y_t is a self-financing portfolio that satisfies

$$\begin{aligned} \mathrm{d}Y_t = r_tY_t\mathrm{d}t &- \Omega^Y_{\overline{T}}(t)\Big[\sigma_{\overline{T}}\lambda_r\mathrm{d}t + \sigma_{\overline{T}}\mathrm{d}W_r(t)\Big] \\ &+ \Omega^Y_S(t)\Big[\big(\sigma_S\lambda_S + \sigma_{Sr}\lambda_r\big)\mathrm{d}t + \sigma_S\mathrm{d}W_S(t) + \sigma_{Sr}\mathrm{d}W_r(t)\Big], \end{aligned} \tag{4.13}$$

where

$$\Omega^Y_S(t) = \Omega_S(t), \quad \Omega^Y_{\overline{T}}(t) = \Omega_{\overline{T}}(t) - \Omega^L_{\overline{T}}(t), \quad \Omega^Y_M(t) = \Omega_M(t) - \Omega^L_M(t). \tag{4.14}$$

Proof. Since $Y_t = X_t - L_t$ by definition,

$$\mathrm{d}Y_t = \mathrm{d}X_t - \mathrm{d}L_t.$$

The dynamics of X_t are given by

$$\begin{aligned}\mathrm{d}X_t = r_t X_t \mathrm{d}t - \Omega_{\overline{T}}(t)\Big[\sigma_{\overline{T}}\lambda_r \mathrm{d}t + \sigma_{\overline{T}}\mathrm{d}W_r(t)\Big] \\ + \Omega_S(t)\Big[\big(\sigma_S\lambda_S + \sigma_{Sr}\lambda_r\big)\mathrm{d}t + \sigma_S\mathrm{d}W_S(t) + \sigma_{Sr}\mathrm{d}W_r(t)\Big] + c(t)\mathrm{d}t.\end{aligned} \tag{4.6 revisited}$$

The dynamics of L_t are given in Equation (4.12). Substitute Equation (4.6) and Equation (4.12) into the above equation and rearrange

$$\begin{aligned}\mathrm{d}Y_t &= r_t\big(Y_t + L_t\big)\mathrm{d}t - \Omega_{\overline{T}}(t)\Big[\sigma_{\overline{T}}\lambda_r\mathrm{d}t + \sigma_{\overline{T}}\mathrm{d}W_r(t)\Big] \\ &\quad + \Omega_S(t)\Big[\big(\sigma_S\lambda_S + \sigma_{Sr}\lambda_r\big)\mathrm{d}t + \sigma_S\mathrm{d}W_S(t) + \sigma_{Sr}\mathrm{d}W_r(t)\Big] + c(t)\mathrm{d}t \\ &\quad - \Big[c(t) + r_t L_t - \lambda_r\Omega^L_{\overline{T}}(t)\sigma_{\overline{T}}\Big]\mathrm{d}t + \Omega^L_{\overline{T}}(t)\sigma_{\overline{T}}\mathrm{d}W_r(t) \\ &= r_t Y_t\mathrm{d}t - \big(\Omega_{\overline{T}}(t) - \Omega^L_{\overline{T}}(t)\big)\Big[\sigma_{\overline{T}}\lambda_r\mathrm{d}t + \sigma_{\overline{T}}\mathrm{d}W_r(t)\Big] \\ &\quad + \Omega_S(t)\Big[\big(\sigma_S\lambda_S + \sigma_{Sr}\lambda_r\big)\mathrm{d}t + \sigma_S\mathrm{d}W_S(t) + \sigma_{Sr}\mathrm{d}W_r(t)\Big].\end{aligned}$$

Denote $\Omega^Y_{\overline{T}}(t) = \Omega_{\overline{T}}(t) - \Omega^L_{\overline{T}}(t)$ and $\Omega_S(t) = \Omega^Y_S(t)$, then

$$\begin{aligned}\mathrm{d}Y_t = r_t Y_t\mathrm{d}t - \Omega^Y_{\overline{T}}(t)\Big[\sigma_{\overline{T}}\lambda_r\mathrm{d}t + \sigma_{\overline{T}}\mathrm{d}W_r(t)\Big] \\ + \Omega^Y_S(t)\Big[\big(\sigma_S\lambda_S + \sigma_{Sr}\lambda_r\big)\mathrm{d}t + \sigma_S\mathrm{d}W_S(t) + \sigma_{Sr}\mathrm{d}W_r(t)\Big].\end{aligned} \tag{4.13 revisited}$$

□

Y_t also satisfies

$$Y_0 = X_0 + \int_0^T c(t)P(0,t)\mathrm{d}t, \quad Y_T = X_T,$$

where X_0 is usually set at 0.

4.2.3 Target

The investment target is modelled using the concept of longevity index discussed in Chapter 3. Since the investment target is the present value of an annuity whose annual payment depends on the post-retirement consumption target, it can be referred to as the target annuitisation level. Let A_T denote the target annuitisation level at the time of retirement, T. Since the real interest rate is assumed to follow a stochastic process, A_T can be written as a series of zero coupon bonds, i.e.

$$A_T = g \sum_{j=0}^{J} P(T, T + j),$$

where g is the annual retirement income, and J represents the annuity term. At any time, t, prior to T, the target annuitisation level is the present value of A_T, given by

$$A_t = g \sum_{j=0}^{J} P(t, T + j).$$

Note that A_t differs from $I_{x,t}$ defined in Equation (3.1). $I_{x,t}$ uses a flat term structure of interest rates, whereas A_t uses the term structure implied by the Vasicek model (4.1). If $I_{x,t}$ is expressed in terms of the zero coupon bond, it is given by

$$I_{x,t} = \sum_{j=0}^{J} P(t, t + j).$$

4.3 Portfolio insurance strategies

Portfolio insurance strategies provide downside protection while keeping upside potential. They are suitable investment strategies in managing pension funds that provide minimum guarantees (Leland, 1980). This section analyses the asset allocation strategies based on the option-based portfolio insurance (OBPI) and constant proportion portfolio insurance (CPPI) strategies for the self-financing portfolio Y_t. I then use the results shown in Equation (4.14) to find the wealth management processes, $\Omega_M(t)$, $\Omega_{\overline{T}}(t)$, and $\Omega_S(t)$ for the target annuitisation fund.

4.3.1 Option-based portfolio insurance strategy

The fund manager aims to provide the fund members with the target annuitisation level at their retirement. One way to achieve this goal is to hold a portfolio consisting of the investment in the equity fund and the options to exchange the amount in the equity fund for the annuity at the time of retirement. The option can be replicated by the available assets. This strategy is similar to a protective put that can insure against unwanted losses.

This section derives the price of the exchange option and the corresponding hedging portfolio. When finding the replicating portfolio for the target annuitisation fund, I first find the portfolio management processes for Y_t, and then obtain the corresponding processes for X_t using Equation (4.14).

4.3.1.1 Option pricing

The value of the portfolio at retirement is given by

$$X_T^{\mathrm{OBPI}} = Y_T^{\mathrm{OBPI}} = nS_T + \big(A_T - nS_T\big)^+ = n\left[S_T + \left(\frac{g}{n}\sum_{j=0}^{J} P(T, T+j) - S_T\right)^+\right],$$

where n represents the number of options, and $(\cdot)^+ = \max(0, \cdot)$. The initial value, Y_0, is determined by n and the value of a single option at time 0, Q_0, as follows

$$Y_0^{\mathrm{OBPI}} = n(S_0 + Q_0). \tag{4.15}$$

The option price at time 0 has a lower bound given by the following equation

$$
\begin{aligned}
Q_0 &= \widetilde{\mathbb{E}}\left[\frac{D(T)}{D(0)}\left(\frac{g}{n}\sum_{j=0}^{J}P(T,T+j)-S_T\right)^+\right] \\
&\geq \widetilde{\mathbb{E}}\left[\frac{D(T)}{D(0)}\left(\frac{g}{n}\sum_{j=0}^{J}P(T,T+j)-S_T\right)\right] \\
&= \widetilde{\mathbb{E}}\left[\frac{D(T)}{D(0)}\left(\frac{g}{n}\sum_{j=0}^{J}P(T,T+j)\right)\right] - \widetilde{\mathbb{E}}\left[\frac{D(T)}{D(0)}S_T\right] \\
&= \frac{g}{n}\sum_{j=0}^{J}P(0,T+j) - S_0 = \frac{g}{n}A_0 - S_0,
\end{aligned}
$$

where $\widetilde{\mathbb{E}}$ is the expectation operator under the risk neutral probability measure $\widetilde{\mathbb{P}}$ that takes the cash fund M as numéraire. Therefore, there needs to be an additional constraint $Y_0 \geq gA_0$ for the option price to be valid.

At any time t prior to time T, the value of the portfolio is given by

$$
Y_t^{\text{OBPI}} = n\left(S_t + Q_t\right),
$$

where Q_t is the value of a single option at time t. Using the risk-neutral pricing formula, the value of the option at time t is given by

$$
Q_t = \widetilde{\mathbb{E}}_t\left[\frac{D(T)}{D(t)}S_T\left(\frac{g}{n}\sum_{j=0}^{J}\frac{P(T,T+j)}{S_T}-1\right)^+\right].
$$

I use the change-of-numéraire technique (Geman et al., 1995), changing the numéraire from cash fund M to equity fund S, to find the option price, Q_t. Let $\widetilde{\mathbb{P}}^{(S)}$ denote the risk neutral measure for the equity fund numéraire. The Radon-Nikodým derivative defining the measure $\widetilde{\mathbb{P}}^{(S)}$ is given by

$$
\frac{\mathrm{d}\widetilde{\mathbb{P}}^{(S)}}{\mathrm{d}\widetilde{\mathbb{P}}} = \frac{S_T M_0}{S_0 M_T} = \exp\left(\sigma_S\widetilde{W}_S(T) - \frac{1}{2}\sigma_S^2 T + \sigma_{Sr}\widetilde{W}_r(T) - \frac{1}{2}\sigma_{Sr}^2 T\right).
$$

The multidimensional Girsanov theorem implies that under $\widetilde{\mathbb{P}}^{(S)}$,

$$\widetilde{W}_S^{(S)}(t) = \widetilde{W}_S(t) - \sigma_S t \quad \text{and} \quad \widetilde{W}_r^{(S)}(t) = \widetilde{W}_r(t) - \sigma_{Sr} t$$

are standard Brownian motions, and that $\widetilde{W}_S^{(S)}$ and $\widetilde{W}_r^{(S)}$ are independent. Therefore, the option price under the risk-neutral measure $\widetilde{\mathbb{P}}^{(S)}$ is given by

$$Q_t = \frac{g}{n} S_t \widetilde{\mathbb{E}}_t^{(S)} \left[\left(\sum_{j=0}^{J} P^{(S)}(T, T+j) - \frac{n}{g} \right)^+ \right], \tag{4.16}$$

where $\widetilde{\mathbb{E}}_t^{(S)}$ denotes the conditional expectation operator given $\mathcal{F}_t$ under the probability measure $\widetilde{\mathbb{P}}^{(S)}$, and $P^{(S)}(T, T+j)$ is the price of the zero-coupon bond denominated in S.

Theorem 4.3.1. *$P^{(S)}(t,T)$ is a martingale under the measure $\widetilde{\mathbb{P}}^{(S)}$. Moreover,*

$$\frac{\mathrm{d}P^{(S)}(t,T)}{P^{(S)}(t,T)} = -\sigma_S \mathrm{d}\widetilde{W}_S^{(S)}(t) - \left(\sigma_r \beta(t,T) + \sigma_{Sr}\right) \mathrm{d}\widetilde{W}_r^{(S)}(t).$$

Proof. See Theorem 9.2.2 of Shreve (2004). □

Remark 4.3.2. Since $\widetilde{W}_S^{(S)}(t)$ and $\widetilde{W}_r^{(S)}(t)$ are independent, I can define a new Brownian motion $\widetilde{W}_P^{(S)}$ such that

$$\frac{\mathrm{d}P^{(S)}(t,T)}{P^{(S)}(t,T)} = \sqrt{\sigma_S^2 + (\sigma_r \beta(t,T) + \sigma_{Sr})^2} \mathrm{d}\widetilde{W}_P^{(S)}(t) \equiv \sigma_P(t,T) \mathrm{d}\widetilde{W}_P^{(S)}(t).$$

So for each $s \leq t$,

$$P^{(S)}(t,T) = P^{(S)}(s,T) \exp\left[\int_s^t \sigma_P(u,T) \mathrm{d}\widetilde{W}_P^{(S)}(u) - \frac{1}{2} \int_s^t \sigma_P^2(u,T) \mathrm{d}u \right]. \tag{4.17}$$

Lemma 4.3.3. *Using the martingale property, the value of the option at time t is given by*

$$\begin{aligned} Q_t = \frac{g}{n} S_t \widetilde{\mathbb{E}}_t^{(S)} \Bigg[\Bigg(\sum_{j=0}^{J} P^{(S)}(t, T+j) \exp\Bigg\{ \varepsilon_{t,T} \sqrt{\int_t^T \sigma_P^2(u, T+j) \mathrm{d}u} \\ - \frac{1}{2} \int_t^T \sigma_P^2(u, T+j) \mathrm{d}u \Bigg\} - \frac{n}{g} \Bigg)^+ \Bigg], \end{aligned} \tag{4.18}$$

where $\varepsilon_{t,T}$ is a random variable that follows a standard normal distribution under the measure $\widetilde{\mathbb{P}}^{(S)}$ for each $t \in [0,T]$, i.e. $\forall t \in [0,T]$: $\varepsilon_{t,T} \sim \mathcal{N}(0,1)$ under $\widetilde{\mathbb{P}}^{(S)}$.

Proof. Comparing Equations (4.18) and (4.15), I want to show that, for each $t \in [0,T]$,

$$P^{(S)}(T,T+j) = P^{(S)}(t,T+j)\exp\left\{\varepsilon_{t,T}\sqrt{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u} - \frac{1}{2}\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u\right\}.$$

For each $s \in [0,t]$,

$$P^{(S)}(t,T) = P^{(S)}(s,T)\exp\left[\int_s^t \sigma_P(u,T)\mathrm{d}\widetilde{W}_P^{(S)}(u) - \frac{1}{2}\int_s^t \sigma_P^2(u,T)\mathrm{d}u\right], \quad \text{(4.16 revisited)}$$

so for each $t \in [0,T]$,

$$P^{(S)}(T,T+j) = P^{(S)}(t,T+j)\exp\left[\int_t^T \sigma_P(u,T+j)\mathrm{d}\widetilde{W}_P^{(S)}(u) - \frac{1}{2}\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u\right]. \tag{4.18}$$

Denote

$$\mathcal{I}(t) = \int_0^t \sigma_P(u,T+j)\mathrm{d}\widetilde{W}_P^{(S)}(u).$$

Since $\sigma_P(t,T)$ is a deterministic function of time t, $\big(\mathcal{I}(T) - \mathcal{I}(t)\big)$ follows a normal distribution with mean zero, and variance $\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u$ under $\widetilde{\mathbb{P}}^{(S)}$, i.e.

$$\mathcal{I}(T) - \mathcal{I}(t) \sim \mathcal{N}\left(0, \int_t^T \sigma_P^2(u,T+j)\mathrm{d}u\right) \text{ under } \widetilde{\mathbb{P}}^{(S)}.$$

Let $\varepsilon_{t,T}$ be a random variable that follows a standard normal distribution under $\widetilde{\mathbb{P}}^{(S)}$ for each $t \in [0,T]$, i.e.

$$\forall t \in [0,T] : \varepsilon_{t,T} \equiv \frac{\mathcal{I}(T) - \mathcal{I}(t)}{\sqrt{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u}} \sim \mathcal{N}(0,1) \text{ under } \widetilde{\mathbb{P}}^{(S)}.$$

Equation (4.18) can therefore be re-written as

$$P^{(S)}(T,T+j) = P^{(S)}(t,T+j)\exp\left[\varepsilon_{t,T}\sqrt{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u} - \frac{1}{2}\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u\right].$$

□

Remark 4.3.4. The integral $\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u$ has a closed-form solution

$$\begin{aligned}
&\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u \\
&= -\frac{2\sigma_r e^{-\kappa j}(1-e^{-\kappa(T-t)})(\sigma_r+\kappa\sigma_{Sr})}{\kappa^3} + \frac{\sigma_r^2 e^{-2\kappa j}(1-e^{-2\kappa(T-t)})}{2\kappa^3} \\
&\quad + \frac{(T-t)(\sigma_r+\kappa\sigma_{Sr})^2}{\kappa^2} + (T-t)\sigma_S^2.
\end{aligned}$$

The closed-form solution will be used for the numerical application.

Lemma 4.3.5. *Using the result from Jamshidian (1989), the value of the option at time t is given by*

$$Q_t = \frac{g}{n} S_t \sum_{j=0}^{J} \widetilde{\mathbb{E}}_t^{(S)} \left[\left(P^{(S)}(T, T+j) - K_j^{(S)}(t) \right)^+ \right], \tag{4.19}$$

where $K_j^{(S)}(t)$ is an appropriate strike price. It is given by

$$K_j^{(S)}(t) = P^{(S)}(t, T+j) \exp\left[\varepsilon_{t,T}^* \sqrt{\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u} - \frac{1}{2}\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u \right], \tag{4.20}$$

where $\varepsilon_{t,T}^*$ satisfies the following equation

$$\sum_{j=0}^{J} P^{(S)}(t, T+j) \exp\left[\varepsilon_{t,T}^* \sqrt{\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u} - \frac{1}{2}\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u \right] = \frac{n}{g}. \tag{4.21}$$

Proof. Jamshidian (1989) proves that pricing an option on a portfolio is equivalent to pricing a portfolio of options with appropriate strike prices, as long as the prices of the portfolio components are all strictly decreasing or increasing with the same state variable. In the present case, each component is a zero-coupon bond (denominated in the equity fund S), which is a monotonic function of $\varepsilon_{t,T}$. It is therefore possible to find an $\varepsilon_{t,T}^*$ such that

$$\sum_{j=0}^{J} P^{(S)}(t, T+j) \exp\left[\varepsilon_{t,T}^* \sqrt{\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u} - \frac{1}{2}\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u \right] = \frac{n}{g}.$$

Let $K_j^{(S)}(t)$ be the price of the zero-coupon bond (denominated in the equity fund S) that corresponds to $\varepsilon^*_{t,T}$, i.e.

$$K_j^{(S)}(t) = P^{(S)}(t,T+j)\exp\left[\varepsilon^*_{t,T}\sqrt{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u} - \frac{1}{2}\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u\right].$$

Following the relationship between $\varepsilon_{t,T}$ and bond prices, it can be shown that

$$\left(\sum_{j=0}^{J} P^{(S)}(T,T+j) - \frac{n}{g}\right)^+ = \sum_{j=0}^{J}\left(P^{(S)}(T,T+j) - K_j^{(S)}(t)\right)^+,$$

which implies that

$$Q_t = \frac{g}{n}S_t\sum_{j=0}^{J}\widetilde{\mathbb{E}}_t^{(S)}\left[\left(P^{(S)}(T,T+j) - K_j^{(S)}(t)\right)^+\right].$$

□

Theorem 4.3.6. *Using the results in Lemma 4.3.3 and Lemma 4.3.5, the value of the option at time t is given by*

$$Q_t = \frac{g}{n}S_t\sum_{j=0}^{J}\left[P^{(S)}(t,T+j)N(-d_{2,t}) - K_j^{(S)}(t)N(-d_{1,t})\right], \tag{4.22}$$

where $N(\cdot)$ represents the cumulative distribution function of the standard normal distribution,

$$d_{1,t} = \frac{1}{\sqrt{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u}}\left(\ln\frac{K_j^{(S)}(t)}{P^{(S)}(t,T+j)} + \frac{1}{2}\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u\right), \tag{4.23}$$

and

$$d_{2,t} = d_{1,t} - \sqrt{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u}. \tag{4.24}$$

Proof. The proof will give an analytical expression to the following expectation

$$\widetilde{\mathbb{E}}_t^{(S)}\left[\left(P^{(S)}(T,T+j) - K_j^{(S)}(t)\right)^+\right]. \tag{4.25}$$

Based on the result of Remark 4.3.2,

$$\ln \frac{P^{(S)}(T,T+j)}{P^{(S)}(t,T+j)}\bigg|\mathcal{F}_t \sim \mathcal{N}\left(-\frac{1}{2}\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u, \int_t^T \sigma_P^2(u,T+j)\mathrm{d}u\right)$$

under $\widetilde{\mathbb{P}}^{(S)}$, so Equation (4.25) is given by

$$\begin{aligned}
&\widetilde{\mathbb{E}}_t^{(S)}\left[\left(P^{(S)}(T,T+j)-K_j^{(S)}(t)\right)^+\right]\\
&=\int_{d_{1,t}}^{\infty}\Bigg\{P^{(S)}(t,T+j)\exp\left[-\frac{1}{2}\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u + y\sqrt{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u}\right]\\
&\qquad - K_j^{(S)}(t)\Bigg\}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\mathrm{d}y,
\end{aligned}$$

where

$$d_{1,t} = \frac{1}{\sqrt{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u}}\left(\ln \frac{K_j^{(S)}(t)}{P^{(S)}(t,T+j)} + \frac{1}{2}\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u\right),$$

and

$$y = \frac{1}{\sqrt{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u}}\left(\ln \frac{P^{(S)}(T,T+j)}{P^{(S)}(t,T+j)} + \frac{1}{2}\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u\right), \quad y|\mathcal{F}_t \sim \mathcal{N}(0,1).$$

The expectation can be re-written as

$$\begin{aligned}
&\widetilde{\mathbb{E}}_t^{(S)}\left[\left(P^{(S)}(T,T+j)-K_j^{(S)}(t)\right)^+\right]\\
&=P^{(S)}(t,T+j)\int_{d_{1,t}}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u + y\sqrt{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u} - \frac{1}{2}y^2\right]\mathrm{d}y\\
&\quad - K_j^{(S)}(t)\int_{d_{1,t}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\mathrm{d}y.
\end{aligned}$$

Note that

$$
\begin{aligned}
&-\frac{1}{2}\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u + y\sqrt{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u} - \frac{1}{2}y^2 \\
&= -\frac{1}{2}\left(y^2 - 2y\sqrt{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u} + \int_t^T \sigma_P^2(u,T+j)\mathrm{d}u\right) \\
&= -\frac{1}{2}\left(y - \sqrt{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u}\right)^2,
\end{aligned}
$$

and

$$
\int_{d_{1,t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\mathrm{d}y = N(-d_{1,t}).
$$

The expectation can be simplified as

$$
\begin{aligned}
&\widetilde{\mathbb{E}}_t^{(S)}\left[\left(P^{(S)}(T,T+j) - K_j^{(S)}(t)\right)^+\right] \\
&= P^{(S)}(t,T+j)\int_{d_{1,t}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(y - \sqrt{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u}\right)^2\right]\mathrm{d}y - K_j^{(S)}(t)N(-d_{1,t}) \\
&= P^{(S)}(t,T+j)\int_{d_{2,t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\mathrm{d}x - K_j^{(S)}(t)N(-d_{1,t}),
\end{aligned}
$$

where

$$
d_{2,t} = d_{1,t} - \sqrt{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u}.
$$

Therefore, the analytical expression is given by

$$
\widetilde{\mathbb{E}}_t^{(S)}\left[\left(P^{(S)}(T,T+j) - K_j^{(S)}(t)\right)^+\right] = P^{(S)}(t,T+j)N(-d_{2,t}) - K_j^{(S)}(t)N(-d_{1,t}).
$$

□

Remark 4.3.7. By comparing Equation (4.23) with Equation (4.20), $d_{1,t} \equiv \varepsilon_{t,T}^*$, so $d_{1,t}$ does not depend on j.

4.3.1.2 Replicating portfolio

Now consider the investment strategy of the target annuitisation funds. The portfolio consists of the investment in the equity fund, and a corresponding hedging portfolio to hedge a short position in the option whose value is given by Equation (4.22).

It is easier to find the hedging portfolio when the numéraire is the equity fund. Divide Equation (4.22) by S_t

$$\frac{Q_t}{S_t} = \frac{g}{n}\sum_{j=0}^{J}\left[P^{(S)}(t,T+j)N(-d_{2,t}) - K_j^{(S)}(t)N(-d_{1,t})\right]. \tag{4.26}$$

Suppose a short position in the option is hedged by holding $\frac{g}{n}N(-d_{2,t})$ units of zero-coupon bond that matures at time $T+j$ $(j = 0, \cdots, J)$ and shorting $\frac{g}{n}\sum_{j=0}^{J} K_j^{(S)}(t)N(-d_{1,t})$ equity fund at each time t. The value of this portfolio agrees with Equation (4.26).

Theorem 4.3.8. *The hedging portfolio in Equation* (4.26) *is self-financing.*

Proof. The differential of the portfolio is given by

$$\begin{aligned}\mathrm{d}\left(\frac{Q_t}{S_t}\right) = \frac{g}{n}\sum_{j=0}^{J}\Big[&N(-d_{2,t})\mathrm{d}P^{(S)}(t,T+j) + P^{(S)}(t,T+j)\mathrm{d}N(-d_{2,t})\\ &+ \mathrm{d}P^{(S)}(t,T+j)\mathrm{d}N(-d_{2,t}) - K_j^{(S)}(t)\mathrm{d}N(-d_{1,t})\\ &- N(-d_{1,t})\mathrm{d}K_j^{(S)}(t) - \mathrm{d}K_j^{(S)}(t)\mathrm{d}N(-d_{1,t})\Big].\end{aligned} \tag{4.27}$$

In order for the portfolio to be self-financing, the change of portfolio value needs to be entirely due to capital gains. In the following I will first show that

$$\sum_{j=0}^{J}\left[N(-d_{1,t})\mathrm{d}K_j^{(S)}(t) + \mathrm{d}K_j^{(S)}(t)\mathrm{d}N(-d_{1,t})\right] = 0, \tag{4.28}$$

and then show that for $j = 0, \cdots, J$

$$P^{(S)}(t,T+j)\mathrm{d}N(-d_{2,t}) + \mathrm{d}P^{(S)}(t,T+j)\mathrm{d}N(-d_{2,t}) - K_j^{(S)}(t)\mathrm{d}N(-d_{1,t}) = 0, \tag{4.29}$$

to prove the portfolio is self-financing.

To show Equation (4.28), I use the results of Remark 4.3.7 and Equation (4.21). In particular, the terms $N(-d_{1,t})$ and $\mathrm{d}N(-d_{1,t})$ do not depend on j due to Remark 4.3.7, and

$$\sum_{j=0}^{J} \mathrm{d}K_j^{(S)}(t) = \mathrm{d}\left(\sum_{j=0}^{J} K_j^{(S)}(t)\right) = \mathrm{d}\left(\frac{n}{g}\right) = 0,$$

due to Equation (4.21). Therefore, the left-hand side of Equation (4.28) is given by

$$N(-d_{1,t})\sum_{j=0}^{J} \mathrm{d}K_j^{(S)}(t) + \mathrm{d}N(-d_{1,t})\sum_{j=0}^{J} \mathrm{d}K_j^{(S)}(t) = 0.$$

To show Equation (4.29), I first derive $\mathrm{d}N(-d_{1,t})$, $\mathrm{d}N(-d_{2,t})$, and $\mathrm{d}P^{(S)}(t, T+j)\mathrm{d}N(-d_{2,t})$ using the Itô's formula. After substituting the derivatives back to the left-hand side of Equation (4.29), I find that the left-hand side consists of functions of $\mathrm{d}t$ and $\mathrm{d}(d_{1,t})$. By showing that the coefficients of both terms are zero, I prove that the left-hand side of Equation (4.29) is equal to the right-hand side. The detailed proof is as follows.

$$d_{1,t} = \frac{1}{\sqrt{\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u}} \left(\ln \frac{K_j^{(S)}(t)}{P^{(S)}(t, T+j)} + \frac{1}{2}\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u\right) \tag{4.23 revisited}$$

and

$$d_{2,t} = d_{1,t} - \sqrt{\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u}, \tag{4.24 revisited}$$

so

$$d_{1,t}^2 - d_{2,t}^2 = 2\ln \frac{K_j^{(S)}(t)}{P^{(S)}(t, T+j)}.$$

Rearranging the above equation gives

$$P^{(S)}(t, T+j)e^{-d_{2,t}^2/2} - K_j^{(S)}(t)e^{-d_{1,t}^2/2} = 0. \tag{4.30}$$

Use the Itô's formula on Equation (4.23)

$$\begin{aligned}
\mathrm{d}(d_{1,t}) &= -\frac{1}{\sqrt{\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u}} \frac{\mathrm{d}P^{(S)}(t, T+j)}{P^{(S)}(t, T+j)} + f(d_{1,t}, t)\mathrm{d}t \\
&= -\frac{1}{\sqrt{\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u}} \sigma_P(t, T+j)\mathrm{d}\widetilde{W}_P^{(S)}(t) + f(d_{1,t}, t)\mathrm{d}t,
\end{aligned}$$

where $f(d_{1,t}, t)$ represents the coefficient of $\mathrm{d}t$. The closed-form solution of $f(d_{1,t}, t)$ is left out since it is irrelevant to the final result. Take derivative on Equation (4.24)

$$\mathrm{d}(d_{2,t}) = \mathrm{d}(d_{1,t}) + \frac{\sigma_P^2(t, T+j)}{2\sqrt{\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u}} \mathrm{d}t.$$

Therefore,

$$\mathrm{d}(d_{2,t})\mathrm{d}(d_{2,t}) = \mathrm{d}(d_{1,t})\mathrm{d}(d_{1,t}) = \frac{\sigma_P^2(t, T+j)}{\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u} \mathrm{d}t.$$

The derivatives of $\mathrm{d}N(-d_{1,t})$ and $\mathrm{d}N(-d_{2,t})$ can be derived using the Itô's formula

$$\mathrm{d}N(-d_{1,t}) = -\frac{1}{\sqrt{2\pi}} e^{-d_{1,t}^2/2} \mathrm{d}(d_{1,t}) + \frac{d_{1,t}}{2\sqrt{2\pi}} e^{-d_{1,t}^2/2} \frac{\sigma_P^2(t, T+j)}{\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u} \mathrm{d}t, \tag{4.31}$$

$$\begin{aligned}
\mathrm{d}N(-d_{2,t}) &= -\frac{1}{\sqrt{2\pi}} e^{-d_{2,t}^2/2} \mathrm{d}(d_{2,t}) + \frac{d_{2,t}}{2\sqrt{2\pi}} e^{-d_{2,t}^2/2} \frac{\sigma_P^2(t, T+j)}{\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u} \mathrm{d}t \\
&= -\frac{1}{\sqrt{2\pi}} e^{-d_{2,t}^2/2} \mathrm{d}(d_{1,t}) + \frac{d_{1,t}}{2\sqrt{2\pi}} e^{-d_{2,t}^2/2} \frac{\sigma_P^2(t, T+j)}{\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u} \mathrm{d}t \\
&\quad - \frac{1}{\sqrt{2\pi}} e^{-d_{2,t}^2/2} \frac{\sigma_P^2(t, T+j)}{\sqrt{\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u}} \mathrm{d}t. \qquad (4.32)
\end{aligned}$$

In addition,

$$\mathrm{d}P^{(S)}(t, T+j)\mathrm{d}(d_{1,t}) = -P^{(S)}(t, T+j) \frac{\sigma_P^2(t, T+j)}{\sqrt{\int_t^T \sigma_P^2(u, T+j)\mathrm{d}u}} \mathrm{d}t.$$

Hence,

$$
\begin{aligned}
\mathrm{d}P^{(S)}(t,T+j)\mathrm{d}N(-d_{2,t}) &= -\frac{1}{\sqrt{2\pi}}e^{-d_{2,t}^2/2}\mathrm{d}P^{(S)}(t,T+j)\mathrm{d}(d_{1,t}) \\
&= \frac{1}{\sqrt{2\pi}}e^{-d_{2,t}^2/2}P^{(S)}(t,T+j)\frac{\sigma_P^2(t,T+j)}{\sqrt{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u}}\mathrm{d}t. \qquad (4.33)
\end{aligned}
$$

Substitute Equations (4.31), (4.32) and (4.33) into the left-hand side of Equation (4.29), and use the result of Equation (4.30). The coefficient of $\mathrm{d}(d_{1,t})$ is given by

$$
-\frac{1}{\sqrt{2\pi}}\left(P^{(S)}(t,T+j)e^{-d_{2,t}^2/2}-K_j^{(S)}(t)e^{-d_{1,t}^2/2}\right)=0.
$$

The coefficient of $\mathrm{d}t$ is given by

$$
\frac{d_{1,t}}{2\sqrt{2\pi}}\frac{\sigma_P^2(t,T+j)}{\int_t^T \sigma_P^2(u,T+j)\mathrm{d}u}\left(P^{(S)}(t,T+j)e^{-d_{2,t}^2/2}-K_j^{(S)}(t)e^{-d_{1,t}^2/2}\right)=0.
$$

Since both coefficients are zero, Equation (4.29) is proved, which implies that the differential of the portfolio is given by

$$
\mathrm{d}\left(\frac{Q_t}{S_t}\right)=\frac{g}{n}\sum_{j=0}^{J}\left[N(-d_{2,t})\mathrm{d}P^{(S)}(t,T+j)\right].
$$

On the other hand, the capital gains differential associated with this portfolio, denominated in units of equity fund, is

$$
\frac{g}{n}\sum_{j=0}^{J}\left[N(-d_{2,t})\mathrm{d}P^{(S)}(t,T+j)\right].
$$

Therefore, the change of value in the portfolio is entirely due to capital gains. This proves Theorem 4.3.8 that the portfolio is self-financing. □

Purchasing n units of the equity fund in addition to the hedging portfolio gives the investment strategy. Therefore, for the self-financing portfolio Y_t, the wealth invested in the equity fund is

$$\Omega_S^{Y,\,\mathrm{OBPI}}(t) = nS_t - gS_t \sum_{j=0}^{J} K_j^{(S)}(t) N(-d_{1,t}), \tag{4.34}$$

and the wealth invested in the zero-coupon bond that matures at time $T + j$ is

$$\Omega_{P(t,T+j)}^{\mathrm{Y,\,OBPI}}(t) = gP(t, T + j)N(-d_{2,t}).$$

Since the bond fund in the investment portfolio has a constant maturity $\overline{T}$, the zero-coupon bonds with various terms of maturity need to be replicated using the cash fund and the bond fund according to Equation (4.3). The wealth invested in the cash fund is

$$\Omega_M^{Y,\,\mathrm{OBPI}}(t) = g \sum_{j=0}^{J} P(t, T + j)N(-d_{2,t}) \left(1 - \frac{\sigma_r \beta(t, T + j)}{\sigma_{\overline{T}}}\right). \tag{4.35}$$

The wealth invested in the bond fund with constant maturity $\overline{T}$ is

$$\Omega_{\overline{T}}^{Y,\,\mathrm{OBPI}}(t) = g \sum_{j=0}^{J} P(t, T + j)N(-d_{2,t}) \frac{\sigma_r \beta(t, T + j)}{\sigma_{\overline{T}}}. \tag{4.36}$$

The target annuitisation fund with employment contributions consists of three parts: *1*) a hedging portfolio that hedges a short position in the option, *2*) n units of the equity fund, and *3*) a loan that represents the future contributions. The first two parts combined constitute the self-financing portfolio, Y_t. Adding the loan component to Equations (4.34), (4.35) and (4.36) gives the investment strategy at the fund level. For the portfolio X_t, the wealth invested in the equity fund, bond fund, cash fund, is respectively

$$\begin{aligned}
\Omega_S^{\mathrm{OBPI}}(t) &= nS_t - gS_t \sum_{j=0}^{J} K_j^{(S)}(t) N(-d_{1,t}), \\
\Omega_{\overline{T}}^{\mathrm{OBPI}}(t) &= g \sum_{j=0}^{J} P(t, T + j)N(-d_{2,t}) \frac{\sigma_r \beta(t, T + j)}{\sigma_{\overline{T}}} + \Omega_{\overline{T}}^{L}(t), \\
\Omega_M^{\mathrm{OBPI}}(t) &= g \sum_{j=0}^{J} P(t, T + j)N(-d_{2,t}) \left(1 - \frac{\sigma_r \beta(t, T + j)}{\sigma_{\overline{T}}}\right) + \Omega_M^{L}(t).
\end{aligned}$$

4.3.2 Constant proportion portfolio insurance strategy

CPPI strategy requires the amount allocated to risky asset as the product of a cushion, $\mathcal{C}_t$, and a multiplier, m (Black and Jones, 1987). The cushion is the portfolio value minus the minimum value one wants to achieve. Since the fund manager aims to provide the fund member with at least A_T at retirement, the cushion is the portfolio value minus the target annuitisation level. If the portfolio value is below the minimum value, the exposure to risky assets is zero. Hence, the exposure to the equity fund at time t is given by

$$\mathcal{E}_t = m\mathcal{C}_t = m(Y_t^{\text{CPPI}} - A_t)^+.$$

Assume that the rest of the assets is invested in a portfolio that replicates A_t. The dynamics of self-financing portfolio value at time t are given by

$$\mathrm{d}Y_t^{\text{CPPI}} = (Y_t^{\text{CPPI}} - \mathcal{E}_t)\frac{\mathrm{d}A_t}{A_t} + \mathcal{E}_t\frac{\mathrm{d}S_t}{S_t}.$$

The replicating portfolio can be regarded as a "safe" asset in that it closely tracks the floor value (Black and Perold, 1992). It is constructed by holding g units of zero-coupon bond that matures at time $T + j$ $(j = 0, \cdots, J)$. Since the bond fund has a constant term to maturity, the zero-coupon bonds with different terms of maturity are replicated using the cash fund and the bond fund. In summary, for the self-financing portfolio (Y_t), the wealth invested in the cash fund is given by

$$\Omega_M^{Y,\,\text{CPPI}}(t) = \frac{g}{A_t}\left(Y_t^{\text{CPPI}} - \mathcal{E}_t\right)\sum_{j=0}^{J} P(t, T+j)\left(1 - \frac{\sigma_r\beta(t, T+j)}{\sigma_{\overline{T}}}\right), \tag{4.37}$$

and the wealth invested in the bond fund is given by

$$\Omega_{\overline{T}}^{Y,\,\text{CPPI}}(t) = \frac{g}{A_t}\left(Y_t^{\text{CPPI}} - \mathcal{E}_t\right)\sum_{j=0}^{J} P(t, T+j)\frac{\sigma_r\beta(t, T+j)}{\sigma_{\overline{T}}}. \tag{4.38}$$

For the fund portfolio (X_t), the wealth invested in each asset can be found using Equations (4.37), (4.38) and the relationships shown in Equation (4.14).

4.4 Numerical application

This section uses a number of numerical applications to analyse the dynamic behaviour of the portfolio insurance strategies derived in the previous section.

4.4.1 Assumptions

Assume the portfolio is rebalanced annually, and that the contribution is made at the beginning of each year for 40 years, representing the period from ages 25 to 65. The loan amount becomes

$$L_t = -\sum_{u=t}^{T} c(u)P(t,u),$$

and $\Omega^L_{\overline{T}}(t)$ given by (4.7) becomes

$$\Omega^L_{\overline{T}}(t) = -\sum_{u=t}^{T} c(u)P(t,u)\frac{\sigma_r\beta(t,u)}{\sigma_{\overline{T}}}.$$

The budget constraint of Y_t is given by

$$\begin{aligned}
Y_{t+1} &= \Omega^{Y,\,\mathrm{PI}}_S(t)\frac{S_{t+1}}{S_t} + \sum_{j=0}^{J}\Omega^{Y,\,\mathrm{PI}}_{P(t,T+j)}(t)\frac{P(t+1,T+j)}{P(t,T+j)} \\
&= \Omega^{Y,\,\mathrm{PI}}_S(t)\frac{S_{t+1}}{S_t} + \sum_{j=0}^{J} gP(t,T+j)N(-d_{2,t})\frac{P(t+1,T+j)}{P(t,T+j)} \\
&= \Omega^{Y,\,\mathrm{PI}}_S(t)\frac{S_{t+1}}{S_t} + \sum_{j=0}^{J} gP(t+1,T+j)N(-d_{2,t}),
\end{aligned}$$

where PI represents either OBPI or CPPI.

4.4.1.1 Parameterisation

The fund balance at retirement is expected to provide the fund member with \$24,000 (in real term) every year for 35 years. The \$24,000 annual retirement benefit is based on the Association of Superannuation Funds of Australia (ASFA) Retirement Standard. It is estimated that an average Australian needs \$24,250 (\$23,754) per annum at around 65 (85) to maintain a modest lifestyle (Association of Superannuation Funds of Australia, 2017). The 35-year horizon covers the period from ages 65 to 100. When constructing the longevity index in Chapter 3, I find that the age beyond which less than 5% of 65-year-old retirees are still alive is close to 100 in all of the four countries using the latest life tables. The relatively long post-retirement horizon largely ensures that retirees can self-insure longevity risk to reflect the thin voluntary annuitisation market around the world.

The parameters that represent the financial market in the base case, shown in Table 4.1, follow the estimation results in Brennan and Xia (2002)[2]. The simulation is done 100,000 times. To investigate the impact of investment horizon, a total of eight cohorts are selected that correspond to 40, 35, $\cdots$, 10, 5 years of pre-retirement investment horizons. In terms of the multiplier in the CPPI strategy, I am interested in cases when $m > 1$, that is, when the payoff function is convex. When $m = 1$, CPPI reduces to buy-and-hold strategies.

Table 4.1. Parameter values for the numerical applications of portfolio insurance strategies in the base case.

Parameter	Value
Real interest rate: $\mathrm{d}r_t = \kappa(\overline{r} - r_t)\mathrm{d}t + \sigma_r \mathrm{d}W_r(t)$	
κ	0.631
$\overline{r}$	0.012
σ_r	0.026
Bond fund return process: $\dfrac{\mathrm{d}P_t^{\overline{T}}}{P_t^{\overline{T}}} = (r_t - \sigma_{\overline{T}}\lambda_r)\mathrm{d}t - \sigma_{\overline{T}}\mathrm{d}W_r(t)$	
$\overline{T}$	20
λ_r	−0.209
Equity fund return process: $\dfrac{\mathrm{d}S_t}{S_t} = (r_t + \sigma_S\lambda_S + \sigma_{Sr}\lambda_r)\mathrm{d}t + \sigma_S\mathrm{d}W_S(t) + \sigma_{Sr}\mathrm{d}W_r(t)$	
σ_S	0.157
σ_{Sr}	−0.020
λ_S	0.343

[2]Brennan and Xia (2002) assume the nominal, rather than the real, stock price follow a geometric Brownian motion. Using their model specification, the dynamics of equity's real price can be obtained by dividing its nominal price by the price index. Since Brennan and Xia (2002) show that the correlation coefficient between stock price and inflation is close to zero, using their parameter estimation results has little impact on the present analysis.

4.4.1.2 Simulation of interest rate and equity fund price

To simulate the real interest rate r_t and the equity fund price S_t, I use the results that r_t follows a normal distribution, and S_t follows a log-normal distribution. In particular, given interest rate r_u where $u < t$, r_t is normally distributed with mean

$$e^{-\kappa(t-u)}r_u + \overline{r}(1 - e^{-\kappa(t-u)}),$$

and variance

$$\frac{\sigma_r^2}{2\kappa}\left(1 - e^{-2\kappa(t-u)}\right).$$

To simulate the real interest rate at times $0 = t_0 < t_1 < \cdots < t_n$, I set

$$r_{t_{i+1}} = e^{-\kappa(t_{i+1}-t_i)}r_{t_i} + \overline{r}\left(1 - e^{-\kappa(t_{i+1}-t_i)}\right) + \sigma_r\sqrt{\frac{1}{2\kappa}\left(1 - e^{-2\kappa(t_{i+1}-t_i)}\right)}Z_{i+1}^r,$$

where Z_{i+1}^r follows the standard normal distribution. To simulate the equity fund price, I set

$$\begin{aligned} S_{t_{i+1}} = S_{t_i}\exp\Bigg[&\left(r_{t_i} + \sigma_S\lambda_S + \sigma_{Sr}\lambda_r - \frac{1}{2}\sigma_S^2 - \frac{1}{2}\sigma_{Sr}^2\right)(t_{i+1} - t_i) \\ &+ \sigma_S\sqrt{t_{i+1} - t_i}Z_{t+i}^S + \sigma_{Sr}\sqrt{t_{i+1} - t_i}Z_{t+i}^r\Bigg], \end{aligned}$$

where Z_{t+i}^S follows the standard normal distribution, and Z_{t+1}^r is independent of Z_{t+1}^S. Figure 4.1 shows that the simulated mean and standard deviation of the interest rate and the equity fund return match well to their theoretical counterparts.

4.4.1.3 Initial values

The equity fund price at time 0 is set at \$1,000. This assumption is without loss of generality. For the OBPI strategy, the initial investment in the equity fund is determined by multiplying the number of options, n, by the equity fund price, S_0. n is obtained by solving Equations (4.15) and (4.22) simultaneously. If S_0 increases (decreases), then n will decrease (increase) accordingly. For the CPPI strategy, the amount invested in the equity

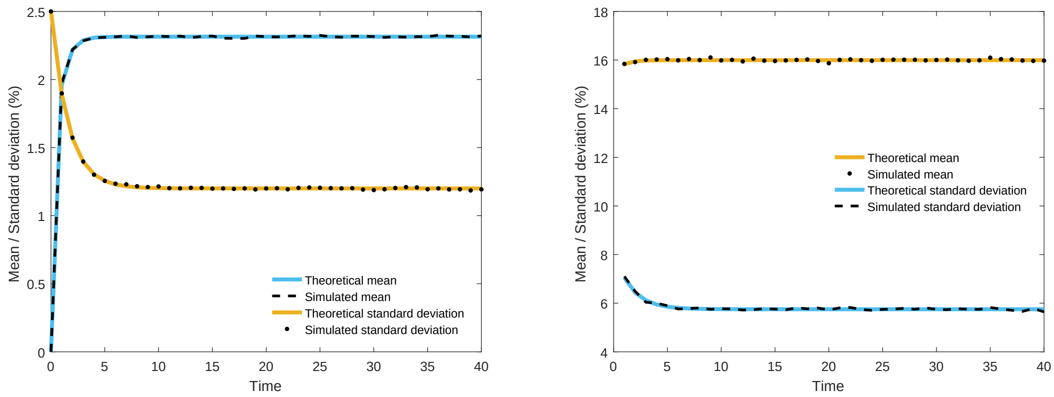


Figure 4.1. Comparison of theoretical and simulated means, and theoretical and simulated standard deviations: (Left Panel) interest rate; (Right Panel) equity fund return.

fund at time 0 equals the product of the multiplier, m, and the cushion amount, $\mathcal{C}_t$, both being independent of S_0.

The real interest rate at time 0 is set at 2.5%, then the target annuitisation level at time 0 can be determined. The results are displayed in the second column of Table 4.2. It is clear that the older the fund member at time 0, the shorter the investment horizon, and consequently the higher the target value. Table 4.2 also shows that the mean and standard deviation of target annuitisation levels at retirement are similar for different cohorts. This is expected as the target annuitisation level is determined by the post-retirement consumption profile, which is the same for all the individuals in the model setup.

Table 4.2. The target annuitisation level at time 0, and the mean and standard deviation (Std) of the target annuitisation level at retirement for different cohorts.

Age at time 0	A_0 (\$000)	A_T (\$000)	
		Mean	Std
25	275.28	619.25	21.00
30	303.87	619.57	20.84
35	335.43	619.17	21.13
40	370.27	619.33	20.87
45	408.73	619.66	20.89
50	451.18	619.00	20.86
55	498.04	619.51	21.00
60	549.78	619.01	20.88

The average target level at retirement is around \$620,000. This figure well approximates the constructed longevity index levels in the previous chapter (Table 3.4) at age 65 given the \$24,000 per annum income target. However, the standard deviations are much smaller than those of constructed indices. This is caused by different assumptions about the interest rate. When constructing the longevity index, the current interest rate, r_t, is assumed to last for the following $T^d_{x,t}$ years. When computing the target annuitisation levels, the interest rate is assumed to follow a mean reverting process. Although the standard deviation of the simulated interest rates is close to the ones used for constructing the indices (Table 4.3), the constructed longevity indices are more sensitive to the volatility in the interest rate, and hence have higher standard deviations.

Table 4.3. The mean and standard deviation (Std) of real interest rate for each country and the simulation.

	Australia	Japan	U.K.	U.S.	Simulation
Mean (%)	2.83	1.75	1.84	2.33	1.26
Std (%)	2.82	2.34	2.26	1.63	2.25

In the numerical application, I assume the contribution is made on an annual basis, and that it increases by 2.5% per annum to reflect productivity growth. The assumption about the first contribution amount, $c(0)$, is shown in Table 4.4. The third column of Table 4.4 shows the present value of future contributions. The buffer above the target annuitisation level affects the extent to which the fund manager can invest in the risky asset. I set the initial fund balance, X_0, such that that the value of the self-financing portfolio, Y_0, is \$33,000 above the target. This would give fund managers adequate room to invest in the risky asset. I later perform the sensitivity analysis on the initial fund balance in Section 4.4.4.

Given S_0, Y_0, and r_0, I can solve for the number of options to be synthesised, and the option price at time 0. Table 4.5 summarises the results. The option is worth less with decreasing investment horizons since the time value of the option decays. As a result, the amount invested in the equity fund increases, so n becomes larger with shorter investment horizons.

Table 4.4. The assumption about initial values, including initial contribution and initial fund balance, used in the base case.

Age at time 0	$c(0)$ ($000)	$-L_0$ ($000)	$Y_0 - A_0$ ($000)	A_0 ($000)	Y_0 ($000)	$X_0 = Y_0 + L_0$ ($000)	$-L_0/X_0$
25	7	308.04	33	275.28	308.28	0.24	1,261.68
30	8	303.92	33	303.87	336.87	32.95	9.22
35	9	289.17	33	335.43	368.43	79.26	3.65
40	10	264.22	33	370.27	403.27	139.05	1.90
45	11	229.48	33	408.73	441.73	212.25	1.08
50	12	185.34	33	451.18	484.18	298.84	0.62
55	13	132.18	33	498.04	531.04	398.86	0.33
60	14	70.37	33	549.78	582.78	512.41	0.14

Table 4.5. Number of options to be replicated and the value of a single option at time 0 in the base case.

	Age at time 0							
	25	30	35	40	45	50	55	60
n	153	170	192	219	254	299	361	455
Q_0 ($)	1,020.01	975.92	915.85	837.90	739.82	618.57	469.10	280.14

4.4.2 Investment strategy

This section discusses the portfolio weights in each asset based on the OBPI and CPPI strategies. For members joining the fund at age 25 (30), the pension fund balance in the first 10 (5) years is relatively low, so the portfolio weights in the equity fund tend to be large positive figures and those in the bond fund tend to be large negative figures. In addition, the portfolio weights are very sensitive to the fund balances. I therefore focus on the results in the last 30 years before retirement for these two cohorts. For the remaining cohorts, the results are shown for the whole pre-retirement period.

Figure 4.2 shows the average portfolio weights in the equity fund for different ages joining the pension fund. If the member joins the fund at a relatively young age, the proportion invested in the equity fund decreases as fund members grow older. This pattern is similar to the lifecycle investment strategy in that the portfolio mix becomes more conservative as members get older. But unlike traditional lifecycle investment strategies that change the portfolio mix in a predetermined way, the portfolio insurance strategies respond dynamically to the investment opportunities. This leads to a higher level of downside risk protection in a bear market and a better upside performance in a bull market compared to the standard lifecycle investment strategy. If, on the other hand, the member joins the fund after mid-30s, the portfolio weight in the equity fund shows an upward trend with age.

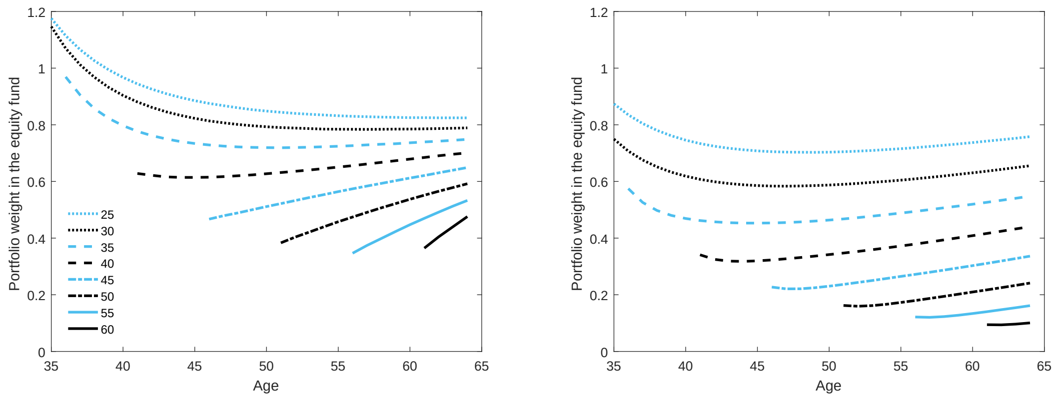


Figure 4.2. Average portfolio weights in the equity fund for the base case: (Left Panel) OBPI; (Right Panel) CPPI with $m = 1.6$.

The difference in portfolio weight between different cohorts is due to the fact the older cohorts have lower amounts of future contributions. For younger cohorts, when they join the fund, the self-financing portfolio, Y_0, is dominated by the safe assets composed of future contributions (see the last column of Table 4.4). This allows fund managers to invest heavily in the risky asset before turning 40. The safe assets gradually deplete as members get older, so the proportion invested in the equity fund diminishes as well. For older cohorts, their initial fund values are significantly higher due to the assumption that the buffer above the target is the same for each cohort when joining the fund. As a result, the present value of future contributions remains a small proportion of the self-financing portfolio. Therefore, the portfolio weights in the equity fund for the fund balance (X_t) follow the trend of the self-financing portfolio (Y_t). The right panel of Figure 4.3 shows that on average, the proportion of the self-financing portfolio invested in the equity fund increases over time. This is due to the nature of the portfolio insurance strategies, which suggest that the greater the portfolio value over the target, the higher the portfolio weight in the risky asset. At time 0, I set Y_0 above A_0 as otherwise the investment problem would become trivial. Since Y_t typically grows faster than A_t, its excess above A_t will generally increase over time. This leads to a higher average proportion of wealth allocated to the equity fund at older ages.

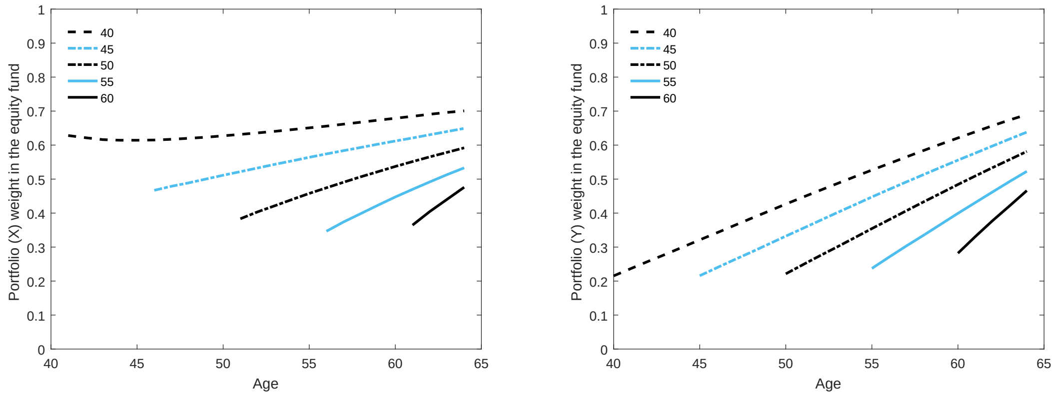
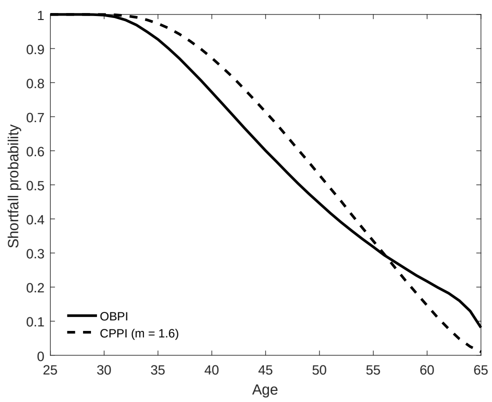


Figure 4.3. Average portfolio weights in the equity fund using the OBPI strategy for the base case: (Left Panel) the target annuitisation fund X_t; (Right Panel) the self-financing portfolio Y_t .

Figure 4.2 also shows that the two portfolio insurance strategies generate different trends in portfolio weight for the younger cohorts after they turn 50. The weight in the equity fund slightly increases using the CPPI strategy while that of the OBPI strategy remains relatively flat. This is related to the CPPI's better downside protection than the OBPI. The downside protection can be measured using the shortfall probability and average shortfall amount. The shortfall probability is defined as the probability that the fund value is below the target annuitisation level, i.e. $\mathbb{P}(X_t < A_t)$. The average shortfall is given by

$$\mathbb{E}\left[X_t - A_t | X_t < A_t\right],$$

where $\mathbb{E}$ is the expectation operator under the real world probability measure $\mathbb{P}$. Figure 4.4 and Figure 4.5 show the shortfall probability and the average shortfall amount (in absolute value), respectively, for the youngest two cohorts. As the fund receives employment contributions and investment returns, both strategies give lower chances and severities of the shortfall. When the fund members approach retirement, both the shortfall probability and the average shortfall amount decrease significantly faster for the CPPI strategy than for the OBPI strategy. Consequently, the CPPI strategy allocates a higher weight to the equity fund. The shortfall protection is discussed in more detail in Section 4.4.3.



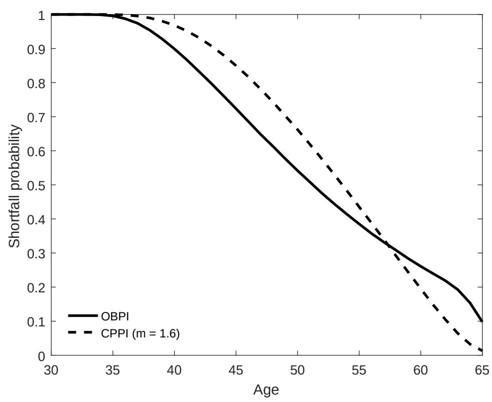


Figure 4.4. The shortfall probability: (Left Panel) members join the fund at age 25; (Right Panel) members join the fund at age 30. The shortfall probability is given by $\mathbb{P}(X_T < A_T)$.

The average portfolio weights in the bond fund and cash fund are shown in Figure 4.6

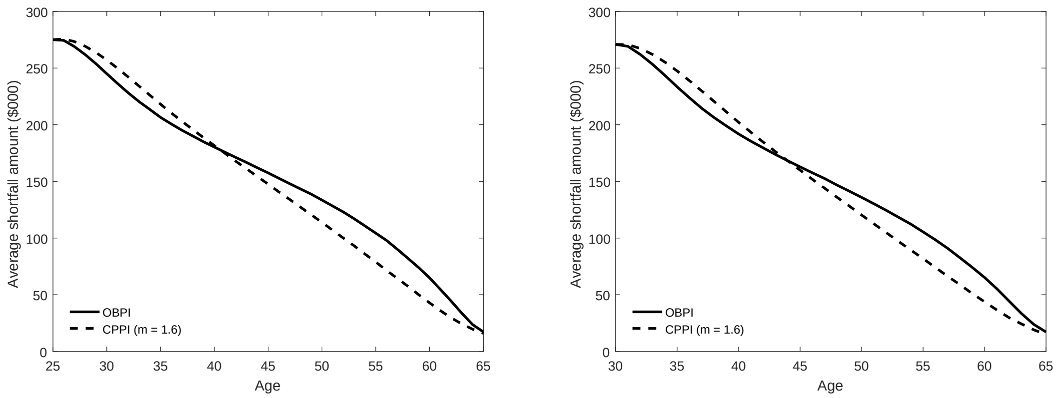


Figure 4.5. The absolute value of average shortfall amount: (Left Panel) members join the fund at age 25; (Right Panel) members join the fund at age 30. Average shortfall amount is given by $\mathbb{E}\left[X_T - A_T | X_T < A_T\right]$.

and Figure 4.7, respectively. The portfolio weights in the bond fund move almost in the opposite direction to those in the equity fund, showing an upward trend for the younger cohorts and a downward trend for the older cohorts. The average portfolio weights in the cash fund show less variation across different cohorts.

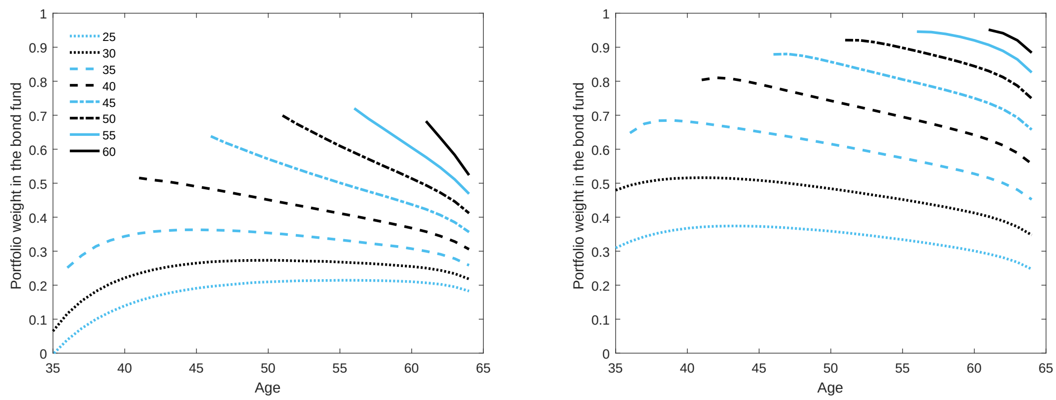


Figure 4.6. Average portfolio weights in the bond fund for the base case: (Left Panel) the OBPI strategy, (Right Panel) the CPPI strategy with $m = 1.6$.

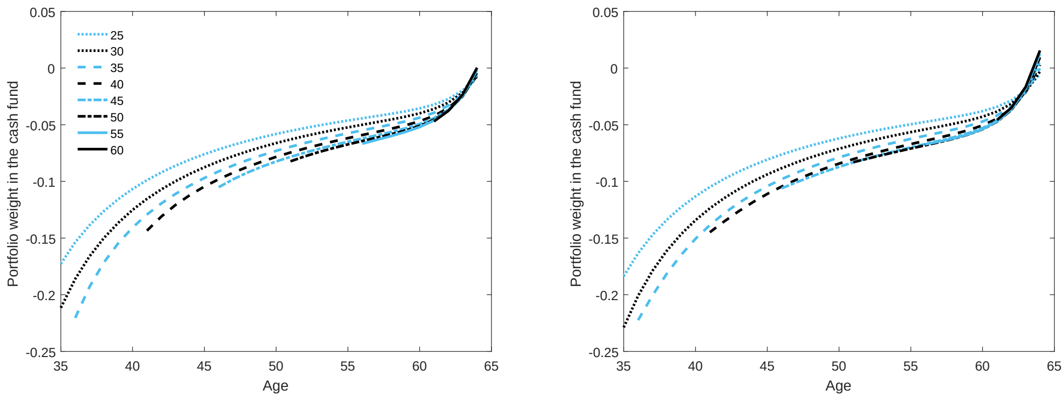


Figure 4.7. Average portfolio weights in the cash fund for the base case: (Left Panel) the OBPI strategy, (Right Panel) the CPPI strategy with $m = 1.6$.

Having discussed the average portfolio weights in each asset, I then investigate their sample paths for those members who are 25 at time 0. I choose the youngest cohort since they have the longest investment horizon. Once again, the portfolio weights in early years can be extremely volatile due to the low fund balance, I focus on the results starting at age 35. Figure 4.8 shows some sample paths and the 95% confidence intervals for the portfolio weights in the equity fund. Although the average weights decrease over age, they have very wide confidence bounds. Some cases observe large year-to-year variability as well. For the CPPI strategy, the 95% confidence intervals become significantly wider as the multiplier increases from 1.2 to 2.0 (Figure 4.9).

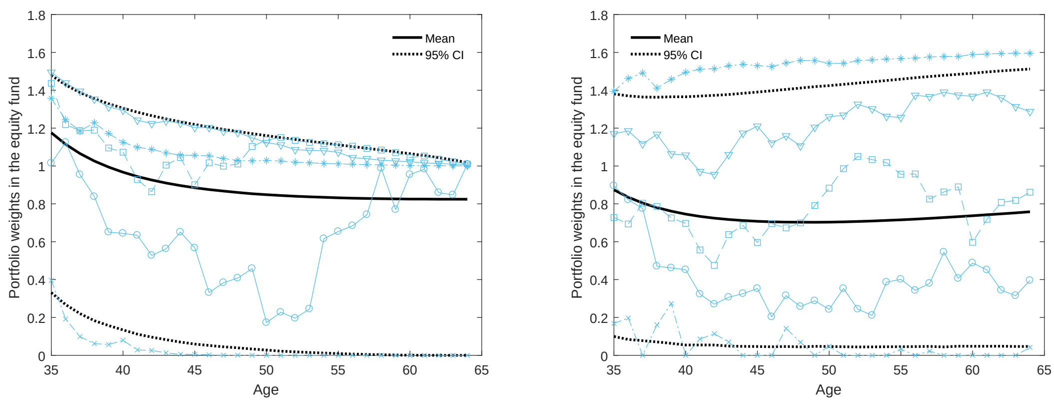


Figure 4.8. Some simulated sample paths (blue lines with markers), mean and 95% confidence intervals of portfolio weights in the equity fund: (Left Panel) the OBPI strategy; (Right Panel) the CPPI strategy with $m = 1.6$. The member joins the fund at age 25.

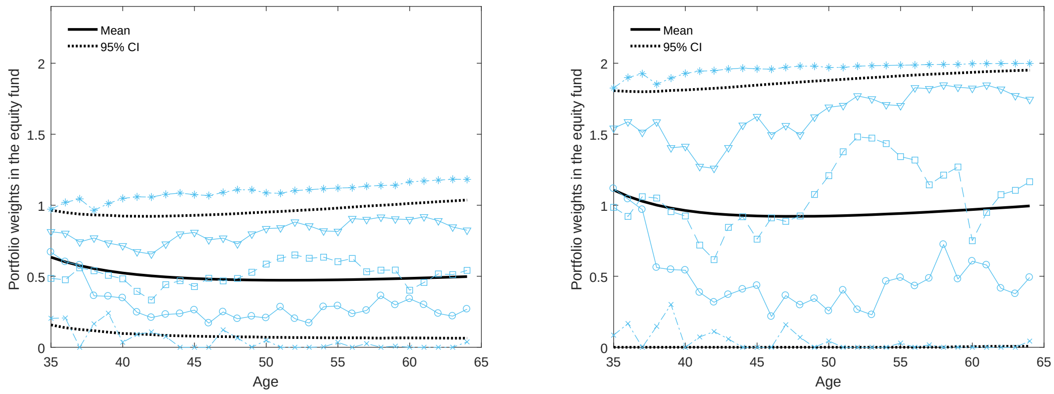


Figure 4.9. Some simulated sample paths (blue lines with markers), mean and 95% confidence intervals of portfolio weights in the equity fund: (Left Panel) the CPPI strategy with $m = 1.2$; (Right Panel) the CPPI strategy with $m = 2$. The member joins the fund at age 25.

4.4.3 Comparison of the payoff

The fund manager uses the portfolio insurance strategies to provide the fund members with a minimum income-based retirement benefit, so I want to examine the portfolio values at retirement. Table 4.6 summarises the mean, median, 95% confidence intervals, shortfall probability, and average shortfall amount of the portfolio values at retirement. The shortfall occurs mainly due to the annual rebalancing assumption, as no shortfall would occur under the assumption of continuous rebalancing. In practice, it is possible to rebalance the portfolio more frequently (e.g. monthly or weekly), but it would also incur higher transaction costs. I do not explicitly consider transaction cost in the study, but it is worth noting that the transaction costs could be very substantial (Boyle and Vorst, 1992). As a consequence, the shortfall probability would not necessarily decrease if the rebalancing frequency increases.

Comparing the two strategies within each single cohort, the CPPI strategy provides a better downside risk protection as indicated by significantly lower shortfall probabilities. The average shortfall amount is comparable between the two strategies, though. In terms of the fund balance at retirement, the OBPI strategy usually gives a higher average amount. Increasing the value of the CPPI multiplier can also increase the average payoff, but this is at the cost of weaker downside protection and the increments in the median

Table 4.6. Mean, median, 95% confidence intervals (CI), shortfall probability, and average shortfall amount of the portfolio values at retirement for the base case.

	Mean ($000)	Median ($000)	95% CI ($000)	Shortfall probability	Average shortfall ($000)	(A_T)
25 years old at time 0						
X_T^{OBPI}	2,616	1,563	(598, 11,223)	0.081	−17.19	−0.027
$X_T^{\text{CPPI}}(m = 1.2)$	1,427	1,023	(660, 4,720)	0.004	−13.85	−0.021
$X_T^{\text{CPPI}}(m = 1.6)$	2,396	1,163	(643, 11,892)	0.010	−15.93	−0.025
$X_T^{\text{CPPI}}(m = 2.0)$	4,445	1,240	(627, 27,280)	0.022	−17.68	−0.027
30 years old at time 0						
X_T^{OBPI}	2,076	1,305	(596, 8,184)	0.098	−17.11	−0.027
$X_T^{\text{CPPI}}(m = 1.2)$	1,157	912	(651, 3,165)	0.006	−13.27	−0.020
$X_T^{\text{CPPI}}(m = 1.6)$	1,697	1,001	(638, 7,057)	0.013	−14.89	−0.023
$X_T^{\text{CPPI}}(m = 2.0)$	2,736	1,054	(624, 14,991)	0.026	−17.17	−0.027
35 years old at time 0						
X_T^{OBPI}	1,664	1,106	(593, 6,000)	0.115	−17.16	−0.027
$X_T^{\text{CPPI}}(m = 1.2)$	975	829	(645, 2,186)	0.009	−12.41	−0.019
$X_T^{\text{CPPI}}(m = 1.6)$	1,270	885	(633, 4,261)	0.018	−14.63	−0.023
$X_T^{\text{CPPI}}(m = 2.0)$	1,784	919	(622, 8,202)	0.032	−16.58	−0.026
40 years old at time 0						
X_T^{OBPI}	1,353	940	(592, 4,401)	0.135	−16.86	−0.027
$X_T^{\text{CPPI}}(m = 1.2)$	853	769	(639, 1,583)	0.014	−13.25	−0.020
$X_T^{\text{CPPI}}(m = 1.6)$	1,010	804	(629, 2,664)	0.024	−15.05	−0.023
$X_T^{\text{CPPI}}(m = 2.0)$	1,259	826	(620, 4,613)	0.039	−16.74	−0.026
45 years old at time 0						
X_T^{OBPI}	1,123	816	(590, 3,258)	0.156	−16.86	−0.027
$X_T^{\text{CPPI}}(m = 1.2)$	773	724	(634, 1,199)	0.022	−12.80	−0.020
$X_T^{\text{CPPI}}(m = 1.6)$	853	745	(626, 1,746)	0.033	−14.67	−0.023
$X_T^{\text{CPPI}}(m = 2.0)$	971	759	(618, 2,650)	0.049	−16.28	−0.025
50 years old at time 0						
X_T^{OBPI}	948	724	(588, 2,374)	0.181	−17.05	−0.027
$X_T^{\text{CPPI}}(m = 1.2)$	718	692	(631, 957)	0.035	−12.65	−0.019
$X_T^{\text{CPPI}}(m = 1.6)$	757	704	(624, 1,208)	0.046	−14.17	−0.022
$X_T^{\text{CPPI}}(m = 2.0)$	808	712	(618, 1,594)	0.062	−15.70	−0.024
55 years old at time 0						
X_T^{OBPI}	821	675	(586, 1,710)	0.201	−17.21	−0.027
$X_T^{\text{CPPI}}(m = 1.2)$	682	669	(629, 808)	0.056	−11.92	−0.018
$X_T^{\text{CPPI}}(m = 1.6)$	698	675	(623, 913)	0.066	−13.02	−0.020
$X_T^{\text{CPPI}}(m = 2.0)$	719	679	(618, 1,058)	0.080	−14.25	−0.022
60 years old at time 0						
X_T^{OBPI}	725	656	(585, 1,188)	0.217	−17.40	−0.028
$X_T^{\text{CPPI}}(m = 1.2)$	657	652	(629, 714)	0.091	−11.84	−0.018
$X_T^{\text{CPPI}}(m = 1.6)$	663	655	(624, 749)	0.095	−12.47	−0.019
$X_T^{\text{CPPI}}(m = 2.0)$	669	657	(619, 791)	0.104	−13.13	−0.020

payoff are marginal. Comparing the payoff across different cohorts for each portfolio insurance strategy, the older the cohort, the lower the payoff level, and consequently the higher the chance of falling short of the target. This is due to the lower present values of future contributions for older cohorts despite their having higher initial fund balances. It is noticeable that the average shortfall amount remains less than 3% of the target annuitisation level across different cohorts, demonstrating the effectiveness of the strategy in meeting the target.

Figure 4.10 compares the fund balance at retirement between two portfolio insurance strategies on a payoff diagram, where the x-axis represents the scaled equity fund price and the y-axis represents the payoff. To facilitate the comparison, both the payoff and the equity fund price are denominated in the target annuitisation level, A_T. The diagram coincides the results displayed in Table 4.6. The table shows that the OBPI strategy performs marginally better than the CPPI strategy with $m = 1.6$ in terms of average and median portfolio amounts, and so does the diagram show that the OBPI strategy performs slightly better if the equity fund performs reasonably well. Table 4.6 shows that the OBPI strategy has a much higher shortfall probability, and Figure 4.10 shows that it has more points below the target annuitisation level (represented by the horizontal line through 1 on the y-axis). These results hold for a lower value of the multiplier (Figure 4.11 (a)), or a shorter investment horizon (Figure 4.11 (c)). If the multiplier becomes larger (Figure 4.11 (b)), however,the payoff of the CPPI strategy is no longer dominated by that of the OBPI strategy, and it also has a wider distribution.

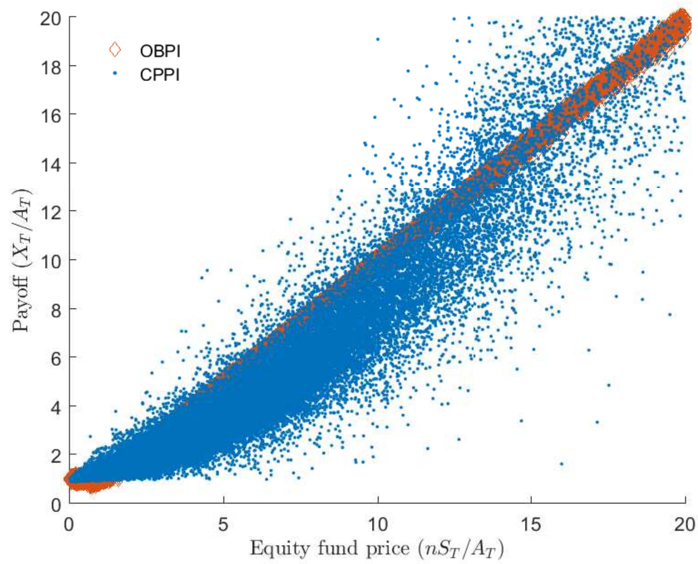
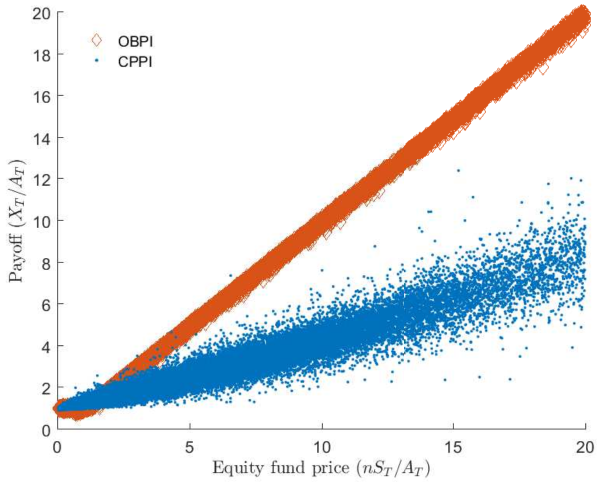
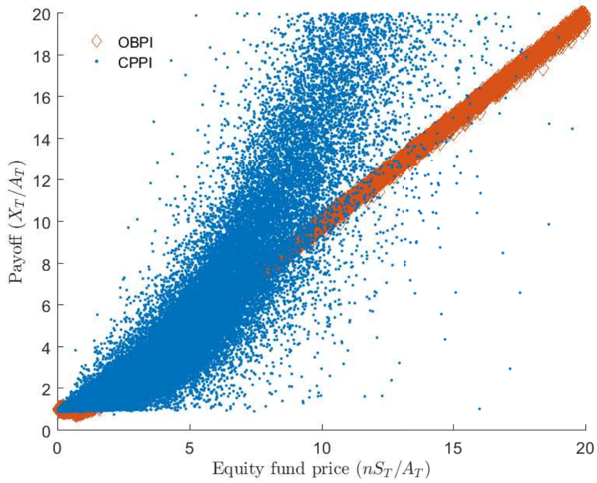


Figure 4.10. Comparison of OBPI and CPPI payoffs. The member joins the fund at age 25 and the CPPI multiplier (m) is 1.6.

(a) 25 years old at time 0, the CPPI multiplier is 1.2



(b) 25 years old at time 0, the CPPI multiplier is 2.0



(c) 50 years old at time 0, the CPPI multiplier is 1.6

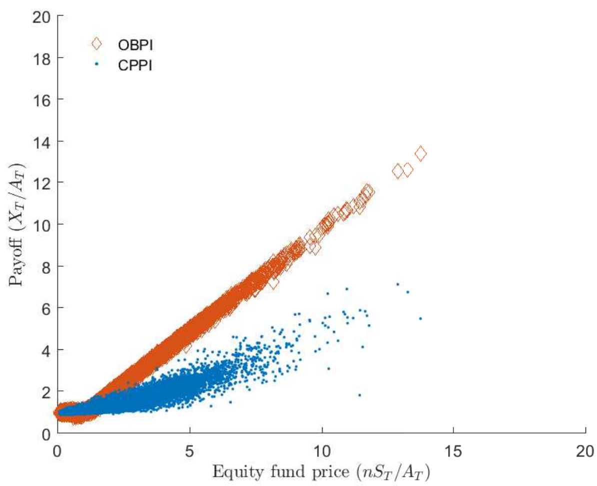


Figure 4.11. Comparison of OBPI and CPPI payoffs in selected scenarios: (a) the CPPI multiplier is close to 1; (b) the CPPI multiplier is relatively large; (c) the investment horizon is relatively short.

4.4.4 Sensitivity analysis

The base case analysis shows that CPPI performs better than OBPI in downside risk protection, and that the average shortfall amount is less than 3% of the target annuitisation level at retirement for both strategies across different cohorts. I perform sensitivity analysis to investigate the extent to which the shortfall probability and the average shortfall amount would increase if *1*) the equity fund price becomes more volatile, or *2*) the initial fund balance, X_0, becomes lower.

4.4.4.1 Equity fund volatility

Fixing all the other parameters as at the base case, I increase the volatility, σ_S, from 0.157 to 0.20. Figure 4.12 shows the payoff diagram. Compared to Figure 4.10, the payoffs of the CPPI strategy become more widespread, whereas there are no noticeable changes to the payoffs of the OBPI strategy.

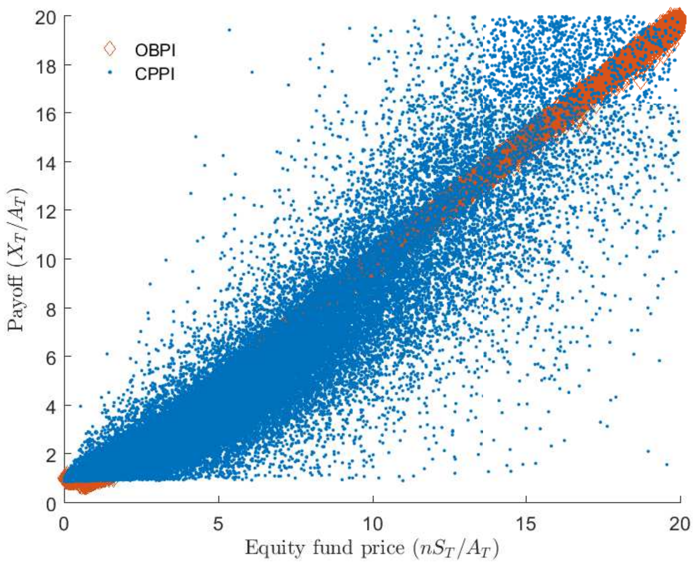


Figure 4.12. Comparison of OBPI and CPPI payoffs when $\sigma_S = 0.2$. The member joins the fund at age 25 and the CPPI multiplier (m) is 1.6.

Table 4.7 compares the shortfall probability and average shortfall amount between the two volatility levels. The CPPI strategy is more sensitive to the change in equity fund volatility. For members joining the fund at age 25, the shortfall probability of the CPPI strategy doubles when the equity fund volatility increases from 0.157 to 0.20. By contrast, the shortfall probability of the OBPI strategy increases by approximately 25%. The average shortfall amount remains a small proportion of the target annuitisation level at retirement. In the worst scenario shown in Table 4.7, the average shortfall amount conditional on its occurrence is less than 5% of the target.

Given the time horizon of the investment, it might be more reasonable to assume stochastic volatility of the equity fund price. Prigent and Bertrand (2003) show that stochastic volatility increases the standard deviation, skewness, and kurtosis of the portfolio returns for the OBPI strategy. However, I would be unable to use the Jamshidian (1989) decomposition method under stochastic volatility due to the introduction of a new source of randomness.

Table 4.7. The shortfall probability and average shortfall amount by different volatility levels of the equity fund.

	Shortfall probability		Average shortfall ($000)		(A_T)	
σ_S	0.157	0.20	0.157	0.20	0.157	0.20
25 years old at time 0						
X_T^{OBPI}	0.081	0.101	−17.19	−18.04	−0.027	−0.029
$X_T^{\text{CPPI}}(m = 1.2)$	0.004	0.010	−13.85	−15.80	−0.021	−0.024
$X_T^{\text{CPPI}}(m = 1.6)$	0.010	0.026	−15.93	−18.12	−0.025	−0.028
$X_T^{\text{CPPI}}(m = 2.0)$	0.022	0.058	−17.68	−28.55	−0.027	−0.045
30 years old at time 0						
X_T^{OBPI}	0.098	0.118	−17.11	−17.73	−0.027	−0.028
$X_T^{\text{CPPI}}(m = 1.2)$	0.006	0.013	−13.27	−14.93	−0.020	−0.023
$X_T^{\text{CPPI}}(m = 1.6)$	0.013	0.031	−14.89	−17.69	−0.023	−0.027
$X_T^{\text{CPPI}}(m = 2.0)$	0.026	0.062	−17.17	−24.15	−0.027	−0.038
35 years old at time 0						
X_T^{OBPI}	0.115	0.136	−17.16	−17.81	−0.027	−0.028
$X_T^{\text{CPPI}}(m = 1.2)$	0.009	0.018	−12.41	−14.48	−0.019	−0.022
$X_T^{\text{CPPI}}(m = 1.6)$	0.018	0.037	−14.63	−17.08	−0.023	−0.027
$X_T^{\text{CPPI}}(m = 2.0)$	0.032	0.070	−16.58	−20.41	−0.026	−0.032
40 years old at time 0						
X_T^{OBPI}	0.135	0.158	−16.86	−17.53	−0.027	−0.028
$X_T^{\text{CPPI}}(m = 1.2)$	0.014	0.024	−13.25	−14.85	−0.020	−0.023
$X_T^{\text{CPPI}}(m = 1.6)$	0.024	0.045	−15.05	−16.82	−0.023	−0.026
$X_T^{\text{CPPI}}(m = 2.0)$	0.039	0.077	−16.74	−19.48	−0.026	−0.030
45 years old at time 0						
X_T^{OBPI}	0.156	0.181	−16.86	−17.61	−0.027	−0.028
$X_T^{\text{CPPI}}(m = 1.2)$	0.022	0.034	−12.80	−14.47	−0.020	−0.022
$X_T^{\text{CPPI}}(m = 1.6)$	0.033	0.055	−14.67	−16.49	−0.023	−0.025
$X_T^{\text{CPPI}}(m = 2.0)$	0.049	0.087	−16.28	−18.69	−0.025	−0.029
50 years old at time 0						
X_T^{OBPI}	0.181	0.206	−17.05	−17.78	−0.027	−0.028
$X_T^{\text{CPPI}}(m = 1.2)$	0.035	0.048	−12.65	−13.89	−0.019	−0.021
$X_T^{\text{CPPI}}(m = 1.6)$	0.046	0.069	−14.17	−15.91	−0.022	−0.025
$X_T^{\text{CPPI}}(m = 2.0)$	0.062	0.098	−15.70	−17.82	−0.024	−0.028
55 years old at time 0						
X_T^{OBPI}	0.201	0.227	−17.21	−17.83	−0.027	−0.028
$X_T^{\text{CPPI}}(m = 1.2)$	0.056	0.068	−11.92	−12.88	−0.018	−0.020
$X_T^{\text{CPPI}}(m = 1.6)$	0.066	0.087	−13.02	−14.47	−0.020	−0.022
$X_T^{\text{CPPI}}(m = 2.0)$	0.080	0.112	−14.25	−16.19	−0.022	−0.025
60 years old at time 0						
X_T^{OBPI}	0.217	0.241	−17.40	−18.46	−0.028	−0.029
$X_T^{\text{CPPI}}(m = 1.2)$	0.091	0.098	−11.84	−12.40	−0.018	−0.019
$X_T^{\text{CPPI}}(m = 1.6)$	0.095	0.110	−12.47	−13.32	−0.019	−0.020
$X_T^{\text{CPPI}}(m = 2.0)$	0.104	0.126	−13.13	−14.44	−0.020	−0.022

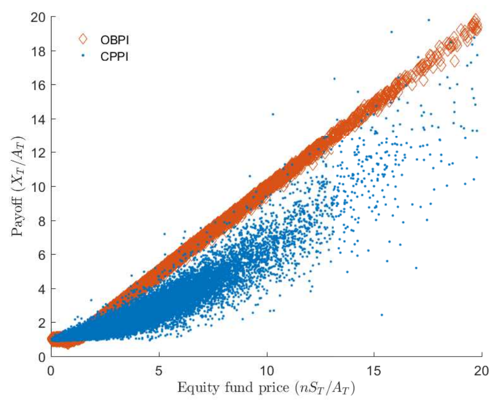
4.4.4.2 Initial fund balance

The base case sets the initial fund balance such that the value of the self-financing portfolio, Y_0, is \$33,000 above the target. I decrease the buffer above the target to \$10,000. The resulting initial fund balance (X_0) for each cohort is shown in Table 4.8. Note that for members joining the fund at age 25, the present value of future contributions is about \$32,760 above the target, A_0. Setting the buffer above the target at \$10,000 would make the initial fund balance negative. I therefore set zero initial fund balance for the youngest cohort.

Table 4.8. Initial fund balance (\$000) for each cohort.

25	30	35	40	45	50	55	60
0.00	9.95	56.26	116.05	189.25	275.84	375.86	489.41

Figure 4.13 compares the payoff to that of the base case for members joining the fund at age 50. When the initial fund balance decreases, the payoff generated by the CPPI strategy is less responsive to the better equity fund performance, whereas the payoff pattern generated by the OBPI strategy has little changes. This shows the robustness of the OBPI strategy to a lower amount of initial contribution to the fund.



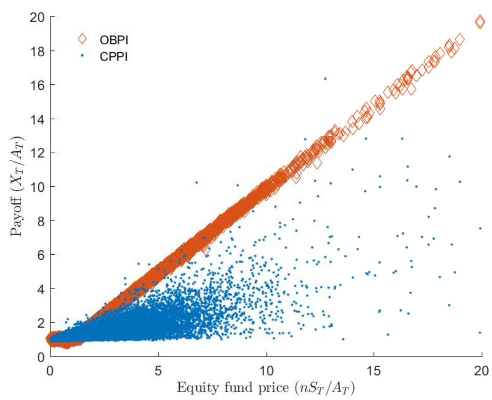


Figure 4.13. Comparison of payoffs between different levels of initial fund balances: (Left Panel) base case; (Right Panel) lower initial fund balance. The member joins the fund at age 40 and the CPPI multiplier (m) is 1.6.

Table 4.9 compares the shortfall probability and the average shortfall amount with those of the base case. The impact on the youngest cohort is minimal since their initial fund balance is reduced by $240 only compared to the base case. For the other cohorts, the shortfall probability increases, and the increment is larger for the CPPI strategy. In the base case, the OBPI strategy leads to a higher shortfall probability across all cohorts. When the initial fund balance is reduced, the OBPI strategy gives lower shortfall probabilities for older cohorts. Another difference compared to the base case is that the shortfall probability of the CPPI strategy decreases rather than increases with the multiplier if members join the fund after mid-30s. Figure 4.14 shows the shortfall probability over the course of accumulation period for members joining the fund at age 40. In the first few years after members' joining the fund, the CPPI strategy with the lowest multiplier has the lowest rate of reduction in the shortfall probability. As members approach retirement, however, this ranking is reversed. The shortfall probability with a lower multiplier declines at a faster pace. And the higher the initial fund balance, the earlier the change occurs. Given enough time, the CPPI strategy with a lower multiplier will lead to a lower shortfall probability.

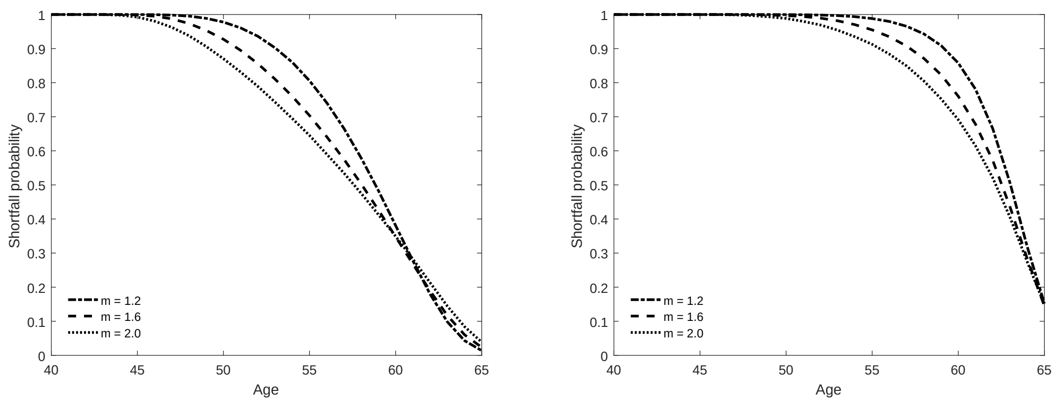


Figure 4.14. Comparison of shortfall probability between different levels of initial fund balances: (Left Panel) base case; (Right Panel) lower initial fund balance. The member joins the fund at age 40. Probability of shortfall is given by $\mathbb{P}(X_T < A_T)$.

Table 4.9. The shortfall probability and average shortfall amount by different initial fund balances (X_0). The initial fund balance for the base case is shown in Table 4.4, and the one for the 'Lower' case is shown in Table 4.8.

	Shortfall probability		Average shortfall ($000)		Average shortfall (A_T)	
X_0	Baseline	Lower	Baseline	Lower	Baseline	Lower
25 years old at time 0						
X_T^{OBPI}	0.081	0.082	−17.19	−17.19	−0.027	−0.027
$X_T^{\text{CPPI}}(m = 1.2)$	0.004	0.005	−13.85	−13.83	−0.021	−0.021
$X_T^{\text{CPPI}}(m = 1.6)$	0.010	0.011	−15.93	−15.91	−0.025	−0.025
$X_T^{\text{CPPI}}(m = 2.0)$	0.022	0.022	−17.68	−17.71	−0.027	−0.027
30 years old at time 0						
X_T^{OBPI}	0.098	0.184	−17.11	−16.58	−0.027	−0.026
$X_T^{\text{CPPI}}(m = 1.2)$	0.006	0.075	−13.27	−15.52	−0.020	−0.024
$X_T^{\text{CPPI}}(m = 1.6)$	0.013	0.080	−14.89	−17.24	−0.023	−0.027
$X_T^{\text{CPPI}}(m = 2.0)$	0.026	0.093	−17.17	−19.05	−0.027	−0.030
35 years old at time 0						
X_T^{OBPI}	0.115	0.206	−17.16	−16.56	−0.027	−0.026
$X_T^{\text{CPPI}}(m = 1.2)$	0.009	0.107	−12.41	−15.80	−0.019	−0.024
$X_T^{\text{CPPI}}(m = 1.6)$	0.018	0.107	−14.63	−17.18	−0.023	−0.027
$X_T^{\text{CPPI}}(m = 2.0)$	0.032	0.117	−16.58	−18.76	−0.026	−0.029
40 years old at time 0						
X_T^{OBPI}	0.135	0.234	−16.86	−16.54	−0.027	−0.026
$X_T^{\text{CPPI}}(m = 1.2)$	0.014	0.152	−13.25	−15.98	−0.020	−0.025
$X_T^{\text{CPPI}}(m = 1.6)$	0.024	0.145	−15.05	−17.08	−0.023	−0.026
$X_T^{\text{CPPI}}(m = 2.0)$	0.039	0.149	−16.74	−18.49	−0.026	−0.029
45 years old at time 0						
X_T^{OBPI}	0.156	0.258	−16.86	−16.62	−0.027	−0.026
$X_T^{\text{CPPI}}(m = 1.2)$	0.022	0.209	−12.80	−16.14	−0.020	−0.025
$X_T^{\text{CPPI}}(m = 1.6)$	0.033	0.193	−14.67	−17.08	−0.023	−0.026
$X_T^{\text{CPPI}}(m = 2.0)$	0.049	0.192	−16.28	−18.08	−0.025	−0.028
50 years old at time 0						
X_T^{OBPI}	0.181	0.288	−17.05	−16.51	−0.027	−0.026
$X_T^{\text{CPPI}}(m = 1.2)$	0.035	0.282	−12.65	−16.48	−0.019	−0.025
$X_T^{\text{CPPI}}(m = 1.6)$	0.046	0.258	−14.17	−17.12	−0.022	−0.027
$X_T^{\text{CPPI}}(m = 2.0)$	0.062	0.249	−15.70	−17.92	−0.024	−0.028
55 years old at time 0						
X_T^{OBPI}	0.201	0.312	−17.21	−16.46	−0.027	−0.026
$X_T^{\text{CPPI}}(m = 1.2)$	0.056	0.363	−11.92	−17.02	−0.018	−0.026
$X_T^{\text{CPPI}}(m = 1.6)$	0.066	0.338	−13.02	−17.34	−0.020	−0.027
$X_T^{\text{CPPI}}(m = 2.0)$	0.080	0.324	−14.25	−17.73	−0.022	−0.028
60 years old at time 0						
X_T^{OBPI}	0.217	0.336	−17.40	−16.54	−0.028	−0.026
$X_T^{\text{CPPI}}(m = 1.2)$	0.091	0.441	−11.84	−17.55	−0.018	−0.027
$X_T^{\text{CPPI}}(m = 1.6)$	0.095	0.424	−12.47	−17.65	−0.019	−0.027
$X_T^{\text{CPPI}}(m = 2.0)$	0.104	0.410	−13.13	−17.82	−0.020	−0.028

In terms of the changes in the absolute value of the average shortfall, the OBPI strategy shows a slight decrease, whereas the CPPI strategy shows some increase. Reducing the initial fund balance decreases the standard deviations of the fund balances at retirement for both strategies (Table 4.10) because the fund manager invests a lower proportion of wealth in the equity fund. However, the reduction in the initial fund balance has different impact on the downside deviation between the two strategies. The downside deviation is given by

$$\sqrt{\mathbb{E}\Big[(X_T - A_T)^2 | X_T < A_T\Big]}. \tag{4.39}$$

Table 4.10 shows that the downside deviation of the OBPI strategy decreases, whereas that of the CPPI strategy increases. As a result, when the shortfall occurs, the CPPI strategy leads to a worse average shortfall amount.

Table 4.10. The standard deviation and the downside deviation (given in Equation (4.39)) by different initial fund balances (X_0). The initial fund balance for the baseline is shown in Table 4.4, and the one for the 'Lower' case is shown in Table 4.8.

X_0	Standard deviation Baseline ($000)	Standard deviation Lower ($000)	Standard deviation Difference (%)	Downside deviation Baseline ($000)	Downside deviation Lower ($000)	Downside deviation Difference (%)
25 years old at time 0						
X_T^{OBPI}	3,340	3,328	−0.4	22	22	0.0
$X_T^{\text{CPPI}}(m=1.2)$	1,401	1,390	−0.8	18	18	0.0
$X_T^{\text{CPPI}}(m=1.6)$	5,345	5,305	−0.7	21	21	0.0
$X_T^{\text{CPPI}}(m=2.0)$	19,997	19,848	−0.7	23	23	0.1
30 years old at time 0						
X_T^{OBPI}	2,355	1,407	−40.2	23	21	−5.3
$X_T^{\text{CPPI}}(m=1.2)$	832	271	−67.5	18	20	13.4
$X_T^{\text{CPPI}}(m=1.6)$	2,785	923	−66.9	20	22	12.3
$X_T^{\text{CPPI}}(m=2.0)$	9,294	3,188	−65.7	22	24	8.9
35 years old at time 0						
X_T^{OBPI}	1,648	1,001	−39.2	22	21	−4.8
$X_T^{\text{CPPI}}(m=1.2)$	487	163	−66.5	17	20	21.1
$X_T^{\text{CPPI}}(m=1.6)$	1,404	476	−66.1	19	22	14.2
$X_T^{\text{CPPI}}(m=2.0)$	3,922	1,369	−65.1	22	24	10.7
40 years old at time 0						
X_T^{OBPI}	1,142	703	−38.5	22	21	−2.5
$X_T^{\text{CPPI}}(m=1.2)$	282	97	−65.5	18	21	17.7
$X_T^{\text{CPPI}}(m=1.6)$	712	243	−65.8	20	22	11.1
$X_T^{\text{CPPI}}(m=2.0)$	1,740	593	−65.9	22	24	7.9
45 years old at time 0						
X_T^{OBPI}	784	486	−38.1	22	21	−2.7
$X_T^{\text{CPPI}}(m=1.2)$	163	59	−63.9	17	21	19.9
$X_T^{\text{CPPI}}(m=1.6)$	361	128	−64.6	20	22	12.5
$X_T^{\text{CPPI}}(m=2.0)$	767	270	−64.9	21	23	8.0
50 years old at time 0						
X_T^{OBPI}	514	317	−38.3	22	22	−4.2
$X_T^{\text{CPPI}}(m=1.2)$	90	35	−61.5	17	21	25.0
$X_T^{\text{CPPI}}(m=1.6)$	173	64	−62.7	19	22	17.0
$X_T^{\text{CPPI}}(m=2.0)$	316	115	−63.6	21	23	11.3
55 years old at time 0						
X_T^{OBPI}	318	193	−39.1	23	21	−5.9
$X_T^{\text{CPPI}}(m=1.2)$	48	20	−58.3	16	22	36.4
$X_T^{\text{CPPI}}(m=1.6)$	81	33	−59.7	17	22	27.7
$X_T^{\text{CPPI}}(m=2.0)$	129	51	−60.7	19	22	20.3
60 years old at time 0						
X_T^{OBPI}	164	96	−41.6	23	22	−7.3
$X_T^{\text{CPPI}}(m=1.2)$	22	10	−54.8	16	22	41.6
$X_T^{\text{CPPI}}(m=1.6)$	33	14	−56.3	16	22	35.5
$X_T^{\text{CPPI}}(m=2.0)$	45	19	−57.6	17	22	29.9

4.4.5 Fund-level asset allocations

Since the target annuitisation strategy is implemented by the fund manager, the portfolio allocation at the fund level is of particular interest. I investigate the asset allocation strategies for a simplified pension fund. At time 0, a total of eight cohorts (ranging from age 25 to age 60) join the pension fund. The fund is closed to new members thereafter. Every five years from time 0, the oldest cohort remaining in the fund retires and withdraws the money from their individual accounts. At time 40, the pension fund closes after the youngest cohort withdraws the money. I assume each cohort contains the same number of individuals, and that each individual's contributions to the fund follows the assumption in Table 4.4.

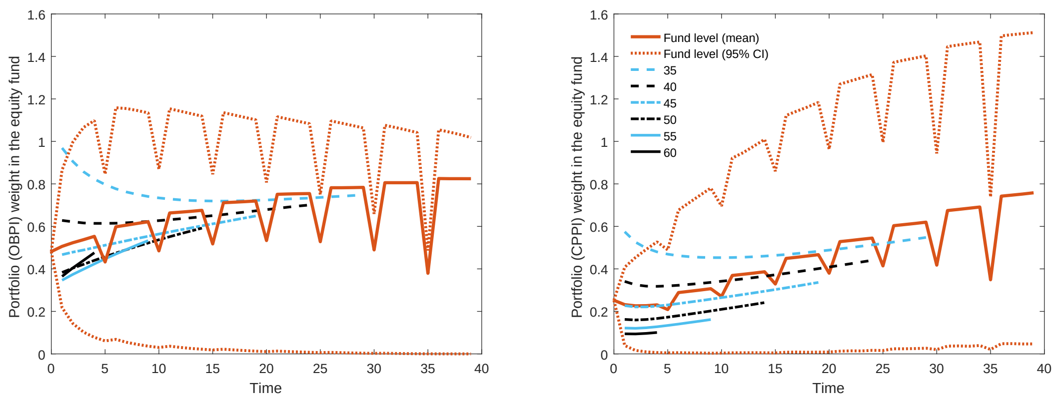


Figure 4.15. Average portfolio weights in the equity fund for the simplified pension fund: (Left Panel) the OBPI strategy, and (Right Panel) the CPPI strategy with $m = 1.6$.

Figure 4.15 shows the average portfolio weights in the equity fund for the simplified pension fund. The weights for several cohorts are also plotted on the graph for comparison. The portfolio weights at the pension fund level show dips every five years. The drops are due to the withdrawals made by retired members. The curves can be smoothed if the fund is open to new members. During the first 10–20 years, the whole pension fund behaves like the middle-aged cohort in terms of the average portfolio weights. As the older cohorts leave the fund, the fund's average portfolio weights move closer to those of the younger cohorts. As a result, the average portfolio weights in the equity fund on both graphs show upward trends. Applying the portfolio insurance strategies to a simplified pension fund could lead to feature studies on how the investment strategies behave in a more realistic setting with, for example, new members joining the fund and members of heterogeneous labour income profiles.

4.5 Conclusions

The design of current DC pension funds usually has insufficient integration between the accumulation and retirement phases. This has been an issue since the accumulated wealth may not be able to provide sustainable income flows in retirement. Target annuitisation funds aim to provide fund members with an amount of retirement benefits that can finance a desired post-retirement consumption path within a confidence interval. They are a possible solution to connecting the accumulation and retirement phases in the DC pension plans, and therefore have attracted increasing attention.

Portfolio insurance strategies provide the investors with the potential to limit downside risk and to participate on the upside. They are suitable investment strategies to achieve the investment objectives of the target annuitisation fund. This chapter investigates the performance of OBPI and CPPI strategies for the target annuitisation fund.

For members joining the fund before mid-30s, the portfolio weights in the equity fund tend to decrease as they get older, but the weights are highly volatile due to the equity market volatility. For members joining the fund at older ages, the average portfolio weights in the equity fund increase as they grow older. The difference is mainly due to the fact that the older cohorts have lower amounts of future contributions, which are in the form of safe assets. This requires relatively high initial fund values to build the buffer above the target annuitisation level upon joining the fund.

In terms of the downside risk protection, the average shortfall amount as a proportion of the target annuitisation level at retirement is minimal for both strategies, and it is robust to a shorter accumulation period, a higher equity market volatility, and a lower initial fund balance. The base case analysis shows the CPPI strategy performs significantly better in reducing the likelihood of shortfall, but the performance is sensitive to the equity market volatility and the initial fund balance. By contrast, the OBPI strategy has about 10% (20%) chance of not meeting the target for members joining the fund before (after) mid-30s in the base case. However, the OBPI strategy typically gives a higher portfolio value at retirement when the equity market is performing reasonably well. And its ability in providing downside risk protection is more robust to the changing equity market volatility and initial fund balance.

The fund manager would be particularly interested in the asset allocation strategies at the fund level, so I also study the fund-level asset allocations for a simplified pension fund that is closed to new members after inception. In terms of the portfolio weights, the pension fund behaves like a member with the average age of the remaining members in the fund. The average portfolio weights in the equity fund show upward trends. There are sudden drops whenever a cohort of members retire and make withdrawals. The weights can be smoothed if the fund is open to new members.

Chapter 5

Housing and retirement financing: When to buy a residential home

5.1 Introduction

Owner-occupied housing serves as a durable consumption good and an investment asset. It plays an important role in retirement planning. A natural question that arises is when to buy a residential home. Compared to renting, owning a house hedges rent risk but introduces the asset price risk at the time when homeowners want to use home equity to fund their retirement. Sinai and Souleles (2005) employ a stylised tenure choice model to assess the extent to which owning trades off the rent and house price risk. To determine the optimal time to become a homeowner, it is not enough to only evaluate the trade-off between these two types of risk because the property purchase decision also depends on the labour earnings and will have a great impact on household consumption and portfolio allocation. To assess how the timing of becoming a homeowner affects savings for retirement, it is also necessary to track the consumption and investment behaviour throughout the working life to derive the wealth level at retirement.

Capturing these features requires a multi-period model of consumption and investment decisions with stochastic labour income, stock returns, and housing cost. The vector autoregressive (VAR) process can model multiple sources of risk and allow for their possible correlations. It is widely used in the context of optimal consumption and asset allocation (see e.g. Barberis, 2000; Campbell and Viceira, 1999; Campbell et al., 2003). I therefore use a VAR process to model the dynamics of the state variables that determine the labour income, investment returns, and housing cost.

To answer the question of when to become a homeowner, I examine how purchasing home property at different ages would affect an individual's pre-retirement consumption level, savings for retirement, and ultimately the lifetime utility level. I follow Ortalo-Magné and Rady (2002) to assume that utility derived from housing-service consumption is the same regardless of the tenure choice. This assumption helps to isolate the trade-off between rent risk and house price risk (Ortalo-Magné and Rady, 2002). The optimal time to purchase a residential home then depends entirely on the expected utility from non-housing consumption and wealth at retirement. The wealth consists of liquid assets (cash and stocks), employment-based pension (e.g. superannuation in Australia), and home equity. The intertemporal consumption and asset allocation decisions are parameterised using the Household, Income and Labour Dynamics in Australia (HILDA) survey data to reflect an average Australian's decisions in his/her age group and housing tenure (renting or owning). In Australia, main residences are generally exempted from capital gain tax, and mortgage interest expenses for the primary residence cannot be claimed as income tax deductions. The present chapter therefore abstracts from tax treatment in the main analysis. I will perform sensitivity analysis in Section 5.6.4 to examine the impact of different tax rates.

The simulation results show that purchasing the property earlier during the working life often leads to a higher level of wealth at retirement due to a higher home equity value and more liquid assets. The higher value of home equity is mainly due to the lower mortgage balance at retirement. In terms of liquid assets, homeowners tend to allocate more to stocks. The investor is therefore able to accumulate more liquid assets. The downside of purchasing the property relates to the dramatic consumption drop that lasts for a few years. A significant proportion of liquid assets are locked in the housing wealth (which is illiquid) after a large amount of down payment is made. The consumption cut results in utility loss, and the earlier the property is purchased, the higher the discounted utility loss. On the other end of the spectrum is to keep renting during the working life. It is unattractive both in terms of retirement wealth and utility level. Individuals who rent the property throughout the working life have to incur high rental costs. This not only constrains the spending on non-housing consumption, which results in a low utility level, but also slows down the wealth accumulation.

The remainder of the chapter is organised as follows. Section 5.2 introduces the model setup. Section 5.3 presents the estimation results of the VAR process. Section 5.4 explains how the HILDA data is used to parameterise the model. Section 5.5 discusses the simulation results, and some robustness checks are performed in Section 5.6. Section 5.7 compares the model predictions with the empirical data. Section 5.8 concludes.

5.2 Model framework

To answer the research question of when to become a homeowner, I investigate how purchasing residential property at different ages would affect the consumption level, savings for retirement, and ultimately the lifetime utility level. Savings for retirement include both liquid and illiquid wealth, whose dynamics are discussed in Section 5.2.1. Section 5.2.2 presents the budget constraints for renters and homeowners. The lifetime utility is defined in Section 5.2.3.

5.2.1 Asset and labour income dynamics

There are two liquid assets available in the market, a riskless asset called cash with gross real return R_f, and a risky asset called stocks (with dividend reinvested) with gross real return $R_{S,t}$. The log return on the risky asset is denoted as $r_{S,t} \equiv \ln(R_{S,t})$. The investor allocates after-consumption liquid wealth between these liquid assets, and receives investment returns. Apart from financial income, the investor receives stochastic labour income at the beginning of each period.

There are two illiquid assets, pension and housing. Given that employment-based pension is compulsory in Australia, the investor is assumed to open a pension account in the first period when he starts working. Throughout his working life, his employer contributes a certain percentage (π) of his labour income to the pension fund at the beginning of each period. The assets in the pension fund are illiquid during his working life, and will become part of retirement savings when he retires. The asset allocation strategy of the pension fund is to invest 70% of the assets in the risky asset, and 30% in the riskless asset. Such a mix is very typical among the default products offered by different pension funds in Australia (Chant et al., 2014). In the model setup, there is only one risky asset and one riskless asset, whereas in practice, there are more types of risky (e.g. private equity, international shares) and safe (e.g. bond) assets. In terms of housing, I assume a homogeneous housing market where the residential properties are of the same economic value. Individuals can either buy or rent these properties. The property capital growth is denoted as $R_{H,t}$.

Following Campbell et al. (2003), I postulate that the dynamics of asset returns and labour income growth follow a first-order vector autoregressive process, or VAR(1). The VAR model is given by

$$\mathbf{x}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{x}_{t-1} + \mathbf{v}_t, \tag{5.1}$$

where $\mathbf{x}_t$ is a vector of state variables, $\mathbf{\Phi}_0$ is a vector of intercepts, $\mathbf{\Phi}_1$ is a matrix of slope coefficients, and $\mathbf{v}_t$ is a vector of shocks to the state variables following an independent and identically distributed multivariate normal distribution with mean $\mathbf{0}$ and variance $\mathbf{\Sigma}_v$.

There are six state variables in the model: log stock returns $r_{S,t}$, property capital growth rate $r_{H,t} \equiv \ln(R_{H,t})$, labour income growth rate $r_{L,t}$, log dividend yield $y_{S,t}$, log rental yield $y_{H,t}$, gross domestic product (GDP) growth rate $r_{G,t}$, i.e.

$$\mathbf{x}_t = \begin{pmatrix} r_{S,t} & r_{H,t} & r_{L,t} & y_{S,t} & y_{H,t} & r_{G,t} \end{pmatrix}'. \tag{5.2}$$

The log dividend yield is included in the model since the dividend-price ratio has been identified as a share return predictor by the empirical research (Campbell and Shiller, 1988; Fama and French, 1988). Similarly, the rent-price ratio has been found to predict house price growth (Gallin, 2008; Campbell et al., 2009), so the log rental yield is included in the state vector. Furthermore, macroeconomic variables are likely to affect the dynamics of house price (see e.g. Abraham and Hendershott, 1996; Muellbauer and Murphy, 1997). Alai et al. (2014) and Shao et al. (2015) include GDP in their VAR models that have house price growth and rental yield in the state vector. Following these studies, the GDP growth rate is included as one of the state variables.

5.2.2 Budget constraints

In the first period, the investor is a renter endowed with liquid wealth B. In each of the following periods $(t > 1)$, the timing of events is as follows. The investor starts the period with total wealth W_t. It consists of liquid wealth (LW_t) brought through from the previous period, pension account balance W_t^R, and home equity (if any). The amount of liquid wealth is given by

$$LW_t = \mathcal{S}_{t-1}R_{S,t} + \mathcal{B}_{t-1}R_f,$$

where $\mathcal{S}_{t-1}$ and $\mathcal{B}_{t-1}$ are the dollar amounts invested in stocks and cash, respectively, in the previous period $t-1$. The beginning-of-period home equity is housing asset net of mortgage balance $\mathcal{M}_t$ before any repayment is made. Then labour income K_t is realised, of which $\pi \times 100\%$ is contributed to the pension fund. After receiving the labour income, the investor must make the rental payment or mortgage payment (down repayment) if he is (becomes) a homeowner. He also needs to decide how much to consume, and how to allocate the remaining liquid wealth between stocks and cash. The next period liquid wealth is to be discussed in the following sections as it depends on the housing tenure. The next period pension account balance before new contributions is given by

$$W_{t+1}^R = (0.3R_f + 0.7R_{S,t+1})(W_t^R + \pi K_t). \tag{5.3}$$

The next period mortgage balance accumulates at the risk-free interest rate

$$\mathcal{M}_{t+1} = (\mathcal{M}_t - RP_t)R_f, \tag{5.4}$$

where RP_t is the amount of repayment made in period t. Additional restrictions on mortgage balances will be imposed to prevent default.

5.2.2.1 Renter

When the investor rents in the previous period, his beginning-of-period wealth consists of liquid assets and pension, i.e.

$$W_t = LW_t + W_t^R. \tag{5.5}$$

A renter can buy a house or continue to rent. The notation $\mathbb{I}_t^{\text{own}}$ is used to indicate whether the investor is a homeowner.

$$\mathbb{I}_t^{\text{own}} = \begin{cases} 0 & \text{if the investor is a renter in period } t, \\ 1 & \text{if the investor is a homeowner in period } t. \end{cases} \tag{5.6}$$

If the renter chooses to rent again, he needs to pay for the rent ($D_{H,t}$) and non-durable consumption goods (C_t), and to decide how to allocate the remaining liquid wealth between cash and stocks. The next period liquid wealth is given by

$$LW_{t+1} = \Big(LW_t + (1-\pi)K_t - C_t - D_{H,t}\Big)\Big(\omega_{S,t}R_{S,t+1} + (1-\omega_{S,t})R_f\Big), \tag{5.7}$$

where $\omega_{S,t}$ is the proportion of after-consumption liquid wealth invested in stocks. The wealth at the start of next period is given by

$$W_{t+1} = LW_{t+1} + W_{t+1}^R. \tag{5.8}$$

The renter may not be able to pay the full rent when there is a large positive shock to the property capital growth and/or the rental yield process. In these circumstances, the government will provide rental assistance to pay for the rent that cannot be covered by the investor. The investor's after-consumption (including rent) wealth will be set at zero.

The individual can choose to purchase a house at ages 30, 40, 50, 60, or over 65.[1] Purchasing the property after 65 means that the individual keeps renting throughout his working life. The house price (P_t^H) is determined by the rental cost ($D_{H,t}$) and rental yield ($Y_{H,t}$) as follows

$$P_t^H = D_{H,t}/Y_{H,t}. \tag{5.9}$$

If the renter chooses to become a homeowner, he needs to make a down payment, DP_t, and finance the rest through a mortgage. The mortgage is of 30-year term, and the loan repayment starts in the next period. The mortgage provider typically requires a minimum deposit, as a proportion ($\underline{\lambda}$) of the property price, so the mortgage balance in the current period must satisfy

$$\mathcal{M}_t \leq (1 - \underline{\lambda}) P_t^H. \tag{5.10}$$

Due to the assumption that the mortgage rate is the same as the risk-free rate, I want to rule out default by adding an additional borrowing constraint (Campbell and Cocco, 2007)

$$\mathcal{M}_t R_f \leq \underline{P_{t+1}^H} + (1 - \pi)\underline{K_{t+1}}, \tag{5.11}$$

where $\underline{P_{t+1}^H}$ and $\underline{K_{t+1}}$ are the lower bounds of house prices and labour income, respectively, in the next period $t+1$. When the individual chooses to become a homeowner, he may not be able to afford the down payment if his after-non-housing-consumption liquid wealth is greater than or equal to the minimum down payment, i.e. if

$$LW_t + (1 - \pi)K_t - C_t \geq P_t^H - \min\left((1 - \underline{\lambda})P_t^H, \frac{1}{R_f}\left(\underline{P_{t+1}^H} + (1 - \pi)\underline{K_{t+1}}\right)\right). \tag{5.12}$$

[1] I assume individuals over the age of 50 can still get mortgage approval if they can meet the minimum down payment requirement. Banks are generally open to approving mortgages for senior borrowers provided that they can give a valid exit strategy that outlines loan repayment plans after retirement. In the present model, individuals typically have sufficient assets in the pension upon retirement to repay the loan (results to be shown in Table 5.10).

In these circumstances, the individual will choose to become a homeowner in 10 years. For example, if a 30-year-old individual cannot afford the down payment, he will keep renting for the next 10 years and choose to become a homeowner again at age 40. If an individual cannot afford the down payment at all possible ages (i.e. 30, 40, 50, and 60), he will keep renting throughout his working life. This means renting throughout working life can either be an active or passive decision. The analysis will treat these two cases separately.

When the investor becomes a homeowner in the current period, he also needs to spend a fraction (φ) of his housing asset value on repairs and maintenance. The next period liquid wealth is given by

$$LW_{t+1} = \Big(LW_t + (1-\pi)K_t - C_t - DP_t - \varphi H_t\Big)\Big(\omega_{S,t}R_{S,t+1} + (1-\omega_{S,t})R_f\Big). \tag{5.13}$$

The total wealth at the beginning of next period is given by

$$W_{t+1} = LW_{t+1} + W^R_{t+1} + (H_t R_{H,t+1} - \mathcal{M}_t R_f). \tag{5.14}$$

5.2.2.2 Homeowner

When the investor is a homeowner in the previous period, I assume he will remain living in the same property. His wealth includes liquid assets, pension and housing asset net of the home loan, i.e.

$$W_t = LW_t + W^R_t + H_t - \mathcal{M}_t. \tag{5.15}$$

The next period liquid wealth is given by

$$LW_{t+1} = \Big(LW_t + (1-\pi)K_t - C_t - RP_t - \varphi H_t\Big)\Big(\omega_{S,t}R_{S,t+1} + (1-\omega_{S,t})R_f\Big). \tag{5.16}$$

The loan repayment normally remains constant once the mortgage balance in the loan origination date is determined. When the property prices fall, however, the investor may need to make additional repayment to keep the loan-to-value ratio below the maximum loan-to-value ratio, $\overline{LVR}$.

5.2.3 Preference

The expected present value of lifetime utility in the first period is given by

$$U_1 = \mathbb{E}_1 \left[\sum_{t=1}^{T} \beta^{t-1} u(C_t) + \beta^T v(W_{T+1}) \right], \tag{5.17}$$

where T is the time of retirement, β the subjective discount factor, $u(\cdot)$ the utility from consuming non-durable goods, and $v(\cdot)$ the utility from accumulating wealth for retirement. In period t, the investor spends C_t on non-durable goods. For a two-period model, it is possible to work with a general utility function that only requires strictly increasing and strict concavity (see e.g. Ortalo-Magné and Rady, 2002; Davidoff, 2006). For a multi-period model specified in (5.17), however, it is necessary to specify a parametric form for the utility function to compare the lifetime utility across different ages of buying a residential home. The power utility function is commonly used to represent individual preference in the literature of portfolio selection with housing (see e.g. Cocco, 2004; Hu, 2005; Yao and Zhang, 2005). I therefore specify both $u(\cdot)$ and $v(\cdot)$ in the form of the power utility function. The utility function $u(\cdot)$ is given by

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}, \tag{5.18}$$

where γ is the coefficient of relative risk aversion. The utility gained from saving for retirement is defined as

$$v(W_{T+1}) = b\frac{W_{T+1}^{1-\gamma}}{1-\gamma}, \tag{5.19}$$

where b measures the strength of retirement saving motive. Assume the investor enters the workforce at age 25 in the first period, and retires at age 65, so $T = 40$.

5.3 Vector autoregressive estimation

5.3.1 Data

The quarterly Australian data shown in Table 5.1 is used to estimate the VAR model. The data available begins in the first quarter of 1992 and ends in the second quarter of 2011. All variables, except for log dividend yield and log rental yield, are deflated using the Australia consumer price index (CPI) and expressed in continuously compounded quarterly rates.

Table 5.1. State variables in the VAR model and the corresponding data source.

Variable	Source Series	Source
$r_{S,t}$	S&P/ASX 200 accumulation index	Australian Bureau of Statistics
$r_{H,t}$	House price index	Residex
$r_{L,t}$	Average ordinary weekly earnings†	Australian Bureau of Statistics
$y_{S,t}$	S&P/ASX 200 index S&P/ASX 200 accumulation index	Australian Bureau of Statistics
$y_{H,t}$	Median rent yield	Residex
$r_{G,t}$	GDP‡	Australian Bureau of Statistics

Note: $r_{S,t}$ = log real return on stocks, $r_{H,t}$ = real property capital growth rate, $r_{L,t}$ = real labour income growth rate, $y_{S,t}$ = log dividend yield, $y_{H,t}$ = log rental yield, $r_{G,t}$ = real GDP growth rate.
† Full time adult males.
‡ Chain volume measure.

The dividend yield is constructed from the S&P/ASX 200 index and S&P/ASX 200 accumulation index following the standard method in the literature (see e.g. Campbell et al., 2003). First construct the dividend payout series using the stock return including dividends and excluding dividends. Then take the dividend series to be the sum of dividend payments over the past 12 months. The dividend yield is dividend series divided by the price index.

Table 5.2 shows the annualised sample statistics of the state variables in the VAR model.

Table 5.2. Sample statistics of the state variables in the VAR model.

Variable	Mean (%)	Std (%)
$r_{S,t}$	6.93	13.85
$r_{H,t}$	4.18	4.97
$r_{L,t}$	1.60	1.26
$Y_{S,t}$	3.91	0.56
$Y_{H,t}$	5.17	1.42
$r_{G,t}$	3.41	0.69

Note: $r_{S,t}$ = log real return on stocks, $r_{H,t}$ = real property capital growth rate, $r_{L,t}$ = real labour income growth rate, $Y_{S,t} \equiv \exp(y_{S,t})$ = dividend yield, $Y_{H,t} \equiv \exp(y_{H,t})$ = rental yield, $r_{G,t}$ = real GDP growth rate.

5.3.2 Estimation results

The maximum likelihood estimation method is employed to estimate parameters in the VAR(1) model. The constant and coefficient estimates (with t-statistics in parentheses) are shown in Table 5.3. Table 5.4 shows the covariance and correlation structure of the innovation in the VAR model. Unexpected log real returns on stocks are highly negatively correlated with shocks to log dividend yield. Campbell et al. (2003), using the U.S. data, also find a high negative correlation between log excess stock returns and shocks to log dividend yield. Unexpected real property capital growth rate is negatively correlated with log rental yield. There is also a positive correlation (about 53%) between real labour income growth rate and real property capital growth rate.

To assess the goodness-of-fit of the estimated parameters, 10,000 sample paths are simulated for each state variable over the sample period 1992–2011, and their probability density functions are compared to those of the historical values. The initial value of the simulated state variables is set to be the earliest observation in the sample, making it consistent with the initial condition used to estimate the VAR model. As shown in Figure 5.1, the density functions of the simulated state variables are largely comparable to their empirical counterparts. The difference comes from the fact that the VAR(1) model imposes a multivariate normal distribution on the vector of the state variables, whereas in practice, the density functions of these state variables do not necessarily have a perfect bell shape.

Table 5.3. VAR estimation results: intercepts and slope coefficients (with t-statistics in parentheses).

	$r_{S,t}$	$r_{H,t}$	$r_{L,t}$	$y_{S,t}$	$y_{H,t}$	$r_{G,t}$
Constant ($\hat{\mathbf{\Phi}}_0$)	64.087	−0.562	4.573	−1.040	−0.227	0.746
	(2.308)	(−0.057)	(1.833)	(−3.482)	(−1.190)	(0.727)
VAR coefficients ($\hat{\mathbf{\Phi}}_1$)						
$r_{S,t+1}$	0.262	0.011	−0.492	19.440	1.025	4.191
	(2.144)	(0.028)	(−0.327)	(2.523)	(0.366)	(1.710)
$r_{H,t+1}$	0.010	0.095	0.645	−1.518	1.477	0.829
	(0.237)	(0.695)	(1.201)	(−0.552)	(1.479)	(0.948)
$r_{L,t+1}$	0.005	−0.041	0.203	1.335	0.032	0.244
	(0.455)	(−1.187)	(1.498)	(1.928)	(0.128)	(1.110)
$y_{S,t+1}$	−0.003	−0.002	0.001	0.699	−0.033	−0.033
	(−2.130)	(−0.567)	(0.078)	(8.434)	(−1.097)	(−1.259)
$y_{H,t+1}$	−0.001	−0.006	0.005	−0.044	0.976	0.013
	(−1.078)	(−2.426)	(0.518)	(−0.827)	(50.655)	(0.787)
$r_{G,t+1}$	0.005	0.001	0.031	0.009	0.146	0.646
	(1.034)	(0.085)	(0.550)	(0.032)	(1.406)	(7.126)

Note: $r_{S,t}$ = log real return on stocks, $r_{H,t}$ = real property capital growth rate, $r_{L,t}$ = real labour income growth rate, $y_{S,t}$ = log dividend yield, $y_{H,t}$ = log rental yield, $r_{G,t}$ = real GDP growth rate.

Table 5.4. VAR estimation results: covariance and correlation matrices of residuals.

	r_S	r_H	r_L	y_S	y_H	r_G
Covariance ($\hat{\mathbf{\Sigma}}_v$)						
r_S	41.675	0.954	0.013	−0.408	−0.008	0.280
r_H		5.302	0.703	−0.027	−0.059	0.109
r_L			0.336	0.002	−0.005	0.023
y_S				0.005	0.000	−0.001
y_H					0.002	−0.002
r_G						0.057
Correlation						
r_S	1.000	0.064	0.004	−0.909	−0.028	0.182
r_H		1.000	0.527	−0.168	−0.580	0.198
r_L			1.000	0.045	−0.212	0.164
y_S				1.000	0.056	−0.088
y_H					1.000	−0.234
r_G						1.000

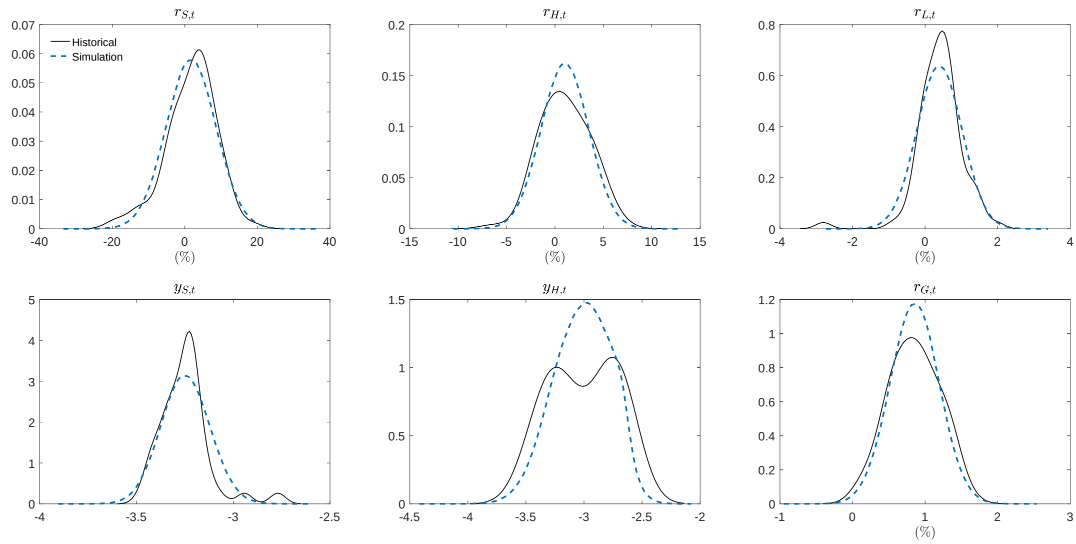


Figure 5.1. Probability density functions of historical and simulated state variables in the VAR model.

5.4 Parameterisation

The parameters related to consumption, asset allocation, and housing are calibrated to the HILDA Survey, which is a household-based panel study that began in 2001 with around 7,700 households and nearly 20,000 individuals. It conducts interviews annually, collecting information about household social and economic conditions. The survey asks questions regarding household finance every fourth wave since wave 2. In terms of the household spending, the survey only collects information about household expenditure on groceries, meals eaten out, and childcare costs before wave 5. Since 2005, the spending categories have been expanded to include more items, such as public transport and taxis, clothing and footwear. Since the interest is in household consumption and portfolio choice decisions, I form an unbalanced panel by selecting individuals who responded to the survey at least once over the period 2006 to 2014. The sample contains 24,091 individuals.

5.4.1 Consumption

I assume the investor spends a certain percentage of disposable income on non-durable goods. An alternative is to assume the investor spends a certain percentage of wealth on non-durable goods, but the consumption-net-worth ratios calculated from the survey data are extremely volatile. The high volatilities are possibly due to the fact that the wealth data is collected three times only over the nine-year (2006 – 2014) period. Therefore the consumption-disposable-income ratio is used.

The household financial year disposable total income is directly available from the survey, and the household expenditure on non-durable goods is approximated as the sum of 17 self-reported spending categories of non-durable goods[2]. Figure 5.2 shows the mean and median consumption-disposable income ratios. It is noticeable that the median values are much more stable, which makes them more suitable to be used in the simulations.

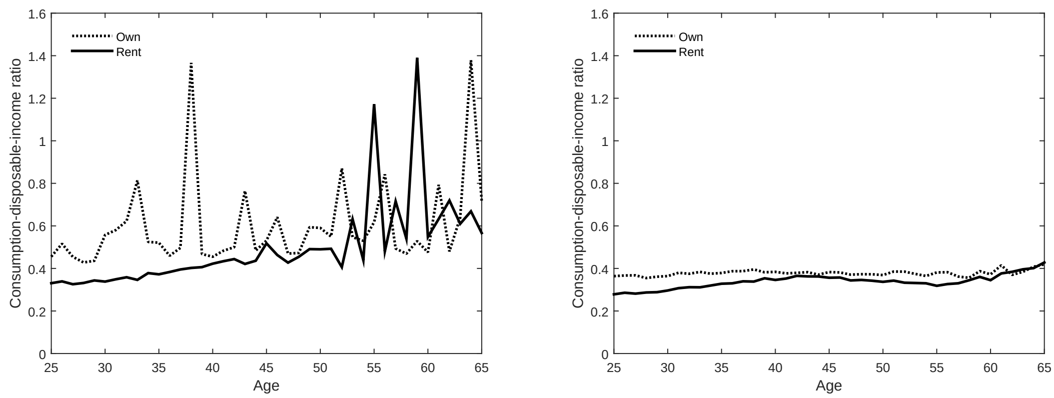


Figure 5.2. Annual expenditure on non-durable goods as a proportion of disposable income: (Left) mean; (Right) median.

Figure 5.3 shows the crude and fitted median annual expenditure on non-durable goods as a proportion of disposable income. The fitted line is obtained by regressing ratios on a polynomial in age. These fitted proportions will be used as the input into the simulation.

[2]This includes spending on *1*) groceries, *2*) alcohol, *3*) cigarettes and tobacco, *4*) public transport and taxis, *5*) meals eaten out, *6*) motor vehicle fuel, *7*) men's clothing and footwear, *8*) women's clothing and footwear, *9*) children's clothing and footwear, *10*) telephone rent and calls, internet charges, *11*) private health insurance, *12*) other insurances, *13*) fees paid to health practitioner, *14*) medicines, prescriptions and pharmaceuticals, *15*) electricity, gas bills and other heating fuel, *16*) motor vehicle repairs and maintenance, and *17*) education fees.

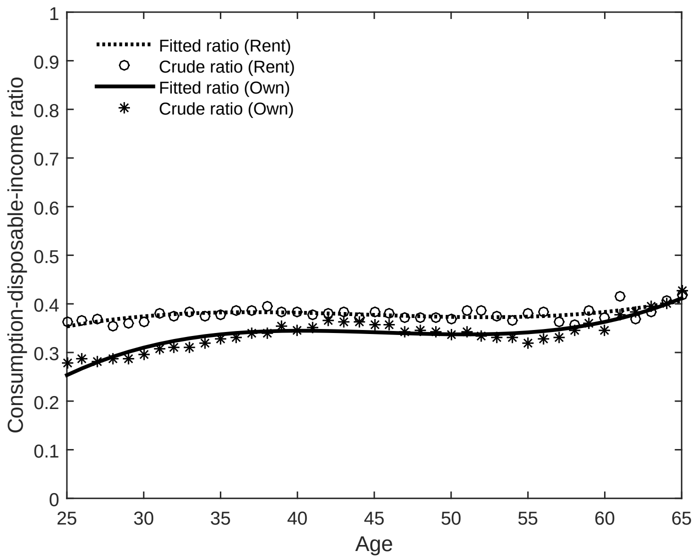


Figure 5.3. Crude and fitted median annual expenditure on non-durable goods as a proportion of disposable income. The fitted line is obtained by regressing ratios on a polynomial in age.

5.4.2 Asset allocation

The investor allocates his after-consumption wealth between cash and stocks. In the HILDA survey, the investment in the risk-free asset consists of cash investments and bank account, and the investments in equity are directly available.

Figure 5.4 shows the equity participation rates, i.e. the proportion of renters or homeowners holding equity investments. Less than 20% of renters participated in the equity market throughout all working ages, while the participation rates increased from 30% at age 25 to above 50% at age 65 for homeowners. Since the majority of individuals did not hold equity investments, they will be treated separately when I analyse the asset allocation decisions. Figure 5.5 shows that conditional on housing tenure, net worth performs better than disposable income in explaining whether an individual holds equity investments. Therefore the wealth level is used to determine whether the investor has positive equity investments in the simulations.

Figure 5.6 shows the average and median proportions of equity investments in liquid assets conditional on participation. The mean and median values have similar ranges,

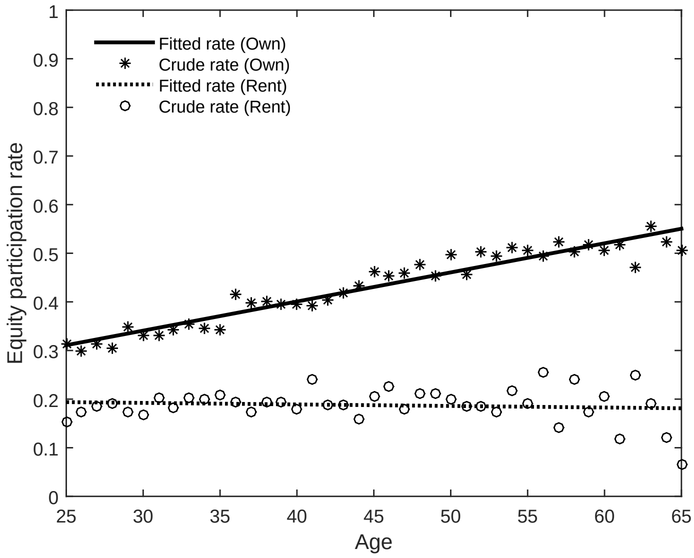


Figure 5.4. Crude and fitted equity participation rates. The fitted line is obtained by regressing rates on a linear function of age.

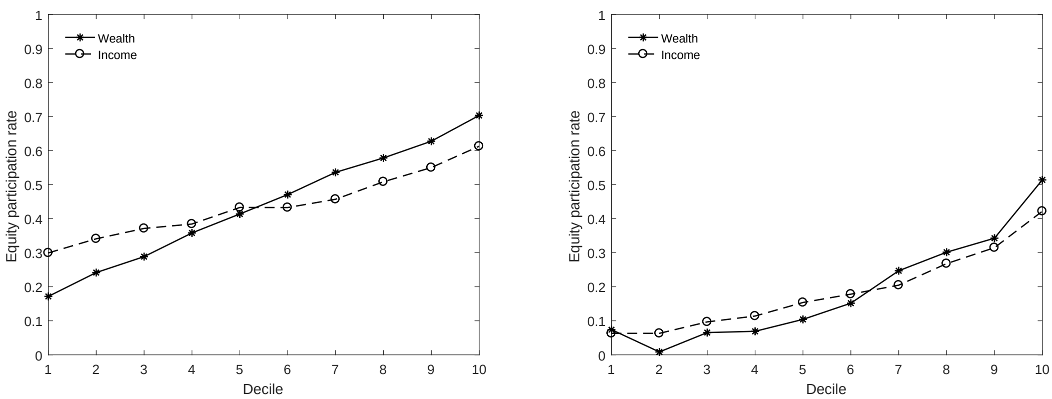


Figure 5.5. Equity participation rate by wealth and income deciles: (Left) owners; (Right) renters.

but the former is more stable. So I will use the fitted (obtained by regressing ratios on a polynomial in age) average equity proportions in liquid assets as the input into the simulation. The crude and fitted average equity proportions in liquid assets are shown in Figure 5.7.

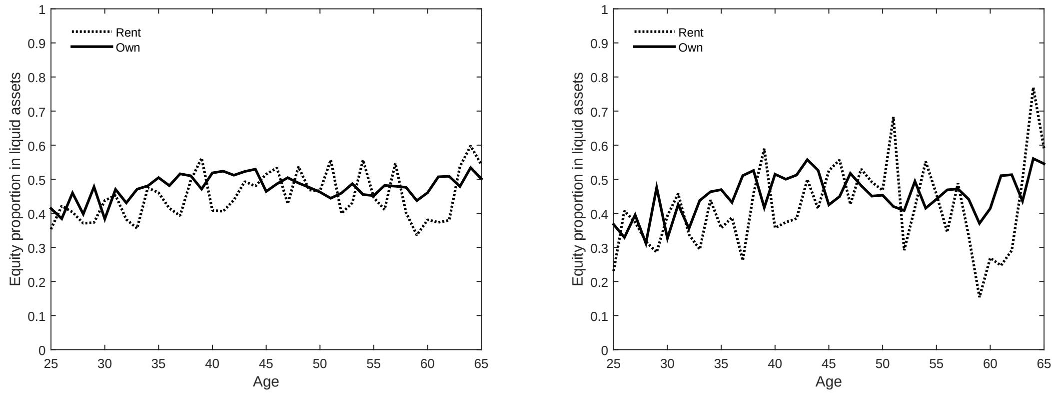


Figure 5.6. Proportion of equity investment in liquid assets: (Left) mean; (Right) median.

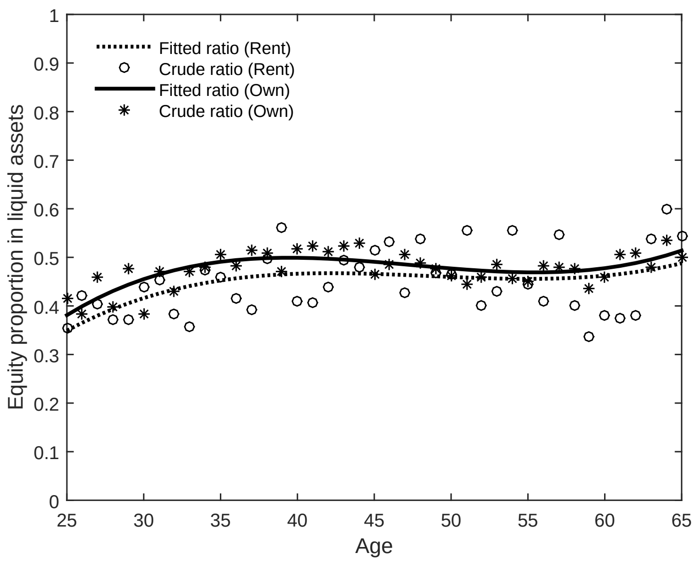


Figure 5.7. Crude and fitted average equity proportions in liquid assets.

5.4.3 Housing

5.4.3.1 Housing tenure transitions

The longitudinal feature of the data makes it possible to track transitions of housing tenure. Table 5.5 summarises the transition counts and exposure years by age group. As the age increases, an increasingly small percentage of homeowners became renters, and the proportion of renters who become homeowners gradually declined. These patterns are confirmed with the log crude transition rates shown in Figure 5.8. The transition rates from renter to homeowner were much higher than those from homeowner to renter, especially after age 35. This backs the assumption in the model setup that individuals cannot become renter again once the home property is purchased.

Table 5.5. Tabulation of raw transition counts from renting to owning and from owning to renting, and approximate exposure years in two housing tenure status (renters and owner-occupiers).

	No. of transitions		Exposure years†	
Age Band	Rent → Own	Own → Rent	Rent	Own
25 – 29	524	343	3,823	3,599
30 – 34	479	270	2,793	4,442
35 – 39	346	267	2,556	5,613
40 – 44	237	190	2,125	6,412
45 – 49	179	135	1,837	7,041
50 – 54	160	144	1,610	6,395
55 – 59	114	88	1,068	5,851
60 – 64	89	66	878	5,241
Total	2128	1,503	16,689	44,592

† The number of exposure years is approximated using the census method, with the rate interval being the calendar year.

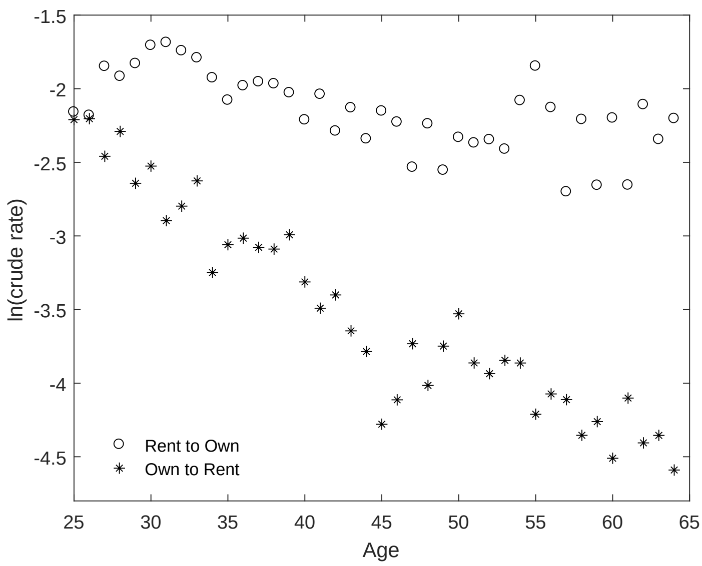


Figure 5.8. The natural logarithm of crude transition rates from renting to owning, and from owning to renting.

5.4.3.2 Housing cost

The investor starts as a renter in the first period, in which the rental payment is set at \$14,328 per annum. This figure is based on the result (Table 5.6) that the average rental payments for those between 25 and 65 is \$1,194 per month (or \$14,328 per annum).

Table 5.6. Average rental payments per month (in 2010 dollars) for renters between ages 25 and 65.

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014
No. Obs	2,273	2,279	2,213	2,411	2,449	3,457	3,444	3,512	3,532
Mean	1,010	1,057	1,099	1,164	1,180	1,260	1,266	1,283	1,264
					Weighted average			1,194	

The maximum loan-to-value ratio on the loan origination date is assumed to be 80%, so the down payment is at least 20% of the property price. The maximum loan-to-value ratio after the loan origination date is set to be 125%. If the investor bought the property just before the collapse of a housing bubble (e.g. the subprime mortgage crisis in the U.S.), it is possible that the home equity become negative. Setting the maximum loan-to-value ratio above 100% means this can happen to some extent.

Homeowners need to pay for the property maintenance and depreciation cost. It is usually expressed as a certain percentage of the market value of the property. Since the market value is not available in the HILDA survey, the self-report home value is used as a proxy. The maintenance and depreciation cost is approximated by the expenditure on home repairs/renovations/maintenance. Table 5.7 shows the average cost as a proportion of home value. The weighted average uses the number of observations in each year as the weights. The annual maintenance and depreciation cost is set at $\varphi = 0.87\%$ of the property value.

Table 5.7. Average maintenance and depreciation cost as a proportion of home asset value for homeowners between ages 25 and 65.

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014
No. Obs	6,263	6,161	6,165	6,260	6,312	8,048	7,999	7,876	7,777
Mean	1.23%	0.86%	0.84%	0.76%	0.94%	0.76%	1.00%	0.74%	0.79%
						Weighted average		0.87%	

5.4.4 Other parameters

The annual discount factor β is set at 0.96, and the coefficient of relative risk aversion γ is set at 5. These two figures are commonly used in the literature. To find a suitable parameter value for b, I follow the assumption in Cauley et al. (2007) that the utility of wealth at retirement is derived from a post-retirement consumption stream.[3] The consumption stream is supported by a term annuity purchased using the total wealth at retirement. The annuity gives a fixed level of payment for J years, starting at the time of retirement. The level payment is given by

$$C_{T+j} = W_{T+1} \frac{R_f - 1}{R_f(1 - R_f^{-J})}, \qquad j = 1, 2, \cdots, J. \tag{5.20}$$

[3] Yao and Zhang (2005) use a similar idea to define the bequest function.

Therefore, the retirement saving utility function can be defined as

$$v(W_{T+1}) = \sum_{j=1}^{J} \beta^{j-1} \frac{C_{T+j}^{1-\gamma}}{1-\gamma} = b \frac{W_{T+1}^{1-\gamma}}{1-\gamma}, \tag{5.21}$$

where

$$b = \frac{1-\beta^J}{1-\beta} \left(\frac{R_f - 1}{R_f(1-R_f^{-J})} \right)^{1-\gamma}. \tag{5.22}$$

The term of the annuity J is set at 33, meaning the annuity provides the consumption till age 98. This age is determined such that the probability of a 65-year-old Australian man living beyond age 98 is less than 5% based on the 2009 Australian Life Tables.

The starting salary is set at $54,000 and the initial wealth is set at $20,000. A later section performs sensitivity analysis on these initial values. Table 5.8 summaries the parameter values used in the simulation.

Table 5.8. Parameter values used in the simulation of housing tenure choice.

Description	Parameter	Value
Subjective discount factor	β	0.96
Coefficient of relative risk aversion	γ	5
Retirement saving motive	b	6,557,300
Risk-free rate	$r_f = \ln(R_f)$	0.02
Maintenance cost	φ	0.87%
Down payment	$\underline{\lambda}$	0.2
Initial wealth	B	$30,000
Maximum loan-to-value ratio	$\overline{LVR}$	1.25

5.5 Numerical results

5.5.1 Financial and economic scenarios

A total of 100,000 sample paths are simulated for each state variable based on the estimated VAR model to represent the financial and economic scenarios faced by individuals. Figure 5.9 shows the simulation results of some of the key variables. The initial house price is obtained from the initial rental payment ($14,328 per annum in the base case) and the simulated rental yield through Equation (5.9).

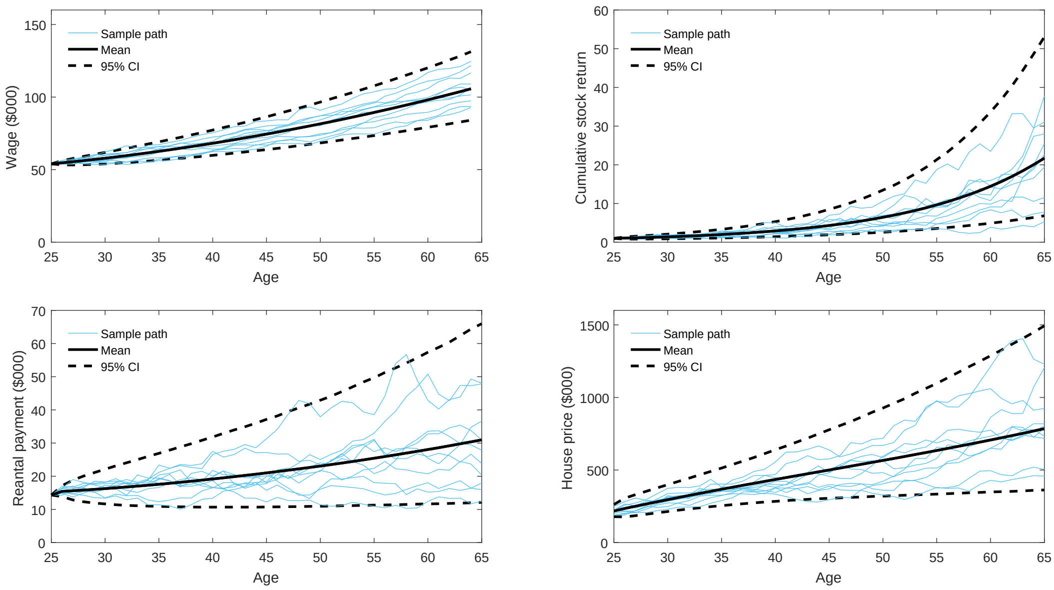


Figure 5.9. Some simulated sample paths, mean, and 95% confidence intervals of the key variables: (Top Left) wage; (Top Right) cumulative stock return; (Bottom Left) rental payment; (Bottom Right) house price.

5.5.2 Expenditure

The expenditure includes non-housing consumption, rental payment if the individual is renting, and down payment, mortgage repayment, and home maintenance cost for homeowners. This section examines how each type of expenditure varies with different ages to become a homeowner.

A significant amount of lifetime utility is derived from non-housing consumption. Figure 5.10 shows the average non-housing consumption paths for a 25-year-old investor who becomes a homeowner at ages 30, 40, 50, 60, or over 65. '65+ (active)' refers to those who actively choose to rent throughout their working life, while '65+ (passive)' refers to those who have to keep renting because they cannot afford to purchase the property at ages 30, 40, 50, and 60. The general pattern for each curve (except for the 65+ ones) is similar. The investor faces a dramatic consumption decrease in the first few years following the property purchase. Afterwards, the average non-housing consumption level quickly recovers and eventually exceeds the pre-purchase level. The consumption cut is due to the liquidity constraint caused by the large amount of down payment, and the quick recovery is due to the fact that homeowners increase the consumption levels faster than the renters. The consumption level rises faster for homeowners because they accumulate the liquid assets at a faster rate, the reason for which will be explained in the next section (Section 5.5.3). The curve that corresponds to purchasing the property at age 30 also indicates that once the mortgage is paid off, the consumption level increments accelerate since the amount of financial commitments is reduced. Furthermore, those who keep renting before retirement maintain a steadily increasing consumption level without dramatic decline or rebound. Although the average levels are almost the same for different groups of individuals before age 30, those who keep renting spend the least on non-durable goods in the years before retirement when the other groups of individuals have become homeowners.

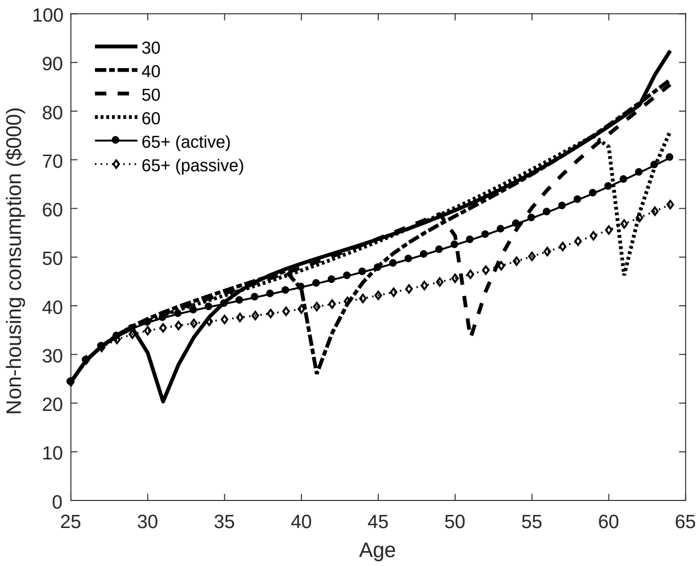


Figure 5.10. Simulated average non-housing consumption paths for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65.

The rental payment is determined by both external factors and the age to become a homeowner. The external factors, including house price and rental yield, determine the rental cost in each period. In addition, the earlier the home property is purchased, the shorter period of rental payment is required. The external factors also affect the age at which an individual is able to become a homeowner since the decision to buy a residential home is constrained by an individual's capacity. Only when the house price is affordable will the individual purchase the property. Figure 5.11 reveals that when the individual purchases the property, the average house price is typically the lowest among all possible cases. Figure 5.12 shows that renting throughout working life, whether actively or passively, leads to one of the highest levels of average rental costs. These sample paths correspond to the relatively high house prices, as shown in Figure 5.11. The housing affordability problem is therefore compounded by the high rental costs, which further reduce the non-housing consumption for renters.

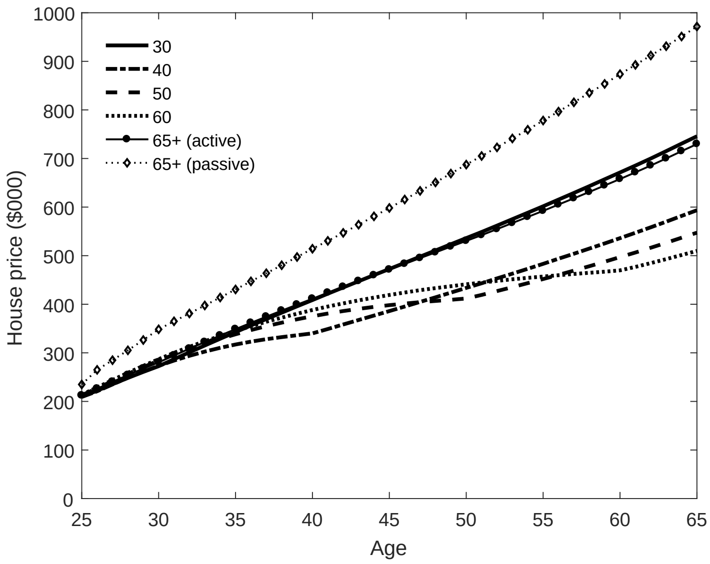


Figure 5.11. Simulated average house price paths for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65.

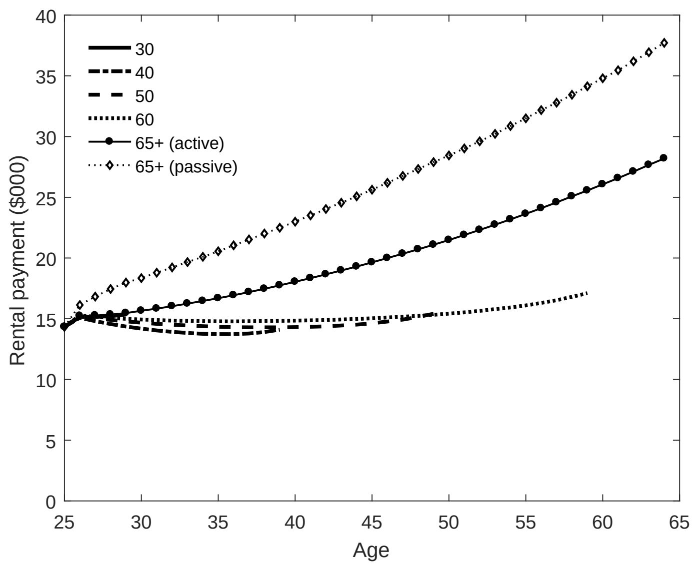


Figure 5.12. Simulated average rental payment paths for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65.

When individuals purchase the property, they need to make a down payment and the remaining is financed by a 30-year fixed rate mortgage. After becoming an homeowner, they need to incur home maintenance costs while repaying the mortgage. Figure 5.13 compares the mortgage repayment plus maintenance cost (housing cost after becoming a homeowner) to the average rental payment (housing cost before becoming a homeowner). Except for those becoming homeowners at age 30, the annual repayment amount is generally higher than the rental costs. Nevertheless, those who keep renting tend to incur the highest cost on housing.

The results so far show the average spending on a single item, such as non-housing consumption, rent, or mortgage repayment. For an easier comparison of cash flows among different ages of purchase, Table 5.9 compares the spending on each single item over the course of an individual's working life discounted at the risk-free rate to the beginning of the first period. For individuals purchasing the property before age 65, their spending on non-housing consumption is largely similar. The individuals who keep renting during working life spend the least on non-housing consumption because they have to incur significantly higher rental costs. The average value exceeds the sum of rental payment,

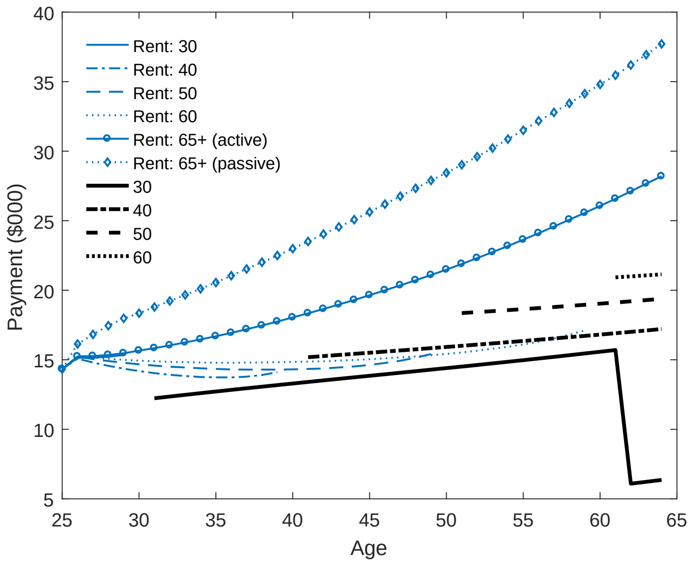


Figure 5.13. Simulated average mortgage repayment plus maintenance cost paths compared to rental payment paths for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65.

down payment, mortgage repayment, and home maintenance cost in any of the other groups who purchase the property before retirement. Their rental payment is much higher than the rest not only because they pay the rent for a longer period, but also due to a higher average level of rental cost.

Table 5.9. The mean and standard deviation (Std) of discounted present value of various consumption items (in $1,000) for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65. The discount rate is the risk-free rate.

	Non-housing consumption		Rental payment		Down payment		Mortgage repayment		Maintenance cost	
	Mean	Std	Mean	Std	Mean	Std	Mean	Std	Mean	Std
30	1,380	108	72	5	49	6	201	23	95	20
40	1,364	104	185	18	50	6	170	21	57	11
50	1,359	109	290	34	50	8	108	17	32	6
60	1,373	114	386	52	47	9	32	6	10	2
65+ (active)	1,278	169	536	128	0	0	0	0	0	0
65+ (passive)	1,149	153	678	135	0	0	0	0	0	0

5.5.3 Wealth

The investor's wealth consists of liquid assets (stocks and cash), home equity (if he is a homeowner), and the assets in his pension account. Assets in residential home and pension account are illiquid during working life as they cannot be used for consumption before retirement. Once the individual enters into retirement stage, assets in his pension account will be used to fund his retirement life, and home equity could be unlocked to provide additional funding.

The general pattern of liquid asset paths (Figure 5.14) resembles that of non-housing consumption due to the assumption that the amount spent on non-durable goods is a certain proportion of disposable income, which consists of liquid assets and labour income. Individuals who do not purchase the property before age 65 maintain a slowly increasing level of liquid assets. On the other hand, the liquid assets for the rest of individuals drop dramatically in the year after the property is purchased, followed by quick recoveries, and eventually exceed the pre-purchase level. Homeowners accumulate the liquid assets at a faster pace than renters because homeowners allocate a higher proportion of after-consumption wealth to stocks.

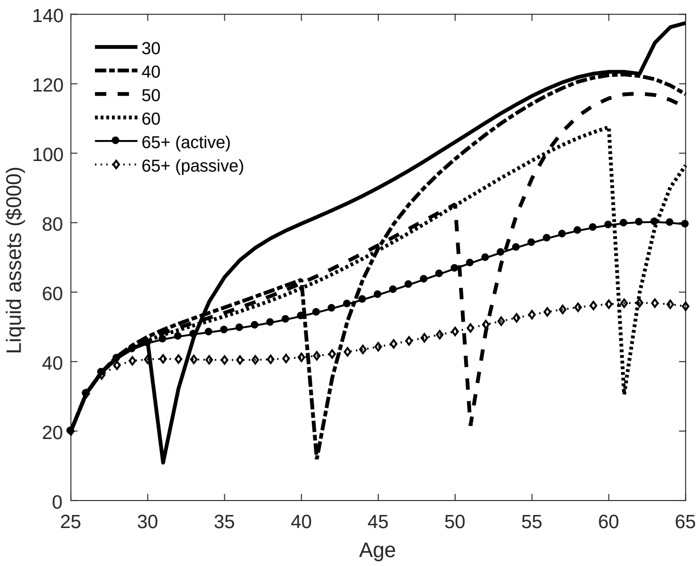


Figure 5.14. Simulated average liquid asset paths for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65.

The pension account balance depends on the amount of contributions to the fund, fund asset allocation strategies, and investment returns. The model assumes 9.5% of labour income is contributed to the fund each year before retirement, and that the pension fund adopts the same investment strategy (i.e. 70% in stocks and 30% in cash) for all fund members. Figure 5.15 shows that different groups of individuals have very similar wage levels and cumulative stock returns. As a result, their average pension account balances are almost the same (Figure 5.16).

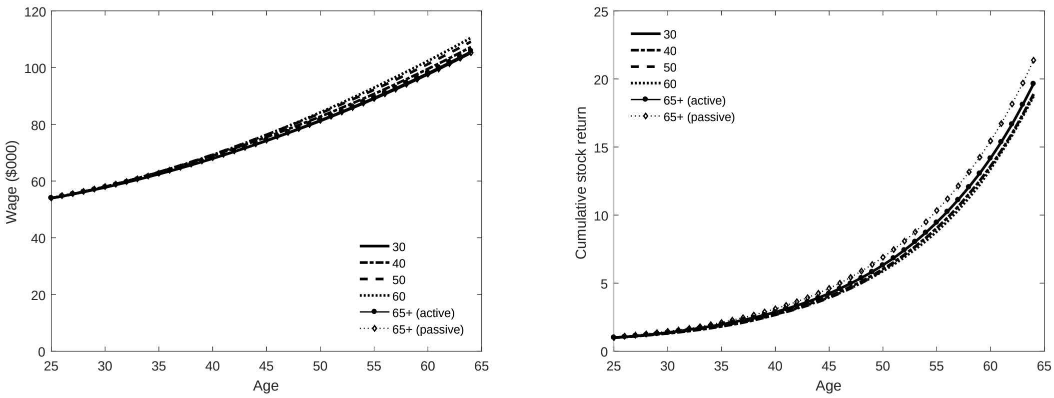


Figure 5.15. (Left Panel) Simulated average wage paths and (Right Panel) average cumulative stock return paths for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65.

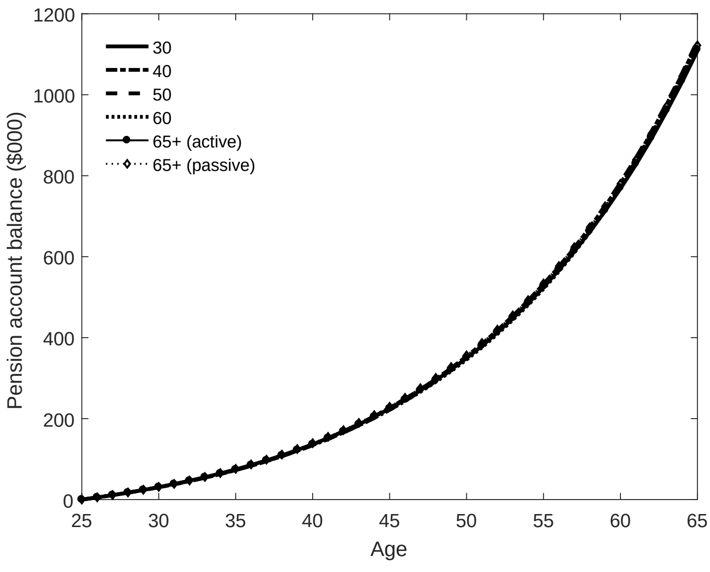


Figure 5.16. Simulated average pension account balance paths for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65.

For homeowners, a significant proportion of their wealth is in the form of home equity. Figure 5.17 displays the average housing asset and home equity paths. Comparing the housing asset at the time of purchase across four cases, the price increases with age since the average rate of property capital growth is positive. Comparing the housing asset at the same age among four cases, the later the property is purchased, the lower the asset value. This is in line with the finding in Figure 5.11 that individuals purchase the property when its price is relatively low. The right panel of Figure 5.17 shows the average housing asset paths net of mortgages. The increase in home equity is driven by both the increase in house price and the decrease in mortgage balance, so its value grows faster than that of the housing asset.

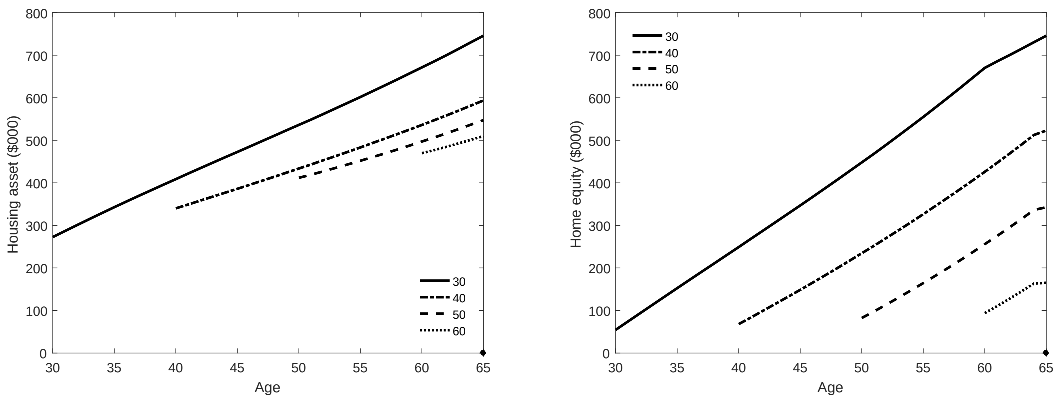


Figure 5.17. (Left Panel) Simulated average housing asset paths and (Right Panel) average home equity paths for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65.

Apart from non-housing consumption during working life, individuals also derive utility from retirement savings. Table 5.10 shows the mean and standard deviation of wealth and its components at retirement. The total wealth is dominated by pension, followed by home equity (if homeowner). The dominance of housing assets in individual's wealth is consistent with the empirical evidence. Besides, the significance of pension is not unexpected given the individual contributes to the pension fund for 40 years, and that there are no tax or investment fees. A later section will briefly examine the impact of tax.

Purchasing the property at a younger age leads to a higher average wealth level at retirement. This is due to higher average values of home equity and liquid assets. The earlier

Table 5.10. The mean and standard deviation (Std) of wealth and its components (in $1,000) at retirement for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65.

	Wealth		Cash		Stocks		Home equity		Pension	
	Mean	Std	Mean	Std	Mean	Std	Mean	Std	Mean	Std
30	1,991	422	98	36	40	38	746	275	1,108	301
40	1,768	380	83	30	34	33	523	189	1,128	308
50	1,589	357	80	28	33	32	343	136	1,133	310
60	1,389	332	68	25	28	27	165	75	1,128	311
65+ (active)	1,191	309	71	35	8	19	0	0	1,112	303
65+ (passive)	1,178	311	50	31	6	14	0	0	1,122	308

the property is purchased, the lower the mortgage balance and the higher the housing asset value at retirement. In terms of the liquid assets, the earlier the investor purchases the house, the higher the equity proportion in liquid assets. A typical investor is therefore able to accumulate more liquid assets.

5.5.4 Certainty equivalent consumption

The optimal time to purchase a residential home is determined by which of the age leads to the highest lifetime utility. Since the power utility function typically gives a negative number, different utility levels are hard to interpret and compare. The corresponding certainty equivalent consumption level is used to measure individual's preference. The certainty equivalent consumption represents the dollar amount of annual consumption that gives the same level of utility as the expected present value of lifetime utility.

Table 5.11 compares the certainty equivalent consumption for a 25-year-old investor who purchases the property at different ages. The certainty equivalent consumption is lowest if the individual purchases a home property at age 30, followed by the 65+ age groups. The highest value occurs if the individual purchases a house at age 60, although the difference between ages 50 and 60 is minimal.

Table 5.11. The certainty equivalent consumption (in $1,000) for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65.

30	40	50	60	65+ (active)	65+ (passive)
33.204	36.264	37.150	37.189	35.590	33.798

There are several reasons behind the results. Firstly, there is a large decrease in the non-housing consumption after purchasing the property. Its impact can be assessed by decomposing the lifetime utility defined in Equation (5.17) into the discounted utility level in each period. Before the last period, each component is given by $\beta^{t-1}u(C_t),\ t = 1, 2, \cdots, T$. In the last period, it is computed as $\beta^T v(W_{T+1})$. Figure 5.18 plots the heat map of the average discounted utility. The y-axis represents the possible ages of purchasing the property, and the x-axis represents the age in each period. The average utility is scaled as follows to avoid interpreting negative numbers

$$\left(\frac{\overline{u}_t \times (1-\gamma)}{\beta^t}\right)^{1/(1-\gamma)}, \quad t = 1, 2, \cdots, T,$$

where

$$\overline{u}_t = \begin{cases} \mathbb{E}_1\left[\beta^{t-1}u(C_t)\right] & t = 1, 2, \cdots, T \\ \mathbb{E}_1\left[\beta^T v(W_{T+1})\right] & \text{otherwise} \end{cases}.$$

Figure 5.18 shows that the impact of sudden decline in consumption is particularly severe for individuals purchasing the property at age 30. Even though their consumption levels gradually increase afterwards (Figure 5.10), and their average amount of retirement wealth is the highest among all groups (Table 5.10), the utility loss incurred early in life cannot be compensated by the utility gained from consumption after mid-30s and retirement wealth. Secondly, renting throughout working life significantly constrains the overall spending on non-housing consumption (Table 5.9) and leads to the lowest average level of retirement wealth (Table 5.10). Thirdly, those individuals who defer property purchasing in working life purchase the property at a relatively affordable price. Besides, the utility loss caused by consumption drop is also deferred and reduced. As a result, they are able to enjoy the benefits of being a homeowner (faster accumulation of

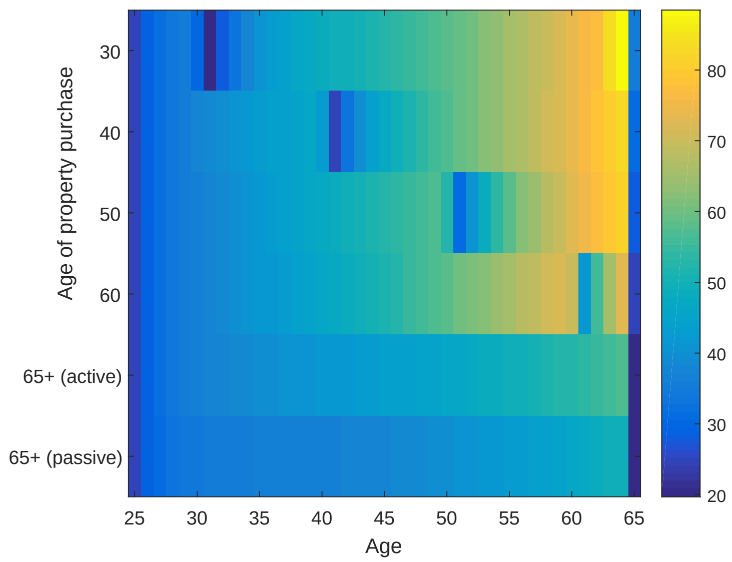


Figure 5.18. Simulated average utility level in each period for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65.

liquid assets) at a lower cost, which leads to a higher expected utility and hence a higher level of certainty equivalent consumption.

5.6 Robustness checks

A number of robustness checks are performed in this section. Section 5.6.1 controls for the housing affordability issue and reruns the simulations. Sections 5.6.2 and 5.6.3 perform sensitivity analysis on initial values and key parameter values, respectively.

5.6.1 Selected sample paths

The simulation analyses done so far do not strictly control for the financial and economic environment. More specifically, individuals who purchase the property at different ages do not share exactly the same simulated sample paths. This is due to the capacity reason that individuals may not be able to afford the property at the pre-specified age in all the simulated sample paths. Different sample paths may raise the concern that differences in wealth levels and certainty equivalent consumption result from different values of the state variables.

This robustness check strictly controls for the financial and economic environment to limit the impact of stock returns, house prices, rental cost, and wage when comparing the lifetime utility and retirement wealth across different groups of homeowners. This is done by selecting the sample paths where investors can afford to purchase the property at all the pre-specified ages: 30, 40, 50, and 60. As a result, those who keep renting before retirement all actively choose to do so. Figure 5.19 shows that the main difference between the full sample and the selected sample lies in the house price. The average house price level is much lower in the selected sample as a result of the selection effect. In addition, the average stock returns and rental yield are generally lower in the selected sample.

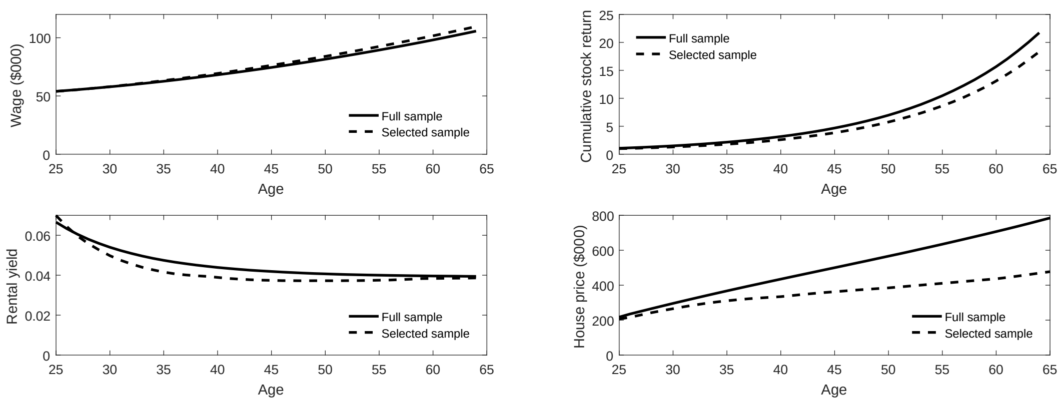


Figure 5.19. Comparison of the simulated average paths between the full simulation sample and the selected simulation sample: (Top Left Panel) wage; (Top Right Panel) cumulative stock return; (Bottom Left Panel) rental yield; (Bottom Right Panel) house price.

Since all the individuals face the same external environment, the rental cost for those who are renting is the same regardless of the age at which the property is purchased. Figure 5.20 compares the average rental cost based on the selected sample to those based on the full sample. On average, the rental payment ranges from just above $12,000 per annum to around $18,000 per annum, which is a lower bound of the average paths shown in Figure 5.12.

Figure 5.21 shows the average non-housing consumption levels based on the selected

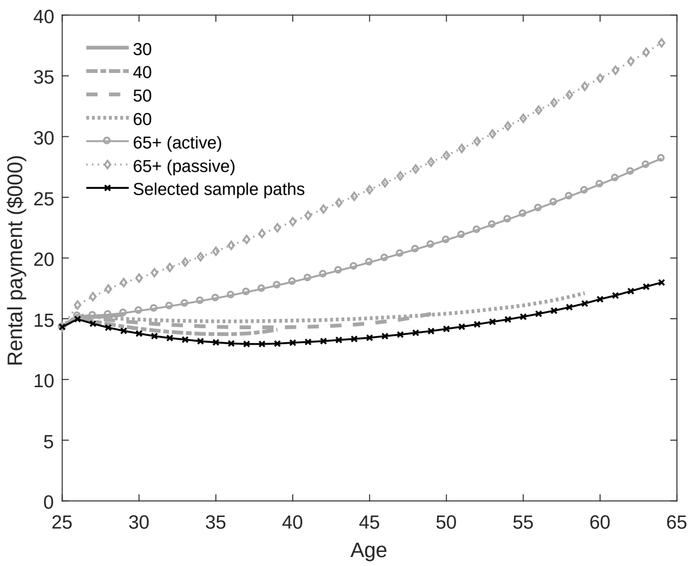


Figure 5.20. The average rental cost based on the selected simulation paths, compared to the average rental payment paths for a 25-year-old investor who chooses to purchase the property at ages 30, 40, 50, 60, and over 65.

simulation paths. The overall patters remain the same compared to the base case analysis (Figure 5.10), except for the 65+ age group. Their consumption levels relative to the other ages are much higher since they no longer face higher average levels of rental cost. Furthermore, discounting the non-housing consumption at the risk-free rate, the 65+ age group enjoys the highest level of non-housing consumption, as shown in Table 5.12.

Table 5.12. Robustness check on the selected sample paths: the mean and standard deviation (Std) of discounted present value of various consumption items (in $1,000) for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65. The discount rate is the risk-free rate.

	Non-housing consumption		Rental payment		Down payment		Mortgage repayment		Maintenance cost	
	Mean	Std	Mean	Std	Mean	Std	Mean	Std	Mean	Std
30	1,451	110	69	4	48	6	196	25	72	9
40	1,397	104	179	17	50	7	167	22	50	7
50	1,394	108	270	29	47	8	101	17	29	5
60	1,407	111	354	42	43	9	29	6	9	2
65+	1,442	112	395	50	0	0	0	0	0	0

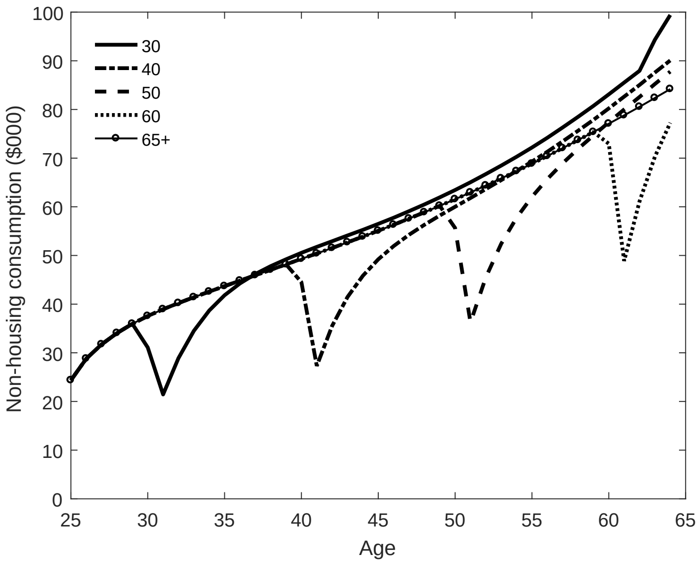


Figure 5.21. Robustness check on the selected sample paths: average non-housing consumption paths for a 25-year-old investor who chooses to purchase the property at ages 30, 40, 50, 60, and over 65.

Comparing the amount of wealth at retirement before and after controlling for the external factors (Table 5.10 and Table 5.13, respectively), the total wealth level decreases for those who are homeowners at retirement. This is mainly due to the lower home equity values caused by the sample selection effect. By contrast, those individuals who keep renting during working life have accumulated slightly more wealth at retirement because they spend less on rent and have more cash.

Table 5.13. Robustness check on the selected sample paths: the mean and standard deviation (Std) of wealth and its components (in $1,000) at retirement for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65.

	Wealth†		Cash		Stocks		Home equity	
	Mean	Std	Mean	Std	Mean	Std	Mean	Std
30	1,760	356	106	35	45	43	477	111
40	1,664	351	87	29	37	35	408	109
50	1,537	345	83	29	35	34	286	100
60	1,389	332	71	26	29	28	157	72
65+	1,243	316	100	27	11	23	0	0

† Wealth includes pension account balance which is the same for all cases. The mean is $1.13 million and the standard deviation is about $312,000.

Table 5.14 shows that the certainty equivalent consumption levels based on the selected simulation paths are similar to the pre-selection case in that the value is lowest for purchasing the property at age 30 and highest for age 60. The main difference between the two cases lies in the 65+ age group. Becoming a homeowner after age 65 is no longer ranked the second worst due to less rental payment.

Table 5.14. Robustness check on the selected sample paths: the certainty equivalent consumption (in $1,000) for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65.

30	40	50	60	65+
34.048	36.629	37.571	37.669	37.397

In summary, the analyses on the selected sample reinforce the results in the base case that the high rental cost prevents individuals from spending more on non-housing consumption and accumulating more liquid assets, leading to a low utility level and wealth level at retirement.

5.6.2 Sensitivity analysis: Initial values

The baseline analysis assumes that in the first period the individual is endowed with $20,000 wealth, earns $54,000 per annum, and that the initial rent is $14,328. This section performs sensitivity analysis on these three initial values. In particular, the initial wealth increases to $100,000 or reduces to zero. The initial wage and the initial rent increases or decreases by a third compared to the base case.

Increasing the level of initial wealth enables individuals to spend more on non-housing consumption (Table 5.15), while having little impact on the amount of savings for retirement except for those who buy a residential property at age 30 (Table 5.16). The exception is mainly driven by a larger amount of home equity. A higher level of initial wealth enables a greater proportion of individuals who choose to become homeowners early in life to afford the property in the years when house price is high. For those who choose to become homeowners at an older age, the value of the property they can afford is mainly determined by their labour earnings, so the initial wealth has a much smaller impact on retirement savings.

Table 5.15. Robustness check on initial wealth: the mean and standard deviation (Std) of discounted present value of various consumption items (in $1,000) for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65. The discount rate is the risk-free rate.

	Non-housing consumption		Rental payment		Down payment		Mortgage repayment		Maintenance cost	
	Mean	Std	Mean	Std	Mean	Std	Mean	Std	Mean	Std
Initial wealth = $100,000										
30	1,450	109	73	5	51	6	209	26	97	20
40	1,444	104	186	18	50	6	170	21	57	11
50	1,439	109	290	34	50	8	108	17	32	6
60	1,453	114	386	52	47	9	32	6	10	2
65+ (active)	1,343	172	552	135	0	0	0	0	0	0
65+ (passive)	1,223	155	693	140	0	0	0	0	0	0
Initial wealth = $0										
30	1,363	108	72	4	49	5	199	22	94	19
40	1,344	104	185	18	50	6	170	21	57	11
50	1,339	109	290	34	50	8	108	17	32	6
60	1,353	114	386	52	47	9	32	6	10	2
65+ (active)	1,263	168	531	126	0	0	0	0	0	0
65+ (passive)	1,130	152	675	134	0	0	0	0	0	0

Table 5.16. Robustness check on initial wealth: the mean and standard deviation (Std) of wealth and its components (in $1,000) at retirement for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65.

	Wealth		Cash		Stocks		Home equity		Pension	
	Mean	Std	Mean	Std	Mean	Std	Mean	Std	Mean	Std
Initial wealth = $100,000										
30	2,011	429	98	36	39	38	763	283	1,110	303
40	1,768	380	83	30	34	33	523	189	1,128	308
50	1,589	357	80	28	33	32	343	136	1,133	310
60	1,389	332	68	25	28	27	165	75	1,128	311
65+ (active)	1,189	309	69	35	8	19	0	0	1,112	304
65+ (passive)	1,185	309	50	32	5	14	0	0	1,130	307
Initial wealth = $0										
30	1,985	420	98	36	40	38	740	273	1,107	301
40	1,768	380	83	30	34	33	523	189	1,128	308
50	1,589	357	80	28	33	32	343	136	1,133	310
60	1,389	332	68	25	28	27	165	75	1,128	311
65+ (active)	1,193	309	72	35	8	19	0	0	1,112	303
65+ (passive)	1,176	311	50	31	6	14	0	0	1,120	307

A higher value of initial wage has a profound effect on both consumption (Table 5.17) and retirement savings (Table 5.18) for all groups of individuals. An $18,000 increase in initial wage raises the expected present value of non-housing consumption by over half a million. Individuals are also paying more for down payment and mortgage repayment because more individuals are able to purchase the property at the desired ages. This is reflected by higher home equity values. Since the rental cost and the house price are positively correlated, the rental payment is also higher. Apart from home equity, individuals also accumulate more wealth in liquid assets and pension accounts as a result of a higher starting wage.

Table 5.17. Robustness check on initial wage: the mean and standard deviation (Std) of discounted present value of various consumption items (in $1,000) for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65. The discount rate is the risk-free rate.

	Non-housing consumption		Rental payment		Down payment		Mortgage repayment		Maintenance cost	
	Mean	Std	Mean	Std	Mean	Std	Mean	Std	Mean	Std
Initial wage = $72,000										
30	1,952	148	74	5	53	8	218	34	100	22
40	1,873	149	207	25	60	10	201	32	68	15
50	1,869	156	325	46	59	11	128	25	39	9
60	1,894	159	430	67	56	12	38	8	12	3
65+ (active)	1,834	206	568	142	0	0	0	0	0	0
65+ (passive)	1,632	189	820	148	0	0	0	0	0	0
Initial wage = $36,000										
30	884	72	63	3	34	3	139	10	72	14
40	890	68	151	11	37	4	125	14	41	7
50	873	69	241	24	37	5	80	11	24	4
60	869	74	325	38	35	5	24	4	7	1
65+ (active)	885	83	364	49	0	0	0	0	0	0
65+ (passive)	666	116	581	137	0	0	0	0	0	0

Table 5.18. Robustness check on initial wage: the mean and standard deviation (Std) of wealth and its components (in $1,000) at retirement for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65.

	Wealth		Cash		Stocks		Home equity		Pension	
	Mean	Std	Mean	Std	Mean	Std	Mean	Std	Mean	Std
Initial wage = $72,000										
30	2,460	522	136	48	56	53	784	294	1,485	405
40	2,284	493	114	41	47	45	634	240	1,490	407
50	2,069	465	110	39	45	44	419	175	1,496	408
60	1,831	435	95	36	39	38	202	94	1,494	409
65+ (active)	1,609	413	111	44	12	28	0	0	1,485	405
65+ (passive)	1,648	452	78	40	8	21	0	0	1,562	451
Initial wage = $36,000										
30	1,386	290	63	23	25	24	572	200	726	202
40	1,211	266	54	19	23	22	368	130	766	210
50	1,090	245	53	19	22	21	242	94	774	212
60	941	226	44	16	18	17	119	52	760	211
65+ (active)	830	214	62	20	7	15	0	0	761	211
65+ (passive)	769	205	24	21	3	8	0	0	742	202

Active and passive renters face different situations when initial rent increases. This is due to the selection effect. As rental cost increases, so does house price. As a result, those who actively choose to rent before retirement are relatively rich (Table 5.20) and are able to spend more on non-housing consumption (Table 5.19). For those who purchase home properties before retirement, an increase in rent slightly reduces their non-housing consumption while improving their retirement savings due to the positive correlation between rental cost and house price.

Table 5.19. Robustness check on initial rent: the mean and standard deviation (Std) of discounted present value of various consumption items (in \$1,000) for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65. The discount rate is the risk-free rate.

	Non-housing consumption		Rental payment		Down payment		Mortgage repayment		Maintenance cost	
	Mean	Std	Mean	Std	Mean	Std	Mean	Std	Mean	Std
Initial rent = \$19,104										
30	1,330	107	88	4	51	4	208	17	104	21
40	1,335	100	214	17	54	6	182	21	60	11
50	1,319	103	340	35	54	8	117	17	35	6
60	1,318	113	455	55	51	8	34	6	11	2
65+ (active)	1,305	151	539	100	0	0	0	0	0	0
65+ (passive)	1,049	170	790	179	0	0	0	0	0	0
Initial rent = \$9,552										
30	1,505	112	49	4	36	6	146	23	67	15
40	1,438	115	142	19	41	7	139	25	48	11
50	1,430	121	224	34	42	9	90	19	27	7
60	1,448	121	297	50	39	9	27	6	9	2
65+ (active)	1,425	148	379	95	0	0	0	0	0	0
65+ (passive)	1,221	163	674	180	0	0	0	0	0	0

Table 5.20. The mean and standard deviation (Std) of wealth and its components (in $1,000) at retirement for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65.

	Wealth		Cash		Stocks		Home equity		Pension	
	Mean	Std	Mean	Std	Mean	Std	Mean	Std	Mean	Std
Initial rent = $19,104										
30	2,057	448	95	35	38	37	832	306	1,092	309
40	1,794	384	82	30	34	32	544	197	1,135	307
50	1,625	369	80	28	33	32	359	140	1,154	319
60	1,404	338	67	24	27	26	174	77	1,136	315
65+ (active)	1,227	321	86	33	9	21	0	0	1,132	315
65+ (passive)	1,158	306	42	32	5	13	0	0	1,112	302
Initial rent = $9,552										
30	1,783	379	103	36	42	40	523	196	1,114	304
40	1,681	364	87	31	36	34	443	173	1,115	304
50	1,534	345	84	30	35	33	297	128	1,118	305
60	1,367	325	73	28	30	29	144	69	1,119	306
65+ (active)	1,213	310	89	32	10	22	0	0	1,114	304
65+ (passive)	1,261	358	55	38	14	31	0	0	1,192	344

Table 5.21 shows the impact of initial values on the certainty equivalent consumption. Increasing (Decreasing) the level of initial wealth leads to a higher (lower) utility level, as expected, but it does not change the result that purchasing the property at age 60 leads to the highest value of certainty equivalent consumption, or lifetime utility. When the initial wage is relatively low, purchasing the property at age 50 becomes slightly more attractive. Comparing the two groups of individuals who purchase the property at age 50 and age 60 given the $36,000 initial wage, they have similar expected present values of non-housing consumption (Table 5.17) because their consumption is constrained by the low wage level. The former group, however, has greater savings for retirement (Table 5.18), which contributes to a higher lifetime utility. When the initial rent is relatively high, purchasing the property at age 50 becomes more attractive. Compared to purchasing the property at age 60, becoming a homeowner at 50 allows individuals to accumulate significantly more wealth without giving up a large amount of non-housing consumption.

Table 5.21. The certainty equivalent consumption (in $1,000) for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65 when initial wealth, initial wage, and initial rent take on different values. The highest value in each column is in bold.

	Initial wealth		Initial wage		Initial rent	
	$100,000	$0	$72,000	$36,000	$19,104	$9,552
30	37.660	29.273	47.570	20.783	31.037	37.925
40	42.481	30.850	48.871	23.503	34.610	38.519
50	44.613	31.230	49.787	**23.631**	**35.074**	39.150
60	**44.747**	**31.244**	**49.923**	23.282	34.810	**39.249**
65+ (active)	40.749	30.568	48.728	23.170	34.292	38.426
65+ (passive)	37.737	29.648	46.500	19.799	30.789	36.154

5.6.3 Sensitivity analysis: Parameters

The lifetime utility, or certainty equivalent consumption, depends on the subjective discount factor (β) and the risk aversion parameter (γ). Table 5.22 shows the certainty equivalent consumption under alternative sets of parameter values.

Table 5.22. The certainty equivalent consumption (in $1,000) for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65 when β and γ take on different values. The highest value in each column is in bold.

	β		γ	
	0.99	0.93	10	2
30	35.828	31.278	26.810	36.943
40	38.319	34.472	31.312	37.649
50	**39.028**	35.177	32.398	**37.811**
60	38.628	**35.226**	**32.519**	37.564
65+ (active)	35.629	34.349	31.922	35.053
65+ (passive)	33.675	32.956	31.038	32.677

A higher value of β means the investor has a lower time preference and therefore puts more emphasis on the utility in the further future. A relatively patient investor finds it more attractive to purchase the property earlier because the utility derived from wealth at retirement has a higher weight in the lifetime utility.

The parameter γ affects individual's preference in two ways. It determines the level of risk aversion, and, if the utility function is of the constant relative risk aversion (CRRA), the elasticity of intertemporal substitution (EIS). In the case of CRRA utility, individuals become more risk tolerant and less willing to substitute consumption over time as γ decreases. An investor with a lower value of γ prefers purchasing the property at a younger age. This could result from a combination of risk tolerance and desire for consumption smoothness. To disentangle these two effects, Table 5.23 shows the volatilities of consumption, rental payment, house prices, and stock index in the base case.

The 50 and 60 age groups take similar levels of risk in rental payment, house price, and stock index, but the age 50 group has more volatile consumption path. Therefore, the changes in the EIS is the main driver behind the impact of γ on certainty equivalent consumption. This suggests the need of separating the risk aversion and EIS in future research by, for example, using the Epstein-Zin-Weil (Epstein and Zin, 1989; Epstein and Zin, 1991; Weil, 1989) type utility functions.

Table 5.23. The volatility† (%) of non-housing consumption, rental payment, house price, and stock index for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65.

	Non-housing consumption	Rental payment	House price	Stock index
30	3.378	6.951	5.816	14.687
40	3.121	7.533	5.827	14.683
50	2.968	7.652	5.826	14.692
60	2.751	7.716	5.835	14.690
65+ (active)	2.647	7.764	-	14.689
65+ (passive)	2.977	7.705	-	14.677

† The volatility is computed as the standard deviation of $\ln(X_t/X_{t-1})$ for $t = 2, 3, \cdots, T$ where X_t denotes the non-housing consumption, rental payment, housing price or cumulative stock return in at time t.

5.6.4 Sensitivity analysis: Tax

The model abstracts from tax on investment returns and capital gains. This section briefly analyses the impact of tax by imposing income taxes on cash and stock returns. Different scenarios are listed in Table 5.24. The superannuation fund in Australia receives tax concessions on investment earnings, so a 15% tax rate inside pension fund is assumed for both 'Low tax' and 'High tax' scenarios. Note that changes in tax rate do not affect the effective mortgage rate since homeowners cannot claim deductions for their mortgage interest expenses.

Table 5.24. The effective tax rates on investment returns inside and outside of pension fund in each scenario.

	Inside pension fund	Outside pension fund
High tax	15%	30%
Low tax	15%	15%
Base case	0%	0%

Compared to the base case, imposing taxes on cash and stock returns has marginal impact on consumption (Table 5.25), but it greatly reduces the wealth level at retirement to a similar extent for all ages (Table 5.26). The reduction mainly comes from a much lower balance in the pension fund. Retirement savings are still dominated by pension, followed by home equity for homeowners, as in the base case. Imposing the tax makes the property returns more attractive than cash and stocks. As a consequence, it becomes slightly more attractive to purchase the property earlier than the base case in terms of lifetime utility (Table 5.27).

Table 5.25. Robustness check on tax: the mean and standard deviation (Std) of discounted present value of various consumption items (in $1,000) for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65. The discount rate is the risk-free rate.

	Non-housing consumption		Rental payment		Down payment		Mortgage repayment		Maintenance cost	
	Mean	Std	Mean	Std	Mean	Std	Mean	Std	Mean	Std
High tax										
30	1,362	104	72	5	49	6	201	22	94	19
40	1,350	100	185	18	50	6	169	21	57	11
50	1,348	106	289	34	50	8	107	17	32	6
60	1,363	112	385	52	46	9	32	6	10	2
65+ (active)	1,269	166	535	128	0	0	0	0	0	0
65+ (passive)	1,143	151	676	135	0	0	0	0	0	0
Low tax										
30	1,371	106	72	5	49	6	201	23	94	19
40	1,357	102	185	18	50	6	169	21	57	11
50	1,353	108	290	34	50	8	108	17	32	6
60	1,368	113	386	52	47	9	32	6	10	2
65+ (active)	1,274	168	536	128	0	0	0	0	0	0
65+ (passive)	1,146	152	677	135	0	0	0	0	0	0

Table 5.26. Robustness check on tax: the mean and standard deviation (Std) of wealth and its components (in $1,000) at retirement for a 25-year-old investor who chooses to purchase the property at ages 30, 40, 50, 60, and over 65.

	Wealth		Cash		Stocks		Home equity		Pension	
	Mean	Std	Mean	Std	Mean	Std	Mean	Std	Mean	Std
High tax										
30	1,752	346	96	36	38	36	744	274	874	197
40	1,523	288	81	30	32	31	520	187	890	201
50	1,345	257	79	28	32	30	340	135	894	201
60	1,149	226	67	25	27	26	164	74	891	202
65+ (active)	956	205	70	34	8	19	0	0	878	198
65+ (passive)	939	205	50	31	6	14	0	0	883	201
Low tax										
30	1,755	347	97	36	39	37	745	275	874	197
40	1,527	290	82	30	33	32	521	188	890	201
50	1,348	259	79	28	32	31	342	136	895	202
60	1,151	228	68	25	28	27	165	74	891	203
65+ (active)	956	205	71	34	8	19	0	0	877	198
65+ (passive)	939	205	50	31	6	14	0	0	884	201

Table 5.27. Robustness check on tax: the certainty equivalent consumption (in $1,000) for a 25-year-old investor who purchases the property at ages 30, 40, 50, 60, and over 65. The highest value in each row is in bold.

	30	40	50	60	65+
Low tax	34.486	39.565	**40.612**	40.443	37.947
High tax	34.424	39.483	**40.522**	40.356	37.868
Base case	34.577	39.772	40.946	**41.015**	38.921

5.7 Comparison with empirical data

The certainty equivalent consumption suggests that the optimal time to buy a residential home is between age 50 and 60. On the other hand, if an individual cares only about the expected wealth at retirement, then purchasing the property early in life is preferred. These two results are robust to varying levels of wealth, labour income, and rental cost, different parameter values, and tax treatment of cash and stock returns. The empirical data in Table 5.28 shows that most Australians purchased their first home before age 45. In the light of the results, this implies that, in reality, people put more emphasis on retirement savings when making the first-home-purchase decision.

Table 5.28. Age distribution (%) of first home buyers with a mortgage from 1995-96 to 2011-12 in Australia.

	1995 | 1996	1996 | 1997	1997 | 1998	1999 | 2000	2000 | 2001	2002 | 2003	2003 | 2004	2005 | 2006	2007 | 2008	2009 | 2010	2011 | 2012
						Proportion of households					
Age of reference person											
15 – 24	9.6	12.6	9.2	9.5	10.5	12.2	10.0	14.7	12.3	11.1	8.8
25 – 34	61.4	56.7	61.5	57.2	65.0	52.9	59.7	53.6	54.4	57.0	58.7
35 – 44	23.4	22.4	22.0	24.8	19.0	25.6	23.2	21.7	26.7	24.0	23.5
45 – 54	*4.2	6.1	5.1	7.5	*4.8	7.6	4.4	7.6	*5.2	7.0	7.0
54 – 64	**0.9	*2.2	*2.1	**1.0	**0.7	*1.7	*2.3	*1.6	**1.2	*0.8	*1.7
65+	**0.5	-	-	-	-	**0.1	**0.4	**0.8	**0.3	*0.1	**0.3
Total	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

* estimate has a relative standard error of 25% to 50% and should be used with caution
** estimate has a relative standard error greater than 50% and is considered too unreliable for general use
Source: Australian Bureau of Statistics (2013b) Table 36 on Page 68

It certainly needs to be acknowledged that the model abstracts from several real world complications that may help to explain the difference between model predictions and empirical data. Ortalo-Magné and Rady (2002) show in a two-period stylised model that an increase in the covariance between income and house price reduces the likelihood of home ownership, and Davidoff (2006) finds empirical evidence to support this finding. I assume a homogeneous housing market, and that all the individuals are exposed to the same level of labour income risk. The resulting correlation between property capital growth rate and labour income growth rate is relatively high. In reality, however, the housing market is heterogeneous and individuals are faced with different levels of labour income risk depending on the nature of their work. Their labour income may not covary with house price as much as what is predicted by the VAR model. Therefore, their optimal time to become a homeowner may be prior to the model prediction. In addition, I assume the transition from renting to owning is irreversible on the basis of the low transition rates in the HILDA survey. This would make purchasing the property early in life unattractive because individuals cannot derive utility from the wealth locked in home equity until age 65. If they were able to sell the property and rent again, they could gain utility from housing wealth before age 65 by selling the property and spending the proceeds on non-housing consumption.

5.8 Conclusions

Residential home insures against rental inflation and is a potential source of retirement income. On the other hand, purchasing a property requires a large amount of liquid assets as down payment, which may constrain the spending on non-durable goods in later years. It also introduces asset price risk when retirees want to unlock home equity. This chapter investigates the optimal time to become a homeowner in a multi-period framework that takes into account various sources of risk, in particular house price risk and rent risk.

The results show that purchasing the property early during working life is likely to result in a higher level of wealth at retirement. Those who become homeowners at a younger age typically have lower mortgage balances and consequently higher home equity values at retirement. They are also able to accumulate more liquid wealth because of higher equity participation rates.

Taking individual's risk aversion and preference for consumption into account, the results show that deferring the purchase to older ages (50 – 60) often leads to a higher lifetime utility level. Purchasing a residential home involves making a significant amount of down payment that would temporarily reduce the consumption in later years. The consumption decline lowers the lifetime utility, and the earlier the decline occurs, the greater the impact on utility. If an investor could defer residential property purchase to older ages, the utility loss due to consumption cut would also be deferred and hence reduced.

Compared to owning a property at retirement, renting throughout working life is relatively unattractive both in terms of retirement savings and lifetime utility levels. Rental cost constrains non-housing consumption and slows down wealth accumulation. For this group of individuals, they either actively choose to rent or are forced to rent because they are unable to afford the down payment. For passive renters, they usually have to pay higher rent because higher house price is often associated with higher rental cost.

Chapter 6

Housing, long-term care insurance, and annuities with recursive utility

6.1 Introduction

Chapter 5 studies the interaction between housing and retirement planning in the pre-retirement phase, focusing on the question of when to purchase a residential property. This chapter moves on to the post-retirement phase, discussing the impact of housing on the demand for two retirement products, namely life annuities and long-term care insurance (LTCI).

I build a multi-period lifecycle model for a single retired homeowner who faces uncertain lifespan, uncertain out-of-pocket health expenditure, and house price risk. Individual preferences are modelled using the Epstein-Zin-Weil-type utility (Epstein and Zin, 1989; Epstein and Zin, 1991; Weil, 1989). The Epstein-Zin model is commonly used in the literature (see e.g. Pang and Warshawsky, 2010; Blake et al., 2014; Yogo, 2016) along with the power utility model. The Epstein-Zin model is preferred over the power utility model for its ability to separately identify the risk aversion and elasticity of intertemporal substitution (EIS). By contrast, the power utility model cannot distinguish the impact of these two factors since the model imposes that one is the inverse of the other. Individuals can choose between an ordinary life annuity and LTCI at the point of retirement. Both products have actuarially fair prices. Home equity will either be bequeathed or liquidated at the point of moving into a long-term care facility. This assumption is based on the empirical evidence that home equity is rarely spent before death except for moving into a nursing home. Davidoff (2009) makes a similar assumption. The probabilities of health state transitions are calibrated to the data from U.S. Health and Retirement Study (HRS), and the other parameters in the lifecycle model take commonly used values in the literature.

The results show that the presence of home equity generally increases the optimal annuitisation rate when retirees have no access to LTCI. Prior literature has shown that bequest motive (Lockwood, 2012) or precautionary savings for healthcare costs (Sinclair and Smetters, 2004; Turra and Mitchell, 2008) can weaken demand for life annuities. For retired homeowners who tend to sell the property at the time of moving into a nursing home, home equity can serve as a bequest and be a form of precautionary savings. As a result, demand for annuities is enhanced in the presence of home equity. When both life annuities and LTCI are available, the presence of home equity can make life annuities more attractive provided that the retiree has sufficient liquid assets. If the amount of liquid assets is low, the spending on purchasing LTCI can impair demand for life annuities. Given retirees tend to liquidate housing wealth in the event of moving to a long-term care facility, home equity typically crowds out demand for LTCI. The sensitivity analysis on preference parameters shows the importance of separately identifying risk aversion and EIS. A higher degree of risk aversion and a lower level of EIS have opposite effects on demand for life annuities and LTCI. Since the power utility model imposes an inverse relationship on risk aversion and EIS, the Epstein-Zin model is more suitable to determine demand for life annuities and LTCI when individuals have various levels of risk aversion and EIS.

The rest of the chapter is organised as follows. Section 6.2 discusses the lifecycle model in detail. Section 6.3 presents the findings from the base case analysis and sensitivity analysis on wealth endowment and preference parameters. Section 6.4 concludes.

6.2 Lifecycle model in retirement

I set up a discrete-time lifecycle model starting at retirement. The model consists of a series of one-year period that is indexed by $t \in \{1, 2, \cdots, T, T+1\}$[1]. The individual retires at $t = 1$ aged 65, and her maximum attainable age is 100, so $T = 36$. All variables are defined in real terms.

6.2.1 Health dynamics and costs

I follow Ameriks et al. (2011) to model the retiree's health status with states '1' (healthy), '2' (mildly disabled), '3' (severely disabled), and '4' (dead). The categorisation of the first three states is based on the number of difficulties in independently performing Activities of Daily Livings (ADLs). There are usually a total of six ADLs: dressing, walking, bathing, eating, transferring and toileting. Mildly disabled state is defined as having 1 – 2 ADL difficulties, and severely disabled state is defined as having 3 – 6 ADL difficulties. The health state at period t is denoted as s_t.

The health state transitions are modelled using a Markov process. Fong et al. (2015) shows a significant proportion of the elderly can recover from disabled state to healthy state. On the other hand, severe disability is usually chronic in nature that substantially reduces the possibility of recovery (Ferri and Olivieri, 2000; Olivieri and Pitacco, 2001). I therefore allow for transition from the mildly disabled state to the healthy state and do not allow for recoveries from the severely disabled state. Figure 6.1 depicts the health state transitions, where σ_{jk} $(j \in \{1, 2, 3\}, k \in \{1, 2, 3, 4\})$ denotes the transition intensity.

[1]Note that the latest possible consumption occurs at $t = T$. The last time index $T + 1$ is for the purpose of bequest only.

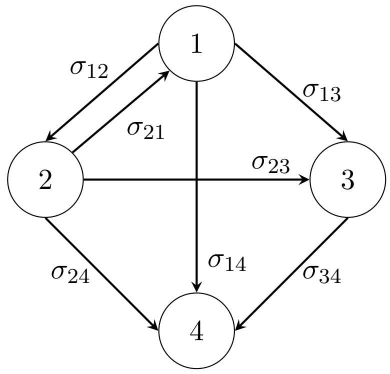


Figure 6.1. Four-state Markov process that models health state transitions.

Given the transition intensities, σ_{jk}, the transition probabilities, $\pi(k|j)$, can be solved for through Kolmogorov equations. In particular, I assume the transition intensities are constant within an integer age. Then the annual transition probabilities for each integer age are given by

$$\begin{pmatrix} \pi(1|1) & \pi(2|1) & \pi(3|1) & \pi(4|1) \\ \pi(1|2) & \pi(2|2) & \pi(3|2) & \pi(4|2) \\ \pi(1|3) & \pi(2|3) & \pi(3|3) & \pi(4|3) \\ \pi(1|4) & \pi(2|4) & \pi(3|4) & \pi(4|4) \end{pmatrix} = \exp\left[\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ 0 & 0 & -\sigma_{34} & \sigma_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}\right],$$

where $\sigma_{11} = -(\sigma_{12} + \sigma_{13} + \sigma_{14})$, $\sigma_{22} = -(\sigma_{21} + \sigma_{23} + \sigma_{24})$, and $\exp[\cdot]$ refers to the matrix exponential.

I follow Ameriks et al. (2011) to model the out-of-pocket health expenditure ($h_t \equiv h(s_t, t)$) as a deterministic process given the health state, s_t. Since the healthcare inflation usually exceeds that of the consumer price index (CPI), it is assumed that the relative price of healthcare increases at a rate of q per annum.

6.2.2 Housing and financial assets

Given that a large majority of retired homeowners have paid off their mortgages, the model assumes the individual lives in a mortgage-free home at retirement. In addition, empirical data shows that housing assets are rarely drawn upon unless the retiree moves to a long-term care facility (see e.g. Venti and Wise, 2004). It is assumed that the retiree will liquidate the house when she becomes severely disabled and subsequently moves to a nursing home. The house has a gross rate of return $R_{H,t+1}$ from time t to time $t+1$, where $\ln(R_{H,t+1})$ follows a normal distribution with mean μ_H and variance σ_H^2. I abstract from the equity market, so the liquid assets are invested in riskless assets only. I assume the risk-free return is constant over time and denote it as R_f.

6.2.3 Retirement products

At retirement, the individual has access to two types of retirement products, life annuities and LTCI, both of which are offered by private companies. The retiree decides the proportion (α) of liquid assets to annuitise and the percentage coverage (λ) of LTCI to purchase. The decisions are made at retirement only. The public offering of similar products is not explicitly considered in the model. Nevertheless, the individual's endowment at retirement can be perceived as including the expected present value of public pension paid during retirement, and the out-of-pocket health expenditure can be seen as net of any publicly funded schemes.

The life annuity is of an ordinary type that provides annual level payment for the remaining lifetime of the annuitant. The payment starts at the beginning of the first period. The annuity is charged at an actuarially fair price. Given an α proportion of liquid assets annuitised at retirement, the annual income from annuity is given by

$$\mathsf{Y} = \frac{\alpha \mathsf{B}}{\sum_{t=1}^{T} R_f^{-(t-1)}{}_{t-1}p_{65,s_1}}, \tag{6.1}$$

where B denotes the initial endowment of liquid assets, ${}_{t-1}p_{65,s_1}$ denotes the probability that a 65-year-old individual with health state s_1 will survive for the next $(t-1)$ years.

The LTCI covers healthcare costs when the policyholder is severely disabled (i.e. health state 3). The premium is assumed to be paid as a lump sum and to exclude any loadings on the product. The actuarially fair premium (P) for a full coverage LTCI policy is given by

$$\mathsf{P} = \sum_{t=2}^{T} R_f^{-(t-1)} \pi(s_t = 3|s_1) h(s_t = 3, t). \tag{6.2}$$

6.2.4 Budget constraints and wealth dynamics

In the first period, the retiree is endowed with liquid wealth of B and housing wealth of $\mathsf{W^H}$, and the retiree is in the healthy state (i.e. health state 1). She then decides the proportion of liquid assets to annuitise and the LTCI coverage to purchase. After that, she receives income from annuity (if any), incurs the healthcare cost, and decides how much to consume. Let B_1 denote the amount of liquid assets available after purchasing the retirement products. It is given by

$$B_1 = (1-\alpha)\mathsf{B} - \lambda\mathsf{P},\ B_1 \geq 0. \tag{6.3}$$

Starting from the second period, the retiree enters the period t with health state s_t and wealth W_t, which consists of housing wealth W_t^H and liquid wealth B_t. Note that W_t, W_t^H, and B_t denote the amount available at the beginning of the period t (i.e. before any action is taken) except for B_1, which is specified otherwise in Equation (6.3). The timing of events is as follows.

1. If $s_t = 4$, the individual is deceased, so the wealth W_t is bequeathed.
2. If $s_t < 4$, one of the following events will occur.
   (a) If $s_t = 3$ and $s_{t-1} \in \{1, 2\}$, the individual will liquidate the home equity and move into a residential care facility.
   (b) If $s_t = 3$ and $s_{t-1} = 3$, the individual will remain staying at the residential care.
   (c) If $s_t < 3$, the individual will remain living at home.

3. If $s_t < 4$, the health costs (net of any LTCI coverage if s_t = 3), are incurred; annuity income, if any, is received; and then a consumption decision (C_t) is made. The remaining liquid assets earn a risk-free return R_f.

The chosen consumption level must not fall below the consumption floor C^f to ensure a minimum standard of living. If the individual's budget cannot support the minimum consumption level, I assume the government will provide subsidy to increase the consumption level to C^f, and that the liquid wealth in the next period will be zero.

The budget constraint for liquid assets B is given by

$$\begin{aligned} & B_2 = \left(B_1 + \mathsf{Y} - h_1 - C_1\right)^+ R_f; \\ & \text{for } t \in \{2, 3, \cdots, T\}, \\ & B_{t+1} = \begin{cases} \left(B_t + \mathsf{Y} - h_t - C_t\right)^+ R_f & \text{if } s_t \in \{1, 2\} \\ \left(B_t + \mathsf{Y} + W_t^H \mathbb{1}_{\{s_{t-1} \in \{1,2\}\}} - (1 - \lambda) h_t - C_t\right)^+ R_f & \text{if } s_t = 3 \end{cases}, \end{aligned} \tag{6.4}$$

where $(\cdot)^+$ is defined as $\max(\cdot, 0)$.

The budget constraint for total wealth W is given by

$$\begin{aligned} & W_2 = B_2 + W_1^H R_{H,2}, \quad \text{where } W_1^H = \mathsf{W}^{\mathsf{H}}; \\ & \text{for } t \in \{2, 3, \cdots, T\}, \\ & W_{t+1} = \begin{cases} B_{t+1} + W_t^H R_{H,t+1} & \text{if } s_t \in \{1, 2\} \\ B_{t+1} & \text{if } s_t = 3 \end{cases}. \end{aligned} \tag{6.5}$$

6.2.5 Preferences

Individuals in the model are assumed to have Epstein-Zin-Weil-type preferences (Epstein and Zin, 1989; Epstein and Zin, 1991; Weil, 1989) over non-housing consumption and a bequest. The housing service consumption is not directly included in the utility function. The housing wealth contributes to the utility through bequest or liquidation of housing that alleviates the budget constraint caused by excessive medical care costs.

The Epstein-Zin model generalises the power utility model in that it can separately identify the risk aversion and EIS. The two elements are intrinsically different. Risk aversion describes an individual's willingness to substitute consumption across different states of the world, whereas EIS describes an individual's willingness to substitute consumption over time. When the individual's EIS (ψ) is the reciprocal of the coefficient of relative risk aversion (γ), the Epstein-Zin model reduces to the power utility model.

The preferences are specified by

$$
\begin{aligned}
V_t &\equiv V(B_t, W_t^H, s_t, t) \\
&= \max_{O_t} \Bigg\{ (1-\beta) C_t^{1-\rho} + \beta \Bigg[\mathbb{E}_t \Big[\sum_{k \neq 4} \pi(s_{t+1} = k | s_t) V(B_{t+1}, W_{t+1}^H, s_{t+1} = k, t+1)^{1-\gamma} \\
&\qquad\qquad + \pi(s_{t+1} = 4 | s_t) b^{\gamma} W_{t+1}^{1-\gamma} \Big] \Bigg]^{\frac{1}{\theta}} \Bigg\}^{\frac{1}{1-\rho}}, \quad \theta = \frac{1-\gamma}{1-\rho}; \\
O_t &= \begin{cases} \{\lambda, \alpha, C_t\}, & \text{for } t = 1; \\ \{C_t\}, & \text{for } t = 2, \cdots, T. \end{cases}
\end{aligned}
\tag{6.6}
$$

The notation V_t is the indirect utility value at time t, β the subjective discount factor, ρ the inverse of EIS (i.e. $\rho = 1/\psi$), $\mathbb{E}$ the expectation operator, b the strength of bequest motive. The subjective discount factor (β) measures an individual's impatience to defer consumption. It takes values between zero and one, with a lower value meaning less willingness to postpone the consumption. The strength of bequest motive (b) takes non-negative values, with a higher value meaning a stronger bequest motive.

6.2.6 Optimisation problem and solution method

Individuals optimise over consumption, annuitisation rate, and insurance coverage to maximise the expected lifetime utility in (6.6), subject to conditions (6.1) to (6.5). I set up grid points on liquid wealth, housing wealth, and current health state to solve the optimisation problem. The method of endogenous grid points (Carroll, 2006) is used to set up the grid points for the liquid assets. The grid points on housing wealth are given exogenously. The lognormal distribution of house price growth is discretised by Gauss-Hermite quadrature. Given that the annuitisation and LTCI coverage decisions have been made in the first period, the optimal consumption in period t is given by

$$\begin{aligned} C_t^* = \Bigg\{\beta R_f \Bigg\{\mathbb{E}_t\Bigg[\sum_{k\neq 4}\pi(s_{t+1}=k|s_t)V_{t+1}^{1-\gamma} + \pi(s_{t+1}=4|s_t)b^\gamma W_{t+1}^{1-\gamma}\Bigg]\Bigg\}^{\frac{1}{\theta}-1} \\ \times\,\mathbb{E}_t\Bigg[\sum_{k\neq 4}\pi(s_{t+1}=k|s_t)V_{t+1}^{\rho-\gamma}C_{t+1}^{-\rho} + \pi(s_{t+1}=4|s_t)\frac{b^\gamma}{1-\beta}W_{t+1}^{-\gamma}\Bigg]\Bigg\}^{-\frac{1}{\rho}}. \end{aligned} \tag{6.7}$$

Proof. The proof shown below builds on the derivations in Chapter 6 of Munk (2013) who solves the optimal consumption problem for an individual with no bequest motive or health risk.

The first-order condition of Equation (6.6) for C_t implies that

$$\begin{aligned} (1-\beta)C_t^{-\rho} = \beta\left\{\mathbb{E}_t\left[\sum_{k\neq 4}\pi(s_{t+1}=k|s_t)V_{t+1}^{1-\gamma} + \pi(s_{t+1}=4|s_t)b^\gamma W_{t+1}^{1-\gamma}\right]\right\}^{\frac{1}{\theta}-1} \\ \times\, R_f\mathbb{E}_t\left[\sum_{k\neq 4}\pi(s_{t+1}=k|s_t)V_{t+1}^{-\gamma}\frac{\partial V_{t+1}}{\partial B_{t+1}} + \pi(s_{t+1}=4|s_t)b^\gamma W_{t+1}^{-\gamma}\right], \end{aligned} \tag{6.8}$$

where $\partial V_{t+1}/\partial B_{t+1}$ can be derived by taking the derivative on the Equation (6.6). For the optimal decision, the equation holds without the maximum, that is

$$\begin{aligned} V_t &\equiv V(B_t, W_t^H, s_t, t) \\ &= \Bigg\{(1-\beta)\left(C_t^*\right)^{1-\rho} + \beta\Bigg[\mathbb{E}_t\Bigg[\sum_{k\neq 4}\pi(s_{t+1}=k|s_t)V(B_{t+1}^*, W_{t+1}^H, s_{t+1}=k, t+1)^{1-\gamma} \\ &\qquad + \pi(s_{t+1}=4|s_t)b^\gamma\left(W_{t+1}^*\right)^{1-\gamma}\Bigg]\Bigg]^{\frac{1}{\theta}}\Bigg\}^{\frac{1}{1-\rho}}, \end{aligned} \tag{6.9}$$

where B_{t+1}^* and W_{t+1}^* denote the next period liquid assets and total wealth, respectively, under the optimal consumption in period t.

Take the derivative of Equation (6.9) w.r.t. B_t

$$
\begin{aligned}
\frac{\partial V_t}{\partial B_t} = V_t^{\rho} \Bigg\{ & (1-\beta)(C_t^*)^{-\rho} \frac{\partial C_t^*}{\partial B_t} \\
& + \beta \left[\mathbb{E}_t \left(\sum_{k \neq 4} \pi(s_{t+1} = k | s_t) V_{t+1}^{1-\gamma} + \pi(s_{t+1} = 4 | s_t) b^{\gamma} (W_{t+1}^*)^{1-\gamma} \right) \right]^{\frac{1}{\theta}-1} \\
& \times \mathbb{E}_t \left[\sum_{k \neq 4} \pi(s_{t+1} = k | s_t) V_{t+1}^{-\gamma} \frac{\partial V_{t+1}}{\partial B_{t+1}^*} \frac{\partial B_{t+1}^*}{\partial B_t} + \pi(s_{t+1} = 4 | s_t) b^{\gamma} (W_{t+1}^*)^{-\gamma} \frac{\partial W_{t+1}^*}{\partial B_t} \right] \Bigg\},
\end{aligned}
\tag{6.10}
$$

where $\partial B_{t+1}^* / \partial B_t$ and $\partial W_{t+1}^* / \partial B_t$ can be derived from the budget constraints (6.4) and (6.5)

$$
\begin{aligned}
\frac{\partial B_{t+1}^*}{\partial B_t} &= \left(1 - \frac{\partial C_t^*}{\partial B_t}\right) R_f, \\
\frac{\partial W_{t+1}^*}{\partial B_t} &= \frac{\partial W_{t+1}^*}{\partial B_{t+1}^*} \frac{\partial B_{t+1}^*}{\partial B_t} = \frac{\partial B_{t+1}^*}{\partial B_t} = \left(1 - \frac{\partial C_t^*}{\partial B_t}\right) R_f.
\end{aligned}
\tag{6.11}
$$

Substitute the Equation (6.11) into Equation (6.10) and then use the first-order condition (6.8)

$$
\begin{aligned}
\frac{\partial V_t}{\partial B_t} = V_t^{\rho} \beta R_f & \left[\mathbb{E}_t \left(\sum_{k \neq 4} \pi(s_{t+1} = k | s_t) V_{t+1}^{1-\gamma} + \pi(s_{t+1} = 4 | s_t) b^{\gamma} (W_{t+1}^*)^{1-\gamma} \right) \right]^{\frac{1}{\theta}-1} \\
& \times \mathbb{E}_t \left[\sum_{k \neq 4} \pi(s_{t+1} = k | s_t) V_{t+1}^{-\gamma} \frac{\partial V_{t+1}}{\partial B_{t+1}^*} + \pi(s_{t+1} = 4 | s_t) b^{\gamma} (W_{t+1}^*)^{-\gamma} \right].
\end{aligned}
\tag{6.12}
$$

Consequently, the first-order condition for C_t can be re-written as

$$
\frac{\partial V_t}{\partial B_t} = (1-\beta) V_t^{\rho} C_t^{-\rho}. \tag{6.13}
$$

This is the envelope condition for the preferences defined in Equation (6.6).

Substitute the envelope condition (6.13) into Equation (6.8). The first-order condition for C_t can be re-stated as

$$(1-\beta)C_t^{-\rho} = \beta \left\{ \mathbb{E}_t \left[\sum_{k \neq 4} \pi(s_{t+1} = k|s_t) V_{t+1}^{1-\gamma} + \pi(s_{t+1} = 4|s_t) b^\gamma W_{t+1}^{1-\gamma} \right] \right\}^{\frac{1}{\theta}-1}$$
$$\times R_f \mathbb{E}_t \left[(1-\beta) \sum_{k \neq 4} \pi(s_{t+1} = k|s_t) V_{t+1}^{\rho-\gamma} C_{t+1}^{-\rho} + \pi(s_{t+1} = 4|s_t) b^\gamma W_{t+1}^{-\gamma} \right]. \tag{6.14}$$

Therefore, the optimal consumption in period t is given by

$$C_t^* = \left\{ \beta R_f \left\{ \mathbb{E}_t \left[\sum_{k \neq 4} \pi(s_{t+1} = k|s_t) V_{t+1}^{1-\gamma} + \pi(s_{t+1} = 4|s_t) b^\gamma W_{t+1}^{1-\gamma} \right] \right\}^{\frac{1}{\theta}-1} \right.$$
$$\left. \times \mathbb{E}_t \left[\sum_{k \neq 4} \pi(s_{t+1} = k|s_t) V_{t+1}^{\rho-\gamma} C_{t+1}^{-\rho} + \pi(s_{t+1} = 4|s_t) \frac{b^\gamma}{1-\beta} W_{t+1}^{-\gamma} \right] \right\}^{-\frac{1}{\rho}}. \tag{6.7 revisited}$$

□

Remark 6.2.1. In the terminal period, $\pi(s_{T+1} = k|s_T) = 0$ for $k \in \{1, 2, 3\}$ and $\pi(s_{T+1} = 4|s_T) = 1$, so the optimal consumption in period T becomes

$$C_T^* = \left(\frac{1}{\beta} - 1\right)^{\frac{1}{\rho}} \left\{ \mathbb{E}_T \left[b^\gamma W_{T+1}^{1-\gamma} \right] \right\}^{\frac{1}{\rho} - \frac{1}{\rho\theta}} \times \left\{ R_f \mathbb{E}_T \left[b^\gamma W_{T+1}^{-\gamma} \right] \right\}^{-\frac{1}{\rho}}. \tag{6.15}$$

The optimisation problem is solved backward, starting from the last period. For the points not lying on the grid, a hybrid interpolation method introduced in Ludwig and Schön (2016) is used to find the optimal consumption and the indirect utility value. Figure 6.2, adapted from Figure 5 of Ludwig and Schön (2016), illustrates the method. Both the before-consumption liquid wealth[2] and housing wealth are used to construct the grid points, denoted by $G^{\cdot,\cdot}$ in the figure. Given the before-consumption wealth w_{t+1} and housing wealth w_{t+1}^H, the following procedures are employed to find the interpolated value.

[2]Before-consumption liquid assets refer to the assets that are ready to be consumed, i.e. after any annuity income, any housing liquidation, and medical expenditure net of LTCI coverage.

First, locate the two rows $G^{\cdot,j}$ and $G^{\cdot,j+1}$ in the exogenous dimension (which is housing wealth) that form the most narrow bracket of w_{t+1}^H. Compute the weights (z and $1 - z$) based on the relative distance to the two rows of grid points. Second, perform linear interpolation in each of the two rows given the before-consumption liquid assets, w_{t+1}. Finally, the interpolated value is the weighted average of the interpolated values from each row, using the weights found in the first step.

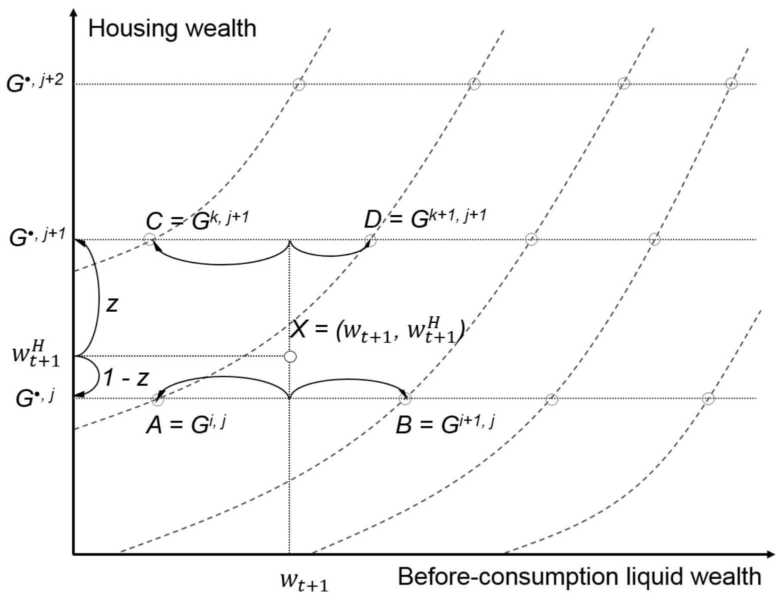


Figure 6.2. Illustration of the hybrid interpolation method used in the backward induction to solve the lifecycle model.

The optimal annuitisation rate and LTCI coverage are solved in the first period using the following steps. First set up the grid points on annuitisation rate and LTCI coverage. On each grid point, solve the optimal consumption and indirect utility levels backwards from the last period to the first period. Given the initial liquid wealth and housing wealth, the indirect utility value in the first period for a healthy individual can be found through the hybrid interpolation method. The optimal annuitisation rate and LTCI coverage are found by searching for the grid point that gives the highest value of indirect utility.

6.2.7 Model parameterisation

6.2.7.1 Health dynamics

The health state transition is estimated using the data from U.S. Health Retirement Study (HRS). HRS surveys a nationally representative sample of Americans over age 50 every two years, starting from 1992. The data from 1998 to 2010 is used due to inconsistent question structure before 1998. The data for female is chosen to calculate the crude transition rates, which are then graduated using Poisson generalised linear model (GLM) (Fong et al., 2015). I choose female data since they face greater challenges in retirement planning. Females have longer life expectancy than males, and they tend to spend more years in disabled state (Fong et al., 2015).

The estimation procedure begins with counting number of transitions and exposure years for each integer age between 50 and 100. The aggregate results in five-year interval are shown in Tables 6.1 and 6.2. The crude transition rates are then graduated using a GLM with the log link function. In particular, the number of transitions at age x is assumed to follow a Poisson distribution with mean (m_x) defined as a polynomial function of age with degree K

$$m_x = e_x \sum_{k=0}^{\mathsf{K}} \eta_k x^k, \tag{6.16}$$

where e_x is the central exposure to risk for x-year-old individuals, η_k the coefficients of the polynomial. The degree of polynomial is selected based on Akaike information criterion corrected for sample size (AICc), Bayesian information criterion (BIC), and the likelihood ratio test. Table 6.3 shows that the three selection criteria give almost identical results of the chosen degree of polynomial for each set of nested models.

Table 6.1. Number of transitions between different health states.

	$1 \to 2$	$1 \to 3$	$1 \to 4$	$2 \to 1$	$2 \to 3$	$2 \to 4$	$3 \to 4$
50 – 54	67	21	8	52	13	2	4
55 – 59	280	40	55	212	69	27	16
60 – 64	458	74	114	436	129	37	36
65 – 69	553	112	193	474	147	86	79
70 – 74	575	107	226	441	178	97	86
75 – 79	579	144	257	349	157	116	171
80 – 84	570	162	315	338	190	166	242
85 – 89	445	172	302	235	211	212	312
90 – 94	218	92	160	86	156	172	296
95 – 100	52	24	51	18	76	75	174
Total	3,797	948	1,681	2,641	1,326	990	1,416

Note: '1' is healthy state, '2' mildly disabled state, '3' severely disabled state, '4' dead state.

Table 6.2. Number of exposure years in different health states.

	Healthy	Mildly disabled	Severely disabled
50 – 54	4,527.18	361.92	121.51
55 – 59	10,816.97	1,136.76	387.61
60 – 64	15,721.89	1,811.16	692.93
65 – 69	16,610.65	2,146.23	802.31
70 – 74	13,975.53	2,079.22	948.19
75 – 79	10,807.98	2,164.77	1,071.76
80 – 84	7,512.86	2,131.81	1,242.44
85 – 89	3,870.87	1,826.11	1,457.01
90 – 94	1,235.42	965.27	1,006.33
95 – 100	235.92	265.35	421.37
Total	85,315.27	14,888.60	8,151.45

Table 6.3. Model selection of the Poisson GLM. The chosen degree of polynomial value is in bold for each set of nested models.

K	AICc	BIC	D_c	ΔD_c
Disability				
σ_{12}: healthy to mildly disabled				
1	334.84	337.96	87.51	
2	304.56	**309.05**	**54.90**	**32.62*****
3	**303.87**	309.61	51.74	3.16*
σ_{13}: healthy to severely disabled				
1	260.49	263.60	64.61	
2	247.74	**252.23**	**49.53**	**15.08*****
3	**246.66**	252.40	45.99	3.54*
σ_{23}: mildly disabled to severely disabled				
1	316.44	319.55	100.70	
2	279.25	**283.74**	**61.17**	**39.52*****
3	**279.14**	284.88	58.60	2.57
Recovery				
σ_{21}: mildly disabled to healthy				
1	301.16	304.27	73.30	
2	**292.57**	**297.06**	**62.38**	**10.92*****
3	294.97	300.72	62.32	0.06
Mortality				
σ_{14}: healthy to dead				
1	272.53	275.64	51.01	
2	**265.01**	**269.50**	**41.16**	**9.85*****
3	267.02	272.77	40.71	0.45
σ_{24}: mildly disabled to dead				
1	246.79	249.90	45.02	
2	**243.68**	**248.18**	**39.58**	**5.44****
3	244.11	249.85	37.54	2.04
σ_{34}: *severely disabled to dead*				
1	**245.02**	**248.13**	**29.59**	
2	247.35	251.85	29.58	0.00
3	247.45	253.20	27.22	2.36

Note: D_c is the residual deviance statistics. ΔD_c denotes the test statistics for the likelihood ratio test. * is for statistic that is significant at the 10% level, ** at the 5% level, *** at the 1% level.

6.2.7.2 Other parameters

The other parameters used in the numerical simulation take the commonly used values in the literature. They are displayed in Table 6.4. The sources of the parameters, unless otherwise specified, are listed in the brackets.

Table 6.4. The parameter values in the lifecycle model used for the base case.

Parameter	Explanation	Value
Asset returns (Yogo, 2016)		
R_f	Risk free rate	1.025
μ_H	Parameters of the lognormal distribution	0.34%
σ_H^2	of house price growth	3.5%
Consumption floor (Ameriks et al., 2011)		
C^f	Floor for healthy and mildly disabled states	\$4,630
	Floor for severely disabled states	\$5,640
Health expenditure (Ameriks et al., 2011)		
$h(s_1, 1)$	Initial cost for healthy state	\$1,000
$h(s_2, 1)$	Initial cost for mildly disabled state	\$10,000
$h(s_3, 1)$	Initial cost for severely disabled state	\$50,000
$q^{\dagger}$	Health expenditure inflation in excess of CPI inflation	1.90%
Preference (Pang and Warshawsky, 2010)		
b	Strength of bequest motive	2
β	Subjective discount factor	0.96
γ	Coefficient of relative risk aversion	5
ψ	Elasticity of intertemporal substitution	0.5

† *Source*: Yogo (2016).

6.3 Results

6.3.1 Base case analysis

In the base case analysis the individual is endowed with \$220,000 liquid wealth and \$280,000 housing wealth at retirement. The \$220,000 liquid wealth is based on the median level of total wealth (consisting of pre-annuitised wealth and liquid financial wealth) for a single woman U.S. household in the HRS estimated by Peijnenburg et al. (2015). The \$280,000 housing wealth leads to a home-equity-to-all-assets ratio of 0.56, which is consistent with the median ratio among homeowners estimated by Davidoff (2009). The individual is healthy at retirement. Based on the estimated health transition probabilities, the annuity costs about \$14.89 for \$1 annual income, and the full coverage of LTCI costs \$94,752.31.

After solving for the optimal decision rules defined on the state space, the time-series profiles of retiree's optimal consumption can be obtained through simulation. Specifically, I first simulate house price growths and health states, and then use the optimal policy rules to calculate the optimal consumption. The simulation is run for 200,000 times. Figure 6.3 shows the simulated housing wealth values in the absence of liquidation. Should the retiree fall into the severely disabled state, the amount of cash from liquidating housing asset alone can support the health expenditure for several years.

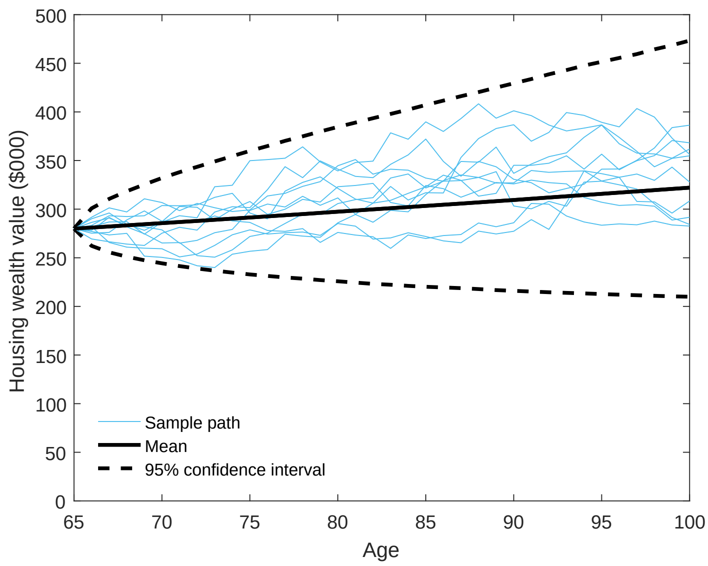


Figure 6.3. Simulated housing asset values in the absence of liquidation. The individual is endowed with \$280,000 housing asset at retirement.

Figure 6.4 shows the survival curve and the simulated proportions of survivors in each health state for individuals who are healthy at retirement. The estimated health transition probabilities predict that a 65-year-old healthy female has about 50% chance of living beyond age 85, and that the probability of becoming severely disabled increases exponentially after age 85. Table 6.5 summarises the number of years spent in each health state and the age of entering into each health state. Conditional upon becoming severely disabled, the average age of occurrence is around 82. The remaining life expectancy after becoming severely disabled is about two years.

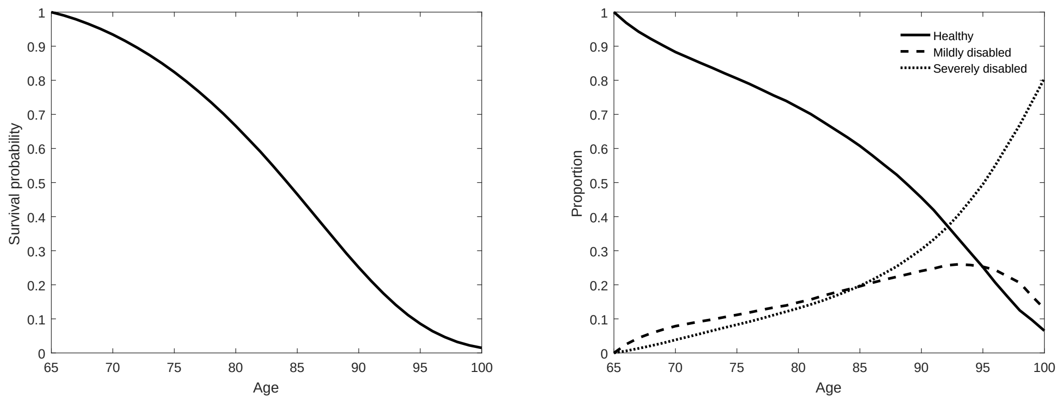


Figure 6.4. (Left Panel) Survival curve and (Right Panel) simulated proportions of survivors in each health state. Individuals are healthy at retirement.

Table 6.5. Number of years spent in each health state and age of entering into each health state conditional upon occurrence: mean, standard deviation (Std), and 95% confidence interval (CI).

Health state	Duration			Starting age		
	Mean	Std	95% CI	Mean	Std	95% CI
Healthy	14.9	7.5	(2, 29)	65	0	(65, 65)
Mildly disabled	2.3	3.3	(0, 11)	76.9	7.3	(66, 92)
Severely disabled	2.1	3.8	(0, 13)	81.8	8.2	(67, 96)

Table 6.6 shows the optimal annuitisation rate and the optimal LTCI coverage for the base case. For the purpose of comparison, the optimal product choices in the absence of housing wealth are also displayed. When annuities alone are available in the market, illiquid housing wealth significantly enhances the demand for annuities. The increased annuitisation rate is related to the dual role of housing wealth in the model. A large proportion of precautionary savings for healthcare costs are held in the form of home equity. If the wealth locked in the home equity is not released, it will be bequeathed to fulfil the bequest motive. Prior research has found that the need for liquidity to cover sizeable health expenditure (Sinclair and Smetters, 2004; Turra and Mitchell, 2008) and bequest motive (Lockwood, 2012) tend to limit demand for annuities. The presence of

home equity therefore lowers the barrier to annuitisation. When LTCI alone is available in the market, illiquid housing wealth reduces demand for LTCI regardless of whether life annuities are available. This confirms the role of home equity as insurance against healthcare costs.

Table 6.6. Optimal annuitisation rate as a proportion of liquid wealth (% Liquid) and as a proportion of total wealth (% Total), and optimal LTCI coverage (LTCI only or LTCI) for the base case.

Wealth ($000)		Single product			Both products		
		Annuity only		LTCI	Annuity		LTCI
Liquid	Housing	% Liquid	% Total	only	% Liquid	% Total	
500	0	0.30	0.30	0.93	0.71	0.71	0.92
220	280	0.94	0.41	0.89	0.65	0.29	0.81

When both products are accessible and retirees have no illiquid home equity, Table 6.6 shows that LTCI significantly increases demand for life annuities because the insurance reduces the need to hold precautionary savings against uncertain healthcare costs. This result is in line with the prior research showing that including elements of LTCI to annuities can enhance the demand for standard life-contingent annuities (see e.g. Ameriks et al., 2008; Wu et al., 2016). When retirees have a significant proportion of wealth locked in illiquid home equity, however, LTCI reduces the optimal annuitisation rate. As a result, it seems that illiquid home equity reduces demand for life annuities when LTCI is also available in the market. In fact, as later to be examined in the sensitivity analysis, whether or not illiquid housing wealth reduces demand for annuities depends on the amount of liquid wealth. In the base case, the amount of liquid wealth available ($220,000) is relatively low, and retirees find it optimal to purchase a substantial coverage of LTCI (which costs about $76,749.37, or 35% of liquid wealth). Therefore the optimal proportion of liquid wealth to be annuitised is reduced.

Figure 6.5 shows the simulated average consumption in four different cases: *1*) no access to either LTCI or life annuities; *2*) access to life annuities only; *3*) access to LTCI only; *4*) access to both LTCI and life annuities. The consumption excludes the healthcare costs.

As discussed in Figure 6.4, the likelihood of becoming severely disabled grows exponentially after age 85. The severely disabled state is associated with expensive healthcare costs which can constrain the consumption if LTCI is not accessible to retirees. On the other hand, purchasing LTCI involves a lump sum payment at retirement, which can reduce the consumption at early retirement. As a result of these two factors, Figure 6.5 shows two intersections at around age 85. Compare individuals with no access to either product (dotted line) to those with access to LTCI only (dash-dot line). The former group, on average, consumes more before 85 and consumes substantially less afterwards. The comparison between the rest two groups shows a similar pattern.

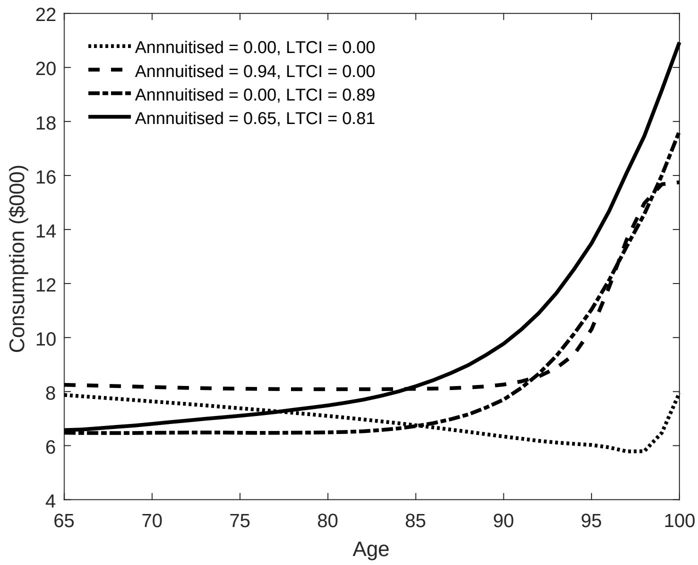


Figure 6.5. Simulated average consumption (excluding healthcare costs) paths at different annuitisation rates and LTCI coverage.

Figure 6.5 also shows a contrasting feature of consumption when no products are available and when at least one product is accessible to retirees. When no products are available, the average consumption shows a downward trend before increasing slightly after age 97. By contrast, when at least one product is accessible, the average consumption remains relatively flat before increasing significantly at early 90s. The home equity liquidation does not materially enhance the average consumption because of the excessive healthcare costs. Figure 6.6 compares the average consumption with and without healthcare costs assuming no products are available in the market. In the absence of healthcare costs, the average consumption increases substantially after around age 90 due to home equity liquidation. Figure 6.5 shows that life annuities alone can also improve the consumption at late retirement. This is due to the mortality premium, which is higher at more advanced ages.

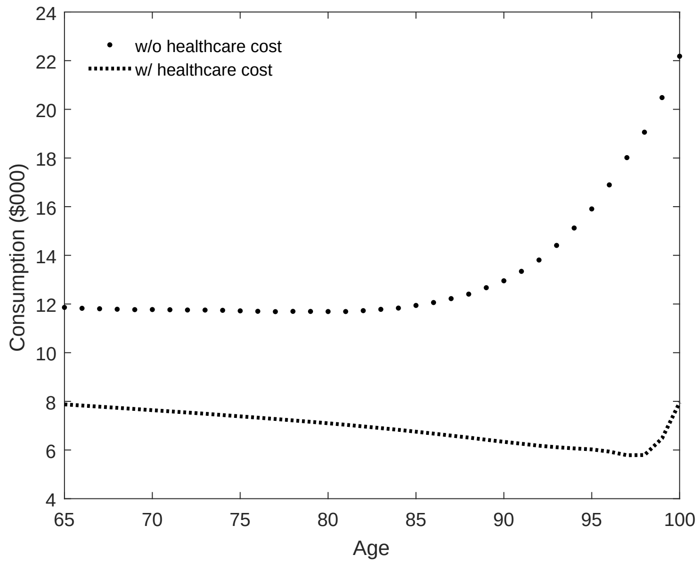


Figure 6.6. Simulated average consumption (excluding healthcare costs) paths with and without healthcare cost. Neither life annuity nor LTCI is available in the market.

Figure 6.7 shows the average liquid asset paths at different annuitisation rates and LTCI coverage. Compare the two cases where only the life annuity is available (dashed line) and where both products are available (solid line). Individuals, on average, accumulate more liquid assets in the former case for the purpose of precautionary savings to cover healthcare expenditure. In addition, the liquid assets tend to decrease at a faster rate when neither product is available (dotted line) compared to the case where only LTCI is available (dash-dot line) because the healthcare cost in the severely disabled state is partially covered by the insurance in the latter case.

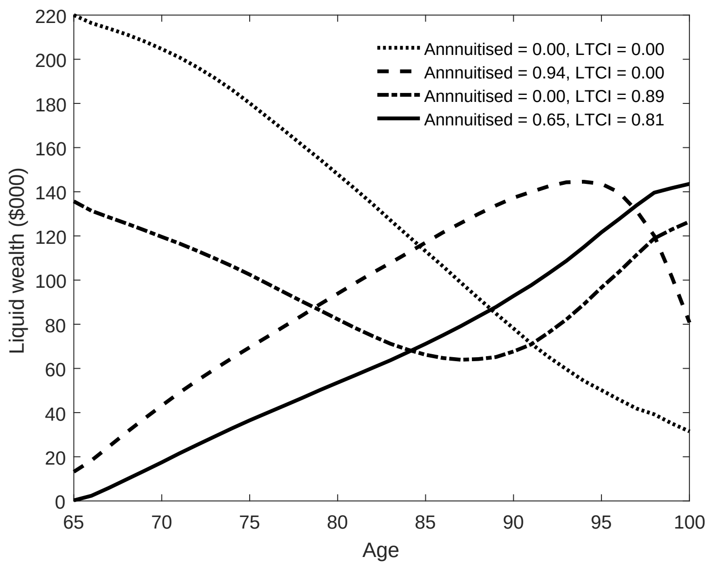


Figure 6.7. Simulated average liquid wealth paths at different annuitisation rates and LTCI coverage.

6.3.2 Sensitivity analysis: Preference parameters

This section performs sensitivity analysis on the values of the parameters that determine an individual's preference. The optimal product choices are shown in Table 6.7. Overall the optimal annuitisation rate and LTCI coverage are relatively robust when both products are accessible. When life annuities alone are available, the optimal annuitisation rate is sensitive to different sets of parameters.

The coefficient of relative risk aversion reflects an individual's attitude towards risk. A lower value means the individual is more risk tolerant, hence requiring less LTCI coverage. The EIS reflects an individual's willingness to substitute consumption over time. A higher value means the individual is less concerned about consumption smoothing year after year, and relatively more concerned about insuring against health risk. A higher value of EIS therefore leads to a stronger demand for the LTCI and weaker demand for life annuity.

The sensitivity analysis on γ and ψ highlights the advantage of the Epstein-Zin model over the power utility model in separating the risk aversion and EIS. The power utility

Table 6.7. Optimal annuitisation rate as a proportion of liquid wealth (% Liquid) and as a proportion of total wealth (% Total), and optimal LTCI coverage (LTCI only or LTCI) for different values of preference parameters.

	Single product			Both products		
	Annuity only		LTCI	Annuity		LTCI
	% Liquid	% Total	only	% Liquid	% Total	
Base case	0.94	0.41	0.89	0.65	0.29	0.81
Coefficient of relative risk aversion						
$\gamma = 2^{\dagger}$	1.00	0.44	0.85	0.68	0.30	0.74
$\gamma = 10$	0.27	0.12	0.91	0.64	0.28	0.83
Elasticity of intertemporal substitution						
$\psi = 0.2^{\dagger}$	1.00	0.44	0.85	0.67	0.29	0.76
$\psi = 0.7$	0.91	0.40	0.95	0.63	0.28	0.85
Strength of bequest motive						
$b = 1$	1.00	0.44	0.78	0.68	0.30	0.74
$b = 4$	0.20	0.41	0.94	0.63	0.28	0.85
Subjective discount factor						
$\beta = 0.93$	0.59	0.26	0.75	0.49	0.22	0.72
$\beta = 0.99$	1.00	0.44	0.96	0.63	0.28	0.85

† When $\gamma = 2$ or $\psi = 0.2$, the Epstein-Zin model defined in Equation (6.6) reduces to the power utility model.

model imposes that the coefficient of relative risk aversion is the inverse of the EIS, so a higher degree of risk aversion inevitably leads to a lower level of EIS. Table 6.7 shows that a higher degree of risk aversion and a lower level of EIS have opposite effects on the optimal LTCI coverage and the optimal annuitisation rate. The power utility model is therefore inadequate in determining the demand for annuities and LTCI when the individual's risk aversion does not coincide with the inverse of her EIS.

Purchasing LTCI transfers the wealth from early retirement to late retirement and to estates. Consequently, a stronger bequest motive leads to a higher LTCI coverage. A lower subjective discount factor means the individual is less willing to postpone the consumption. Since purchasing LTCI reduces the consumption at early retirement, a lower subjective discount factor reduces demand for LTCI.

6.3.3 Sensitivity analysis: Wealth endowment

The section investigates the impact of wealth endowment and its composition on the optimal product choice. The wealth endowment doubles and halves compared to the base case, and the ratio of home equity to total wealth varies from less than 30% to over 80% to capture a wide range of household portfolio compositions. Table 6.8 shows the optimal annuitisation rate and LTCI coverage in each scenario. For comparison purposes, the case without illiquid home equity in each wealth level is also shown in the table.

Housing wealth generally crowds out the demand for the LTCI except when the total wealth level is too low, e.g. $250,000. Compare the optimal LTCI coverage between the scenarios with and without home equity when the total wealth is $500,000 or $1,000,000. The presence of illiquid home equity reduces the optimal insurance coverage regardless of the availability of annuities. The only exception is when retirees are endowed with $500,000 total wealth and they can only purchase LTCI. The optimal insurance coverage increases slightly from 93% to 97% when the endowment includes $140,000 housing wealth. The small increment is related to the fact that the liquid wealth left after purchasing

Table 6.8. Optimal annuitisation rate as a proportion of liquid wealth (% Liquid) and as a proportion of total wealth (% Total), and optimal LTCI coverage (LTCI only or LTCI) for different wealth endowments.

Wealth ($000)			Single product			Both products		
			Annuity only		LTCI	Annuity		LTCI
Total	Liquid	Housing	% Liquid	% Total	only	% Liquid	% Total	
250	250	0	0.26	0.26	0.00	0.50	0.50	0.98
250	180	70	0.35	0.25	1.00	0.48	0.35	0.98
250	110	140	0.00	0.00	1.00	0.13	0.06	1.00
250	40	210	0.00	0.00	0.42	0.00	0.00	0.42
500	500	0	0.30	0.30	0.93	0.71	0.71	0.92
500	360	140	0.54	0.39	0.97	0.78	0.56	0.83
500	220	280	0.94	0.41	0.89	0.65	0.29	0.81
500	80	420	1.00	0.16	0.84	0.00	0.00	0.84
1,000	1,000	0	1.00	1.00	1.00	0.73	0.73	0.96
1,000	720	280	1.00	0.72	0.82	0.89	0.64	0.83
1,000	440	560	1.00	0.44	0.60	0.88	0.39	0.55
1,000	160	840	1.00	0.16	0.30	0.84	0.13	0.27

LTCI can also hedge against uncertain healthcare costs. When housing wealth endowment is relatively low, its hedging effectiveness is marginally inferior to that of liquid wealth. Retirees therefore want to purchase more LTCI coverage. Compare the optimal LTCI coverage among the scenarios with home equity when the total wealth is $500,000 or $1,000,000. It decreases as home equity endowment increases, except when the retiree is extremely cash poor and asset rich (e.g. endowed with $80,000 liquid wealth out of $500,000 total wealth).

When individuals are endowed with a relatively low level of total wealth, i.e. $250,000, they purchase nearly the full LTCI coverage subject to their budget constraints[3]. The government subsidy that guarantees a minimum level of consumption plays a role in the high take-up of insurance. LTCI primarily severs to transfer the consumption from healthy state to severely disabled state. When the amount of liquid wealth is low, a higher LTCI coverage has no material impact on the consumption levels in the healthy state as

[3]When individuals are endowed with $40,000 liquid wealth, 42% is the maximum LTCI coverage they can afford.

they remain close to the consumption floor (the left panel of Figure 6.8). On the other hand, a higher insurance coverage can significantly improve the consumption level in the severely disabled state (the right panel of Figure 6.8), lifting the lifetime utility. As a result, retirees are willing to purchase a high insurance coverage when their liquid wealth is very limited.

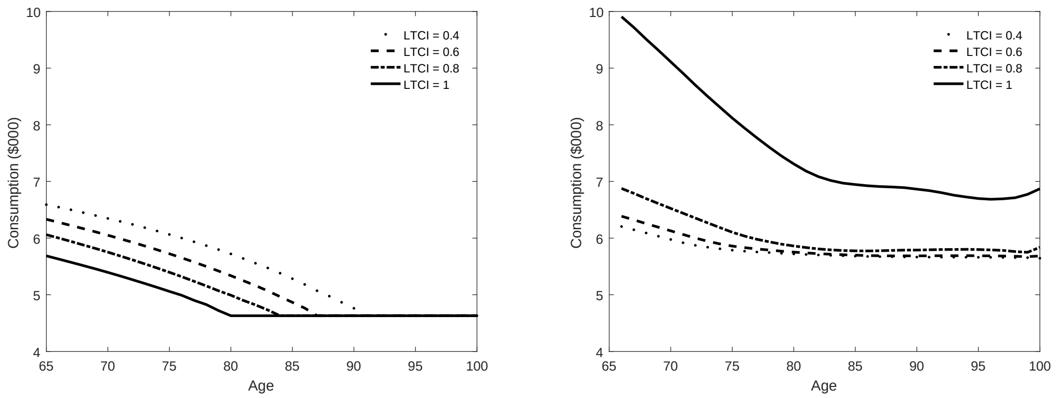


Figure 6.8. Simulated average consumption (excluding healthcare costs) paths of individuals at different health states: (Left Panel) healthy; (Right Panel) severely disabled. The annuitisation rates are zero in both panels. Individuals are endowed with $180,000 liquid wealth and $70,000 housing wealth.

In terms of demand for annuities, Table 6.8 shows that when life annuities alone are accessible to retirees, home equity increases the optimal proportion of liquid wealth to be annuitised unless the liquid wealth is too low (where no annuitisation is optimal) or total wealth is high (where full annuitisation is optimal even without housing wealth). This finding is consistent with the base case analysis. When both annuities and LTCI are available in the market, home equity can increase or decrease demand for life annuities depending on the amount of liquid wealth. Compare the optimal annuitisation rates between scenarios with and without home equity. When liquid wealth is sufficiently high (e.g. $360,000 liquid wealth out of $500,000 total wealth), the presence of home equity increases the optimal proportion of liquid assets to be annuitised and vice versa. In the model the risk of uncertain healthcare costs is more severe than the risk of outliving one's financial resources, so retirees value LTCI more than life annuity. When allocating the liquid assets between annuities and LTCI, they are willing to satisfy the demand for LTCI at the cost of a lower annuitisation level. The presence of home equity for a given level of total wealth decreases the amount of liquid wealth available, so retirees might reduce the optimal annuitisation rate (as a percentage of liquid wealth) to fulfil demand for LTCI. Compare the demand for annuities among scenarios with home equity controlling for the level of total wealth. The optimal annuitisation rate decreases as home equity value increases. An increasing home equity reduces both the spending on LTCI and the amount of liquid wealth. The net effect is that the spending on LTCI as a proportion of liquid wealth increases, so the optimal annuitisation rate, as a percentage of liquid wealth, decreases.

6.4 Conclusions

The high home ownership rate among the elderly and the significance of home equity in household portfolios among retired homeowners suggest the importance of home equity in retirement planning. I study the impact of housing wealth on the demand for life annuities and LTCI in a lifecycle framework. The individual chooses an annuitisation rate and LTCI coverage at retirement, and consumption over the course of retirement. Upon becoming severely disabled, the retired homeowner will liquidate her home equity and move to a long-term care facility. I use the Epstein-Zin utility model to separately identify an individual's risk aversion and EIS. Retirees face multiple sources of risk from uncertain healthcare costs, uncertain lifespan, and house price shocks.

The results show that the presence of home equity typically increases the optimal annuitisation rate when life annuities alone are available in the market. Prior studies show that precautionary savings for sizeable health expenditures (Sinclair and Smetters, 2004; Turra and Mitchell, 2008) and bequest motive (Lockwood, 2012) are among a number of factors that can dampen demand for life annuities. For retired homeowners who tend to sell the property at the time of moving into a nursing home, home equity is both a form of precautionary savings and bequest. The presence of home equity therefore lowers the barrier to annuitisation. When retirees have access to both life annuities and LTCI, the presence of home equity can enhance demand for annuities if the retiree has sufficient liquid assets. Otherwise, the spending on LTCI can impair demand for life annuities.

The demand for LTCI is generally crowded out by home equity since the liquidation of housing wealth tends to be highly correlated with the payment of LTCI.

It is important to separately identify risk aversion and EIS. A higher degree of risk aversion and a lower level of EIS have opposite effects on the demand for life annuities and LTCI. Since the power utility model imposes an inverse relationship on risk aversion and EIS, the model reflects preference of only a small group of individuals, putting a large majority of consumers at risk of mis-allocating their wealth for retirement products. It is therefore more appropriate to apply the Epstein-Zin model to individuals with various levels of risk aversion and EIS.

This chapter has practical implications on the offering of retirement products. Both life annuities and LTCI are effective instruments to manage post-retirement risks and to maintain living standard at retirement. For a given wealth level, the proportion of illiquid home equity in the portfolio can have a large impact on demand for annuities and LTCI. It is therefore important to differentiate between homeowners and non-homeowners when providing the products.

Chapter 7

Conclusions

Demographic changes and pension scheme transitions have exposed individuals to great challenges in financing their retirement. This thesis studies two important sources of savings for retirement, namely employment-based pension and housing. The prevalence of defined contribution (DC) plans in occupational pension schemes puts members at risk of accumulating insufficient wealth in the fund. Portfolio insurance strategies explicitly protect against downside risk, and prove to be optimal under certain conditions. Several papers have applied portfolio insurance strategies to DC pension management, and a number of papers have explored the optimal DC pension management that provides a minimum guarantee. However, little research has been done on how portfolio insurance strategies perform in a DC pension fund that targets an inflation- and longevity-protected annuity.

Chapter 4, building on some of the empirical results in Chapter 3, applies the option-based and constant proportion portfolio insurance strategies to managing target annuitisation funds. The portfolio weights in the equity fund show a downward trend for members joining the fund before mid-30s, in line with the lifestyle investment strategy. Members joining the fund after mid-30s have lower amounts of contributions, which are in the form of safe assets. Their portfolio weights in the equity fund tend to increase over time. Overall, the portfolio weights are highly volatile due to the volatility of the equity fund. The numerical simulations assume annual contributions to the fund and rebalancing annually, so it is possible that the terminal fund value falls short of the target. The expected shortfall amount remains under 5% of the target for both strategies in different scenarios of equity fund volatility and contribution amount. The constant proportion strategy gives lower probabilities of shortfall in the baseline analysis, while the option-based strategy provides more robust level of protection in unfavourable scenarios. In addition, the option-based strategy often leads to higher terminal values at retirement.

Among savings outside of occupational pension, housing is usually the most important part. Owner-occupied property serves a dual purpose: a consumption good and an investment asset. The role of consumption goods insures homeowners against rental fluctuations, and home equity can be unlocked to fund non-housing consumption and healthcare costs. However, housing tenure choice is not often considered in conjunction with retirement planning, and little is known on how housing wealth influences retirees' consumption and demand for financial products.

Chapter 5 studies housing tenure choice during pre-retirement phase while bearing in mind the importance of housing in retirement planning. It uses a vector autoregressive (VAR) process to generate economic scenarios, including house prices, rental costs, and labour income. Monte Carlo simulations are performed to compare how different ages of purchasing a residential property affect one's consumption, savings for retirement, and lifetime utility.

Purchasing the property earlier typically leads to greater savings at retirement due to lower mortgage balances and higher proportions of liquid wealth invested in risky assets. On the other hand, deferring the property purchase to an older age is more attractive in terms of lifetime utility. Purchasing a home property involves transferring a substantial amount of liquid assets to illiquid housing wealth. This reduces non-housing consumption, impairing lifetime utility. And the negative impact is more significant if property is bought at a younger age. Rental costs constrain spending on non-housing consumption and slow down wealth accumulation, so renting throughout working life leads to low savings for retirement and has large welfare costs.

Chapter 6 examines how presence of home equity affects demand for life annuities and long-term care insurance (LTCI) in a lifecycle model with an Epstein-Zin utility function. A retired homeowner chooses LTCI coverage and annuitisation rate at the point of retirement, and chooses consumption level over the course of retirement. Her health transition is modelled by a Markov process with four state: healthy, mildly disabled, severely disabled, and dead. Becoming severely disabled is associated with liquidating the home equity and moving to a long-term care facility. Apart from health shocks, she also faces the risk of uncertain lifespan and uncertain house prices.

The presence of housing lowers the barrier to annuitisation, since home equity serves as precautionary savings to cover uncertain healthcare costs and is a form of bequest. When retirees have access to life annuities only, home equity increases annuitisation rate. When LTCI is also available in the market, the impact of home equity depends on the amount of liquid assets as spending on LTCI could reduce demand for annuities. Given home property is usually liquidated in the event of becoming severely disabled, the presence of home equity reduces optimal LTCI coverage regardless of the availability of annuities. The sensitivity analysis on preference parameters shows the importance of separating risk aversion and elasticity of intertemporal substitution (EIS). A lower level of risk aversion implies a lower demand for LTCI and a higher optimal annuitisation rate. Individuals with a lower level of EIS are less willing to substitute consumption over time, demanding more annuities and less LTCI. A power utility function, imposing an inverse relationship between risk aversion and EIS, would mis-specify demand for annuities and LTCI since individuals tend to have relative risk aversion greater than the reciprocal of the EIS (Brown and Kim, 2013).

This thesis can be extended in several directions. The contribution to DC pension funds is usually a proportion of one's salary that is subject to random fluctuations. The study on DC pension fund management can incorporate labour income risk to assess whether the risk management feature of portfolio insurance strategies is sensitive to alternative labour income profiles. Chapter 4 studies the asset allocation strategies from the perspective of a DC pension fund manager. It would also be interesting to investigate the optimal funding problem from a fund member's perspective. Making contributions to the pension fund creates a trade-off between pre-retirement and post-retirement consumption since savings in the fund are usually illiquid until reaching retirement. Future work can consider how much a member should contribute to maximise his lifetime utility. Chapter 5 studies a housing tenure choice problem by comparing the consumption, wealth, and utility outcome among predetermined ages of purchasing a home property. This greatly simplifies the analysis and gives an intuitive result presentation, but it does not allow individuals to respond dynamically to changing economic and financial circumstances. Future work can be done to address this issue. Chapter 6 assumes that home equity is liquidated once the retiree moves to a long-term care facility, so home equity is both a form of precautionary savings and bequest, lowering the barrier to annuitisation. Future work can be done to endogenise the decision of selling the property. This would help to determine which of the two roles has a larger impact.

Appendix A

HILDA Disclaimer Notice

The Household, Income and Labour Dynamics in Australia (HILDA) Survey was initiated and is funded by the Australian Government Department of Social Services (DSS), and is managed by the Melbourne Institute of Applied Economic and Social Research (Melbourne Institute). The findings and views based on these data should not be attributed to either DSS or the Melbourne Institute.

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